Practical Method of Calculation for Vierendeel Truss Bridges.

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The proper treatment of frames with stiff connecting joints, composed of rectangular elements, is of constantly increasing importance in structural engineering.

The Vierendeel trusses are an important example of the structures with rigid connections, especially now a days as the electric-arc-welding process is of increasing current use and the analysis of statically indeterminate stresses has become very simplified and practical.

The calculation method here proposed is of three kinds: (A) The Mechanical Tabulation Method applied to the analysis of Vierendeel trusses of parallel chords, (B) The Slope Distribution Method applied to the same, (C) The X-Distribution Method for the analysis of Vierendeel trusses of polygonal or parallel chords.

A. The Mechanical Tabulation Method Applied to the Analysis of Vierendeel Trusses of Parallel Chords.

The analysis in this paper is based upon the following assumptions:

(1). The connections between the verticals and chord members are perfectly rigid.

(2). The length of a member is not changed by direct stress, and the deformation of a member due to the internal shearing stress is zero.

(3). The settlement of the foundations is disregarded.

According to these assumptions all the verticals and chord members, in strained state, which intersect at one point are subjected to an equal change in joint-rotation angle (or slope) and the moment at an end of a vertical or a chord member is expressed as a function of the changes in the slopes and of the deflection of one end of the member relative to the other end (or member revolution angle).

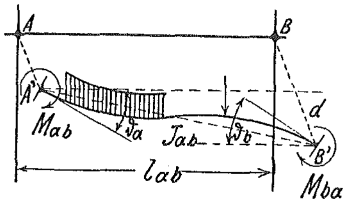


Fig. 1.

Generally if a member AB restrained at the ends in flexure is subjected to any system of intermediate loads, the moments at the ends of the member are expressed by the well known equation: (Fig. 1)

$$M_{ab} = 2EK_{ab}(2\theta_a + \theta_b - 3R_{ab}) - C_{ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In this equation we denote by

M_{ab} the resisting moments at A,
E the modulus of elasticity of the material,
θ_a, θ_b the changes in the slope of the tangent to the elastic curve at A and B,
$K_{ab} = \frac{J_{ab}}{l_{ab}}$, in which J_{ab} = moment of inertia of the section of the member AB; l_{ab} = length of the member AB,
$R_{ab} = \frac{\delta_{ab}}{l_{ab}}$, in which δ_{ab} = deflection of the end B from its initial position,
$C_{ab} = \frac{2A}{l_{ab}^2}\{3\xi - l_{ab}\}$, in which A = area of the moment diagram of a simple beam AB due to the intermediate loads; ξ = distance of the centroid of the area A from the end B.

The conventional signs of the quantities used in the equation require further explanation as follows:

The resisting moment on a section is considered positive when the resisting couple acts in a clockwise direction upon the portion of the member considered and by this conventional rule each resisting moment of the portion of the member considered must always be indicated by the subscript M_{ab}.

The moment of an external force is positive when it tends to cause a clockwise rotation.

The change in slope is considered positive when the tangent to the elastic line of a member has been turned clockwise, measured from its initial position.

The member-revolution angle is considered always positive to the rotation in the clockwise direction from the initial position of member.

The intersections of the neutral axes of the chord members with the neutral axes of the verticals are denoted by 1, 2, 3, 4 etc. for the upper-chord-panel joints and by $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$ etc. for the lower chord panel joints, beginning at the left and reading toward the right.

The value of K(*i. e.* $J:l$) of chord members in the way of reading are denoted by the subscript of the number of the intersection from which the member considered begins in the direction as mentioned above.

The values of K of verticals are denoted by K' with subscript of the number of the intersection under which the neutral axis of the vertical considered hangs, e.g., K_1', K_2', K_3', etc. (*Fig. 2*)

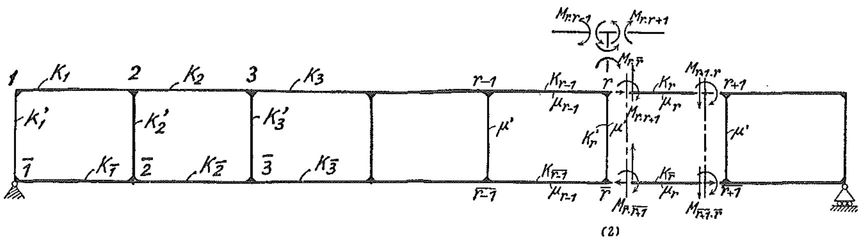


Fig. 2.

For ordinal trusses, the load term C_{ab} vanishes in equation (1), because all vertical loads are assumed to be applied on panel joints.

Putting in equation (1)

$$2E\theta_a = \varphi_a\,, \quad 2E\theta_b = \varphi_b\,, \quad -6ER_{ab} = \mu_{ab}\,, \quad C_{ab} = 0$$

we have :

$$M_{ab} = K_{ab}\{2\varphi_a + \varphi_b + \mu_{ab}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

For the member which has no member-revolution angle, the term μ_{ab} vanishes and we have :

$$M_{ab} = K_{ab}\{2\varphi_a + \varphi_b\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

I. Number of unknowns and number of equations.

Fig. 2 shows a parallel chord truss of Vierendeel type and the assumptions adopted make the value μ equal both for an upper chord member and the lower chord member in the same panel. That is to say, the number of μ for all chord members coincides with the number of panels.

Thus, the value of μ for all the vertical members is equal and so the number of μ for all verticals is one.

Therefore, as a whole, the total number of μ of a parallel chord truss of Vierendeel type is denoted by $(n+1)$, in which $n=$ total number of panel of the given truss.

Next, the assumptions adopted give one unknown slope at each panel joint and the total number of φ for all the panel joints of the truss considered is indicated by m, which denotes the total number of panel joints.

Summing up the above two kinds of unknowns we have $(n+m+1)$ unknowns in total, which are to be determined from three kind of equations, i.e., from joint-, panel-, and span-equilibrium equations.

II. Joint-Equilibrium Equation.

At panel joint r in *Fig. 2*, we have from equation (2):

$$M_{r\cdot r+1} = K_r\{2\varphi_r+\varphi_{r+1}+\mu_r\}\ ,$$

$$M_{r\cdot\bar{r}} = K'_r\{2\varphi_r+\varphi_{\bar{r}}+\mu'\}\ ,$$

$$M_{r\cdot r-1} = K_{r-1}\{2\varphi_r+\varphi_{r-1}+\mu_{r-1}\}\ .$$

From the equilibrium condition

$$M_{r\cdot r+1}+M_{r\cdot\bar{r}}+M_{r\cdot r-1}=0\ ,$$

we obtain the following equation:

$$\left.\begin{aligned}\varphi_r &= -(\varphi_{r-1}+\mu_{r-1})\gamma_{r-1}-(\varphi_{r+1}+\mu_r)\gamma_r-(\varphi_{\bar{r}}+\mu')\gamma'_r\\ &= -(\varphi_{\text{left}}+\mu_{\text{left}})\gamma_{\underset{(r)}{\text{left}}}-(\varphi_{\text{right}}+\mu_{\text{right}})\gamma_{\underset{(r)}{\text{right}}}\\ &\quad-(\varphi_{\text{lower}}+\mu_{\text{lower}})\gamma_{\underset{(r)}{\text{lower}}}\end{aligned}\right\}\ \ldots\ldots\ (4)$$

where

$$\left.\begin{aligned}&\gamma_{r-1}=\frac{K_{r-1}}{\rho_r}\ ,\quad \gamma_r=\frac{K_r}{\rho_r}\ ,\quad \gamma'_r=\frac{K'_r}{\rho_r}\\ &\rho_r=2(K_r+K_{r-1}+K'_r)\end{aligned}\right\}\ \ldots\ldots\ (5)$$

or

$$\left.\begin{aligned}\gamma_{\underset{(r)}{\text{left}}} &= \frac{K_{\text{left}}}{2\times(\text{Sum of } K \text{ of all the members which intersect at } r)}\\ \gamma_{\underset{(r)}{\text{right}}} &= \frac{K_{\text{right}}}{2\times(\text{Sum of } K \text{ of all the members which intersect at } r)}\\ \gamma_{\underset{(r)}{\text{lower}}} &= \frac{K_{\text{lower}}}{2\times(\text{Sum of } K \text{ of all the members which intersect at } r)}\end{aligned}\right\} . \quad (6)$$

In a similar way, we have the joint-equilibrium equation at the lower panel joint $\bar{r}$ as follows:

$$\varphi_{\bar{r}} = -(\varphi_{\text{left}}+\mu_{\text{left}})\gamma_{\underset{(\bar{r})}{\text{left}}}-(\varphi_{\text{right}}+\mu_{\text{right}})\gamma_{\underset{(\bar{r})}{\text{right}}} -(\varphi_{\text{upper}}+\mu_{\text{upper}})\gamma_{\underset{(\bar{r})}{\text{upper}}} \quad \ldots\ldots\ (7)$$

and μ_{upper} in (7) is absolutely the same quantity as μ_{lower} in (4).

We call these equations (4) and (7) " Joint-Equilibrium Equations" and there are obtained as many equations as the number of panel joints.

III. Panel-Equilibrium Equations.

As shown in *Fig. 2*, imazining two vertical sections very near by panel joints r and $r+1$, we have the equilibrium condition of the upper and lower chord moments:

$$M_{r,r+1}+M_{r+1,r}+M_{\bar{r},\overline{r+1}}+M_{\overline{r+1},\bar{r}}+\mathfrak{M}_r = 0 \quad \ldots\ldots\ldots\ (A)$$

where $\mathfrak{M}_r$ denotes the bending moment induced by panel shear for the r^{th} panel from the left end support;

and

$$M_{r,r+1} = K_r(2\varphi_r+\varphi_{r+1}+\mu_r)\ ,$$

$$M_{r+1,r} = K_r(2\varphi_{r+1}+\varphi_r+\mu_r)\ .$$

Therefore $\quad M_{r,r+1}+M_{r+1,r} = 3K_r\left\{\varphi_r+\varphi_{r+1}+\frac{2}{3}\mu_r\right\}\ .$

Similarly: $\quad M_{\bar{r},\overline{r+1}}+M_{\overline{r+1},\bar{r}} = 3K_{\bar{r}}\left\{\varphi_{\bar{r}}+\varphi_{\overline{r+1}}+\frac{2}{3}\mu_r\right\}\ .$

We get therefore from equation (A):

$$
\left.\begin{aligned}
&\mu_r = -D_r - (\varphi_r + \varphi_{r+1})t_r - (\varphi_{\bar{r}} + \varphi_{\overline{r+1}})t_{\bar{r}} \\
\text{where}\quad &D_r = \frac{\mathfrak{M}_r}{2(K_r + K_{\bar{r}})}, \quad t_r = \frac{3K_r}{2(K_r + K_{\bar{r}})} \\
&t_{\bar{r}} = \frac{3K_{\bar{r}}}{2(K_r + K_{\bar{r}})}
\end{aligned}\right\} \quad \ldots\ldots \quad (8)
$$

Each panel gives one equation of this kind and equation (8) is of the r^{th} panel, as mentioned above, and there are obtained as many equations as the number of panels.

IV. Span-Equilibrium Equation.

We imagine further one horizontal section near by the upper and lower panel joints, as shown in *Fig. 3,* considering each vertical member as a free body.

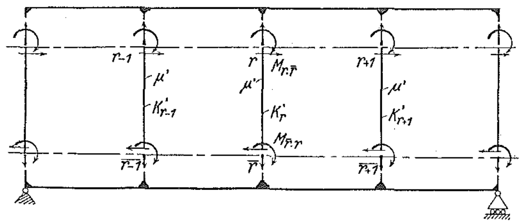


Fig. 3.

The equilibrium condition under the action of the moments at the two extremities of all the verticals in the given truss gives the following equation, considering together all of the verticals as a free body:

$$
\sum_{r=1}^{r=n}(M_{r,\bar{r}} + M_{\bar{r},r}) + Wh = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(B)}
$$

where n denotes the number of verticals, W the horizontal concentrated load applied at the upper panel joint in the direction from left to right, and h the height of verticals, i.e. the truss height.

If there exist no horizontal loads, the term Wh naturally vanishes. In equation (B)

$$
M_{r.\bar{r}} + M_{\bar{r}.r} = K_r'\{3\varphi_r + 3\varphi_{\bar{r}} + 2\mu'\}
$$

and from equation (B) we get:

$$3\sum_{r=1}^{r=n} K'_r\{\varphi_r+\varphi_{\bar{r}}\}+2\mu'\{K'_1+K'_2+\cdots+K'_n\}+Wh=0 \quad \ldots. \quad (9)$$

or putting

$$\left.\begin{array}{l} 2(K'_1+K'_2+\cdots+K'_n)=T \\ \dfrac{3K'_1}{T}=u'_1 \qquad \dfrac{3K'_2}{T}=u'_2 \cdots \dfrac{3K'_n}{T}=u'_n \end{array}\right\} \cdots\cdots\cdots (9)_0$$

we have from equation (9):

$$\mu'=-\frac{Wh}{T}-(\varphi_1+\varphi_{\bar{1}})u'_1-(\varphi_2+\varphi_{\bar{2}})u'_2-\cdots-(\varphi_n+\varphi_{\bar{n}})u'_n \quad \ldots.. \quad (10)$$

One span, i.e. one truss, gives one equation of this kind.

