# Diagrams of Temperature Distribution in a Thick Plate.

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(1946, 3, 1.)

## Synopsis.

*The temperature distributions in a thick plate, when one surface is heated, are numerically* calculated and represented by diagrams as the functions of time and distance from the surface.

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## 1. Introduction.

Practically it may be necessary for engineers to know the influences, which the abrupt change of the temperature of the surface yields. When a plane surface of a semi-infinite domain is heated and kept at a constant temperature for some duration $m$, the temperature in the domain will increase at first and decrease gradually. The temperature distribution must be a function of time $t$ and the distance $x$ from the surface. As the first example, the calculations are carried out about this case; the time duration $m$ of heating being taken as a parameter. As the second example, the heat distributions in the thick plate are calculated; the one of the surfaces is heated and the other is thermally insulated. Practically the temperature distribution in most cases can be looked upon as the intermediated case of these two extreme examples.

## 2. Heat Conduction of the Semi-infinite Solid.

In this paragraph the temperature distributions are calculated when the solid is bounded by the plane $x=0$ and extends to infinity in the direction of $x$ positive, and the isothermal surfaces are planes parallel to $x=0$ and the flow of heat is linear, the lines of flow being parallel to the axis of $x$.

Let the initial temperature be given by $u=0$, and the surface temperature vary with the time and the temperature at infinite boundary be zero. Then, in the semi-infinite solid, where $u$ has to satisfy

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad \cdots\cdots (1)$$

$u = 0$ when $t = 0,$

$u = f(t)$ at $x = 0,$

and $u = 0$ at $x = \infty,$

the solution is given by

$$u = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{f(\tau)\, e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau . \quad \cdots\cdots (2)$$

The constant $a^2$ is called the diffusivity of that substance, or its thermometric conductivity, and it is denoted by

$$a^2 = \frac{\kappa}{c\rho},$$

where $\kappa =$ thermal conductivity of the substance,

$c =$ specific heat of it,

and $\rho =$ density of it.

Now let the temperature $u=f(t)$ at $x=0$ be in such a state that temperature keeps constant for a certain duration and then it falls, as shown in Fig-1.,

$f(t) = k$ when $0 \leqq t \leqq m$

and $f(t) = \dfrac{k}{n-m}(n-t)$ when $m \leqq t \leqq n$ .

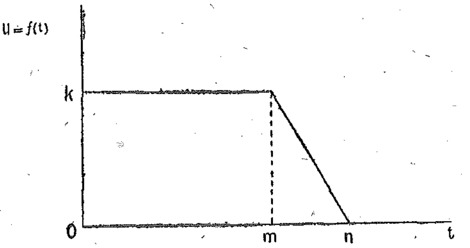

Fig. 1

In this case, in order to obtain the relations between time $t$ and temperature $u$ at a certain depth $x$, we must investigate the following four cases:

(1) when the surface temperature $f(t) = k$ continues infinitely, temperature at $t$ is given by

$$u_1 = \frac{kx}{2a\sqrt{\pi}} \int_0^t \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau . \quad \cdots\cdots (3)$$

(2) when the surface temperature $f(t) = k$ breaks off at time $m$, temperature at $t(>m)$ is given by

$$u_2 = \frac{kx}{2a\sqrt{\pi}} \int_0^m \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau . \quad (t > m) \quad \cdots\cdots (4)$$

(3) when the surface temperature decreases with time from $t = m$ and becomes zero at $t = n$, temperature at $t$ (between $m$ and $n$) is given by

$$u_3 = \frac{kx}{2a\sqrt{\pi}} \int_0^m \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau + \frac{kx}{2a\sqrt{\pi}} \int_m^t \frac{n-\tau}{n-m} \cdot \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau .$$

$$= u_2 + u_3' . \quad (m > t > n) \quad \cdots\cdots (5)$$

(4) after the surface temperature becomes zero at $t = n$, temperature at $t$ ($>n$) is given by

$$u_4 = \frac{kx}{2a\sqrt{\pi}} \int_0^m \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau + \frac{kx}{2a\sqrt{\pi}} \int_m^n \frac{n-\tau}{n-m} \cdot \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} d\tau .$$

$$= u_2 + u_4' . \quad (t > n) \quad \cdots\cdots (6)$$

When we calculate the above each formula (3) ~ (6), we had better transform the variable and integrate

| | | | | |
|---|---|---|---|---|
| from | $t - \tau = 0$ | to | $t - \tau = t$ | in (3), |
| from | $t - \tau = t - m$ | to | $t - \tau = t$ | in (4), |
| from | $t - \tau = 0$ | to | $t - \tau = t - m$ | in (5), |
| from | $t - \tau = t - \tau$ | to | $t - \tau = t - m$ | in (6), |

and about $\frac{n-\tau}{n-m}$ in (5), (6) calculate the value

$$\frac{n-\tau}{n-m} = \frac{n-t+(t-\tau)}{n-m} = r .$$

In this problem we assume that the substance is concrete

$$a^2 = 0.0064 \quad \text{(C.G.S. — unit)} ,$$

and we shall show the temperature change at 1.0 cm depth from the surface.

Then

$$\frac{x}{2a\sqrt{\pi}} = 3.52 , \qquad \frac{x^2}{4a^2} = 39.1$$

And the value of

$$\frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} = \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$$

for various value of $t-\tau$ are shown as Table 1.

Relation of $t-\tau$ and $\frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}$ is illustrated in Fig. 2. It is showed from $t-\tau=0$ till $t-\tau=180$ in cases of $x = 0.5$ cm, 1.0 cm, and 1.5 cm.

When we calculate the integral value of (3) by the above, we can get the value of $u_1$ as Table 2.

This value $u_1$ is illustrated in Fig. 5 by thick full lines. In this diagram axis of abscissa shows the time $t$ (sec.), and axis of ordinate shows the value of $\frac{u}{k}$. Then, we show the value of $u_2$ for various $m$ by (4) in following Table 3. This value $u_2$ is illustrated in Fig. 5 by broken lines.

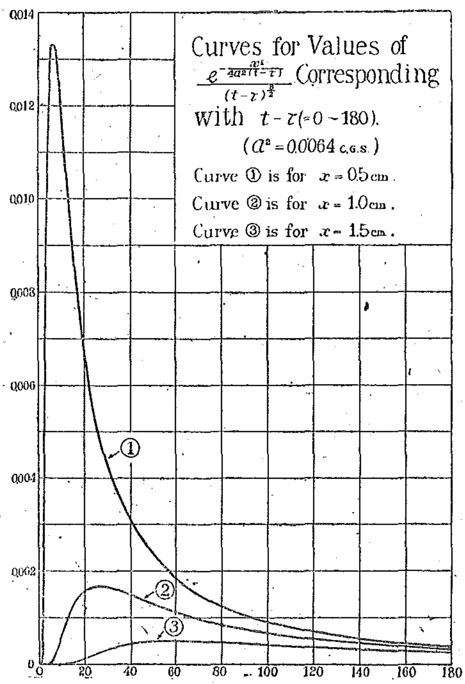


Fig. 2

Table 1.

| $t-\tau$ | $\dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | $t-\tau$ | $\dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | $t-\tau$ | $\dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | $t-\tau$ | $\dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | $t-\tau$ | $\dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 80 | 0.000856 | 1020 | 0.0000295 | 2100 | 0.0000102 | 3180 | 0.00000551 |
| 3 | 0.000000 | 100 | 0.000676 | 1080 | 0.0000272 | 2160 | 0.00000978 | 3200 | 0.00000545 |
| 4 | 0.000006 | 120 | 0.000548 | 1100 | 0.0000264 | 2200 | 0.00000952 | 3240 | 0.00000537 |
| 5 | 0.000036 | 140 | 0.000456 | 1140 | 0.0000251 | 2220 | 0.00000939 | 3300 | 0.00000521 |
| 6 | 0.000102 | 160 | 0.000386 | 1200 | 0.0000233 | 2280 | 0.00000903 | 3360 | 0.00000501 |
| 8 | 0.000331 | 180 | 0.000333 | 1260 | 0.0000217 | 2300 | 0.00000891 | 3400 | 0.00000498 |
| 10 | 0.000633 | 200 | 0.000290 | 1300 | 0.0000207 | 2340 | 0.00000869 | 3420 | 0.00000494 |
| 12 | 0.000922 | 240 | 0.000228 | 1320 | 0.0000202 | 2400 | 0.00000837 | 3480 | 0.00000482 |
| 14 | 0.001169 | 300 | 0.000169 | 1380 | 0.0000190 | 2460 | 0.00000807 | 3500 | 0.00000477 |
| 16 | 0.001361 | 360 | 0.000131 | 1400 | 0.0000185 | 2500 | 0.00000788 | 3540 | 0.00000470 |
| 18 | 0.001498 | 400 | 0.000113 | 1440 | 0.0000178 | 2520 | 0.00000778 | 3600 | 0.00000458 |
| 20 | 0.001575 | 420 | 0.000106 | 1500 | 0.0000168 | 2580 | 0.00000752 | 3800 | 0.00000423 |
| 22.5 | 0.001643 | 480 | 0.0000877 | 1560 | 0.0000158 | 2600 | 0.00000744 | 3900 | 0.00000407 |
| 25 | 0.001660 | 500 | 0.0000827 | 1600 | 0.0000153 | 2640 | 0.00000726 | 4000 | 0.00000391 |
| 27 | 0.001672 | 540 | 0.0000741 | 1620 | 0.0000150 | 2700 | 0.00000703 | 4200 | 0.00000364 |
| 28.5 | 0.001669 | 600 | 0.0000638 | 1680 | 0.0000142 | 2760 | 0.00000680 | 4500 | 0.00000328 |
| 30 | 0.001657 | 660 | 0.0000556 | 1700 | 0.0000140 | 2800 | 0.00000665 | 4800 | 0.00000299 |
| 32 | 0.001629 | 700 | 0.0000510 | 1740 | 0.0000135 | 2820 | 0.00000659 | 5100 | 0.00000273 |
| 34 | 0.001598 | 720 | 0.0000490 | 1800 | 0.0000128 | 2880 | 0.00000638 | 5400 | 0.00000250 |
| 36 | 0.001561 | 780 | 0.0000437 | 1860 | 0.0000122 | 2900 | 0.00000632 | 5600 | 0.00000238 |
| 38 | 0.001525 | 800 | 0.0000421 | 1900 | 0.0000118 | 2940 | 0.00000619 | 6000 | 0.00000214 |
| 40 | 0.001487 | 840 | 0.0000392 | 1920 | 0.0000116 | 3000 | 0.00000601 | 6400 | 0.00000194 |
| 44 | 0.001410 | 900 | 0.0000355 | 1980 | 0.0000111 | 3060 | 0.00000583 | 6600 | 0.00000185 |
| 50 | 0.001293 | 960 | 0.0000323 | 2000 | 0.0000110 | 3100 | 0.00000572 | 6800 | 0.00000177 |
| 60 | 0.001119 | 1000 | 0.0000304 | 2040 | 0.0000106 | 3120 | 0.00000567 | 7200 | 0.00000163 |

