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Title	Application of the Maximum Principle to a Continuous Path Determination Problem
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Citation	Memoirs of the Faculty of Engineering, Hokkaido University, 12(1), 1-31
Issue Date	1967-02
Doc URL	https://hdl.handle.net/2115/37852
Type	departmental bulletin paper
File Information	12(1)_1-32.pdf



Application of the Maximum Principle to a Continuous Path Determination Problem

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1. Introduction

Recently some technical methods based on the maximum principle have been developed. Most of them, however, seem to require a considerably extensive computer system to determine the initial conditions of the auxiliary vectors in comparison with the conventional PID control method, and seem to lack in investigations on the relationship between the cost required for such a computer system and the benefit derived from this type of control system.

To clarify these problematic points, it is necessary to investigate the problems: i. e. the operational time of the computer system, identification of the plant, non-linearity of control elements, disturbances to the control process, correction of errors resulting from component apparatus of the system, etc.

The authors have developed a unique method by which the optimal control technique in the linear system by Pearson and Chaudhuri et al.^{3,4)} is applied to the general solution of the system derived by the Pontryagin's existence theorem and thereby the time (T) required for the condition of the plant to be changed optimally to the target value is obtained directly. That is, a method has been developed by which T is increased or decreased in accordance with the determination of error function[†] as to its sign which varies with varying control process and thereby the optimal value is obtained, in an attempt to simplify the computer system involved. As a result, the optimal condition can be computed in a comparatively simple manner on a digital computer. Furthermore, it was found that an optimal control is possible, in principle, also with the analog technique without necessarily resorting to the digital technique and with no loss of generality.

Then, as an example, the application of this control system to the submarine depth control problem was shown, together with investigations made to verify the validity. The following is a detailed report of this subject.

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† $\varepsilon(T)$ in the report.

2. System description and analysis

The maximum principle was established basically by Pontryagin in 1956. Since then many theories concerning this principle have been developed. In practice, however, most of them require an extensive computer system to determine the initial values of auxiliary variable vectors which determine optimal values of the system and can not replace the PID control system in the general process control.

The purpose of this paper is to present a method to obtain by the zero method, initial conditions of auxiliary variable vectors which are encountered in the application of the maximum principle to the general process control, to eliminate the complexity of the computer system and to derive a procedure for controlling with sequential correction of any deviation of the path which may occur during control operation.

The characteristic of the plant is assumed to be described by an n -th order linear process variable vector as shown in Eq. (1).

$$\left. \begin{aligned} \dot{X} &= AX + BU = f(X, U) \\ \text{or} \quad \dot{x}_i &= \sum_{j=1}^n a_{ij}x_j + \sum_{j=1}^n b_{ij}u_j = f_i(X, U), \quad i=1, 2, \dots, n \end{aligned} \right\} \quad (1)$$

Now consider the case where the process changes between $t=0$ and $t=T$. Since the generality of the theory is not lost by taking $X(T)$ at $t=T$ at the origin of the coordinates, let $X(T)=0$, and the cost function is generally given as

$$J = \int_0^T F(X, U) dt \quad (2)$$

Then, control U that makes J minimum can be obtained by controlling H given by the following equation so that it is always maximum by the maximum principle.

$$H = \sum_{i=0}^n \phi_i f_i(X, U) - F(X, U) \quad (3)$$

$$\text{where} \quad \dot{\phi}_i = -\frac{\partial H}{\partial x_i}, \quad i=1, 2, 3, \dots, n$$

Eq. (3) is rewritten in vector form as

$$H = \Psi^{TR} f(X, U) - F(X, U) \quad (4)$$

The cost function J is given as

$$J = \int_0^T (X^{TR}CX + \lambda \|U\|^2) dt \quad (5)$$

where X^{TR} : transposed matrix of X
 C : constant matrix
 λ : constant
 $\|U\|^2 = U_1^2 + U_2^2 + \dots + U_m^2$

Substituting Eq. (1) and Eq. (5) into Eq. (4) gives

$$H = \Psi^{TR}(AX + BU) - (X^{TR}CX + \lambda \|U\|^2) \quad (6)$$

The necessary condition for making H maximum with respect to U is obtained by differentiating Eq. (6) by U and finding such U that satisfies $\partial H / \partial U = 0$. Therefore, the optimal value of U , U^0 , should satisfy

$$U^0 = \frac{1}{2\lambda} B^{TR}\Psi \quad (7)$$

On the other hand, from Eq. (4)

$$\dot{\Psi} = -A^{TR}\Psi + 2CX \quad (8)$$

Denote initial conditions of X and Ψ by $X(0)$ and $\Psi(0)$ and let $\Phi(t)$ and $R(t)$ be the fundamental matrixes of $\dot{X} = AX$ and $\dot{\Psi} = -A^{TR}\Psi$, respectively, then the following relations hold true.

$$\dot{\Phi}(t) = A\Phi(t), \quad \Phi(0) = [1], \quad X(t) = \Phi(t)X(0) \quad (9)$$

$$\dot{R}(t) = -A^{TR}R(t), \quad R(0) = [1], \quad \Psi(t) = R(t)\Psi(0) \quad (10)$$

$$\Phi^{-1}(t) = R^{TR}(t) \quad (11)$$

Denoting the solution of Eq. (1) by the fundamental matrixes (1) gives

$$X(t) = \Phi(t) \left(X(0) + \int_0^t \Phi^{-1}(t) BU^0(t) dt \right) \quad (12)$$

Under the optimal control, $X(T) = 0$ at $t = T$, hence the following equation

$$-\Phi(T)X(0) = \Phi(T) \int_0^T \Phi^{-1}(t) BU^0(t) dt \quad (13)$$

The scalar product of Eq. (13) and $X(0)\Psi^{-1}(T)$ is given as

$$-X(0) \cdot X(0) = \int_0^T X(0) \cdot \Phi^{-1}(t) BU^0(t) dt \quad (14)$$

Substituting Eq. (7) and Eq. (11) into this gives

$$-X(0) \cdot X(0) = \frac{1}{2\lambda} \int_0^T X(0) \cdot R^{TR}(t) BB^{TR}\Psi(t) dt \quad (15)$$

Substitute Eq. (7) into Eq. (1) and solve it simultaneously with Eq. (8), then we have

$$\begin{bmatrix} \dot{X} \\ \Psi \end{bmatrix} = \begin{bmatrix} A & \frac{1}{2\lambda} BB^{TR} \\ 2C & -A^{TR} \end{bmatrix} \begin{bmatrix} X \\ \Psi \end{bmatrix} \quad (16)$$

Denote the fundamental matrix of this equation by $V(t)$, and Eq. (17) holds true

$$\begin{bmatrix} X(t) \\ \Psi(t) \end{bmatrix} = V(t) \begin{bmatrix} X(0) \\ \Psi(0) \end{bmatrix} \quad (17)$$

where $V(t)$ is represented by a matrix of $2n \times 2n$. Denote each element of $V(t)$ by v_{ij} and put $V(t)$ as

$$\left. \begin{aligned} v_{ij}^{xx} &= v_{ij}, & v_{ij}^{x\phi} &= v_{ij+n} \\ v_{ij}^{\phi x} &= v_{i+nj}, & v_{ij}^{\phi\phi} &= v_{i+nj+n}, \end{aligned} \quad i, j = 1, 2, 3, \dots, n \right\} \quad (18)$$

and

$$V(t) = \begin{bmatrix} V_{xx}(t) & V_{x\phi}(t) \\ V_{\phi x}(t) & V_{\phi\phi}(t) \end{bmatrix} \quad (19)$$

then

$$X(t) = V_{xx}(t) X(0) + V_{x\phi}(t) \Psi(0) \quad (20)$$

$$\Psi(t) = V_{\phi x}(t) X(0) + V_{\phi\phi}(t) \Psi(0) \quad (21)$$

Putting $t=T$ in Eq. (20)

$$\begin{aligned} V_{x\phi}(T) \Psi(0) &= -V_{xx}(T) X(0) \\ \Psi(0) &= -V_{x\phi}^{-1}(T) V_{xx}(T) X(0) \end{aligned} \quad (22)$$

as $X(T)=0$, which shows the relation between $\Psi(0)$ and $X(0)$.

Substituting Eq. (21) into Eq. (15) gives Eq. (23).

$$-X(0) \cdot X(0) = \frac{1}{2\lambda} \int_0^T X(0) \cdot R^{TR}(t) BB^{TR} (V_{\phi\phi}(t) X(0) + V_{\phi\phi}(t) \Psi(0)) dt \quad (23)$$

Now put $R(t)=[r_{ij}]$ and Eq. (24) holds true.

$$\begin{aligned} -2\lambda \sum_{j=1}^n (x_j(0))^2 &= \int_0^T \sum_{i=1}^n x_i(0) \left(\sum_{j=1}^n x_j(0) \sum_{m=1}^n v_{mj}^{\phi x} \sum_{l=1}^n r_{li} \sum_{k=1}^n b_{lk} \cdot b_{mk} \right. \\ &\quad \left. + \sum_{j=1}^n \phi_j(0) \sum_{m=1}^n v_{mj}^{\phi\phi} \sum_{l=1}^n r_{li} \sum_{k=1}^n b_{lk} \cdot b_{mk} \right) dt \end{aligned} \quad (24)$$

Therefore, putting

$$w_{ij}^{\phi x}(T) = \int_0^T \sum_{m=1}^n v_{mj}^{\phi x} \sum_{l=1}^n r_{li} \sum_{k=1}^n b_{lk} b_{mk} dt \quad (25)$$

$$w_{ij}^{\phi \phi}(T) = \int_0^T \sum_{m=1}^n v_{mj}^{\phi \phi} \sum_{l=1}^n r_{li} \sum_{k=1}^n b_{lk} b_{mk} dt \quad (26)$$

$$[w_{ij}(T)] = V_{x\phi}^{-1}(T) V_{xx}(T) \quad (27)$$

gives Eq. (28) and Eq. (29).

$$-2\lambda \sum_{j=1}^n (x_j(0))^2 = \sum_{i=1}^n x_i(0) \left(\sum_{j=1}^n x_j(0) w_{ij}^{\phi x}(T) + \sum_{j=1}^n \phi_j(0) w_{ij}^{\phi \phi}(T) \right) \quad (28)$$

$$\phi_j(0) = - \sum_{p=1}^n x_p(0) w_{jp}(T) \quad (29)$$

Since Eqs. (25) through (27) are determined, as the characteristic of the plant is determined, substituting Eqs. (25) through (27) for the characteristic of the plant is permitted. Thus, the initial conditions based on the maximum principle can be obtained easily by obtaining T or $\Psi_j(0)$ at the time when the solution $\Psi_j(0)$ of Eq. (29) satisfies Eq. (28) as T is increased.

