On Beams and Columns of Uniform Strength Solved by the Method of Calculus of Variation.

By

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Contents.

Introduction.

The beam of uniform strength which has the variable cross section to keep the maximum fibre stress a constant, was sometimes studied from earlier time to use most efficiently the material.

This problem is usually solved by the well knwon relation "$\sigma=\frac{M}{I}h=\text{const.}$", where σ means the maximum fibre stress, M the bending moment due to external loads, I the moment of inertia of a cross section and h the half depth of the beam.

In this paper the same problem and its related problems are more generally discussed at a different point of view.

§ 1. General Theory.

It is the well known "Isoperimetric Problem"[1] in Calculus of Variations to find the functional relation between independent and dependent variables in the integrant, which minimizes (or maximizes) a definite integral, keeping another definite integral constant.

In these problems it is general that the integrants contain simultaneously independent and dependent variables and their derivatives in themselves.

But if no derivatives are invo'ved in the integrants the problem is greatly simplified as follows.

Now we take two integrals

$$A=\int_0^l g(x,y)\,dx, \qquad B=\int_0^l f(x,y)\,dx.$$

Let it be required to make the integral B a minimum (or a maximum), keeping the integral A constant.

Since the integral A is constant, our problem is equivalent to minimizing (or maximizing) the following integral,

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1) O. Bolza "Lectures on the Calculus of Variations." p. 206, 1904.

$$I=\int_0^l \left\{f(x,y)+\lambda g(x,y)\right\} dx, \quad \cdots\cdots \quad (1)$$

where λ is any constant multiplier which must later be determined.

Denote any small quantity by ε and an arbitrary function of y vanishing when $x=0$ and $x=l$, by η.

We can consider $\varepsilon\eta$ a variation of y, so that ΔI, a variation of I due to that of y, is represented by

$$\begin{aligned}\Delta I &= \int_0^l \left[\left\{ f(x,y+\varepsilon\eta)+\lambda g(x,y+\varepsilon\eta)\right\} - \left\{ f(x,y)+\lambda g(x,y)\right\}\right] dx \\ &= \int_0^l \varepsilon\eta \frac{\partial (f+\lambda g)}{\partial y} dx + \int_0^l \frac{\varepsilon^2\eta^2}{2} \frac{\partial^2 (f+\lambda g)}{\partial y^2} dx + \cdots\cdots . \quad \cdots\cdots \quad (2)\end{aligned}$$

If ε is sufficiently small, it is obvious that the sign of ΔI will be determined only by that of the first term on the right-hand side of this equation.

Therefore if I is to take a minimum (or a maximum) value which is a stationary value, it is obvious that the first term of (2) vanishes for all possible small variations of y.

Consequently, it follows that

$$\int_0^l \varepsilon\eta \frac{\partial (f+\lambda g)}{\partial y} dx = 0.$$

Since η is an arbitrary function of y in the above equation, we get from the integral theorem

$$\frac{\partial (f+\lambda g)}{\partial y} = 0 , \quad \cdots\cdots \quad (3)$$

This equation corresponds to the so-called "Euler's differential equation.", and in this case it becomes a simple algebraic equation because the functions f and g in (3) contain no derivative.

In the next place, from the sign of second term of (2), we can see whether the above obtained relation minimizes or maximizes the integral I.

On the other hand, if it is required to make the integral A a minimum (or a maximum) while keeping the integral B equal to a constant, we have

$$\frac{\partial (g+\lambda f)}{\partial y} = 0 , \quad \cdots\cdots \quad (4)$$

where the constant multiplier must be determined from the definite integral B.

Since (4) is obtained by replacing λ by $\frac{1}{\lambda}$ in (3), it is the only different point between the both cases which integral is taken for determining the value of λ.

If the integrants f and g contain many independent variables such as y, z, $\cdots\cdots$ in themselves, from the same consideration as before we have

$$\frac{\partial (f+\lambda g)}{\partial y} = 0 ,$$

$$\frac{\partial (f+\lambda g)}{\partial z} = 0 ,$$

$$\cdots\cdots\cdots ,$$

$$\cdots\cdots\cdots .$$

Solving these equations simultaneously, we obtain y, z, ········ represented as the function of x.

§ 2. An Apprication to Beams.

Neglecting the effect of shearing stresses, the strain energy of the beam is

$$W = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx\,.$$

where

M : the bending moment,

E : Young's modulus of the material,

I : the moment of inertia of a cross section of the beam,

l : the length of the beam.

Denote the sectional area of the beam by A, then the total volume of the beam V is represented by

$$V = \int_0^l A\,dx\,.$$

Let us consider the form of the beam which has the minimum value of the strain energy W, keeping its total volume to a constant value.

From the privious discussion, we have

$$\frac{\partial}{\partial y}\left(\frac{M^2}{EI}\right) + \lambda\frac{\partial A}{\partial y} = 0\,, \qquad \cdots\cdots (5)$$

where y stands for one dimension concerning the sectional area of the beam.

If P denotes a concentrated load, it follows in an equilibrium state of the beam that

$$\frac{1}{2}P\eta = W\,,$$

where η is the deflection of the loaded point.

Therefore we can find that Eq. (5) gives the form of the beam minimizing the deflection of a loaded point.

If we denote by ζ the deflection of any point of the beam, we have

$$\zeta = \int_0^l \frac{MM'}{EI}\,dx$$

in which, M' means $\frac{\partial M}{\partial X}$, and X an actual load acting on the point considered or a virtual load when there is no actual loads.

Let us now require the form of the beam minimizing ζ, then it follows:

$$\frac{\partial}{\partial y}\left(\frac{MM'}{EI}\right) + \lambda\frac{\partial A}{\partial y} = 0\,. \qquad \cdots\cdots (6)$$

If in the equation (6) we require the deflection of the point under an external concentrated load, it is obvious that Eq. (6) coincides with Eq. (5).

By the way, we may also consider that the form of the beam minimizing its curvature $\frac{1}{\rho}$ in the meaning of the least square is derived from the condition.

$$\int_0^l \left(\frac{1}{P}\right)^2 dx = \int_0^l \left(\frac{M}{EI}\right)^2 dx = \min.$$

§ 3. Beams of Uniform Strength.

(1) Simple beams of the circular cross section.

Denoting by r the rudius of the cross section, by A its cross sectional area, and by I the moment of inertia of it, and taking the axis of the beam as x-coordinate axis, we have

$$A = \pi r^2,$$

$$I = \frac{1}{4} \pi r^4.$$

Substituting these values in Eq. (5) and writing r instead of y,

$$\frac{M^2}{E} \frac{\partial}{\partial r} \left(\frac{4}{\pi r^4} \right) + \lambda \frac{\partial \pi r^4}{\partial r} = 0,$$

then

$$\frac{8}{\lambda \pi^2 E} M^2 = r^6. \qquad \text{(7)}$$

From this the maximum fibre stress of the beam is expressed as follows:

$$\sigma_{\max} = \frac{M}{I} r = \frac{4M}{\pi r^4} r = \sqrt{2\lambda E}.$$

Since $\sigma_{\max}$ is free of x, it will be seen that the beam with r represented by Eq. (7) has a uniform strength.

a) The case of a single concentrated load.

At the load condition as shown in Fig. 1, bending moments is

$$M = \frac{Pb}{l} x \qquad \text{when} \quad 0 < x < a,$$

$$M = \frac{Pa}{l} (l - x) \qquad \text{when} \quad a < x < l.$$

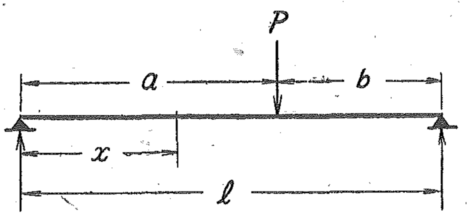

Fig. 1

Hence by Eq. (7), the form of the beam is represented as follows:

$$r^6 = \frac{8P^2b^2x^2}{\lambda E \pi^2 l^2} \qquad \text{when} \quad 0 < x < a.$$

or

$$r = \left(\frac{8P^2l^2}{\lambda E \pi^2} \right)^{\frac{1}{6}} \left(\frac{b}{l} \right)^{\frac{1}{3}} \left(\frac{x}{l} \right)^{\frac{1}{3}};$$

Then, writing

$$\frac{8P^2l^2}{\lambda E\pi^2}=K^6 ,$$

we have

$$r=K\left(\frac{b}{l}\right)^{\frac{1}{3}}\left(\frac{x}{l}\right)^{\frac{1}{3}}, \qquad \text{(8, 1)}$$

Moreover

$$V_1=\int_0^a \pi r^2 dx ,$$

so that we shall obtain

$$V_1=\frac{3}{5}\pi K^2 l\left(\frac{b}{l}\right)^{\frac{2}{3}}\left(\frac{a}{l}\right)^{\frac{5}{3}}.$$

In the same manner, if $a<x<l$, we have

$$r=K\left(\frac{a}{l}\right)^{\frac{1}{3}}\left(1-\frac{x}{l}\right)^{\frac{1}{3}}. \qquad \text{(8, 2)}$$

