On the Yield-line Analysis of Orthotropically Reinforced Concrete Slabs with Variable Thickness

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Abstract Johansen's yield-line theory is extended to be applicable to the slabs with variable resisting moments. Some examples of the solution for rectangular slabs with linearly varying thickness are shown.

1. Introduction

Nowadays in our country an elastic analysis is exclusively used in the design of concrete structures. While this method is valid for the calculation of serviceability limit states, the reasonable factor of safety against failure of statically indeterminate structures can only be estimated by the analysis considering plastic deformation. Yield-line theory[1,2,3] systematized by K. W. Johansen is one of the plastic theories which gives an upper bound solution for the collapse load of reinforced concrete slabs. However it is generally recognized that the actual strength of slabs is greater than that calculated by the yield-line theory because of additional resistances which are neglected in the theory. Therefore it is useful for the practical design of slabs as far as adequate mechanisms are postulated.

In this paper Johansen's yield line theory is developed to be applicable to the slabs with variable thickness and solutions for the collapse load of rectangular slabs with linearly varying thickness for various edge conditions are shown.

2. Assumptions

In the following analysis a so-called stepped yield criterion is used. Effects of membrane action and kinking of the reinforcement are neglected. The increase of moment after yielding due to strain hardening of reinforcing bars and rising of neutral axis is also neglected. Only plastic rotation along yield-lines are considered as deformation. Therefore plastic regions are assumed to be formed of straight lines (yield lines) or their combination (e. g. fan mechanism). Rotation capacity of any section is assumed to be infinite.

3. Analysis by Work Method

In the slab with variable thickness yield moment is not constant even when the reinforcing bars are arranged at regular intervals. In the slab reinforced in the directions of x and y yield moments are expressed by

$$\begin{aligned} m_x(x, y) &= m_0 g_x(x, y) \\ m_y(x, y) &= m_0 g_y(x, y) \end{aligned} \tag{1}$$

where m_0 represents an arbitrarily selected standard value of yield moment. Then the normal moment per unit length for a particular yield line, $y=k_jx+l_j$, making an angle ϕ_j with y axis is determined as follows;

$$m_{n,j}(x) = m_0 g_{n,j}(x, \lambda_i) \tag{2}$$

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where

$$g_{n,j}(x, \lambda_i) = g_x(x, k_jx+l_j)\cos^2\phi_j + g_y(x, k_jx+l_j)\sin^2\phi_j \tag{3}$$

and λ_i, $i=1, 2, \ldots$, is the unknown to define the yield-line pattern postulated.

When a virtual displacement rate $\dot{\delta}_0$ is assumed at a certain point in the slab, the corresponding displacement rate at any position and the corresponding rotation rate along any yield-line are determinable.

$$\dot{\delta}(x, y) = \dot{\delta}_0 h(x, y, \lambda_i) \tag{4}$$

$$\dot{\theta}_{n,i}(x) = \dot{\delta}_0 q_j(x, \lambda_i) \tag{5}$$

The expenditure of energy by the external load, $w=w_0p(x, y)$, is then

$$\iint w\dot{\delta}dxdy = w_0\dot{\delta}_0F(\lambda_i) \tag{6}$$

where

$$F(\lambda_i) = \iint p(x, y)h(x, y, \lambda_i)dxdy \tag{7}$$

The internal energy dissipation rate is also determined in the following form ;

$$\sum_j \int m_{n,j}\dot{\theta}_{n,j}dx = m_0\dot{\delta}_0G(\lambda_i) \tag{8}$$

where

$$G(\lambda_i) = \sum_j \int g_{n,j}(x, \lambda_i)q(x, \lambda_i)dx \tag{9}$$

From equating eq. s (6) and (8) we get the relation between yield moment and collapse load ;

$$m_0/w_0 = F(\lambda_i)/G(\lambda_i) \tag{10}$$

The critical value of λ_i can be determined so as to give the maximum value of m_0/w_0. In the case in which a stationary maximum value is valid in the slab we obtain

$$\frac{\partial}{\partial\lambda_i}\left(\frac{F}{G}\right)=0 \quad \text{or} \quad \frac{F}{G}=\frac{\partial F/\partial\lambda_i}{\partial G/\partial\lambda_i} \tag{11}$$

If the solution of the above equation is out of the valid region of the slab, the physically determined limiting value of λ_i must be used.

