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Title	Maximal Circuit-free and Cutset-free Subgraphs
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Citation	Memoirs of the Faculty of Engineering, Hokkaido University, 14(1), 101-106
Issue Date	1975-03
Doc URL	https://hdl.handle.net/2115/37933
Type	departmental bulletin paper
File Information	14(1)_101-106.pdf



Maximal Circuit-free and Cutset-free Subgraphs

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(Received July 31, 1974)

Abstract

A maximal circuit-free subgraph and a maximal cutset-free subgraph have important meanings in a graph. That is to say, a maximal circuit-free subgraph of a graph is a tree and a maximal cutset-free subgraph of a graph is a cotree. In this paper, the properties of a maximal circuit-free and cutset-free subgraph are investigated. And it is shown that a maximal circuit-free and cutset-free subgraph h_m of a graph is a certain hybrid tree and the number of edges of a hybrid tree containing the maximum number of edges changes according to the partition (E_y, E_z) of the set of edges E and its minimum value m_f is equal to the number of edges of h_m . The number m_f coincides with the topological degrees of freedom of the graph.

I. Introduction

Circuits (or loops) and cutsets are basic concepts in graph theory. Some fundamental concepts of a graph are defined by using these basic concepts. The rank of a graph is defined to be the number of linearly independent cutsets in the graph. The nullity of a graph is the number of linearly independent circuits in the graph. A tree on a graph is a maximal circuit-free^{*1} subgraph and a cotree on a graph is a maximal cutset-free subgraph^{2),3)} and the number of edges of a tree or a cotree equals the rank or the nullity of the graph, respectively. These show that a maximal circuit-free subgraph and a maximal cutset-free subgraph have important meanings in a graph. Then, what meaning does a maximal circuit-free and cutset-free subgraph have? In this paper, the properties of a maximal circuit-free and cutset-free subgraph are investigated and the relations among the subgraphs and the hybrid trees are presented.

II. Preliminaries

In order to clarify the relations among a maximal circuit-free and cutset-free subgraph and hybrid trees, first we shall present the definition of hybrid trees and some properties which will be necessary for the following discussions.

G is an undirected and connected graph. Let E be the set of edges of G and partition E into two subsets E_y and E_z such that $E_y \cup E_z = E$ and $E_y \cap E_z = \phi$.

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*1 A subgraph of G is said to be circuit-free or cutset-free in G if it contains no circuits or no cutsets respectively.

Definition 1: ε_z and $\bar{\varepsilon}_z$ ($\bar{\varepsilon}_z = E_z - \varepsilon_z$) are subsets of E_z in such a manner that the subgraphs^{*2} ε_z and $\bar{\varepsilon}_z$ are cutset-free and circuit-free in G respectively. ε_y and $\bar{\varepsilon}_y$ ($\bar{\varepsilon}_y = E_y - \varepsilon_y$) are subsets of E_y in such a way that the subgraphs ε_y and $\bar{\varepsilon}_y$ are circuit-free and cutset-free respectively in the graph G_y derived from G by open-circuiting^{*3} all the edges of ε_z and short-circuiting all the edges of $\bar{\varepsilon}_z$. Then the subgraph $\varepsilon_y \cup \varepsilon_z$ is called a hybrid tree in G with respect to the partition (E_y, E_z) of E ^{*4}.

In this definition, if $E_y \neq \phi$ and $E_z = \phi$, then since $G = G_y$ and ε_y and $\bar{\varepsilon}_y$ are circuit-free and cutset-free in G_y respectively, the hybrid tree ε_y is a tree of G ^{*5}. If $E_y = \phi$ and $E_z \neq \phi$, then since $G_y = \phi$ and ε_z and $\bar{\varepsilon}_z$ are cutset-free and circuit-free in G respectively, the hybrid tree ε_z is a cotree of G . This means that a hybrid tree of G becomes a tree or a cotree of G according to a partition of E . In a hybrid tree $\varepsilon_y \cup \varepsilon_z$, the subgraphs ε_y and ε_z form a part of a tree and a cotree in G respectively. So it may safely be said that a hybrid tree has properties of both a tree and a cotree.

