



HOKKAIDO UNIVERSITY

Title	Surface symmetries, homology representations, and group cohomology
Author(s)	Akita, Toshiyuki
Description	RIMS 研究集会報告集
Relation	有限群のコホモロジー論の研究 = Cohomology Theory of Finite Groups and Related Topics
Citation	京都大学数理解析研究所講究録, 1581, 103-108
Issue Date	2008-02
Doc URL	https://hdl.handle.net/2115/38888
Type	journal article
File Information	akita.pdf



SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY

TOSHIYUKI AKITA
秋田利之 (北海道大学大学院理学研究院)

Given a finite group G of automorphisms of a compact Riemann surface, we discuss a relation between Mumford-Morita-Miller classes of odd indices and the homology representation of G . Since most participants were group theorists rather than topologists, I separate the algebraic and the topological ingredients and explain the former in detail.

1. SURFACE SYMMETRIES

1.1. The Grieder group of a finite group. Let G be a finite group and $\hat{\gamma}$ the conjugacy class of $\gamma \in G$. We denote by $\langle \hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_q \rangle$ an unordered q -tuple ($q \geq 0$) of conjugacy classes of nontrivial elements of G satisfying $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$, and $\mathcal{M}_G$ the set of all such q -tuples. We can define an abelian monoid structure on $\mathcal{M}_G$ by

$$\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle + \langle \hat{\gamma}_{q+1}, \dots, \hat{\gamma}_r \rangle = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q, \hat{\gamma}_{q+1}, \dots, \hat{\gamma}_r \rangle.$$

The identity element is the empty tuple $\langle \rangle$. We call $\mathcal{M}_G$ the *Grieder monoid* of G . Now let $\mathcal{M}'_G$ be the submonoid generated by $\langle \hat{\gamma}, \hat{\gamma}^{-1} \rangle$ ($\gamma \in G$) and set $\mathcal{A}_G := \mathcal{M}_G / \mathcal{M}'_G$. The quotient $\mathcal{A}_G$ is an abelian group. The inverse element is given by

$$-\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle = \langle \hat{\gamma}_1^{-1}, \dots, \hat{\gamma}_q^{-1} \rangle \text{ in } \mathcal{A}_G.$$

We call $\mathcal{A}_G$ the *Grieder group* of G . As the names suggest, $\mathcal{M}_G$ and $\mathcal{A}_G$ were introduced and studied by Grieder [5, 6] to study surface symmetries. First of all, $\mathcal{A}_G$ is finitely generated:

Proposition 1 ([5]). $\mathcal{A}_G \cong \mathbb{Z}^m \oplus \mathbb{Z}_2^n$ for some $m, n \geq 0$.

A homomorphism $f: H \rightarrow G$ of groups induces a homomorphism $f_*: \mathcal{A}_H \rightarrow \mathcal{A}_G$ of abelian groups by $f_*(\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle) = \langle f(\hat{\gamma}_1), \dots, f(\hat{\gamma}_q) \rangle$ so that the assignment $G \mapsto \mathcal{A}_G$ is a covariant functor. In addition, for an inclusion $i: H \hookrightarrow G$, one can also define the restriction $i^*: \mathcal{A}_G \rightarrow \mathcal{A}_H$ via surface symmetries. Grieder [5] verified the double coset formula and hence proved the following proposition:

Proposition 2. *The assignment $G \mapsto \mathcal{M}_G$ is a Mackey functor.*

1.2. Ramification data. By a *surface symmetry* we mean a pair (G, C) , where C is a compact Riemann surface of genus $g \geq 2$, and G is a finite group of automorphisms of C . For each $x \in C$, let G_x be the isotropy subgroup at x . Note that G_x is necessarily cyclic. Set $S = \{x \in C \mid G_x \neq 1\}$, and let $S/G = \{x_1, x_2, \dots, x_q\}$ be a set of representatives of G -orbits of elements of S . For each $x_i \in S/G$, choose a generator γ_i of G_{x_i} such that γ_i acts on the holomorphic tangent space $T_{x_i}C$ by $z \mapsto \exp(2\pi\sqrt{-1}/|G_{x_i}|)z$ with respect to a suitable local coordinate z at x_i . The *ramification data* of (G, C) , abbreviated by $\delta(G, C)$, is the unordered q -tuple $\langle \hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_q \rangle$. It satisfies $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$, and hence $\delta(G, C)$ is an element of the Grieder monoid $\mathcal{M}_G$. Conversely, we have the following proposition.