From equation (10) we know that the most effective term on μ' is the first term of the right-hand member of equation (10), i.e. $\frac{Wh}{T}$, because the numerical values of u'_1, u'_2 etc. are very small quantities relative to the value of $\frac{Wh}{T}$ and when there exist no horizontal loads, the value of μ' may be assumed to be approximately zero, putting $u'=0$ in it, and the true value is obtained by Repeated Trials.

V. Special Case.

When the chord sections are designed in order that the value of K for the upper chord member may be equal to that of the lower chord member in the same panel, the values of φ at both extremities of each vertical member become equal and the inflexion point of the elastic curve of each vertical member takes its position in the middle of its length.

The verification is as follows:
In equations (4) and (7), we put

$$\gamma_{\underset{(r)}{\text{left}}}=\gamma_{\underset{(\bar{r})}{\text{left}}}, \quad \gamma_{\underset{(r)}{\text{right}}}=\gamma_{\underset{(\bar{r})}{\text{right}}}, \quad \gamma_{\underset{(r)}{\text{lower}}}=\gamma_{\underset{(\bar{r})}{\text{upper}}}$$

or in other notations

$$\gamma_{r-1}=\gamma_{\overline{r-1}}, \quad \gamma_r=\gamma_{\bar{r}}, \quad \gamma'_r=\gamma'_{\bar{r}}.$$

Therefore (4)—(7) gives:

$$\varphi_r - \varphi_{\bar{r}} = (\varphi_{\overline{r-1}} - \varphi_{r-1})\gamma_{r-1} + (\varphi_{\overline{r+1}} - \varphi_{r+1})\gamma_r + (\varphi_r - \varphi_{\bar{r}})\gamma'_r$$

or

$$(\varphi_r - \varphi_{\bar{r}})(1-\gamma'_r) + (\varphi_{r-1} - \varphi_{\overline{r-1}})\gamma_{r-1} + (\varphi_{r+1} - \varphi_{\overline{r+1}})\gamma_r = 0 \,.$$

We get such n equations for n verticals and solving those n equations simultaneously, whose unknowns are $(\varphi_1 - \varphi_{\bar{1}})$, $(\varphi_2 - \varphi_{\bar{2}})\cdots\cdots$ $(\varphi_n - \varphi_{\bar{n}})$ gives $\varphi_1 = \varphi_{\bar{1}}$, $\varphi_2 = \varphi_{\bar{2}}$, $\cdots\cdots\varphi_n = \varphi_{\bar{n}}$, because the right-hand member of each equation is zero.

In such a special case as mentioned above, the treatment of the problem becomes very simplified and thus the problem may be solved only by treating either the elastic deformations for the upper half truss or those for the lower half.

VI. Mechanical Tabulation Method.

We explain the Mechanical Tabulation Method by a parellel chord truss of five equal panels, as shown in *Fig. 4*.

Combining equations (4) and (5) we have:

$$(\varphi_{r-1} + \mu_{r-1})K_{r-1} + \varphi_r\rho_r + (\varphi_{r+1} + \mu_r)K_r + (\varphi_{\bar{r}} + \mu')K'_r = 0 \ldots \quad (11)$$

At upper panel joint 1, putting $r=1$ in equation (11) gives:

(1) $\varphi_1\rho_1 + (\varphi_2 + \mu_1)K_1 + (\varphi_{\bar{1}} + \mu')K'_1 = 0$

At upper panel joint 2 we have:

(2) $(\varphi_1 + \mu_1)K_1 + \varphi_2\rho_2 + (\varphi_3 + \mu_2)K_2 + (\varphi_{\bar{2}} + \mu')K'_2 = 0$

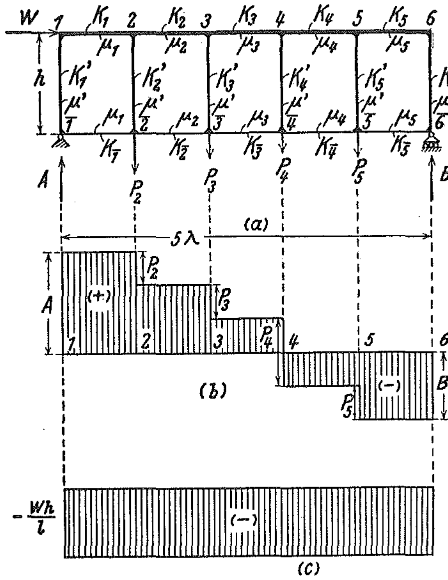


Fig. 4.

In a similar way, we get further four joint-equilibrium equations for the upper panel joints and furthermore six joint-equilibrium equations for the lower panel joints.

Table 1.

Equation	Left-Hand Member of Equation																		Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	$\varphi_{\bar{6}}$	$\varphi_{\bar{5}}$	$\varphi_{\bar{4}}$	$\varphi_{\bar{3}}$	$\varphi_{\bar{2}}$	$\varphi_{\bar{1}}$	μ_1	μ_2	μ_3	μ_4	μ_5	μ'	
(1)	ρ_1	K_1										K_1'	K_1					K_1'	0
(2)	K_1	ρ_2	K_2								K_2'		K_1	K_2				K_2'	0
(3)		K_2	ρ_3	K_3						K_3'				K_2	K_3			K_3'	0
(4)			K_3	ρ_4	K_4				K_4'						K_3	K_4		K_4'	0
(5)				K_4	ρ_5	K_5		K_5'								K_4	K_5	K_5'	0
(6)					K_5	ρ_6	K_6'										K_5	K_6'	0
(7)						K_6'	$\rho_{\bar{6}}$	$K_{\bar{5}}$									$K_{\bar{5}}$	K_6'	0
(8)					K_5'		$K_{\bar{5}}$	$\rho_{\bar{5}}$	$K_{\bar{4}}$							$K_{\bar{4}}$	$K_{\bar{5}}$	K_5'	0
(9)				K_4'				$K_{\bar{4}}$	$\rho_{\bar{4}}$	$K_{\bar{3}}$					$K_{\bar{3}}$	$K_{\bar{4}}$		K_4'	0
(10)			K_3'						$K_{\bar{3}}$	$\rho_{\bar{3}}$	$K_{\bar{2}}$			$K_{\bar{2}}$	$K_{\bar{3}}$			K_3'	0
(11)		K_2'								$K_{\bar{2}}$	$\rho_{\bar{2}}$	$K_{\bar{1}}$	$K_{\bar{1}}$	$K_{\bar{2}}$				K_2'	0
(12)	K_1'										$K_{\bar{1}}$	$\rho_{\bar{1}}$	$K_{\bar{1}}$					K_1'	0
(13)	t_1	t_1									$t_{\bar{1}}$	$t_{\bar{1}}$	1						$-D_1$
(14)		t_2	t_2							$t_{\bar{2}}$	$t_{\bar{2}}$			1					$-D_2$
(15)			t_3	t_3					$t_{\bar{3}}$	$t_{\bar{3}}$					1				$-D_3$
(16)				t_4	t_4			$t_{\bar{4}}$	$t_{\bar{4}}$							1			$-D_4$
(17)					t_5	t_5	$t_{\bar{5}}$	$t_{\bar{5}}$									1		$-D_5$
(18)	u_1'	u_2'	u_3'	u_4'	u_5'	u_6'	u_6'	u_5'	u_4'	u_3'	u_2'	u_1'						1	$-Wh/T$

Those equations are shown tabulated in *Table 1*, (1) to (12).

Next, panel-equilibrium equation (8) gives for the first panel:

$$\mu_1+(\varphi_1+\varphi_2)t_1+(\varphi_{\bar{1}}+\varphi_{\bar{2}})t_{\bar{1}} = -D_1$$

which is shown in Table 1, (13).

For the second panel:

$$\mu_2+(\varphi_2+\varphi_3)t_2+(\varphi_{\bar{2}}+\varphi_{\bar{3}})t_{\bar{2}} = -D_2$$

which is shown in Table 1, (14)

In a similar manner, we have further three panel-equilibrium equations, which are shown tabulated in *Table 1*, (15) to (17).

Finally, span-equilibrium equation (10) gives:

$$\mu'+(\varphi_1+\varphi_{\bar{1}})u_1'+(\varphi_2+\varphi_{\bar{2}})u_2'+\cdots\cdot+(\varphi_6+\varphi_{\bar{6}})u_6' = -\frac{Wh}{T}\,.$$

This equation is tabulated in *Table 1.* (18).

Table 1 shows a quite systematic arrangement of coefficients ρ, K, t, u, D and Wh/T.

The coefficient ρ finds itself in the table on a continuous line which may be expressed as a diagonal line of a square formed by columns of φ and rows of general equations, φ being taken horizontally and the number of the equations vertically.

This diagonal ρ-line forms a quite symmetrical axis of the table for the position of the coefficients tabulated.

We may thus obtain quite mechanically the simultaneous equations which solve problems of Vierendeel trusses with parallel chords, ithout starting from equations (4), (8) and (10).

VII. Effect of Vertical Loads.

If the horizontal joint load W shown in *Fig. 4* (a) vanishes, the term $-Wh/T$ disappears in the right-hand member of the equations in *Table 1*, and in the same table, D expresses:

$$\left.\begin{aligned} &D_1 = \frac{\mathfrak{M}_1}{2(K_1+K_{\bar{1}})}, \quad D_2 = \frac{\mathfrak{M}_2}{2(K_2+K_{\bar{2}})}, \quad D_3 = \frac{\mathfrak{M}_3}{2(K_3+K_{\bar{3}})}, \\ &D_4 = \frac{\mathfrak{M}_4}{2(K_4+K_{\bar{4}})}, \quad D_5 = \frac{\mathfrak{M}_5}{2(K_5+K_{\bar{5}})}. \end{aligned}\right\} \quad \dots (12)$$

In equation (12), $\mathfrak{M}_1, \mathfrak{M}_2, \cdots\cdots \mathfrak{M}_5$ denote the bending moments due to panel shears for the panels 1, 2, $\cdots\cdots$ 5 respectively and are expressed by the following expressions:

$$\left.\begin{aligned} &\mathfrak{M}_1 = A\lambda, \quad \mathfrak{M}_2 = (A-P_2)\lambda, \quad \mathfrak{M}_3 = (A-P_2-P_3)\lambda \\ &\mathfrak{M}_4 = (A-P_2-P_3-P_4)\lambda, \quad \mathfrak{M}_5 = (A-P_2-P_3-P_4-P_5)\lambda = -B\lambda \\ &\text{where} \qquad A = \text{Reaction at the support } A. \end{aligned}\right\} \quad \dots (13)$$

VIII. Effect of Horizontal Loads.

For the effect of horizontal joint load W, we have:

$$\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3 = \mathfrak{M}_4 = \mathfrak{M}_5 = -\frac{Wh}{l}\lambda.$$

(*see Fig. 4* (c))

IX. Mechanical Tabulation for the case, when the value of K is equal, both for the upper and lower chords in each panel.

In the case mentioned above, the joint-equilibrium equation, for joint r, is as follows:

$$\left.\begin{aligned} &\varphi_{r-1}K_{r-1}+\varphi_r q_r+\varphi_{r+1}K_r+\mu_r K_r+\mu_{r-1}K_{r-1}+\mu' K'_r=0 \\ \text{where}\qquad &q_r=2K_{r-1}+3K'_r+2K_r\,. \end{aligned}\right\}\quad\ldots\ldots\ (14)$$

Substituting $K_1=K_{\bar{1}}$, $K_2=K_{\bar{2}}\cdots\cdot K_5=K_{\bar{5}}$ in the truss shown in *Fig. 4* (*a*) and from equation (14) we get:

At panel joint 1

(1) $\qquad \varphi_1 q_1+\varphi_2 K_1+\mu_1 K_1+\mu' K'_1=0$

At panel Joint 2

(2) $\qquad \varphi_1 K_1+\varphi_2 q_2+\varphi_3 K_2+\mu_2 K_2+\mu_1 K_1+\mu' K'_2=0$

In a similar manner, we have six equations which are shown tabulated in *Table 2.*

And the panel-equilibrium equation gives in this case:

$$\mu_r=-D_r-1.5(\varphi_r+\varphi_{r+1})\ \ldots\ldots\ldots\ldots\ldots\ldots\ (15)$$

The first panel-equilibrium equation gives:

(7) $\qquad \mu_1=-D_1-1.5(\varphi_1+\varphi_2)$

The second panel-equilibrium equation gives:

(8) $\qquad \mu_2=-D_2-1.5(\varphi_2+\varphi_3)$

Table 2.

Equation	Left-Hand Member of Equation												Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3	μ_4	μ_5	μ'	
(1)	q_1	K_1					K_1					K'_1	0
(2)	K_1	q_2	K_2				K_1	K_2				K'_2	0
(3)		K_2	q_3	K_3				K_2	K_3			K'_3	0
(4)			K_3	q_4	K_4				K_3	K_4		K'_4	0
(5)				K_4	q_5	K_5				K_4	K_5	K'_5	0
(6)					K_5	q_6					K_5	K'_6	.0
(7)	1.5	1.5					1						$-D_1$
(8)		1.5	1.5					1					$-D_2$
(9)			1.5	1.5					1				$-D_3$
(10)				1.5	1.5					1			$-D_4$
(11)					1.5	1.5					1		$-D_5$
(12)	V'_1	V'_2	V'_3	V'_4	V'_5	V'_6						1	$-Wh/T$

Table 3.