Let HT , T and CT be the sets of hybrid trees, trees and cotrees in G respectively. It is clear from the definition of hybrid trees that $\varepsilon_y \cup \bar{\varepsilon}_z$ and $\bar{\varepsilon}_y \cup \varepsilon_z$ (which is the complement of $\varepsilon_y \cup \bar{\varepsilon}_z$) are circuit-free and cutset-free in G respectively and thus $\varepsilon_y \cup \bar{\varepsilon}_z$ and $\bar{\varepsilon}_y \cup \varepsilon_z$ are a tree and a cotree of G respectively. Hence we obtain,

$$T = \{t \mid t = ht \oplus E_z, ht \in HT\} \quad (1)$$

$$CT = \{\bar{t} \mid \bar{t} = ht \oplus E_y, ht \in HT\} \quad (2)$$

where $\oplus$ denotes the ring sum. These show the relations among HT , T and CT and when $E_y \neq \phi$ and $E_z = \phi$, $T = HT$ and when $E_y = \phi$ and $E_z \neq \phi$, $CT = HT$.

The number of edges in a tree and a cotree of G is identical with the rank r and the nullity μ of G respectively. The number of edges in a hybrid tree of G , however, is not always equal to r or μ and every hybrid tree of G does not contain the same number of edges. Let hH_k be a subset of HT in such a way that

$$hT_k = \{ht \mid |ht| = k, ht \in HT\} \quad (3)$$

Theorem 1: Let HT be the set of hybrid trees of G with respect to the partition (E_y, E_z) of E . Then HT is classified as follows,

^{*2} The subgraph specified by the subset of edges ε is defined as the subgraph whose set of edges is ε and whose set of vertices is the set of all the vertices that are incident with the members of ε .

Hereafter, the subgraph with the suffix "y" or "z" denotes that of E_y or E_z , respectively.

^{*3} The term "open-circuiting an edge" is used here in the meaning of removing the edge and the term "short-circuiting an edge" is used in the meaning of identifying the two end vertices of the edge and then removing it.

^{*4} If $\varepsilon_z = \phi$, $\bar{\varepsilon}_z \neq \phi$ and G_y consists of some self-loops only, then the hybrid tree is denoted by ϕ .

^{*5} The subgraphs g_1 and g_2 ($g_2 = E - g_1$) of G are a tree and a cotree respectively, if g_1 and g_2 are circuit-free and cutset-free in G , respectively.

$$HT = \{hT_m, hT_{m-2}, \dots, hT_{l+2}, hT_l\} \quad (4)$$

where $hT_k \not\equiv \phi$, $m \geq k \geq l$ and m, k and l are nonnegative integers.

Proof: Let $ht_k \in hT_k$ be a hybrid tree in such a way that $ht_k = \varepsilon_y \cup \varepsilon_z$, $\bar{\varepsilon}_z = E_z - \varepsilon_z$ and $\bar{\varepsilon}_y = E_y - \varepsilon_y$. If $|\varepsilon_z| = j$, then $|\varepsilon_y| = k - j$. The rank of the graph G_y derived from G by open-circuiting all the edges of ε_z and short-circuiting all the edges of $\bar{\varepsilon}_z$ is $k - j$, since ε_y is a tree of G_y . If E_z is partitioned into two subsets ε'_z and $\bar{\varepsilon}'_z$ are such that $|\varepsilon'_z| = j - 1$ and ε'_z and $\bar{\varepsilon}'_z$ are cutset-free and circuit-free in G respectively, the rank of G'_y derived from G by open-circuiting all the edges of ε'_z and short-circuiting all the edges of $\bar{\varepsilon}'_z$ is $k - j - 1$. Let ε'_y be a subset of E_y in such a way that ε'_y and $\bar{\varepsilon}'_y (= E_y - \varepsilon'_y)$ are circuit-free and cutset-free in G'_y respectively. Then $|\varepsilon'_y| = k - j - 1$ and the subgraph $\varepsilon'_y \cup \varepsilon'_z$ is a hybrid tree and $|\varepsilon'_y \cup \varepsilon'_z| = k - 2$. If $\varepsilon'_z = j + 1$, then $|\varepsilon'_y \cup \varepsilon'_z| = k + 2$. This means that in any partition of E_z , hybrid trees whose number of edges are $k + 1$ or $k - 1$ are not obtained.