3. Application to the actual control system

For the purpose of this article, a control system may be divided into blocks as shown in Fig. 1. The problem considered here is that the coefficient of Eq. (1) should be obtained from the signal given by the input of the plant within the block marked Identification, and the system be controlled to eliminate the difference between the output $X(t)$ and the set value $X_I(0)$ when the process of computing $w_{ij}^{\phi x}(T)$, $w_{ij}^{\phi \phi}(T)$, $w_{ij}(T)$ (where $i, j = 1, 2, 3, \dots, n$,

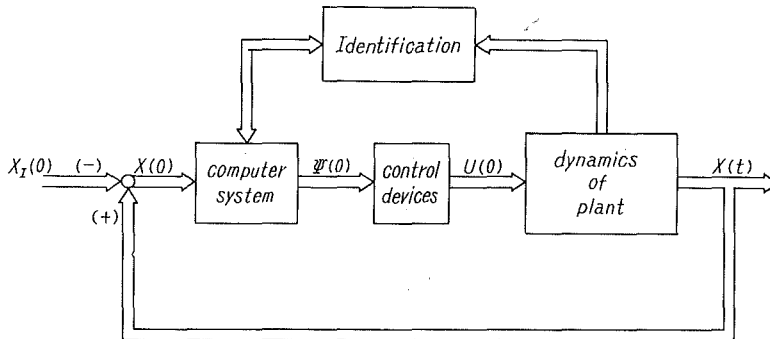


Fig. 1. Block diagram of general control system

Note: $X(0) = X(t) - X_I(0)$
 $X(t)$: out put vector
 $X_I(t)$: in put vector

$T=0, \Delta t, 2\Delta t, \dots, p\Delta t, \dots, n\Delta t$) is in effect. Since the generality of error $X(0)$ which occurs when the set value $X_r(0)$ is given is not lost if the end value of $X(t)$, $X(T)$, is taken at the origin of the coordinates, it follows that

$$X(0) = -(X_r(0) - X(t)) \quad (30)$$

If $X(0)$ is given as the input to the computer system, $\Psi_j(0)$ is obtained by simultaneous solution of Eqs. (28) and (29), and by calculating it as the initial condition for Eq. (16), the system is controlled to satisfy Eq. (7), then it is the optimal control. However, the actual control is affected by indention of the plant, non-linearity of control elements, disturbances, etc. and it is very difficult in many cases to carry out the control as computed. At this point, the problem of path determination arises which carries out the correction of initial condition $X(0)$ sequentially. The block diagram of Fig. 1 illustrates this method. The computer speed is increased to determine control $U(0)$ by the initial condition $\Psi(0)$ at the present time, and then similarly, by a new initial condition at the next time. This will be described in the following.

3.1. Determination of optimal time by error function

If the optimal time T^0 is given in Eq. (29), it is immediately possible to obtain $\Psi(0)$ from $X(0)$. In general, however, the value of T^0 which must satisfy Eq. (28) can not be obtained easily. Now, put arbitrary T in Eq. (23) and denote the resulting difference between the right and left side (hereinafter referred to as error function) by $\varepsilon(T)$. Let $\varepsilon(T)$ be

$$\varepsilon(T) = X(0)X(0) + \frac{1}{2\lambda} \int_0^T X(0) \cdot R^{TR}(t) BB^{TR} (V_{\phi x}(t)X(0) + V_{\phi\psi}(t)\Psi(0)) dt \quad (31)$$

then, $\varepsilon(T)$ is zero $T=T^0$ and $\varepsilon(T)$ must not be equal to zero for any T meeting the relation

$$0 < T < T^0 \quad (32)$$

Because, if $T=T_1$ which satisfies Eq. (32) exists and $\varepsilon(T_1)=0$, it means that a control with a control interval shorter than T^0 is possible and this fact is inconsistent with the optimal time T^0 . If, therefore, T is taken for the abscissa and $\varepsilon(T)$ for the ordinate as shown in Fig. 2, $\varepsilon(T)$ intersects the abscissa only at $T=T^0$ for any T which satisfies Eq. (32). Therefore, the sign of $\varepsilon(T)$ is inverted between $T=T^0+\Delta t$, that is, when T is changed slightly in the positive direction from optimal time $T=T^0$ and $T=T^0-\Delta t$, that is, when the change is in the negative direction. The sign of $\varepsilon(T)$ is not inverted for $0 < T < T^0$ as mentioned above. Thus, the following equation holds true.

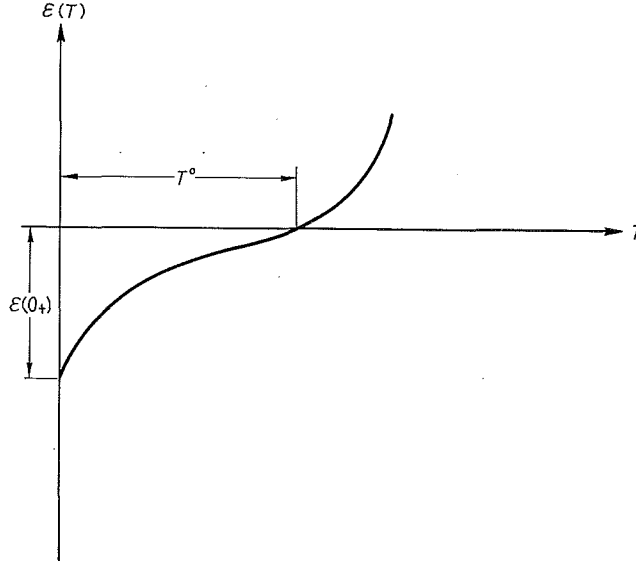


Fig. 2. Characteristics of error index $\varepsilon(T)$

$$\text{sign} \{ \varepsilon(T) \} = -\text{sign} \{ \varepsilon(T^0 + \Delta t) \} \quad (33)$$

for $0 < T < T^0$.

Since Eq. (33) must hold true even if T approaches zero infinitely, it can be put that

$$\text{sign} \left\{ \lim_{T \rightarrow 0+} \varepsilon(T) \right\} = -\text{sign} \{ \varepsilon(T^0 + \Delta t) \} \quad (34)$$

On the other hand, the value of $\lim_{T \rightarrow 0+} \varepsilon(T)$ when T of $\varepsilon(T)$ approaches zero infinitely is obtained as follows. First, the procedure for obtaining of T^0 from Eq. (31) will be shown. Give $T = T_i$ and find $\Psi(0)$ from Eq. (22).

$$\Psi(0) = -V_{x\psi}^{-1}(T_i) V_{xx}(T_i) X(0) \quad (35)$$

Substitute this into Eq. (31).

$$\varepsilon(T_i) = X(0) \cdot X(0) + \frac{1}{2\lambda} \int_0^{T_i} X(0) \cdot R^{TR}(t) B B^{TR} (V_{\phi x}(t) X(0) + V_{\phi\psi}(t) \Psi(0)) dt \quad (36)$$

Let T be increased by the relation $T_i = T_i + \Delta t$ until $\varepsilon(T_i) = 0$, then the following relation holds true for $\varepsilon(\Delta t)$.

$$\varepsilon(\Delta t) \doteq \varepsilon(0) + \Delta t \varepsilon'(0)$$

increase or decrease T in accordance with the determination of $\varepsilon(T)$ as to

$$\begin{aligned}
&\doteq X(0) \cdot X(0) + \lim_{\delta \rightarrow 0} \frac{1}{2\lambda} \delta X(0) \cdot R^{RR}(\delta) BB^{RR} \left(V_{\phi x}(\delta) X(0) \right. \\
&\quad \left. - V_{\phi\phi}(\delta) V_{x\phi}^{-1}(\delta) V_{xx}(\delta) X(0) \right) + \frac{1}{2\lambda} \Delta t X(0) \cdot R^{RR}(0) \\
&\quad \times BB^{RR} \left(V_{\phi x}(0) X(0) - V_{\phi\phi}(0) V_{x\phi}^{-1}(\Delta t) V_{xx}(\Delta t) X(0) \right) \quad (37)
\end{aligned}$$

In this equation, from the definition of fundamental matrix,

$$R^{RR}(0) = [1], \quad V_{\phi x}(0) = [0], \quad V_{\phi\phi}(0) = [1] \quad (38)$$

Since the value of $V_{xx}(\Delta t)$ is the value at $t = \Delta t$ given as the solution of Eq. (16),

$$V_{xx}(\Delta t) \doteq [1] + A\Delta t \quad (39)$$

The same applies to $V_{xx}(\delta)$.