Table 2.

| $t$ | $\int_0^t \dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}d\tau$ | $u_1$ | $t$ | $\int_0^t \dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}d\tau$ | $u_1$ | $t$ | $\int_0^t \dfrac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}d\tau$ | $u_1$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0. | 0. | 1500 | 0.2308 | $k\times$ 0.813 | 4200 | 0.2513 | $k\times$ 0.884 |
| 60 | 0.0704 | $k\times$ 0.248 | 1800 | 0.2352 | 〃 0.829 | 4500 | 0.2523 | 〃 0.888 |
| 120 | 0.1174 | 〃 0.414 | 2100 | 0.2387 | 〃 0.840 | 4800 | 0.2533 | 〃 0.891 |
| 180 | 0.1430 | 〃 0.504 | 2400 | 0.2414 | 〃 0.850 | 5100 | 0.2541 | 〃 0.894 |
| 240 | 0.1595 | 〃 0.562 | 2700 | 0.2437 | 〃 0.857 | 5400 | 0.2550 | 〃 0.897 |
| 300 | 0.1713 | 〃 0.604 | 3000 | 0.2457 | 〃 0.865 | 6000 | 0.2563 | 〃 0.902 |
| 600 | 0.2020 | 〃 0.711 | 3300 | 0.2474 | 〃 0.871 | 6600 | 0.2575 | 〃 0.906 |
| 900 | 0.2163 | 〃 0.762 | 3600 | 0.2488 | 〃 0.876 | 7200 | 0.2585 | 〃 0.910 |
| 1200 | 0.2250 | 〃 0.793 | 3900 | 0.2501 | 〃 0.880 | | | |

Table 3.

| $m$ | $t$ | $t-m$ | $\int_0^m \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}d\tau$ | $u_2$ |
|---|---|---|---|---|
| 60 | 120 | 60 | 0.0470 | $k \times$ 0.165 |
| | 180 | 120 | 0.0256 | 〃 0.090 |
| | 240 | 180 | 0.0165 | 〃 0.058 |
| | 300 | 240 | 0.0118 | 〃 0.041 |
| | 360 | 300 | 0.0091 | 〃 0.032 |
| | 600 | 540 | 0.0041 | 〃 0.015 |
| | 900 | 840 | 0.0022 | 〃 0.008 |
| | 1200 | 1140 | 0.0015 | 〃 0.005 |
| 120 | 180 | 60 | 0.0726 | $k \times$ 0.256 |
| | 240 | 120 | 0.0421 | 〃 0.148 |
| | 300 | 180 | 0.0283 | 〃 0.100 |
| | 600 | 480 | 0.0090 | 〃 0.032 |
| | 900 | 780 | 0.0047 | 〃 0.017 |
| | 1200 | 1080 | 0.0030 | 〃 0.011 |
| 180 | 240 | 60 | 0.0890 | $k \times$ 0.314 |
| | 300 | 120 | 0.0539 | 〃 0.190 |
| | 360 | 180 | 0.0373 | 〃 0.131 |
| | 480 | 300 | 0.0222 | 〃 0.078 |
| | 600 | 420 | 0.0151 | 〃 0.053 |
| | 900 | 720 | 0.0076 | 〃 0.027 |
| | 1200 | 1020 | 0.0047 | 〃 0.017 |
| 240 | 300 | 60 | 0.1009 | $k \times$ 0.355 |
| | 600 | 360 | 0.0222 | 〃 0.078 |
| | 900 | 660 | 0.0107 | 〃 0.038 |
| | 1200 | 960 | 0.0066 | 〃 0.023 |
| 300 | 360 | 60 | 0.1100 | $k \times$ 0.387 |
| | 420 | 120 | 0.0702 | 〃 0.247 |
| | 480 | 180 | 0.0505 | 〃 0.178 |
| | 600 | 300 | 3.0312 | 〃 0.110 |
| | 900 | 600 | 0.0143 | 〃 0.050 |
| | 1200 | 900 | 0.0086 | 〃 0.030 |
| | 1500 | 1200 | 0.0059 | 〃 0.021 |
| | 1800 | 1500 | 0.0044 | 〃 0.015 |
| | 2100 | 1800 | 0.0034 | 〃 0.012 |
| | 2400 | 2100 | 0.0028 | 〃 0.010 |
| | 2700 | 2400 | 0.0023 | 〃 0.008 |
| | 3000 | 2700 | 0.0020 | 〃 0.007 |
| | 3300 | 3000 | 0.0017 | 〃 0.006 |
| | 3600 | 3300 | 0.0015 | 〃 0.005 |
| 600 | 900 | 300 | 0.0455 | $k \times$ 0.160 |
| | 1200 | 600 | 0.0229 | 〃 0.081 |
| | 1500 | 900 | 0.0146 | 〃 0.051 |
| | 1800 | 1200 | 0.0103 | 〃 0.036 |
| | 2100 | 1500 | 0.0078 | 〃 0.028 |
| | 2400 | 1800 | 0.0062 | 〃 0.022 |
| | 2700 | 2100 | 0.0051 | 〃 0.018 |
| | 3000 | 2400 | 0.0043 | 〃 0.015 |
| 900 | 960 | 60 | 0.1479 | $k \times$ 0.521 |
| | 1080 | 180 | 0.0788 | 〃 0.277 |
| | 1200 | 300 | 0.0541 | 〃 0.191 |
| | 1500 | 600 | 0.0288 | 〃 0.101 |
| | 1800 | 900 | 0.0189 | 〃 0.067 |
| | 2100 | 1200 | 0.0138 | 〃 0.048 |
| | 2400 | 1500 | 0.0106 | 〃 0.037 |
| | 2700 | 1800 | 0.0085 | 〃 0.030 |
| | 3000 | 2100 | 0.0070 | 〃 0.025 |
| 1200 | 1500 | 300 | 0.0601 | $k \times$ 0.211 |
| | 1800 | 600 | 0.0332 | 〃 0.117 |
| | 2100 | 900 | 0.0224 | 〃 0.079 |
| | 2400 | 1200 | 0.0165 | 〃 0.058 |
| | 2700 | 1500 | 0.0129 | 〃 0.045 |
| | 3000 | 1800 | 0.0105 | 〃 0.037 |
| 1500 | 1800 | 300 | 0.0645 | $k \times$ 0.227 |
| | 2100 | 600 | 0.0367 | 〃 0.129 |
| | 2400 | 900 | 0.0252 | 〃 0.089 |
| | 2700 | 1200 | 0.0188 | 〃 0.066 |
| | 3000 | 1500 | 0.0149 | 〃 0.052 |
| 1800 | 1860 | 60 | 0.1656 | $k \times$ 0.583 |
| | 1980 | 180 | 0.0939 | 〃 0.331 |
| | 2100 | 300 | 0.0679 | 〃 0.239 |
| | 2400 | 600 | 0.0394 | 〃 0.139 |
| | 2700 | 900 | 0.0275 | 〃 0.097 |

| | | | | |
|---|---|---|---|---|
| | 3000 | 1200 | 0.0208 | 〃 0.073 |
| | 3300 | 1500 | 0.0165 | 〃 0.058 |
| | 3600 | 1800 | 0.0136 | 〃 0.048 |
| | 3900 | 2100 | 0.0114 | 〃 0.040 |
| | 4200 | 2400 | 0.0099 | 〃 0.035 |
| | 4500 | 2700 | 0.0086 | 〃 0.021 |
| | 5400 | 3600 | 0.0062 | 〃 0.020 |
| 2100 | 2400 | 300 | 0.0707 | $k\times$ 0.249 |
| | 2700 | 600 | 0.0417 | 〃 0.147 |
| | 3000 | 900 | 0.0294 | 〃 0.103 |
| | 3300 | 1200 | 0.0225 | 〃 0.079 |
| | 3600 | 1500 | 0.0180 | 〃 0.063 |
| 2400 | 2700 | 300 | 0.0730 | $k\times$ 0.257 |
| | 3000 | 600 | 0.0437 | 〃 0.154 |
| | 3300 | 900 | 0.0311 | 〃 0.109 |
| | 3600 | 1200 | 0.0240 | 〃 0.084 |
| | 2760 | 60 | 0.1737 | $k\times$ 0.611 |
| | 2880 | 180 | 0.1019 | 〃 0.359 |
| | 3000 | 300 | 0.0749 | 〃 0.264 |
| | 3300 | 600 | 0.0454 | 〃 0.160 |
| | 3600 | 900 | 0.0325 | 〃 0.115 |