Writing

$$V_2=\int_a^l \pi r^2 dx ,$$

we have from (8, 2)

$$V_2=\frac{3}{5}\pi K^2 l\left(\frac{a}{l}\right)^{\frac{2}{3}}\left(\frac{b}{l}\right)^{\frac{5}{3}}.$$

When the total volume of the beam V is given,

$$V=V_1+V_2=\frac{3}{5}\pi K^2 l\left\{\left(\frac{a}{l}\right)^{\frac{5}{3}}\left(\frac{b}{l}\right)^{\frac{2}{3}}+\left(\frac{a}{l}\right)^{\frac{2}{3}}\left(\frac{b}{l}\right)^{\frac{5}{3}}\right\}=\frac{3}{5}\pi K^2 l\left(\frac{ab}{l^2}\right)^{\frac{2}{3}} . \qquad (9)$$

From this, we have

$$K=\left(\frac{5V}{3\pi l}\right)^{\frac{1}{2}}\left(\frac{l^2}{ab}\right)^{\frac{1}{3}}.$$

Substituting K in Eq. (8, 1), (8, 2), we find that the form of the beam is determined by

$$\left.\begin{aligned} r&=\left(\frac{5}{3\pi}\frac{V}{l}\right)^{\frac{1}{2}}\left(\frac{l}{a}\right)^{\frac{1}{3}}\left(\frac{x}{l}\right)^{\frac{1}{3}}=0.7284\left(\frac{l}{a}\right)^{\frac{1}{3}}\left(\frac{V}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)^{\frac{1}{3}}\text{when } 0<x<a,\\ r&=\left(\frac{5}{3\pi}\frac{V}{l}\right)^{\frac{1}{2}}\left(\frac{l}{b}\right)^{\frac{1}{3}}\left(\frac{l-x}{l}\right)^{\frac{1}{3}}=0.7284\left(\frac{l}{b}\right)^{\frac{1}{3}}\left(\frac{V}{l}\right)^{\frac{1}{2}}\left(\frac{l-x}{l}\right)^{\frac{1}{3}}\text{when } a<x<l.\end{aligned}\right\} \qquad (10)$$

At the loaded point it becomes

$$r=0.7284\left(\frac{V}{l}\right)^{\frac{1}{2}}.$$

This means that r is constant regardless of position of load.

Let us consider now two cases of a beam with a concentrated load; one loaded at (a, b), a, b being the distances of the load from the both ends, and the other loaded at (a', b') as shown in Fig-2. For the former case

$$r_1=C\left(\frac{l}{a}\right)^{\frac{1}{3}}\left(\frac{x_1}{l}\right)^{\frac{1}{3}}=C\left(\frac{x_1}{a}\right)^{\frac{1}{3}} \qquad \text{when} \quad 0<x_1<a ,$$

$$r_2=C\left(\frac{l}{b}\right)^{\frac{1}{3}}\left(\frac{l-x_2}{l}\right)^{\frac{1}{3}}=C\left(\frac{l-x_2}{b}\right)^{\frac{1}{3}} \qquad \text{when} \quad a<x_2<l .$$

For the later case

$$r_1' = C\left(\frac{x_1'}{a'}\right)^{\frac{1}{3}} \qquad \text{when} \qquad 0 < x_1' < a' \,,$$

$$r_2' = C\left(\frac{l-x_2'}{b'}\right)^{\frac{1}{3}} \qquad \text{when} \qquad a' < x_2' < l \,.$$

We will compare these cases concerning the position of the same sectional area of the beams Equating r_1 and r_1', we have

$$\frac{x_1}{a} = \frac{x_1'}{a'} \,.$$

Similarly from $r_2 = r_2'$

$$\frac{l-x_2}{b} = \frac{l-x_2'}{b'} \qquad \text{or} \qquad \frac{l-x_2}{l-a} = \frac{l-x_2'}{l-a'} \,,$$

from $r_1 = r_2$

$$\frac{x_1}{a} = \frac{l-x_2}{b} = \frac{l-x_2}{l-a} \,.$$

And then, we have

$$\frac{x_1-x_1'}{a-a'} = \frac{x_2-x_2'}{a-a'} \,.$$

From the above relation, we can simply and graphically obtain the form of the beam loaded at any point by using the form of the beam of a certain load condition.

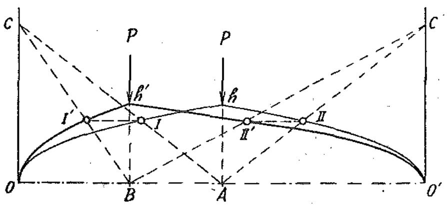

Fig. 2

Denote in Fig. 2 by the curve 0–I–h–II–$0'$ the form of the beam loaded at A, by C (or C') any point on the vertical through the end point 0 (or 0') and by I (or II) the point of intersection of the curve 0–I–h–II–$0'$ and the straight joining the two points A and C (or C'), then we shall obtain a point I' (or II') if we bring the parallel line to x–axis passing through I (or II) into intersection with the line CB (or $C'B$), this is the required point on the form of the new beam loaded at B.

We have tried above to obtain an expression of the form of the beam that has the value of its volume as a numerical factor.

On the other hand, let it be required to obtain another expression with the factor concerning the stresses.

If r is given at the position where the bending moment is maximum, the form of the beam can be determined since the numerical factor K is obtained from Eq. (8, 1) or (8, 2) and we can find the volume of the beam from Eq. (9).

Some further calculations give

$$r=\left(\frac{4P}{\pi\sigma}\right)^{\frac{1}{3}} b^{\frac{1}{3}} \left(\frac{x}{l}\right)^{\frac{1}{3}} \qquad \text{when} \quad 0<x<a\,,$$

$$r=\left(\frac{4P}{\pi\sigma}\right)^{\frac{1}{3}} a^{\frac{1}{3}} \left(\frac{l-x}{l}\right)^{\frac{1}{3}} \qquad \text{when} \quad a<x<l\,.$$

These expresstions are the usual formulae of the beam of uniform strength.

Next, let it be required to obtain the form of the beam whose volume is a minimum among the various forms of the beams having a constant strain energy or deflection at the loaded point.

For this purpose, we must determine the factor K in Eq. (8) from the integral of the strain energy.

By the similar calculations as before, we find the relations

$$\left.\begin{aligned} r&=\left(\frac{12Pl^3}{5\pi E\eta}\right)^{\frac{1}{4}}\left(\frac{a}{l}\right)^{\frac{1}{6}}\left(\frac{b}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)^{\frac{1}{3}}=0.9349\left(\frac{Pl^3}{E\eta}\right)^{\frac{1}{4}}\left(\frac{a}{l}\right)^{\frac{1}{6}}\left(\frac{b}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)^{\frac{1}{3}} \text{when } 0<x<a\,,\\ r&=0.9349\left(\frac{Pl^3}{E\eta}\right)^{\frac{1}{4}}\left(\frac{a}{l}\right)^{\frac{1}{2}}\left(\frac{b}{l}\right)^{\frac{1}{6}}\left(\frac{l-x}{l}\right)^{\frac{1}{3}} \qquad \text{when } a<x<l\,. \end{aligned}\right\}\cdots\cdots \tag{11}$$

which represent the form of the beam of uniform strength.

Consequently we know that the above obtained various formulae represent the same form of a beam but respectively with the different factor.

Calculating the deflection of the loaded point of the beam represented by Eq. (10), we have

$$\eta=\frac{108}{125}\frac{\pi}{E}\left(\frac{l}{V}\right)^2\frac{Pa^2b^2}{l}\,.$$

The similar deflection of a cylindrical beam with the some volume is shown as follows:

$$\eta'=\frac{4}{3}\frac{\pi}{E}\left(\frac{l}{V'}\right)^2\frac{Pa^2b^2}{l}\,. \quad \cdots\cdots\cdots\cdots \tag{12}$$

Therefore, we obtain

$$\eta=0.648\,\eta'\,.$$

Hence, it will be concluded that the deflection η amounts about 65% of η' regardless of the position of a external load.

Next, the volume of the beam shown by Eq. (11) is equal to

$$V=1.6475\left(\frac{Pl}{E\eta}\right)^{\frac{1}{2}} ab\,,$$

while that of the cylindrical beam is

$$V'=2.0466\left(\frac{Pl}{E\eta'}\right)^{\frac{1}{2}} ab\,.$$

by (12).

Equating the deflections for two cases, we can compare their volumes.

that is $\quad V=0.805\,V'$.

b) The case of a uniformly distributed load.

In this case the form of the beam will evidently becomes symmetrical about the middle point of the beam. By similar calculations by Eq. (7), we have

$$r = K\left(1-\frac{x}{l}\right)^{\frac{1}{3}}\left(\frac{x}{l}\right)^{\frac{1}{3}} \qquad \text{when} \qquad 0 < x < \frac{l}{2},$$

where K is a numerical factor which can be expressed bv V or $\sigma_{\max}$ and so on to meet a need as previous exsamples.

c) The case of any kinds of load.

