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Analysis of Rectangular Building Frames and Contour Line Representation of Loading Effects.

By

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With the increasing complexity of modern frame work with stiff connecting joints composed of rectangular forms, the relative practical importance of the proper treatment of indeterminate stresses has greatly increased.

The analysis of these stresses, when viewed from a theoretical standpoint, is so complicated that for many years there have been unknown factors in the scope of the problems of high built structures.

In order to overcome this difficulty it seems necessary to develop some practical and workable method that is simple in its theoretical analysis. At the same time it should furnish a means of calculation, that both lessens the mathematical difficulties and shortens the tedious labour of detailed computations.

The speed and accuracy of the Mechanical Tabulation Method herewith presented provides a substantial contribution towards the attainment of this much desired result.

In the following pages the author describes some important properties with regard to the elastic distortion of the frame work, composed of rectangular elements; to the best of the author's knowledge the contour line representation of the loading effects is essentially new.

A little familiarity with the Mechanical Tabulation Method, when followed by the process of Repeated Trials will be sufficient to show its most striking advantages and to establish its superiority over the old methods of solution.

1. Fundamental Equations—In a previous paper the author developed an exact solution of building frames composed of any number of bays and stories under any system of vertical loads on girders and horizontal

loads at joints on the left side of the structure.

This solution may have been very instructive to show how easy it is mechanically to obtain and solve the simultaneous equations from which the redundancies are to be computed.

With the systematic arrangement of the suffix-number of the proposed nomenclature and the symmetrical property of the places where the coefficients of unknown quantities are to be written, the proposed method makes no mistakes in writing the table of equations. That is to say, it gives the entire group of equilibrium equations of unknown quantities without any computation and solves a great variety of statically-indeterminate problems with remarkable speed and accuracy of calculation.

Indeed in general it may be considered the first step of computation to obtain all the simultaneous equations from which the redundancies are to be determined and it seems to be the second to find the answers, solving the equations simultaneously.

The Mechanical Tabulation Method contributes to the former and the process of Repeated Trials proposed by the author, especially in the case of horizontal loadings, is very serviceable in respect to the latter, showing a process accurate enough to obtain practically true values of redundant stresses. If a member $A-B$ restrained at the ends and subjected to flexure carries any system of intermediate loads, the moment at the end A is expressed by the well known equation:

$$(1) \quad M_{ab} = 2EK \left\{ 2\theta_a + \theta_b - 3 \frac{d}{L} \right\} \mp C_{ab},$$

in this equation, we denote by M_{ab} the moment at the end A of the member AB ; θ_a and θ_b the joint-rotation angle of the ends A and B , respectively, of the member; d the deflection of one end of the member AB relative to the other; L the length of the member AB ; C_{ab} the fixed-ended moment at the end A of the member AB , fixed at both ends; E the modulus of elasticity of the material; K the moment of inertia of the section divided by the length of the member AB . For different systems of loads, C_{ab} is given in Memoirs Vol. I, No. 3, Tables B and C.

The resisting moment on a section is considered positive when the couple acts in a clockwise direction upon the portion of the member considered; the joint-rotation angle θ is considered positive when the angle has been turned clockwise, measured from its initial position; the member-revolution angle d/L is also positive to the revolution in the clockwise

direction from the initial position of the member.

For the analysis of high built frames by the Mechanical Tabulation Method, we introduce the following equation:

$$(2) \quad M_{ab} = K \{ 2\varphi_a + \varphi_b + \mu \} \mp C_{ab}.$$

In this equation,

$$(3) \quad \begin{cases} \varphi_a = 2E\theta_a, \\ \varphi_b = 2E\theta_b, \\ \mu = -6E \frac{d}{L}. \end{cases}$$

For a frame with equal K , we express M_{ab} in the following form:

$$(4) \quad \left\{ \begin{array}{l} M_{ab} = 2\varphi_a + \varphi_b + \mu \mp C_{ab} \\ \text{where} \\ \varphi_a = 2EK\theta_a, \quad \varphi_b = 2EK\theta_b, \\ \mu = -6EK \frac{d}{L}. \end{array} \right.$$

For the purpose of obtaining M_{ab} , it is more practical to consider φ and μ rather than θ and d/L as unknowns.

In this paper we treat only the problem of building frames with no yieldings at the foundation. Therefore all the girders remain always horizontal, while the columns deflect due to the system of asymmetrical vertical loads and horizontal ones.

To obtain the equations from which the unknown slope and deflection terms can be determined, we have two kinds of statical conditions of equilibrium: one of these is of the equilibrium under the action of the moments at every point of intersection of the neutral axes of girders and columns, considering the point of intersection as a free body; the other condition is of the equilibrium under the action of the shears and moments at the two extremities of all the columns in a story, considering together all of the columns in a story as a free body.

From the former condition of equilibrium, each joint gives one equation; from the latter each story gives one equation, i.e. for an n -storied bent with m -points of intersection of the neutral axes of girders and columns, the number of equations obtained, from which the slopes and the deflections can be determined, is $m+n$. The unknown quantities in the

equations obtained are the changes in the slopes and the horizontal deflections of the columns. The changes in the slope at a joint, by the rigid connection, are equal for all of the members which meet at the joint; and the horizontal deflections of the columns in a story are equal in the case in question. We obtain therefore $m+n$ equations in which $m+n$ unknowns are contained. Solving these equations we can determine the unknowns and then the moments required.

2. Joint-Equilibrium Equations—For the joint S shown in Fig. 1, the equilibrium condition of the total moment gives:

$$(5) \quad \begin{cases} \Sigma M_S = 0, \\ \text{or} \\ M_{Sa} + M_{Sb} + M_{Sc} + M_{Sd} = 0. \end{cases}$$

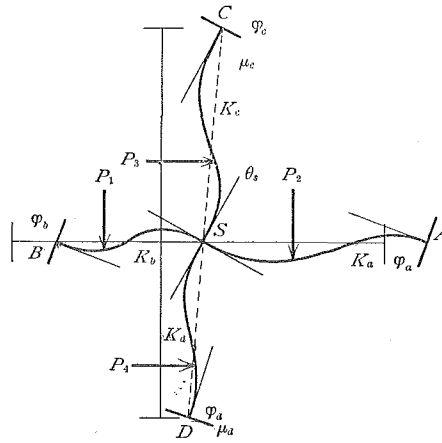


Fig. 1.

Substituting equation (2) in (5) gives:

$$(6) \quad \begin{cases} \varphi_s \rho_s + \varphi_a K_a + \varphi_b K_b + \varphi_c K_c + \varphi_d K_d + \mu_c K_c + \mu_d K_d \\ = (C_{Sa} - C_{Sb}) + (C_{Sc} - C_{Sd}) \\ \text{where} \\ \rho_s = 2(K_a + K_b + K_c + K_d). \end{cases}$$

We call the above equation "Joint-Equilibrium Equation" which is memorized easily by the following graphical representation:

$$(7) \quad \begin{aligned} & \varphi_c K_c + \mu_c K_c \\ & + \\ & \varphi_b K_b + \varphi_s \rho_s + \varphi_a K_a = C_{sa} - C_{sb} \\ & + \\ & \varphi_d K_d + \mu_d K_d \end{aligned}$$

(See Fig. 1)

If the horizontal intermediate loads vanish

$$C_{s2} = C_{sa} = 0,$$

and equation (7) becomes:

$$(8) \quad \left\{ \begin{aligned} & \varphi_c K_c + \mu_c K_c \\ & + \\ & \varphi_b K_b + \varphi_s \rho_s + \varphi_a K_a = \tau_s, \\ & + \\ & \varphi_d K_d + \mu_d K_d \end{aligned} \right. \quad \begin{aligned} & \text{where} \\ & \tau_s = C_{s\text{-right-hand}} - C_{s\text{-left-hand}} \end{aligned}$$

For symmetrical frames with symmetrical loadings, μ_c and μ_d vanish and equation (8) becomes:

$$(9) \quad \begin{aligned} & \varphi_c K_c \\ & + \\ & \varphi_b K_b + \varphi_s \rho_s + \varphi_a K_a = \tau_s. \\ & + \\ & \varphi_d K_d \end{aligned}$$

For a frame with equal K , we obtain, from equation (4), the following equations (10) and (11) which correspond to equations (8) and (9) respectively:

$$(10) \quad \left\{ \begin{aligned} & \varphi_c + \mu_c \\ & + \\ & \varphi_b + \varphi_s(2n) + \varphi_a = \tau_s, \\ & + \\ & \varphi_d + \mu_d \end{aligned} \right. \quad \begin{aligned} & \text{where} \\ & n = \text{no. of the members which come in at the joint } S. \end{aligned}$$

$$(11) \quad \left\{ \begin{aligned} & \varphi_c \\ & + \\ & \varphi_b + \varphi_s(2n) + \varphi_a = \tau_s. \\ & + \\ & \varphi_d \end{aligned} \right.$$

3. Story-Equilibrium Equations—We consider two horizontal sections at both ends of the columns at the r -th. story of a building frame

subjected to any system of vertical loads and horizontal loads concentrated at any joints on the left side of the structure (Fig. 2).

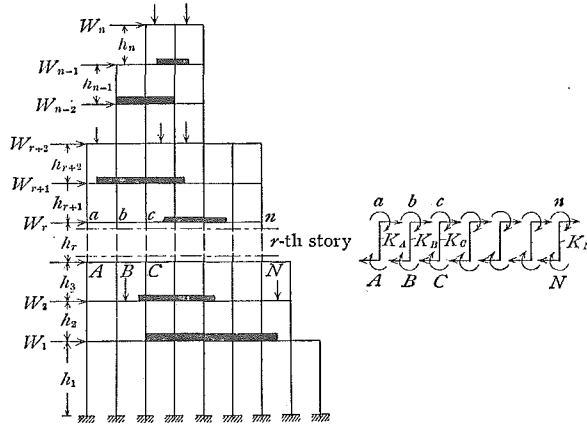


Fig. 2.