Since $V_{x\phi}^{-1}(\Delta t)$ is the inverse matrix of $V_{x\phi}(\Delta t)$ and $V_{x\phi}(\Delta t)$ is the solution of Eq. (16),

$$V_{x\phi}(\Delta t) \doteq \Delta t \frac{1}{2\lambda} BB^{RR}$$

Therefore, it can be put that

$$V_{x\phi}^{-1}(\Delta t) \doteq \frac{2\lambda}{\Delta t} [BB^{RR}]^{-1} \quad (40)$$

Substituting Eqs. (38), (39) and (40) into Eq. (37), we have

$$\varepsilon(\Delta t) = -X(0) \cdot X(0) - \Delta t X(0) \cdot AX(0) \quad (41)$$

If Δt is selected sufficiently small in Eq. (41),

$$\varepsilon(\Delta t) = -X(0) \cdot X(0) \leq 0 \quad (42)$$

The equality holds true only when $X(0) \cdot X(0) = 0$. However, $X(t)$ is initially zero, and therefore, there is no need of control. Then,

$$-\text{sign} \left\{ \varepsilon(T^0 + \Delta t) \right\} = \text{sign} \left\{ \lim_{T \rightarrow 0+} \varepsilon(T) \right\} = \text{sign} \left\{ \lim_{\Delta t \rightarrow 0} \varepsilon(\Delta t) \right\} < 0 \quad (43)$$

From the result so far obtained, a method is conceived to seek the optimal time T^0 of T , by which T is increased or decreased toward T^0 based on a judgement

$$\left. \begin{aligned}
\varepsilon(T) < 0 : \quad T &= T + \Delta t \\
\varepsilon(T) > 0 : \quad T &= T - \Delta t \\
\varepsilon(T) = 0 : \quad T &= T^0
\end{aligned} \right\} \quad (44)$$

or an important relation is derived by which a servo mechanism which can

positive or negative sign, if constructed, will enable a continuous obtaining of T^0 . Throughout the description given above, $\varepsilon(T)$ was assumed to intersect abscissa T at $T=T^0$ as shown in Fig. 2. A special case is conceivable where $\varepsilon(T)$ is tangential to abscissa T at $T=T^0$. Under this condition,

$$\left. \begin{array}{l} \varepsilon(T^0 + \Delta t) < 0 \quad \text{and} \quad \varepsilon'(T^0 + \Delta t) < 0 \\ \text{or} \\ \varepsilon(T^0) = 0 \quad \text{and} \quad \varepsilon'(T^0) = 0 \end{array} \right\} \quad (45)$$

That is, since the condition of $\varepsilon'(T)$ that satisfies Eq. (44) is reciprocal to Eq. (45), the following relation must hold true when T' is taken in the vicinity of T^0 .

$$\varepsilon'(T') > 0 \quad (46)$$

That is, the conditions of Eq. (47) must be satisfied.

$$X(0) R^{TR}(T') B B^{TR} (V_{\phi x}(T') + V_{\phi \phi}(T') V_{x\phi}^{-1}(T') V_{x\alpha}(T')) X(0) > 0 \quad (47)$$

The authors checked Eq. (47) for mere confirmation in making the judgement on Eq. (44), but have no experience with a case reciprocal to Eq. (47), that is, the problem of tangential (T), in controlling many initial conditions of the submarine mentioned later. This special case is of considerable interest to us. In practice, however, it presents no significant trouble in the control because scanning is made from $T=0$ and such $T=T^0$ that makes $\varepsilon(T)=0$ is obtained.

3.2. Computing procedure

A linear plant has the characteristic given in the form of Eq. (1) in general. When non-linear elements are involved, the plant is controllable by the present method if Eq. (1) is given approximately as an equation of small perturbation. When Eq. (1) is given, the values of A and B are determined. Thus, by solving,

$$\frac{d}{dt} \begin{bmatrix} r_{11}(t) & r_{12}(t) & \cdots & r_{1n}(t) \\ & r_{22}(t) & & \\ & & \ddots & \\ r_{n1}(t) & & & r_{nn}(t) \end{bmatrix} = - \begin{bmatrix} a_{11} & & & a_{1n} \\ & \ddots & & \\ & & a_{ij} & \\ & & & \ddots \\ a_{n1} & & & a_{nn} \end{bmatrix} \begin{bmatrix} r_{11}(t) & r_{12}(t) & \cdots & r_{1n}(t) \\ & r_{22}(t) & & \\ & & \ddots & \\ r_{n1}(t) & & & r_{nn}(t) \end{bmatrix} \quad (48)$$

for the initial conditions

$$\begin{bmatrix} r_{11}(0) & & & r_{1n}(0) \\ & \ddots & & \\ & & r_{ij}(0) & \\ & & & \ddots \\ r_{n1}(0) & & & r_{nn}(0) \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

we can obtain $r_{ij}(t)$, ($t=0, \Delta t, 2\Delta t, n\Delta t$).

Next, rewrite Eq. (16) in the form of Eq. (49) and obtain v_{ij} by solving Eq. (49)

$$\begin{aligned} & \frac{d}{dt} \begin{bmatrix} v_{11}(t) & v_{12}(t) & \cdots & v_{1n}(t) \\ & v_{ij}(t) & & \\ & & & v_{nn}(t) \\ v_{n1}(t) & & & \end{bmatrix} \\ &= \begin{bmatrix} \begin{array}{c} a_{11}a_{12} \cdots a_{1n} \\ \vdots \\ a_{ni} \cdots a_{nn} \end{array} & \begin{array}{c} d_{11}d_{12} \cdots d_{1n} \\ \vdots \\ d_{ni} \cdots d_{nn} \end{array} \\ \hline \begin{array}{c} 2C_{11}2C_{12} \cdots 2C_{1n} \\ \vdots \\ 2C_{ni} \cdots 2C_{nn} \end{array} & \begin{array}{c} -a_{11} \cdots -a_{ni} \\ \vdots \\ -a_{12} \cdots -a_{nn} \end{array} \end{bmatrix} \begin{bmatrix} v_{11}(t) & v_{12}(t) & \cdots & v_{1n}(t) \\ & v_{ij}(t) & & \\ & & & v_{nn}(t) \\ v_{n1}(t) & & & \end{bmatrix} \end{aligned} \quad (49)$$

for the initial conditions

$$[v_{ij}(0)] = [1]$$

$$[d_{ij}] = \frac{1}{2\lambda} [b_{ij}] [b_{ji}]$$

Classify $v_{ij}(t)$ ($t=0, \Delta t, 2\Delta t, 3\Delta t, \dots, n\Delta t$) obtained from Eq. (49) as shown by Eq. (18) and obtain $v_{ij}^{xx}(t)$, $v_{ij}^{x\phi}(t)$, $v_{ij}^{\phi x}(t)$ and $v_{ij}^{\phi\phi}(t)$. Using the result, calculate $w_{ij}^{\phi x}(T)$, $w_{ij}^{\phi\phi}(T)$ and $w_{ij}(T)$ from Eqs. (25), (26) and (27). The authors have

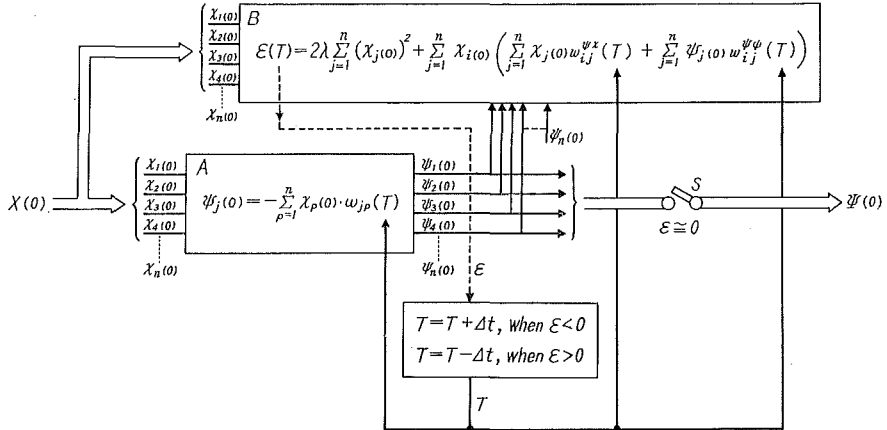


Fig. 3. Computer system for optimal control

applied the Runge-Kutta-Gill method and made the calculation of Eqs. (25) through (27) by the trapezoidal rule, that is, $\int_a^b f(x) dx = \frac{1}{2}(f_0 + f_1)$. The block Identification shown in Fig. 1 represents the process up to the obtaining of w . In Fig. 1, if a difference $X(0)$ between the set value $X_r(0)$ and output $X(t)$ occurs, it is sent to the computer where the optimal control $U(0)$ corresponding to $X(0)$ is calculated. This block is shown in detail in Fig. 3. Referring to the figure, when $X(0)$ is given, a certain value of T is set and in block A , $\phi_i(0)$ is calculated by Eq. (29) and the result is used in block B to calculate $\varepsilon(T)$. The calculation is repeated by the relation $T = T + \Delta t$ or $T = T - \Delta t$ depending on whether $\varepsilon(T) < 0$ or $\varepsilon(T) > 0$ until $\varepsilon(T) \approx 0$, that is, $T = T^0$ is obtained. Now, close the switching circuit S and send $\Psi(0)$ to the next stage, then $\Psi(0)$ becomes the initial value of the initial variable vector for $X(0) \cdot \Psi(0)$ given in Fig. 3 is once held in Fig. 4.