Equation	Left-Hand Member of Equation												Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3	μ_4	μ_5	μ'	Coeff : $1/6K$
(1)	5	1					1					1	0
(2)	1	7	1				1	1				1	0
(3)		1	7	1				1	1			1	0
(4)			1	7	1				1	1		1	0
(5)				1	7	1				1	1	1	0
(6)					1	5					1	1	0
(7)	1	1					$2/3$						$-\mathfrak{M}_1$
(8)		1	1					$2/3$					$-\mathfrak{M}_2$
(9)			1	1					$2/3$				$-\mathfrak{M}_3$
(10)				1	1					$2/3$			$-\mathfrak{M}_4$
(11)					1	1					$2/3$		$-\mathfrak{M}_5$
(12)	1	1	1	1	1	1						$n/3$	$-Wh$

Further up to the sixth panel-equilibrium equation they are tabulated in the same table.

Finally, the span-equilibrium equation gives:

$$\mu' = -\frac{Wh}{T} - 2\varphi_1 u_1' - 2\varphi_2 u_2' - 2\varphi_3 u_3' - \cdots - 2\varphi_n u_n'$$

or putting $2u_r' = \nu_r'$ gives:

$$\nu_1' = \frac{3K_1'}{K_1' + K_2' + \cdots + K_n'}, \quad \nu_2' = \frac{3K_2'}{K_1' + K_2' + \cdots + K_n'} \quad \text{etc.}$$

And the span-equilibrium equation takes the following form:

$$\mu' = -\frac{Wh}{T} - \varphi_1 \nu_1' - \varphi_2 \nu_2' - \cdots - \varphi_n \nu_n' \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

This equation is also very easy to memorize, because it has a very simple and regular form.

Applying equation (16) to the truss of *Fig. 4 (a)*, we have:

$$(12) \qquad \mu' = -\frac{Wh}{T} - \varphi_1 \nu_1' - \varphi_2 \nu_2' - \cdots - \varphi_6 \nu_6'.$$

The twelve equations calculated above are shown in tabular form in *Table 2*, which shows the rules in the tabulation and properties of the table for the case, when the value of K is equal, both for the upper and lower chords in each panel.

X. Mechanical Tabulation for the case, when the value of K for all of the members is equal.

If the value of K for all the members of the whole truss shows an equal amount, the expressions q, K, and ν in *Table 2* become as follows:

$$q_1 = q_6 = 5K, \quad q_2 = q_3 = q_4 = q_5 = 7K,$$

$$K_1 = K_2 = \cdots K_1' = K_2' = \cdots = K$$

$$\nu_1' = \nu_2' = \cdots = \frac{3}{n} = \nu$$

and from *Table 2* we get *Table 3*.

XI. Calculation of Approximate Value of φ.

From equations (4) and (5) we get:

$$\varphi_r\rho_r = (\varphi_{r-1}+\mu_{r-1})K_{r-1}-(\varphi_{r+1}+\mu_r)K_r-(\varphi_{\bar{r}}+\mu')K'_r \;\ldots\ldots\; \text{(a)}$$

and equation (7) gives:

$$\varphi_{\bar{r}}\rho_{\bar{r}} = -(\varphi_{\overline{r-1}}+\mu_{r-1})K_{\overline{r-1}}-(\varphi_{\overline{r+1}}+\mu_r)K_{\bar{r}}-(\varphi_r+\mu')K'_r \;\ldots\; \text{(b)}$$

Considering further $\varphi_{r-1} = \varphi_{\overline{r-1}} = \varphi_r = \varphi_{\bar{r}} = \varphi_{r+1} = \varphi_{\overline{r+1}}$ and summing up the above two equations gives:

$$\begin{aligned}3\varphi_r(K_{r-1}+K_r+K_{\overline{r-1}}+K_{\bar{r}}+2K'_r)+\mu_{r-1}(K_{r-1}+K_{\overline{r-1}})\\ +\mu_r(K_r+K_{\bar{r}})+2\mu'K'_r = 0 \;\ldots\ldots\ldots\; \text{(c)}\end{aligned}$$

Panel-equilibrium equation (8) gives likewise:

$$\mu_r = -D_r-2\varphi_r(t_r+t_{\bar{r}}),$$

$$\mu_{r-1} = -D_{r-1}-2\varphi_r(t_{r-1}+t_{\overline{r-1}})$$

and from these two equations we have:

$$\mu_r(K_r+K_{\bar{r}}) = -\frac{\mathfrak{M}_r}{2}-3\varphi_r(K_r+K_{\bar{r}}),$$

$$\mu_{r-1}(K_{r-1}+K_{\overline{r-1}}) = -\frac{\mathfrak{M}_{r-1}}{2}-3\varphi_r(K_{r-1}+K_{\overline{r-1}}).$$

Summing up these two gives:

$$\begin{aligned}\mu_{r-1}(K_{r-1}+K_{\overline{r-1}})+\mu_r(K_r+K_{\bar{r}}) = -\frac{1}{2}(\mathfrak{M}_{r-1}+\mathfrak{M}_r)\\ -3\varphi_r(K_{r-1}+K_{\overline{r-1}}+K_r+K_{\bar{r}}) \;\ldots\ldots\; \text{(d)}\end{aligned}$$

From equation (10), putting in it $u' = 0$, we get approximately:

$$\mu' \doteqdot -\frac{Wh}{T} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(e)}$$

If there are no horizontal loads, it may be then assumed

$$\mu' \doteqdot 0.$$

Substituting equations (d) and (e) in (c) gives:

$$\varphi_r = \frac{\mathfrak{M}_{r-1}+\mathfrak{M}_r}{12K'_r}+\frac{Wh}{6(K'_1+K'_2+K'_3+\cdots\cdots+K'_n)} \quad \cdots\cdots\cdots \quad (17)$$

When W vanishes, the above equation becomes:

$$\varphi_r = \frac{\mathfrak{M}_{r-1}+\mathfrak{M}_r}{12K'_r}$$

XII. Calculation of Approximate Value of μ.

(A). μ of Chord Members.

The approximate value of μ_r (say $\mu_r^{(0)}$) is obtained from equation (8):

$$\left.\begin{aligned} \mu_r^{(0)} &= -D_r-(\varphi_r^{(0)}+\varphi_{r+1}^{(0)})(t_r+t_{\bar{r}}) \\ &= -D_r-\frac{3}{2}(\varphi_r^{(0)}+\varphi_{r+1}^{(0)}) \\ \text{where} \qquad D_r &= \frac{\mathfrak{M}_r}{2(K_r+K_{\bar{r}})} \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (19)$$

In equation (19) $\varphi_r^{(0)}$ and $\varphi_{r+1}^{(0)}$ are to be found by means of equation (17) or (18).

(B). μ of Vertical Members.

We denote by $\mu'^{(0)}$ the approximate value of verticals and from equation (10) we get:

$$\mu'^{(0)} = -\frac{Wh}{T}-2\varphi_1^{(0)}u'_1-2\varphi_2^{(0)}u'_2-2\varphi_3^{(0)}u'_3-\cdots\cdots-2\varphi_n^{(0)}u'_n \quad \cdots\cdots \quad (20)$$

The value worked out from equation (20) is more accurate than that from equation (e).

XIII. Numerical Examples.

Example 1. The horizontal parallel chord truss of Vierendeel type with five equal panels as shown in *Fig. 5* carries a single concentrated load P at the lower panel joint $\bar{5}$. This truss is assumed to have equal values of K for all members of the truss.

It is required to find all the bending moments at each panel joint.

Looking at *Table 3* quite mechanically we get *Table 4* which gives the simultaneous elastic equations, from which the unknown φ and μ are to be determined.

Here one has:

$$\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3 = \mathfrak{M}_4 = \frac{1}{5} P\lambda\,,$$

$$\mathfrak{M}_5 = -\frac{4}{5} P\lambda\,.$$

(*Table 3*)

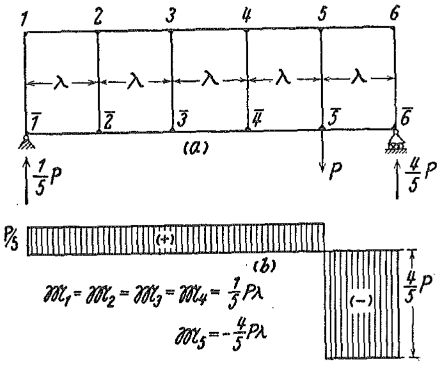


Fig. 5.

Table 4.

Equation	Left-Hand Member of Equation												Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3	μ_4	μ_5	μ'	Coeff : $\frac{P\lambda}{30K}$
(1)	5	1					1					1	0
(2)	1	7	1				1	1				1	0
(3)		1	7	1				1	1			1	0
(4)			1	7	1				1	1		1	0
(5)				1	7	1				1	1	1	0
(6)					1	5					1	1	0
(7)	1	1					2/3						−1
(8)		1	1					2/3					−1
(9)			1	1					2/3				−1
(10)				1	1					2/3			−1
(11)					1	1					2/3		4
(12)	1	1	1	1	1	1						2	0

Next we have from equation (18):

$$\varphi_1^{(0)} = \frac{\mathfrak{M}_1}{12K} = \frac{P\lambda}{5}\frac{1}{12K} = \frac{P\lambda}{30K}\left(\frac{1}{2}\right),$$

$$\varphi_2^{(0)} = \frac{\mathfrak{M}_1 + \mathfrak{M}_2}{12K} = \frac{P\lambda}{30K}(1)\,,$$

$$\varphi_3^{(0)} = \varphi_4^{(0)} = \frac{P\lambda}{30K}(1)\,,$$

$$\varphi_5^{(0)} = \frac{\mathfrak{M}_4+\mathfrak{M}_5}{12K} = \frac{\frac{1}{5}+\left(-\frac{4}{5}\right)}{12K}P\lambda = \frac{P\lambda}{30K}\left(-\frac{3}{2}\right),$$

$$\varphi_6^{(0)} = \frac{\mathfrak{M}_5}{12K} = \frac{-\frac{4}{5}P\lambda}{12K} = \frac{P\lambda}{30K}(-2)\,.$$

Regarding $P\lambda/(30K)$ as a coefficient and disregarding it one has;

$$\varphi_1^{(0)} = 0.5\,, \qquad \varphi_2^{(0)} = 1.0\,, \qquad \varphi_3^{(0)} = 1.0\,,$$

$$\varphi_4^{(0)} = 1.0\,, \qquad \varphi_5^{(0)} = -1.5\,, \qquad \varphi_6^{(0)} = -2.0\,.$$

Putting $K_r = K_{\bar{r}} = K$ in equation (19) gives:

$$D_1 = \frac{\mathfrak{M}_1}{4K} = \frac{1}{5}P\lambda\left(\frac{1}{4K}\right) = \frac{P\lambda}{20K} = \frac{P\lambda}{30K}\left(\frac{3}{2}\right),$$

$$D_2 = D_3 = D_4 = \frac{P\lambda}{30K}\left(\frac{3}{2}\right),$$

$$D_5 = \frac{1}{4K}\left(-\frac{4}{5}P\lambda\right) = \frac{P\lambda}{30K}(-6)\,.$$

and therefore

$$\mu_1^{(0)} = -D_1-\frac{3}{2}(\varphi_1^{(0)}+\varphi_2^{(0)}) = -1.5-1.5(0.5+1.0) = -3.75\,,$$

$$\mu_2^{(0)} = -D_2-\frac{3}{2}(\varphi_2^{(0)}+\varphi_3^{(0)}) = -1.5-1.5(1.0+1.0) = -4.5\,,$$

$$\mu_3^{(0)} = -D_3-\frac{3}{2}(\varphi_3^{(0)}+\varphi_4^{(0)}) = -4.5\,,$$

$$\mu_4^{(0)} = -0.75\,, \qquad \mu_5^{(0)} = 11.25\,.$$

and

$$u_1' = u_2' = u_3' = u_4' = u_5' = u_6' = \frac{3K}{2\times 6K} = 0.25\,,$$

$$\mu'^{(0)} = -2u'(\varphi_1^{(0)}+\varphi_2^{(0)}+\varphi_3^{(0)}+\varphi_4^{(0)}+\varphi_5^{(0)}+\varphi_6^{(0)})$$

$$= -2\times 0.25(0.5+1.0+1.0+1.0-1.5-2.0) = 0\,.$$

Using the values of $\varphi^{(0)}$ and $\mu^{(0)}$ calculated above, we find the approximate values of φ and μ as follows.
From *Table 4*, (1) we get the first approximate value of φ_1:

$$\varphi_1^{(1)} = \frac{1}{5}(-\varphi_2^{(0)} - \mu_1^{(0)} - \mu'^{(0)}) = 0.2(-1.0+3.75-0) = 0.55\,.$$

From *Table 4*, (2) we get the first approximate value of φ_2:

$$\varphi_2^{(1)} = \frac{1}{7}(-\varphi_1^{(1)} - \varphi_3^{(0)} - \mu_1^{(0)} - \mu_2^{(0)} - \mu'^{(0)})$$

$$= \frac{1}{7}(-0.55-1.0+3.75+4.5-0) = 0.957\,.$$

Similarly we have from *Table 4*, (3):

$$\varphi_3^{(1)} = \frac{1}{7}(-0.957-1.0+4.5+4.5-0) = 1.006\,.$$

From (4), (5) and (6) one has:

$$\varphi_4^{(1)} = 0.82\,, \qquad \varphi_5^{(1)} = 0.383\,, \qquad \varphi_6^{(1)} = -2.33\,.$$