The equilibrium condition of the r -th. story as a whole gives:

$$(12) \left\{ \begin{array}{l} M_{aA} + M_{bB} + M_{cC} + \dots + M_{nN} \\ M_{Aa} + M_{Bb} + M_{Cc} + \dots + M_{NN} \end{array} \right\} + l_r \sum_{k=r}^n W_k = 0.$$

Substituting equation (2) in equation (12) gives:

$$(13) \left\{ \begin{array}{l} \varphi_a K_A + \varphi_b K_B + \varphi_c K_C + \dots + \varphi_n K_N \\ \varphi_A K_A + \varphi_B K_B + \varphi_C K_C + \dots + \varphi_N K_N \end{array} \right\} + \mu_r X_r = R_r,$$

where

$$\left\{ \begin{array}{l} X_r = \frac{2}{3} (K_A + K_B + K_C + \dots + K_N), \\ R_r = -\frac{l_r}{3} \sum_{k=r}^n W_k. \end{array} \right.$$

We call this equation of equilibrium "Story-Equilibrium Equation" which is memorized easily by the above representation (see Fig. 2).

4. Solution of Simultaneous Elastic Equations—In the course of the entire calculation of the multiple redundancies, it may be said that it is the first step of the work to determine the equilibrium equations to be used in order to find the unknowns and it is the second and final step of the work to solve the simultaneous equations and get the results.

The Mechanical Tabulation Method serves, as mentioned frequently,

for the first step and the process of Repeated Trials which will be explained in this paragraph, serves for the second.

For a comparatively complex problem of multiple storied bent with horizontal loads, the ordinary process of successive substitution in current application may not give good results and it takes very much time to get values which may be considered approximately true; while for the case of all conditions of symmetry under vertical loads, the ordinary method of successive substitution succeeds in getting at approximately true values very easily. Yet naturally, for the latter case, the proposed process furnishes a more rapid method to obtain the practically true value of unknowns. Here, however, it may be noticed that for the case of few redundancies, the process of exact elimination is good. In the case of a very large number of unknowns, the proposed method with which familiarity is easily gained offers one of the best means of solution, always getting good results with little effort of work.

a. METHOD OF SOLUTION I, EQUATIONS CONTAINING
 φ AND μ .

In the case of a rectangular building frame, the maximum number of unknowns contained in joint-equilibrium equations may be considered five slope and two deflection terms; in story equilibrium equations, there may be considered one deflection term and as many slope terms as the number of extremities of the columns, except in the first story.

For the sake of simplicity, we take the following equations:

$$A_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 = m_1. \dots\dots\dots (1)$$

$$b_1 X_1 + B_2 X_2 + b_3 X_3 + b_4 X_4 + b_5 X_5 = m_2. \dots\dots\dots (2)$$

$$c_1 X_1 + c_2 X_2 + C_3 X_3 + c_4 X_4 + c_5 X_5 = m_3. \dots\dots\dots (3)$$

$$d_1 X_1 + d_2 X_2 + d_3 X_3 + D_4 X_4 + d_5 X_5 = m_4. \dots\dots\dots (4)$$

$$e_1 X_1 + e_2 X_2 + e_3 X_3 + e_4 X_4 + E_5 X_5 = m_5. \dots\dots\dots (5)$$

These equations are obtained from Table I which may be directly determined by the Mechanical Tabulation Method. In actual cases X_1 to X_3 represent the slope terms φ ; X_4 and X_5 represent the deflection terms μ ; to explain the method of solution of such simultaneous equations is the object of this paragraph.

Let the five coefficients represented by large letter A, B, C, D, E be assumed greater than the remaining coefficients of small letters represented

by a, b, c, d, e .

TABLE I.

Equation	Left-Hand Member of Equation					Right-Hand Member of Equation
	X_1	X_2	X_3	X_4	X_5	
(1)	A	a_2	a_3	a_4	a_5	m_1
(2)	b_1	B_2	b_3	b_4	b_5	m_2
(3)	c_1	c_2	C_3	c_4	c_5	m_3
(4)	d_1	d_2	d_3	D_4	d_5	m_4
(5)	e_1	e_2	e_3	e_4	E_5	m_5

Substituting X_1' into all the unknowns in equation (1), except X_4 and X_5 gives:

$$A_1 X_1' + a_2 X_1' + a_3 X_1' = m_1 - a_4 X_4 - a_5 X_5,$$

from which we get:

$$\left. \begin{aligned} X_1' &= \gamma_1 - \delta_1 X_4 - \eta_1 X_5, \\ \gamma_1 &= \frac{m_1}{A_1 + a_2 + a_3}, \\ \delta_1 &= \frac{a_4}{A_1 + a_2 + a_3}, \\ \eta_1 &= \frac{a_5}{A_1 + a_2 + a_3}. \end{aligned} \right\} \dots \dots \dots (a)$$

where

In a similar way

$$b_1 X_2' + B_2 X_2' + b_3 X_2' = m_2 - b_4 X_4 - b_5 X_5,$$

which gives

$$\left. \begin{aligned} X_2' &= \gamma_2 - \delta_2 X_4 - \eta_2 X_5, \\ \gamma_2 &= \frac{m_2}{b_1 + B_2 + b_3}, \\ \delta_2 &= \frac{b_4}{b_1 + B_2 + b_3}, \\ \eta_2 &= \frac{b_5}{b_1 + B_2 + b_3}. \end{aligned} \right\} \dots \dots \dots (b)$$

where

Similarly

$$\left. \begin{aligned} X_3' &= \gamma_3 - \delta_3 X_4 - \eta_3 X_5, \\ \text{where } \gamma_3 &= \frac{m_3}{c_1 + c_2 + C_3} \\ \delta_3 &= \frac{c_4}{c_1 + c_2 + C_3} \\ \eta_3 &= \frac{c_5}{c_1 + c_2 + C_3} \end{aligned} \right\} \dots\dots\dots (c)$$

While, substituting these values of X' into equation (4) gives:

$$d_1 X_1' + d_2 X_2' + d_3 X_3' + D_4 X_4 + d_5 X_5 = m_4$$

or with the results above obtained, this equation gives:

$$\begin{aligned} d_1(\gamma_1 - \delta_1 X_4 - \eta_1 X_5) + d_2(\gamma_2 - \delta_2 X_4 - \eta_2 X_5) \\ + d_3(\gamma_3 - \delta_3 X_4 - \eta_3 X_5) + D_4 X_4 + d_5 X_5 = m_4. \end{aligned}$$

From this equation we get:

$$X_4 \left\{ D_4 - \sum_{r=1}^3 d_r \delta_r \right\} + X_5 \left\{ d_5 - \sum_{r=1}^3 d_r \eta_r \right\} = m_4 - \sum_{r=1}^3 d_r \gamma_r \dots\dots\dots (d)$$

In a similar way, from equation (5) we get:

$$e_1 X_1' + e_2 X_2' + e_3 X_3' + e_4 X_4 + E_5 X_5 = m_5$$

Or substituting the values of X' above obtained gives:

$$\begin{aligned} e(\gamma_1 - \delta_1 X_4 - \eta_1 X_5) + e_2(\gamma_2 - \delta_2 X_4 - \eta_2 X_5) \\ + e_3(\gamma_3 - \delta_3 X_4 - \eta_3 X_5) + e_4 X_4 + E_5 X_5 = m_5 \end{aligned}$$

Or from this equation we have:

$$X_4 \left\{ e_4 - \sum_{r=1}^3 e_r \delta_r \right\} + X_5 \left\{ E_5 - \sum_{r=1}^3 e_r \eta_r \right\} = m_5 - \sum_{r=1}^3 e_r \gamma_r \dots\dots\dots (e)$$

Applying the method of substitution for equation (d) gives:

$$X_4' \left\{ D_4 - \sum_{r=1}^3 d_r \delta_r \right\} + X_4' \left\{ d_5 - \sum_{r=1}^3 d_r \eta_r \right\} = m_4 - \sum_{r=1}^3 d_r \gamma_r$$

This equation determines X_4' , viz.

$$X_4' = \frac{m_4 - \sum_{r=1}^3 d_r \gamma_r}{D_4 + d_5 - \sum_{r=1}^3 d_r (\delta_r + \eta_r)} = A_4 \dots\dots\dots (f)$$

Similarly from equation (e) we get:

$$X_5' \left\{ e_4 - \sum_{r=1}^3 e_r \delta_r \right\} + X_5' \left\{ E_5 - \sum_{r=1}^3 e_r \eta_r \right\} = m_5 - \sum_{r=1}^3 e_r \gamma_r$$

Or

$$X_5' = \frac{m_5 - \sum_{r=1}^3 e_r \gamma_r}{e_4 + E_5 - \sum_{r=1}^3 e_r (\delta_r + \eta_r)} = \Delta_5 \dots \dots \dots (g)$$

With these values of X_4' and X_5' we may therefore determine X_1' , X_2' , X_3' from equations (a), (b) and (c):

$$\left. \begin{aligned} X_1' &= \gamma_1 - \delta_1 X_4' - \eta_1 X_5' = \gamma_1 - \delta_1 \Delta_4 - \eta_1 \Delta_5 \\ X_2' &= \gamma_2 - \delta_2 X_4' - \eta_2 X_5' = \gamma_2 - \delta_2 \Delta_4 - \eta_2 \Delta_5 \\ X_3' &= \gamma_3 - \delta_3 X_4' - \eta_3 X_5' = \gamma_3 - \delta_3 \Delta_4 - \eta_3 \Delta_5 \end{aligned} \right\} \dots \dots \dots (h)$$

We call the values above obtained the first approximate value of X' , with which the process of Repeated Trials may be continued as follows:

Representing the second approximate values of X with X'' , we find them from the following equations:

$$A_1 X_1' + a_2 X_2' + a_3 X_3' + a_4 X_4' + a_5 X_5' = m_1 \dots \dots \dots (A)$$

$$b_1 X_1'' + B_2 X_2' + b_3 X_3' + b_4 X_4' + b_5 X_5' = m_2 \dots \dots \dots (B)$$

$$c_1 X_1'' + c_2 X_2'' + C_3 X_3' + c_4 X_4' + c_5 X_5' = m_3 \dots \dots \dots (C)$$

$$d_1 X_1'' + d_2 X_2'' + d_3 X_3'' + D_4 X_4' + d_5 X_5' = m_4 \dots \dots \dots (D)$$

$$e_1 X_1'' + e_2 X_2'' + e_3 X_3'' + e_4 X_4'' + E_5 X_5' = m_5 \dots \dots \dots (E)$$

That is to say, from equation (A) we get X_1'' . Similarly we may find X_2'' from equation (B), by using the value of X_1'' above calculated and also by using X' which are known quantities. Just in a similar way, we find X_3'' to X_5'' from equations (C), (D) and (E), considering X that has the largest coefficient as the unknown quantity.

