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On a Solution of Non-linear Differential Equation

$$\frac{\partial u}{\partial t} - \alpha u^2 \frac{\partial u}{\partial x} + \gamma \frac{\partial^3 u}{\partial x^3} = 0$$

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Résumé

Non-linear partial differential equation :

$$\frac{\partial u}{\partial t} - \alpha u^2 \frac{\partial u}{\partial x} + \gamma \frac{\partial^3 u}{\partial x^3} = 0,$$

with $\alpha\gamma > 0$, has a solution: $u(x, t) = A \cdot p(b(x + vt))$, where $p(z)$ is Weierstraß' p -function. $A^2 = 360 \gamma b^4 / \alpha$, b , and v are constants.

The Weierstraßian p -function multiplied by the squared Jacobian sn -function is found to be a solution of the Korteweg-de Vries equation.

The general evolution equation can be written as :

$$\frac{\partial}{\partial t} u = - \frac{\partial}{\partial x} F(t, x, u, u_x, u_{xx}, \dots), \quad (1)$$

with the conserved density $u = u(x, t)$ and flux F . The subscript denotes differentiation.

If we take :

$$F = \frac{\delta I}{\delta u}, \quad (2)$$

and

$$I = \int_D \left\{ \frac{\varepsilon}{2} u^2 + \frac{\zeta}{6} u^3 + \frac{\alpha}{12} u^4 + \frac{\beta}{2} (u_x)^2 + \frac{\gamma}{2} (u_{xx})^2 \right\} dx, \quad (3)$$

where $\delta I / \delta u$ means the functional derivative of I with regard to u , and $\alpha, \beta, \gamma, \varepsilon$, and ζ are constants.

Assuming that all the functions u, u_x, u_{xx} , and u_{xxx} vanish at the upper and the lower boundaries of domain D , we have

$$\frac{\delta I}{\delta u} = \varepsilon u + \frac{\zeta}{2} u^2 + \frac{\alpha}{3} u^3 - \beta u_{xx} + \gamma u_{xxx}. \quad (4)$$

From (1), (2), and (4), we obtain

$$u_t + \varepsilon u_x + \zeta uu_x + \alpha u^2 u_x - \beta u_{xxx} + \gamma u_{xxxxx} = 0. \quad (5)$$

While, the solution of the higher order Korteweg-de Vries equation :

$$u_t - \alpha u^n u_x + \eta (uu_{xx})_x + \eta' u_x u_{xx} - \beta u_{xxx} + \gamma u_{xxxxx} = 0, \quad (6)$$

was discussed by several authors. *e.g.* For $n=2$, $\alpha=-45$, $\eta=15$, $\eta'=\beta=0$, and $\gamma=1$, soliton solutions of eq. (6) was found by Hirota's method^{1,2)}. For $n=2$, $\alpha=-30$, $\eta=\eta'=10$, $\beta=0$, $\gamma=1$, eq. (6) has also soliton solutions^{2,3)}.

An oscillatory solitary wave was observed⁴⁾ in a non-linear electric circuit for $n=1$, $\alpha=-1$, $\eta=\eta'=\beta=0$, and $\gamma=-1$ in (6), with reference to the computation⁵⁾, although the exact soliton solution for this equation seems to have not yet been found.

In the present paper we shall consider eq. (5) with $\alpha=-\alpha$, $\varepsilon=\zeta=\beta=0$, or eq. (6) with $n=2$ and $\eta=\eta'=\beta=0$, *i. e.*

$$u_t - \alpha u^2 u_x + \gamma u_{xxxxx} = 0. \quad (\alpha\gamma > 0) \quad (7)$$

Assuming a travelling solution with velocity v :

$$u(x, t) = u(x + vt) \equiv u(\xi), \quad (8)$$

with

$$\xi = x + vt, \quad (v = \text{const}) \quad (9)$$

and putting (8) into eq. (7), we obtain

$$vu_\xi - \alpha u^2 u_\xi + \gamma u_{\xi\xi\xi\xi\xi} = 0, \quad (10)$$

which can be integrated to give

$$vu - \frac{\alpha}{3} u^3 + \gamma u_{\xi\xi\xi\xi} = -C, \quad (11)$$

with an integration constant C .