In this case we can plot the form of the beam of uniform strength depending on Eq. (7), when the bending moment at every point of the beam is given, and we can also represent λ by V, σ, P, η, and so on respectively as before.

In the same way, we can easily study a cantilever beam by using Eq. (7).

The form of a cantilever acted by a single concentrated load on its free end will coincides with a half form of a simple beam with a load on its middle point, since the external load in the former case can be regarded as a reaction in the later.

By this reason we can apply Eq. (10) or Eq. (11) to the present case.

(2) Simple beams with rectangular cross section.

Denote

w : the width of a cross section,

$2h$: the depth of a cross section,

$A = 2hw$: the area of a cross section,

$I = \frac{2h^3w}{3}$: the moment of inertia of a cross section.

Regarding w, h, as independent variables, we have following equations as already explained:

$$\frac{\partial}{\partial h}\left(\frac{M^2}{EI}\right) + \lambda\frac{\partial A}{\partial h} = 0,$$

$$\frac{\partial}{\partial w}\left(\frac{M^2}{EI}\right) + \lambda\frac{\partial A}{\partial w} = 0.$$

Therefore, the form of the beam must be represented by following equations:

$$\frac{9M^2}{Ewh^4} = 4\lambda w \qquad \cdots\cdots (13, 1),$$

$$\frac{3M^2}{Ew^2h^3} = 4\lambda h \qquad \cdots\cdots (13, 2).$$

It is obvious that these equations can not be satisfied simultaneously, in other words, considering w and h as independent variables, we can not determine the form of the beam.

And then, if the width w is constant, we can determine the hight h by Eq. (13, 1) and vice versa.

a) The case of a single concentrated load.

Using the same notations as in Fig. 1, we have for a beam of constant width

$$h = K\left(\frac{b}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)^{\frac{1}{2}} \qquad \text{when} \qquad 0 < x < a,$$

$$h = K\left(\frac{a}{l}\right)^{\frac{1}{2}}\left(1-\frac{x}{l}\right)^{\frac{1}{2}} \qquad \text{when} \qquad a < x < l,$$

and for that of constant hight

$$w = K\left(\frac{b}{l}\right)\left(\frac{x}{l}\right) \qquad \text{when} \quad 0 < x < a\ ,$$

$$w = K\left(\frac{a}{l}\right)\left(1-\frac{x}{l}\right) \qquad \text{when} \quad a < x < l\ ,$$

in which a numerical factor K is to be determined as before.

It is easy to show that these beams have uniform strength.

b) The case of a uniformly distributed load.

Since the form of the beam becomes symmetrical about its middle point in this case, it is permitted to consider about only one half of the beam.

Putting

$$w = \text{const.}\ ,$$

we have

$$h = K\left(1-\frac{x}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)^{\frac{1}{2}} \qquad \text{when} \quad 0 < x < \frac{l}{2}\ ,$$

while for constant h we have

$$w = K\left(1-\frac{x}{l}\right)\left(\frac{x}{l}\right) \qquad \text{when} \quad 0 < x < \frac{l}{2}\ .$$

These expressions represent the form of the beam of uniform strength with a factor K which will be determined later.

In the case where the bending moment due to loads can be resolved into two components M and M' acting in the plane of the two principal axes of the cross section, the form of the beam of the least strain energy is given by the equations

$$\frac{\partial}{\partial h}\left(\frac{M^2}{EI} + \frac{M'^2}{EI'}\right) + \lambda\frac{\partial A}{\partial h} = 0\ ,$$

$$\frac{\partial}{\partial w}\left(\frac{M^2}{EI} + \frac{M'^2}{EI'}\right) + \lambda\frac{\partial A}{\partial w} = 0\ ,$$

or

$$\left.\begin{aligned} \frac{9}{2h^4w^2}M^2 + \frac{6}{h^2w^4}M'^2 = 2\lambda E\ , \\ \frac{3}{2h^4w^2}M^2 + \frac{18}{h^2w^4}M'^2 = 2\lambda E\ . \end{aligned}\right\} \qquad \cdots\cdots\cdots \quad (14)$$

since the moments of inertia with respect to two principal axes of the cross section are $I = \frac{2h^3w}{3}$, and $I' = \frac{hw^3}{6}$, respectively

It follows from this that

$$\frac{M}{M'} = \frac{2h}{w}\ .$$

Therefore we find that the magnitude of each side of the cross section is respectively proportional to the bending moment acting parallel to the corresponding side.

Using the above relation and Eq. (14), we obtain

$$\frac{M}{h^2 w} = \sqrt{\frac{\lambda E}{3}} ,$$

Hence,

$$\left.\begin{aligned} h &= \left(\frac{M^2}{2M'}\right)^{\frac{1}{3}} \left(\frac{3}{\lambda E}\right)^{\frac{1}{6}} , \\ w &= \left(\frac{4M'^2}{M}\right)^{\frac{1}{3}} \left(\frac{3}{\lambda E}\right)^{\frac{1}{6}} , \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (15)$$

in which λ is the numerical factor that will suitably be determined in case of need, and also it is easy to show that these expressions give the form of the beam of uniform strength.

That is, writing

$$\sigma = \frac{M}{I} h = \frac{3M}{2h^2 w} , \qquad \sigma' = \frac{M'}{I'} \frac{w}{2} = \frac{3M'}{hw^2} ,$$

we find

$$\sigma + \sigma' = \frac{3M}{2h^2 w} + \frac{3M'}{hw^2} = \frac{3M}{2h^2 w} + \frac{3}{hw^2} \frac{wM}{2h} = \frac{3M}{h^2 w} = \sqrt{3\lambda E} = \text{const.} \quad \cdots\cdots\cdots\cdots (16)$$

From this, λ will be determined, if the maximum fibre stress along an edge line of the beam is given.

As a matter of course, if either one of w and h is given as a constant, we must use the corresponding one in Eq. (14). But in this case we do not obtain the beam of uniform strength in such a meaning as before.

§ 4. Beams with the Least Deflection at Any Particular Point.

(1) Beams of the circular cross section.

a) The case of a single concentrated load.

In the case as shown in Fig. 3, let us consider the form of a beam of the least deflection at the position G.

Using the formula (6), we obtain

$$\frac{8}{\lambda E \pi^2} MM' = r^6 , \quad \cdots\cdots\cdots\cdots (17)$$

in which M' represent the bending moment due to a virtual unit load acting at G.

Since the bending moments are expressed by

$$M = \frac{Pb}{l} x \qquad M' = \frac{b'}{l} x \qquad \text{when} \quad 0 < x < a ,$$

$$M = \frac{Pa(l-x)}{l} \qquad M' = \frac{b'}{l} x \qquad \text{when} \quad a < x < a' ,$$

$$M = \frac{Pa(l-x)}{l} \qquad M' = \frac{a'(l-x)}{l} \qquad \text{when} \quad a' < x < l ,$$

the required form of the beam, is determined by

$$r^6 = \frac{8Pbb'}{\lambda E\pi^2}\left(\frac{x}{l}\right)^2 \quad \text{or} \quad r = \left(\frac{8Pbb'}{\lambda E\pi^2}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{3}} \qquad \text{when} \quad 0 < x < a ,$$

Writing

$$\frac{8Pl^2}{\lambda E\pi^2} = K^6 ,$$

we have

$$r = K\left(\frac{bb'}{l^2}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{3}} . \qquad \cdots\cdots (18,\ 1)$$

Substituting this in

$$V_1 = \int_0^a \pi r^2 dx ,$$

we obtain

$$V_1 = \frac{3}{5}\pi K^2\left(\frac{bb'}{l^2}\right)^{\frac{1}{3}} l\left(\frac{a}{l}\right)^{\frac{5}{3}} .$$

In the same manner, when $a < x < a'$

$$r^6 = \frac{8Pab'}{\lambda E\pi^2}\left(1-\frac{x}{l}\right)\left(\frac{x}{l}\right) \quad \text{or} \quad r = K\left(\frac{ab'}{l^2}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{6}} . \qquad \cdots\cdots (18,\ 2)$$

