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A Generalization of the Crossed Extensor

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Abstract

Extensor analysis, a generalization of tensor analysis, offers many mathematical models designed to meet requirement of application to physical science and/or engineering. Since extensor analysis is concerned with the calculation of higher order differentials, it also plays an important role in the study of higher order space.

The purpose of this paper is to show the existence of a new extensor, referred to as a hybrid extensor, and to investigate its properties. It is a possible generalization of the crossed extensor which has been studied elsewhere.

Contents

Abstract	141
1. Introduction	141
2. Notations and Preliminaries	141
3. Hybrid Extensor	143
4. Contractions and Reduced Range	148
5. Conclusions	150
References	150

1. Introduction

The concept of extensor with several important properties has been established by H. V. Craig¹⁾ and A. Kawaguchi^{2),3),4)}. Craig has also shown the existence of a certain special extensor in his work⁵⁾, which he refers to as the crossed extensor.

Since then, works on crossed extensor have been done by some investigators^{6),7),9)}. Among them, Y. Katsurada extended the concept of crossed extensor in the manifold of line-elements to that in the manifold of surface-elements of a higher order, and she introduced the concept of generalized crossed extensor⁶⁾. She, however, treats either covariant or contravariant components of it. Therefore, in this paper the authors will supplement and explain in detail the generalized crossed extensor from the point of view of a new extensor which will be called a hybrid extensor. Although we discuss only the case of one parameter, it can easily be extended to that of multiparameter. Also it will be shown that this new extensor contains an ordinary extensor, crossed extensor and generalized crossed extensor as a special case.

2. Notations and Preliminaries

The notations used in this paper are mostly the same as those defined in the

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previous paper⁹⁾. The line-elements of order M with respect to the curve $x^i = x^i(t)$ in an N -dimensional differentiable manifold are represented by

$$(x^i, x^{(1)i}, x^{(2)i}, \dots, x^{(M)i}),$$

and Craig's expoint transformation by

$$\begin{cases} x^\alpha = x^\alpha(x^i) \\ x^{(\alpha)a} = X_{(\rho-1)i}^{(\alpha-1)a} x^{(\rho)i}. \quad (\alpha, \rho = 1, 2, \dots, M) \end{cases}$$

The coefficients $X_{(\rho)i}^{(\alpha)a}$ of expoint transformation have the following available relation :

$$\begin{aligned} X_{(\rho)i}^{(\alpha)a} &= \binom{\alpha}{\rho} X_i^{a(\alpha-\rho)} = \binom{\alpha}{\rho} \left(\frac{\partial x^a}{\partial x^i} \right)^{(\alpha-\rho)} && \text{(for } \alpha \geq \rho) \\ &= 0, && \text{(for } \alpha < \rho) \end{aligned}$$

where $\binom{\alpha}{\rho}$ is a binomial coefficient. In connection with these coefficients, we get the following identities

$$\begin{aligned} \sum_{\beta=\mu}^{\alpha} X_{(\mu)i}^{(\beta)a} X_{(\beta)j}^{(\alpha)b} &= \binom{\alpha}{\mu} (X_i^a X_j^b)^{(\alpha-\mu)}, \\ \sum_{\beta=\mu}^{\nu} X_{(\mu)i}^{(\beta)a} X_{(\beta)j}^{(\nu)b} &= \binom{\nu}{\mu} (X_i^a X_j^b)^{(\nu-\mu)}, \end{aligned}$$

from which we can deduce the following general form :

$$\begin{aligned} \sum_{\alpha_1=\mu}^{\nu} \dots \sum_{\alpha_{r+s-1}=\mu}^{\nu} X_{(\mu)i_1}^{(\alpha_1)a_1} X_{(\alpha_1)i_2}^{(\alpha_2)a_2} X_{(\alpha_2)i_3}^{(\alpha_3)a_3} \dots X_{(\alpha_{r-1})i_r}^{(\alpha_r)a_r} X_{(\alpha_r)i_{b_1}}^{(\alpha_{r+1})j_1} X_{(\alpha_{r+1})i_{b_2}}^{(\alpha_{r+2})j_2} \dots X_{(\alpha_{r+s-1})i_{b_s}}^{(\nu)j_s} \\ = \binom{\nu}{\mu} (X_{i_1}^{a_1} X_{i_2}^{a_2} X_{i_3}^{a_3} \dots X_{i_r}^{a_r} X_{b_1}^{j_1} X_{b_2}^{j_2} \dots X_{b_s}^{j_s})^{(\nu-\mu)}. \end{aligned}$$