And next from (7):

$$\mu_1^{(1)} = \frac{3}{2}(-1-\varphi_1^{(1)}-\varphi_2^{(1)}) = \frac{3}{2}(-1-0.55-0.957) = -3.76\,.$$

From (8):

$$\mu_2^{(1)} = \frac{3}{2}(-1-\varphi_2^{(1)}-\varphi_3^{(1)}) = 1.5(-1-0.957-1.006) = -4.44\,.$$

From (9):

$$\mu_3^{(1)} = 1.5(-1-\varphi_3^{(1)}-\varphi_4^{(1)}) = 1.5(-1-1.006-0.82) = -4.24\,.$$

From (10):

$$\mu_4^{(1)} = 1.5(-1-0.82-0.383) = -3.3\,.$$

From (11):

$$\mu_5^{(1)} = 1.5(4-0.383+2.33) = 8.92\,.$$

From (12):

$$\mu' = \frac{1}{2}(-\varphi_1^{(1)}-\varphi_2^{(1)}-\varphi_3^{(1)}-\varphi_4^{(1)}-\varphi_5^{(1)}-\varphi_6^{(1)})$$

$$= 0.5(-0.55-0.957-1.006-0.82-0.383+2.33) = -0.693\,.$$

The further calculation with these results gives in a similar way:

$$\varphi_1^{(2)} = 0.7\,, \qquad \varphi_2^{(2)} = 1.03\,, \qquad \varphi_3^{(2)} = 1.08\,,$$

$$\varphi_4^{(2)} = 1.01\,, \qquad \varphi_5^{(2)} = -0.52\,, \qquad \varphi_6^{(2)} = -1.54\,,$$

$$\mu_1^{(2)} = -4.1\,, \qquad \mu_2^{(2)} = -4.67\,, \qquad \mu_3^{(2)} = -4.64\,,$$

$$\mu_4^{(2)} = -2.24\,, \qquad \mu_5^{(2)} = 9.09\,, \qquad \mu' = -0.88\,.$$

Table 5 shows the calculation results up to the fifth approximation.

Table 5.

Coefficient: $\dfrac{P\lambda}{30K}$

	Approximate calculation	First approximation	Second approximation	Third approximation	Fourth approximation	Fifth approximation
φ_1	0.5	0.55	0.7	0.79	0.83	0.846
φ_2	1.0	0.957	1.03	1.11	1.17	1.2
φ_3	1.0	1.006	1.08	1.15	1.19	1.22
φ_4	1.0	0.82	1.01	1.02	1.02	1.01
φ_5	-1.5	0.383	-0.52	-0.78	-0.88	-0.93
φ_6	-2.0	-2.33	-1.54	-1.49	-1.53	-1.56
μ_1	-3.75	-3.76	-4.1	-4.35	-4.5	-4.57
μ_2	-4.5	-4.44	-4.67	-4.89	-5.04	-5.13
μ_3	-4.5	-4.24	-4.64	-4.76	-4.82	-4.85
μ_4	-0.75	-3.3	-2.24	-1.86	-1.71	-1.62
μ_5	11.25	8.92	9.09	9.4	9.62	9.7
μ'	0	-0.693	-0.88	-0.9	-0.9	-0.89

With these fifth approximate values of φ and μ, we find the joint moments from equation (2):

$$M_{12} = K\{2\varphi_1 + \varphi_2 + \mu_1\}$$

$$= K\{2(0.846) + 1.2 - 4.57\}\frac{P\lambda}{30K} = -0.055\,P\lambda\,,$$

$$M_{23} = K\{2\varphi_2 + \varphi_3 + \mu_2\}$$

$$= K\{2(1.2) + 1.22 - 5.13\}\frac{P\lambda}{30K} = -0.05\,P\lambda\,,$$

$$M_{34} = -0.047\,P\lambda\,, \quad M_{45} = -0.018\,P\lambda\,, \quad M_{56} = +0.21\,P\lambda\,.$$

In a similar way

$$M_{21} = -0.04\,P\lambda\,, \qquad M_{32} = -0.05\,P\lambda\,, \qquad M_{43} = -0.054\,P\lambda\,,$$

$$M_{54} = -0.082\,P\lambda\,, \quad M_{65} = 0.19\,P\lambda\,,$$

and the joint moment at the upper end of each vertical is as follows:

$$M_{1\bar{1}} = K\{3\varphi_1+\mu'\} = K\{3(0.846)-0.89\} = 0.055\,P\lambda\,,$$

$$M_{2\bar{2}} = 0.09\,P\lambda\,, \qquad M_{3\bar{3}} = 0.09\,P\lambda\,, \qquad M_{4\bar{4}} = 0.071\,P\lambda\,,$$

$$M_{5\bar{5}} = -0.123\,P\lambda\,, \qquad M_{6\bar{6}} = -0.19\,P\lambda\,.$$

Example 2. The horizontal parallel chord truss of Vierendeel type with five equal panels as shown in *Fig. 6* carries a single concentrated load P at the lower panel joint 4. This truss is assumed to have equal values of K for all members of the truss.

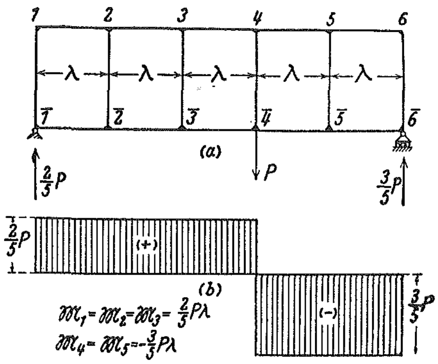


Fig. 6.

It is required to determine all the joint moments of the truss. In a similar way as for *Table 4*, we get quite mechanically *Table 6* which gives the simultaneous elastic equations from which the unknowns φ and μ are to be determined.

Table 6.

Equation	Left-Hand Member of Equation												Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	μ_1	μ_2	μ_3	μ_4	μ_5	μ'	Coeff : $\frac{P\lambda}{30K}$
(1)	5	1					1					1	0
(2)	1	7	1				1	1				1	0
(3)		1	7	1				1	1			1	0
(4)			1	7	1				1	1		1	0
(5)				1	7	1				1	1	1	0
(6)					1	5					1	1	0
(7)	1	1					2/3						−2
(8)		1	1					2/3					−2
(9)			1	1					2/3				−2
(10)				1	1					2/3			3
(11)					1	1					2/3		3
(12)	1	1	1	1	1	1						2	0

Here we have :

$$\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3 = \frac{2}{5}P\lambda\,, \quad \mathfrak{M}_4 = \mathfrak{M}_5 = -\frac{3}{5}P\lambda\,.$$

Quite analogous to the former example, we secure the calculation results up to the fourth approximation which are tabulated in *Table 7*.

Table 7.

Coefficient : $\frac{P\lambda}{30K}$

	Approximate calculation	First approximation	Second approximation	Third approximation	Fourth approximation
φ_1	1.0	1.1	1.11	1.11	1.11
φ_2	2.0	1.9	1.86	1.85	1.84
φ_3	2.0	1.8	1.72	1.69	1.67
φ_4	-0.5	-0.47	-0.48	-0.49	-0.49
φ_5	-3.0	-2.7	-2.6	-2.56	-2.55
φ_6	-1.5	-1.7	-1.7	-1.69	-1.68
μ_1	-7.5	-7.5	-7.46	-7.44	-7.42
μ_2	-9.0	-8.5	-8.37	-8.31	-8.26
μ_3	-5.25	-5.0	-4.86	-4.8	-4.77
μ_4	9.75	9.3	9.12	9.08	9.06
μ_5	11.25	11.1	10.95	10.88	10.85
μ'	0	0.035	0.045	0.045	0.05

With the fourth approximate values of φ and μ there are worked out the joint moments as follows :

$$M_{12} = -0.11\,P\lambda\,, \qquad M_{1\bar{1}} = 0.11\,P\lambda\,;$$

$$M_{23} = -0.1\,P\lambda\,, \qquad M_{21} = -0.09\,P\lambda\,, \qquad M_{2\bar{2}} = 0.19\,P\lambda\,;$$

$$M_{34} = -0.06\,P\lambda\,, \qquad M_{32} = -0.10\,P\lambda\,, \qquad M_{3\bar{3}} = 0.17\,P\lambda\,;$$

$$M_{45} = 0.18\,P\lambda\,, \qquad M_{43} = -0.14\,P\lambda\,, \qquad M_{4\bar{4}} = -0.04\,P\lambda\,;$$

$$M_{56} = 0.136\,P\lambda\,, \qquad M_{54} = 0.116\,P\lambda\,, \qquad M_{5\bar{5}} = -0.253\,P\lambda\,;$$

$$M_{65} = 0.165\,P\lambda\,, \qquad M_{6\bar{6}} = -0.166\,P\lambda\,.$$

Example 3. For the Vierendeel truss shown in *Fig. 6*, there are assumed in the problem the following conditions : the value of K for the upper chord members and verticals is assumed to be equal and to be K; the value of K for the lower chord members is assumed to be $0.5\,K$. The load condition is the same as in the previous example 2 and it is required to find all the joint moments of the truss.

Looking at *Table 1* we have *Table 8* quite mechanically, from which the unknown φ and μ are to be determined.

For the determination of the unknowns φ and μ from *Table 8*, we apply the resulting values of φ and μ, which have been obtained in the previous example, as the assumed values, because for both trusses the load condition as well as the stiffness condition are closely similar.

Table 9 shows the calculation results up to the fourth approximate values, with which joint moments have been calculated and written in *Fig. 7* in the corresponding places of every panel joints.

Table 8.

Equation	Left-Hand Member of Equation																		Right-Hand Member
	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	$\varphi_{\bar{6}}$	$\varphi_{\bar{5}}$	$\varphi_{\bar{4}}$	$\varphi_{\bar{3}}$	$\varphi_{\bar{2}}$	$\varphi_{\bar{1}}$	μ_1	μ_2	μ_3	μ_4	μ_5	μ'	Coeff: $\frac{P\lambda}{30K}$
(1)	4	1										1	1					1	0
(2)	1	6	1								1		1	1				1	0
(3)		1	6	1						1				1	1			1	0
(4)			1	6	1				1						1	1		1	0
(5)				1	6	1		1								1	1	1	0
(6)					1	4	1										1	1	0
(7)						1	3	0,5									0,5	1	0
(8)					1		0,5	4	0,5							0,5	0,5	1	0
(9)				1				0,5	4	0,5					0,5	0,5		1	0
(10)			1						0,5	4	0,5			0,5	0,5			1	0
(11)		1								0,5	4	0,5	0,5	0,5				1	0
(12)	1										0,5	3	0,5					1	0
(13)	1	1									0,5	0,5	1						−4,0
(14)		1	1							0,5	0,5			1					−4,0
(15)			1	1					0,5	0,5					1				−4,0
(16)				1	1			0,5	0,5							1			6,0
(17)					1	1	0,5	0,5									1		6.0
(18)	1	1	1	1	1	1	1	1	1	1	1	1						4	0

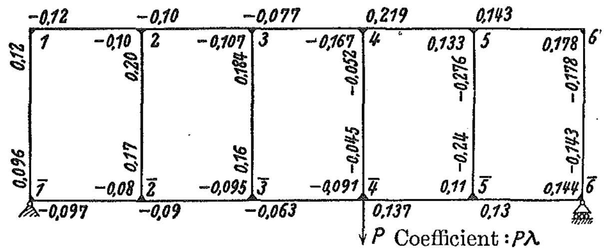


Fig. 7.

Table 9.

Coefficient: $\frac{P\lambda}{30K}$

	Approximate calculation	First approximation	Second approximation	Third approximation	Fourth approximation
φ_1	1.11	1.34	1.41	1.43	1.44
φ_2	1.84	2.04	2.14	2.19	2.21
φ_3	1.67	1.92	2.03	2.08	2.10
φ_4	-0.49	-0.6	-0.63	-0.62	-0.615
φ_5	-2.55	-2.95	-3.1	-3.17	-3.19
φ_6	-1.68	-1.94	-2.06	-2.12	-2.15
$\varphi_{\bar{6}}$	-1.68	-0.84	-1.0	-1.08	-1.10
$\varphi_{\bar{5}}$	-2.55	-1.7	-1.8	-1.93	-1.98
$\varphi_{\bar{4}}$	-0.49	-0.42	-0.34	-0.37	-0.40
$\varphi_{\bar{3}}$	1.67	1.02	1.16	1.23	1.25
$\varphi_{\bar{2}}$	1.84	1.22	1.32	1.41	1.46
$\varphi_{\bar{1}}$	1.11	0.64	0.63	0.67	0.69
μ_1	-7.42	-7.88	-8.36	-8.59	-8.70
μ_2	-8.26	-8.63	-9.2	-9.49	-9.63
μ_3	-4.77	-5.48	-5.73	-5.83	-5.89
μ_4	9.06	10.10	10.62	10.88	10.98
μ_5	10.85	11.50	12.29	12.66	12.83
μ'	0.05	0.045	0.055	0.07	0.072

B. The Slope Distribution Method Applied to the Analysis of Vierendeel Trusses of Parallel Chords.

The assumptions in the present method are the same as in the Mechanical Tabulation Method or in the Slope Deflection Method. Namely, (1) the connections between the verticals and chord members are perfectly rigid, (2) the length of a member is not changed by direct stress, and the deformation of a member due to the internal shearing stress is zero and (3) the settlement of the foundations is disregarded.