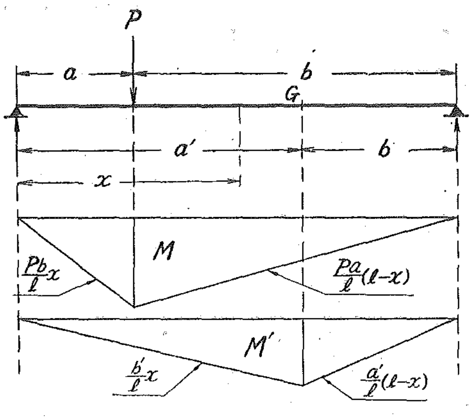

Fig. 3

Substituting this in the expression

$$V_2 = \int_a^{a'} \pi r^2 dx ,$$

we obtain

$$V_2 = \pi K^2\left(\frac{ab'}{l^2}\right)^{\frac{1}{3}} l\int_{\frac{a}{l}}^{\frac{a'}{l}} t^{\frac{1}{3}}\left(1-t\right)^{\frac{1}{3}} dt = \pi K^2\left(\frac{ab'}{l^2}\right)^{\frac{1}{3}} l\cdot S ,$$

in which

$$S = 0.75\left\{\left(\frac{a'}{l}\right)^{\frac{4}{3}}-\left(\frac{a}{l}\right)^{\frac{4}{3}}\right\}-0.142\,857\left\{\left(\frac{a'}{l}\right)^{\frac{7}{3}}-\left(\frac{a}{l}\right)^{\frac{7}{3}}\right\}-0.033\,333\left\{\left(\frac{a'}{l}\right)^{\frac{10}{3}}-\left(\frac{a}{l}\right)^{\frac{10}{3}}\right\}$$
$$-0.014\,245\left\{\left(\frac{a'}{l}\right)^{\frac{13}{3}}-\left(\frac{a}{l}\right)^{\frac{13}{3}}\right\}-0.007\,716\left\{\left(\frac{a'}{l}\right)^{\frac{16}{3}}-\left(\frac{a}{l}\right)^{\frac{16}{3}}\right\} \qquad \cdots\cdots (18,\ 2')$$

Similarly, when $a' < x < l$

$$r = \left(\frac{8Pad'}{\lambda E\pi^2}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{3}} \quad \text{or} \quad r = K\left(\frac{aa'}{l^2}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{3}} . \qquad \cdots\cdots\cdots\cdots (18,\ 3)$$

Substituting this value of r in

$$V_3 = \int_{a'}^{l} \pi r^2 dx ,$$

we obtain

$$V_3 = \frac{3}{5}\pi K^2\left(\frac{aa'}{l^2}\right)^{\frac{1}{3}} l\left(1-\frac{a'}{l}\right)^{\frac{5}{3}} = \frac{3}{5}\pi K^2\left(\frac{aa'}{l^2}\right)^{\frac{1}{3}} l\left(\frac{b'}{l}\right)^{\frac{5}{3}} .$$

If the total volume of the beam V is given, we find the following relation:

$$V = V_1 + V_2 + V_3 = K^2\pi l\left\{\frac{3}{5}\left(\frac{bb'}{l^2}\right)^{\frac{1}{3}}\left(\frac{a}{l}\right)^{\frac{5}{3}} + \frac{3}{5}\left(\frac{aa'}{l^2}\right)^{\frac{1}{3}}\left(\frac{b'}{l}\right)^{\frac{5}{3}} + \left(\frac{ab'}{l^2}\right)^{\frac{1}{3}} S\right\} .$$

Therefore

$$K = \left(\frac{V}{\pi C l}\right)^{\frac{1}{2}} ,$$

where

$$C = \frac{3}{5}\left\{\left(\frac{bb'}{l^2}\right)^{\frac{1}{3}}\left(\frac{a}{l}\right)^{\frac{5}{3}} + \left(\frac{aa'}{l^2}\right)^{\frac{1}{3}}\left(\frac{b'}{l}\right)^{\frac{5}{3}}\right\} + \left(\frac{ab'}{l^2}\right)^{\frac{1}{3}} . S .$$

Accordingly, the form of the beam may be determined by

$$\left.\begin{aligned} r &= \left(\frac{V}{\pi C l}\right)^{\frac{1}{2}}\left(\frac{bb'}{l^2}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{3}} && \text{when} \quad 0 < x < a , \\ r &= \left(\frac{V}{\pi C l}\right)^{\frac{1}{2}}\left(\frac{ab'}{l^2}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{6}} && \text{when} \quad a < x < a' , \\ r &= \left(\frac{V}{\pi C l}\right)^{\frac{1}{2}}\left(\frac{aa'}{l^2}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{3}} && \text{when} \quad a' < x < l . \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots (19)$$

On the other hand, the numerical factor K can be so determined that the maximum fibre stress in Eq. (18) keeps a given value.

For this purpose, let us investigate the stress distribution in the beam of such kind.

Since the maximum fibre stress is expressed by

$$\sigma_{\max} = \frac{M}{I} r = \frac{4M}{\pi r^3} ,$$

we have from Eq. (18, 1) and (18, 2)

$$\sigma_1 = \frac{4Pl}{K^3\pi}\left(\frac{b}{b'}\right)^{\frac{1}{2}} = \text{const.} \qquad \text{when} \quad 0 < x < a ,$$

$$\sigma_2 = \frac{4Pl}{K^3\pi}\left(\frac{a}{b'}\right)^{\frac{1}{2}}\left(\frac{l-x}{x}\right)^{\frac{1}{2}} \qquad \text{when} \quad a < x < a' .$$

From this, it is known that the fibre stress never becomes constant in the interval $a<x<a'$. By the condition $\frac{d\sigma}{dx} \neq 0$ it lies between the limits

$$\frac{4Pl}{K^3\pi}\left(\frac{b}{b'}\right)^{\frac{1}{2}} \gtreqqless \sigma \gtreqqless \frac{4Pl}{K^3\pi}\left(\frac{a}{a'}\right)^{\frac{1}{2}} .$$

Next, from Eq. (18, 3)

$$\sigma_3 = \frac{4Pl}{K^3\pi}\left(\frac{a}{a'}\right)^{\frac{1}{2}} = \text{const.} \qquad \text{when} \quad a' < x < l\ .$$

Therefore we find that either σ_1 or σ_3 is the maximum, where the relation

$$\sigma_1 : \sigma_3 = \left(\frac{b}{b'}\right)^{\frac{1}{2}} : \left(\frac{a}{a'}\right)^{\frac{1}{2}},$$

exists between them.

Requiring the form of the beam with a given deflection η and the least volume, we must determine the factor K in Eq. (18) due to the following equation:

$$\eta = \int_0^l \frac{MM'}{EI}\,dx = \text{const.}\ .$$

Then we have

$$\eta = \frac{4Pl^3}{\pi EK^4}C\ ,$$

in which the notation C has the same meaning as before, and then

$$K = \left(\frac{4C}{\pi E}\right)^{\frac{1}{4}}\left(\frac{Pl^3}{\eta}\right)^{\frac{1}{4}}.$$

Therefore, we obtain

$$\left.\begin{aligned} r &= \left(\frac{4C}{\pi E}\right)^{\frac{1}{4}}\left(\frac{Pl^3}{\eta}\right)^{\frac{1}{4}}\left(\frac{bb'}{l^2}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{3}} && \text{when} \quad 0 < x < a\ ,\\ r &= \left(\frac{4C}{\pi E}\right)^{\frac{1}{4}}\left(\frac{Pl^3}{\eta}\right)^{\frac{1}{4}}\left(\frac{ab'}{l^2}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{6}} && \text{when} \quad a < x < a'\ ,\\ r &= \left(\frac{4C}{\pi E}\right)^{\frac{1}{4}}\left(\frac{Pl^3}{\eta}\right)^{\frac{1}{4}}\left(\frac{aa'}{l^2}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{3}} && \text{when} \quad a' < x < l\ . \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (20)$$

The relation between the deflection at the point G and the volume of the beam can be obtained from the above equations; that is

$$\eta = \frac{4Pl^3C}{\pi EK^4} = \frac{4\pi Pl^5C^3}{EV^2} = \frac{4\pi Pl^5}{EV^2}\left[\frac{3}{5}\left\{\left(\frac{a}{l}\right)^{\frac{5}{3}}\left(\frac{bb'}{l^2}\right)^{\frac{1}{3}} + \left(\frac{b'}{l}\right)^{\frac{5}{3}}\left(\frac{aa'}{l^2}\right)^{\frac{1}{3}}\right\} + \left(\frac{ab'}{l^2}\right)^{\frac{1}{3}}S\right]^3.$$

The similar relation of a cylindrical beam is given by the following equation:

$$\eta' = \frac{4\pi Pl^5}{EV'^2}\,\frac{1}{6}\left[2\left(\frac{a}{l}\right)^2\left(\frac{bb'}{l^2}\right) + \left(\frac{b}{l}\right)^2\left(\frac{ab'}{l^2}\right) - \left(\frac{b'}{l}\right)^2\left(\frac{ab'}{l^2}\right)\right].$$

Equating η and η', we can compare the volumes of these two kinds of beams. On the other hand we must equate V and V' in order to compare their deflection.

Eq. (18) are of the symmetrical form with respect to (a, b) and (a', b') so that the form of the beam expressed by Eq. (18) coincides with that of the beam under a load at G which gives the least deflection at the formerly loaded point, and then these deflections in both cases may become equal to each other by Maxwell's reciprocal theorem.