($\mu, \nu = 0, 1, 2, \dots, M; \nu \geq \mu$)

Here, we shall put, for the sake of simplicity, as follows :*

$$\begin{aligned} X_{i_1 i_2 \dots i_r b_1 b_2 \dots b_s}^{a_1 a_2 \dots a_r j_1 j_2 \dots j_s} &\equiv X_{i_1}^{a_1} X_{i_2}^{a_2} \dots X_{i_r}^{a_r} X_{b_1}^{j_1} X_{b_2}^{j_2} \dots X_{b_s}^{j_s}, \\ X_{(\mu) i_1 i_2 \dots i_r b_1 b_2 \dots b_s}^{a_1 a_2 \dots a_r j_1 j_2 \dots j_s} &\equiv \sum_{\alpha_1=\mu}^{\nu} \dots \sum_{\alpha_{r+s-1}=\mu}^{\nu} X_{(\mu) i_1}^{(\alpha_1) a_1} X_{(\alpha_1) i_2}^{(\alpha_2) a_2} X_{(\alpha_2) i_3}^{(\alpha_3) a_3} \dots X_{(\alpha_{r-1}) i_r}^{(\alpha_r) a_r} \\ &\times X_{(\alpha_r) b_1}^{(\alpha_{r+1}) j_1} X_{(\alpha_{r+1}) b_2}^{(\alpha_{r+2}) j_2} \dots X_{(\alpha_{r+s-1}) b_s}^{(\nu) j_s}. \end{aligned}$$

Then we find the following propositions :

Proposition 1. We assume that (x^a, x^b, x^c, \dots) , (x^i, x^j, x^k, \dots) and (x^r, x^s, x^t, \dots) are three different coordinate systems. Then the equalities

$$X_{(\sigma) i b j}^{a t c(\rho)} X_{(\rho) r k t}^{i s j(\lambda)} = X_{(\sigma) r b t}^{a s c(\lambda)}$$

hold, and they are generally represented by

$$X_{(\mu) i_1 \dots i_r b_1 \dots b_s}^{a_1 \dots a_r j_1 \dots j_s(\nu)} X_{(\nu) t_1 \dots t_r j_1 \dots j_s}^{i_1 \dots i_r r_1 \dots r_s(\omega)} = X_{(\mu) t_1 \dots t_r b_1 \dots b_s}^{a_1 \dots a_r r_1 \dots r_s(\omega)}.$$

Proposition 2. In regard to two different coordinate systems (x^a, x^b, x^c, \dots) and (x^i, x^j, x^k, \dots) , it follows that

* We find these notations in 5).

$$X_{(a)ik}^{ajc(\rho)} X_{(e)df}^{iek(\lambda)} = \delta_a^i \delta_b^j \delta_c^e \delta_d^f$$

and generally

$$X_{(\mu)\tilde{i}_1 \dots \tilde{i}_r, b_1 \dots b_s}^{a_1 \dots a_r, j_1 \dots j_s(\nu)} X_{(\nu)c_1 \dots c_r, j_1 \dots j_s}^{i_1 \dots i_r, a_1 \dots a_s(\omega)} = \delta_\mu^\nu \delta_{c_1 \dots c_r}^{a_1 \dots a_r} \delta_{b_1 \dots b_s}^{i_1 \dots i_s},$$

where

$$\delta_{c_1 \dots c_r, b_1 \dots b_s}^{a_1 \dots a_r, i_1 \dots i_s} \equiv \delta_{c_1}^{a_1} \dots \delta_{c_r}^{a_r} \delta_{b_1}^{i_1} \dots \delta_{b_s}^{i_s}.$$

These two propositions suggest that we can define a certain extensor which has $X_{(\mu)\tilde{i}_1 \dots \tilde{i}_r, b_1 \dots b_s}^{a_1 \dots a_r, j_1 \dots j_s(\nu)}$ for the coefficients of transformation.