In order to prove that $hT_k \not\equiv \phi$ when $hT_{k-2} \neq \phi$ and $hT_{k+2} \not\equiv \phi$, let us assume $hT_k = \phi$. Let $ht' \in hT_{k-2}$, $ht'' \in hT_{k+2}$, $ht' = \varepsilon'_y \cup \varepsilon'_z$, $ht'' = \varepsilon''_y \cup \varepsilon''_z$, $\bar{\varepsilon}'_z = E_z - \varepsilon'_z$ and $\bar{\varepsilon}''_z = E_z - \varepsilon''_z$. From the assumption, for any edge $e_z \in \varepsilon''_z$, the subgraph $e_z \cup \bar{\varepsilon}'_z$ contains a circuit. Let $\bar{\varepsilon}' = \bar{\varepsilon}'_z \cap \bar{\varepsilon}'_z$ and $\bar{\varepsilon}'' = \bar{\varepsilon}''_z - \bar{\varepsilon}'$, then $\bar{\varepsilon}'_z$ consists of $\bar{\varepsilon}'$ and $|\bar{\varepsilon}''| + 2$ edges of ε''_z . Since the subgraph $\bar{\varepsilon}'_z$ contains no circuits, the subgraph $\bar{\varepsilon}'_z \cup \bar{\varepsilon}''$ contains at most $|\bar{\varepsilon}''|$ circuits. The subgraph $\bar{\varepsilon}'_z \cup \bar{\varepsilon}''$ however contains $|\bar{\varepsilon}''| + 2$ circuits, because the subgraph $\bar{\varepsilon}'_z \cup \bar{\varepsilon}''$ consists of $\bar{\varepsilon}'_z$ and $|\bar{\varepsilon}''| + 2$ edges of ε''_z . This contradicts the assumption. Q. E. D.

The values of m and l in the classification (4) of HT vary with the partition (E_y, E_z) of E . If the graph G is separable and E is partitioned into two subsets E_y and E_z such that E_z consists of all the edges of some separable components and $E_y = E - E_z$, then $m = l$. If the graph G is nonseparable, then $m = l$ if and only if HT coincides with T or CT then $m = l = r$ or μ .

III. Maximal Circuit-Free and Cutset-Free Subgraphs

Let H be the set of subgraphs of G satisfying the following condition: the graph derived from G by short-circuiting some edges of a subgraph $h \in H$ and open-circuiting all the other edges of h contains no circuits but self-loops. Then there exists the minimum subgraph satisfying the above condition. Let $H_m \subset H$ be the set of the minimum subgraphs. Then we obtain the following theorem.

Theorem 2: Let h_m be a maximal circuit-free and cutset-free subgraph of G . Then $h_m \in H_m$.

Proof. For any edge $e_i \in (E - h_m)$, ($i = 1, 2, \dots$), the subgraph $h_m \cup e_i$ contains a circuit or a cutset of G . Let ε_c be the set of all the edges contained in such circuits and ε_1 be the subset of h_m in such a way that $\varepsilon_1 \subset \varepsilon_c$ and $\varepsilon_2 (= h_m - \varepsilon_1) \not\subset \varepsilon_c$. Clearly, the graph derived by short-circuiting all the edges of ε_1 and open-circuiting all the edges of ε_2 contains no circuits but self-loops, namely, $h_m \in H$.

Consider a subgraph g of G which is circuit-free and cutset-free and is not maximal. Then there exists at least one edge e_j of $E - g$ in such a way that

$g \cup e_j$ contains neither circuits nor cutsets of G . Therefore, g is not an element of H . This means that if a subgraph $h \in H$ is circuit-free and cutset-free, it must be a maximal circuit-free and cutset-free subgraph of G . In order to show that $h_m \in H_m$, we assume that $h_m \notin H_m$ and g_h is an element of H_m in such a way that $g_h = g_1 \cup g_2$, $g_1 \cap g_2 = \emptyset$ and the graph derived from G by short-circuiting all the edges of g_1 and open-circuiting all the edges of g_2 contains no circuits but only self-loops. From the above explanations, g_h must contain circuits or cutsets. Assuming that circuits are contained in g_h and let L_p be one of them. Then, L_p contains at least two edges of g_2 because if L_p consists of only edges of g_1 or one edge of g_2 and edges of g_1 , the subgraph $g_h - e_p$, ($e_p \in L_p$) belongs to H and g_h is not a minimum subgraph. Let $L_{p2} = L_p \cap g_2$ and G_p be the graph derived from G by short-circuiting all the edges of g_1 and open-circuiting all the edges of $g_2 - L_{p2}$. In G_p , L_{p2} forms circuits and since $g_h \in H_m$, there exists an edge e_p of L_{p2} in such a way that e_p and the edges of $\bar{g}_h (= E - g_h)$ construct a circuit L_s consisting of more than three edges. Let $e_s \in L_s$, ($e_s \notin L_{p2}$) and $g'_2 = (g_2 - e_p) \cup e_s$. Then, $g_1 \cup g'_2$ belongs to H and does not contain the circuit L_p . Applying this procedure to all the circuits of g_h , we obtain the subgraph that contains no circuits and consists of the same number of edges as g_h . Similarly, when g_h contains cutsets or both circuits and cutsets, we obtain the subgraph g'_h from g_h in such a way that $|g'_h| = |g_h|$, $g_h \in H$ and g'_h contains neither circuits nor cutsets. This contradicts the assumption that $h_m \notin H_m$.

