Azimuthal-Velocity-Prebunching Effects on Electron-Wave Interactions in CEF-Type Devices

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Abstract

This paper is concerned with the azimuthal-velocity-prebunching effect on electron-wave interactions in CEF-type devices in a special case where $b=0$, $d=0$, $Q=0$, $\beta_e \approx 20$, and $C \approx 0.05$.

The small-signal forward-wave output power was given by

$$P(\theta) = \frac{1}{2} G_{\rm eq} \, |r_0^2 \Omega_0 \Omega_1(0)/\eta|^2 ,$$

and the equivalent conductance $G_{\rm eq}$ was yielded by

$$G_{\rm eq} = \frac{9}{8} \frac{1}{k_c^2 \beta_e^4} \frac{1}{K} \left[2 \left(\cosh \frac{1}{2} \sqrt{\beta_e C}\, \phi - \cos \frac{\sqrt{2}}{\beta_e C} \phi \right)^2 + \sin^2 \frac{\sqrt{2}}{\beta_e C} \phi \right] .$$

The term of $\cosh(\sqrt{\beta_e C}\,\phi/2)$ is due to the growing and decreasing waves in the device and the terms of $\cos(\sqrt{2}\,\phi/\beta_e C)$ and $\sin(\sqrt{2}\,\phi/\beta_e C)$ are due to the characteristic ripple in the CEF-type focusing systems. The ripple effect on the equivalent conductance in the azimuthal-velocity-prebunching case is larger than that on the equivalent resistance in the radial-current prebunching case.

Contents

1. Introduction

Recently, Nunn and Rowe[1),2)] have presented analyses of the centrifugal-electrostatic focusing or CEF-type devices, in which the centrifugal force of the electrons in the interaction region is balanced by an equal and opposite radial electric field, as shown in Fig. 1. More recently, Sakuraba and Rowe[3),4),5)] have shown that certain types of photodemodulators may make effective use of CEF-type electron beams. A review of the literature reveals that the prebunching effects in CEF-type systems have received relatively little analytical attention[6),7)].

This paper deals with the azimuthal-velocity-prebunching effect on electron-wave interactions in CEF-type forward-wave devices. The derivation will be based on Sakuraba and Rowe's analysis for forward- and backward-wave CEF-type devices, which treats the electron- wave interaction problem from a viewpoint of the equivalent circuit analysis. The equivalent conductance of demodulating output will be expressed in terms of the growing wave, the decreasing wave and the characteristic rippling.

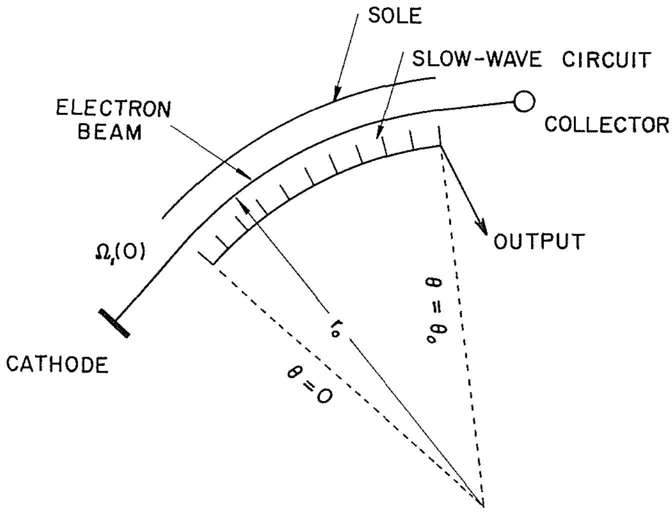

Fig. 1. Model for analysis of CEF-type forward-wave devices.

2. The Electronic Equations

The equations of motion for an electron revolving about the center of the CEF-type forward-wave device from the Lagrange function for a single particle

$$\ddot{r} - r\dot{\theta}^2 = -\eta\left[E_r + E(r)\right], \qquad (1)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -\eta E_\theta, \qquad (2)$$

where

E_r = radial r-f electric field intensity,

$E(r)$ = static radial electric field intensity which balances the centrifugal force of the electron at radius r,

E_θ = azimuthal r-f electric field intensity,

η = magnitude of the electron's charge-to-mass ratio.

The dots appearing above the quantities in the foregoing equations signify the total time derivation of these terms. The azimuthal velocity and radial dependence can be expressed in tems of their unperturbed values by means of the relations

$$r = r_0 + r_1, \qquad (3)$$

$$\dot{\theta} = \frac{d\theta}{dt} = \Omega_0 + \Omega_1 , \tag{4}$$

where

r_0 = radius of the center-of-the-beam radius,

r_r = radial perturbation funciton,

Ω_0 = unperturbed spatial angular velocity of an electron at radius r_0,

Ω_1 = azimuthal velocity perturbation function.

It is important to observe that while r_1 and Ω_1 depend on the spatial angle θ, and on time, both of these functions must be independent of radial variations. This condition follows in the wake of the Eulerian fluid hypothesis and prohibits the crossing of electron trajectories. The requirement of zero rate of change of Ω_1 with respect to r results from the Brillouin flow assumption which asserts that the spatial angular velocity of all particles in the electron stream must remain invariant.