3. Hybrid Extensor

Definition 3.1. The quantities $T^{\tilde{i}_1 \dots \tilde{i}_r(\mu)}$ and $T_{j_1 \dots j_r(\nu)}$ are called the components of an excontravariant and the components of an excovariant blocked extensors of range M respectively, when they are transformed as follows:

$$\begin{aligned} T^{\tilde{i}_1 \dots \tilde{i}_r(\mu)} &= X_{(\alpha)\tilde{a}_1 \dots \tilde{a}_r}^{\tilde{i}_1 \dots \tilde{i}_r(\mu)} T^{a_1 \dots a_r(\alpha)}, \\ T_{j_1 \dots j_r(\nu)} &= X_{(\nu)b_1 \dots b_r(\beta)}^{j_1 \dots j_r(\nu)} T_{b_1 \dots b_r(\beta)} \\ &\quad (r \geq 1; \alpha, \beta, \mu, \nu = 0, 1, 2, \dots, M) \end{aligned}$$

under the expoint transformation.

Definition 3.2. If the quantities $T^{\tilde{i}_1 \dots \tilde{i}_r(\mu)}$ and $T_{j_1 \dots j_r(\nu)}$ are transformed

$$\begin{aligned} T^{\tilde{i}_1 \dots \tilde{i}_r(\mu)} &= X_{(\alpha)\tilde{a}_1 \dots \tilde{a}_r}^{a_1 \dots a_r(\mu)} T^{a_1 \dots a_r(\alpha)}, \\ T_{j_1 \dots j_r(\nu)} &= X_{(\nu)\tilde{b}_1 \dots \tilde{b}_r}^{j_1 \dots j_r(\beta)} T_{b_1 \dots b_r(\beta)} \\ &\quad (r \geq 1; \alpha, \beta, \mu, \nu = 0, 1, 2, \dots, M) \end{aligned}$$

under the expoint transformation, then we shall call these quantities the components of an excontravariant generalized crossed extensor* of range M and the components of an excovariant one, respectively.

Definition 3.3. Under the expoint transformation, the components $T^{\tilde{i}_1 \dots \tilde{i}_r, j_1 \dots j_s(\mu)}$ of an excontravariant hybrid extensor of range M and the components $T^{k_1 \dots k_r, \tilde{l}_1 \dots \tilde{l}_s(\nu)}$ of an excovariant one are varied in the rules

$$\begin{aligned} T^{\tilde{i}_1 \dots \tilde{i}_r, j_1 \dots j_s(\mu)} &= X_{(\alpha)\tilde{a}_1 \dots \tilde{a}_r, b_1 \dots b_s(\mu)}^{\tilde{i}_1 \dots \tilde{i}_r, j_1 \dots j_s(\mu)} T^{a_1 \dots a_r, b_1 \dots b_s(\alpha)}, \\ T^{k_1 \dots k_r, \tilde{l}_1 \dots \tilde{l}_s(\nu)} &= X_{(\nu)c_1 \dots c_r, \tilde{d}_1 \dots \tilde{d}_s(\beta)}^{k_1 \dots k_r, \tilde{l}_1 \dots \tilde{l}_s(\nu)} T^{c_1 \dots c_r, \tilde{d}_1 \dots \tilde{d}_s(\beta)}. \\ &\quad (r, s \geq 0, r+s \geq 1; \alpha, \beta, \mu, \nu = 0, 1, 2, \dots, M) \end{aligned}$$