Q. E. D.

This theorem shows that a maximal circuit-free and cutset-free subgraph is one of the minimum subgraphs in H . As an example, let us consider a graph G of Fig. 1. Let $h_m = \{e_1, e_4, e_7\}$, then h_m is a maximal circuit-free and cutset-free subgraph and the graph derived from G by open-circuiting the edge e_7 and short-circuiting the edges e_1 and e_4 contains no circuits but only has self-loops.

A tree t and a cotree $\bar{t}$ are elements of H , since the graph derived from G by short-circuiting all the edges of t consists only of self-loops, and the graph derived from G by open-circuiting all the edges of $\bar{t}$ contains no circuits. We obtain the following lemma concerning hybrid trees.

Lemma 1: Let ht be a hybrid tree of G with respect to the partition (E_y, E_z) of E . Then $ht = \varepsilon_y \cup \varepsilon_z$ is an element of hT_m , if and only if the graph G' derived from G by open-circuiting all the edges of ε_z and short-circuiting all the edges of ε_y contains no circuits but only has self-loops.

Proof: Let $ht_m \in hT_m$, $ht_m = \varepsilon_y \cup \varepsilon_z$, $\bar{\varepsilon}_y = E_y - \varepsilon_y$ and $\bar{\varepsilon}_z = E_z - \varepsilon_z$. From the defi-

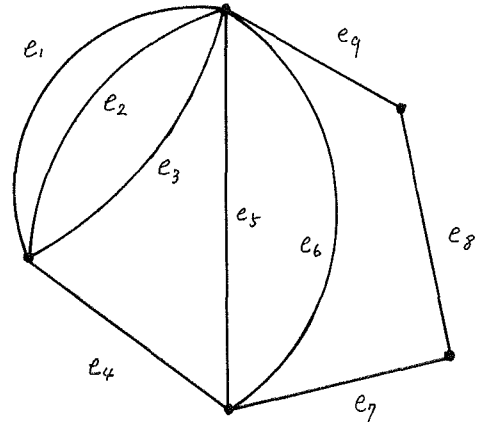


Fig. 1. A graph G .

inition of hybrid trees, ε_z must be a maximal cutset-free subgrape of E_z and then the subgraphs $\bar{\varepsilon}_z$ and $\bar{\varepsilon}_z \cup E_y$ contain no circuits. Let G' be the graph derived from G by open-circuiting all the edges of ε_z . In the graph G' , the subgraph $\varepsilon_y \cup \bar{\varepsilon}_z$ contains no circuits and the endpoints of an edge in $\bar{\varepsilon}_y$ are contained in the subgraph ε_y . Therefore, the graph derived from G' by short-circuiting all the edges of ε_y contains no circuits but only has self-loops.

Let $ht_k \in HT$, $ht_k \notin hT_m$, $ht_k = \varepsilon'_y \cup \varepsilon'_z$, $\bar{\varepsilon}'_y = E_y - \varepsilon'_y$ and $\bar{\varepsilon}'_z = E_z - \varepsilon'_z$. Since ε'_z is not a maximal cutset-free subgraph of E_z in G and $\bar{\varepsilon}'_z$ contains no circuits of G , there exists some circuits consisting of edges of $\bar{\varepsilon}'_z$ and E_y . Let one of them be L' . Since $\bar{\varepsilon}'_z \cup \varepsilon'_y$ contains no circuits, L' contains edges of $\bar{\varepsilon}'_y$. Therefore, the graph G' derived from G by open-circuiting all the edges of ε_z and short-circuiting all the edges of ε_y contains some circuits which are not self-loops.

Q. E. D.

This lemma shows that only the element $ht_m = \varepsilon_y \cup \varepsilon_z$ of hT_m in HT has the properties which are such that the graph derived from G by short-circuiting all the edges of ε_y and open-circuiting all the edges of ε_z contains no circuits but self-loops. From lemma 1 and theorem 2, we obtain the following theorem.

Theorem 3: The minimum value m_f of m in the classification (4) of HT equals the number of edges contained in a maximal circuit-free and cutset-free subgraph in G .