As an exsample, putting for the loaded point

$$\frac{a}{l} = \frac{1}{4}\ , \qquad \frac{b}{l} = \frac{3}{4}\ ,$$

and for the position G

$$\frac{a'}{l}=\frac{5}{8}, \qquad \frac{b'}{l}=\frac{3}{8},$$

we have the deflection at G

$$\eta=0.008\,89\left(\frac{4\pi Pl^5}{EV^2}\right).$$

Requiring the deflection of a cylindrical beam of the equal volume at the same position, we have

$$\eta'=0.012\,45\left(\frac{4\pi Pl^5}{EV^2}\right).$$

At the corresponding position of the beam of uniform strength, we have

$$\eta''=0.010\,30\left(\frac{4\pi Pl^5}{EV^2}\right).$$

The forms of the beams and the corresponding defelction curves are shown in Fig. 4, in which the curve A represents the form of the beam minimixing

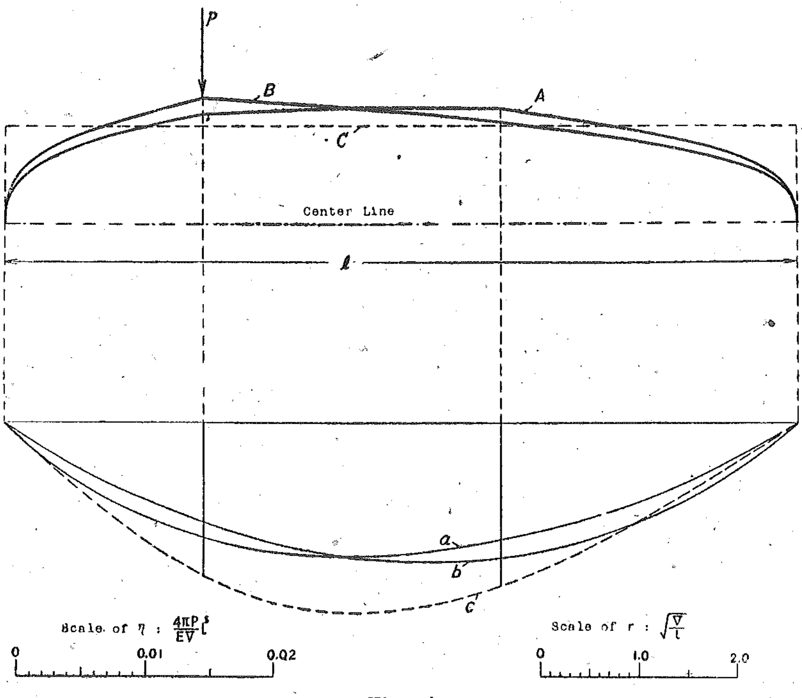

Fig. 4

the deflection at G and the curve a the corresponding deflection curve, the curve C shows te form of the cylindrical one of the equal volume and c the defection curve, the curve B shows the form of the beam of uniform strength of the equal volume and b the deflection curve.

b) The case of a uniformly distributed load.

Denoting by q the intensity of the load, we obtain the form of the beam which minimizes the deflection at G keeping its volume a given value, or minimizes the volume keeping the deflection at G a constant as follows:

$$\left.\begin{aligned} r &= K\left(\frac{b'}{l}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{3}}\left(1-\frac{x}{l}\right)^{\frac{1}{6}} && \text{when} \quad 0 < x < a' , \\ r &= K\left(\frac{a'}{l}\right)^{\frac{1}{6}}\left(\frac{x}{l}\right)^{\frac{1}{6}}\left(1-\frac{x}{l}\right)^{\frac{1}{3}} && \text{when} \quad a' < x < l . \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots \quad (21)$$

Considering the former case, we find as the expression of K:

$$K = \left(\frac{V}{\pi l S}\right)^{\frac{1}{2}} ;$$

for the later case,

$$K = \left(\frac{2qS}{E\pi\eta}\right)^{\frac{1}{4}} l ,$$

in which

$$\begin{aligned} S = 0.666\,666 &\left\{\left(\frac{b'}{l}\right)^{\frac{1}{3}}\left(\frac{a'}{l}\right)^{\frac{5}{3}}+\left(\frac{a'}{l}\right)^{\frac{1}{3}}\left(\frac{b'}{l}\right)^{\frac{5}{3}}\right\} - 0.125\,000\left\{\left(\frac{b'}{l}\right)^{\frac{1}{3}}\left(\frac{a'}{l}\right)^{\frac{8}{3}}+\left(\frac{a'}{l}\right)^{\frac{1}{3}}\left(\frac{b'}{l}\right)^{\frac{8}{3}}\right\} \\ -0.030\,303 &\left\{\left(\frac{b'}{l}\right)^{\frac{1}{3}}\left(\frac{a'}{l}\right)^{\frac{11}{3}}+\left(\frac{a'}{l}\right)^{\frac{1}{3}}\left(\frac{b'}{l}\right)^{\frac{11}{3}}\right\} - 0.013\,228\left\{\left(\frac{b'}{l}\right)^{\frac{1}{3}}\left(\frac{a'}{l}\right)^{\frac{14}{3}}+\left(\frac{a'}{l}\right)^{\frac{1}{3}}\left(\frac{b'}{l}\right)^{\frac{14}{3}}\right\} \\ -0.006\,859 &\left\{\left(\frac{b'}{l}\right)^{\frac{1}{3}}\left(\frac{a'}{l}\right)^{\frac{17}{3}}+\left(\frac{a'}{l}\right)^{\frac{1}{3}}\left(\frac{b'}{l}\right)^{\frac{17}{3}}\right\} - \cdots\cdots\cdots\cdots . \end{aligned}$$

Equating K's for both cases, we obtain the relation between the deflection η and the volume V. That is

$$\frac{V}{\pi l S} = \left(\frac{2qS}{E\pi\eta}\right)^{\frac{1}{2}} l^2 , \qquad \text{or} \qquad V^2 = 2\pi S^3 \frac{q l^6}{E}\frac{1}{\eta} .$$

(2) Beams of the rectangular cross section.

In this case, it is also impossible, as in the case of the beams of uniform strength, to consider both the depth and the width of the cross section as independent variables.

Supposing one of them to be a constant, we find that the other is represented by following equalions.

Denote by w the width of the cross section and by $2h$ its depth.

a) The case of a single concentrated load.

a. 1) In the case where w is given as a constant, we have

$$\left.\begin{aligned} h &= K\left(\frac{bb'}{l^2}\right)^{\frac{1}{4}}\left(\frac{x}{l}\right)^{\frac{1}{2}} && \text{when} \quad 0 < x < a , \\ h &= K\left(\frac{ab'}{l^2}\right)^{\frac{1}{4}}\left(1-\frac{x}{l}\right)^{\frac{1}{4}}\left(\frac{x}{l}\right)^{\frac{1}{4}} && \text{when} \quad a < x < a' , \\ h &= K\left(\frac{aa'}{l^2}\right)^{\frac{1}{4}}\left(1-\frac{x}{l}\right)^{\frac{1}{2}} && \text{when} \quad a' < x < l . \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (22)$$

in which, for the beam having the least deflection

$$K = \frac{V}{2wlC} \ ,$$

while for the beam having the least volume

$$K = \left(\frac{3PC}{2Ew\eta}\right)^{\frac{1}{3}} l \ ,$$

with

$$C = \frac{2}{3}\left\{\left(\frac{bb'}{l^2}\right)^{\frac{1}{4}}\left(\frac{a}{l}\right)^{\frac{3}{2}} + \left(\frac{aa'}{l^2}\right)^{\frac{1}{4}}\left(\frac{b'}{l}\right)^{\frac{3}{2}}\right\} + \left(\frac{ab'}{l^2}\right)^{\frac{1}{4}} S \ ,$$

and

$$S = 0.8\left\{\left(\frac{a'}{l}\right)^{\frac{5}{4}} - \left(\frac{a}{l}\right)^{\frac{5}{4}}\right\} - 0.111\,111\left\{\left(\frac{a'}{l}\right)^{\frac{9}{4}} - \left(\frac{a}{l}\right)^{\frac{9}{4}}\right\} - 0.028\,846\left\{\left(\frac{a'}{l}\right)^{\frac{13}{4}} - \left(\frac{a}{l}\right)^{\frac{13}{4}}\right\}$$
$$- 0.012\,868\left\{\left(\frac{a'}{l}\right)^{\frac{17}{4}} - \left(\frac{a}{l}\right)^{\frac{17}{4}}\right\} - 0.007\,162\left\{\left(\frac{a'}{l}\right)^{\frac{21}{4}} - \left(\frac{a}{l}\right)^{\frac{21}{4}}\right\} - \cdots\cdots\cdots .$$