Remark: The concepts of blocked and generalized crossed extensors are involved in one of the hybrid extensors as a special case, since we have the components of excontravariant blocked or generalized crossed extensors $T^{\tilde{i}_1 \dots \tilde{i}_r(\mu)}$ or $T_{j_1 \dots j_s(\nu)}$, putting $s=0$ or $r=0$ for the components of excontravariant hybrid extensor $T^{\tilde{i}_1 \dots \tilde{i}_r, j_1 \dots j_s(\mu)}$. Also the blocked extensors involve an ordinary extensor, i.e. if we put $r=1$ or $s=1$ for the components of blocked extensors $T^{\tilde{i}_1 \dots \tilde{i}_r(\mu)}$ or $T_{j_1 \dots j_s(\nu)}$, then

* Although there is a slight difference between the generalized crossed extensor in the sense of Y. Katsurada and our extensor defined here, the latter being more limited than the former, we shall call our extensor a generalized crossed extensor.

we can regard $T^{\dot{i}_1^{(\mu)}}$, $T_{j_1^{(\nu)}}$ as the components of ordinary extensor from the definition of the coefficients of transformation. Similarly, we see that the crossed extensor defined in 9) is involved in the generalized crossed extensor.

Theorem 3.1. If the quantities $T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}^{(\mu)}$ are the components of any excontravariant hybrid extensor of range M , then the quantities

$$\binom{M}{\nu} T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}^{(M-\nu)} \quad (\nu = 0, 1, 2, \dots, M)$$

are the components of excovariant hybrid extensor of range M .

Proof. Under the expoint transformation, we have

$$\begin{aligned} & \binom{M}{\nu} T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}^{(M-\nu)} \\ &= \binom{M}{\nu} X_{(a) a_1 \dots a_r, j_1 \dots j_s}^{\dot{i}_1 \dots \dot{i}_r, b_1 \dots b_s (M-\nu)} T^{a_1 \dots a_r}_{b_1 \dots b_s}^{(\alpha)} \\ &= \binom{M}{\nu} \binom{M-\nu}{\alpha} (X_{a_1 \dots a_r, j_1 \dots j_s}^{\dot{i}_1 \dots \dot{i}_r, b_1 \dots b_s})^{(M-\nu-\alpha)} T^{a_1 \dots a_r}_{b_1 \dots b_s}^{(\alpha)} \\ &= \binom{M}{\nu} \binom{M-\nu}{M-\beta} (X_{a_1 \dots a_r, j_1 \dots j_s}^{\dot{i}_1 \dots \dot{i}_r, b_1 \dots b_s})^{(\beta-\nu)} T^{a_1 \dots a_r}_{b_1 \dots b_s}^{(M-\beta)} \\ & \hspace{15em} (\text{putting } \alpha = M-\beta) \\ &= \binom{\beta}{\nu} (X_{a_1 \dots a_r, j_1 \dots j_s}^{\dot{i}_1 \dots \dot{i}_r, b_1 \dots b_s})^{(\beta-\nu)} \binom{M}{\beta} T^{a_1 \dots a_r}_{b_1 \dots b_s}^{(M-\beta)} \\ &= X_{(\nu) a_1 \dots a_r, j_1 \dots j_s}^{\dot{i}_1 \dots \dot{i}_r, b_1 \dots b_s (\beta)} \binom{M}{\beta} T^{a_1 \dots a_r}_{b_1 \dots b_s}^{(M-\beta)} \end{aligned}$$

Q. E. D.

and the theorem is proved.

Corollary 3.1. If two quantities $T^{\dot{i}_1 \dots \dot{i}_r, (\mu)}$ and $T_{k_1 \dots k_s}^{(\nu)}$ are the components of any excontravariant blocked and excontravariant generalized crossed extensor of range M respectively, then the quantities

$$\binom{M}{\rho} T^{\dot{i}_1 \dots \dot{i}_r, (M-\rho)}, \quad \binom{M}{\sigma} T_{k_1 \dots k_s}^{(M-\sigma)}$$

are the components of the excovariant generalized crossed and of the excovariant blocked extensors of range M respectively.