In the current method of successive substitutions, we generally apply the mere first approximate values, without using the second approximate values which have already become known. While, for the purpose of rapidly getting at true values, we mix the first and second approximate values, as shown in equations (A) to (E), where we find X'' below the line $A-B-C-D-E$ and X' above the same diagonal line. This process is available by slide-rules and is repeated successively.

b. METHOD OF SOLUTION II, EQUATIONS
CONTAINING φ ONLY.

For the analysis of symmetrical building frames with symmetrical vertical loads, there come in only the slope terms and no deflection terms, and therefore the process of solution is very much simplified, considering in this case one kind of unknowns i.e. slope terms instead of two kinds as were concerned in the former case.

To obtain the first approximate values, we get from equation (1):

$$A_1 X_1' + a_2 X_1' + a_3 X_1' + a_4 X_1' + a_5 X_1' = m_1$$

which gives:

$$X_1' = \frac{m_1}{A_1 + a_2 + a_3 + a_4 + a_5}.$$

Similarly from equation (2) we get:

$$X_2' = \frac{m_2}{b_1 + B_2 + b_3 + b_4 + b_5}$$

etc.

The first approximate values are obtained by solving each equation, considering the unknown which has the largest coefficients as the unknown to be solved, and dividing the right-hand member of the equation by the total sum of the coefficients in the left-hand member of the same.

On having determined the first approximate values of unknowns, we continue the method of substitutions, quite in the same way as mentioned in the method of solution I.

EXAMPLE FOR THE METHOD OF SOLUTION I.

Fig. 3 shows a symmetrical three-span frame, twenty stories high. The frame is subjected to a horizontal load at each joint on the left side of the structure. It is required to find the joint moments.

The values of K of all the members are written in Fig. 3 on the corresponding members. The Mechanical Tabulation Method gives Table II from which the simultaneous equations to be used in determining the unknowns may be written without any computations.

Calculating all the coefficients of the unknowns in the left-hand member of the equations and also all the constant terms in the right-hand member of the equations, and substituting those values above calculated

in Table II, we have Table III from which the unknowns are to be determined.

Applying the method of solution I, we get the following equation from the first equation of Table III:

$$183.8 \theta_1 + 30.5 \theta_2 + 35.6 \theta_4 = -25.8 \mu_1 - 35.6 \mu_2$$

or assuming:

$$183.8 \theta_1' + 30.5 \theta_1' + 35.6 \theta_1' = -25.8 \mu_1 - 35.6 \mu_2$$

which gives

$$\theta_1' = -0.10324 \mu_1 - 0.14246 \mu_2.$$

(coefficient: 0.0001 being disregarded)

In a similar way from the second equation of Table III we get:

$$\theta_2' = -0.07131 \mu_1 - 0.09840 \mu_2.$$

In the same way we continue the calculation until we obtain θ_{40}' . The calculation results are tabulated in Table IV.

With the values of θ' in Table IV, from equation (41) in Table III we get:

$$25.8 (-0.10324 \mu_1 - 0.14246 \mu_2) + 25.8 (-0.07131 \mu_1 - 0.09840 \mu_2) + 34.4 \mu_1 = -58.5$$

or

$$29.90 \mu_1 - 6.21 \mu_2 = -58.5.$$

In a similar way, we get from equation (42):

$$35.6 (\theta_1' + \theta_2' + \theta_3' + \theta_4') + 47.5 \mu_2 = -39.4$$

or substituting the values of θ' in Table IV gives:

$$-6.21 \mu_1 + 30.88 \mu_2 - 8.01 \mu_3 = -39.4.$$

In just the same way we continue the calculation until we obtain the equation which contains μ_{19} and μ_{20} . The calculation results are shown in Table V.

Next, from the first equation in Table V we have:

$$29.90 \mu_1 - 6.21 \mu_2 = -58.5$$

or

$$29.9 \mu_1' - 6.21 \mu_1' = -58.5,$$

which gives:

$$\mu_1' = -2.46$$

In a similar way, from the second equation in Table V we get:

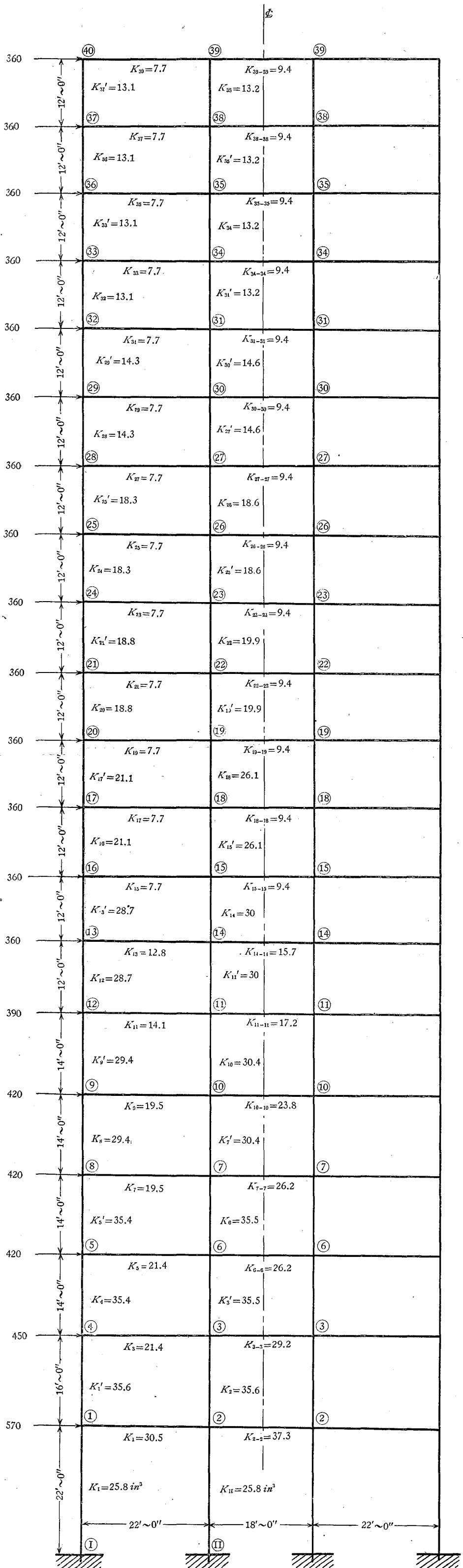


Fig. 3.

Table II.

[illegible]

N.B. In this table we denote by

$$\mu = -3R, \quad J = 2 \sum \left(\frac{I_g}{l} + \frac{I_c}{h} \right) \quad \text{for all columns and girders which come in a joint,}$$

$$(J) = J + \left\{ \frac{I}{I} \quad \text{for the central girder} \right\},$$

$$q_r = -\frac{Q_r \cdot h_r}{6E}.$$

θ , K and X are represented in equations (1) and (13) respectively.

Table III.

[illegible]

Table IV.

Coefficient of 0.0001

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}	μ_{20}
θ_1'	0.10324	0.14246																		
θ_2'	0.07131	0.09840																		
θ_3'		0.09751	0.09723																	
θ_4'		0.12843	0.12771																	
θ_5'			0.12798	0.12798																
θ_6'			0.09978	0.09978																
θ_7'				0.10603	0.09080															
θ_8'				0.13998	0.11625															
θ_9'					0.12516	0.12516														
θ_{10}'					0.09734	0.09734														
θ_{11}'						0.11051	0.10905													
θ_{12}'						0.13573	0.13250													
θ_{13}'							0.13628	0.13628												
θ_{14}'							0.11299	0.11299												
θ_{15}'								0.13661	0.11885											
θ_{16}'								0.16638	0.12232											
θ_{17}'									0.14095	0.14095										
θ_{18}'									0.12554	0.12554										
θ_{19}'										0.13788	0.10512									
θ_{20}'										0.14776	0.13165									
θ_{21}'											0.13834	0.13834								
θ_{22}'											0.11658	0.11658								
θ_{23}'												0.11930	0.11151							
θ_{24}'												0.13988	0.13616							
θ_{25}'													0.13770	0.13770						
θ_{26}'													0.11418	0.11418						
θ_{27}'														0.12326	0.09675					
θ_{28}'														0.15136	0.11828					
θ_{29}'															0.13131	0.13131				
θ_{30}'															0.10511	0.10511				
θ_{31}'																0.10839	0.09800			
θ_{32}'																0.13580	0.12441			
θ_{33}'																	0.12881	0.12881		
θ_{34}'																	0.10115	0.10115		
θ_{35}'																		0.10115	0.10115	
θ_{36}'																		0.12881	0.12881	
θ_{37}'																			0.12881	0.12881
θ_{38}'																			0.10115	0.10115
θ_{39}'																				0.14521
θ_{40}'																				0.20994





Table V.