The azimuthal velocity and radial perturbation function are thus dependent only on θ and t, thus when employing the usual procedure for partial differentiation, the equations of motion become

$$(\Omega_0+\Omega_1)\left[(\Omega_0+\Omega_1)\frac{\partial^2 r_1}{\partial\theta^2}+\left(\frac{\partial r_1}{\partial\theta}\right)\left(\frac{\partial\Omega_1}{\partial\theta}\right)+\frac{\partial^2 r_1}{\partial\theta\cdot\partial t}\right]+\left[\frac{\partial^2 r_1}{\partial t^2}+\left(\frac{\partial r_1}{\partial\theta}\right)\left(\frac{\partial\Omega_1}{\partial t}\right)\right.$$
$$\left.+(\Omega_0+\Omega_1)\frac{\partial^2 r_1}{\partial\theta\cdot\partial t}\right]-(r_0+r_1)(\Omega_0+\Omega_1)^2=-\eta E_r-\frac{(r_0\Omega_0)^2}{r_0+r_1}, \tag{5}$$

$$2(\Omega_0+\Omega_1)\left[(\Omega_0+\Omega_1)\frac{\partial r_1}{\partial\theta}+\frac{\partial r_1}{\partial t}\right]+(r_0+r_1)\left[(\Omega_0+\Omega_1)\frac{\partial\Omega_1}{\partial\theta}+\frac{\partial\Omega_1}{\partial t}\right]=-\eta E_\theta . \tag{6}$$

The last term of Eq. (5) results from the condition in which the radial electric force at r just balances the centrifugal force of the electron at the same radius, so that the particle possesses a stable motion in the interaction space. This condition may be described mathematically as

$$-\eta E(r) = -\frac{v_{\theta e}^2}{r} = \frac{-\eta(V_c - V_s)}{r\ln(r_s/r_c)}$$
$$= -\frac{(r_0\Omega_0)^2}{(r_0+r_1)} , \tag{7}$$

where

$v_{\theta e}$ = azimuthal component of the equilibrium velocity of the electron at radius r,

V_c = potential of the circuit measured relative to the cathode,

V_s = potential of the sole electrode measured relative to the cathode,

r_c = radius of the circuit,

r_s = radius of the sole.

Equations (5) and (6) may be written in a form for which derivatives do not appear by noting that $\partial f/\partial\theta = -j\beta f$, and $\partial f/\partial t = j\omega f$, where f designates a function of θ and time. The $-j\beta$ and $j\omega$ operators are a consequence of the characteristic variation

$$e^{j(\omega t-\beta_0\theta)}$$

that is found to apply to all field components in a cold circuit. It is important to observe, however, that the cold-circuit circular propagation constant β_0 is replaced by the propagation constant β which holds in the presence of an energy-carrying electron beam.

Subject to these transformations Eqs. (5) and (6) can be written

$$r_1\Big[(\Omega_0+\Omega_1)(\beta\omega-\Omega_0\beta^2-2\Omega_1\beta^2)+\omega(\Omega_0\beta-\omega+2\Omega_1\beta)-2\Omega_0(\Omega_0+\Omega_1)-\Omega_1^2\Big] -\Omega_1 r_0(2\Omega_0+\Omega_1)=-\eta E_r\,, \tag{8}$$

$$2r_1(\Omega_0+\Omega_1)\Big[\omega-\beta(\Omega_0+\Omega_1)\Big]+\Omega_1(r_0+r_1)\Big[\omega-\beta(\Omega_0+\Omega_1)\Big]=j\eta E_\theta\,, \tag{9}$$

where r_1/r_0 is assumed to be very much less than unity.

The azimuthal current equation has the from

$$I_\theta=(\tau_0+\tau_1)(r_0+r_1)(\Omega_0+\Omega_1)=-I_\theta+\tau_\theta\,, \tag{10}$$

where

$$-I_0=\Omega_0 r_0\tau_0\,, \tag{11}$$

$$i_\theta=\Omega_0\tau_0 r_1+r_0\tau_0\Omega_1+\Omega_0 r_0\tau_1+\tau_0 r_1\Omega_1+\Omega_0 r_1\tau_1+r_0\Omega_1\tau_1+r_1\Omega_1\tau_1\,, \tag{12}$$

where

τ_0 = d–c component of ring charge density due to the steady motion of the beam,

τ_1 = a-c component of ring charge density due to the space-charge bunching of the beam.

The ring charge density of the beam may be expressed as

$$\tau=\rho h\sigma \tag{13}$$

where

ρ = volume charge density of the beam,

h = height of the beam parallel to the z-axis,

σ = radial width of the beam.

The radial current equation is given by

$$i_r=(\tau_0+\tau_1)\frac{d}{dt}(r_0+r_1)=(\tau_0+\tau_1)\left[(\Omega_0+\Omega_1)\frac{\partial r_1}{\partial\theta}+\frac{\partial r_1}{\partial t}\right] =jr_1(\omega\tau_0-\beta\Omega_0\tau_0-\beta\tau_0\Omega_1+\omega\tau_1-\beta\Omega_0\tau_1-\beta\Omega_1\tau_1)\,, \tag{14}$$

A comparison of Eqs. (12) and (14) reveals that the azimuthal current has a steady component given by I_0, in addition to the a–c component given by i_θ, whereas the radial current contains only the alternating component. This is a direct consequence of the fact that the steady radial motion of the beam, resulting from interaction with the circuit wave, cannot be accounted for by the present analysis. The steady component of azimuthal current, on the other hand, exists because the electron beam is assumed to enter the interaction region with an initial linear tangential velocity of v_0 at radius r_0. The negative sign accompanying I_0 in Eqs. (10) and (11) is necessitated by the unfortunate notion that a current flow is positive in the direction opposite to that of election movement.

The continuity equation has the well-known from

$$\nabla\cdot(\bar{l}_r i_r+\bar{l}_\theta I_\theta)=-\frac{\partial}{\partial t}(\tau_0+\tau_1)\,, \tag{15}$$

where

$\bar{l}_r$ = unit radial vector,

$\bar{l}_\theta$ = unit azimuthal vector.