A solution of eq. (11) can be obtained as

$$u(\xi) = A \cdot \wp(b\xi), \quad (A : \text{real positive}) \quad (12)$$

with a real constant b , and $A^2 = 360 \gamma b^4 / \alpha$. $\wp(z) = \wp(z | 2\omega_1, 2\omega_3)$ is the Weierstraßian $\wp$ -function with fundamental periods $2\omega_1$ and $2\omega_3$:

$$2\omega_1 = 2 \int_{e_1}^{+\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \quad (13)$$

and

$$2\omega_3 = 2i \int_{-\infty}^{e_3} \frac{dz}{\sqrt{-(4z^3 - g_2 z - g_3)}}. \quad (14)$$

$2\omega_1$ is real and $2\omega_3$ is purely imaginary, and $e_i = \wp(\omega_i)$ ($i=1, 2, 3$), are all real quantities, for

$$g_2^3 - 27g_3^2 > 0. \quad (15)$$

We put here $\omega_2 = \omega_1 + \omega_3$. e_i 's satisfy the following cubic equation :

$$4z^3 - g_2 z - g_3 \equiv 4(z - e_1)(z - e_2)(z - e_3) = 0, \quad (16)$$

with $e_3 < e_2 < e_1$.

By means of relations :

$$e_1 + e_2 + e_3 = 0, \quad (17)$$

$$g_2 = -4(e_1 e_2 + e_2 e_3 + e_3 e_1) = \frac{v}{18\gamma b^4} = 20 \frac{v}{\alpha A^2}, \quad (18)$$

and

$$g_3 = 4e_1 e_2 e_3 = \frac{C}{12A\gamma b^4} = 30 \frac{C}{\alpha A^3}, \quad (19)$$

eq. (16) is written as :

$$4z^3 - 20 \frac{v}{\alpha A^2} z - 30 \frac{C}{\alpha A^3} = 0. \quad (20)$$

Function $\wp(z)$ is an even function of z and has a pole of order 2 in any primitive period-parallellogram on the complex z -plane. So, the solution (12) is a real solution $A \cdot \wp(b\xi)$ with $v < 0$, travelling towards $+x$ -direction for $\alpha < 0$ and $\gamma < 0$, while it is also a real solution with $v > 0$, travelling towards $-x$ -direction for $\alpha > 0$ and $\gamma > 0$.

The function $\wp(z)$ is expressed as

$$\wp(z) = e_3 + \frac{e_1 - e_3}{sn^2(z\sqrt{e_1 - e_3}, k)}, \quad (21)$$

with the Jacobian elliptic function $sn(z, k)$ of modulus k :

$$k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad (22)$$

and we can find that the function $\phi(\xi)$ defined by :

$$\begin{aligned} \phi(\xi) &= A \cdot \wp(b\xi) \cdot sn^2(b\xi\sqrt{e_1 - e_3}, k) \\ &= A \left\{ (e_1 - e_3) + e_3 \cdot sn^2(b\xi\sqrt{e_1 - e_3}, k) \right\}, \end{aligned} \quad (23)$$

satisfies the following Korteweg - de Vries equation :

$$\frac{\partial \phi}{\partial t} - \alpha_0 \phi \frac{\partial \phi}{\partial x} + \gamma_0 \frac{\partial^3 \phi}{\partial x^3} = 0, \quad (24)$$

with

$$\alpha_0 = \frac{v(e_2 - e_3)}{A(e_3^2 + e_1 e_2)}, \quad (25)$$

and

$$\gamma_0 = \frac{v e_3}{12b^2(e_3^2 + e_1 e_2)}. \quad (26)$$

A) If we tend $k \rightarrow 0$, then we have:

$$2\omega_1 \rightarrow \pi/\sqrt{-3e_3}, \quad 2\omega_3 \rightarrow i\infty, \quad (27)$$

$$\left. \begin{aligned} e_1 &\rightarrow \frac{2}{A} \sqrt{\frac{5v}{3\alpha}}, \\ e_2 = e_3 &\rightarrow -\frac{1}{A} \sqrt{\frac{5v}{3\alpha}}, \end{aligned} \right\} \quad (28)$$

and the integration constant C should be chosen to be:

$$C = \frac{4v}{9} \sqrt{\frac{5v}{3\alpha}}, \quad (29)$$

so that eq. (7) could have a real solution. And the solution (12) takes the form:

$$u(x, t) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{ 3 \operatorname{cosec}^2 \left(b\xi \sqrt{\frac{1}{A} \sqrt{\frac{15v}{\alpha}}} \right) - 1 \right\}. \quad (30)$$