Equation	Left-Hand Member of Equation																				Right-Hand Member of Equation Coefficient of 0.0001
	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}	μ_{20}	
1	29.90	-6.21																			-58.5
2	-6.21	30.88	-8.01																		-39.4
3		-8.01	31.15	-8.07																	-52.3
4			-8.07	30.41	-7.34																-30.3
5				-7.34	27.08	-6.64															-28.2
6					-6.64	25.91	-7.21														-26.2
7						-7.21	24.82	-7.30													-20.9
8							-7.30	23.01	-7.08												-19.4
9								-7.08	19.57	-6.25											-17.9
10									-6.25	18.53	-5.52										-16.4
11										-5.52	16.31	-4.92									-14.9
12											-4.92	15.88	-4.78								-13.4
13												-4.78	15.39	-4.64							-11.9
14													-4.64	14.89	-3.96						-10.4
15														-3.96	12.78	-3.41					-8.94
16															-3.41	12.36	-3.21				-7.45
17																-3.21	11.65	-3.02			-5.96
18																	-3.02	11.55	-3.02		-4.47
19																		-3.02	11.55	-3.02	-2.98
20																			-3.02	9.91	-1.49



Table VI.

Coefficient of 0.0001

Approximation	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}	μ_{20}
First Approximate Value	-2.46	-2.36	-2.05	-2.02	-2.15	-2.17	-2.02	-2.24	-2.86	-2.42	-2.53	-2.16	-1.99	-1.65	-1.65	-1.29	-1.09	-0.81	-0.54	-0.21
Second Approximate Value	-2.46	-2.36	-2.16	-2.06	-2.13	-2.15	-2.08	-2.34	-2.70	-2.55	-2.45	-2.21	-1.97	-1.73	-1.60	-1.33	-1.09	-0.82	-0.54	-0.23
Third Approximate Value	-2.45	-2.36	-2.18	-2.07	-2.12	-2.16	-2.13	-2.35	-2.65	-2.53	-2.44	-2.20	-1.99	-1.75	-1.59	-1.33	-1.10	-0.822	-0.545	-0.272
True Value	-2.4513	-2.3631	-2.1855	-2.0796	-2.1279	-2.1591	-2.1441	-2.3601	-2.6523	-2.5296	-2.4420	-2.1990	-1.9890	-1.7550	-1.6035	-1.3344	-1.0965	-0.8226	-0.5508	-0.2928

N.B. The calculation results shown in this table were obtained by using slide-rule.

Table VII.

Coefficient of 0.0001

Approximation	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}	θ_{19}	θ_{20}	θ_{21}	θ_{22}	θ_{23}	θ_{24}	θ_{25}	θ_{26}	θ_{27}	θ_{28}	θ_{29}	θ_{30}	θ_{31}	θ_{32}	θ_{33}	θ_{34}	θ_{35}	θ_{36}	θ_{37}	θ_{38}	θ_{39}	θ_{40}
First Approximate Value	0.59	0.40	0.42	0.56	0.52	0.40	0.40	0.53	0.54	0.42	0.46	0.56	0.58	0.48	0.64	0.72	0.74	0.66	0.60	0.69	0.65	0.54	0.48	0.57	0.50	0.41	0.36	0.44	0.38	0.31	0.25	0.31	0.24	0.19	0.14	0.17	0.10	0.08	0.03	0.04
Second Approximate Value	0.63	0.39	0.43	0.59	0.53	0.40	0.40	0.55	0.56	0.41	0.46	0.58	0.57	0.46	0.67	0.76	0.77	0.68	0.59	0.70	0.67	0.54	0.48	0.58	0.51	0.41	0.36	0.45	0.40	0.31	0.24	0.32	0.25	0.19	0.14	0.18	0.10	0.07	0.02	0.03
Third Approximate Value	0.624	0.383	0.440	0.594	0.545	0.41	0.42	0.546	0.540	0.403	0.467	0.585	0.586	0.471	0.662	0.748	0.750	0.664	0.604	0.700	0.654	0.534	0.480	0.584	0.515	0.418	0.362	0.453	0.393	0.304	0.246	0.321	0.252	0.189	0.136	0.179	0.101	0.076	0.025	0.036
True Value	0.6227	0.3809	0.4424	0.5967	0.5506	0.4160	0.4041	0.5508	0.5408	0.4014	0.4730	0.5894	0.6044	0.4846	0.6586	0.7446	0.7390	0.6526	0.6018	0.7051	0.6508	0.5346	0.4789	0.5842	0.5218	0.4224	0.3647	0.4578	0.3950	0.3030	0.2477	0.3219	0.2527	0.1903	0.1361	0.1801	0.1085	0.0835	0.0338	0.0517

N.B. The Calculation results shown in this table were obtained by using slide-rule.

Table XI.

Approximation	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}	θ_{19}	θ_{20}	θ_{21}	θ_{22}	θ_{23}	θ_{24}	θ_{25}	θ_{26}	θ_{27}	θ_{28}	θ_{29}	θ_{30}	θ_{31}	θ_{32}	θ_{33}	θ_{34}	θ_{35}	θ_{36}	θ_{37}	θ_{38}	θ_{39}	θ_{40}
First Approximate Value	810.3	172.5	166.7	730.5	732.1	169.9	181.4	800.7	862.0	194.1	217.1	934.9	961.5	224.1	265.4	1173.9	1127.2	244.3	269.5	1178.4	1241.7	300.6	308.1	1255.5	1269.7	315.9	342.7	1395.7	1549.5	374.5	387.0	1602.5	1659.2	400.4	400.4	1659.2	1659.2	400.4	671.7	2996.3
Second Approximate Value	931.6	98.1	115.8	780.7	784.4	117.9	127.4	876.0	943.0	130.0	164.6	1014.5	990.8	164.9	231.2	1296.2	1174.4	186.3	214.5	1250.5	1332.0	241.1	242.5	1229.1	1329.5	245.1	262.5	1497.7	1694.1	279.8	282.9	1736.1	1813.2	291.1	289.1	2122.9	1543.9	247.9	435.3	3847.7
Third Approximate Value	934.3	90.8	131.4	752.6	766.5	128.9	137.9	857.2	921.9	138.0	178.8	997.3	954.9	177.6	247.3	1282.6	1138.0	200.7	231.4	1226.6	1327.6	256.1	266.9	1311.5	1320.1	265.8	281.5	1466.1	1657.7	296.9	302.9	1660.8	1707.2	310.2	286.6	1807.4	1307.1	310.8	360.3	3927.8

Coefficient of E^{-1}

$$\mu_2' = -2.36$$

All the unknowns μ' are calculated in the same way and the results are tabulated in Table VI.

Substituting these values of μ' in Table VI in the equations of Table IV, determines the first approximate values of θ' , which are shown in Table VII.

In Tables VI and VII, there are tabulated the first, second and third approximate values of unknowns and also the true values.

These calculations have been made by a single person within ten hours, using a slide-rule.

For the final purposes of finding the joint moments, the calculation is as follows:

At joint 1 we get:

$$\begin{aligned} M_{12} &= 2EK_1(2\theta_1 + \theta_2) \\ &= 2(29000000)(30.5)\{2(0.624) + 0.383\}(0.0001) \\ &= 288000 \text{ in. lb.} \\ \\ M_{1I} &= 2EK_I(2\theta_1 + \mu_1) \\ &= 2(29000000)(25.8)\{2(0.624) - 2.45\}(0.0001) \\ &= -180000 \text{ in. lb.} \\ \\ M_{14} &= 2EK_1'(2\theta_1 + \theta_4 + \mu_2) \\ &= 2(29000000)(35.6)\{2(0.624) + 0.594 - 2.36\}(0.0001) \\ &= -107000 \text{ in. lb.} \end{aligned}$$

That is to say, the total sum of the negative moment is -287000 in lb. and the total sum of the positive moment is 288000 in lb.

For practical purposes, this discrepancy is not effective, but if it is necessary to obtain more accurate results, the additional computations should be made, corresponding to the required necessity. For the verification of the exactness of the proposed method, the Bulletin of the University of Illinois, No. 80 was very important. The author of the present paper gratefully acknowledges indebtedness to the authors of the Bulletin. In Table VIII, the joint moments are tabulated, where the values with bracket show the values quoted from the Bulletin.

TABLE VIII.