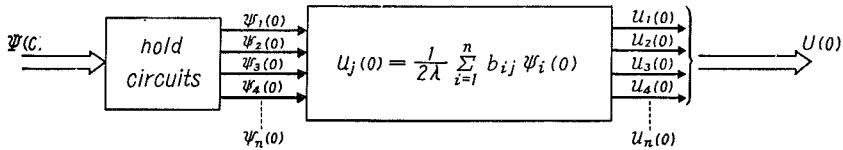


Fig. 4. Network of control devices

If control u_j is given by

$$u_j(0) = \frac{1}{2\lambda} \sum_{i=1}^n b_{ij} \psi_i(0) \quad (50)$$

the optimal value of u_j is obtained. The action of u_j is fed back to $X(0)$ through the characteristic of the plant and is given as input in Fig. 3, thus completing a closed control loop. This means that a chance is given for repeated correction of errors of the entire system including the computer errors. On the other hand, a certain optimal value T_1^0 is given by the judgement on Eq. (44) and thereby an optimal control $U_{ij}(T_1^0)$ is given. Thereafter, give $T_2 = T_1^0 - \Delta t$ as the estimated value of the initial value of T_2 for the second time onward, and the operation time becomes shortest. Generally, it is possible to limit the number of repetitions of calculation of Eqs. (28) and (29) to zero or 1 so that practically no repeated calculation is needed, by using

$$T_p = T_{p-1}^0 - \Delta t, \quad (p=1, 2, \dots, n) \quad (51)$$

for the initial value of T_p .

4. Application to the 4th order linear model for submarine pitching motion control

It is already well known that the pitching motion of the submarine can be represented by the 4th order linear equations⁹⁾. To be noted is that the equations of motions of the submarine have coefficients which are a function of ship speed and so vary frequently. When a digital computer is introduced in the control system, Eqs. (25) through (27) can be calculated easily and no problem arises in the calculation of these equations necessitated by any change of data given by the Identification. In the case of analog control, however, it is difficult to change the values of Eqs. (25) through (27) as described later and this may make the control impossible when the characteristic of the plant changes largely. In the case of the submarine, it may be assumed fixed as a result of normalization of the motion⁹⁾, and generally the following expression may be used.

$$\left. \begin{aligned} A_1\ddot{h} + A_2\dot{h} + A_3\ddot{\theta} + A_4\dot{\theta} + A_5\theta &= A_6\beta_{ea} - A_7\beta_{ef} \\ B_1\ddot{h} - B_2\dot{h} + B_3\ddot{\theta} + B_4\dot{\theta} - B_5\theta &= B_6\beta_{ea} + B_7\beta_{ef} \end{aligned} \right\} \quad (52)$$

where θ : pitching angle in degree
 h : depth in meter
 β_{ea} : angle of stern plane in degree
 β_{ef} : angle of bow plane in degree

Now put

$$\left. \begin{aligned} x = \dot{h}, \quad x_2 = h, \quad x_3 = \dot{\theta}, \quad x_4 = \theta \\ u_1 = \beta_{ea}, \quad u_2 = \beta_{ef} \end{aligned} \right\} \quad (53)$$

then the following relations hold true.

$$\left. \begin{aligned} a_{11} &= -\frac{1}{A} \left(\frac{A_2}{A_1} + \frac{A_3B_2}{A_1B_3} \right), & a_{31} &= \frac{1}{A} \left(\frac{B_2}{B_3} + \frac{B_1A_2}{B_3A_1} \right) \\ a_{13} &= -\frac{1}{A} \left(\frac{A_4}{A_1} - \frac{A_3B_4}{A_1B_3} \right), & a_{33} &= -\frac{1}{A} \left(\frac{B_4}{B_3} - \frac{B_1A_4}{B_3A_1} \right) \\ a_{14} &= -\frac{1}{A} \left(\frac{A_5}{A_1} + \frac{A_3B_5}{A_1B_3} \right), & a_{34} &= \frac{1}{A} \left(\frac{B_5}{B_3} + \frac{B_1A_5}{B_3A_1} \right) \end{aligned} \right\} \quad (54)$$

$$\left. \begin{aligned} b_{11} &= \frac{1}{A} \left(\frac{A_6}{A_1} - \frac{A_3B_6}{A_1B_3} \right), & b_{31} &= \frac{1}{A} \left(\frac{B_6}{B_3} - \frac{B_1A_6}{B_3A_1} \right) \\ b_{12} &= -\frac{1}{A} \left(\frac{A_7}{A_1} + \frac{A_3B_7}{A_1B_3} \right), & b_{32} &= \frac{1}{A} \left(\frac{B_7}{B_3} + \frac{B_1A_7}{B_3A_1} \right) \end{aligned} \right\} \quad (55)$$

where $A = \begin{pmatrix} 1 - \frac{B_1 A_3}{A_1 B_3} \end{pmatrix}$

Take for example a 1,000 ton class submarine to show values of A and B in the following.

$$A = \begin{pmatrix} -0.03739 & 0 & -0.02431 & -0.001671 \\ 1.0 & 0 & 0 & 0 \\ 0.1361 & 0 & -0.1113 & -0.002615 \\ 0 & 0 & 0 & 1.0 \end{pmatrix} \quad (56)$$

$$B = \begin{pmatrix} 0.0003664 & -0.0003043 \\ 0 & 0 \\ 0.002267 & 0.001067 \\ 0 & 0 \end{pmatrix} \quad (57)$$

$R(t)$ shown in Eq. (10) is the fundamental matrix of the equation expressed as

$$\Psi(t) = -A^T \Psi(t), \quad (58)$$

where T : transposed matrix of A

Now let $\Psi_1(0)$, $\Psi_2(0)$, $\Psi_3(0)$ and $\Psi_4(0)$ be

$$\Psi_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Psi_4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (59)$$

then, $[r_{i1}]$ ($i=1, 2, 3, 4$) is the solution of Eq. (58) for the initial condition taken as $\Psi_1(0)$. Generally, $[r_{ij}]$ ($i=1, 2, 3, 4$) is the solution of Eq. (41) for the initial condition as $\Psi_j(0)$ ($\Psi_j(0)=1$ if $j=i$, $\Psi_j(0)=0$ if $j \neq i$). For $[v_{ij}]$, the solution can be obtained similarly from the eight initial conditions and Eq. (16). Fig. 6 shows the difference between the conventional automatic steering and the steering on the maximum principle presented when the ship is to change its depth 5 m in a horizontal travelling conditions. The control conditions are as follows.

(1) Automatic steering

In Eq. (52), let the relation between β_{ea} and β_{ef} be

$$\beta_{ef} = 2\beta_{ea} \quad (59)$$

Assume a feedback to be represented by

$$\beta_{ea} = -k_\theta \theta + k_h h - k_\delta \dot{\theta} \quad (60)$$

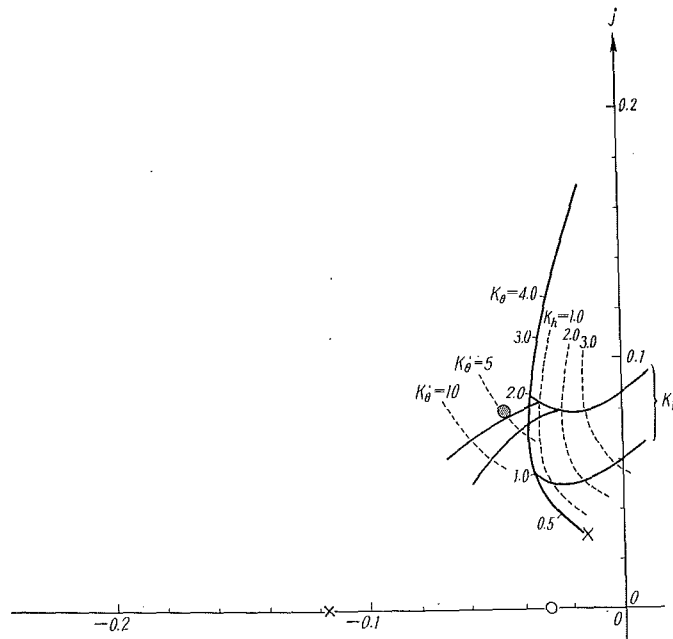


Fig. 5. Root locus for submarine control system

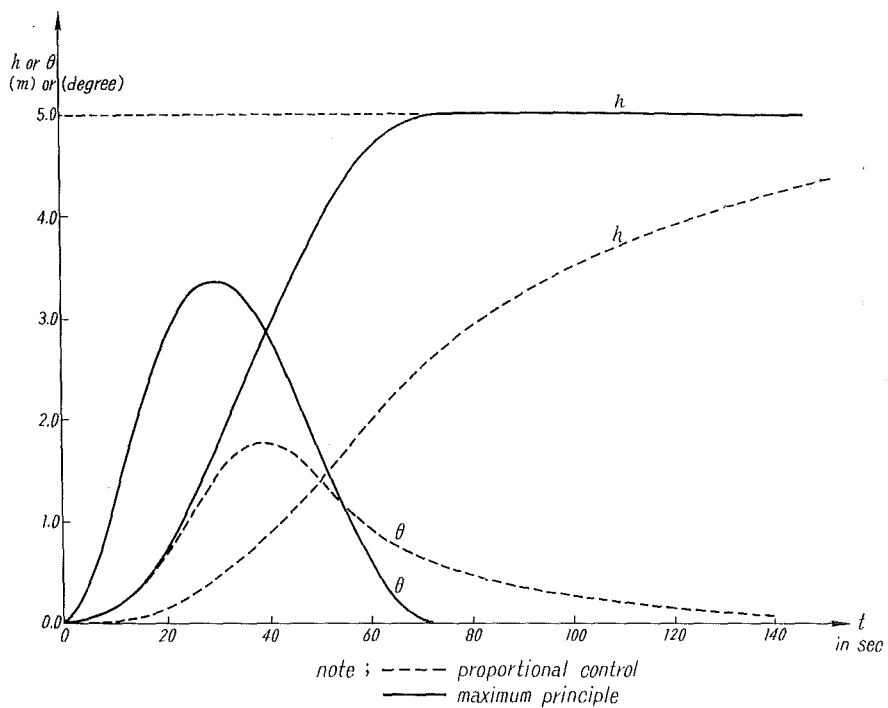


Fig. 6. Comparison of proportional control and optimal control for submarine in longitudinal motion