The relation between V and η in this case is

$$V^3 = 12\, C^4 \frac{Pl^6 w^2}{E} \frac{1}{\eta} \ .$$

a. 2) In the case where h is given, we have

$$\left.\begin{aligned} w &= K\left(\frac{bb'}{l^2}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right) && \text{when} \quad 0 < x < a \ , \\ w &= K\left(\frac{ab'}{l^2}\right)^{\frac{1}{2}}\left(1-\frac{x}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)^{\frac{1}{2}} && \text{when} \quad a < x < a' \ , \\ w &= K\left(\frac{aa'}{l^2}\right)^{\frac{1}{2}}\left(1-\frac{x}{l}\right) && \text{when} \quad a' < x < l \ , \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (23)$$

in which, for the beam having the least deflection

$$K = \frac{V}{2hlC} \ ,$$

while for the beam having the least volume

$$K = \left(\frac{3PC}{2Eh^3\eta}\right) l^3 \ ,$$

with

$$C = \frac{1}{2}\left\{\left(\frac{bb'}{l^2}\right)^{\frac{1}{2}}\left(\frac{a}{l}\right)^2 + \left(\frac{aa'}{l^2}\right)^{\frac{1}{2}}\left(\frac{b'}{l}\right)^2\right\} + \left(\frac{ab'}{l^2}\right)^{\frac{1}{2}} S \ ,$$

and

$$S = 0.333\,333\left\{\left(\frac{a'}{l}\right)^{\frac{3}{2}} - \left(\frac{a}{l}\right)^{\frac{3}{2}}\right\} - 0.2\left\{\left(\frac{a'}{l}\right)^{\frac{5}{2}} - \left(\frac{a}{l}\right)^{\frac{5}{2}}\right\} - 0.013\,889\left\{\left(\frac{a'}{l}\right)^{\frac{9}{2}} - \left(\frac{a}{l}\right)^{\frac{9}{2}}\right\}$$
$$- 0.007\,102\left\{\left(\frac{a'}{l}\right)^{\frac{11}{2}} - \left(\frac{a}{l}\right)^{\frac{11}{2}}\right\} - \cdots\cdots\cdots .$$

In this case the relation between V and η becomes

$$V = 3C^3 \frac{Pl^4}{Eh^2} \frac{1}{\eta} \ .$$

b) The case of a uniformly distributed load.

b. 1) In the case where w is given as a constant, we have

$$\left.\begin{aligned} h &= K\left(\frac{b'}{l}\right)^{\frac{1}{4}}\left(\frac{x}{l}\right)^{\frac{1}{2}}\left(1-\frac{x}{l}\right)^{\frac{1}{4}} && \text{when} \quad 0 < x < a' , \\ h &= K\left(\frac{a'}{l}\right)^{\frac{1}{4}}\left(\frac{x}{l}\right)^{\frac{1}{4}}\left(1-\frac{x}{l}\right)^{\frac{1}{2}} && \text{when} \quad a' < x < l , \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (24)$$

in which for the beam having the least deflection

$$K = \frac{V}{2wlS} ,$$

while for the beam having the least volume

$$K = \left(\frac{3qS}{4Ew\eta}\right)^{\frac{1}{3}} l^{\frac{4}{3}} ,$$

with

$$\begin{aligned} S = {} & 0.666\,666 \left\{\left(\frac{b'}{l}\right)^{\frac{1}{4}}\left(\frac{a'}{l}\right)^{\frac{3}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{4}}\left(\frac{b'}{l}\right)^{\frac{3}{2}}\right\}-0.1\left\{\left(\frac{b'}{l}\right)^{\frac{1}{4}}\left(\frac{a'}{l}\right)^{\frac{5}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{4}}\left(\frac{b'}{l}\right)^{\frac{5}{2}}\right\} \\ & -0.026\,786\left\{\left(\frac{b'}{l}\right)^{\frac{1}{4}}\left(\frac{a'}{l}\right)^{\frac{7}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{4}}\left(\frac{b'}{l}\right)^{\frac{7}{2}}\right\}-0.012\,153\left\{\left(\frac{b'}{l}\right)^{\frac{1}{4}}\left(\frac{a'}{l}\right)^{\frac{9}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{4}}\left(\frac{b'}{l}\right)^{\frac{9}{2}}\right\} \\ & -0.006\,836\left\{\left(\frac{b'}{l}\right)^{\frac{1}{4}}\left(\frac{a'}{l}\right)^{\frac{11}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{4}}\left(\frac{b'}{l}\right)^{\frac{11}{2}}\right\} - \cdots\cdots\cdots\cdots . \end{aligned}$$

The relation between V and η in this case is

$$V^3 = 6C^4 \frac{ql^7w^2}{E} \frac{1}{\eta} .$$

b. 2) In the case where h is given as a constant, we have

$$\begin{aligned} w &= K\left(\frac{b'}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)\left(1-\frac{x}{l}\right)^{\frac{1}{2}} && \text{when} \quad 0 < x < a' , \\ w &= K\left(\frac{a'}{l}\right)^{\frac{1}{2}}\left(\frac{x}{l}\right)^{\frac{1}{2}}\left(1-\frac{x}{l}\right) && \text{when} \quad a' < x < l , \end{aligned}$$

in which, for the beam having the least deflection

$$K = \frac{V}{2hlS} ,$$

or, for the beam having the least volume

$$K = \left(\frac{3qS}{4Eh^3\eta}\right) l^4 ,$$

with

$$\begin{aligned} S = {} & 0.5\left\{\left(\frac{a'}{l}\right)^{2}\left(\frac{b'}{l}\right)^{\frac{1}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{2}}\left(\frac{b'}{l}\right)^{2}\right\}-1.666\,666\left\{\left(\frac{a'}{l}\right)^{3}\left(\frac{b'}{l}\right)^{\frac{1}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{2}}\left(\frac{b'}{l}\right)^{3}\right\} \\ & -0.031\,250\left\{\left(\frac{a'}{l}\right)^{4}\left(\frac{b'}{l}\right)^{\frac{1}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{2}}\left(\frac{b'}{l}\right)^{4}\right\}-0.012\,5\left\{\left(\frac{a'}{l}\right)^{5}\left(\frac{b'}{l}\right)^{\frac{1}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{2}}\left(\frac{b'}{l}\right)^{5}\right\} \\ & -0.006\,51\left\{\left(\frac{a'}{l}\right)^{6}\left(\frac{b'}{l}\right)^{\frac{1}{2}}+\left(\frac{a'}{l}\right)^{\frac{1}{2}}\left(\frac{b'}{l}\right)^{6}\right\} - \cdots\cdots\cdots\cdots . \end{aligned}$$

The relation between V and η is in this case

$$V = 1.5\, S^2 \frac{ql^5}{Eh^2} \frac{1}{\eta} .$$

§ 5. A Consideration about the Modification of the Form of Beams of Uniform Strength.

Since only the effect of the bending momet is generally considered in the practical theory of the beam, the area of the cross section of beams discussed above vanishes at the end point where the bending moment becomes zero.

The beams of uniform strength can not resist the shearing stresses at the end point, therefore the support should be constructed as shown in Fig. 5 in order to resist the shearing stresses.

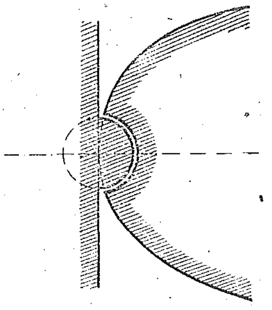

Fig. 5

In a general way, the effects of shearing stresses are covered by a practically simple modification of the form of uniform strength.

By the fact that the beam of uniform strength is one that has the least strain energy, we may introduce a new modification as follows.

Supposing that the distribution of shearing stresses on a sectional area of the beam of uniform strength can be replaced by that of the beam with a constant sectional area without much error in the total effect, though the former is generally different from the later, we can use the following expression for the total strain energy including the effect of the shearing stress:

$$W = \frac{M^2}{2EI} + \frac{\kappa Q^2}{2GA} ,$$

in which

κ : the numerical factor varying with the shape of the cross sections:

1.2 for the rectangular cross section,

1.19 for the circular cross section.

Let us consider as an exsample the beam with the constant width w and the variable depth $2h$, in which h is a variable.

Putting

$$I = \frac{2}{3} wh^3 , \qquad A = 2wh ,$$

we have as the expression of the form of the beam having the least strain energy

$$\frac{\partial W}{\partial h} + \lambda \frac{\partial A}{\partial h} = 0 .$$

Therefore,

$$\frac{9M^2}{4Ewh^4} + \frac{0.3\, Q^2}{Gwh^2} - 2\lambda w = 0 ,$$

in which G is the modulus of rigidity.

Using the well known relation

$$G = \frac{m}{2(1+m)} E \ , \qquad (m: \text{Poisson's number.})$$

we see that the above equation is transformed into

$$\frac{1}{wE}\left\{\frac{9M^2}{4h^4} + 0.6\left(\frac{1+m}{m}\right)\frac{Q^2}{h^2}\right\} - 2\lambda w = 0 \ .$$

Taking $m = 3$ for steel, we have

$$\frac{9}{4} M^2 + \frac{4}{5} Q^2h^2 - 2\lambda Ew^2h^4 = 0 \ . \qquad \text{(25)}$$

At the position where the bending moment is zero, Eq. (25) becomes

$$\frac{4}{5} Q^2 - 2\lambda Ew^2h^2 = 0 \ , \qquad \text{or} \qquad \frac{Q}{2wh} = \left(\frac{5}{8}\lambda E\right)^{\frac{1}{2}} .$$

Denoting by τ_m the mean shearing stress on the cross section, we have

$$\tau_m = \frac{Q}{2wh} = \left(\frac{5}{8}\lambda E\right)^{\frac{1}{2}} .$$

Since the factor λ can be determined from the above relation, the form of the beam is represented by Eq. (25).