As an example of the hybrid extensor, we shall consider the following quantities. If the quantities $T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}$ are the differentiable components of any tensor field defined along an arc of class C^M , $x^i = x^i(t)$, then assuming the required differentiability, the quantities which are the μ -th derivative of $T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}$ with respect to t

$$T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}^{(\mu)} \quad (\mu = 0, 1, 2, \dots, M)$$

are the components of the excontravariant hybrid extensor, and if we represent the quantities which are the $(M-\nu)$ -th derivative of $T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}$ with respect to t , multiplying by the binomial coefficient $\binom{M}{\nu}$, as

$$T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s^{(\nu)}} \equiv \binom{M}{\nu} T^{\dot{i}_1 \dots \dot{i}_r}_{j_1 \dots j_s}^{(M-\nu)}, \quad (\nu = 0, 1, 2, \dots, M)$$

then they are the components of excovariant hybrid extensor.

*Theorem 3.2.** If $T^{\ell_1 \dots \ell_r, k_1 \dots k_p}_{j_1 \dots j_s}^{(\mu)}$ and $T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\nu)}$ are the components of two arbitrary excontravariant hybrid extensors of range M , then the quantities

$$T^{\ell_1 \dots \ell_r, k_1 \dots k_p}_{j_1 \dots j_s, l_1 \dots l_q}^{(\omega)} \equiv \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} T^{\ell_1 \dots \ell_r, j_1 \dots j_s}^{(\omega-\nu)} T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\nu)} \quad (\omega = 0, 1, 2, \dots, M)$$

are also the components of excontravariant hybrid extensor of range M .

Proof. By making use of the transformation of the hybrid extensor, we can calculate as follows :

$$\begin{aligned} & \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} T^{\ell_1 \dots \ell_r, j_1 \dots j_s}^{(\omega-\nu)} T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\nu)} \\ &= \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} X_{(\alpha) a_1 \dots a_r, b_1 \dots b_s}^{\ell_1 \dots \ell_r, j_1 \dots j_s} X_{(\beta) c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, a_1 \dots a_q} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q}^{(\beta)} \\ &= \sum_{\nu=0}^{\omega} \sum_{\beta=0}^{\nu} \sum_{\alpha=0}^{\omega-\nu} \binom{\omega}{\nu} \binom{\omega-\nu}{\alpha} \binom{\nu}{\beta} (X_{a_1 \dots a_r, j_1 \dots j_s}^{\ell_1 \dots \ell_r, b_1 \dots b_s})^{(\omega-\nu-\alpha)} \\ & \quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, a_1 \dots a_q})^{(\nu-\beta)} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q}^{(\beta)} \\ &= \sum_{\beta=0}^{\omega} \sum_{\alpha=0}^{\omega-\beta} \sum_{\nu=0}^{\omega-\alpha} \binom{\omega}{\alpha+\beta} \binom{\alpha+\beta}{\beta} \binom{\omega-(\alpha+\beta)}{\nu-\beta} (X_{a_1 \dots a_r, j_1 \dots j_s}^{\ell_1 \dots \ell_r, b_1 \dots b_s})^{(\omega-\nu-\alpha)} \\ & \quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, a_1 \dots a_q})^{(\nu-\beta)} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q}^{(\beta)} \\ &= \sum_{\beta=0}^{\omega} \sum_{\gamma=\beta}^{\omega-\beta} \sum_{\nu=\beta}^{\omega-(\gamma-\beta)} \binom{\omega}{\gamma} \binom{\gamma}{\beta} \binom{\omega-\gamma}{\nu-\beta} (X_{a_1 \dots a_r, j_1 \dots j_s}^{\ell_1 \dots \ell_r, b_1 \dots b_s})^{(\omega-\nu-(\gamma-\beta))} \\ & \quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, a_1 \dots a_q})^{(\nu-\beta)} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\gamma-\beta)} T^{c_1 \dots c_p, d_1 \dots d_q}^{(\beta)} \\ & \hspace{15em} (\text{putting } \alpha = \gamma - \beta) \\ &= \sum_{\beta=0}^{\omega} \sum_{\gamma=\beta}^{\omega} \binom{\omega}{\gamma} \binom{\gamma}{\beta} (X_{a_1 \dots a_r, j_1 \dots j_s}^{\ell_1 \dots \ell_r, b_1 \dots b_s} X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, a_1 \dots a_q})^{(\omega-\gamma)} \\ & \quad \times T^{a_1 \dots a_r, b_1 \dots b_s}^{(\gamma-\beta)} T^{c_1 \dots c_p, d_1 \dots d_q}^{(\beta)} \\ &= X_{(\gamma) a_1 \dots a_r, c_1 \dots c_p, j_1 \dots j_s, l_1 \dots l_q}^{\ell_1 \dots \ell_r, k_1 \dots k_p, b_1 \dots b_s, d_1 \dots d_q}^{(\omega)} \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\gamma-\beta)} T^{c_1 \dots c_p, d_1 \dots d_q}^{(\beta)}. \end{aligned}$$