4. Analysis by Equilibrium Method

An alternative method to the work method is that of considering the equilibrium of the forces acting to the individual element into which the slab is divided by the yield-lines or yield-lines and slab edges. The forces to be considered generally consist of bending and twisting moments and shear forces along the yield-lines and external loads on the slab surface. The equilibrium method has become available after Johansen found the theorem of replacing the twisting moment and shear forces acting along yield-lines by the statically equivalent nodal forces at the ends of the yield-lines. Now Johansen's nodal force theory is extended to be applicable to the slab with variable yield moments as follows.

Consider two yield-lines, $y=k_1x$ and $y=k_2x$, meeting at a point as shown in Fig. 1. The origin of the co-ordinate is taken at the intersection point. When we take a point e on the yield -line $\overline{ab}$ at a short distance from the point a the line $\overline{ec}$ is expressed by

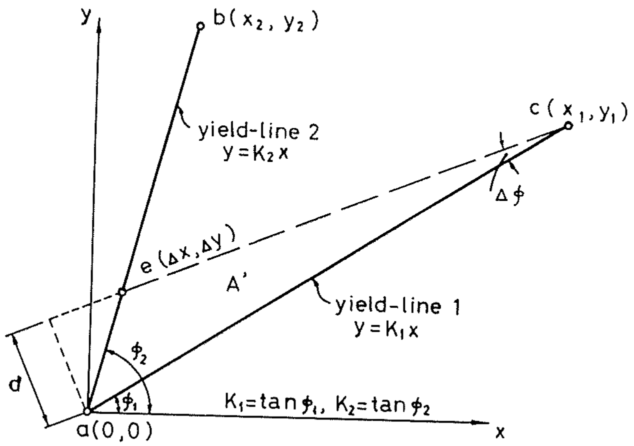

Fig. 1

$$y-y_1=k'(x-x_1), \qquad k'=k_1-(k_1-k_2)\Delta x/x_1 \tag{12}$$

and the length d is related to Δx as follows;

$$d=\frac{k_2-k'}{k'}\Delta x\sin(\phi_1-\Delta\phi)=(k_2-k')\Delta x\cos(\phi_1-\Delta\phi) \tag{13}$$

If the yield-lines $\overline{ac}$ and $\overline{ab}$ are governed by the meshes with yield moments

$$m_x=F_1(x,y), \qquad m_y=G_1(x,y) \tag{14}$$

and

$$m_x=F_2(x,y), \qquad m_y=F_2(x,y) \tag{15}$$

respectively, the resultant moments along $\overline{ac}$ and $\overline{ae}$ are found from the following equations

$$M_{x1}=\int_0^{y_1}F_1\left(\frac{y}{k_1},y\right)dy, \qquad M_{y1}=\int_0^{x_1}G_1(x,k_1x)\,dx \tag{16}$$

and

$$M_{x2}=\int_0^{\Delta y}F_2\left(\frac{y}{k_2},y\right)dy, \qquad M_{y2}=\int_0^{\Delta x}G_2(x,k_2x)\,dx \tag{17}$$

respectively. Similarly the resultant moments along $\overline{ec}$ are obtained by

$$M_x'=\int_{\Delta y}^{y_1}\left(F_1+\frac{\partial F_1}{\partial x}\delta x\right)_{x=y/k'}dy, \qquad M_y'=\int_{\Delta x}^{x_1}\left(G_1+\frac{\partial G_1}{\partial y}\delta y\right)_{y=k'x}dx \tag{18}$$

where

$$\delta x=\frac{(k_1-k_2)(y_1-y)}{k_1y_1}\Delta x, \qquad \delta y=\frac{(k_2-k_1)(x_1-x)}{x_1}\Delta x \tag{19}$$