Proof: Let ht_m be a hybrid tree of hT_{m_f} with respect to the partition (E'_y, E'_z) of E which gives the minimum value m_f of m . Then from lemma 1 and theorem 2, ht_m belongs to H and $|ht_m| \geq |h_m|$, where h_m is a maximal circuit-free and cutset-free subgraph. Therefore, we have only to prove that h_m can be a hybrid tree in G . For any edge e_i of $\bar{h}_m (= E - h_m)$, the subgraph $h_m \cup e_i$ contains a circuit or a cutset. Let E_y be the set of all the edges contained in such circuits and E_z be $E - E_y$. And let $h_m = \varepsilon_y \cup \varepsilon_z$, $\varepsilon_y \subset E_y$, $\varepsilon_z \subset E_z$, $\bar{\varepsilon}_y = E_y - \varepsilon_y$ and $\bar{\varepsilon}_z = E_z - \varepsilon_z$. Then $\bar{\varepsilon}_z$ contains no circuits and since $h_m \cup e_i$, ($e_i \in \bar{\varepsilon}_z$) contains a cutset of G , $\bar{\varepsilon}_z$ contains no circuits. In the graph G_y derived from G by short-circuiting all the edges of $\bar{\varepsilon}_z$ and open-circuiting all the edges of ε_z , ε_y contains no circuits and since $h_m \cup e_i$, ($e_i \in \bar{\varepsilon}_y$) contains a circuit of G , $\bar{\varepsilon}_y$ contains no cutsets. This means that h_m is a hybrid tree of G with respect to the partition (E_y, E_z) .

Q. E. D.

As an example, let us consider the graph G of Fig. 1 again. In the graph G , a subgraph $h_m = \{e_1, e_4, e_7\}$ is a maximal circuit-free and cutset-free one. Let $E_z = \{e_7, e_8, e_9\}$ and $E_y = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Then $h_m = \varepsilon_y \cup \varepsilon_z$, ($\varepsilon_z = \{e_7\}$, $\varepsilon_y = \{e_1, e_4\}$) is a hybrid tree and $|h_m| (= 3)$ is a minimum value of m .

In the proof of theorem 3, a method of the partition of E minimizing “ m ” was shown. The method, however, is not unique. Because if the subgraph $h_m \cup e_i$, ($e_i \in \bar{h}_m$) contains both a circuit and a cutset of G , e_i may be an element of either E_y or E_z .

Since a maximal circuit-free and cutset-free subgraph h_m in G contains neither circuits nor cutsets, h_m can be a subset of both a tree and a cotree of G .

Thus h_m may be also regarded as the non-common branches (or chords) of a pair of maximally distant trees of $G^{(4),5)}$. And the number of edges in a maximal circuit-free and cutset-free subgraph in G coincides with the topological degrees of freedom of G .

IV. Conclusion

The properties of a maximal circuit-free and cutset-free subgraph are investigated and the relationships among the subgraph and hybrid trees are presented. And it is shown that the number of edges of a hybrid tree containing the maximum number of edges changes according to the partition (E_y, E_z) of E and its minimum value is equal to the number of edges in a maximal circuit-free and cutset-free subgraph of G . (Note that the number of edges in a tree or a cotree of G is constant and is equal to the number of edges in a maximal circuit-free subgraph or a maximal cutset-free subgraph of G , respectively). These results are important not only as a basic theory of a graph, but also as an application to network theory, since the number of edges of a maximal circuit-free and cutset-free subgraph gives the degrees of freedom of the physical system (graph).

References

- 1) F. Harary: Graph Theory, Addison-Wesley, Inc. 1969.
- 2) G. J. Minty: "On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network-programming". J. Math. Mech. 15, pp. 485-520, 1966.
- 3) M. Iri: Network Flow, Transportation and Scheduling, Theory and Algorithms, Academic Press, Inc. 1969.
- 4) G. Kishi and Y. Kajitani: "Maximally distant trees and principal partition of a linear graph," IEEE Trans. Circuit Theory, Vol. CT-16, pp. 323-330, August 1969.
- 5) T. Ohtsuki, Y. Ishizaki and H. Watanabe: "Topological degrees of freedom and mixed analysis of electrical networks," IEEE Trans. Circuit Theory, Vol. CT-17, pp. 491-499, November 1970.
- 6) S. Okada and R. Onodera: "A unified treatise on the topology of networks and algebraic electromagnetism," RAAG Memoirs I, 1955.
- 7) M. Sengoku, T. Kurobe and Y. Ogawa: "On the realization of a set of hybrid trees", Trans. IECE Japan, Vol. 55-A, No. 2, pp. 60-66 (1972).