Proposition 3 (see [5]). *For any element $\mu \in \mathcal{M}_G$, there exists a surface symmetry (G, C) whose ramification data coincides with μ .*

2. GROUP COHOMOLOGY

2.1. The first Chern class. Let $\langle \gamma \rangle$ be a cyclic group of order m generated by γ and $\rho_\gamma : \langle \gamma \rangle \rightarrow \mathbb{C}^\times$ a linear character defined by $\gamma \mapsto \exp(2\pi i/m)$. For any finite group G , we have natural isomorphisms

$$\mathrm{Hom}(G, \mathbb{C}^\times) \cong H^1(G, \mathbb{C}^\times) \cong H^2(G, \mathbb{Z}).$$

Here, the latter isomorphism is the connecting homomorphism associated to the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^\times \rightarrow 0$. Define $c(\gamma) \in H^2(\langle \gamma \rangle, \mathbb{Z})$ to be the image of ρ_γ under the isomorphism $\mathrm{Hom}(\langle \gamma \rangle, \mathbb{C}^\times) \cong H^2(\langle \gamma \rangle, \mathbb{Z})$. The cohomology class $c(\gamma)$ is sometimes called the *first Chern class* of ρ_γ .

2.2. MMM classes (algebra). For each element $\mu = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle$ of $\mathcal{M}_G$, define a series of cohomology classes $e_k(\mu) \in H^{2k}(G, \mathbb{Z})$ ($k \geq 1$) by

$$e_k(\mu) := \sum_{i=1}^q \mathrm{Tr}_{\langle \gamma_i \rangle}^G(c(\gamma_i)^k) \in H^{2k}(G, \mathbb{Z}),$$

where $\mathrm{Tr}_{\langle \gamma \rangle}^G : H^*(\langle \gamma \rangle, \mathbb{Z}) \rightarrow H^*(G, \mathbb{Z})$ is the transfer. We call $e_k(\mu)$ the *k-th Mumford-Morita-Miller class* of μ (MMM class in short). The definition of $e_k(\mu)$ is motivated by topology, as will be explained in the next subsection. Observe that the assignment $\mu \mapsto e_k(\mu)$ defines a well-defined homomorphism $\mathcal{M}_G \rightarrow H^{2k}(G, \mathbb{Z})$ of abelian monoids. For k odd, it induces a well-defined homomorphism $\mathcal{A}_G \rightarrow H^{2k}(G, \mathbb{Z})$ of abelian groups, for we have $c(\gamma^{-1}) = -c(\gamma)$. In addition, we can prove the following proposition:

Proposition 4. For odd $k \geq 1$, the homomorphism $\mathcal{A}_G \rightarrow H^{2k}(G, \mathbb{Z})$ is a natural transformation of Mackey functors.

2.3. MMM classes (topology). The definition of $e_k(\mu)$ is inspired by a result of Kawazumi and Uemura [8] concerning of characteristic classes of oriented surface bundles. Let Σ_g be the closed oriented surface of genus $g \geq 2$. Let $\pi : E \rightarrow B$ an oriented Σ_g -bundle, $T^\vee E$ the tangent bundle along the fiber of π , and $e \in H^2(E; \mathbb{Z})$ the Euler class of $T^\vee E$. Define $e_k^{\text{top}}(\pi) \in H^{2k}(B; \mathbb{Z})$ by $e_k^{\text{top}}(\pi) := \pi_!(e^{k+1})$ where $\pi_! : H^*(E; \mathbb{Z}) \rightarrow H^{*-2}(B; \mathbb{Z})$ is the Gysin homomorphism (the superscript “top” stands for “topology”). $e_k^{\text{top}}(\pi)$ is called the k -th Mumford-Morita-Miller class of π , as it was introduced in [11, 10, 9].