While, function $\phi(\xi)$ defined in (23) reads:

$$\phi(\xi) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{ 3 - \sin^2 \left(b\xi \sqrt{\frac{1}{A} \sqrt{\frac{15v}{\alpha}}} \right) \right\}. \quad (31)$$

B) If we take $k \rightarrow 1$, then from (13), (14), (17), (18), (19), and (22), we have:

$$2\omega_1 \rightarrow +\infty, \quad 2\omega_3 \rightarrow i\pi/\sqrt{3e_1}, \quad (32)$$

$$\left. \begin{aligned} e_1 = e_2 &\rightarrow \frac{1}{A} \sqrt{\frac{5v}{3\alpha}}, \\ e_3 &\rightarrow -\frac{2}{A} \sqrt{\frac{5v}{3\alpha}}, \end{aligned} \right\} \quad (33)$$

and the integration constant C should be chosen to be:

$$C = -\frac{4v}{9} \sqrt{\frac{5v}{3\alpha}}. \quad (34)$$

Solution (12) turns to be:

$$u(x, t) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{ 3 \coth^2 \left(b\xi \sqrt{\frac{1}{A} \sqrt{\frac{15v}{\alpha}}} \right) - 2 \right\}, \quad (35)$$

which gives a solitary pulse at $\xi = x + vt = 0$. While, function $\phi(\xi)$ for $k \rightarrow 1$ reads:

$$\phi(\xi) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{ 3 - 2 \tanh^2 \left(b\xi \sqrt{\frac{1}{A} \sqrt{\frac{15v}{\alpha}}} \right) \right\}. \quad (36)$$

C) If $0 < k < 1$, function $\wp(b\xi)$ is real for the values of C , which lies in the region:

$$-\frac{4}{9} \cdot |v| \cdot \sqrt{\frac{5v}{3\alpha}} < C < \frac{4}{9} \cdot |v| \cdot \sqrt{\frac{5v}{3\alpha}}. \quad (37)$$

When we take $C=0$, then eq. (11) reads :

$$vu - \frac{\alpha}{3}u^3 + \gamma u_{\xi\xi\xi\xi} = 0, \quad (38)$$

and eq. (20) becomes to be

$$z\left(z^2 - 5\frac{v}{\alpha A^2}\right) = 0. \quad (39)$$

Then, solution (12) can be reduced to :

$$u(x, t) = \sqrt{\frac{5v}{\alpha}} \cdot \left\{ \frac{2}{sn^2\left(b\xi\sqrt{\frac{2}{A}\sqrt{\frac{5v}{\alpha}}}, \sqrt{\frac{1}{2}}\right)} - 1 \right\}, \quad (40)$$

with

$$\left. \begin{aligned} g_3 &= 4e_1e_2e_3 = 0, \\ g_2 &= \frac{v}{18\gamma b^4} = \frac{20v}{\alpha A^2} > 0, \end{aligned} \right\} \quad (41)$$

i. e.

$$\left. \begin{aligned} e_2 &= 0, \\ e_1 &= -e_3 = \frac{1}{A}\sqrt{\frac{5v}{\alpha}}, \end{aligned} \right\} \quad (42)$$

and

$$k = \sqrt{\frac{-e_3}{e_1 - e_3}} = \sqrt{\frac{1}{2}}. \quad (43)$$

While, function $\phi(\xi)$ for $C=0$, reads :

$$\phi(\xi) = \sqrt{\frac{5v}{\alpha}} \cdot \left\{ 2 - sn^2\left(b\xi\sqrt{\frac{2}{A}\sqrt{\frac{5v}{\alpha}}}, \sqrt{\frac{1}{2}}\right) \right\}. \quad (44)$$

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