In cylindrical coordinates the continuity equation becomes

$$\frac{i_r}{r_0+r_1}+\frac{\partial i_r}{\partial r}+\left(\frac{1}{r_0+r_1}\right)\frac{\partial i_\theta}{\partial\theta}=-\frac{\partial\tau_1}{\partial t}\,, \tag{16}$$

because the time rate of change of the steady component of ring charge density τ_0 is zero. Furthermore, the second term in the left member of the above equation is also zero in view of the requirement that follows from the Eulerian fluid hypothesis. Accordingly, the continuity equation assumes the general form

$$\frac{r_1}{r_0+r_1}\Big[\tau_0(\omega-\beta\Omega_0)+\tau_1(\omega-\beta\Omega_0)-\beta\tau_0\Omega_1-\beta\Omega_1\tau_1\Big]$$
$$-\frac{\beta}{r_0+r_1}\Big[\tau_0 r_1(\Omega_0+2\Omega_1)+\tau_0 r_0\Omega_1+\Omega_0\tau_1(r_0+2r_1)+2r_0\Omega_1\tau_1+3r_1\Omega_1\tau_1\Big]=-\omega\tau_1\,, \tag{17}$$

where the appropriate differential operators, and the substitutions for i_θ and i_r from Eqs. (12) and (14), have been employed.

While the above expressions satisfy the Brillouin flow and the Eulerian fluild hypothesis described earlier, they must be simplfied further in order to keep the mathematical development within reasonable bounds of complexity. This is accomplished through the use of the small-signal assumption which imposes the condition that all a–c quantities are small in proportion to the d–c or steady quantities, and that squares and products of perturbation functions may be neglected in comparison to the first power of these terms. Within the scope of these limitations the equations of motion and the continuity equation take the form

$$r_1\Omega_0^2\Big[(\beta_e-\beta)^2+2\Big]+\Omega_1(2r_0\Omega_0)=\eta E_r\,, \tag{18}$$

$$2r_1\Omega_0^2(\beta_e-\beta)+\Omega_0\Omega_1 r_0(\beta_e-\beta)=j\eta E_\theta\,, \tag{19}$$

$$r_1(\beta_e-2\beta)+\Omega_1\left(\frac{-\beta r_0}{\Omega_0}\right)+\tau_1\left[\frac{r_0(\beta_e-\beta)}{\tau_0}\right]=0\,, \tag{20}$$

where

$$\beta_e = \frac{\omega}{\Omega_0}, \qquad \text{electrical radian/spatial radian.} \tag{21}$$

The quantity β_e is an analogue of the linear electron phase constant, which may be considered as the circular propagation constant of a disturbance traveling with the same azimuthal velocity as the electron stream.

The perturbation functions r_1, Ω_1, and τ_1 are obtained from the solution of Eqs. (18), (19) and (20), by means of Cramer's rule. The determinant of the homogeneous system of these equations Δ_s is found to be

$$\Delta_s = \frac{\Omega_0^3 r_0^2(\beta_e-\beta)^2}{\tau_0}\left[(\beta_s-\beta)^2-2\right]. \tag{22}$$

Following the customary procedure, the perturbation functions are readily computed

$$r_1 = \frac{\eta\Omega_0 r_0^2}{\tau_0\Delta_s}(\beta_e-\beta)\left[E_r(\beta_e-\beta)-j2E_\theta\right], \tag{23}$$

$$\Omega_1 = \frac{\eta\Omega_0^2 r_0}{\tau_0\Delta_s}(\beta_e-\beta)\left[2E_r(\beta-\beta_e)+jE_\theta\left((\beta_e-\beta)^2+2\right)\right], \tag{24}$$

$$\tau_1 = \frac{\eta\Omega_0 r_0}{\Delta_s}(\beta_e-\beta)\left[-E_r\beta_e+jE_\theta\left((\beta_e-\beta)\beta+2\right)\right]. \tag{25}$$

Under the small-signal assumption the a–c components of azimuthal and radial current, given by Eqs. (12) and (13), become

$$i_\theta = \Omega_0\tau_0 r_1 + r_0\tau_0\Omega_1 + \Omega_0 r_0\tau_1\,, \tag{26}$$

$$i_r = j\tau_0\Omega_0 r_1(\beta_e-\beta)\,. \tag{27}$$

Equations (23), (24) and (25) may be used in conjunction with Eqs. (26) and (27) to obtain the azimuthal and radial ballistic equations,

$$i_\theta = \frac{\eta\tau_0}{\Omega_0(\beta_e-\beta)}\left[\frac{-E_r(2\beta_e-\beta)+jE_\theta((\beta_e-\beta)\beta_e+2)}{(\beta_e-\beta)^2-2}\right], \tag{28}$$

$$i_r = \frac{j\eta\tau_0}{\Omega_0}\left[\frac{E_r(\beta_e-\beta)-j2E_\theta}{(\beta_e-\beta)^2-2}\right]. \tag{29}$$

If we apply the small-signal approximation to the continuity equation, the expression for the a–c component of ring charge density τ_1 takes the form

$$\tau_1 = \left[\frac{\beta}{\omega r_0}i_\theta + j\frac{i_r}{\omega r_0}\right]. \tag{30}$$

By substituting Eqs. (28) and (29) into Eq. (30), the ring charge density may be explicity written in terms of the r–f fields

$$\tau_1 = \frac{\eta\tau_0}{\Omega_0^2 r_0(\beta_e-\beta)}\left[\frac{-\beta_e E_r+jE_\theta((\beta_e-\beta)\beta+2)}{(\beta_e-\beta)^2-2}\right]. \tag{31}$$