If moments are taken about the line $\overline{ec}$ for the equilibrium of the small triangle A' we get

$$(M_{x1}-M_{x2}-M_x')\sin(\phi_1-\Delta\phi)+(M_{y1}-M_{y2}-M_y')\cos(\phi_1-\Delta\phi)+K_{A'a}d=0 \tag{20}$$

where $K_{A'a}$ represents a nodal force acting at the point a in the element A'. Substituting eq. s (16), (17) and (18) into eq. (20) and letting $\Delta x\to 0$, we find

$$K_{A'a}=\frac{k_1k_2}{k_2-k_1}\{F_2(0,0)-F_1(0,0)\}+\frac{1}{k_2-k_1}\{G_2(0,0)-G_1(0,0)\}$$
$$-\frac{1}{y_1}\int_0^{y_1}\frac{\partial F_1}{\partial x}\left(\frac{y}{k_1},y\right)(y_1-y)\,dy+\frac{1}{x_1}\int_0^{x_1}\frac{\partial G_1}{\partial y}(x,k_1x)(x_1-x)\,dx \tag{21}$$

Next, according to the procedure by Johansen we consider three yield-lines meeting at a point as shown in Fig. 2 and define the notations in the similar manner to those mensioned above. If we consider the equilibrium of the elements A' and B', and eliminate the component of nodal forces for the line $\overline{ae}$, we get

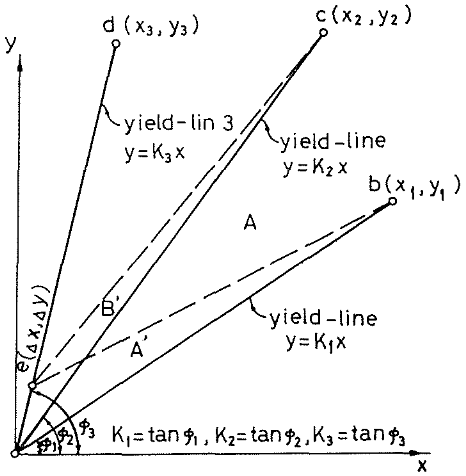

Fig. 2

$$
\begin{aligned}
K_{\Delta a}=&\frac{k_1k_3}{k_3-k_1}\{F_3(0,0)-F_1(0,0)\}+\frac{1}{k_3-k_1}\{G_3(0,0)-G_1(0,0)\}\\
&-\frac{k_2k_3}{k_3-k_2}\{F_3(0,0)-F_2(0,0)\}-\frac{1}{k_3-k_2}\{G_3(0,0)-G_2(0,0)\}\\
&-\frac{1}{y_1}\int_0^{y_1}\frac{\partial F_1}{\partial x}\left(\frac{y}{k_1},y\right)(y_1-y)\,dy+\frac{1}{y_2}\int_0^{y_2}\frac{\partial F_2}{\partial x}\left(\frac{y}{k_2},y\right)(y_2-y)\,dy\\
&+\frac{1}{x_1}\int_0^{x_1}\frac{\partial G_1}{\partial y}(x,k_1x)(x_1-x)\,dx-\frac{1}{x_2}\int_0^{x_2}\frac{\partial G_2}{\partial y}(x,k_2x)(x_2-x)\,dx \qquad (22)
\end{aligned}
$$

This is a generalized equation for the calculation of nodal forces in an orthotropically reinforced slab and coincides with Johansen's equation if each yield moment is taken to be constant.