| | | | | |
|---|---|---|---|---|
| 2700 | 3900 | 1200 | 0.0251 | 〃 0.089 |
| | 4200 | 1500 | 0.0205 | 〃 0.072 |
| | 4500 | 1800 | 0.0171 | 〃 0.061 |
| | 4800 | 2100 | 0.0146 | 〃 0.051 |
| | 5400 | 2700 | 0.0113 | 〃 0.039 |
| | 6000 | 3300 | 0.0089 | 〃 0.031 |
| | 7200 | 4500 | 0.0062 | 〃 0.022 |
| 3000 | 3300 | 300 | 0.0766 | $k\times$ 0.270 |
| | 3600 | 600 | 0.0468 | 〃 0.165 |
| 3300 | 3600 | 300 | 0.0781 | $k\times$ 0.275 |
| 3600 | 3660 | 60 | 0.1787 | $k\times$ 0.629 |
| | 3780 | 180 | 0.1066 | 〃 0.375 |
| | 3900 | 300 | 0.0788 | 〃 0.277 |
| | 4200 | 600 | 0.0493 | 〃 0.173 |
| | 4500 | 900 | 0.0360 | 〃 0.127 |
| | 4800 | 1200 | 0.0284 | 〃 0.100 |
| | 5100 | 1500 | 0.0233 | 〃 0.082 |
| | 5400 | 1800 | 0.0200 | 〃 0.069 |
| | 6000 | 2400 | 0.0149 | 〃 0.052 |
| | 6600 | 3000 | 0.0118 | 〃 0.042 |
| | 7200 | 3600 | 0.0097 | 〃 0.034 |

Next, for determining $u_3$ and $u_4$ by (5) and (6), we assume the following values of $m$ and $n$:

$m=60,\ n=180$; $m=300,\ n=600$; $m=1800,\ n=3600$; $m=3600,\ n=3900$;
$m=60,\ n=360$; $m=900,\ n=1200$; $m=2700,\ n=3000$; $m=3600,\ n=4500$;
$m=180,\ n=360$; $m=900,\ n=1800$; $m=2700,\ n=3600$; $m=3600,\ n=5400$;
$m=180,\ n=480$; $m=1800,\ n=2100$; $m=2700,\ n=4500$; $m=3600,\ n=7200$.
$m=300,\ n=420$; $m=1800,\ n=2700$; $m=2700,\ n=5400$;

Now, as the preparation of obtaining the integration of $u_3'$ in (5) we must decide the value of

$$\frac{n-\tau}{n-m}\cdot\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}=r\cdot\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}},$$

corresponding to the various value of $t-\tau$. As one example, we show that value at $t=240$, 300, and 360 in cases of $m=180$ and $n=360$. It is shown in the table 4. In this case the relation between $t-\tau$ and $r\cdot\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ is represented by the curves is Fig. 3, and then we shall be able to require the integral of $u_3'$. Similarly, we must decide the value of

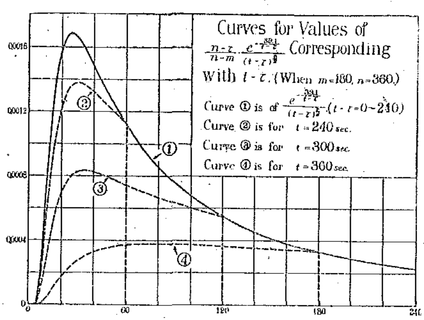


Fig. 3.

$$\frac{n-\tau}{n-m}\cdot\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}=r\cdot\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$$

for $u_4'$. As before, temperatures at $t=480$, 600, 900, 1800 and 2700 in cases of $m=180$ and $n=360$ are shown in Table 5.

In this case the relation between $t-\tau$ and $r\cdot\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ is illustrated by the curves in Fig. 4, then we shall be able to obtain the integral of $u_4'$.

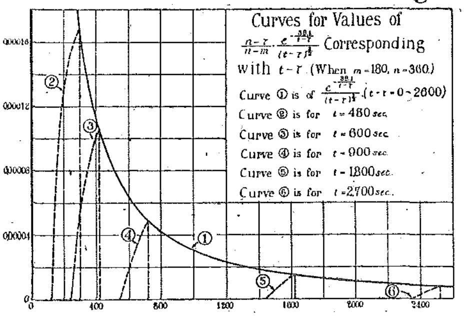


Fig. 4.

Thus by obtaining the integrals for the various time $t$ for a certain $m$ and $n$, the values of $u_3'$ and $u_4'$ will be calculated and finally $u_3$ and $u_4$. It is as shown in Table 6. This results is illustrated by Fig. 5, by fine full lines. In the same way the results are illustrated by Fig. 6, 7 and 8 for $x=2$ cm, 3 cm, and 4 cm respectively.

Table 4.

$m=180$, $n=360$, $n-m=180$

| $t-\tau$ | $t=240$, $t-m=60$, $r=\frac{120+(t-\tau)}{180}$ | | $t=300$, $t-m=120$, $r=\frac{60+(t-\tau)}{180}$ | | $t=360$, $t-m=180$, $r=\frac{t-\tau}{180}$ | |
|---|---|---|---|---|---|---|
| | $r$ | $r\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | $r$ | $r\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | $r$ | $r\frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ |
| 0 | 0.6667 | 0. | 0.3333 | 0. | 0. | 0. |
| 3 | 0.6833 | 0.000000 | 0.3500 | 0.000000 | 0.0167 | 0.000000 |
| 4 | 0.6889 | 0.000004 | 0.3556 | 0.000002 | 0.0222 | 0.000000 |
| 5 | 0.6944 | 0.000025 | 0.3611 | 0.000012 | 0.0278 | 0.000001 |
| 6 | 0.7000 | 0.000071 | 0.3667 | 0.000037 | 0.0333 | 0.000003 |
| 8 | 0.7111 | 0.000235 | 0.3778 | 0.000125 | 0.0444 | 0.000015 |
| 10 | 0.7222 | 0.000457 | 0.3889 | 0.000246 | 0.0556 | 0.000035 |
| 12 | 0.7333 | 0.000676 | 0.4000 | 0.000369 | 0.0667 | 0.000061 |
| 14 | 0.7444 | 0.000870 | 0.4111 | 0.000481 | 0.0778 | 0.000091 |
| 16 | 0.7556 | 0.001028 | 0.4222 | 0.000575 | 0.0889 | 0.000121 |
| 18 | 0.7667 | 0.001148 | 0.4333 | 0.000649 | 0.1000 | 0.000150 |
| 20 | 0.7778 | 0.001225 | 0.4444 | 0.000700 | 0.1111 | 0.000175 |
| 22.5 | 0.7917 | 0.001301 | 0.4583 | 0.000753 | 0.1250 | 0.000205 |
| 25 | 0.8056 | 0.001354 | 0.4722 | 0.000794 | 0.1389 | 0.000233 |
| 27 | 0.8167 | 0.001365 | 0.4833 | 0.000808 | 0.1500 | 0.000251 |
| 28.5 | 0.8250 | 0.001377 | 0.4917 | 0.000821 | 0.1583 | 0.000264 |
| 30 | 0.8333 | 0.001381 | 0.5000 | 0.000829 | 0.1667 | 0.000276 |
| 32 | 0.8444 | 0.001376 | 0.5111 | 0.000833 | 0.1778 | 0.000290 |
| 34 | 0.8556 | 0.001367 | 0.5222 | 0.000835 | 0.1889 | 0.000302 |
| 36 | 0.8667 | 0.001353 | 0.5333 | 0.000833 | 0.2000 | 0.000312 |
| 38 | 0.8778 | 0.001339 | 0.5444 | 0.000830 | 0.2111 | 0.000322 |
| 40 | 0.8889 | 0.001322 | 0.5556 | 0.000826 | 0.2222 | 0.000330 |
| 50 | 0.9444 | 0.001221 | 0.6111 | 0.000790 | 0.2778 | 0.000359 |
| 60 | 1.0000 | 0.001119 | 0.6667 | 0.000746 | 0.3333 | 0.000373 |
| 80 | | | 0.7778 | 0.000666 | 0.4444 | 0.000380 |
| 100 | | | 0.8889 | 0.000601 | 0.5556 | 0.000376 |
| 120 | | | 1.0000 | 0.000548 | 0.6667 | 0.000365 |
| 140 | | | | | 0.7778 | 0.000355 |
| 160 | | | | | 0.8889 | 0.000343 |
| 180 | | | | | 1.0000 | 0.000333 |

Table 5.