3. The Forward-Wave Circuit Equation

In a recent book Rowe[8] has treated the problem of a sheet beam passing parallel to the face of a biperiodic structure. Exprtssed in cylindrical coordinates, the general equation for a lossless anisotropic transmission line takes the form

$$\frac{Z_{0,0}}{Z_{0,1}}(v_{0,0}v_{0,1})\frac{\partial^2 V}{\partial z^2}+\left(\frac{v_{0,0}}{r_c}\right)\frac{\partial^2 V}{\partial \theta^2}-\left(1+\frac{v_{0,0}}{v_{0,1}}\frac{Z_{0,0}}{Z_{0,1}}\right)\frac{\partial^2 V}{\partial t^2}$$
$$=-(v_{0,0}Z_{0,0})\frac{\partial^2(k_c\tau_1)}{\partial t^2}, \tag{32}$$

where

- V = r–f voltage on the biperiodic structure, which is a function of z, θ and t,
- k_c = coupling factor between the ribbon beam and the biperiodic circuit,
- $v_{0,0}$ = linear tangential phase velocity of the azimuthal-directed wave at the surface of the r–f circuit,
- $v_{0,1}$ = axial component of the wave phase velocity at the surface of the r–f circuit,
- $Z_{0,0}$ = azimuthal component of the circuit impedance at the surface of the slow-wave structure,
- $Z_{0,1}$ = axial component of the circuit impedance at the surface of the slow-wave structure.

The approximation for E_r/E_θ, evaluated at the radius of the center-of-the-beam electron, is replaced by $-j1.0$, according to the Nunn and Rowe's[1),2)] discussion. The azimuthal r–f electric field intensity E_θ at the center-of-the-beam radius r_0 is related to the scalar r–f voltage V, which is a function of z, θ, and t, by means of the expression

$$E_\theta=-\frac{1}{r_0}\frac{\partial}{\partial\theta}(k_cV)=j\beta k_c\frac{V}{r_0}. \tag{33}$$

The magnitude of the effective interaction impedance K at the center-of-the-beam radius r_0 defined as

$$K=\left|\frac{k_c^2r_cZ_{0,0}}{r_0}\right|. \tag{34}$$

The electromagnetic analysis presented by Nunn and Rowe[1)] may be used to show that

$$v_{0,0}=\omega r_c/\beta_0, \qquad v_{0,1}=\omega/\gamma, \tag{35}$$

$$\beta_c=\omega/\Omega_0, \tag{36}$$

where γ is the axial propagation constant of the r–f wave in the presence and absence of electrons. Therefore, the general circuit equation applicable to forward-wave devices may be written

$$E_\theta=+\frac{j\beta\beta_0\omega K}{\beta_0^2-\beta^2}\tau_1. \tag{37}$$

4. The Determinantal Equation

The basic relations necessary for the derivation of the determinantal equation are the ballistic equation, given by Eq. (31), and the circuit equation obtained above. Turning first to a consideration of Eqs. (31) and (33) and $E_\theta/E_r \approx -j1.0$, the azimuthal r–f electric field intensity must be replaced by the voltage V. Since the circuit fields have been assumed to be excited primarily by the circular motion of the electron beam, the azimuthal r–f electric field intensity at the center of the beam is related to the circuit voltage according to the expression Eq. (33). The ring charge density and circuit voltage can be eliminated from Eqs. (31), (33) and (37) so as to obtain the secular equation

$$1 = \left(\frac{-\eta\tau_0 K}{\Omega_0 r_0}\right)\left(\frac{\beta\beta_0\beta_e}{\beta_0^2-\beta^2}\right)\left(\frac{\beta}{\beta_e-\beta}\right)\left[\frac{\beta(\beta_e-\beta)+\beta_e+2}{\beta(\beta_e-\beta)^2-2\beta}\right]. \tag{38}$$

The quantity C is the gain parameter whose mathematical definition is

$$C^3 \equiv \frac{-\eta\tau_0 K}{\Omega_0 r_0} = \frac{KI_0}{2V_0}, \tag{39}$$

in which K is the magnitude of the effective azimuthal interaction impedance prescribed by Eq. (34). Therefore the secular equation may be written

$$\frac{C^3\beta^2\beta_0\beta_e}{(\beta_0^2-\beta^2)(\beta_e-\beta)}\left[\frac{\beta(\beta_e-\beta)+\beta_e+2}{\beta(\beta_e-\beta)^2-2\beta}\right] = 1 . \tag{40}$$

It is appropriate at this point to introduce the incremental propagation constants following the method of Nunn and Rowe[1]. Thus, the expression for the circular propagation constants in the presence and in the absence of the electron beam are defined as

$$\beta = \beta_e(1+jC\delta), \qquad \beta_0 = \beta_e, \tag{41}$$

where

δ = incremental propagation constant (mathematically defined below). It was assumed that the spatial angular velocities of the circuit wave and the electron beam at radius r_0 are in synchronism and the loss parameter is zero. The incremental propagation constant δ is defined as

$$\delta = x + jy , \tag{42}$$

where

x = real part of the incremental propagation constant,

y = imaginary part of the incremental propagation constant.

It follows that

$$e^{-j\beta\theta} = e^{-j\beta_e\theta(1-Cy)}e^{\beta_e C\theta x} . \tag{43}$$

An examination of this expression shows that a positive value of x yields an exponentially increasing function of the spatial angle θ, associated with growing waves. A negative value of x is associated with waves whose amplittudes exponentially decay with increasing spatial angle, while a zero value of x indicates the presence of unattenuated waves. Upon setting x epual to zero, the remaining exponential in Eq. (43) leads to

$$\frac{\omega}{\Omega_{W1}} = \frac{\omega}{\Omega_0}(1-Cy), \tag{44}$$

where Ω_{W1} is the spatial angular velocity of r–f wave in the presence of the beam. This relation follows directly from Eq. (36) and from the fact that the circular propagation constant in the presence of the beam is equal to the left member of the above equation. Inasmuch as $Cy \ll 1$, the spatial angular velocity of the wave in the presence of the beam becomes

$$\Omega_{W1} = \Omega_0(1+Cy). \tag{45}$$

It is apparent from a study of this expression that $y>0$ for a wave whose spatial angular velocity exceeds that of an electron at radius r_0, while $y<0$ for a wave whose spatial angular velocity is less than that of an electron at radius r. This condition in which y is exactly equal to zero describes the case of synchronism between the wave and electron spatial angular velocities at radius r_0. The expressions for the circular propagation constants, given by Eqs. (41) and (42), may now be substituted into Eq. (40) to obtain,

$$(1+jC\delta)\Big[(-j\beta_e^2C^3)\delta^5+(-2\beta_e^2C^2)\delta^4+jC(\beta_e^2C^3-2)\delta^3$$
$$+2(\delta_e^2C^3-2)\delta^2+jC^2(-\beta_e^2+\beta_e+2)\delta+C(\beta_e+2)\Big]=0\ . \tag{46}$$