Then, for various values of elements k_θ , k_h and k_δ of this feedback, the root locus for the control system is as shown in Fig. 5. If the values of k_θ , k_h and k_δ are selected to correspond to the dot, that is, $k_\theta=2$, $k_h=1$ and $k_\delta=5$, then the result of control of this system is as shown by the dotted line in Fig. 6.

(2) Steering on the maximum principle

The solid line in Fig. 6 shows the result of control obtained by the calculating method shown in Fig. 3 and Fig. 4 for such data as $[C]=[1]$, $\lambda=1$ and $\Delta t=2.0$ s in Fig. 6. Since the value of $\Psi(0)$ calculated for $X(0)$ in Fig. 3 is once held before being supplied as $U(0)$ in Fig. 4, the steering shown in Fig. 7 takes a stepped form, and the path shown in Fig. 6 is an approximate solution of the optimal path.

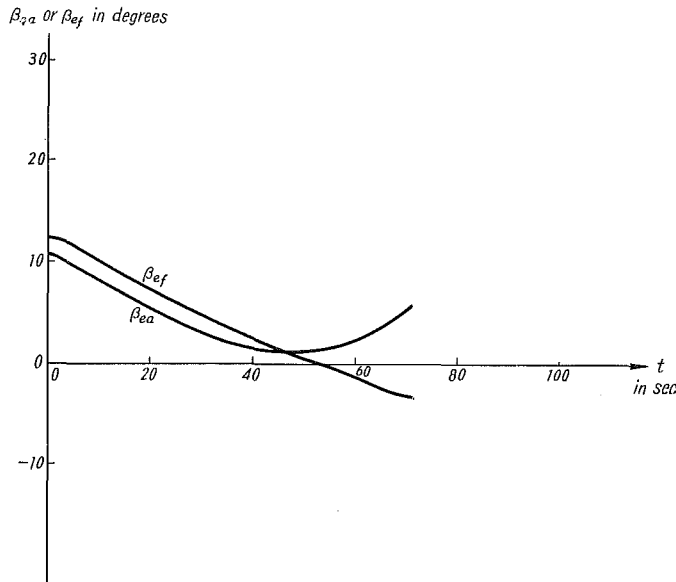


Fig. 7. Values of stern plane and bow plane

5. Extension of the scope, and control mode discussions

In the application of the maximum principle to the general process control, Eq. (5) may be construed as follows.

- (1) Normal control
$$J_1 = \int_0^T \left(\sum_{j=1}^4 x_j^2(t) + \sum_{i=1}^2 u_i^2(t) \right) dt$$
- (2) Minimum deviation control
$$J_2 = \int_0^T \sum_{j=1}^4 x_j^2(t) dt$$

- (3) Minimum u control $J_3 = \int_0^T \sum_{i=1}^2 u_i^2(t) dt$
- (4) Shortest time control $J_4 = \int_0^T dt$

The normal control and the minimum u control can be determined uniquely by the method described in 3, whereas the minimum deviation control and the shortest time control can not be determined simply because of the restriction imposed in the derivation of Eq. (7). If such a control in which λ in Eq. (5) approaches zero as near as possible, is called the minimum deviation control, a considerably precise approximation of it can be obtained by the path correction method described in Fig. 3. The minimum deviation control in the strict sense shows up as a fluttering control, while the minimum deviation control modified as defined above to have λ which is selected very small allows the restriction on u to be alleviated so that calculated value of u is

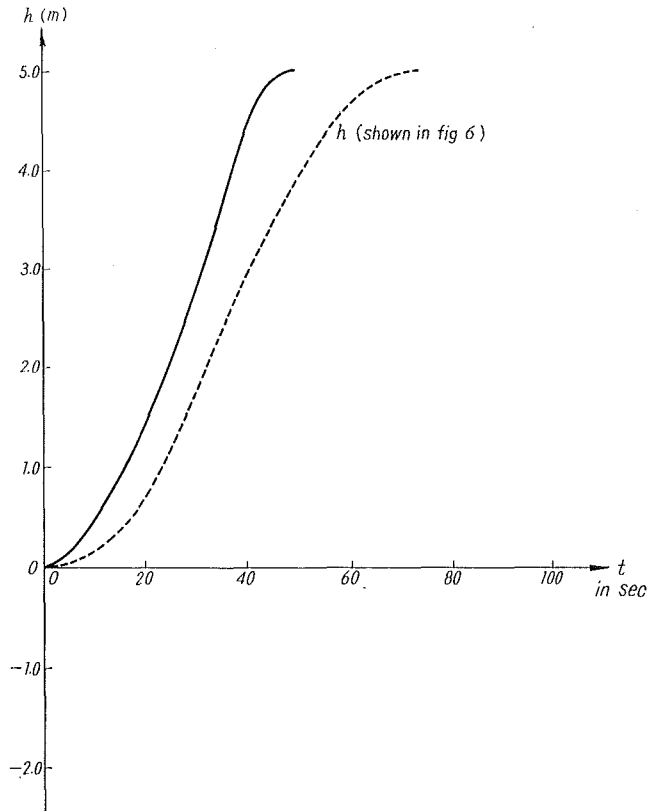


Fig. 8. Sub-optimal minimum error control

accordingly large and a saturation may be considered for it. Thus, the difference between the calculated value of u and the saturated value $u_{\max}$ can be corrected by the path correction to a considerably precise approximation. As an application of this concept to the submarine. Fig. 8 shows the result of control for the data

$$\left. \begin{aligned} C &= [1.0], \quad \lambda = 0.001 \\ u_{1\max} &= u_{2\max} = 25.0 \text{ degrees} \end{aligned} \right\} \quad (61)$$

when the submarine changes its depth 5 m.

This figure indicates that the control speed is considerably faster than the path of Fig. 6 shown by the dotted line and that each control becomes saturated. The minimum u control corresponds to $C=[0]$ in Eq. (5) and the control method equivalent to the normal steering may be used for this control. Fig. 9 shows the result of this control for the data $C=[0.0]$ and $\lambda=0.001$. In the case of the shortest time control, it is impossible to apply the path correction in this limited condition where λ approaches zero and other methods should be resorted to. Generally for the actual processes, however, the shortest time control should hardly be needed if the minimum deviation control

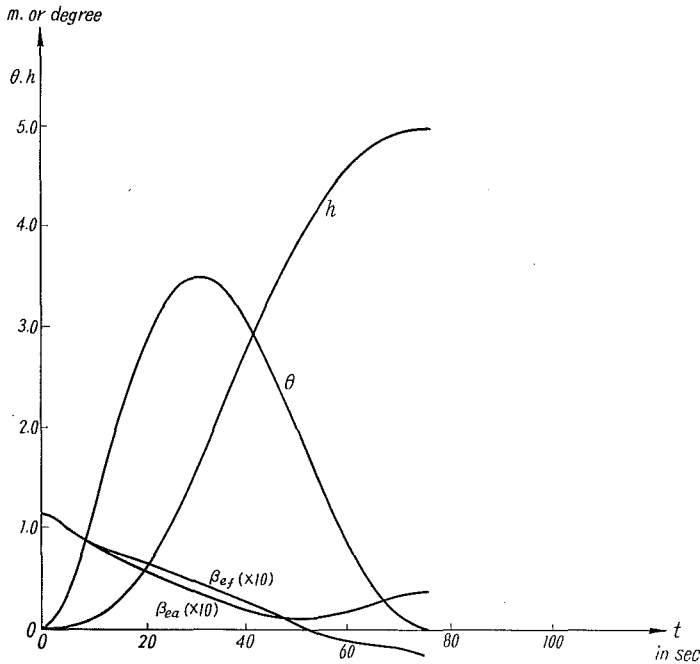


Fig. 9. U minimum control

and the minimum u control are possible. Especially in the case of a submarine, there is absolutely no need of this control, because depth hunting must be avoided for safety purpose, thus the meaning of the optimal condition becomes ambiguous in this limited case where the characteristics change severely, and the fluttering control is always accompanied by hazards: it is only necessary to change the mode of normal steering between specific ranges of λ . This description, however, does not mean that the shortest time control is meaningless, but that for the purpose of simplifying the computer system and offering a generalized approach in place of the conventional PID system, as intended by the authors, the shortest time control may be discounted from the consideration. From the discussion given above, it is seen that the computer system shown in Fig. 3 and Fig. 4 can be applied in the same form to normal control, minimum deviation control and minimum u control, and that the generality of the control exists including the cases where saturable control elements are involved and measurement errors of the plant are present. Especially, in the latter case of the measurement errors ϵ , very precise control is obtained if the characteristic of the plant is simulated precisely by the Runge-Kutta-Gill method and w is calculated by the first approximation of the Newton-Cotes' integration formula, that is, the trapezoidal rule. The last problem left in the application of the control method to general process control lies in the mode of simplification of the computer system shown in Fig. 3 and Fig. 4. At this point, the authors considered an analog computer system and have proposed a system for continuous control.