Joint Moments in 1000 in. lb.		
$M_{1-2} = 288 (287.8)$	$M_{9-10} = 168 (167.7)$	$M_{17-18} = 96 (95)$
$M_{1-4} = -107 (-107.5)$	$M_{9-12} = -83 (-83.2)$	$M_{17-20} = -41 (-42.4)$
$M_{1-11} = -180 (-178.8)$	$M_{9-8} = -85 (-84.5)$	$M_{17-16} = -50 (-52.6)$
$M_{2-2} = 248 (247.5)$	$M_{10-10} = 167 (166)$	$M_{18-18} = 109 (106.8)$
$M_{2-1} = 246 (245)$	$M_{10-9} = 152 (152)$	$M_{18-17} = 93 (91.1)$
$M_{2-3} = -239 (-239)$	$M_{10-11} = -155 (-155.7)$	$M_{18-19} = -91 (-94.2)$
$M_{2-11} = -252 (-251)$	$M_{10-7} = -161 (-162)$	$M_{18-15} = -100 (-104.2)$
$M_{3-3} = 225 (226)$	$M_{11-11} = 140 (141.5)$	$M_{19-19} = 99 (98.5)$
$M_{3-4} = 184 (184)$	$M_{11-12} = 125 (125.6)$	$M_{19-20} = 85 (85)$
$M_{3-6} = -183 (-182)$	$M_{11-14} = -126 (-124)$	$M_{19-22} = -81 (-81.2)$
$M_{3-2} = -227 (-226.5)$	$M_{11-10} = -144 (-143.3)$	$M_{19-18} = -100 (-102.2)$
$M_{4-3} = 202 (203)$	$M_{12-11} = 135 (135)$	$M_{20-19} = 89 (89.6)$
$M_{4-5} = -91 (-90.7)$	$M_{12-13} = -62 (-60)$	$M_{20-21} = -42 (-41.5)$
$M_{4-1} = -113 (-113)$	$M_{12-9} = -75 (-75)$	$M_{20-17} = -47 (-46.5)$
$M_{5-6} = 187 (187.1)$	$M_{13-14} = 125 (125.9)$	$M_{21-22} = 82 (81.9)$
$M_{5-8} = -90 (-87.8)$	$M_{13-16} = -70 (-67.7)$	$M_{21-24} = -33 (-34.1)$
$M_{5-4} = -101 (-100.3)$	$M_{13-12} = -62 (-57.6)$	$M_{21-20} = -47 (-47.5)$
$M_{6-6} = 187 (189.6)$	$M_{14-14} = 130 (132.5)$	$M_{22-22} = 88 (87.5)$
$M_{6-5} = 170 (171.7)$	$M_{14-13} = 114 (117)$	$M_{22-21} = 77 (76.6)$
$M_{6-7} = -175 (-173.5)$	$M_{14-15} = -130 (-127.5)$	$M_{22-23} = -74 (-75.1)$
$M_{6-3} = -189 (-187.5)$	$M_{14-11} = -125 (-122)$	$M_{22-19} = -89 (-89)$
$M_{7-7} = 184 (184.4)$	$M_{15-15} = 108 (107.8)$	$M_{23-23} = 79 (78.3)$
$M_{7-8} = 153 (153.6)$	$M_{15-16} = 92 (92)$	$M_{23-24} = 69 (68.6)$
$M_{7-10} = -161 (-162)$	$M_{15-18} = -100 (-103.3)$	$M_{23-22} = -81 (-81.5)$
$M_{7-6} = -177 (-176.1)$	$M_{15-14} = -96 (-97.2)$	$M_{23-20} = -65$
$M_{8-7} = 170 (170.5)$	$M_{16-15} = 96 (95.8)$	$M_{24-23} = 74 (73.4)$
$M_{8-9} = -83 (-82.8)$	$M_{16-17} = -50 (-51.9)$	$M_{24-21} = -41 (-41.4)$
$M_{8-5} = -89 (-87.8)$	$M_{16-13} = -45 (-44.4)$	$M_{24-25} = -32$

EXAMPLE OF THE METHOD OF SOLUTION II.

Fig. 4 shows a symmetrical four-span bent twenty stories high. The bent is subjected to a system of symmetrical vertical loads. It is required



Table IX.

Equation	Left Hand Member of Equation																																						Right-Hand Member of Equation Coefficient $(2E)^{-1}$			
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}	θ_{19}	θ_{20}	θ_{21}	θ_{22}	θ_{23}	θ_{24}	θ_{25}	θ_{26}	θ_{27}	θ_{28}	θ_{29}	θ_{30}	θ_{31}	θ_{32}	θ_{33}	θ_{34}	θ_{35}	θ_{36}	θ_{37}	θ_{38}	θ_{39}	θ_{40}		
1	f_1	K_1		K_1'																																					C_{12}	
2	K_1'	f_2	K_2																																							$C_{20}-C_{21}$
3		K_2	f_3	K_3		K_3'																																				$C_{30}-C_{31}$
4	K_1'		K_3	f_4	K_4																																					C_{43}
5			K_4'	f_5	K_5		K_5'																																			C_{50}
6			K_5'		K_5	f_6	K_6																																			$C_{60}-C_{65}$
7					K_6	f_7	K_7		K_7'																																	$C_{70}-C_{78}$
8					K_5'		K_7	f_8	K_8																																	C_{87}
9							K_8	f_9	K_9		K_9'																															$C_{9,10}$
10						K_5'		K_9	f_{10}	K_{10}																															$C_{10,0}-C_{10,9}$	
11								K_{10}	f_{11}	K_{11}		K_{11}'																													$C_{11,0}-C_{11,12}$	
12								K_9'		K_{11}	f_{12}	K_{12}																													$C_{12,11}$	
13										K_{12}	f_{13}	K_{13}		K_{13}'																											$C_{13,14}$	
14										K_{11}'		K_{13}	f_{14}	K_{14}																											$C_{14,0}-C_{14,13}$	
15												K_{14}	f_{15}	K_{15}		K_{15}'																									$C_{15,0}-C_{15,16}$	
16												K_{13}'		K_{15}	f_{16}	K_{16}																									$C_{16,15}$	
17													K_{16}	f_{17}	K_{17}		K_{17}'																								$C_{17,18}$	
18													K_{15}'	K_{17}	f_{18}	K_{18}																									$C_{18,0}-C_{18,17}$	
19														K_{18}	f_{19}	K_{19}		K_{19}'																							$C_{19,0}-C_{19,20}$	
20														K_{17}'		K_{19}	f_{20}	K_{20}																							$C_{20,19}$	
21																K_{20}	f_{21}	K_{21}		K_{21}'																					$C_{21,22}$	
22																K_{19}'		K_{21}	f_{22}	K_{22}																					$C_{22,0}-C_{22,21}$	
23																	K_{22}	f_{23}	K_{23}		K_{23}'																				$C_{23,0}-C_{23,24}$	
24																	K_{21}'		K_{23}	f_{24}	K_{24}																				$C_{24,23}$	
25																		K_{24}	f_{25}	K_{25}		K_{25}'																			$C_{25,20}$	
26																			K_{25}	f_{26}	K_{26}																				$C_{26,0}-C_{26,25}$	
27																				K_{26}	f_{27}	K_{27}		K_{27}'																	$C_{27,0}-C_{27,28}$	
28																				K_{25}'		K_{27}	f_{28}	K_{28}																	$C_{28,27}$	
29																					K_{28}	f_{29}	K_{29}		K_{29}'																$C_{29,30}$	
30																					K_{27}'		K_{29}	f_{30}	K_{30}																$C_{30,0}-C_{30,29}$	
31																						K_{30}	f_{31}	K_{31}		K_{31}'															$C_{31,0}-C_{31,32}$	
32																						K_{29}'		K_{31}	f_{32}	K_{32}															$C_{32,31}$	
33																							K_{32}	f_{33}	K_{33}		K_{33}'														$C_{33,34}$	
34																								K_{31}'		K_{33}	f_{34}	K_{34}													$C_{34,0}-C_{34,33}$	
35																										K_{34}	f_{35}	K_{35}		K_{35}'											$C_{35,0}-C_{35,36}$	
36																											K_{32}'		K_{35}	f_{36}	K_{36}										$C_{36,35}$	
37																												K_{36}	f_{37}	K_{37}		K_{37}'									$C_{37,38}$	
38																													K_{35}'		K_{37}	f_{38}	K_{38}								$C_{38,0}-C_{38,37}$	
39																														K_{38}	f_{39}	K_{39}									$C_{39,0}-C_{39,40}$	
40																															K_{37}'		K_{39}	f_{40}							$C_{40,39}$	



Table X.

Equation	Left-Hand Member of Equation																																								Right-and Member of Equation Coefficient of E ⁻¹		
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}	θ_{19}	θ_{20}	θ_{21}	θ_{22}	θ_{23}	θ_{24}	θ_{25}	θ_{26}	θ_{27}	θ_{28}	θ_{29}	θ_{30}	θ_{31}	θ_{32}	θ_{33}	θ_{34}	θ_{35}	θ_{36}	θ_{37}	θ_{38}	θ_{39}	θ_{40}			
1	183.8	30.5		35.6																																					202500		
2	30.5	258.4	35.6																																							56000	
3		35.6	243.4	21.4		35.5																																				56000	
4	35.6		21.4	184.8	35.4																																					202500	
5				35.4	184.4	21.4		35.4																																		202500	
6			35.5		21.4	237.2	35.5																																			56000	
7						35.5	223.2	19.5		30.4																																56000	
8				35.4		19.5	168.6	29.4																																		202500	
9						30.4		29.4	156.6	19.5		29.4																														202500	
10						39.4		19.5	208.2	30.4																																56000	
11									30.4	183.4	14.1		30.0																													56000	
12										14.1	144.4	28.7																															202500
13											28.7	140.4	12.8		28.7																											202500	
14										30.0		12.8	177.0	30.0																												56000	
15													30.0	146.4	7.7		26.1																									56000	
16												28.7		7.7	115.0	21.1																										202500	
17															21.1	99.8	7.7		21.1																							168750	
18														26.1		7.7	138.6	26.1																								48500	
19																	26.1	126.2	7.7		19.9																					48500	
20																21.1		7.7	95.6	18.8																						168750	
21																			18.8	90.6	7.7		18.8																			168750	
22																		19.9		7.7	113.8	19.9																				48500	
23																				19.9	111.2	7.7		18.6																		48500	
24																				18.8		7.7	89.6	18.3																		168750	
25																							18.3	88.6	7.7		18.3															168750	
26																						18.6		7.7	108.6	18.6																48500	
27																								18.6	100.6	7.7		14.6														48500	
28																								18.3		7.7	80.6	14.3														168750	
29																										14.3	72.6	7.7		14.3												168750	
30																									14.6		7.7	92.6	14.6													48500	
31																											14.6	89.8	7.7		13.2											48500	
32																												14.3		7.7	70.2	13.1										168750	
33																													13.1	67.8	7.7		13.1									168750	
34																														13.2		7.7	87.0	13.2								48500	
35																															13.2	87.0	7.7		13.2							48500	
36																																7.7	67.8	13.1								168750	
37																																	13.1	67.8	7.7		13.1					168750	
38																																		13.2		7.7	87.0	13.2				48500	
39																																					13.2	60.6	7.7				54750
40																																						13.1		7.7	41.6		186975



to find the jointmoments. The values of K of all the members are written in Fig. 4 on the corresponding members.