This is the general determinantal equation for CEF-type traveling-wave devices in the case where the spatial angular velocities of the circuit wave and the electron beam at radius r_0 are in synchronism and the space charge parameter is zero.

An extremely wide range of digital computer results given by Nunn and Rowe[1] have shown that, under all circumstances applicable to both forward- and backward-wave devices, the two incremental propagation constants are accurately given by

$$\delta_6 = j2/C, \tag{47}$$

$$\delta_5 = j/C. \tag{48}$$

Equation (47) shows that the wave associated with the incremental propagation constant δ_6 possesses an azimuthal phase delay that numerically increases in the direction opposite to the remaining four waves exhibiting azimuthal propagation. This wave, which arises on the r–f circuit from the charge-induction fields of the electron beam, is identified as the backward wave. It is especially cautioned not to regard the source of this wave as arising from reflections occuring at a mismatched output transducer. The wave actually results from the negative azimuthal propagation predicted by the second solution of the one dimensional wave equation. It therefore appears regardless of the state of the output match, although reflections may contribute additional components. Equation (48) implies that the wave associated with the incremental propagation constant δ_5 possesses no azimuthally-oriented propagation and is thus constrained to exhibit only radial motion, if any.

Although Eq. (46) is the general expression for the small-signal equation, its significance can be more readily appreciated through the introduction of appropriate simplifications. In particular, it will be assumed that

$$\beta_e \approx 20\,, \quad \text{and} \quad C \approx 0.05\,.$$

Upon making the assumption the expressions for incremental propagation constants take the appropriate form

$$\delta_{1,2} \approx \pm \frac{1}{2}\sqrt{\beta_e C} - j\frac{3}{4\beta_e^2 C}\,, \tag{49}$$

$$\delta_{3,4} \approx j\left(\frac{\sqrt{2}}{\beta_e C} \mp j\frac{3}{4\beta_e^2 C}\right). \tag{50}$$

The real part of the incremental propagation constant x_1, which is greater than zero, is associated with the wave that exponentially increases with increasing spatial angle, while x_2 is associated with the exponentially decreasing wave. The imaginary parts of the incremental propagation constant y_1, and y_2 possess the same value. Inasmuch as Eq. (45) reveals that y is positive for a wave whose spatial angular velocity is greater than the unperturbed velocity of the electron stream, and negative for a wave whose spatial angular velocity is less than that of the beam, it follows that y_1 and y_2 are slow space-charge waves. By the same reasoning, y_3 is a fast and y_4 is a slow space-charge wave. The δ_1 and δ_2 waves are due to the interaction between a synchronous space-charge wave and a forward wave. The δ_3 and δ_4 waves are due to the characteristic ripple in the CEF-type focusing system. The electron motion in radial and linear tangential perturbations is a combination of circular motion with an angular velocity Ω_0 and a simple harmonic motion in a radial direction of angular frequency $\sqrt{2}\,\Omega_0$. The spatial angle corresponding to one period in radial and linear tangential perturbing influences is the characteristic rippling angle. The circular propagation constant for the characteristic ripple, which is the ratio of angular frequency of simple harmonic motion in the radial direction to angular velocity of circular motion, is equal to $\sqrt{2}$.

5. The Demodulating Output and the Equivalent Conductance

In order to investigate the demodulating characteristics of the CEF-type devices it is necessary to determine the initial wave amplitude set up at the input boundary. Since the solution of the determinantal equation involves six roots, each of which may possess real and imaginary parts, a total of six waves are excited by an energy-carrying electron beam. Only four of these waves are found to be important, however, as a careful examination of the numerical results indicates.

It is therefore essential to prescribe only four independent conditions which must be satisfied at the input. These are:

1. The total circuit voltage applied is zero at the input.
2. The total azimuthal convection current is zero at the input.
3. The radial current is zero at the input.
4. The total azimuthal velocity variation of the electron beam applied at the input is equal to the sum of the azimuthal velocity variation associated with each wave.

Stated mathematically, these conditions become at $\theta=0$ and $r=r_0$

$$\sum_{n=1}^{4} V_n = 0\,, \tag{51}$$

$$\sum_{n=1}^{4} i_{\theta n} = 0\,, \tag{52}$$

$$\sum_{n=1}^{4} i_{rn} = 0\,, \tag{53}$$

$$\sum_{n=1}^{4} \Omega_{1n} = \Omega_1(0)\,. \tag{54}$$

where

V_n = the amplitude of the circuit voltage of the nth wave at $\theta=0$,

$i_{\theta n}$ = the amplitude of the azimuthal convection current of the nth wave at $\theta=0$ and $r=r_0$,

i_{rn} = the amplitude of the radial current of the nth wave at $\theta=0$ and $r=r_0$,

Ω_{1n} = the amplitude of the azimuthal velocity perturbation of the nth wave at $\theta=0$ and $r=r_0$,

$\Omega_1(0)$ = total azimuthal velocity perturbation applied to the electron beam at $\theta=0$ and $r=r_0$.