6. Control by analog computation

Taking for example the control of a submarine, it means that the control on the maximum principle can be carried out entirely by analog computation. Fig. 10 illustrates the technique for a fourth linear control system. A total of 48 potentiometers to generate function w_{ij} , $w_{ij}^{\phi x}$ and $w_{ij}^{\phi\phi}$ ($i, j = 1, 2, 3, 4$) are prepared (the number 48 will be used here for the clear explanation of the relationship between Fig. 3 and Fig. 4 although only 32 are actually needed) and mounted in such a manner as to interlock with each other on the same shaft which is driven by a motor supplied from the comparator which detects the polarity (+ or -) of $\epsilon(T)$. Since the angular displacement of the potentiometer shaft is proportional to T , potentiometers provide outputs $w_{ij}(T)$, $w_{ij}^{\phi x}(T)$ and $w_{ij}^{\phi\phi}(T)$. As can be seen in the figure, composite output of potentiometers w_{11} , w_{12} , w_{13} , w_{14} , appears in the output of amplifier F_1 and corresponds to $\phi(0)$ which satisfies Eq. (29). Similarly, composite outputs of w_{21} , w_{22} , w_{23} , w_{24} , ..., and w_{41} , w_{42} , w_{43} , w_{44} appear in the outputs of amplifier

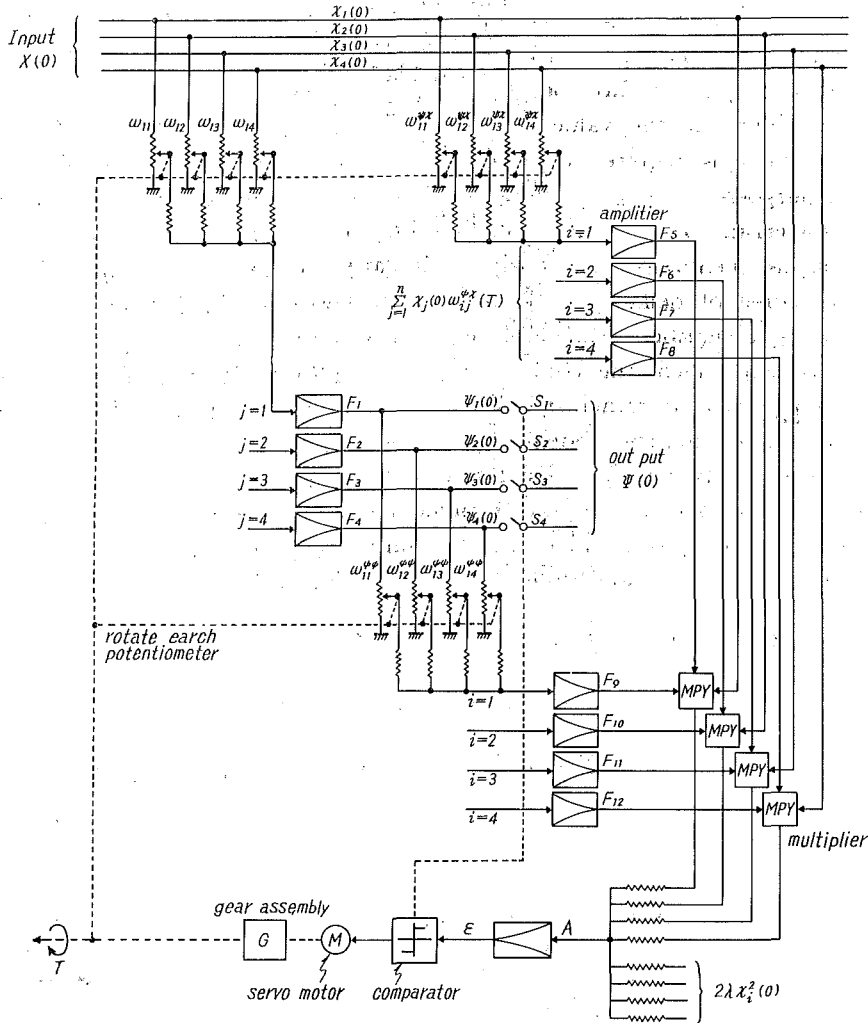


Fig 10. Analog computation technique

F_2 , F_3 and F_4 and correspond to $\phi_2(0)$, $\phi_3(0)$ and $\phi_4(0)$, respectively. On the other hand, outputs of amplifier F_5 , F_6 , F_7 and F_8 obtained from $(w_{11}^{\phi x}, w_{12}^{\phi x}, w_{13}^{\phi x}, w_{14}^{\phi x})$, $\dots$, $(w_{41}^{\phi x}, w_{42}^{\phi x}, w_{43}^{\phi x}, w_{44}^{\phi x})$ and outputs of amplifiers F_9 , F_{10} , F_{11} and F_{12} obtained from $(w_{11}^{\phi \phi}, w_{12}^{\phi \phi}, w_{13}^{\phi \phi}, w_{14}^{\phi \phi})$, $\dots$, $(w_{41}^{\phi \phi}, w_{42}^{\phi \phi}, w_{43}^{\phi \phi}, w_{44}^{\phi \phi})$ are summed appropriately in the output of multiplier MPY where sums are multiplied by input $x_1(0)$, $x_2(0)$, $\dots$, $x_4(0)$. Multiplier outputs are summed together in amplifier A to which another input $2\lambda(x_1^2(0) + x_3^2(0) + x_4^2(0))$ is also applied. Then, amplifier A provides output $\epsilon(T)$ which satisfies Eq. (28) and is sent to the determinant

mechanism where it is used to shift the potentiometer shaft so that T for $\varepsilon \cong 0$, that is, the optimal control time T^0 is obtained. When switch groups S_1, S_2, S_3 and S_4 are turned on, the calculated result $\Psi(0)$ is provided at the output to determine the value of control properly. The potentiometer used for this control is required only to have an ordinary class of precision for practical purposes: a slight difference between the potentiometer setting and the characteristic of the plant, if present, may be corrected by the path correction as mentioned previously. Similarly, a servo system having an ordinary degree of delay may be used with almost no inconvenience for the control of a submarine. Furthermore, as mentioned in the preceding section, once T is determined, the control may be continued in quite the same way as a general servo, or rather smoothly as compared against the digital system.

The analog control system, as compared with the digital system, is characterized by its very low cost for new installation: in this regard, it may be considered as one of the conventional mechanical servo systems of single function type, being convenient for the control of a submarine. To simplify the system of Fig. 10, substitute Eq. (29) into Eq. (28) and we obtain

$$\begin{aligned}
 -2\lambda \sum_{j=1}^n (x_j(0))^2 &= \sum_{i=1}^n x_i(0) \left(\sum_{j=1}^n x_j(0) w_{ij}^{\phi x}(T) \right. \\
 &\quad \left. - \sum_{j=1}^n w_{ij}^{\phi \phi}(T) \sum_{p=1}^n x_j(0) w_{jp}(T) \right) \\
 &= \sum_{i=1}^n x_i(0) \sum_{j=1}^n x_j(0) \left(w_{ij}^{\phi x}(T) - \sum_{p=1}^n w_{ip}^{\phi \phi}(T) w_{pj}(T) \right)
 \end{aligned} \tag{62}$$

Noting that

$$w_{ij}^0(T) = w_{ij}^{\phi x}(T) - \sum_{p=1}^n w_{ip}^{\phi \phi}(T) w_{pj}(T) \tag{63}$$

it is possible and yet practical to simulate by means of two sets of potentiometers of $w_{ij}(T)$ and $w_{ij}^0(T)$, $w_{ij}^0(T)$ being used for the generation of functions by potentiometers.

7. Conclusion

In summary, on-line application of a control device based on the maximum principle in place of the general PID control device is not so difficult, and especially upon the development of analog control systems, it has been found that the principle of conventional servo systems seeking continuously for the optimal value by zero-method can be applied also in this case. Whether the PID control device or the control device based on the maximum principle is

superior can not be said generally, but, most PID control systems require various tests which are repeated after the device is manufactured to determine the optimal value and moreover, it is impossible to determine at present whether the result of control is truly optimal or not. This difficulty increases in intensity in a complicated multi-variable control system. In contrast, the application of the maximum principle is advantageous in that most procedures are eliminated and that optimal control is insured even for a complicated system.