Now let (G, C) be a surface symmetry as in Section 1.2. Associated with (G, C) , there is an oriented surface bundle $\pi : EG \times_G C \rightarrow BG$ called the *Borel construction*, where $EG \rightarrow BG$ is the universal G -bundle. We denote by $e_k^{\text{top}}(G, C) \in H^{2k}(G, \mathbb{Z})$ the k -th MMM class of the Borel construction π . A result of Kawazumi and Uemura [8] implies the following result:

Theorem 5. We have $e_k^{\text{top}}(G, C) = e_k(\delta(G, C))$ where $\delta(G, C)$ is the ramification data of (G, C) .

3. HOMOLOGY REPRESENTATIONS

3.1. Algebra. In what follows, we denote by $R(G)$ the complex representation ring (or the character ring) of a finite group G . Let $\langle \gamma \rangle$ be a cyclic group of order m generated by γ and $\rho_\gamma : \langle \gamma \rangle \rightarrow \mathbb{C}^\times$ a linear character as in Section 2.1. Define $\Delta_\gamma \in R(\langle \gamma \rangle) \otimes \mathbb{Q}$ by

$$\Delta_\gamma := 2 \sum_{k=1}^{m-1} \left(\frac{k}{m} - \frac{1}{2} \right) \rho_\gamma^{\otimes k} = \frac{2}{m} \sum_{k=1}^{m-1} k \rho_\gamma^{\otimes k} - r_{\langle \gamma \rangle} + 1_{\langle \gamma \rangle},$$

where $r_{\langle \gamma \rangle}$ is the regular representation and $1_{\langle \gamma \rangle}$ is the trivial 1-dimensional representation of $\langle \gamma \rangle$. Now, for each element $\mu = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle$ of $\mathcal{M}_G$, define the G -signature $\sigma(\mu)$ of μ by

$$\sigma(\mu) := \sum_{k=1}^q \text{Ind}_{\langle \gamma_k \rangle}^G (\Delta_{\gamma_k}) \in R(G) \otimes \mathbb{Q}.$$

Proposition 6. $\sigma(\mu) \in R(G)$ for every G and $\mu \in \mathcal{M}_G$.

See the next section for the proof. Note that, in case $\mu \in \mathcal{M}_G$ consists of a single conjugacy class ($\mu = \langle \hat{\gamma} \rangle$ for $\gamma \in [G, G]$), Proposition 6 was proved by T. Yoshida [13].

The assignment $\mu \mapsto \sigma(\mu)$ yields a homomorphism $\mathcal{M}_G \rightarrow R(G)$ of monoids, which induces a well-defined homomorphism $\mathcal{A}_G \rightarrow R(G)$ of abelian groups. In addition, we can prove the following proposition:

Proposition 7. $\mathcal{A}_G \rightarrow R(G)$ is a natural transformation of Mackey functors.

3.2. Topology. Let (G, C) be a surface symmetry, and H_C the space of holomorphic 1-forms on C . Note that $\dim_{\mathbb{C}} H_C = g$ where g is the genus of the Riemann surface C . Then G acts on H_C and hence H_C is a complex representation of G . A virtual representation $\sigma^{\text{top}}(G, C) := H_C - \overline{H}_C \in R(G)$ is called the G -signature of (G, C) , where $\overline{H}_C$ is the complex conjugate.

Proposition 8. We have $\sigma^{\text{top}}(G, C) = \sigma(\delta(G, C))$ where $\delta(G, C)$ is the ramification data of (G, C) .