When the yield moments vary linearly, that is when

$$m_{x,j}=a_j+b_jx+c_jy, \qquad m_{y,j}=\bar{a}_j+\bar{b}_jx+\bar{c}_jy \qquad (23)$$

then

$$K_{\Delta a}=\frac{k_1k_3}{k_3-k_1}(a_3-a_1)+\frac{1}{k_3-k_1}(\bar{a}_3-\bar{a}_1)-\frac{k_2k_3}{k_3-k_2}(a_3-a_2)-\frac{1}{k_3-k_2}(\bar{a}_3-\bar{a}_2)-\frac{1}{2}(b_1y_1-b_2y_2-c_1x_1+c_2x_2) \qquad (24)$$

Furthermore when all yield-lines which meet are governed by the same mesh eq. (24) reduces to

$$K_{\Delta a}=-\frac{1}{2}b(y_1-y_2)+\frac{1}{2}\bar{c}(x_1-x_2) \qquad (25)$$

From the above equation it is found that the theorem demonstrated by Johansen

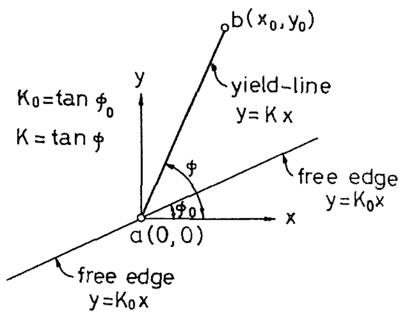

Fig. 3

that at the junction of yield-lines governed by the same mesh each of the nodal forces is zero is only applicable to the slab with constant yield moments.

When a yield-line meets a free edge of the slab as shown in Fig. 3 nodal forces can be calculated by considering the free edge as a yield-line with zero yield moment, that is from eq. (24) we get

$$K_{Aa}=\frac{1}{k_0-k}(kk_0a+\bar{a})+\frac{1}{2}(by_0-\bar{c}x_0) \tag{26}$$

If an x-axis is taken on a free edge eq. (26) reduces to

$$K_{Aa}=-\frac{\bar{a}}{k}+\frac{1}{2}(by_0-\bar{c}x_0) \tag{27}$$

Moreover when a yield-line intersects a free edge at right angles we find

$$K_{Aa}=\frac{1}{2}by_0 \tag{28}$$

As seen from the process of deriving the above equations they can only applicable when the actual moment divided by the yield moment has a stationary maximum along the yield lines. If it is not the case the magnitude of nodal forces must be determined by the equilibrium conditions, generally of vertical forces.

5. Typical Examples of Solution of Rectangular Slabs with Linearly Varying Thickness

In the following examples we consider only when a linearly varying distributed load is applied and yield moments vary in one direction only, that is

for a positive yield-line

$$m_x=m_0(1+\gamma y/\alpha L), \qquad m_y=\xi m_0(1+\gamma y/\alpha L) \tag{29}$$

for a negative yield-line

$$m_x=-\zeta m_0(1+\gamma y/\alpha L), \qquad m_y=-\mu m_0(1+\gamma y/\alpha L) \tag{30}$$

(1) Rectangular slab supported on two opposite sides

If each of the supported edges are fixed (Fig. 4)

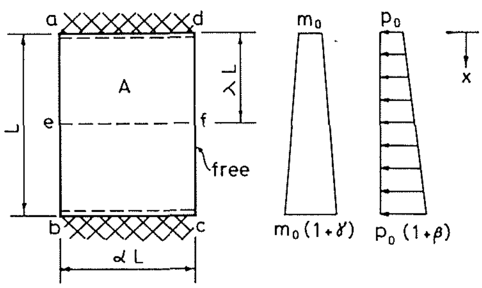

Fig. 4

$$\frac{m_0}{p_0L^2}=\frac{1}{6}\,\frac{(3+2\beta\lambda)\lambda^2}{1+\zeta} \tag{31}$$

where the value of λ can be determined by the following equation.