The general form of the nth component of the above perturbation expressions can be obtained by substituting Eq. (41) into Eqs. (24), (27) and (28), thus yielding

$$C_{\theta n} V_n = i_{\theta n}\left(\frac{-r_0\Omega_0}{\eta\tau_0 k_c\beta_e}\right), \tag{55}$$

$$C_{rn} V_n = i_{rn}\left(\frac{jr_0\Omega_0}{\eta\tau_0 k_c\beta_e^2}\right), \tag{56}$$

$$C_{\Omega n} V_n = \Omega_{1n}\left(\frac{-r_0^2\Omega_0}{\eta k_c\beta_e^2}\right), \tag{57}$$

where the coefficients $C_{\theta n}$, C_{rn} and $C_{\Omega n}$ are given by

$$C_{\theta n} = \frac{(1+jC\delta_n)}{j\beta_e C\delta_n}\left[\frac{\beta_e(1-jC\delta_n)-j\beta_e^2 C\delta_n+2}{(\beta_e C\delta_n)^2+2}\right], \tag{58}$$

$$C_{rn} = \frac{(1+jC\delta_n)}{\beta_e}\left[\frac{j\beta_e C\delta_n-2}{-(\beta_e C\delta_n)^2-2}\right], \tag{59}$$

$$C_{\Omega n} = \frac{(1+jC\delta_n)}{j\beta_e^2 C\delta_n}\left[\frac{-j2\beta_e C\delta_n-(\beta_e C\delta_n)^2+2}{(\beta_e C\delta_n)^2+2}\right]. \tag{60}$$

The input boundary conditions, given by Eqs. (51)–(54), can now be expressed using coefficients in the compact from of matrix algebra

$$\begin{Vmatrix} 1 & 1 & 1 & 1 \\ C_{\theta 1} & C_{\theta 2} & C_{\theta 3} & C_{\theta 4} \\ C_{r1} & C_{r2} & C_{r3} & C_{r4} \\ C_{\Omega 1} & C_{\Omega 2} & C_{\Omega 3} & C_{\Omega 4} \end{Vmatrix} \cdot \begin{Vmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ A_v \Omega_1(0) \end{Vmatrix}, \tag{61}$$

where

$$A_v = \frac{-r_0^2 \Omega_0}{\eta k_c \beta_e^2}. \tag{62}$$

The inverse of the C-matrix leads to the desired set of initial wave amplitude. Thus,

$$\begin{Vmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{Vmatrix} = \begin{Vmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{Vmatrix} \cdot \begin{Vmatrix} 0 \\ 0 \\ 0 \\ A_v \Omega_1(0) \end{Vmatrix} = \begin{Vmatrix} A_v & D_{14} & \Omega_1(0) \\ A_v & D_{24} & \Omega_1(0) \\ A_v & D_{34} & \Omega_1(0) \\ A_v & D_{44} & \Omega_1(0) \end{Vmatrix}. \tag{63}$$

The D-matrix is the symbolic from of the desired inverse of the C-matrix, while the elements D_{14}, D_{24}, D_{34} and D_{44} are the initial wave amplitudes appropriate to the stipulated boundary conditions. The real and imaginary parts of the D-elements are given by

$$D_{n4} = u_{n4} + j v_{n4}, \tag{64}$$

where

u_{n4} = real part of the initial amplitude of the nth wave,

v_{n4} = imaginary part of the initial amplitude of the nth wave.

In the case where $\beta_e \approx 20$ and $C \approx 0.05$, the expressions for the coefficients $C_{\theta n}$, C_{rn}, $C_{\Omega n}$, u_{n4} and v_{n4} take the approximate form

$$C_{\theta 1} \approx \frac{-\beta_e C \sqrt{\beta_e C} - j2}{2C\sqrt{\beta_e C}}, \tag{65}$$

$$C_{\theta 2} \approx \frac{-\beta_e C \sqrt{\beta_e C} + j2}{2C\sqrt{\beta_e C}}, \tag{66}$$

$$C_{\theta 3} \approx \frac{-\beta_e^2 (1+\sqrt{2})}{3}, \tag{67}$$

$$C_{\theta 4} \approx \frac{-\beta_e^2 (1-\sqrt{2})}{3}, \tag{68}$$

$$C_{r1} \approx \frac{4\sqrt{\beta_e C} - j\beta_e^2 C^2}{4\beta_e \sqrt{\beta_e C}}, \tag{69}$$

$$C_{r2} \approx \frac{4\sqrt{\beta_e C} + j\beta_e^2 C^2}{4\beta_e \sqrt{\beta_e C}}, \tag{70}$$

$$C_{r3} \approx \frac{2(1+\sqrt{2})}{3}, \tag{71}$$

$$C_{r4} \approx \frac{2(1-\sqrt{2})}{3}, \tag{72}$$

$$C_{\Omega 1} \approx \frac{-\beta_e^2 C^2 - j2\sqrt{\beta_e C}}{\beta_e^3 C^2}, \tag{73}$$

$$C_{\Omega 2} \approx \frac{-\beta_e^2 C^2 + j2\sqrt{\beta_e C}}{\beta_e^3 C^2}, \tag{74}$$

$$C_{\Omega 3} \approx \frac{-4(1+\sqrt{2})}{3\sqrt{2}}, \tag{75}$$

$$C_{\Omega 4} \approx \frac{4(1-\sqrt{2})}{3\sqrt{2}}, \tag{76}$$

$$u_{14} \approx \frac{3}{4}, \tag{77}$$

$$v_{14} = 0, \tag{78}$$

$$u_{24} \approx \frac{3}{4}, \tag{79}$$

$$v_{24} = 0, \tag{80}$$

$$u_{34} \approx \frac{-3(2-\sqrt{2})}{8}, \tag{81}$$

$$v_{34} = 0, \tag{82}$$

$$u_{44} \approx \frac{-3(2+\sqrt{2})}{8}, \tag{83}$$

$$v_{44} = 0. \tag{84}$$

The total voltage of the r–f circuit becomes at $\theta=\theta$

$$V_R(\theta) = \sum_{n=1}^{4} V_n e^{-j\beta_n \theta}, \tag{85}$$

where

$$V_n = A_v D_{n4} \Omega_1(0), \qquad n = 1, 2, 3, 4. \tag{86}$$

Upon substituting Eq. (86) into Eq. (85) it follows that

$$V_R(\theta) = A_v \left(\sum_{n=1}^{4} D_{n4} e^{\phi\delta_n} \right) \Omega_1(0) e^{-j2\pi N}, \tag{87}$$