The character of $\sigma^{\text{top}}(G, C)$ is given by the Eichler trace formula (see [4] for instance). The proposition can be verified by comparing characters of $\sigma^{\text{top}}(G, C)$ and $\sigma(\delta(G, C))$. An alternative proof was given by N. Kawazumi (unpublished manuscript). Since every $\mu \in \mathcal{M}_G$ can be realized as a ramification data of a surface symmetry, Proposition 6 follows from the last proposition. The following fact is an easy consequence of Proposition 8.

Corollary 9. If all the complex characters of G are $\mathbb{R}$ -valued, then $\sigma(\mu) = 0$ for all $\mu \in \mathcal{M}_G$.

Proof. Choose a surface symmetry (G, C) with $\delta(G, C) = \mu$. Then we have $\sigma(\mu) = \sigma^{\text{top}}(G, C) = H_C - \overline{H}_C = 0$ since $H_C = \overline{H}_C$ by the assumption. $\square$

4. A RELATION OF $e_k(\mu)$ AND $\sigma(\mu)$

Theorem 10. Let G be a finite group and $\mu, \nu \in \mathcal{M}_G$.

- (1) If $\sigma(\mu) = \sigma(\nu)$ then $e_k(\mu) = e_k(\nu)$ for all odd $k \geq 1$.
- (2) If $\sigma(\mu) = 0$ then $e_k(\mu) = 0$ for all odd $k \geq 1$.

Since $R(G)$ is free as an abelian group, the homomorphism $\mathcal{A}_G \rightarrow R(G)$ in Section 3.1 induces $\phi_1 : \mathcal{A}_G / \text{Tor}(\mathcal{A}_G) \rightarrow R(G)$, where $\text{Tor}(\mathcal{A}_G)$ is the torsion subgroup of $\mathcal{A}_G$. For odd $k \geq 1$, let $\phi_2 : \text{Tor}(\mathcal{A}_G) \rightarrow H^{2k}(G, \mathbb{Z})$ be the restriction of the homomorphism $\mathcal{A}_G \rightarrow H^{2k}(G, \mathbb{Z})$ in Section 2.2. The proof of Theorem 10 is based on the following two facts:

Theorem 11. For any finite group G and any odd $k \geq 1$,

- (1) The homomorphism $\phi_1 : \mathcal{A}_G / \text{Tor}(\mathcal{A}_G) \rightarrow R(G)$ is injective.
- (2) The homomorphism $\phi_2 : \text{Tor}(\mathcal{A}_G) \rightarrow H^{2k}(G, \mathbb{Z})$ is trivial.

The first statement is proved by using a result of Edmonds and Ewing [3], while the second statement is proved by considering the cohomology of metacyclic 2-groups. The detail will appear elsewhere. Theorem 10 and Corollary 9 imply the following corollary:

Corollary 12. *If all the complex characters of G are $\mathbb{R}$ -valued, then $e_k(\mu) = 0$ for all $\mu \in \mathcal{M}_G$ and odd $k \geq 1$.*

Define $\mathcal{R}_G$ to be the image of $\phi_1 : \mathcal{A}_G / \text{Tor}(\mathcal{A}_G) \rightarrow R(G)$. In view of Theorem 11, there exists a series of homomorphisms $\Phi_k : \mathcal{R}_G \rightarrow H^{2k}(G, \mathbb{Z})$ (k odd) which assigns $e_k(\mu)$ to $\sigma(\mu)$. Let $c : \text{Hom}(G, \mathbb{C}^\times) \rightarrow H^2(G, \mathbb{Z})$ be the natural isomorphism as in Section 2.1 and $\det : R(G) \rightarrow \text{Hom}(G, \mathbb{C}^\times)$ the determinant homomorphism (see [13] for precise). Then the homomorphism Φ_1 is determined by the following proposition:

Proposition 13. $e_1(\mu) = 6 \cdot c(\det(\sigma(\mu)))$ for all $\mu \in \mathcal{M}_G$.