The Mechanical Tabulation Method gives Table IX from which the simultaneous equations to be used to determine the unknowns may be written without any computations.

From the values of K shown in Fig. 4 we determine the values of J as follows:

$$J_1 = 2\{35.6 + 30.5 + 25.8\} = 183.8 \text{ in.}^3$$

$$J_2 = 2\{30.5 + 25.8 + 37.3 + 35.6\} = 258.4 \text{ in.}^3$$

In a similar way

$$J_3 = 243.4 \text{ in.}^3 \quad J_4 = 184.8 \text{ in.}^3 \quad J_5 = 184.4 \text{ in.}^3$$

etc.

For the given load

$$C_{ab} = C_{ba} = \frac{l}{16} \{3P_1 + 2P_2\} + \frac{wl^2}{12}.$$

This expression gives:

$$C_{12} = C_{21} = \dots = \frac{18(12)}{16} \{3(4000) + 2(6000)\}$$

$$+ \frac{1}{12} \{250(18)(18)(12)\}$$

$$= 405000 \text{ in. lb.}$$

$$C_{20} = C_{02} = \dots = \frac{22(12)}{16} \{3(4000) + 2(6000)\}$$

$$+ \frac{1}{12} \{250(22)(22)(12)\}$$

$$= 517000 \text{ in. lb.}$$

etc.

Substituting these values above obtained in Table IX gives Table X.

Applying the method of solution II, we obtain the unknowns as shown in Table XI. This complete calculation may be made by a single person within about six hours, using a slide-rule.

To find the joint moments, the calculation is as follows: At the joint 1 we get:

$$M_{12} = 2EK_1 \{2\theta_1 + \theta_2\} - C_{12}$$

TABLE XII.

(a)

Joint Moments in 1000 in. lb.			
Joint Moment	Total Positive Moment	Total Negative Moment	Discrepancy
$M_{12} = -285$ $M_{14} = 187$ $M_{1I} = 96$	283	-285	-2
$M_{21} = 473$ $M_{20} = -503$ $M_{23} = 22$ $M_{2II} = 9$	504	-503	+1
$M_{34} = 448$ $M_{32} = -501$ $M_{36} = 28$ $M_{3I} = 25$	501	-501	0
$M_{43} = -335$ $M_{41} = 174$ $M_{45} = 161$	335	-335	0
$M_{56} = -334$ $M_{58} = 169$ $M_{54} = 162$	331	-334	-3
$M_{65} = 449$ $M_{62} = -503$ $M_{67} = 28$ $M_{63} = 28$	505	-503	+2
$M_{78} = 449$ $M_{70} = -503$ $M_{7-10} = 25$ $M_{7-6} = 29$	503	-503	0
$M_{8-7} = -333$ $M_{8-9} = 155$ $M_{8-5} = 176$	331	-333	-2
$M_{9-10} = -328$ $M_{9-12} = 167$ $M_{9-8} = 159$	326	-328	-2
$M_{10-c} = -504$ $M_{10-9} = 452$ $M_{10-11} = 28$ $M_{10-7} = 25$	505	-504	+1

(b)

Joint Moments in 1000 in. lb.			
Joint Moment	Total Positive Moment	Total Negative Moment	Discrepancy
$M_{11-c} = -505$ $M_{11-12} = 443$ $M_{11-14} = 32$ $M_{11-10} = 30$	505	-505	0
$M_{12-11} = -344$ $M_{12-13} = 169$ $M_{12-9} = 171$	340	-344	-4
$M_{13-14} = -352$ $M_{13-16} = 183$ $M_{13-12} = 167$	350	-352	-2
$M_{14-c} = -506$ $M_{14-13} = 438$ $M_{14-15} = 36$ $M_{14-11} = 32$	506	-506	0
$M_{15-c} = -508$ $M_{15-16} = 432$ $M_{15-18} = 36$ $M_{15-14} = 40$	508	-508	0
$M_{16-15} = -362$ $M_{16-17} = 156$ $M_{16-13} = 202$	358	-362	-4
$M_{17-18} = -299$ $M_{17-20} = 148$ $M_{17-10} = 150$	298	-299	-1
$M_{18-c} = -427$ $M_{18-17} = 361$ $M_{18-19} = 33$ $M_{18-15} = 34$	428	-427	+1
$M_{19-c} = -426$ $M_{19-20} = 363$ $M_{19-22} = 28$ $M_{19-18} = 35$	426	-426	0
$M_{20-19} = -296$ $M_{20-21} = 142$ $M_{20-17} = 152$	294	-296	-2

(c)

Joint Moments in 1000 in. lb.			
Joint Moment	Total Positive Moment	Total Negative Moment	Discrepancy
$M_{21-22} = -293$ $M_{21-24} = 149$ $M_{21-20} = 146$	295	-293	+2
$M_{22-c} = -425$ $M_{22-21} = 366$ $M_{22-23} = 31$ $M_{22-19} = 29$	426	-425	+1
$M_{23-c} = -425$ $M_{23-24} = 366$ $M_{23-26} = 30$ $M_{23-22} = 31$	427	-425	+2
$M_{24-23} = -293$ $M_{24-25} = 144$ $M_{24-21} = 149$	293	-293	0
$M_{25-26} = -293$ $M_{25-28} = 150$ $M_{25-24} = 145$	295	-293	+2
$M_{26-c} = -425$ $M_{26-25} = 366$ $M_{26-27} = 30$ $M_{26-23} = 30$	426	-425	+1
$M_{27-c} = -424$ $M_{27-28} = 369$ $M_{27-30} = 25$ $M_{27-26} = 31$	425	-424	+1
$M_{28-27} = -288$ $M_{28-29} = 131$ $M_{28-25} = 156$	287	-288	-1
$M_{29-30} = -282$ $M_{29-32} = 142$ $M_{29-28} = 137$	279	-282	-3
$M_{30-c} = -423$ $M_{30-29} = 372$ $M_{30-31} = 26$ $M_{30-27} = 25$	423	-423	0

(d)

Joint Moments in 1000 in. lb.			
Joint Moment	Total Positive Moment	Total Negative Moment	Discrepancy
$M_{31-c} = -423$ $M_{31-32} = 372$ $M_{31-34} = 24$ $M_{31-30} = 26$	422	-423	-1
$M_{32-31} = -282$ $M_{32-33} = 132$ $M_{32-29} = 142$	274	-282	-8
$M_{33-34} = -280$ $M_{33-36} = 137$ $M_{33-32} = 133$	270	-280	-10
$M_{34-c} = -423$ $M_{34-33} = 373$ $M_{34-35} = 24$ $M_{34-31} = 24$	420	-423	-3
$M_{35-c} = -424$ $M_{35-36} = 374$ $M_{35-38} = 23$ $M_{35-34} = 23$	420	-424	-4
$M_{36-35} = -277$ $M_{36-37} = 129$ $M_{36-33} = 139$	268	-277	-9
$M_{37-38} = -290$ $M_{37-40} = 170$ $M_{37-36} = 120$	290	-290	0
$M_{38-c} = -423$ $M_{38-37} = 367$ $M_{38-39} = 26$ $M_{38-35} = 24$	417	-423	-6
$M_{39-c} = -470$ $M_{39-40} = 445$ $M_{39-38} = 27$	472	-470	+2
$M_{40-39} = -247$ $M_{40-37} = 240$	240	-247	-7

$$= 2(30 \cdot 5) \{ 2(934 \cdot 3) + 90 \cdot 8 \} - 405000$$

$$= -285000 \text{ in. lb.}$$

$$M_{14} = 187000 \text{ in. lb.}$$

$$M_{11} = 96000 \text{ in. lb.}$$

etc.

The total sum of the positive moment is 283000 in. lb. and that of the negative moment -285000 in. lb. For practical purposes, this discrepancy of the equilibrium condition may not be effective, but if it is required to have more accurate results, the number of repetitions are to be increased, corresponding to the required necessity.

In Table XII there are shown the joint moments, the sum of the positive and negative moments and the discrepancies of the latter two.

5. Contour Line Representation of Vertical Load Effects—Fig. 5 represents a three storied building frame with three bays. All legs of the frame are fixed at the foundation. This frame carries any system of vertical loads which are symmetrical, as shown in the sketch, with respect to the center line of the structure.

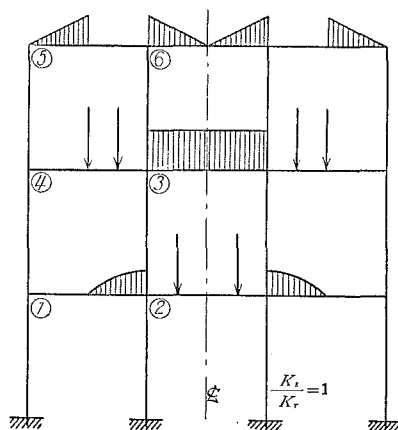


Fig. 5.