$m = 180, \quad n = 360, \quad n-m = 180$

| $t-\tau$ | $t=480, \begin{smallmatrix} t-m=300, \\ t-n=120, \end{smallmatrix} r=\frac{(t-\tau)-120}{180}$ | | $t-\tau$ | $t=600, \begin{smallmatrix} t-m=420, \\ t-n=240, \end{smallmatrix} r=\frac{(t-\tau)-240}{180}$ | | $t-\tau$ | $t=900, \begin{smallmatrix} t-m=720, \\ t-n=540, \end{smallmatrix} r=\frac{(t-\tau)-540}{180}$ | |
|---|---|---|---|---|---|---|---|---|
| | $r$ | $r \cdot \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | | $r$ | $r \cdot \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | | $r$ | $r \cdot \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ |
| 120 | 0. | 0. | 240 | 0. | 0. | 540 | 0. | 0. |
| 140 | 0.1111 | 0.000051 | 300 | 0.3333 | 0.000056 | 600 | 0.3333 | 0.000021 |
| 160 | 0.2222 | 0.000086 | 400 | 0.8889 | 0.000101 | 700 | 0.8889 | 0.000045 |
| 180 | 0.3333 | 0.000111 | 420 | 1.0000 | 0.000106 | 720 | 1.0000 | 0.000049 |
| 200 | 0.4444 | 0.000129 | | | | | | |
| 240 | 0.6667 | 0.000152 | | | | | | |
| 300 | 1.0000 | 0.000169 | | | | | | |

| $t-\tau$ | $t=1800, \begin{smallmatrix} t-m=1620, \\ t-n=1440, \end{smallmatrix} r=\frac{(t-\tau)-1440}{180}$ | | $t-\tau$ | $t=2700, \begin{smallmatrix} t-m=2520, \\ t-n=2340, \end{smallmatrix} r=\frac{(t-\tau)-2340}{180}$ | |
|---|---|---|---|---|---|
| | $r$ | $r \cdot \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ | | $r$ | $r \cdot \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}}$ |
| 1440 | 0. | 0. | 2340 | 0. | 0. |
| 1500 | 0.3333 | 0.000006 | 2400 | 0.3333 | 0.000003 |
| 1600 | 0.8889 | 0.000014 | 2500 | 0.8889 | 0.000007 |
| 1620 | 1.0000 | 0.000015 | 2520 | 1.0000 | 0.000008 |

Table 6.

| $m$ | $n$ | $t$ | $\int_m^t r \cdot \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}} d\tau$ | $u_3'$ | $u_3$ | $\int_m^n r \cdot \frac{e^{-\frac{39.1}{t-\tau}}}{(t-\tau)^{\frac{3}{2}}} d\tau$ | $u_4'$ | $u_4$ |
|---|---|---|---|---|---|---|---|---|
| 60 | 180 | 120 | 0.0563 | $k\times$ 0.198 | $k\times$ 0.364 | | | |
| | | 180 | 0.0542 | 〃 0.191 | 〃 0.281 | | | |
| | | 240 | | | | 0.0289 | $k\times$ 0.102 | $k\times$ 0.160 |
| | | 300 | | | | 0.0179 | 〃 0.063 | 〃 0.104 |
| | | 600 | | | | 0.0048 | 〃 0.017 | 〃 0.031 |
| | | 900 | | | | 0.0025 | 〃 0.009 | 〃 0.017 |
| | 360 | 120 | 0.0657 | $k\times$ 0.231 | $k\times$ 0.400 | | | |
| | | 180 | 0.0930 | 〃 0.328 | 〃 0.418 | | | |
| | | 240 | 0.0920 | 〃 0.324 | 〃 0.382 | | | |
| | | 300 | 0.0779 | 〃 0.274 | 〃 0.316 | | | |
| | | 360 | 0.0562 | 〃 0.198 | 〃 0.230 | | | |
| | | 600 | | | | 0.0155 | $k\times$ 0.055 | $k\times$ 0.069 |
| | | 900 | | | | 0.0072 | 〃 0.025 | 〃 0.033 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 180 | 360 | 240 | 0.0615 | $k\times$ 0.217 | $k\times$ 0.530 | | | |
| | | 300 | 0.0758 | 〃 0.267 | 〃 0.456 | | | |
| | | 360 | 0.0571 | 〃 0.201 | 〃 0.332 | | | |
| | | 480 | | | | 0.0216 | $k\times$ 0.076 | $k\times$ 0.154 |
| | | 600 | | | | 0.0116 | 〃 0.041 | 〃 0.094 |
| | | 900 | | | | 0.0049 | 〃 0.017 | 〃 0.044 |
| | 480 | 240 | 0.0657 | $k\times$ 0.231 | $k\times$ 0.545 | | | |
| | | 300 | 0.0860 | 〃 0.327 | 〃 0.517 | | | |
| | | 480 | 0.0562 | 〃 0.198 | 〃 0.276 | | | |
| | | 600 | | | | 0.0254 | $k\times$ 0.089 | $k\times$ 0.143 |
| | | 900 | | | | 0.0092 | 〃 0.032 | 〃 0.059 |
| 300 | 420 | 360 | 0.0563 | $k\times$ 0.198 | $k\times$ 0.586 | | | |
| | | 420 | 0.0542 | 〃 0.191 | 〃 0.438 | | | |
| | | 480 | | | | 0.0289 | $k\times$ 0.102 | $k\times$ 0.279 |
| | | 600 | | | | 0.0122 | 〃 0.043 | 〃 0.153 |
| | | 900 | | | | 0.0040 | 〃 0.014 | 〃 0.064 |
| | | 1800 | | | | 0.0010 | 〃 0.004 | 〃 0.019 |
| | 600 | 360 | 0.0657 | $k\times$ 0.231 | $k\times$ 0.618 | | | |
| | | 480 | 0.0920 | 〃 0.324 | 〃 0.501 | | | |
| | | 600 | 0.0562 | 〃 0.198 | 〃 0.308 | | | |
| | | 900 | | | | 0.0128 | $k\times$ 0.045 | $k\times$ 0.095 |
| | | 1200 | | | | 0.0064 | 〃 0.023 | 〃 0.053 |
| | | 1800 | | | | 0.0028 | 〃 0.010 | 〃 0.025 |
| | | 2700 | | | | 0.0013 | 〃 0.005 | 〃 0.013 |
| 900 | 1200 | 960 | 0.0657 | $k\times$ 0.231 | $k\times$ 0.752 | | | |
| | | 1080 | 0.0920 | 〃 0.324 | 〃 0.601 | | | |
| | | 1200 | 0.0562 | 〃 0.198 | 〃 0.388 | | | |
| | | 1500 | | | | 0.0128 | $k\times$ 0.045 | $k\times$ 0.147 |
| | | 1800 | | | | 0.0064 | 〃 0.023 | 〃 0.089 |
| | | 2700 | | | | 0.0021 | 〃 0.007 | 〃 0.037 |