The proposition follows from the Grothendieck-Riemann-Roch theorem and a result of Harer [7] (see also [1, Proposition 6]). Proposition 13 can be generalized to larger k , provided G is cyclic. Recall that, for every finite group G , there is a series of homomorphisms $s_k : R(G) \rightarrow H^{2k}(G, \mathbb{Z})$ ($k \geq 0$) of abelian groups, which satisfies the following properties:

- (1) $s_1(\rho) = c(\det \rho)$ for all $\rho \in R(G)$.
- (2) If ρ is a linear character, then $s_k(\rho) = c(\rho)^k$.

$s_k(\rho)$ is called the k -th Newton class of $\rho \in R(G)$. See [12] for further details. Let B_{2k} be the $2k$ -th Bernoulli number and N_{2k}, D_{2k} coprime integers satisfying $B_{2k}/k = N_{2k}/D_{2k}$. Then a result of the author and Kawazumi [2] implies the following result:

Theorem 14. *If G is cyclic, then $N_{2k} \cdot e_{2k-1}(\mu) = D_{2k} \cdot s_{2k-1}(\sigma(\mu))$ holds for all $\mu \in \mathcal{M}_G$ and $k \geq 1$.*

Now let G be a cyclic group of order m , and suppose that N_{2k} is prime to m . Choose an integer N_{2k}^* satisfying $N_{2k} \cdot N_{2k}^* \equiv 1 \pmod{m}$. Under these assumptions, we have

$$e_{2k-1}(\mu) = N_{2k}^* D_{2k} \cdot s_{2k-1}(\sigma(\mu))$$

for all $\mu \in \mathcal{M}_G$, and hence determining Φ_{2k-1} for these cases. In particular, we have $e_1(\mu) = 6 \cdot s_1(\sigma(\mu))$, $e_3(\mu) = -60 \cdot s_3(\sigma(\mu))$, $e_5(\mu) = 126 \cdot s_5(\sigma(\mu))$, $e_7(\mu) = -120 \cdot s_7(\sigma(\mu))$ for any cyclic group G and $\mu \in \mathcal{M}_G$, since $N_{2k} = 1$ for $1 \leq k \leq 4$.

REFERENCES

1. T. Akita, Nilpotency and triviality of mod p Morita-Mumford classes of mapping class groups of surfaces, *Nagoya Math. J.* **165** (2002) 1-22.
2. T. Akita, N. Kawazumi, Integral Riemann-Roch formulae for cyclic subgroups of mapping class groups, *Math. Proc. Cambridge Phil. Soc.* (to appear).
3. A. L. Edmonds, J. H. Ewing, Remarks on the cobordism group of surface diffeomorphisms, *Math. Ann.* **259** (1982) 497-504.
4. H. M. Farkas, I. Kra, *Riemann Surfaces* (2nd ed.), GTM Vol. 71, Springer-Verlag, New York-Heidelberg-Berlin, 1992.
5. R. Grieder, Geometric representation theory and G -signature, arXiv:math/9811102.
6. ———, G -actions on Riemann surfaces and the associated group of singular orbit data, arXiv:math/9902048.
7. J. L. Harer, The second homology group of the mapping class group of an oriented surface, *Invent. Math.* **72** (1983) 221-239.
8. N. Kawazumi, T. Uemura, Riemann-Hurwitz formula for Morita-Mumford classes and surface symmetries, *Kodai Math. J.* **21** (1998) 372-380.
9. E. Y. Miller, The homology of the mapping class group, *J. Differential Geom.* **24** (1986) 1-14.
10. S. Morita, Characteristic classes of surface bundles, *Invent. Math.* **90** (1987) 551-577.
11. D. Mumford, Towards an enumerative geometry of the moduli space of curves, in *Arithmetic and Geometry* Vol. II, Birkhäuser, Boston (1983) 271-328.
12. C. B. Thomas, *Characteristic Classes and Cohomology of Finite Groups*, Cambridge Studies in Advanced Mathematics 9, Cambridge University Press, Cambridge, 1986.
13. T. Yoshida, Character-theoretic transfer (II), *J. Algebra* **118** (1988) 498-527.

DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO, 060-0810 JAPAN

E-mail address: akita@math.sci.hokudai.ac.jp