Notation

- J : Cost function
- X : Process vector, $X=[x_j]$, $j=1, 2, \dots, n$
- X^T : Transposed matrix of X
- C : Constant vector
- λ : Constant
- U : Control vector, $U=[u_j]$, $j=1, 2, \dots, n$
- A : Constant of the plant, $A=[a_{ij}]$, $i, j=1, 2, \dots, n$
- B : Constant of the plant, $B=[b_{ij}]$, $i, j=1, 2, \dots, n$
- T : Optimal time required for the process change
- t : Time
- Ψ : Auxiliary vector, $\Psi=[\phi_j]$, $j=1, 2, \dots, n$
- H : Hamiltonian
- A^T : Transposed matrix of A
- B^T : Transposed matrix of B
- Φ : Fundamental matrix of $\dot{X}=AX$
- R : Fundamental matrix of $\dot{\Psi}=-A^T\Psi$
- V : Fundamental matrix which satisfies Eq. (16)
- $w_{ij}^{\phi x}$: Characteristic equation represented by Eq. (25)
- $w_{ij}^{\phi \phi}$: Characteristic equation represented by Eq. (26)
- w_{ij} : Characteristic equation represented by Eq. (27)
- ε : Error of the determinant
- A_i : Coefficient of the equation of motion of the submarine, $i=1, 2, \dots, 7$
- B_i : Coefficient of the equation of motion of the submarine, $i=1, 2, \dots, 7$
- θ : Pitching angle in degrees
- $\dot{\theta}$: $d\theta/dt$ (deg/sec)
- h : Depth in meters
- $\dot{h}$: dh/dt (m/sec)
- β_{en} : Angle of stern plane in degrees
- β_{ef} : Angle of bow plane in degrees

k_θ : Pitching angle gain, β_{ea}/θ

k_h : Depth gain, β_{ea}/h

$k_{\dot{\theta}}$: Pitching angle velocity gain, $\beta_{ea}/\dot{\theta}$

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Appendix

(1) Flow chart for computation

The flow chart is shown in Fig. a-1. The coefficients a_{ij} and b_{ij} , and initial conditions $x_j(0)$ must be stored automatically from external sources.

(2) Symbols

Symbols employed in the computer are shown in Table 1.

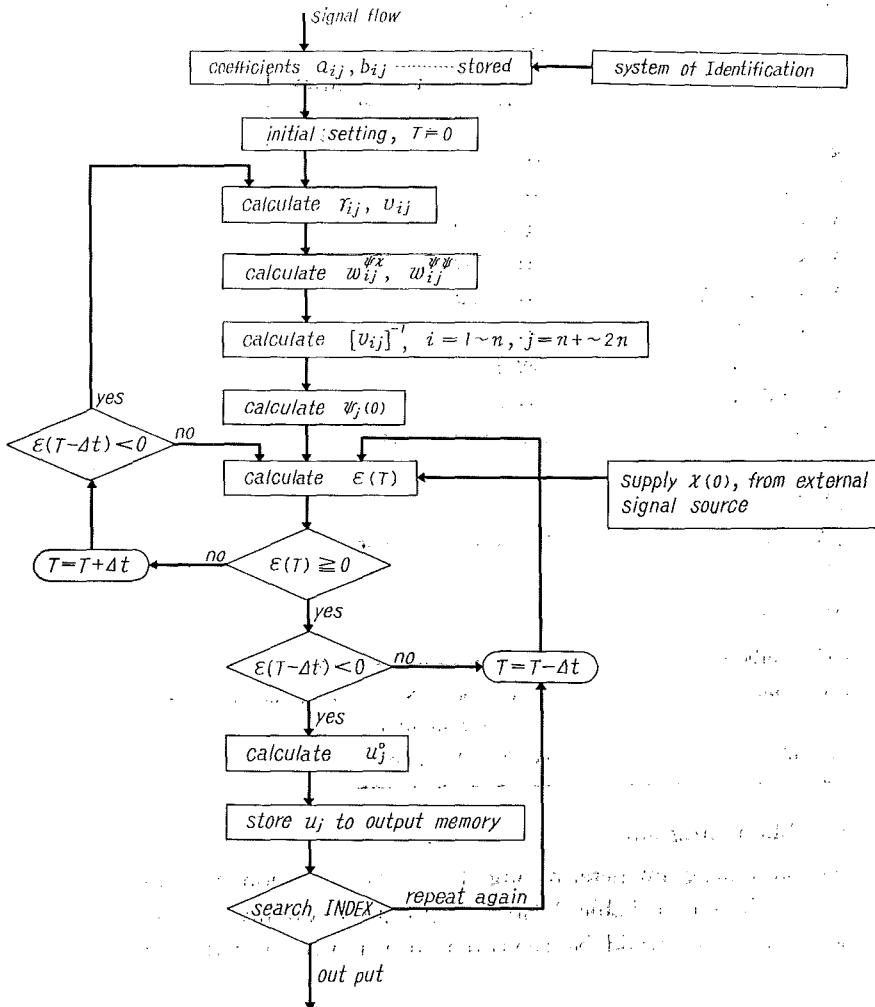


Fig. a-1. Flow chart

TABLE 1. Table of symbols employed in the computer

T	t
DELTAT	Δt
A(I, J)	a_{ij}
B(I, J)	b_{ij}
X(J)	x_j
U(J)	u_j
X $\bar{O}$ (J)	$x_j(0)$
F $\bar{O}$ (J)	$\phi_j(0)$
D(I, J)	d_{ij}
	$[d_{ij}] = \begin{bmatrix} A & \frac{1}{2\lambda} BB^T R \\ 2C & -A \end{bmatrix}$
R(I, J)	r_{ij}
V(I, J)	v_{ij}
AUXW(1, I, J)	$w_{ij}^{\phi x}(t + \Delta t)$
AUXW(2, I, J)	$w_{ij}^{\phi y}(t + \Delta t)$
AUXW(3, I, J)	$w_{ij}(t + \Delta t)$
AUXW $\bar{O}$ (1, I, J)	$w_{ij}^{\phi x}(t)$
AUXW $\bar{O}$ (2, I, J)	$w_{ij}^{\phi y}(t)$
AUXW $\bar{O}$ (3, I, J)	$w_{ij}(t)$
UMAX(J)	upper limit of u_j
QR(I, J)	} auxiliary variables
QV(I, J)	
QA(I, J)	
QX(I, J)	
AUXV(I, J)	
DIF symbols	employed in subroutines
Computer	NEAC 2800 (NEC's scientific and engineering purpose computer)
Compiler	AUTOMATH 3800

(3) Main program

The main program determining the initial condition $\Psi(0)$ and the optimal control U is shown in Table 2. (as this program contains repeats itself causing a waste of time, it should be rewritten into a well-formed chart.)

IFN	FFN	PROGRAM: SUBS	JOB: CENTER	PAGE: 01
0001		DIMENSION FO(4),A(4,4),B(4,4),XO(4),D(8,8) 1,P(8,4),QR(4,4),V(8,8),QV(8,8),AUXMO(3,4,4) 2,AUXW(3,4,4),AUXV(4,4),DIFK(8),DIFR(8),DIFU(8) 3,DIFA(8,8),DIFB(8,8),DIFY(8),DIFQ(8),DIF 4,A(8,8),DIFA(8),DIFC(8,8),DIFD(4,4) 5,UO(4),UMAX(4),UX(4) DELTAT=2.0 (= Δt) TO=2.0 ALFA=0.1 RAMDA=1.0 (= λ) A(1,1)=0.03739 A(1,2)=0.0 A(1,3)=-0.02431 A(1,4)=-0.001671 A(2,1)=1.0 A(2,2)=0.0 A(2,3)=2.0 A(2,4)=0.0 A(3,1)=0.1361 A(3,2)=0.0 A(3,3)=-0.1113 A(3,4)=-0.002615 A(4,1)=0.0 A(4,2)=0.0 A(4,3)=1.0 A(4,4)=0.0 B(1,1)=0.0003664 B(1,2)=-0.0003043 B(1,3)=0.0 B(1,4)=0.0 B(2,1)=0.0 B(2,2)=0.0 B(2,3)=0.0 B(2,4)=0.0 B(3,1)=0.002267 B(3,2)=0.001067 B(3,3)=0.0 B(3,4)=0.0 B(4,1)=0.0 B(4,2)=0.0 B(4,3)=0.0 B(4,4)=0.0 UMAX(1)=1000000.0 UMAX(2)=1000000.0 UMAX(3)=1000000.0 UMAX(4)=1000000.0 TSPD=0.001 1001 DEPTH=2.5 1002 DO 1003 I=1,4 XO(I)=0.0 UO(I)=0.0		
0002				
0003				
0004				
0005				
0006				
0007				
0010				
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0014				
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0016				
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0020				
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0040				
0041				
0042				
0043				
0044				
0045				
0046				
0047				
0050				
0051				
0052				
0053	1001	DEPTH=2.5		
0054	1002	DO 1003 I=1,4		
0055		XO(I)=0.0		
0056		UO(I)=0.0		