where

$$\phi = \beta_e C \theta = 2\pi C N. \tag{88}$$

The quantity N specifies the number of "wave angle" contained in a total spatial angle of θ radians. The electrical angle ϕ is proportional to the total number of wave angles along the r-f structure and to the gain parameter. Finally, the voltage of the circuit at $\theta=\theta$ may be used to obtain the demodulating output of the CEF-type devices by means of the formula

$$P(\theta)=\frac{1}{2K}|V_R(\theta)|^2=\frac{1}{2K}|A_v D_v \Omega_1(0)|^2$$
$$=\frac{|D_v|^2}{2Kk_e^2\beta_e^4}\left|\frac{r_0^2\Omega_0\Omega_1(0)}{\eta}\right|^2, \tag{89}$$

where K is the effective azimuthal interaction impedance for forward mode. It should be noted that the quantity $r_0^2\Omega_0\Omega_1(0)/\eta$ is analogous to an a-c voltage. It will be convenient to define the equivalent conductance as

$$G_{\text{eq}}\equiv\frac{|D_v|^2}{Kk_e^2\beta_e^4}, \tag{90}$$

Using this definition, Eq. (89) becomes

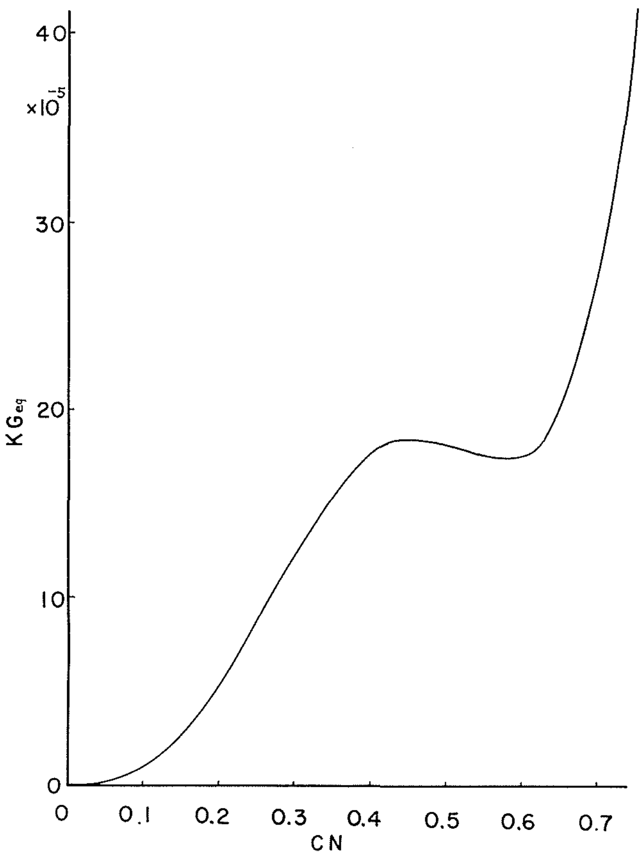

Fig. 2. KG_{eq} vs. CN in the case where $d=0$, $b=0$, $Q=0$, $\beta_e\approx 20$ and $C\approx 0.05$.

$$P(\theta)=\frac{1}{2}G_{\mathrm{eq}}\left|\frac{r_0^2\Omega_0\Omega_1(0)}{\eta}\right|^2. \tag{91}$$

In the reference condition in which $\beta_e \approx 20$ and $C \approx 0.05$, the equivalent conductance can be written by

$$G_{\mathrm{eq}}=\frac{9}{8}\frac{1}{k_c^2\beta_e^4 K}\left[2\left(\cosh\frac{1}{2}\sqrt{\beta_e C}\,\phi-\cos\frac{\sqrt{2}}{\beta_e C}\phi\right)^2+\sin^2\frac{\sqrt{2}}{\beta_e C}\phi\right]. \tag{92}$$

The term of cosh $(\sqrt{\beta_e C}\,\phi/2)$ is due to the growing and decreasing waves in the device and the terms of $\cos(\sqrt{2}\,\phi/\beta_e C)$ and $\sin(\sqrt{2}\,\phi/\beta_e C)$ are due to the characteristic ripple in the CEF-type focusing system. The equivalent conductance in the case where $\beta_e \approx 20$, $C \approx 0.05$ and $k_c = 0.8$ is shown in Fig. 2. A study of Fig. 2 shows that the conductance possesses a dependency on the waves associated with incremental propagation constants δ_3 and δ_4. The total voltage, which is the sum of four components δ_1, δ_2, δ_3 and δ_4, must be zero at $\theta=0$ and $r=r_0$ where

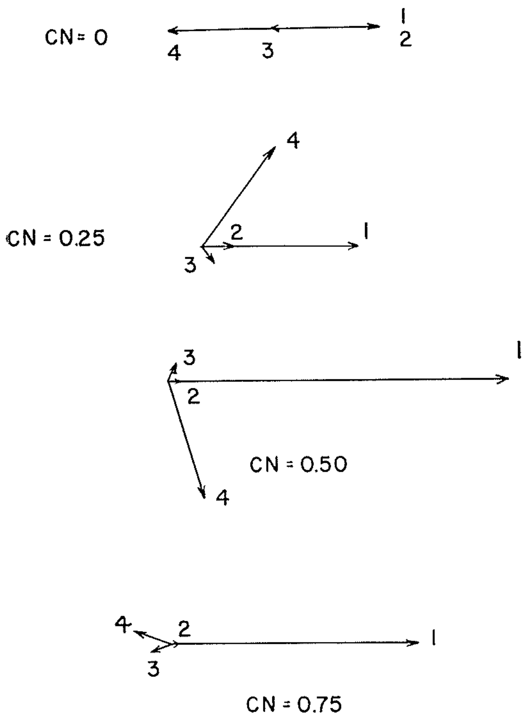

Fig. 3. Relative amounts of the four wave components of circuit voltage at various distances down the device. Values at $CN=0.75$ are drawn on a scale of one to four.