It is required to find the vertical load effects in terms of the value “ τ ”, which is given in equation (8). That is, the joint-rotation angles are to be found in terms of the difference of the fixed-ended moments at each joint of the frame, considering the members as beams fixed at both ends. As the frame shown in Fig. 5 has an axis of symmetry and carries vertical loads symmetrical about the vertical center line of the structure,

no horizontal deflections come in calculation; therefore the equations are very much simplified and expressed only by terms of the slope.

TABLE XIII.

Equation	Left-Hand Member of Equation						Right-Hand Member of Equation
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	
(1)	6	1		1			τ_1
(2)	1	7	1				τ_2
(3)		1	7	1		1	τ_3
(4)	1		1	6	1		τ_4
(5)				1	4	1	τ_5
(6)			1		1	5	τ_6

Equation (11) gives the general equations to be used in determining the unknowns φ in a building frame with equal K , under symmetrical conditions.

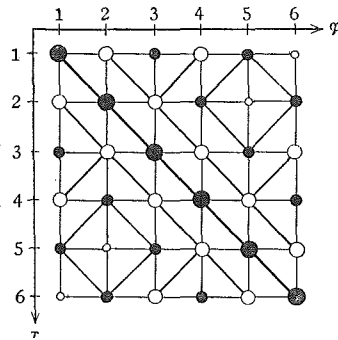


Fig. 6.

For joint 1, equation (11) gives:

$$\begin{array}{c} \varphi_4 \\ + \\ \varphi_1(6) + \varphi_2 = \tau_1. \end{array}$$

For joint 2, equation (11) gives:

$$\begin{array}{c} \varphi_3 \\ + \\ \varphi_1 + \varphi_2(8) + (-\varphi_2) = \tau_2. \end{array}$$

For joint 3, equation (11) gives:

$$\begin{array}{c} \varphi_6 \\ + \\ \varphi_4 + \varphi_3(8) + (-\varphi_3) = \tau_3. \\ + \\ \varphi_2 \end{array}$$

etc.

Table XIII shows the total equations, obtained likewise as shown above, to be used in determining the unknown quantities φ_1 to φ_6 in function of load terms τ_1 to τ_6 .

Solving the six equations simultaneously gives:

$$\begin{aligned} \varphi_1 &= 0.177 \tau_1 - 0.027 \tau_2 + 0.009 \tau_3 - 0.032 \tau_4 + 0.009 \tau_5 - 0.004 \tau_6 \\ \varphi_2 &= -0.027 \tau_1 + 0.150 \tau_2 - 0.024 \tau_3 + 0.009 \tau_4 - 0.004 \tau_5 + 0.005 \tau_6 \\ \varphi_3 &= 0.009 \tau_1 - 0.024 \tau_2 + 0.156 \tau_3 - 0.030 \tau_4 + 0.016 \tau_5 - 0.034 \tau_6 \\ \varphi_4 &= -0.032 \tau_1 + 0.009 \tau_2 - 0.030 \tau_3 + 0.186 \tau_4 - 0.050 \tau_5 + 0.016 \tau_6 \\ \varphi_5 &= 0.009 \tau_1 - 0.004 \tau_2 + 0.016 \tau_3 - 0.050 \tau_4 + 0.277 \tau_5 - 0.059 \tau_6 \\ \varphi_6 &= -0.004 \tau_1 + 0.005 \tau_2 - 0.034 \tau_3 + 0.016 \tau_4 - 0.059 \tau_5 + 0.219 \tau_6 \end{aligned}$$

TABLE XIV.

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6
τ_1	0.177	-0.027	0.009	-0.032	0.009	-0.004
τ_2	-0.027	0.150	-0.024	0.009	-0.004	0.005
τ_3	0.009	-0.024	0.156	-0.030	0.016	-0.034
τ_4	-0.032	0.009	-0.030	0.186	-0.050	0.016
τ_5	0.009	-0.004	0.016	-0.050	0.277	-0.059
τ_6	-0.004	0.005	-0.034	0.016	-0.059	0.219

Table XIV shows the above values in tabular form. The relation between the coefficients in the table explains the following fact:

- Load term τ_3 at joint 3 has the same effect as that of τ_5 at joint 5 with respect to the slope term φ_1 .
- Effect of the load term τ_5 with respect to the slope term φ_1 is equal to that of τ_4 with respect to the slope term φ_2 .
- Effect of the load terms τ_2 and τ_6 with respect to the slope term φ_1 is equal to that of τ_1 and τ_5 respectively with respect to the slope term φ_2 .

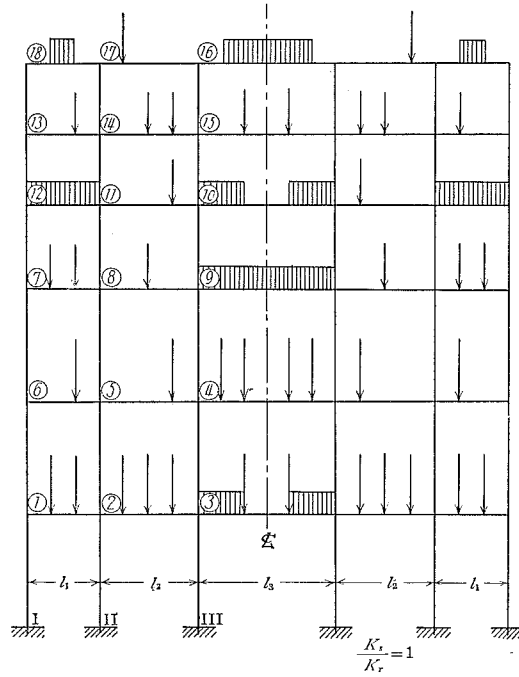


Fig. 7.

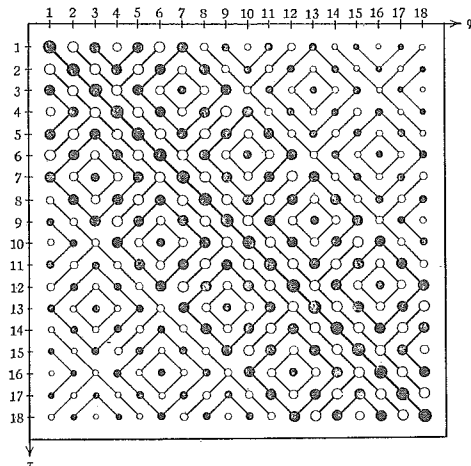


Fig. 8.

These properties are obvious in Table XIV, considering a diagonal line, from the top left corner to the bottom right corner of the table, as a symmetrical axis of the table.

- d. The essential part of the load term, which controls the slope terms, lies on the diagonal line above mentioned. In other words, the largest primary effect of the load terms is expressed by the coefficients on the diagonal line above mentioned.

It has positive effects (Fig. 6).

- e. The secondary effect is expressed by the coefficients on the lines which lie on both sides of the diagonal and by the coefficients on the lines perpendicular to the diagonal. It has negative effects (Fig. 6).

In a similar way, we can represent the effect of vertical loads on the joint-rotation angles by contour-line-figures. This representation makes obvious the effect of vertical loads on the unknowns and leads to the solution of the problem with speed and accuracy. Fig. 8 shows the contour-line-representation of the vertical load effects with respect to the building frame shown in Fig. 7 (Compare Table XV with Fig. 8).

Table XV.

Equation	Left-Hand Member of Equation																		Right-Hand Member of Equation
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}	
(1)	6	1				1													τ_1
(2)	1	8	1		1														τ_2
(3)		1	7	1															τ_3
(4)			1	7	1				1										τ_4
(5)		1		1	8	1		1											τ_5
(6)	1				1	6	1												τ_6
(7)						1	6	1				1							τ_7
(8)					1		1	8	1		1								τ_8
(9)				1				1	7	1									τ_9
(10)									1	7	1				1				τ_{10}
(11)								1		1	8	1		1					τ_{11}
(12)							1				1	6	1						τ_{12}
(13)												1	6	1				1	τ_{13}
(14)											1		1	8	1		1		τ_{14}
(15)										1				1	7	1			τ_{15}
(16)															1	5	1		τ_{16}
(17)														1		1	6	1	τ_{17}
(18)													1				1	4	τ_{18}

6. Contour Line Representation of Horizontal Load Effects—

Fig. 9 represents a three storied building frame with three bays. This frame carries a horizontal load concentrated at each joint on the vertical left side of the structure whose rigidity ratio is $K_s:K_r=1$.

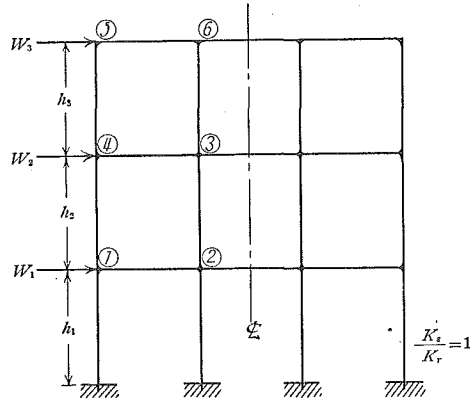


Fig. 9.

It is required to find the horizontal load effects in terms of the value R , which is given in equation (13). Namely, the unknown distortion terms are to be found in functions of R_1 , R_2 and R_3 .

As the frame shown in Fig. 9 has an axis of symmetry of the building and carries a horizontal load at each joint on the left side of the structure, the horizontal deflection of the columns as well as the joint-rotation angles come in calculation and the frame has nine unknowns μ_1 to μ_3 and φ_1 to φ_6 .

Equation (10) determines six joint-equilibrium equations and equation (13) three story-equilibrium equations.