The method is divided into two parts: one of which treats the problems of different K for upper and lower chord members in a panel and the other part treats the problems of a Vierendeel truss, which has the same values of K for upper and lower chord members in a panel.

I. The Slope Distribution Method Applied to the Problems with Different Value of K for Upper and Lower Chord Members.

With respect to the truss and loading conditions given in Fig. 8, there are worked out the values of ρ from equation (5):

$$\rho_1 = \rho_6 = 2(K+K) = 4K\ , \quad \rho_2 = \rho_3 = \rho_4 = \rho_5 = 6K\ ,$$

$$\rho_{\bar{1}} = \rho_{\bar{6}} = 2(K+0.5K) = 3K\ , \quad \rho_{\bar{2}} = \rho_{\bar{3}} = \rho_{\bar{4}} = \rho_{\bar{5}} = 4K\ .$$

With these values of ρ we have from equation (5):
at panel joint 1

$$\gamma_{\substack{\text{right}\\(1)}} = \frac{K_1}{\rho_1} = \frac{K}{4K} = 0.25\ , \quad \gamma_{\substack{\text{lower}\\(1)}} = \frac{K_1'}{\rho_1} = \frac{K}{4K} = 0.25\ ;$$

at panel joint $\bar{1}$

$$\gamma_{\substack{\text{right}\\(\bar{1})}} = \frac{K_{\bar{1}}}{\rho_{\bar{1}}} = \frac{0.5K}{3K} = 0.1667\ , \quad \gamma_{\substack{\text{upper}\\(\bar{1})}} = \frac{K_1'}{\rho_{\bar{1}}} = \frac{K}{3K} = 0.3333\ ,$$

at panel joint 2

$$\gamma_{\substack{\text{right}\\(2)}} = \frac{K_2}{\rho_2} = \frac{K}{6K} = 0.1667\ , \quad \gamma_{\substack{\text{left}\\(2)}} = \frac{K_1}{\rho_2} = \frac{K}{6K} = 0.1667\ ,$$

$$\gamma_{\substack{\text{lower}\\(2)}} = \frac{K_1'}{\rho_2} = \frac{K}{6K} = 0.1667\ .$$

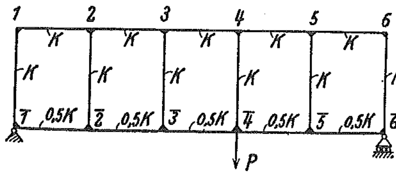
at panel joint $\bar{2}$

$$\gamma_{\substack{\text{right}\\(\bar{2})}} = \frac{K_{\bar{2}}}{\rho_{\bar{2}}} = \frac{0.5}{4K} = 0.125\ , \quad \gamma_{\substack{\text{left}\\(\bar{2})}} = \frac{K_{\bar{2}}}{\rho_{\bar{2}}} = \frac{0.5K}{4K} = 0.125\ ,$$

$$\gamma_{\substack{\text{upper}\\(\bar{2})}} = \frac{K_2'}{\rho_{\bar{2}}} = \frac{K}{4K} = 0.25\ .$$

In a similar way, there are calculated all the values of γ in every panel joint.

Next we calculate t from equation (8):
for the first panel



Coefficient: $P\lambda$

Fig. 8.

$$t_1 = \frac{3K_1}{2(K_1+K_{\bar{1}})} = \frac{3K}{2(K+0.5K)} = 1\,,$$

$$t_{\bar{1}} = \frac{3K_{\bar{1}}}{2(K_1+K_{\bar{1}})} = \frac{3(0.5K)}{2(K+0.5K)} = 0.5\,;$$

for the second panel

$$t_2 = \frac{3K_2}{2(K_2+K_{\bar{2}})} = \frac{3K}{2(K+0.5K)} = 1\,,$$

$$t_{\bar{2}} = \frac{3K_{\bar{2}}}{2(K_2+K_{\bar{2}})} = \frac{3(0.5K)}{2(K+0.5K)} = 0.5\,.$$

In a similar way, there are calculated all the values of t for all the panels.

Further we work out u' from equation $(9)_0$:
for the first vertical

$$u'_1 = \frac{3K'_1}{T} = \frac{3K'_1}{2(K'_1+K'_2+K'_3+\cdots+K'_6)} = \frac{3K}{2(6K)} = 0.25\,.$$

In a similar way we have

$$u'_2 = u'_3 = u'_4 = u'_5 = u'_6 = 0.25\,.$$

Then we work out $\mathfrak{M}$ in order to find D in equation (8). As shown in Fig. 6, the value of $\mathfrak{M}$ is given by the following expression:

$$\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}_3 = \frac{2}{5}P\lambda\,, \quad \mathfrak{M}_4 = \mathfrak{M}_5 = -\frac{3}{5}P\lambda\,.$$

Therefore from equation (8):

$$D_1 = \frac{\mathfrak{M}_1}{2(K_1+K_{\bar{1}})} = \frac{2P\lambda}{5\times 2(K+0.5K)} = \frac{2P\lambda}{15K}\,.$$

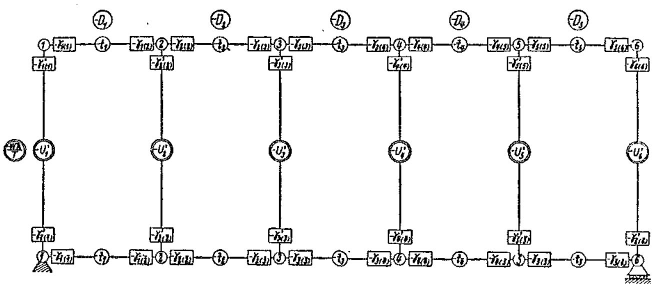


Fig. 9.

For the sake of convenience to compare with the previous example, we use the multiplier $P\lambda/(30K)$ and the expression of D_1 is as follows :

$$D_1 = \frac{2P\lambda}{15K} = \frac{P\lambda}{30K}(4) .$$

In a similar way

$$D_2 = D_3 = \frac{P\lambda}{30K}(4) , \quad D_4 = D_5 = \frac{P\lambda}{30K}(-6) .$$

The calculating operation, in this proposed method of slope distribution, is worked out on the given truss-schema, writting down these numericel values of γ, t, u and D, whose writing positions are as shown in Fig. 9. For convenience of calculation there are considered always minus signs as shown in Fig. 9 and 10.

First Calculation of Approximate Values.

The ordinary method of calculation is used to determine the true values of φ and μ by solving elastic equations simultaneously, while the propesed method is to show, step by step, the approach to the true values by our special procedure of calculation, which is carried out on the truss-schema itself. It is necessary, however, for the proposed method firstly to have the approximate values of φ and μ, which are the nucleus, so to speak, of the true values.

As the proposed example has no horizontal loading, equation (18) gives :

$$\varphi_r = \frac{\mathfrak{M}_{(\text{left})} + \mathfrak{M}_{(\text{right})}}{12K_r'}$$

and therefore

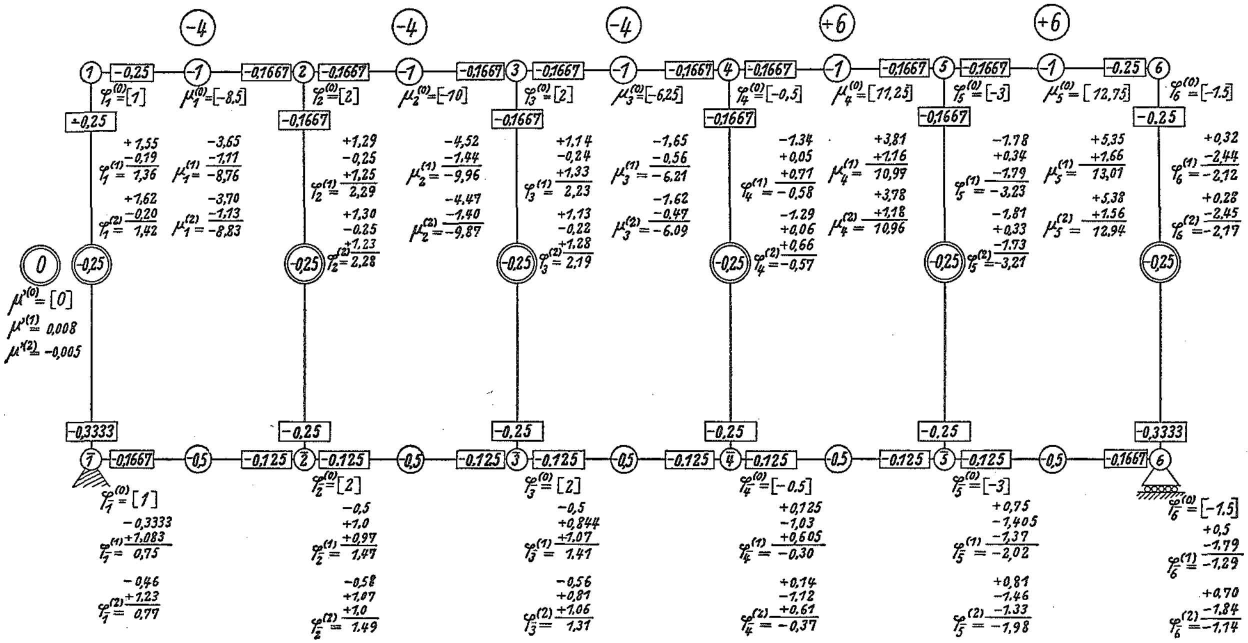
$$\varphi_1^{(0)} = \frac{2P\lambda}{5(12K)} = \frac{P\lambda}{30K}(1) , \quad \varphi_{\bar{1}}^{(0)} = \frac{2P\lambda}{5(12K)} = \frac{P\lambda}{30K}(1) ,$$

$$\varphi_2^{(0)} = \varphi_{\bar{2}}^{(0)} = \frac{4P\lambda}{5(12K)} = \frac{P\lambda}{30K}(2) , \quad \varphi_3^{(0)} = \varphi_{\bar{3}}^{(0)} = \frac{P\lambda}{30K}(2) ,$$

$$\varphi_4^{(0)} = \varphi_{\bar{4}}^{(0)} = \frac{(-3+2)P\lambda}{5(12K)} = \frac{P\lambda}{30K}(-0.5) ,$$

$$\varphi_5^{(0)} = \varphi_{\bar{5}}^{(0)} = \frac{-6P\lambda}{5(12K)} = \frac{P\lambda}{30K}(-3) ,$$

$$\varphi_6^{(0)} = \varphi_{\bar{6}}^{(0)} = \frac{-3P\lambda}{5(12K)} = \frac{P\lambda}{30K}(-1.5) .$$



Coefficient: $\frac{P\lambda}{30K}$

Fig. 10.

The approximate values of μ are determined from equation (8):

$$\mu_r^{(0)} = -D_r-(\varphi_r+\varphi_{r+1})t_r-(\varphi_{\bar{r}}+\varphi_{\overline{r+1}})t_{\bar{r}}$$

and therefore

$$\mu_1^{(0)} = -D_1-(\varphi_1^{(0)}+\varphi_2^{(0)})t_1-(\varphi_{\bar{1}}^{(0)}+\varphi_{\bar{2}}^{(0)})t_{\bar{1}}$$

$$= -4-(1+2)(+1)-(1+2)(+0.5) = -8.5\ .$$

In a similar way

$$\mu_2^{(0)} = -D_2-(\varphi_2^{(0)}+\varphi_3^{(0)})t_2-(\varphi_{\bar{2}}^{(0)}+\varphi_{\bar{3}}^{(0)})t_{\bar{2}}$$

$$= -4-(2+2)(+1)-(2+2)(0.5) = -10\ ,$$

$$\mu_3^{(0)} = -4-(2-0.5)(1)-(2-0.5)(0.5) = -6.25\ ,$$

$$\mu_4^{(0)} = +6-(-0.5-3)(1)-(-0.5-3)(0.5) = 11.25\ ,$$

$$\mu_5^{(0)} = +6-(-3-1.5)(1)-(-3-1.5)(0.5) = 12.75\ .$$

The approximate value of μ' is determined from equation (10):

$$\mu'^{(0)} = -(\varphi_1^{(0)}+\varphi_{\bar{1}}^{(0)})u_1'-(\varphi_2^{(0)}+\varphi_{\bar{2}}^{(0)})u_2'-(\varphi_3^{(0)}+\varphi_{\bar{3}}^{(0)})u_3'-\cdots$$

$$-(\varphi_6^{(0)}+\varphi_{\bar{6}}^{(0)})u_6' = (1+1)(-0.25)+(2+2)(-0.25)$$

$$+(2+2)(-0.25)+(-0.5-0.5)(-0.25)+(-3-3)(-0.25)$$

$$+(-1.5-1.5)(-0.25) = (-0.25)(0) \doteq 0\ .$$

That the calculation result gives zero shows that the effect of vertical loading is negligible upon the value of μ', as we have already proved in the paragraph on Span-Equilibrium Equation.

All these approximate values of $\varphi^{(0)}$ are written in the corresponding places under each joint as shown in Fig. 10. The similar values of $\mu^{(0)}$ are written in the corresponding panel as shown in the same figure, and that of $\mu'^{(0)}$ is written in the space of the left outside of the truss schema.

These are the preparatory calculations of the proposed method. The proposed calculation process will be explained in the following article.

Method of Calculation on Truss Schema itself.