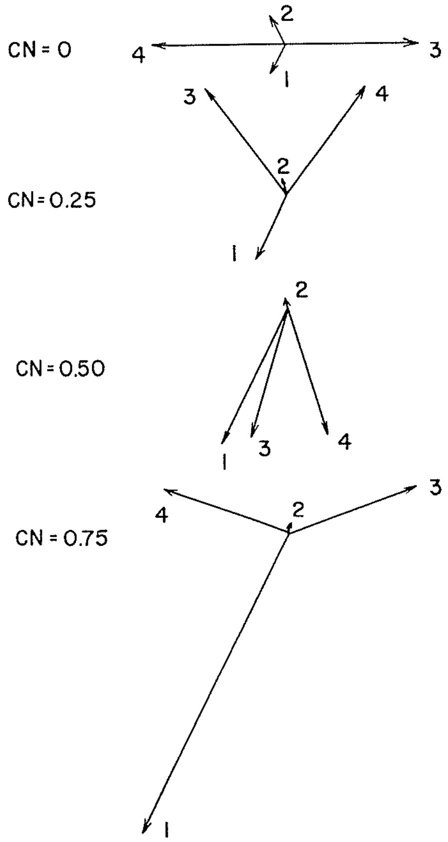

Fig. 4. Relative amounts of four wave components of azimuthal current at various distances down the devive.

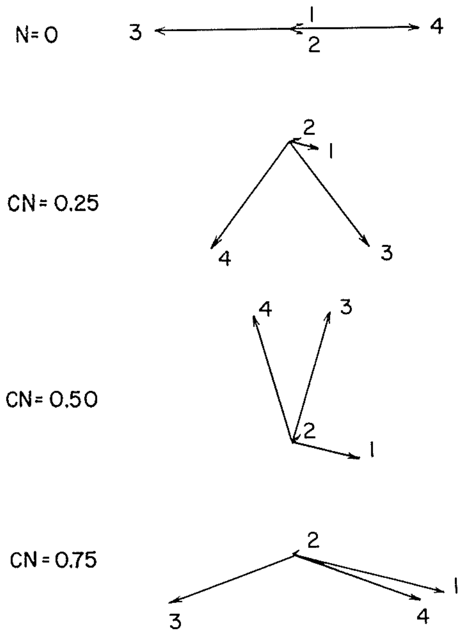

Fig. 5. Relative amounts of four wave components of radial current at various distances down the device.

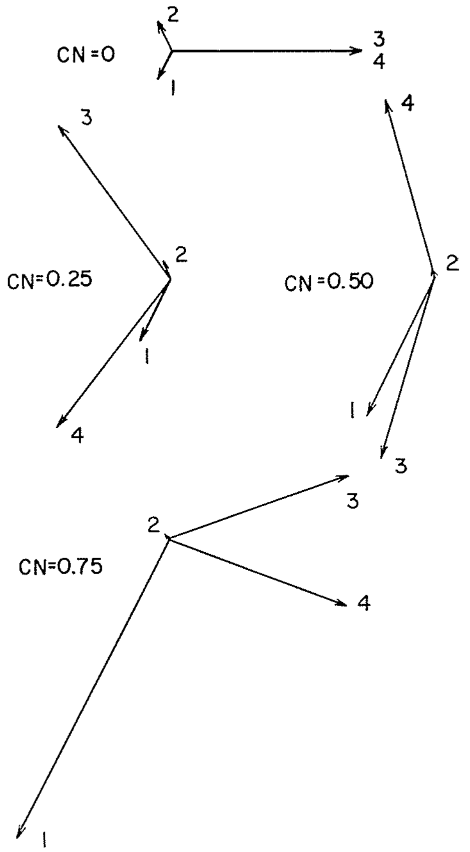

Fig. 6. Relative amounts of four wave components of azimuthal velocity at various distances down the device.

the d–c stream enters. The total azimuthal convection current and the radial current are zero $\theta=0$ and $r=r_0$. On the otherhand, the total azimuthal velocity variation of the electron-beam applied at the input is equal to the azimuthal velocity variation associated with each wave. Each components of the circuit voltage, the total azimuthal convection current, the radial current or the total azimuthal velocity variation is modified by the factor $\exp[-j\beta_e\theta(1-Cy)]\ \exp[\beta_e C\theta x]$ as it progresses down the interaction region. With a knowledge of how much of such a wave was started, it is a simple matter to find the variation of components with space angle in Figs. 3, 4, 5 and 6. The actual field, current, or velocity is the sum of four components.

6. Conclusions

The expression for the demodulating power output and the equivalent conductance was given in the CEF-type device in a special case where $b=0$, $d=0$, $Q=0$, $\beta_e\approx 20$, $C\approx 0.05$ and azimuthal velocity variation was applied at the input.

The equivalent conductance is due to growing and decreasing waves and the characteristic ripple in CEF-type focusing system. The ripple effect on the equivalent conductance in the azimuthal-velocity-prebunching case is larger than that on the equivalent resistance in the radial-current-prebunching case.

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