$$2\beta\gamma\lambda^3+(3\gamma-\beta)\lambda^2+2(3+2\beta)\lambda-(3+\beta)=0 \tag{32}$$

If the edge $\overline{ad}$ is simply supported

$$\begin{gathered}\frac{m_0}{p_0L^2}=\frac{1}{6}(3+2\beta\lambda)\lambda^2\\ 2\beta(\gamma+\zeta+\gamma\zeta)\lambda^3+3(\gamma+\zeta+\gamma\zeta+\beta)\lambda^2+6\lambda-(3+\beta)=0\end{gathered} \tag{33}$$

If the edge $\overline{bc}$ is also simply supported

$$\frac{m_0}{p_0L^2}=\frac{1}{6}(3+2\beta\lambda)\lambda^2,\qquad 2\beta\gamma\lambda^3+3(\gamma+\beta)\lambda^2+6\lambda-(3+\beta)=0 \tag{34}$$

(2) Rectangular slab suported on three sides, and collapse mechanism shown in Fig. 5

If each of the supported edges are fixed

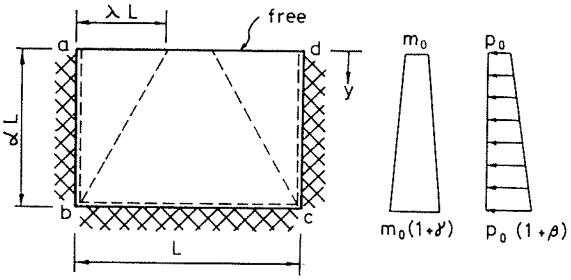

Fig. 5

$$\frac{m_0}{p_0L^2}=\frac{1}{12}\frac{(4+\beta)\lambda^2}{(2+\gamma)(1+\zeta-\xi\lambda^2/\alpha^2)},\qquad \frac{2\xi(2+\gamma)(3+\beta)+\mu(1+\gamma)(4+\beta)}{2\alpha^2(1+\zeta)(2+\gamma)}\lambda^2+(4+\beta)\lambda-(3+\beta)=0 \tag{35}$$

If the edges $\overline{ab}$ and $\overline{cd}$ are simply supported or the edge $\overline{bc}$ is simply supported a solution can be obtained by substituting $\zeta=0$ or $\mu=0$ in eq. (35), respectively.

(3) Rectangular slab supported on three sides, and collapse mechanism shown in Fig. 6

If each of the supported edges are fixed

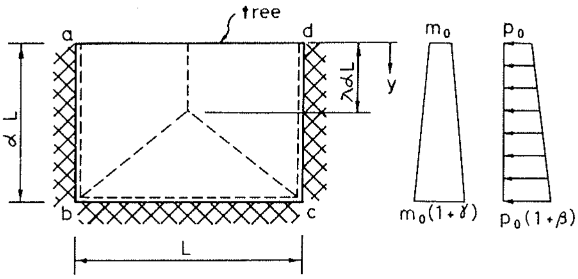

Fig. 6

$$\left.\begin{aligned}&\frac{m_0}{p_0L^2}=\frac{1}{12}\alpha^2\lambda^2\frac{2+2\beta-\beta\lambda}{(\xi+\mu)(1+\gamma)}\\&\phi\beta\lambda^3+\{3\beta-2\phi(1+\beta)\}\lambda^2-8(1+\beta)\lambda+6(2+\beta)=0\\&\phi=\{4\alpha^2(2+\gamma)(1+\zeta)-\xi\gamma\}/(\xi+\mu)(1+\gamma)\end{aligned}\right\} \tag{36}$$

If the edges $\overline{ab}$ and $\overline{cd}$ or the edge $\overline{bc}$ is simply supported $\zeta=0$ or $\mu=0$ is substituted in the above equations.

References

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3) Jones, L. L., Wood, R. H., Yield-Line Analysis of Slabs, Thames & Hudson and Chatto & Windus, London, 1967.