In the case where a concentrated load P acts on the middle point of the span of the beam, the bending moment is

$$\frac{Px}{2} .$$

Putting.

$$\frac{2\lambda Ew^2l^2}{P^2} = K \ ,$$

the form of the beam becomes

$$2.25\left(\frac{x}{l}\right)^2 + 0.8\left(\frac{h}{l}\right)^2 - 4K\left(\frac{h}{l}\right)^4 = 0 \ . \qquad \text{(i)}$$

Neglecting the effect of the own weight of the beam and choosing l = 60 cm, P = 2400 kg, w = 5 cm and $\tau_{\max}$ = 900 kg/cm², we obtain as usual

$$\tau_m = \frac{2}{3}\tau_{\max} = 600 \text{ kg/cm}^2 \ ,$$

and then

$$\frac{1200}{600} = 2\,(\text{cm}^2) \ , \qquad \text{or} \qquad h = 0.2 \text{ cm} \ ,$$

for the required cross sectional area at the position where M is zero.

Since $M = 0$ for $x = 0$, we have from Eq. (i)

$$0.8 - 4K\left(\frac{0.2}{60}\right)^2 = 0 \ , \qquad \text{or} \qquad K = 18000 \ .$$

Let us investigate the maximum fibre stress $\sigma_{\max}$ at the cross section on which the maximum bending moment takes place, then it follows from Eq. (i),

$$2.25\left(\frac{1}{2}\right)^2 + 0.8\left(\frac{h}{l}\right)^2 - 4 \times 18000\left(\frac{h}{l}\right)^4 = 0 \ ,$$

and then

$$\left(\frac{h}{l}\right)^2 = 0.002\,81 \ , \qquad \text{or} \qquad h = 3.175 \text{ cm} \ .$$

Therefore,

$$\sigma_{\max} = 1071 \text{ kg/cm}^2 \ .$$

Comparing this with the beam calculated with no modification which has the equal depth at the middle point of the span, we see that the differences are very small at the portion near the support as shown in table 1. in which the values of $\sigma_{\max}$ for the modified beam are also shown for comparision.

Table 1.

x/l	h of the beam without the modification	h of the modified beam	differences	$\sigma_{\max}$ of the modified beam
0	0 cm	0.200 cm	0.200 cm	0 kg/cm²
1/200	0.318	0.350	0.032	881
1/100	0.449	0.471	0.022	972
1/16	1.123	1.129	0.006	1059
1/8	1.588	1.592	0.003	1065
2/8	2.245	2.247	0.002	1069
3/8	2.750	2.751	0.001	1070
1/2	3.175	3.175	0	1071

Let us next consider the case of a uniformly distributed load, whose intensity is q kg/cm. Considering the following relations:

$$M = \frac{qlx}{2}\left(1 - \frac{x}{l}\right) \ , \qquad Q = ql\left(\frac{1}{2} - \frac{x}{l}\right) \ ,$$

we find, from Eq., (25), that the form of the beam is represented as follows:

$$\frac{9}{16}\left(\frac{x}{l}\right)^2\left(1 - 2\frac{x}{l} + \frac{x^2}{l^2}\right) + \frac{4}{5}\left(\frac{1}{4} - \frac{x}{l} + \frac{x^2}{l^2}\right)\left(\frac{h}{l}\right)^2 - K\left(\frac{h}{l}\right)^4 = 0 \ . \quad \cdots\cdots \quad \text{(ii)}$$

Put

$$q = 40 \text{ kg/cm}, \qquad l = 60 \text{ cm}, \qquad w = 5 \text{ cm}, \qquad \tau_{\max} = 900 \text{ kg/cm}^2,$$

then we have

$$\tau_m = 600 \text{ kg/cm}^2 \ ,$$

It follows from (ii) for $h = 0.2$ cm at $x = 0$ and $x = l$,

$$K = 18000 \ ;$$

For the special value $\frac{x}{l} = \frac{1}{2}$, h and the maximum fibre stress $\sigma_{\max}$ is respectively calculated as follows:

$$h = 2.243 \ , \qquad \sigma_{\max} = 1073 \text{ kg/cm}^2 \ .$$

The effect of the own weight may commonly be neglected in problems of the beam of uniform-strength, but the cases where the own weight must be especially considered were studied by Dr. K. Aichi[2].

2) K. Aichi, Jour. of the Nippon Mechanical Eng. Society. Vol. 22, No. 54, J. Prescott "Applied Elasticity." p. 67.

§ 6. Columns of Uniform Strength acted by Axial Loads.

Let us now consider the column acted by the axial load from the same point of view as before. As an exsample we take a cylindrical column with one end free and the other fixed.

The deflection curve must be initially assumed in order to find the critical load by means of energy method. If we use for this purpose the deflection curve corresponding to the slight buckling of a ordinary column of constant cross section as the curve for this case, we have, referring the rectangular coordinate axes having the fixed end as their origin, the following expression:

$$y = \delta\left(1 - \cos\frac{\pi x}{2l}\right), \qquad (26)$$

where δ denotes the maximum deflection at the free end and l the length of the column.

Since the bending moment M at any point of the column is represented by

$$M_x = P\delta\cos\frac{\pi x}{2l}.$$

The strain energy becomes

$$W = \int_0^l \frac{M^2_x}{2EI}\,dx = \frac{2P^2\delta^2}{E\pi}\int_0^l \frac{\cos^2\left(\dfrac{\pi x}{2l}\right)}{r^4}\,dx. \qquad (27)$$

Now the volume of the column is

$$V = \int_0^l \pi r^2\,dx. \qquad (27')$$

Considering δ as an arbitrary constant, we have for the condition that the strain energy becomes minimum

$$\frac{\partial}{\partial r}\left\{\frac{1}{r^4}\cos^2\left(\frac{\pi x}{2l}\right) + \lambda\pi r^2\right\} = 0, \qquad (28)$$

containing in λ all factors of the integral (27).

When the column is slightly buckled by an axial load P, the work done by P is[3)]

$$T = \frac{P\delta^2\pi^2}{16l},$$

Now by the principle of energy

$$T = W,$$

from which

$$\frac{E\pi^3}{32lP} = \int_0^l \frac{\cos^2\left(\dfrac{\pi x}{2l}\right)}{r^4}\,dx. \qquad (29)$$

It is seen from the preceding consideration that Eq. (28) represents the form of the column minimizing the quantity expressed by (29); in other words it is the form of the column bearing the maximum critical load.

From Eq. (28) we obtain as the first approximation for the form of the column

3) S. Timoshenko "Theory of Elastic Stability." p. 79. 1936

$$r=\left(\frac{2}{\lambda\pi}\right)^{\frac{1}{6}}\left(\cos\frac{\pi x}{2l}\right)^{\frac{1}{3}}. \qquad \cdots\cdots (30)$$

Substituting the above expression (30) into Eq. (27′), we have

$$V=\left(\frac{2}{\lambda\pi}\right)^{\frac{1}{3}}\pi\int_0^l\left(\cos\frac{\pi x}{2l}\right)^{\frac{2}{3}}dx=\left(\frac{2}{\lambda\pi}\right)^{\frac{1}{3}}\sqrt{\pi}\,l\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)}.$$

Using the notation

$$\left(\frac{2}{\lambda\pi}\right)^{\frac{1}{3}}=K,$$

we have

$$K=\frac{V\Gamma\left(\frac{4}{3}\right)}{l\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)}=0.4452\frac{V}{l},$$

and accordingly

$$r=K^{\frac{1}{2}}\left(\cos\frac{\pi x}{2l}\right)^{\frac{1}{3}}=0.667\,23\left(\frac{V}{l}\right)^{\frac{1}{2}}\left(\cos\frac{\pi x}{2l}\right)^{\frac{1}{3}}. \qquad \cdots\cdots (31)$$

Since the deflection curve is originaly assumed roughly, a better approximate critical load must be found.

For this purpose, the method of successive approximations can be applied, but it is tedious and difficult to obtain the second approximate deflection curve by the numerical calculations, and then a graphical method can conveniently used[4].

Now assuming the deflection curve to be expressed by

$$y=\delta\cdot\left\{1-f\left(\frac{x}{l}\right)\right\}$$

we find that Eq. (30) becomes

$$r=\left(\frac{2}{\lambda\pi}\right)^{\frac{1}{6}}\cdot f^{\frac{1}{3}}\left(\frac{x}{l}\right)=K^{\frac{1}{2}}\cdot f^{\frac{1}{3}}. \qquad \cdots\cdots (32)$$

The elastic load is in this case

$$w=\frac{P\cdot\delta\cdot f}{EI}.$$

Denote $I=I_0$, $r=r_0$ when $\frac{x}{l}=0$,

then

$$w=\frac{P\cdot\delta}{EI_0}\left(\frac{f}{\mu}\right), \qquad \text{where } \frac{I}{I_0}=\frac{\pi r^4}{\pi r_0^4}=\left(\frac{r}{r_0}\right)^4=\mu. \qquad \cdots\cdots (33)$$

If we show the new deflection curve by drawing, and measure f at different points of the curve, we can find values of r corresponding to these points.