| | 1800 | 1200 | 0.1367 | $k\times$ 0.481 | $k\times$ 0.672 | | | |
| | | 1500 | 0.1009 | 〃 0.355 | 〃 0.457 | | | |
| | | 1800 | 0.0449 | 〃 0.158 | 〃 0.198 | | | |
| | | 2100 | | | | 0.0182 | $k\times$ 0.064 | $k\times$ 0.113 |
| | | 2700 | | | | 0.0079 | 〃 0.028 | 〃 0.058 |
| | | 3600 | | | | 0.0038 | 〃 0.013 | 〃 0.031 |
| | | 1860 | 0.0657 | $k\times$ 0.231 | $k\times$ 0.814 | | | |
| | | 1980 | 0.0920 | 〃 0.324 | 〃 0.654 | | | |
| | | 2100 | 0.0562 | 〃 0.198 | 〃 0.437 | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 2100 | 2400 | | | | 0.0128 | k× 0.045 | k× 0.184 |
| | | 2700 | | | | 0.0064 | 〃 0.023 | 〃 0.119 |
| | | 3600 | | | | 0.0021 | 〃 0.007 | 〃 0.055 |
| | | 2100 | 0.1367 | k× 0.481 | k× 0.720 | | | |
| | | 2400 | 0.1009 | 〃 0.355 | 〃 0.494 | | | |
| | | 2700 | 0.0449 | 〃 0.158 | 〃 0.255 | | | |
| | 2700 | 3000 | | | | 0.0182 | k× 0.064 | k× 0.137 |
| 1800 | | 3600 | | | | 0.0079 | 〃 0.028 | 〃 0.076 |
| | | 4500 | | | | 0.0038 | 〃 0.013 | 〃 0.035 |
| | | 2100 | 0.1544 | k× 0.543 | k× 0.782 | | | |
| | | 2400 | 0.1525 | 〃 0.537 | 〃 0.675 | | | |
| | | 2700 | 0.1313 | 〃 0.462 | 〃 0.559 | | | |
| | | 3000 | 0.1031 | 〃 0.363 | 〃 0.436 | | | |
| | 3600 | 3300 | 0.0702 | 〃 0.247 | 〃 0.305 | | | |
| | | 3600 | 0.0359 | 〃 0.126 | 〃 0.174 | | | |
| | | 3900 | | | | 0.0189 | k× 0.067 | k× 0.107 |
| | | 4200 | | | | 0.0133 | 〃 0.047 | 〃 0.081 |
| | | 5400 | | | | 0.0056 | 〃 0.020 | 〃 0.040 |
| | | 2760 | 0.0657 | k× 0.231 | k× 0.843 | | | |
| | | 2880 | 0.0920 | 〃 0.324 | 〃 0.681 | | | |
| | | 3000 | 0.0562 | 〃 0.198 | 〃 0.462 | | | |
| | 3000 | 3300 | | | | 0.0128 | k× 0.045 | k× 0.205 |
| | | 3600 | | | | 0.0064 | 〃 0.023 | 〃 0.138 |
| | | 4500 | | | | 0.0021 | 〃 0.007 | 〃 0.068 |
| | | 3000 | 0.1367 | k× 0.481 | k× 0.745 | | | |
| | | 3300 | 0.1009 | 〃 0.355 | 〃 0.515 | | | |
| | | 3600 | 0.0449 | 〃 0.158 | 〃 0.273 | | | |
| | 3600 | 3900 | | | | 0.0182 | k× 0.064 | k× 0.153 |
| | | 4200 | | | | 0.0112 | 〃 0.039 | 〃 0.111 |
| 2700 | | 5400 | | | | 0.0038 | 〃 0.013 | 〃 0.053 |
| | | 3000 | 0.1544 | k× 0.543 | k〃 0.807 | | | |
| | | 3300 | 0.1525 | 〃 0.537 | 〃 0.696 | | | |
| | | 3600 | 0.1313 | 〃 0.462 | 〃 0.577 | | | |
| | | 3900 | 0.1031 | 〃 0.363 | 〃 0.452 | | | |
| | 4500 | 4200 | 0.0702 | 〃 0.247 | 〃 0.319 | | | |
| | | 4500 | 0.0359 | 〃 0.126 | 〃 0.186 | | | |
| | | 4800 | | | | 0.0189 | k× 0.067 | k× 0.118 |
| | | 5400 | | | | 0.0101 | 〃 0.036 | 〃 0.075 |
| | | 6000 | | | | 0.0067 | 〃 0.024 | 〃 0.055 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5400 | 3000 | 0.1599 | k× 0.563 | k× 0.827 | | | |
| | | 3300 | 0.1696 | 〃 0.597 | 〃 0.757 | | | |
| | | 3600 | 0.1608 | 〃 0.566 | 〃 0.681 | | | |
| | | 4200 | 0.1246 | 〃 0.439 | 〃 0.511 | | | |
| | | 4800 | 0.0799 | 〃 0.281 | 〃 0.332 | | | |
| | | 5400 | 0.0307 | 〃 0.108 | 〃 0.147 | | | |
| | | 6000 | | | | 0.0137 | k× 0.048 | k× 0.079 |
| | | 7200 | | | | 0.0066 | 〃 0.023 | 〃 0.045 |
| 3600 | 3900 | 3660 | 0.0657 | k× 0.221 | k× 0.850 | | | |
| | | 3780 | 0.0920 | 〃 0.324 | 〃 0.699 | | | |
| | | 3900 | 0.0562 | 〃 0.198 | 〃 0.475 | | | |
| | | 4200 | | | | 0.0128 | k× 0.045 | k× 0.219 |
| | | 4500 | | | | 0.0064 | 〃 0.023 | 〃 0.149 |
| | | 5400 | | | | 0.0021 | 〃 0.007 | 〃 0.077 |
| | 4500 | 3900 | 0.1367 | k× 0.481 | k× 0.759 | | | |
| | | 4200 | 0.1009 | 〃 0.355 | 〃 0.529 | | | |
| | | 4500 | 0.0449 | 〃 0.158 | 〃 0.285 | | | |
| | | 4800 | | | | 0.0182 | k× 0.064 | k× 0.164 |
| | | 5400 | | | | 0.0079 | 〃 0.038 | 〃 0.097 |
| | | 6000 | | | | 0.0047 | 〃 0.016 | 〃 0.058 |
| | 5400 | 3900 | 0.1544 | k× 0.543 | k× 0.821 | | | |
| | | 4200 | 0.1525 | 〃 0.537 | 〃 0.710 | | | |
| | | 4500 | 0.1313 | 〃 0.462 | 〃 0.589 | | | |
| | | 4800 | 0.1031 | 〃 0.363 | 〃 0.463 | | | |
| | | 5100 | 0.0702 | 〃 0.247 | 〃 0.329 | | | |
| | | 5400 | 0.0359 | 〃 0.126 | 〃 0.196 | | | |
| | | 6000 | | | | 0.0133 | k× 0.047 | k× 0.088 |
| | | 7200 | | | | 0.0056 | 〃 0.020 | 〃 0.054 |
| | 7200 | 3900 | 0.1632 | k× 0.574 | k× 0.852 | | | |
| | | 4200 | 0.1780 | 〃 0.627 | 〃 0.800 | | | |
| | | 4800 | 0.1648 | 〃 0.580 | 〃 0.680 | | | |
| | | 5400 | 0.1361 | 〃 0.479 | 〃 0.548 | | | |
| | | 6000 | 0.1022 | 〃 0.360 | 〃 0.412 | | | |
| | | 6600 | 0.0704 | 〃 0.231 | 〃 0.272 | | | |
| | | 7200 | 0.0280 | 〃 0.097 | 〃 0.133 | | | |