For joint 1, equation (10) gives:

$$\begin{aligned} &\varphi_4 + \mu_2 \\ &+ \\ &\varphi_1(6) + \varphi_2 = 0. \\ &+ \\ &\mu_1 \end{aligned}$$

For joint 2, equation (10) gives:

$$\begin{aligned} &\varphi_3 + \mu_2 \\ &+ \\ &\varphi_1 + \varphi_2(9) = 0 \\ &+ \\ &\mu_1 \end{aligned}$$

etc.

For the first story, equation (13) gives:

$$2\{\varphi_1 + \varphi_2\} + \mu_1 X_1 = R_1,$$

in this equation

$$X_1 = \frac{2}{3} (4) = \frac{8}{3}.$$

TABLE XVI.

Equation	Left-Hand Member of Equation									Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3	
(1)	6	1		1			1	1		0
(2)	1	9	1				1	1		0
(3)		1	9	1		1		1	1	0
(4)	1		1	6	1			1	1	0
(5)				1	4	1			1	0
(6)			1		1	7			1	0
(7)	1	1					4/3			$R_1/2$
(8)	1	1	1	1				4/3		$R_2/2$
(9)			1	1	1	1			4/3	$R_3/2$

For the second story, equation (13) gives:

$$2\{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4\} + \mu_2 X_2 = R_2,$$

in this equation

$$X_2 = \frac{2}{3} (4) = \frac{8}{3}$$

etc.

These equilibrium equations are recorded in Table XVI. Solving the nine equations simultaneously gives:

$$\varphi_1 = -0.0903 R_1 - 0.096 R_2 - 0.0064 R_3$$

$$\varphi_2 = -0.056 R_1 - 0.0643 R_2 - 0.0076 R_3$$

$$\varphi_3 = -0.0074 R_1 - 0.0649 R_2 - 0.066 R_3$$

$$\varphi_4 = -0.0048 R_1 - 0.098 R_2 - 0.0953 R_3$$

$$\varphi_5 = -0.00115 R_1 - 0.0074 R_2 - 0.1182 R_3$$

$$\varphi_6 = -0.00013 R_1 - 0.0092 R_2 - 0.064 R_3$$

$$\mu_1 = 0.485 R_1 + 0.120 R_2 + 0.0105 R_3$$

$$\mu_2 = 0.118 R_1 + 0.619 R_2 + 0.131 R_3$$

$$\mu_3 = 0.0095 R_1 + 0.137 R_2 + 0.632 R_3$$

Table XVII shows the above calculation results in tabular form. If the number of the joints as shown in Fig. 10 be transposed, the equation table becomes as shown in Table XVIII and the calculation results are entered in Table XIX.

The relation between the coefficients in the table shows the following facts (Table XIX):

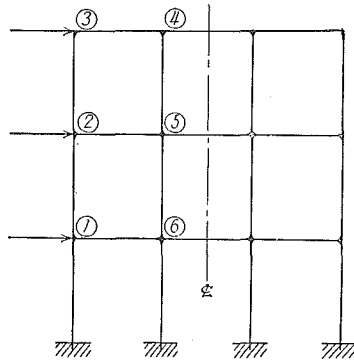


Fig. 10.

- a. Load terms R_1 and R_2 are more effective than load term R_3 with respect to the slope term φ_1 .
- b. Load terms R_2 and R_3 are more effective than load term R_1 with respect to the slope term φ_2 .
- c. Load term R_3 is more effective than load terms R_1 and R_2 with respect to the slope term φ_3 . These properties are obvious in Table XIX, on noticing the stepped line of greatest negative values as shown in the same table. It is very interesting to find the coefficients of φ in equations (7) to (9) of Table XVIII in an exactly similar arrangement to the coefficients of R in Table XIX. The very same properties are obvious likewise regarding Table XX and Fig. 11. In other words, the largest primary effect of horizontal load terms is expressed by the coefficient on the stepped line above mentioned and has negative effects (Really, R expresses negative quantities and therefore this is a positive effect).

TABLE XVII.

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3
R_1	-0.0903	-0.056	-0.0074	-0.0048	-0.00115	-0.00013	0.485	0.118	0.0095
R_2	-0.096	-0.0643	-0.0649	-0.098	-0.0074	-0.0092	0.120	0.619	0.137
R_3	-0.0064	-0.0076	-0.066	-0.0953	-0.1182	-0.064	0.0105	0.131	0.632

TABLE XVIII.

Equation	Left-Hand Member of Equation									Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3	
(1)	6	1				1	1	1		0
(2)	1	6	1		1			1	1	0
(3)		1	4	1					1	0
(4)			1	7	1				1	0
(5)		1		1	9	1		1	1	0
(6)	1				1	9	1	1		0
(7)	<u>1</u>					<u>1</u>	4 / 3			$R_1 / 2$
(8)	<u>1</u>	<u>1</u>			<u>1</u>	<u>1</u>		4 / 3		$R_2 / 2$
(9)		<u>1</u>	<u>1</u>	<u>1</u>	<u>1</u>				4 / 3	$R_3 / 2$

TABLE XIX.

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3
R_1	<u>-0.090</u>	-0.0048	-0.00115	-0.00013	-0.0074	<u>-0.056</u>	0.485	0.118	0.0095
R_2	<u>-0.096</u>	<u>-0.098</u>	-0.0074	-0.0092	<u>-0.0649</u>	<u>-0.064</u>	0.120	0.619	0.137
R_3	-0.006	<u>-0.095</u>	<u>-0.1182</u>	<u>-0.064</u>	<u>-0.066</u>	-0.0076	0.0105	0.131	0.632

- d. The secondary effect is expressed by the coefficients on the lines which lie on both sides of the stepped line and has negative effects (Really for the same reason as mentioned above, it is a positive effect).
- e. For the deflection terms, the relation between μ_1 , μ_2 , μ_3 and R_1 , R_2 , R_3 are similar to the X-line in Table XVIII. That is to say,

the largest primary effects are along the diagonal line, from the top left corner, composed by μ and R , to the bottom right corner of the square.

f. The secondary effects occur along both sides of the diagonal.

In a similar way, we can represent the effect of horizontal loads on the joint-rotation angles and the member-revolution angles by contour line figures.

This representation makes the effect of horizontal loads on the unknowns obvious and leads to the solution of the problem with speed and accuracy.

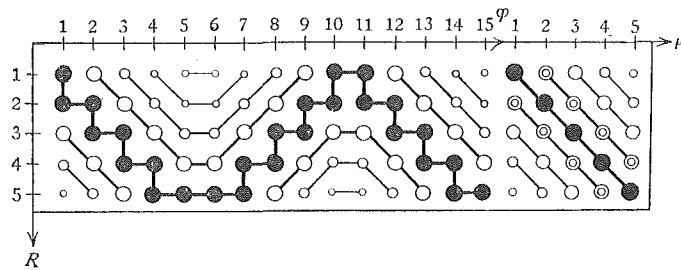


Fig. 11.

Fig. 11 shows the contour line representation of the horizontal load effects with respect to the building frame shown in Fig. 12.

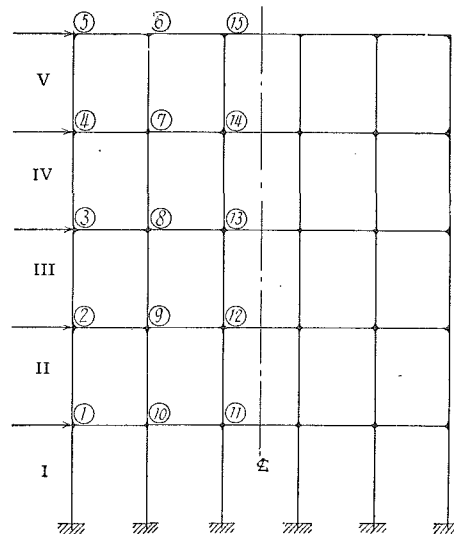


Fig. 12.

Table XX.

Equation	Left-Hand Member of Equation																				Right-Hand Member of Equation
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	μ_1	μ_2	μ_3	μ_4	μ_5	
1	6	1								1						1	1				0
2	1	6	1						1								1	1			0
3		1	6	1				1										1	1		0
4			1	6	1		1												1	1	0
5				1	4	1														1	0
6					1	6	1								1					1	0
7				1		1	8	1						1					1	1	0
8			1				1	8	1				1						1	1	0
9		1						1	8	1		1					1	1			0
10	1								1	8	1					1	1				0
11									1	9	1					1	1				0
12									1		1	9	1				1	1			0
13								1				1	9	1				1	1		0
14							1						1	9	1				1	1	0
15						1								1	7					1	0
16	1									1	1					2					$\frac{R_1}{2}$
17	1	1							1	1	1	1					2				$\frac{R_2}{2}$
18		1	1					1	1			1	1					2			$\frac{R_3}{2}$
19			1	1			1	1					1	1					2		$\frac{R_4}{2}$
20				1	1	1	1							1	1					2	$\frac{R_5}{2}$

Summary and Conclusions.

The general conclusions to be drawn from the investigations described in this paper are included in the resumé below:

Tables obtained by the Mechanical Tabulation Method determine the statically-indeterminate quantities with speed and accuracy and furnish at the same time the load effect line; in other words, the vertical load effects are similar to the arrangement of the coefficients of unknown slopes in the systematic arrangement of the joint-equilibrium equations; the horizontal load effects are similar to the configuration of the coefficients of unknowns in the story-equilibrium equations.

All the positions of the figures in the table (Compare Table XV with Fig. 8 and Table XX with Fig. 11) are important, serving to indicate the tendency of the primary load effects, leading easily to the estimation of the secondary effects.

These properties are important to the investigation of the statically-indeterminate stresses and further to the similar problems in structural mechanics.

Moreover, from a pure mathematical stand point, the relation found from the concrete examples above mentioned between unknowns and constants may be a new problem, which will afford in future a field of mathematical study.