IFN	FFN	PROGRAM: SUB	JOB: CENTER	PAGE: 02
0057		QX(I)=0.0		
0060	1003	CONTINUE		
0061		DEPTH=2.0*DEPTH		
0062		XO(2)=DEPTH		
0063		IF (DEPTH=50.0) 1,1,1024		
0064	1004	STOP		
0065	1	T=0.0		
0066		DO 11 J=1,4		
0067		DO 8 I=1,4		
0070		R(I,J)=0.0		
0071		QR(I,J)=0.0		
0072	5	D(I,J)=A(I,J)		
0073		D(I+4,J)=0.0		
0074		D(I+4,J+4)=-A(J,1)		
0075	8	CONTINUE		
0076		D(J+4,J)=2.0		
0077	10	R(I,J)=1.0		
0100	11	CONTINUE		
0101		DO 20 J=1,4		
0102		DO 19 I=1,4		
0103		AW=0.0		
0104	15	DO 17 K=1,4		
0105		AW=AW+R(I,K)*R(J,K)		
0106	17	CONTINUE		
0107		D(I,J+4)=AW/(2.0*RAMDA)		
0110	19	CONTINUE		
0111	20	CONTINUE		
0112		DO 27 J=1,8		
0113		DO 25 I=1,8		
0114		V(I,J)=0.0		
0115		QV(I,J)=0.0		
0116	25	CONTINUE		
0117		V(J,J)=1.0		
0120	27	CONTINUE		
0121	115	DO 122 J=1,4		
0122		DO 121 I=1,4		
0123		DO 120 K=1,3		
0124		AUXW(K,I,J)=0.0		
0125		AUXW(K,I+J)=0.0		
0126	120	CONTINUE		
0127	121	CONTINUE		
0130	122	CONTINUE		
0131	200	DO 205 J=1,4		
0132		DO 205 I=1,4		
0133		DIFA(I,J)=-A(J,I)		
0134		DIFB(I,J)=R(I,J)		
0135		DIFQA(I,J)=QR(I,J)		
0136	205	CONTINUE		
0137		DIFXA=T		
0140		DIFXOA=DELTAT		
0141		NDIFA=4		

} SUBROUTINE INITIAL SETTING

IFN	FFN	PROGRAM: SUB5	JOB: CENTER	PAGE: 03
0142		KDIFA=1	SUBROUTINE INITIAL SETTING	
0143	210	GO TO 9840	SUBROUTINE FOR CALCULATION OF HOMOGENOUS EQUATION	
0144	9051	DO 215 J=1,4		
0145		DO 215 I=1,4		
0146		QR(I,J)=DIFQA(I,J)		
0147		R(I,J)=DIFB(I,J) (SOLUTION OF γ_{ij})		
0150	215	CONTINUE		
0151		DO 221 J=1,8		
0152		DO 221 I=1,8		
0153		DIFA(I,J)=D(I,J)		
0154		DIFB(I,J)=V(I,J)		
0155	220	UIFQA(I,J)=QV(I,J)	INITIAL SETTINGS FOR SUBROUTINE	
0156	221	CONTINUE		
0157		DIFXA=T		
0160		DIFXA=DELTA		
0161		NDIFA=8		
0162	225	KDIFA=2		
0163		GO TO 9840	SUBROUTINE FOR CALCULATION OF HOMOGENOUS EQUATION	
0164	9052	DO 231 J=1,8		
0165		DO 231 I=1,8		
0166		QV(I,J)=UIFQA(I,J)		
0167	230	V(I,J)=DIFB(I,J) (SOLUTION OF γ_{ij})		
0170	231	CONTINUE		
0171	303	DO 321 KV=1,2		
0172		DO 304 J=1,4		
0173		JJJJ=J+4*(KV-1)		
0174		DO 304 I=1,4		
0175		IIII=I+4		
0176		AUXV(I,J)=V(IIII,JJJJ)		
0177	304	CONTINUE		
0200	305	DO 321 J=1,4		
0201		DO 321 I=1,4		
0202		AW1=0,0		
0203		DO 318 M=1,4		
0204		AW2=0,0		
0205	310	DO 316 L=1,4		
0206		AW3=0,0		
0207		DO 314 K=1,4		
0210		AW3=AW3+B(L,K)*B(M,K)		
0211	314	CONTINUE		
0212	315	AW2=AW2+AW3*R(L,I)		
0213	316	CONTINUE		
0214		AW1=AW1+AW2*AUXV(M,J)		
0215	318	CONTINUE		
0216		AUXW(KV,I,J)=AUXW(KV,I,J)+(AUXW0(KV,I,J)+AW1)*DELTA/2,0		
0217	320	AUXW0(KV,I,J)=AW1		
0220	321	CONTINUE		
0221		T=DIFXA		
0222		IF(T-TO*DELTA/2,0) 200,200,320		
0223	328	A=A		
0224	400	DO 403 J=1,4	INITIAL SETTINGS FOR SUBROUTINE	

IFM	EFM	PROGRAM: SUBS	JOB: CENTER	PAGE: 04
0225	DO 403 I=1,4	} INITIAL SETTINGS FOR SUBROUTINE		
0226	DIFC(I,J)=V(I,J+4)			
0227	CONTINUE			
0230	NDIFC=4			
0231	405 KDIFC=1			
0232	GO TO 9830	← SUBROUTINE FOR CALCULATION OF INVERSE MATRIX		
0233	9041 DO 410 J=1,4			
0234	DO 410 I=1,4			
0235	AUXV(I,J)=DIFC(I,J)	SOLUTION OF $[V_{ij}]^{-1}$		
0236	410 CONTINUE			
0237	DO 418 J=1,4			
0240	DO 418 I=1,4			
0241	AW=0.0			
0242	DO 416 K=1,4			
0243	415 AW=AW+AUXV(I,K)*V(K,J)			
0244	416 CONTINUE			
0245	AUXW(3,I,J)=AW			
0246	418 CONTINUE			
0247	DO 425 I=1,4			
0250	420 AW=0.0			
0251	DO 423 J=1,4			
0252	AW=AW+AUXW(3,I,J)*X0(J)			
0253	423 CONTINUE			
0254	FO(I)=AW			
0255	425 CONTINUE			
0256	500 AW1=0.0			
0257	DO 507 I=1,4			
0260	AW2=0.0			
0261	DO 505 J=1,4			
0262	AW2=AW2+AUXW(1,I,J)*X0(J)+AUXW(2,I,J)*FO(J)			
0263	505 CONTINUE			
0264	AW1=AW1+AW2*X0(I)+2.0*RAMDA*X0(I)**2			
0265	507 CONTINUE			
0266	GO TO 515			
0267	509 IF (ERROR1) 200,514,514			
0270	514 GO TO 600			
0271	515 ERROR1=AW1 (- E(T))			
0272	AW=0.0			
0273	DO 519 J=1,4			
0274	AW=AW+AUXW(3,J,2)*AUXW(2,2,J)			
0275	519 CONTINUE			
0276	ERROR2=2.0*RAMDA+AUXW(1,2,2)-AW			
0277	GO TO 509			
0300	600 GO TO 601			
0301	601 GO TO 602			
0302	602 TFIN=T*TSPO			
0303	PRINT604,T,TFIN,ERROR1			
0304	604 FORMAT(1H4,5X,3E12.4)			
0305	PRINT606,(FU(J),J=1,4)			
0306	606 FORMAT(1H2,5X,4E12.4)			
0307	PRINT608,(X0(J),J=1,4)			

IFN	EFN	PROGRAM	SUBS	JOBI	CENTER	PAGE
0310	608	FORMAT(1H2,5X,4E12.4)				05
0311		PRINT610,(UO(J),J=1,4)				
0312	610	FORMAT(1H2,5X,4E12.4)				
0313	611	GO TO 700				
0314	700	T=DELTAT				
0315	701	DO 708 J=1,4				
0316		DO 705 I=1,4				
0317		DIFA(I,J)=A(I,J)				
0320		DIFB(I,J)=B(I,J)				
0321	705	CONTINUE				
0322		DIFY(J)=AQ(J)				
0323		DIFQ(J)=QX(J)				
0324	708	CONTINUE				
0325		A=A				
0326	710	DIFX=T				
0327		DIFXO=DELTAT				
0330		MDIF=1				
0331		MDIF=1				
0332		KDIF=1				
0333		GO TO 9000				
0334	9015	T=DIFX				
0335		IF(T+DELTAT/2.0-TFINE)710,710,718				
0336	718	DO 725 I=1,4				
0337		AW=0.0				
0340		DO 722 J=1,4				
0341		AW=AW+B(J,I)*FO(J)				
0342	722	CONTINUE				
0343		UO(I)=AW/(2.0*RAMDA)				
0344		XO(I)=DIFY(I)				
0345		QX(I)=DIFQ(I)				
0346	725	CONTINUE				
0347	726	T=TFINE/TSPD				
0350		IF(T-2.0-DELTAT/2.0)728,728,727				
0351	727	GO TO 1				
0352	728	GO TO 1002				
0353	9011	DO 735 J=1,4				
0354		IF(ABS(UO(J))-ABS(UHAX(J)))732,732,734				
0355	732	DIFU(J)=UO(J)				
0356		GO TO 737				
0357	734	DIFU(J)=SIGN(UHAX(J),UO(J))				
0360	737	A=A				
0361	735	CONTINUE				
0362	736	GO TO 9001				
0363	9021	PAUSE 9021				
0364	9022	PAUSE 9022				
0365	9023	PAUSE 9023				
0366	9024	PAUSE 9024				
0367	9025	PAUSE 9025				
0370	9026	PAUSE 9026				
0371	9027	PAUSE 9027				
0372	9031	PAUSE 9031				

INITIAL SETTINGS FOR SUBROUTINE

SUBROUTINE TO SOLVE DIFFERENTIAL EQUATION

CALLED FROM SUBROUTINE 9000

PAGE 06

SUBPROGRAMS REFERENCED

INTRINSIC FUNCTION	ABS
INTRINSIC FUNCTION	SIGN
INTRINSIC FUNCTION	SQRT

OVERLAY SIZE	1790
IR SPAN SIZE	178

CENTER

PROGRAM NAME	OVERLAY SIZE	IR-SPAN	NC-ARRAY SIZE
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COMMON	BLOCK NAME	SIZE
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LOGICAL UNIT S/D BUFFER SIZE

3 DOUBLE 23
TOTAL AMOUNT OF MEMORY REQUIRED FOR JOB 005457