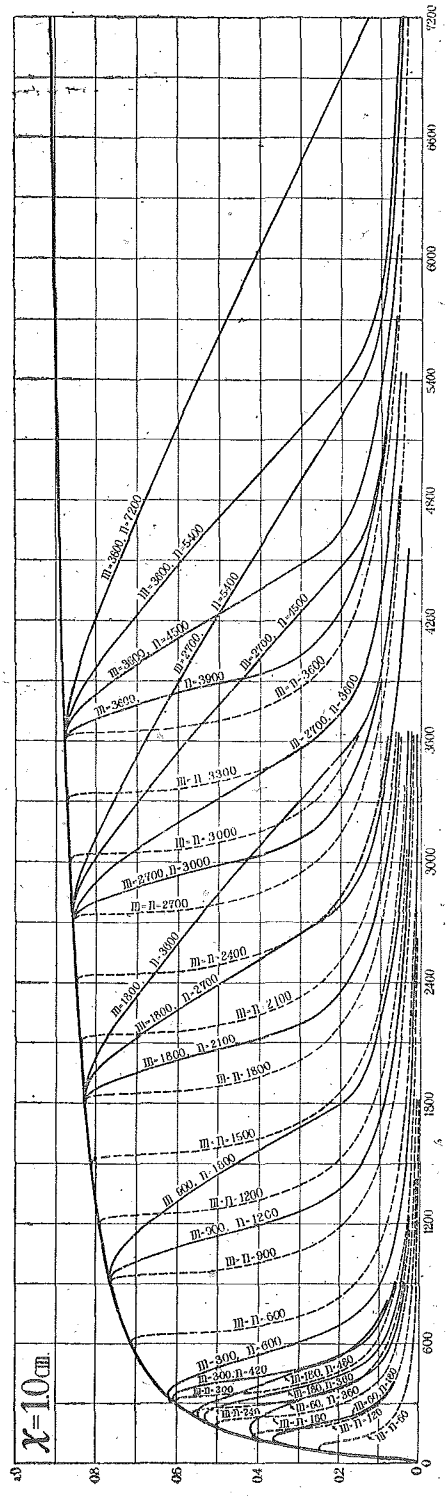


Fig. 5

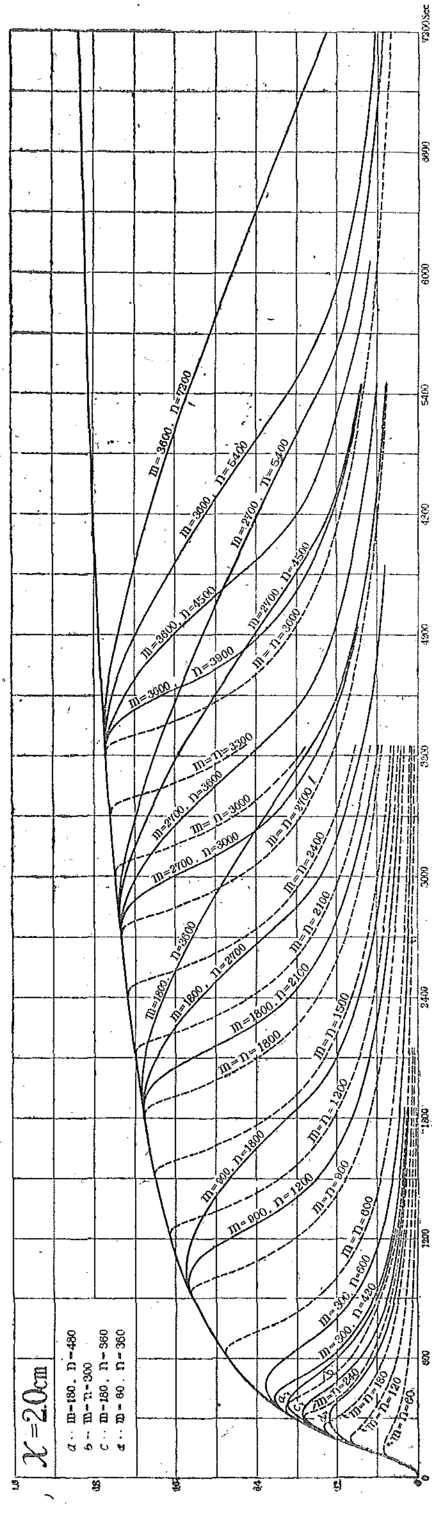


Fig. 6

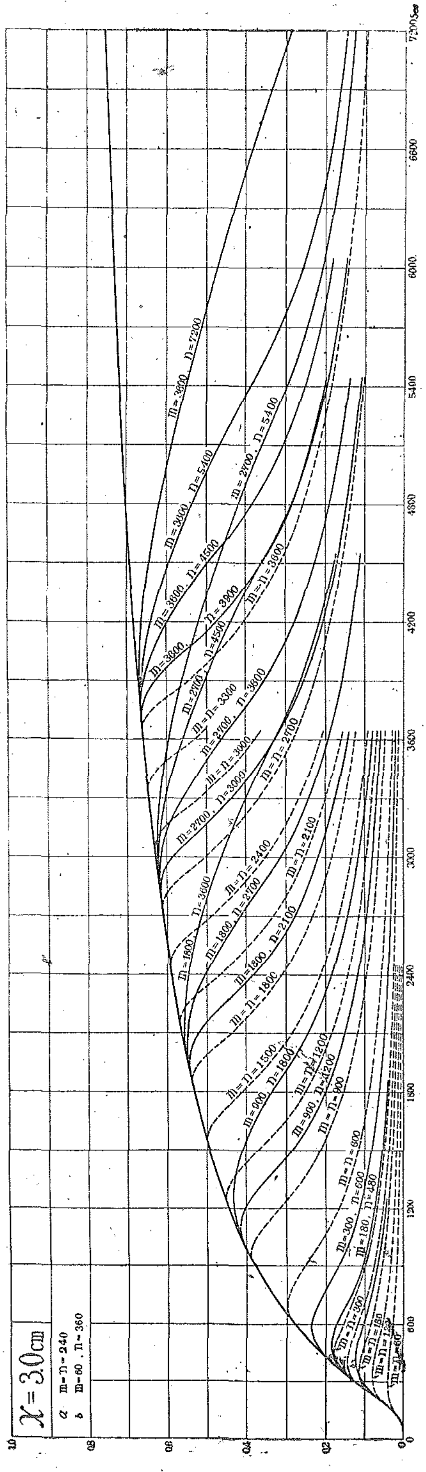


Fig. 7

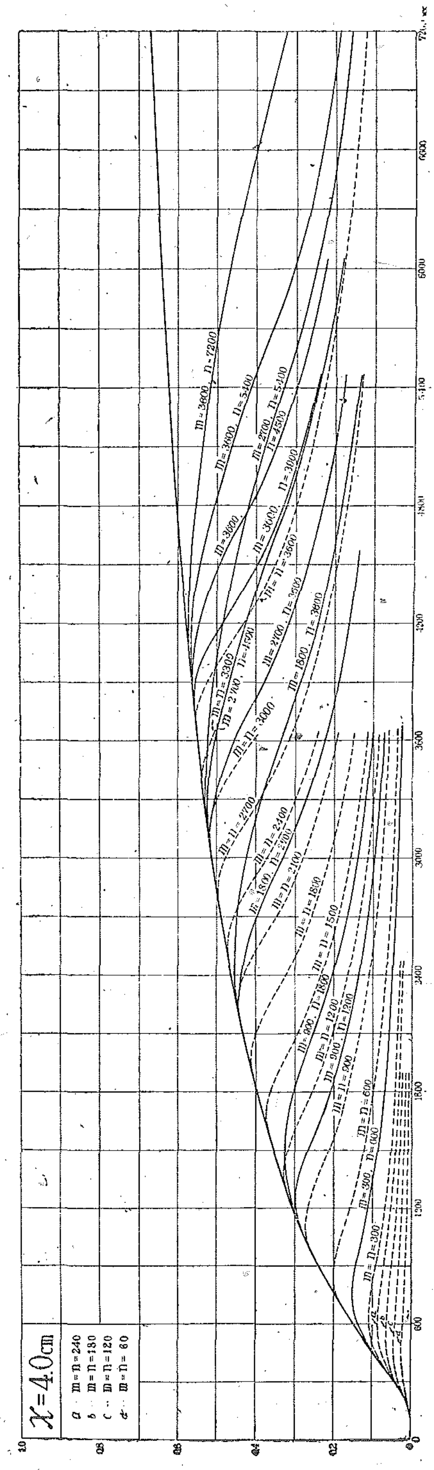


Fig. 8

## 3. Heat Conduction of the Solid Bounded by two Parallel Planes.

In this paragraph the temperature distribution is calculated when the range of $x$ is limited to the interval $0 \leqq x \leqq C$. Let the solid be bounded by the planes $x=0$ and $x=C$, initial temperature being given by $u=0$, and surface temperature varying with the time, and back plane $x=C$ being thermally insulated. In this plate, where $u$ has to satisfy

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (7)$$

$$\left.\begin{array}{lll} u=0 & \text{when} & t=0\,, \\ u=f(t) & \text{at} & x=0\,, \\ \dfrac{\partial u}{\partial x}=0 & \text{at} & x=C\,, \end{array}\right\}$$

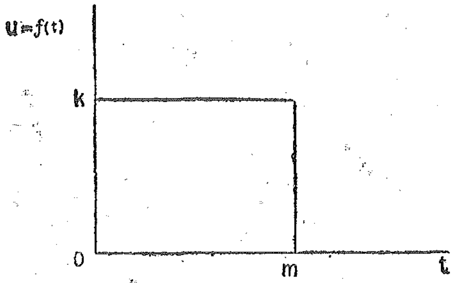


Fig. 9

the solution is given by

$$u = \frac{2}{C}\sum_{n=0}^{\infty} \sin\frac{2n+1}{2C}\pi x\, \frac{a^2(2n+1)\pi}{C}\int_0^t f(\tau)\, e^{-a^2\left(\frac{2n+1}{2C}\pi\right)^2 (t-\tau)}\, d\tau\,. \qquad \cdots\cdots\cdots (8)$$

Now, especially the temperature change at the thermally insulated back plane $x=C$ alone will be calculated numerically. It follows from (8)

$$u = \frac{2a^2}{C}\sum_{n=0}^{\infty}(-1)^n \frac{2n+1}{2C}\pi\int_0^t f(\tau)\, e^{-a^2\left(\frac{2n+1}{2C}\pi\right)^2 (t-\tau)}\, d\tau\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (9)$$

As the last paragraph let the temperature $u=f(t)$ at $x=0$ be in such a state that steady temperature continues a certain duration, then it breaks off as shown by Fig. 9,

$$f(t)=k \qquad \text{between} \qquad t=0 \qquad \text{and} \qquad t=m,$$

$$\text{and} \qquad f(t)=0 \qquad \text{when} \qquad t>m.$$

Then we shall be able to think the following two cases:

(1) when $t<m$, from (9) it follows that

$$u_1 = \frac{2k}{C}\sum_{n=0}^{\infty}(-1)^n \frac{2n+1}{2C}\pi a^2 \frac{1}{a^2\left(\frac{2n+1}{2C}\pi\right)^2}\left[e^{-a^2\left(\frac{2n+1}{2C}\pi\right)^2 (t-\tau)}\right]_0^t$$

$$= k\left\{1-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\, e^{-a^2\left(\frac{2n+1}{2C}\pi\right)^2 t}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (10)$$

(2) when $t>m$, from (9) it follows that

$$u_2 = \frac{4k}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{e^{-a^2\left(\frac{2n+1}{2C}\pi\right)^2 (t-m)} - e^{-a^2\left(\frac{2n+1}{2C}\pi\right)^2 t}\right\} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (11)$$

Writing $b=\dfrac{a^2\pi^2}{4C^2}$ and expanding (10) and (11), it follows that

$$u_1=k\left[1-\frac{4}{\pi}\left(e^{-bt}-\frac{1}{3}e^{-9bt}+\frac{1}{5}e^{-25bt}-\frac{1}{7}e^{-49bt}+\frac{1}{9}e^{-81bt}-\cdots\right)\right](t<m), \quad\cdots\cdots \quad (12)$$

and $$u_2=\frac{4k}{\pi}\left[\left(e^{-b(t-m)}-\frac{1}{3}e^{-9b(t-m)}+\cdots\cdots\right)-\left(e^{-bt}-\frac{1}{3}e^{-9bt}+\cdots\right)\right](t>m). \quad\cdots\cdots \quad (13)$$

In the above fomulae when $t$ is large enough, we may neglect the quantity under $\frac{1}{3}e^{-9bt}$.

Also in this problem, we assume $a^2=0.0064$ (C.G.S.—unit), and we shall show the temperature change at $C=4.0$ cm. Then we get $b=0.000987$.

Therefore as the values of $u$ for the various $t$, we obtain the Table 7, from (12).

These values of $u$ are illustrated in Fig. 10 by full line. In this diagram the axis of abscissa shows the time $t$ (sec.) and the axis of ordinate shows the value of $\frac{u}{k}$.

When we require the values of $u_2$ for various $t$ and $m$ from (13), we can directly apply Table 7. Therefore we obtain the Table 8.