The factor K must be determined from the condition that the value of V is given.

Since

$$V=\int_0^l \pi r^2 dx=\int_0^l \pi K f^{\frac{2}{3}}dx=\pi K\sum_n\left\{f_n^{\frac{2}{3}}\cdot\Delta x\right\},$$

4) S. Timoshenko "Theory of Elastic Stability." p. 132.

we can calculate K from the following expression:

$$K=\frac{V}{\pi\sum_{n}\left\{f_n^{\frac{2}{3}}\cdot\Delta x\right\}}=\frac{0.3183}{\sum_{n}\left\{f_n^{\frac{2}{3}}\cdot\left(\frac{\Delta x}{l}\right)\right\}}\cdot\frac{V}{l}, \qquad (34)$$

In order to obtain the higher accuracy of the value of K, it is advantageous to apply Simpson's one third rule since the ordinary summation in the denominator is used instead of the integration.

Thus making the second approximation, we get the new deflection curve and obtain a higher accuracy.

The auther concludes from his experience, that the third approximate form of the beam is not neccessary since the third approximate deflection curve drawn from the second approximate form coincides actually with the second approximate deflection curve.

These procedures are shown in Fig. 6 and Table 2.

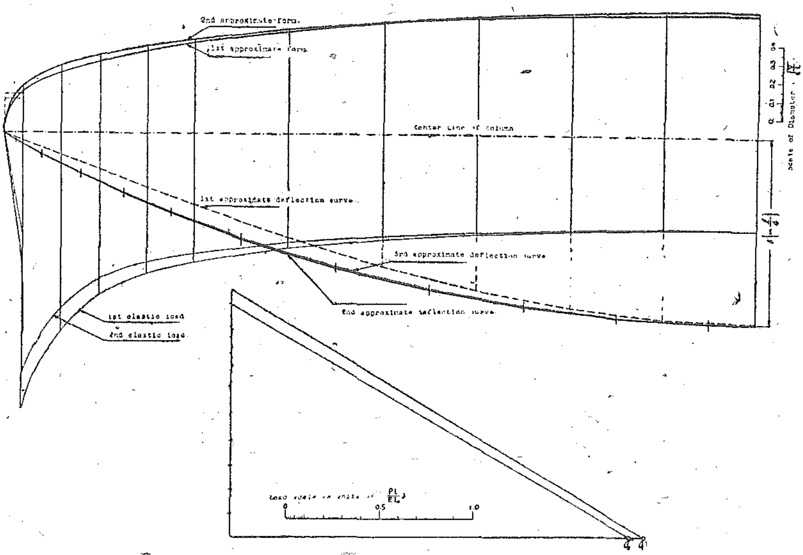

Fig. 6.

Table 2.

x/l	1st. approximate f calculated by Eq. (26)	1st. approximate r	2nd. approximate f measured on the Draw.	2nd. approximate r
0	1.000 δ	0.667 $\sqrt{\frac{V}{l}}$	1.000 δ	0.649 $\sqrt{\frac{V}{l}}$
1/8	0.980	0.663	0.985	0.646
2/8	0.924	0.650	0.944	0.637
3/8	0.831	0.627	0.875	0.621
4/8	0.707	0.594	0.775	0.596
5/8	0.554	0.548	0.640	0.560
6/8	0.383	0.484	0.470	0.505
7/8	0.194	0.386	0.265	0.417
39/40	0.038	0.224	0.060	0.254
79/80	0.019	0.178	0.030	0.202
1	0	0	0	0

In the method explained above, the drawing becomes difficult near the top of the column because the elastic load w is infinite when $\frac{x}{l}=1$, and therefore the moments of inertia I near the top — in the interval $\frac{39}{40} \leqq \frac{x}{l} \leqq 1$ in this exsample — were replaced by the mean value of I.

Then, measuring the pole distance in the ray polygon with respect to the second deflection curve, we have

$$2.14 \frac{Pl}{EI_0}\,\delta = 2.14\left(\frac{\delta}{l}\right)\left(\frac{Pl^2}{EI_0}\right) = 0.535\left(\frac{Pl^2}{EI_0}\right) .$$

Therefore we can find the value of the critical load by the equation

$$0.535\,\frac{Pl^2}{EI_0}\,\delta = \delta .$$

From this

$$P_{cr.} = 1.871\,\frac{EI_0}{l^2} ,$$

But

$$I_0 = \frac{\pi r_0^4}{4} = 0.044\,\frac{\pi V^2}{l^2} ,$$

and we have

$$P_{cr.} = 0.083\,\frac{\pi E V^2}{l^4} . \qquad (35)$$

Taking the critical load of the cylindrical column for a comparison,

$$P'_{cr.} = \frac{\pi^2 EI}{4l^2} = 0.0625\,\frac{\pi E V'}{l^4} . \qquad (36)$$

We will compare the volumes of the columns of the both cases for the same critical load.

Putting $P_{cr.} = P'_{cr.}$ or $0.083\,\frac{\pi E V^2}{l^4} = 0.0625\,\frac{\pi E V'^2}{l^4}$,

we get

$$\frac{V}{V'} = 0.867 .$$

The above obtained figure actually coincides with the figure : $\frac{\sqrt{3}}{2} = 0.866$ found by Clausen[5].

On the other hand, when P is given, r is required by proceeding such the same procedure as explained before since V or $\sqrt{\frac{V}{l}}$ is determined with Eq. (35).

Next, the fact that the columns obtained above are those of uniform strength can be explained as follows.

A bending moment due to the deflection curve $y=\delta\cdot f$ is

$$M = P\cdot\delta\cdot f .$$

5) I. Todhunter, K. Pearson, "A History of the Theory of Elasticity and of the Strength of Materials." Vol. 2, Part. 1, p. 327.

S. Timoshenko "Theory of Elastic Stability." p. 132.

A. Ono "On the Stability of Long Struts of Variable Cross Section." Memoirs of the College of Eng., Kyushu Imp. University. Vol. 1, p. 404.

From Eq. (32)

$$I = \frac{\pi r^4}{4} = \frac{\pi}{4} K^2 \cdot f^{\frac{4}{3}} ,$$

and then the maximum fibre stress at any position of the beam is

$$\sigma_{\max} = \frac{M}{I} r = \frac{4P\cdot\delta\cdot f}{\pi K^2 \cdot f^{\frac{4}{3}}} K^{\frac{1}{2}} \cdot f^{\frac{1}{3}} = \frac{4P\cdot\delta}{\pi\cdot K^{\frac{3}{2}}} .$$

Therefore, since $\sigma_{\max}$ is free from f or $\frac{x}{l}$, we find that the above obtained column is of uniform strength.

The results of the actual calculation are shown in table 3 in which we see that the error is about 3% in the portion near the top of the column because we used a mean moment of inertia.

Table 3.

x/l	$\sigma_{\max}$
0	3.66 $\left[\delta/\left(\frac{V}{l}\right)^{\frac{3}{2}}\right]\left[\frac{4P}{\pi}\right]$
1/8	3.65
2/8	3.66
3/8	3.66
4/8	3.66
5/8	3.64
6/8	3.64
7/8	3.63
39/40	3.75
79/80	3.75
1	—

Finally, putting $V = V'$, we have from Eq. (35) and Eq. (36)

$$P_{cr.} = 1.328 \, P'_{cr.} \, .$$

Dr. K. Aichi found that at the point $\frac{x}{l} = 0$ the width of the ordinary column is three fourth times of that of the colum of uniform strength with the same maximum fibre stress as the former, and the proportion of the critical loads of these two cases is[6)]

$$\frac{P_{cr.}}{P'_{cr.}} = 2.37 \, .$$

Now, in table 2, we see that r is equal to $0.649\sqrt{\frac{V}{l}}$ at $\frac{x}{l} = 0$, and therefore the volume of the ordinary column with the radius of $\frac{3}{4}$ times of that is

$$V' = \pi r^2 l = \pi \left(\frac{3}{4} \times 0.649\right)^2 \frac{V}{l} \cdot l ,$$

or

$$\frac{V}{V'} = 1.343 \, .$$

From the above relation with Eq. (35) and Eq. (36)

$$\frac{P_{cr.}}{P'_{cr.}} = 1.328 \left(\frac{V}{V'}\right)^2 = 2.39$$

Therefore we find that the both figures actually coincide with each other.

It is obvious that the method explained above can also be applied to beams of square, rectangle and orther cross section, and it will be seen that the errors are negligibly small in practice.

In the case of the column hinged at both ends, it is evident from symmetry that each half of the column is in the same condition as the foregoing case.

Conclusion.

In this paper the auther have generally discussed the old problems concerning the most effective form of the beam and the column by the application of a simple theory of "Calculus of Variations" and made it clear that the beams of the least strain energy, of the least deflection or of the least volume are quite the same ones which have the uniform strength throughout their lengths.

6) K. Aichi, Jour. of the Nippon Mech. Eng. Society. Vol. 22, p. 54.