It is required, in this section, to find the statically indeterminate quantities φ and μ. This calculation is not made by solving elastic equations simultaneously, but is, in the present method, to be worked out quite mechanically on the truss schema itself.

We begin the calculation at the lower panel joint $\overline{1}$ and then $\overline{2}$ and so on up to joint $\overline{6}$; from the last lower panel joint the calculation is carried out along the upper panel joints from the right to the left, i.e. the calculation order for the upper panel joints is of the joints 6, 5, 4, 3, 2 and 1. Next, the calculation of $\mu_1^{(1)}$ is worked out in the space of the first panel and then the calculation of $\mu_2^{(1)}$ in the second panel and so on. The calculation of $\mu'^{(1)}$ is to be carried out in the space of the left outside of the truss schema. Such calculation procedure is to be repeated in the same way as mentioned above and the detailed calculation at each panel joint or panel space is as follows.

Calculation at Joint $\overline{1}$.

Firstly required the sum of φ and μ' which gives $(1+0)=1$. The numerical value of this sum is multiplied by the numerical value in the parenthesis which is found above the joint considered, i.e., $(-0.3333).(1+0)=-0.3333$.

This calculation result is written under the panel joint $\overline{1}$ as shown in Fig. 10.

Next, in a similar way, required the sum of φ and μ with respect to the right element: this sum is multipled by the value in the parenthesis to the right of the joint $\overline{1}$. This calculation is carried out by slide-rule computation as follows: $(-0.1667)\{2+(-8.5)\}=+1.083$. This calculation result is written down under the previous calculation result (-0.3333). See Fig. 10.

And then required the sum of these two.

$$\begin{array}{ll} \text{This gives:} & -0.3333 \\ & \underline{+1.083} \\ \text{Approximately} & +0.75 \end{array}$$

This is the first approximate value of $\varphi_{\overline{1}}$.

Calculation at Joint $\overline{2}$.

The calculation at joint $\overline{1}$ has been of the joint composed of two members, while the other joints are composed of two chord members and one vertical; therefore the summation as well as the multiplication are to be carried out in three directions. The numerical calculation procedure is shown in Fig. 10.

For upwards: $(-0.25)(2+0)=-0.5$,
For the right: $(-0.125)(2-10)=+1.0$,
For the left: $(-0.125)(0.75-8.5)=+0.97$.

The summation of these three gives 1.47 as shown in Fig. 10 which is the first approximate value of $\varphi_{\bar{2}}$, i.e. $\varphi_{\bar{2}}^{(1)}$.

In a similar way, the calculation is carried out at each panel joint in the lower chord member as well as in the upper.

That is to say, as shown in Fig. 10, for example:

$$\varphi_{\bar{3}}^{(1)} = 1.41 , \qquad \varphi_{\bar{6}}^{(1)} = -1.29 , \qquad \varphi_{6}^{(1)} = -2.12 ,$$

$$\varphi_{4}^{(1)} = -0.58 , \qquad \varphi_{1}^{(1)} = 1.36 .$$

Calculation in Panel 1.

For the determination of chord-member-revolution angle, the numerical value in the small circle shown in the midst of each chord member is to be multiplied by the sum of the values φ which are found at both extremities of the chord member considered. This calculation is carried out for the upper as well as for lower chord for one panel under consideration; for example

For the first panel or panel 1:

For the upper chord member:

$$(-1)(1.36+2.29) = -3.65,$$

For the lower chord member:

$$(-0.5)(0.75+1.47) = -1.11.$$

Next, add the numerical value written in the circle above the upper chord member to the sum of the above two and find the total of all these. For the case under consideration, this total sum results -8.76. This represents the first approximate value of μ_1 i.e. $\mu_1^{(1)}$.

Calculation in Panel 2.

In a similar way, in the second panel we have

$$\begin{array}{r} (-1)(2.29+2.23) = -4.52 \\ (-0.5)(1.47+1.41) = -1.44 \\ -D_2 = -4.0 \\ \hline \text{Total sum:} \quad -9.96 \end{array}$$

This gives $\mu_2^{(1)}$.

In a similar way there are calculated

$$\mu_3^{(1)} = -6.21\,, \qquad \mu_4^{(1)} = 10.97\,, \qquad \mu_5^{(1)} = 13.01\,.$$

Calculation for μ' of Verticals.

For the determination of μ' for verticals, we multiply the numerical values written in the double circle at the middle of each vertical member by the sums of φ which are found at the extremities of the vertical under consideration. Such product is worked out with respect to all the verticals and their total sum is found. This calculation gives the first approximate value of μ'.

For the present numerical example, however, each vertical has the same value for $-\mu'$ (i.e. -0.25), and therefore the calculation is simplified by the sum of total φ multiplied by -0.25.

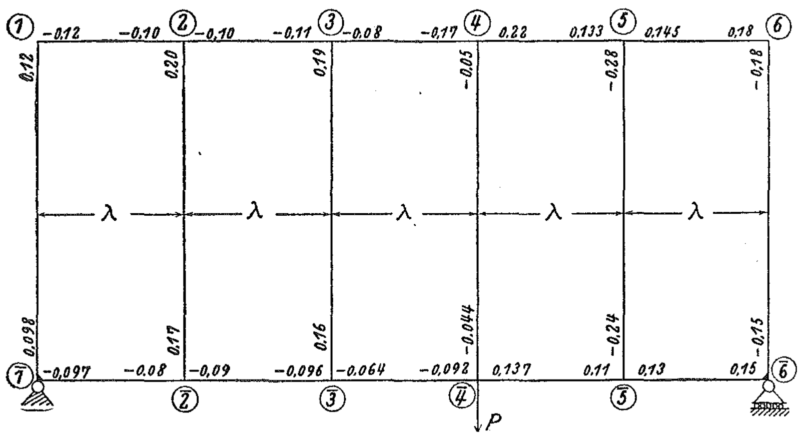
The summation of φ is as follows:

For upper chord-joint:	For lower chord-joint:
1.36	0.75
2.29	1.47
2.23	1.41
−0.58	−0.30
−3.23	−2.02
−2.12	−1.29
Total: −0.05	Total: +0.02

Accordingly there results: $(-0.05+0.02)(-0.25) \doteqdot 0.008$.

This is, $\mu'^{(1)} = 0.008$.

By the calculation above explained, all the first approximate values of φ and μ have been determined. This calculation procedure is quite



Coeff.: $P\lambda$

Fig. 11.

mechanical and systematic; there is no tedious labour. Next, the same calculation procedure is to be repeated as shown in Fig. 10, where we see the procedure twice repeated.

It seems necessary to repeat the procedure at least three times, and for the grade of necessity the number of repetitions is to be increased.

Fig. 11 shows the joint moment distribution worked out from the second approximate values above calculated. The moment values worked out by using the fourth approximate values are shown in Fig. 7.

II. The Slope Distribution Method Applied to the Problems of Vierendeel Truss with the Same Values of K for Upper and Lower Chord in a Panel.

In the case where the value of K for the upper chord is the same as that of the lower chord in each panel, the joint-rotation angles take the same value at both extremities of each vertical; namely $\varphi_{\bar{r}} = \varphi_r$.

For this reason, it is therefore sufficient only to determine all the joint-rotation angles for the upper chord joints or for the lower chord ones. Here in this paper it is required to find all the joint-rotation angles for the upper chord joints.

As in equation (14), we have at joint r:

$$q_r = 2(K_{\text{left}} + K_{\text{right}}) + 3K_{(\text{vertical})} = \rho_r + K'_{r(\text{vertical})}$$

and for the truss shown in Fig. 8, putting K in all the values of K for the lower chords, we get:

$$q_1 = \rho_1 + K = 2(K+K) + K = 5K,$$

$$q_2 = \rho_2 + K = 2(K+K+K) + K = 7K,$$

$$q_3 = q_4 = q_5 = 7K, \quad q_6 = 5K.$$

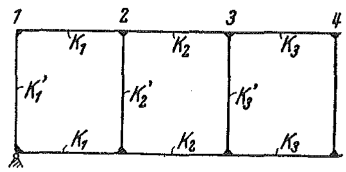


Fig. 12.

In general, for the truss shown in Fig. 12, we have:

$$q_1 = \rho_1 + K'_1 = 2(K_1 + K'_1) + K'_1,$$

$$q_2 = \rho_2 + K'_2 = 2(K_1 + K_2 + K'_2) + K'_2,$$

$$q_3 = \rho_3 + K'_3 = 2(K_2 + K_3 + K'_3) + K'_3.$$

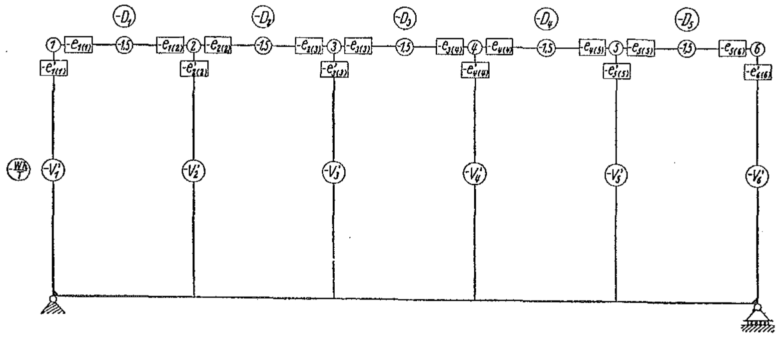


Fig. 13.

At each panel joint we calculate the value of q and then divide each K, which meets at the joint, by the value of q. This quotient is expressed by e as shown in the following:

$$\left.\begin{aligned}
&\text{at joint 1:}\\
&\qquad e_{1(1)} = \frac{K_1}{q_1}, \qquad e'_{1(1)} = \frac{K'_1}{q_1};\\
&\text{at joint 2:}\\
&\qquad e_{2(2)} = \frac{K_2}{q_2}, \qquad e'_{2(2)} = \frac{K'_2}{q_2}, \qquad e_{1(2)} = \frac{K_1}{q_2};\\
&\text{at joint 3:}\\
&\qquad e_{3(3)} = \frac{K_3}{q_3}, \qquad e'_{3(3)} = \frac{K'_3}{q_3}, \qquad e_{2(3)} = \frac{K_2}{q_3}.
\end{aligned}\right\} \quad \ldots (21)$$

and so on.

The writing position of e is shown in Fig. 13; it is in a similar relation with γ in Fig. 9.

Next we use in the case in question the numerical constant value (-1.5) in the place of $(-t)$ which has been used in Fig. 9, because from equation (8) we have:

$$\left.\begin{aligned}
\mu_r &= -D_r - 2(\varphi_r + \varphi_{r+1})t_r\\
&= -D_r + (\varphi_r + \varphi_{r+1})(-2t_r)\\
\text{in which}\qquad\qquad&\\
-2t_r &= \frac{-3K_r}{(K_r + K_r)} = -1.5\,.
\end{aligned}\right\} \quad \ldots\ldots (22)$$

In other words, we have, independent of the panel, $-2t_r = -1.5$, which is written down in the circle in the middle of the upper chord members.

From equation (10)

$$\left.\begin{aligned}\mu' &= -\frac{Wh}{T}-(2\varphi_1 u_1'+2\varphi_2 u_2'+\cdots\cdots+2\varphi_n u_n')\\ &= -\frac{Wh}{T}-(\varphi_1\nu_1'+\varphi_2\nu_2'+\cdots\cdots+\varphi_n\nu_n')\\ &\text{in which}\\ \nu_r' &= 2u_r' = \frac{3K_r'}{K_1'+K_2'+\cdots\cdots+K_n'}\end{aligned}\right\} \quad \cdots\cdots (23)$$

and for the first vertical we have

$$\nu_1' = \frac{3K_1'}{K_1'+K_2'+\cdots\cdots+K_n'},$$

for the second vertical

$$\nu_2' = \frac{3K_2'}{K_1'+K_2'+\cdots\cdots+K_n'}$$

and so on.

These numerical values are to be written in the circle mid-way in each vertical. See Fig. 13.

III. Numerical Example.

The horizontal parallel chord truss of Vierendeel type with five equal panels as shown in Fig. 14 carries a single load P at the lower panel joint $\bar{4}$. This truss is assumed to have equal values of K for all the upper and lower chord members and $K/3$ for all the varticals. It is required to find all the joint moments by the Slope-Distribution Method.

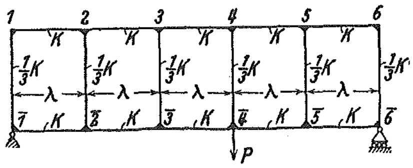


Fig. 14.

(*I*). *Preparatory Calculation.*

Firstly we find q as follows:

$$q_1 = q_6 = 2K+3\left(\frac{K}{3}\right) = 3K,$$

$$q_2 = q_3 = q_4 = q_5 = 2K+2K+3\left(\frac{K}{3}\right) = 5K$$

and therefrom we get e,

at joint 1:

$$e_{1(1)}=\frac{K}{3K}=\frac{1}{3}\,,\quad e'_{1(1)}=\frac{\frac{1}{3}K}{3K}=\frac{1}{9}\,;$$

at joint 2:

$$e_{1(2)}=\frac{K}{5K}=\frac{1}{5}\,,\quad e_{2(2)}=\frac{K}{5K}=\frac{1}{5}\,,\quad e'_{2(2)}=\frac{\frac{1}{3}K}{5K}=\frac{1}{15}$$

and so on.