Table 7.

| $t$ | $e^{-bt}$ | $-\frac{1}{3}e^{-9bt}$ | $\frac{1}{5}e^{-25bt}$ | $-\frac{1}{7}e^{-49bt}$ | $\frac{1}{9}e^{-81bt}$ | $\sum_{n=0}^{\infty}\frac{(-1)^n e^{-(2n+1)^2bt}}{2n+1}$ | $\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n e^{-(2n+1)^2bt}}{2n+1}$ | $u_1$ |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | 1. | $k\times 0.$ |
| 60 | 0.9425 | − 0.1956 | 0.0455 | − 0.0078 | 0.0009 | 0.7854 | 1.000 | 〃 0.000 |
| 120 | 0.8883 | − 0.1148 | 0.0102 | − 0.0004 | 0.0000 | 0.7833 | 0.997 | 〃 0.003 |
| 300 | 0.7437 | − 0.0232 | 0.0001 | − 0.0000 | — | 0.7206 | 0.918 | 〃 0.082 |
| 360 | 0.7010 | − 0.0136 | 0.0000 | — | — | 0.6874 | 0.875 | 〃 0.125 |
| 420 | 0.6607 | − 0.0080 | — | — | — | 0.6527 | 0.831 | 〃 0.169 |
| 600 | 0.5531 | − 0.0016 | — | — | — | 0.5515 | 0.702 | 〃 0.298 |
| 660 | 0.5213 | − 0.0009 | — | — | — | 0.5204 | 0.663 | 〃 0.337 |
| 720 | 0.4913 | − 0.0006 | — | — | — | 0.4908 | 0.625 | 〃 0.375 |

| $t$ | $e^{-bt}$ | $\frac{4}{\pi}e^{-bt}$ | $u_1$ | $t$ | $e^{-bt}$ | $\frac{4}{\pi}e^{-bt}$ | $u_1$ | $t$ | $e^{-bt}$ | $\frac{4}{\pi}e^{-bt}$ | $u_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 900 | 0.4114 | 0.524 | $k\times$ 0.476 | 2400 | 0.0936 | 0.119 | $k\times$ 0.881 | 3900 | 0.0213 | 0.027 | $k\times$ 0.973 |
| 960 | 0.3877 | 0.494 | 〃 0.506 | 2460 | 0.0882 | 0.112 | 〃 0.888 | 4200 | 0.0158 | 0.020 | 〃 0.980 |
| 1020 | 0.3654 | 0.465 | 〃 0.535 | 2520 | 0.0832 | 0.106 | 〃 0.894 | 4500 | 0.0118 | 0.015 | 〃 0.985 |
| 1200 | 0.3060 | 0.390 | 〃 0.610 | 2700 | 0.0696 | 0.089 | 〃 0.911 | 4800 | 0.0088 | 0.011 | 〃 0.989 |
| 1260 | 0.2884 | 0.367 | 〃 0.633 | 2760 | 0.0656 | 0.084 | 〃 0.916 | 5100 | 0.0065 | 0.008 | 〃 0.992 |
| 1320 | 0.2718 | 0.346 | 〃 0.654 | 2820 | 0.0618 | 0.079 | 〃 0.921 | 5400 | 0.0049 | 0.006 | 〃 0.994 |
| 1500 | 0.2275 | 0.290 | 〃 0.710 | 3000 | 0.0514 | 0.065 | 〃 0.935 | 5460 | 0.0046 | 0.006 | 〃 0.994 |
| 1560 | 0.2145 | 0.273 | 〃 0.727 | 3060 | 0.0488 | 0.062 | 〃 0.938 | 5520 | 0.0043 | 0.005 | 〃 0.995 |
| 1620 | 0.2021 | 0.257 | 〃 0.743 | 3120 | 0.0460 | 0.059 | 〃 0.941 | 5700 | 0.0036 | 0.005 | 〃 0.995 |
| 1800 | 0.1692 | 0.215 | 〃 0.785 | 3300 | 0.0385 | 0.049 | 〃 0.951 | 6000 | 0.0027 | 0.003 | 〃 0.997 |
| 1860 | 0.1595 | 0.203 | 〃 0.797 | 3360 | 0.0363 | 0.046 | 〃 0.954 | 6300 | 0.0020 | 0.003 | 〃 0.997 |
| 1920 | 0.1503 | 0.191 | 〃 0.809 | 3420 | 0.0342 | 0.044 | 〃 0.956 | 6600 | 0.0015 | 0.002 | 〃 0.998 |
| 2100 | 0.1259 | 0.160 | 〃 0.840 | 3600 | 0.0286 | 0.036 | 〃 0.964 | 6900 | 0.0011 | 0.001 | 〃 0.999 |
| 2160 | 0.1186 | 0.151 | 〃 0.849 | 3660 | 0.0270 | 0.034 | 〃 0.966 | 7200 | 0.0008 | 0.001 | 〃 0.999 |
| 2220 | 0.1118 | 0.142 | 〃 0.858 | 3720 | 0.0254 | 0.032 | 〃 0.968 | | | | |

Table 8.

| $m$ | $t$ | $t-m$ | $u_2$ | $m$ | $t$ | $t-m$ | $u_2$ | $m$ | $t$ | $t-m$ | $u_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 300 | 360 | 60 | $k\times$ 0.125 | | 2700 | 2100 | 〃 0.072 | 1200 | 1620 | 420 | 〃 0.574 |
| | 420 | 120 | 〃 0.166 | | 3000 | 2400 | 〃 0.054 | | 1800 | 600 | 〃 0.487 |
| | 600 | 300 | 〃 0.215 | | 3300 | 2700 | 〃 0.040 | | 2100 | 900 | 〃 0.363 |
| | 660 | 360 | 〃 0.213 | 900 | 960 | 60 | $k\times$ 0.506 | | 2400 | 1200 | 〃 0.270 |
| | 900 | 600 | 〃 0.179 | | 1020 | 120 | 〃 0.532 | | 2700 | 1500 | 〃 0.201 |
| | 1200 | 900 | 〃 0.134 | | 1200 | 300 | 〃 0.528 | | 3000 | 1800 | 〃 0.150 |
| | 1500 | 1200 | 〃 0.100 | | 1500 | 600 | 〃 0.412 | | 3300 | 2100 | 〃 0.111 |
| | 1800 | 1500 | 〃 0.074 | | 1800 | 900 | 〃 0.308 | | 3600 | 2400 | 〃 0.083 |
| | 2100 | 1800 | 〃 0.055 | | 2100 | 1200 | 〃 0.229 | | 3900 | 2700 | 〃 0.062 |
| | 2400 | 2100 | 〃 0.041 | | 2400 | 1500 | 〃 0.171 | | 4200 | 3000 | 〃 0.045 |
| 600 | 660 | 60 | $k\times$ 0.337 | | 2700 | 1800 | 〃 0.127 | 1500 | 1560 | 60 | $k\times$ 0.727 |
| | 720 | 120 | 〃 0.372 | | 3000 | 2100 | 〃 0.095 | | 1620 | 120 | 〃 0.740 |
| | 900 | 300 | 〃 0.394 | | 3300 | 2400 | 〃 0.070 | | 1800 | 300 | 〃 0.702 |
| | 960 | 360 | 〃 0.382 | | 3600 | 2700 | 〃 0.052 | | 1920 | 420 | 〃 0.640 |
| | 1200 | 600 | 〃 0.313 | | 3900 | 3000 | 〃 0.038 | | 2100 | 600 | 〃 0.542 |
| | 1500 | 900 | 〃 0.234 | | 1260 | 60 | $k\times$ 0.633 | | 2400 | 900 | 〃 0.403 |
| | 1800 | 1200 | 〃 0.174 | | 1320 | 120 | 〃 0.651 | | 2700 | 1200 | 〃 0.301 |
| | 2100 | 1500 | 〃 0.129 | | 1500 | 300 | 〃 0.628 | | 3000 | 1500 | 〃 0.224 |
| | 2400 | 1800 | 〃 0.096 | | | | | | 3300 | 1800 | 〃 0.166 |

| | | | |
|---|---|---|---|
| | 3600 | 2100 | 〃 0.124 |
| | 3900 | 2400 | 〃 0.092 |
| | 4200 | 2700 | 〃 0.068 |
| | 4500 | 3000 | 〃 0.050 |
| 1800 | 1860 | 60 | k× 0.797 |
| | 1920 | 120 | 〃 0.806 |
| | 2100 | 300 | 〃 0.757 |
| | 2220 | 420 | 〃 0.689 |
| | 2400 | 600 | 〃 0.583 |
| | 2700 | 900 | 〃 0.435 |
| | 3000 | 1200 | 〃 0.324 |
| | 3300 | 1500 | 〃 0.241 |
| | 3600 | 1800 | 〃 0.179 |
| | 3900 | 2100 | 〃 0.133 |
| | 4200 | 2400 | 〃 0.099 |
| | 4500 | 2700 | 〃 0.074 |
| | 4800 | 3000 | 〃 0.054 |
| | 5100 | 3300 | 〃 0.041 |
| 2100 | 2160 | 60 | k× 0.849 |
| | 2220 | 120 | 〃 0.855 |
| | 2400 | 300 | 〃 0.798 |
| | 2520 | 420 | 〃 0.725 |
| | 2700 | 600 | 〃 0.614 |
| | 3000 | 900 | 〃 0.458 |
| | 3360 | 1200 | 〃 0.341 |
| | 3600 | 1500 | 〃 0.253 |
| | 3900 | 1800 | 〃 0.188 |
| | 4200 | 2100 | 〃 0.140 |
| | 4500 | 2400 | 〃 0.104 |
| | 4800 | 2700 | 〃 0.077 |
| | 5100 | 3000 | 〃 0.057 |
| | 5400 | 3300 | 〃 0.043 |
| 2400 | 2460 | 60 | k× 0.888 |
| | 2520 | 120 | 〃 0.891 |
| | 2700 | 300 | 〃 0.829 |
| | 2820 | 420 | 〃 0.752 |
| | 3000 | 600 | 〃 0.637 |
| | 3300 | 900 | 〃 0.475 |
| | 3600 | 1200 | 〃 0.353 |
| | 3900 | 1500 | 〃 0.263 |
| | 4200 | 1800 | 〃 0.195 |
| | 4500 | 2100 | 〃 0.145 |
| | 4800 | 2400 | 〃 0.108 |
| | 5100 | 2700 | 〃 0.080 |
| | 5400 | 3000 | 〃 0.059 |

| | | | |
|---|---|---|---|
| | 5700 | 3300 | 〃 0.044 |
| 2700 | 2760 | 60 | k× 0.916 |
| | 2820 | 120 | 〃 0.919 |
| | 3000 | 360 | 〃 0.852 |
| | 3120 | 420 | 〃 0.772 |
| | 3300 | 600 | 〃 0.673 |
| | 3420 | 720 | 〃 0.581 |
| | 3600 | 900 | 〃 0.487 |
| | 3900 | 1200 | 〃 0.362 |
| | 4200 | 1500 | 〃 0.270 |
| | 4500 | 1800 | 〃 0.200 |
| | 4800 | 2100 | 〃 0.149 |
| | 5100 | 2400 | 〃 0.111 |
| | 5400 | 2700 | 〃 0.082 |
| | 5700 | 3000 | 〃 0.061 |
| | 6000 | 3300 | 〃 0.046 |
| 3000 | 3060 | 60 | k× 0.938 |
| | 3120 | 120 | 〃 0.939 |
| | 3300 | 300 | 〃 0.869 |
| | 3420 | 420 | 〃 0.787 |
| | 3600 | 600 | 〃 0.666 |
| | 3720 | 720 | 〃 0.592 |
| | 3900 | 900 | 〃 0.497 |
| | 4290 | 1200 | 〃 0.369 |
| | 4500 | 1500 | 〃 0.275 |
| | 4800 | 1800 | 〃 0.204 |
| | 5100 | 2100 | 〃 0.152 |
| | 5400 | 2400 | 〃 0.113 |
| | 5700 | 2700 | 〃 0.084 |
| | 6000 | 3000 | 〃 0.062 |
| | 6300 | 3300 | 〃 0.046 |
| 3300 | 3360 | 60 | k× 0.954 |
| | 3420 | 120 | 〃 0.954 |
| | 3600 | 300 | 〃 0.881 |
| | 3720 | 420 | 〃 0.798 |
| | 3900 | 600 | 〃 0.675 |
| | 4200 | 900 | 〃 0.503 |
| | 4500 | 1200 | 〃 0.375 |
| | 4800 | 1500 | 〃 0.279 |
| | 5100 | 1800 | 〃 0.207 |
| | 5400 | 2100 | 〃 0.154 |
| | 5700 | 2400 | 〃 0.115 |
| | 6000 | 2700 | 〃 0.085 |
| | 6300 | 3000 | 〃 0.063 |
| | 6600 | 3300 | 〃 0.047 |

| | | | |
|---|---|---|---|
| 3600 | 3660 | 60 | k× 0.966 |
| | 3720 | 120 | 〃 0.965 |
| | 3900 | 300 | 〃 0.890 |
| | 4200 | 600 | 〃 0.682 |
| | 4500 | 900 | 〃 0.509 |
| | 4800 | 1200 | 〃 0.378 |
| | 5100 | 1500 | 〃 0.281 |
| | 5400 | 1800 | 〃 0.209 |
| | 5700 | 2100 | 〃 0.156 |
| | 6000 | 2400 | 〃 0.116 |
| | 6300 | 2700 | 〃 0.086 |
| | 6600 | 3000 | 〃 0.064 |
| | 6900 | 3300 | 〃 0.048 |
| 4200 | 4500 | 300 | k× 0.903 |
| | 4800 | 600 | 〃 0.691 |
| | 5100 | 900 | 〃 0.515 |
| | 5400 | 1200 | 〃 0.384 |
| | 5700 | 1500 | 〃 0.285 |
| | 6600 | 1800 | 〃 0.212 |
| | 6300 | 2100 | 〃 0.158 |
| | 6600 | 2400 | 〃 0.117 |
| | 6000 | 2700 | 〃 0.087 |
| | 7200 | 3000 | 〃 0.064 |
| 4800 | 5100 | 300 | k× 0.909 |
| | 5400 | 600 | 〃 0.696 |
| | 5700 | 900 | 〃 0.519 |
| | 6000 | 1200 | 〃 0.386 |
| | 6200 | 1500 | 〃 0.287 |
| | 6600 | 1800 | 〃 0.214 |
| | 6900 | 2100 | 〃 0.159 |
| | 7200 | 2400 | 〃 0.118 |
| 5400 | 5460 | 60 | k× 0.994 |
| | 5520 | 120 | 〃 0.992 |
| | 5700 | 300 | 〃 0.913 |
| | 6000 | 600 | 〃 0.699 |
| | 6300 | 900 | 〃 0.521 |
| | 6600 | 1200 | 〃 0.388 |
| | 6900 | 1500 | 〃 0.288 |
| | 7200 | 1800 | 〃 0.214 |
| 6000 | 6300 | 300 | k× 0.915 |
| | 6600 | 600 | 〃 0.700 |
| | 6900 | 900 | 〃 0.522 |
| | 7200 | 1200 | 〃 0.389 |

These values of $u$ are illustrated in Fig. 10 by broken lines.

In the same way the result of calculation for $C = 10.0$ cm is illustrated by Fig. 11.

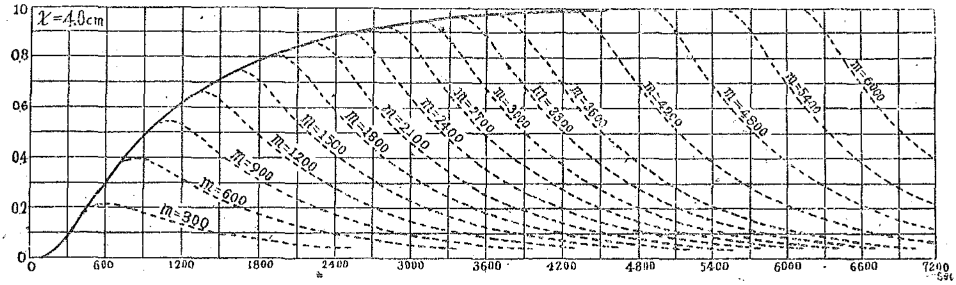


Fig. 10.

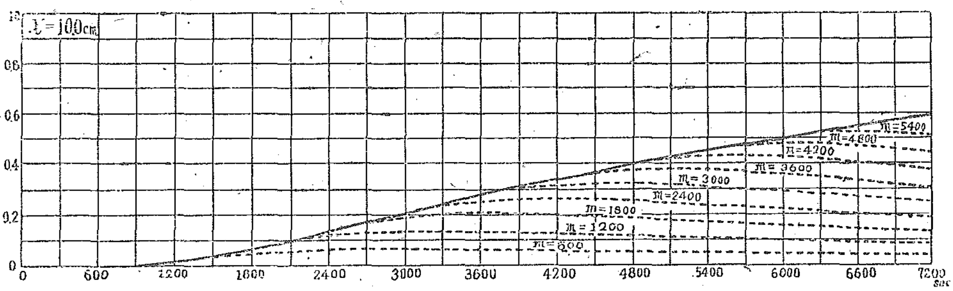


Fig. 11 (a).

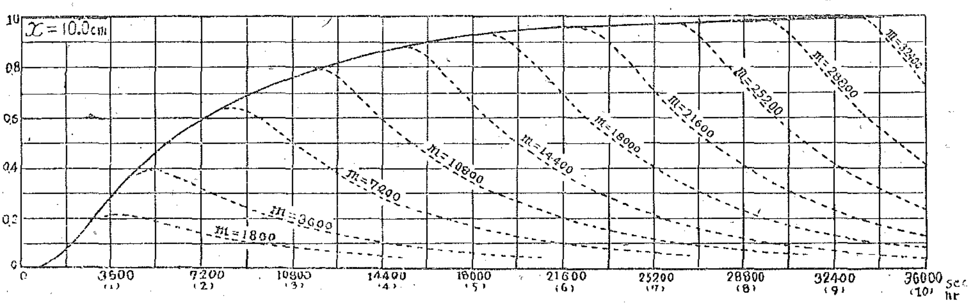


Fig. 11 (b).

## 4. Application.

Now we shall show the similarity of heat conduction between the geometrically similar two solids. Let the two similar solids have the relation as Table 9., and let $\varphi = \dfrac{l_1}{l_2}$ be the dimension ratio, $\theta = \dfrac{u_1}{u_2}$ be the temperature ratio, and $\delta = \dfrac{t_1}{t_2}$ be the time ratio. As the above examples we think about the direction of $x$—axis. Then the differential equations of the heat conduction are respectivelly as follows

Table 9.

| | Solid I. | Solid II. |
|---|---|---|
| Diffusivity | $a_1^2$ | $a_2^2$ |
| Dimension | $l_1$ | $l_2$ |
| Temperature | $u_1$ | $u_2$ |
| Time | $t_1$ | $t_2$ |

$$\frac{\partial u_1}{\partial t_1} = a_1^2 \frac{\partial^2 u_1}{\partial x_1^2}, \qquad (14)$$

$$\frac{\partial u_2}{\partial t_2} = a_2^2 \frac{\partial^2 u_2}{\partial x_2^2}. \qquad (15)$$

In the case of similar state, we obtain

$$\frac{\theta}{\delta} = \frac{a_1^2}{a_2^2} \cdot \frac{\theta}{\varphi^2}$$

or $$\delta = \varphi^2 \frac{a_2^2}{a_1^2} \qquad (16)$$

from (14) and (15).

According to (16), if the relation between temperature and time about certain dimension $l_1$ for the material having the diffusivity $a_1^2$ is given, we shall be able to obtain the relation about any dimension $l_2$ for other material having the diffusivity $a_2^2$.

Accordingly, in conformity with above examples of calculation in paragraph 2 and 3, we can apply the similarity of heat conduction to the various problems.

And then in the above two paragraphs we have shown the state of heat conduction in the semi-infinite solid and the solid plate. We shall be able to think that the temperature distribution in the most cases may be looked upon as the one in the intermediate case of these two extreme cases. Therefore the state of temperature distribution and change for the real substance can be estimated roughly from above two calculations.

At the end of this treatise I express my sense of gratitude to Prof. Y. Ikeda and Prof. K. Otubo.