These values are to be written down in the corresponding places as shown in Fig. 15. Also as mentioned previously, we write down (-1.5) in the circle at the mid-point of every upper chord member. If the multiplier $P\lambda/(30K)$ be disregarded, then we have:

$$-D_1=-D_2=-D_3=-3\,,\qquad -D_4=-D_5=+4.5\,.$$

These numerical values are written in Fig. 15.

Next we calculate ν as follows:

$$-\nu'_1=-\nu'_2=-\nu'_3=-\nu'_4=-\nu'_5=-\nu'_6$$

$$=\frac{-3\left(\frac{1}{3}K\right)}{6\left(\frac{1}{3}K\right)}=-\frac{1}{2}\,.$$

These values are written at the mid-point of every vertical. See Fig. 15.

(a) *First Calculation of Approximate Values of φ.*

From equation (18)

$$\varphi_r=\frac{\mathfrak{M}_{(\mathrm{left})}+\mathfrak{M}_{(\mathrm{right})}}{12K'_r}$$

in which we have in the given case

$$\mathfrak{M}_1=\mathfrak{M}_2=\mathfrak{M}_3=\frac{2}{5}P\lambda\,,\qquad \mathfrak{M}_4=\mathfrak{M}_5=-\frac{3}{5}P\lambda\,,$$

$$K'_1=K'_2=\cdots\cdots=K'_6=\frac{1}{3}K\,.$$

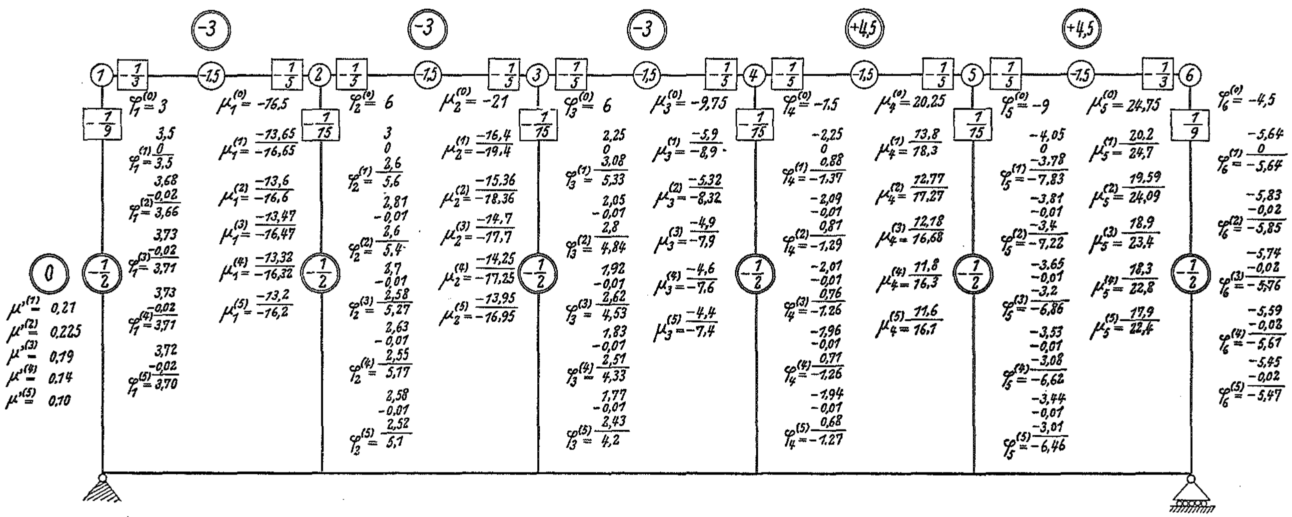


Fig. 15.

Therefore there are obtained

$$\varphi_1^{(0)} = \frac{\frac{2}{5}P\lambda}{12\left(\frac{1}{3}K\right)} = \frac{P\lambda}{30K}(3)\,,$$

$$\varphi_2^{(0)} = \frac{\frac{2}{5}P\lambda+\frac{2}{5}P\lambda}{12\left(\frac{1}{3}K\right)} = \frac{P\lambda}{30K}(6)\,,$$

$$\varphi_3^{(0)} = \frac{P\lambda}{30K}(6)\,, \quad \varphi_4^{(0)} = \frac{\frac{2}{5}P\lambda-\frac{3}{5}P\lambda}{12\left(\frac{1}{3}K\right)} = \frac{P\lambda}{30K}\left(-\frac{3}{2}\right),$$

$$\varphi_5^{(0)} = \frac{-\frac{3}{5}P\lambda-\frac{3}{5}P\lambda}{12\left(\frac{1}{3}K\right)} = \frac{P\lambda}{30K}(-9)\,,$$

$$\varphi_6^{(0)} = \frac{-\frac{3}{5}P\lambda}{12\left(\frac{1}{3}K\right)} = \frac{P\lambda}{30K}\left(-\frac{9}{2}\right).$$

These numerical values are written in the corresponding places in Fig. 15. (the multiplier $P\lambda/30K$) is disregarded).

(b) *First Calculation of Approximate Values of* μ.

From equation (8)

$$\mu_r = -D_r-2(\varphi_r+\varphi_{r+1})t_r = -D_r+(\varphi_r+\varphi_{r+1})(-2t_r)$$
$$= -D_r+(\varphi_r+\varphi_{r+1})(-1.5)\,.$$

Therefore for the first panel

$$\mu_1^{(0)} = -D_1+(\varphi_1^{(0)}+\varphi_2^{(0)})(-1.5) = -3+(3+6)(-1.5) = -16.5$$

Similarly

$$\mu_2^{(0)} = -D_2+(\varphi_2^{(0)}+\varphi_3^{(0)})(-1.5) = -3+(6+6)(-1.5) = -21\,,$$

$$\mu_3^{(0)} = -D_3+(\varphi_3^{(0)}+\varphi_4^{(0)})(-1.5) = -3+\left(6-\frac{3}{2}\right)(-1.5) = -9.75\,,$$

$$\mu_4^{(0)} = -D_4+(\varphi_4^{(0)}+\varphi_5^{(0)})(-1.5) = +4.5+\left(-\frac{3}{2}-9\right)(-1.5) = 20.25\,,$$

$$\mu_5^{(0)} = +4.5+\left(-9-\frac{9}{2}\right)(-1.5) = 24.75\,.$$

These numerical values are written in the corresponding places in Fig. 15.

(II) Method of Calculation on Truss Schema itself.

We begin the calculation at joint 1 working toward joint 6, from which we get $\varphi_1^{(1)}$ to $\varphi_6^{(1)}$.

Next the calculation is continued from the first panel to the fifth panel, from which we get $\mu_1^{(1)}$ to $\mu_5^{(1)}$.

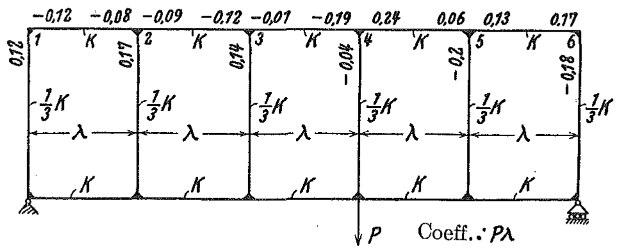


Fig. 16.

Then multiply by $(-1/2)$ the total sum of φ and get $\mu'^{(1)}$. All the intermediate calculations are shown in Fig. 15 and as the fifth approximate values of φ and μ we have:

$$\varphi_1^{(5)} = 3.7\,, \quad \varphi_2^{(5)} = 5.1\,, \quad \varphi_3^{(5)} = 4.2\,, \quad \varphi_4^{(5)} = -1.27\,,$$

$$\varphi_5^{(5)} = -6.46\,, \quad \varphi_6^{(5)} = -5.47\,, \quad \mu_1^{(5)} = -16.2\,, \quad \mu_2^{(5)} = -16.95\,,$$

$$\mu_3^{(5)} = -7.4\,, \quad \mu_4^{(5)} = 16.1\,, \quad \mu_5^{(5)} = 22.4\,, \quad \mu'^{(5)} = 0.10\,.$$

With these values of φ and μ there are calculated all the joint moments which are denoted in the corresponding joints and places as shown in Fig. 16.

C. The X-Distribution Method for the Analysis of Vierendeel Trusses of Polygonal or Parallel Chords.

The types of Vierendeel truss bridges are generally divided, as shown in Fig. 17, into four kinds. The curved chord type seems to be, superior to the parallel chord type, both from the point of view of the concern of stress state as well as of economy of materials.

Type A in Fig. 17 is completely, from one end to the other, composed of rectangular elements, while Types B and D are of trianglar for each end panel and Type C of trapezoid elements.

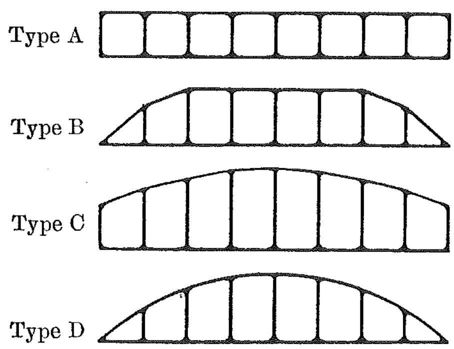


Fig. 17.

In the present opinion of the author, the rectangular element of each end panel for a parallel chord truss, as a through bridge, seems to have the excess materials, compared with the triangular or trapezoidal element of the other types, with respect to the aspect of stress distribution, because the excess construction of end posts and end upper chords seems to cause the structure to become more rigidly resistant to bending moment and therefrom the actual stress by joint moments may be increased.

I. Preparatory Calculation.

As in the case of continuous beams or bents, rigid frame problems require, for all the members, the value of $J/s = K$ (s: length of member considered and J: moment of inertia of the section).

The joints, as shown in Fig. 18, are numbered from the left end to the right and each value K of the upper chord member is assumed or constructed actually to be equal, for each panel, to that of the lower chord member; all the values of K are firstly to be calculated, whose suffix numbers are as shown in the figure. The length of each vertical is expressed by h with the same suffix as the joint which belongs to the extremities of the vertical.

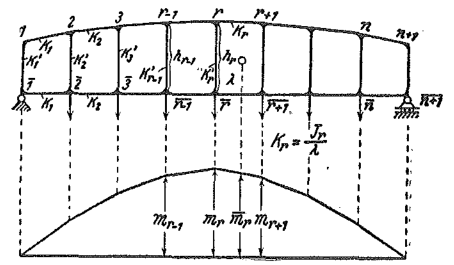


Fig. 18.

Considering the given span length of the truss as the total length of a simple girder, we draw, as shown in Fig. 18, a bending moment diagram due to the given load system and the bending moment at the panel joint r is expressed by m_r.

All the values of K, h and m, mentioned above, are necessarily to be given as known values and with these numerical quantities the following calculation is to be prepared for the determination of the statically indeterminate stresses, to which the problems of Vierendeel Trusses are exposed.

(*1*). *Calculation of* t_r.

For every panel, it is required to work out the value h_r/h_{r+1}, i.e. the length of a vertical divided by the length of the next vertical, Or

$$t_r = \frac{h_r}{h_{r+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

(*2*). *Calculation of* a_r.

For every panel, it is required to work out the value $t_r^2 K_r/K'_r$, in which t_r is given by the calculation above and K_r for the chord member and K'_r for the vertical in consideration (see Fig. 18).

$$a_r = \frac{K_r}{K'_r} t_r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

(*3*). *Calculation of* b_r.

For every panel, it is required to work out the value:

$$b_r = 2(1 + t_r + t_r^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

(*4*). *Calculation of* c_r.

For every panel, it is required to work out the value:

$$c_r = \frac{K_r}{K'_{r+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

(5). *Calculation of B_r.*

From the above calculation we prepare the following summation:

$$B_r = a_r + b_r + c_r \qquad \text{(28)}$$

(6). *Calculation of G_r.*

For every panel, it is required to work out the following numerical value:

$$G_r = \frac{1}{h_{r+1}}\left\{m_r(1+2t_r) + m_{r+1}(2+t_r)\right\} \qquad \text{(29)}$$

in which m_r and m_{r+1} are to be found from the diagram of Fig. 18. As in the other method, we express by $\overline{m}_r$ the bending moment, as shown in Fig. 18, under the centroid of the trapezoid, which is composed of two verticals h_r and h_{r+1} and two chord members $(r, r+1)$ and $(\bar{r}, \overline{r+1})$. Then G_r is expressed as follows:

$$G_r = \frac{3\overline{m}_r}{h_{r+1}}(1+t_r) \qquad \text{(30)}$$

II. Fundamental Equations.

For the r^{th} panel numbered from the left support of a Vierendeel truss, we denote by X_r the horizontal component of the resultant direct stress, which is induced in the upper or lower chord member in the panel (see Fig. 19) and likewise we denote by X_{r-1} and X_{r+1} those for the $(r-1)^{\text{th}}$ and $(r+1)^{\text{th}}$ panels respectively. There exists then the well known following relation between these X_{r-1}, X_r, X_{r+1} and the numerical values a, B, c, G above calculated and prepared:

$$X_{r-1}a_r - X_r B_r + X_{r+1}c_r = -G_r \qquad \text{(I)*}$$

For a truss which has n panels, substituting $r = 1, 2, 3 \cdots\cdots n$ for r in equation (I), we get n equations from which n unknown quantities are to be determined. These n equations are given in tabular from in Table 10.

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* This equation is given in another form of representation in Bleich, Theorie und Berechnung der eisernen Brücken, p. 513.

Table 10.

Equation	Left-Hand Member of Equation										Right-Hand Member
	X_1	X_2	X_3	X_4	X_5	-	-	-	-	X_n	
(1)	$-B_1$	c_1									$-G_1$
(2)	a_2	$-B_2$	c_2								$-G_2$
(3)		a_3	$-B_3$	c_3							$-G_3$
(4)			a_4	$-B_4$	c_4						$-G_4$
⋮				\	\	\					⋮
⋮					\	\	\				⋮
⋮						\	\	\			⋮
⋮							\	\	\		⋮
⋮								\	\	\	⋮
(n)									a_n	$-B_n$	$-G_n$

In Table 10, each coefficient B which is found on the diagonal line of the table is very much greater than a and c, when compared with each other; we can therefore easily determine the unknown X by the method of iteration.

From equation (I) we have:

$$X_r = \pi_r + X_{r-1}A_r + X_{r+1}C_r \quad \text{(II)}$$

in which

$$\left.\begin{aligned} \pi_r &= \frac{G_r}{B_r} \\ A_r &= \frac{a_r}{B_r} \\ C_r &= \frac{c_r}{B_r} \end{aligned}\right\} \quad \text{(III)}$$

If we had by preparatory calculation the numerical values of a, B, c and G, we can easily determine, from equation (III), the numerical values π, A, C.

III. X-Distribution Method.

For n panels, n equations contain n unknowns X_1 to X_n. X-Distribution Method is the method of determination of X, bringing to balance the unknown X, which finds itself in unbalanced state, step by step by the process of repetition, distributing the amount of unbalance.

For each panel, we write down the numerical values π, A, C in each definite place as indicated in Fig. 20 and the calculation of X is worked out on the truss schema as shown in Fig. 22 whose given conditions are indicated in Fig. 21.

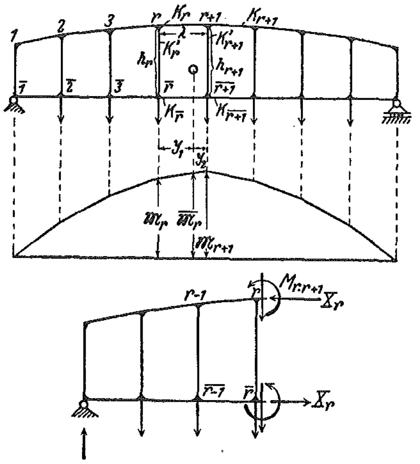


Fig. 19.

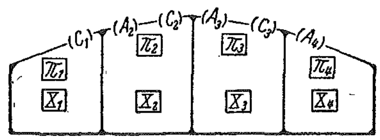


Fig. 20.

The calculation of X is as follows:

On the scale of slide-rule, we put the value of C_1 and multiply by π_2 (because at first we assume π_2 as the approximate value of X_2) and the product $C_1 \times \pi_2$ is to be written down in the space of the first panel and then add π_1 to it. This calculation gives the first approximate value of X_1 and we denote it by $X_1^{(1)}$.

This calculation is worked out in the first panel.

Next, we shift the calculation to the second panel and find, by slide-rule, the product $A_2 \times X_1^{(1)}$, where A_2 is given in parenthesis at the left end of the upper chord member of the panel, while $X_1^{(1)}$ is given by the previous calculation in the first panel. This product is written down in the space of the second panel and the product $C_2 \times \pi_3$ is worked out too. We write it down in the space under the last product, then add π_2 to it and find the sum of them. This calculation gives $X_2^{(1}$.

Similar calculations are to be continued up to the last panel. After completing such calculations one time, the similar procedure by multiplication and addition is repeated and so on.

The following numerical example shows the calculation process in Fig. 22.

When we compare the above procedure on truss schema with equation (II), the mathematical meaning of the method will become obvious.

IV. Joint Moment in terms of X and *m*.

We make an imaginary vertical section near to joint r as shown in Fig. 19 and cut the upper and lower chords very near to the vertical h_r. Equilibrium condition for the bending moment then gives :

$$M_{r,r+1}=\frac{1}{2}(m_r-X_rh_r) \qquad \text{(IV)}$$

In a similar manner we have for panel joint $r+1$:

$$M_{r+1\cdot r}=\frac{1}{2}(X_rh_{r+1}-m_{r+1}) \qquad \text{(V)}$$

These two equations of joint moments are given in terms of X and m.

V. Joint Moment in terms of X_r and X_{r-1}.

There are made two imaginary vertical sections near the panel joints r and $\bar{r}$, at both sides of the vertical member h_r, and the equilibrium of forces acting on the vertical member h_r, considering $M_{r,\bar{r}}+M_{r,r-1}+M_{r,r+1}=0$, gives :

$$M_{r,\bar{r}}=\frac{1}{2}(X_r-X_{r-1})h_r \qquad \text{(VI)}$$

The equation of the joint moment at the extremity of the vertical is thus given in terms of X_r and X_{r-1}.

VI. Numerical Example.

It is required to find the joint moments of a Vierendeel truss, whose stiffness relation is as shown in Fig. 21, carrying a single vertical load P at the panel joint $\bar{4}$.

Firstly we determine X by X-Distribution Method and then joint moments by equations (IV), (V) and (VI).

Table 11 shows the results of the preparatory calculation for t, a, b, c, B, m and G. With these numerical values of a, B, c and G, the coefficients π, A and C are worked out as shown in Table 12; these values are written down in the definite places in the truss schema as shown in Fig. 22, on which the X-Distribution method is carried out. Thus we get from Fig. 22:

$$X_1 = 0.362\,, \quad X_2 = 0.638\,,$$
$$X_3 = 0.867\,, \quad X_4 = 0.828\,,$$
$$X_5 = 0.498 \text{ (Coeff.: } P\lambda/h) .$$

These values X_1 to X_5 are obtained, as shown in Fig. 22, by six repetitions of the X-Distribution Method.

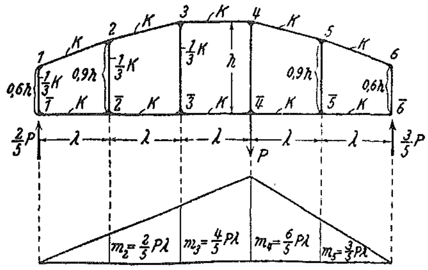


Fig. 21.

Table 11.

Panel	$r=1$	2	3	4	5
h_r	$0.6\,h$	$0.9\,h$	h	h	$0.9\,h$
$t_r = \frac{h_r}{h_{r+1}}$	0.67	0.9	1.0	1.11	1.5
t_r^2	0.45	0.81	1.0	1.23	2.25
$1+t_r+t_r^2$	2.12	2.71	3.0	3.34	4.75
b_r	4.24	5.42	6.0	6.68	9.50
a_r	1.35	2.43	3.0	3.69	6.75
c_r	3	3	3	3	3
B_r	8.59	10.85	12.0	13.37	19.25
m_r	0	$0.4\,P\lambda$	$0.8\,P\lambda$	$1.2\,P\lambda$	$0.6\,P\lambda$
m_{r+1}	$0.4\,P\lambda$	$0.8\,P\lambda$	$1.2\,P\lambda$	$0.6\,P\lambda$	0
G_r	$1.19\frac{P\lambda}{h}$	$3.44\frac{P\lambda}{h}$	$6.0\frac{P\lambda}{h}$	$6.37\frac{P\lambda}{h}$	$4.0\frac{P\lambda}{h}$

Table 12.

Panel	$r=1$	2	3	4	5
π_r	$0.139\frac{P\lambda}{h}$	$0.317\frac{P\lambda}{h}$	$0.5\frac{P\lambda}{h}$	$0.476\frac{P\lambda}{h}$	$0.208\frac{P\lambda}{h}$
A_r	0.157	0.224	0.25	0.276	0.351
C_r	0.35	0.277	0.25	0.224	0.156

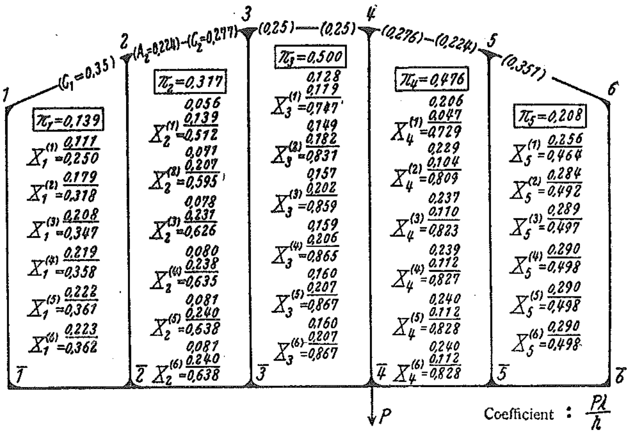


Fig. 22.

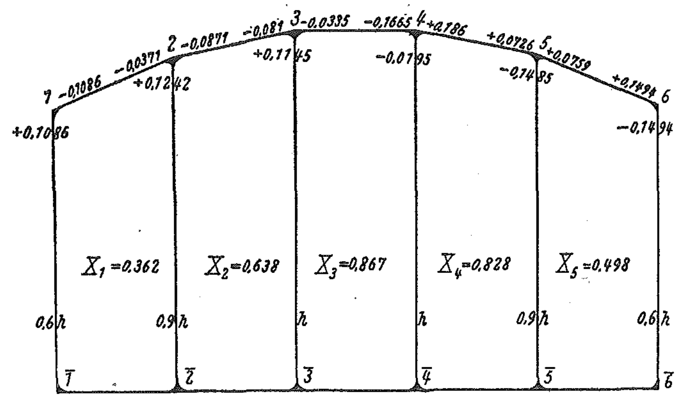
Next we work out $M_{r,\,r+1}$ and $M_{r+1,\,r}$, substituting X_1 to X_5 in equation (IV) and (V). The results of this calculation are tabulated in Table 13.

Table 13.

Panel No.	1	2	3	4	5
$X_r h_r$	$0.2172\ P\lambda$	$0.5742\ P\lambda$	$0.867\ P\lambda$	$0.828\ P\lambda$	$0.4482\ P\lambda$
$X_r h_{r+1}$	$0.3258\ P\lambda$	$0.638\ P\lambda$	$0.867\ P\lambda$	$0.7452\ P\lambda$	$0.2988\ P\lambda$
m_r	0	$0.4\ P\lambda$	$0.8\ P\lambda$	$1.2\ P\lambda$	$0.6\ P\lambda$
$m_r - X_r h_r$	$-0.2172\ P\lambda$	$-0.1742\ P\lambda$	$-0.067\ P\lambda$	$+0.372\ P\lambda$	$+0.1518\ P\lambda$
$M_{r\cdot r+1}$	$-0.1086\ P\lambda$	$-0.0871\ P\lambda$	$-0.0335\ P\lambda$	$+0.186\ P\lambda$	$+0.0759\ P\lambda$
$X_r h_{r+1} - m_{r+}$	$-0.0742\ P\lambda$	$-0.162\ P\lambda$	$-0.333\ P\lambda$	$+0.1452\ P\lambda$	$+0.2988\ P\lambda$
$M_{r+1\cdot r}$	$-0.0371\ P\lambda$	$-0.081\ P\lambda$	$-0.1665\ P\lambda$	$+0.0726\ P\lambda$	$+0.1494\ P\lambda$

Table 14.

Verticals	$1-\bar{1}$	$2-\bar{2}$	$3-\bar{3}$	$4-\bar{4}$	$5-\bar{5}$	$6-\bar{6}$
$X_r - X_{r-1}$	$+0.362\dfrac{P\lambda}{h}$	$+0.276\dfrac{P\lambda}{h}$	$+0.229\dfrac{P\lambda}{h}$	$-0.039\dfrac{P\lambda}{h}$	$-0.330\dfrac{P\lambda}{h}$	$-0.498\dfrac{P\lambda}{h}$
$(X_r - X_{r-1})h_r$	$+0.2172\ P\lambda$	$+0.2484\ P\lambda$	$+0.229\ P\lambda$	$-0.039\ P\lambda$	$-0.297\ P\lambda$	$-0.2988\ P\lambda$
$M_{r\bar{r}}$	$+0.1086\ P\lambda$	$+0.1242\ P\lambda$	$+0.1145\ P\lambda$	$-0.0195\ P\lambda$	$-0.1485\ P\lambda$	$-0.1494\ P\lambda$



Coeff. of X: $\dfrac{P\lambda}{h}$

Coeff. of M: $P\lambda$

Fig. 23.

The joint moment at the extremity of vertical member is found from equation (VI). The results of this calculation are shown in Table 14. All the joint moments, thus worked out, are written in the corresponding places in Fig. 23, to facilitate the location of each one and also to check if the calculated moments satisfy the equilibrium condition $\Sigma M_r = 0$.

At joint 1, the positive moment is $0.1086\,P\lambda$ and the negative moment $-0.1086\,P\lambda$; the equilibrium condition here is satisfied.

At joint 2, the sum of the negative moment is $-0.1242\,P\lambda$ and the positive moment $0.1242\,P\lambda$.

At joint 3, the sum of the negative moment is $-0.1145\,P\lambda$ and the positive moment $0.1145\,P\lambda$.

If there occurs a certain amount of discrepancy of equilibrium condition, the number of repetitions should be increased, corresponding to the required necessity.