Q. E. D.

Theorem 3.3. If $T^{\ell_1 \dots \ell_r, j_1 \dots j_s}^{(\sigma)}$ and $T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\rho)}$ are the components of two arbitrary excovariant hybrid extensors of range M , then the quantities

$$T^{\ell_1 \dots \ell_r, k_1 \dots k_p}_{j_1 \dots j_s, l_1 \dots l_q}^{(\tau)} \equiv \sum_{\beta=0}^{M-\tau} \binom{M}{\beta}^{-1} \binom{\gamma+\beta}{\beta} T^{\ell_1 \dots \ell_r, j_1 \dots j_s}^{(\tau+\beta)} T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\beta)} \quad (\gamma = 0, 1, 2, \dots, M)$$

are also the components of excovariant hybrid extensor of range M .

Proof. Under successive applications of the transformation we have the following equalities :

$$\sum_{\beta=0}^{M-\gamma} \binom{M}{\beta}^{-1} \binom{\gamma+\beta}{\beta} T^{\ell_1 \dots \ell_r, j_1 \dots j_s}^{(\tau+\beta)} T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\beta)}$$

* This theorem corresponds to the Theorem 1.5 in 6).

$$\begin{aligned}
&= \sum_{\beta=0}^{M-\gamma} \binom{M}{\beta}^{-1} \binom{\gamma+\beta}{\beta} X_{(\gamma+\beta)a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s(\alpha)} X_{(M-\beta)c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q(\delta)} T^{a_1 \dots a_r, b_1 \dots b_s(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q(\delta)} \\
&= \sum_{\beta=0}^{M-\gamma} \sum_{\alpha=\gamma+\beta}^M \sum_{\delta=M-\beta}^M \binom{M}{\beta}^{-1} \binom{\gamma+\beta}{\beta} \binom{\alpha}{\gamma+\beta} \binom{\delta}{M-\beta} (X_{a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s})^{(\alpha-\gamma-\beta)} \\
&\quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q})^{(\delta-M+\beta)} T^{a_1 \dots a_r, b_1 \dots b_s(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q(\delta)} \\
&= \sum_{\beta=0}^{M-\gamma} \sum_{\alpha=\gamma+\beta}^M \sum_{\rho=0}^{\beta} \binom{M}{\beta}^{-1} \binom{\alpha}{\beta} \binom{\alpha}{\gamma+\beta} \binom{M-\rho}{M-\beta} (X_{a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s})^{(\alpha-\gamma-\beta)} \\
&\quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q})^{(\beta-\rho)} T^{a_1 \dots a_r, b_1 \dots b_s(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\rho)} \quad (\text{putting } \delta = M-\rho) \\
&= \sum_{\alpha=\gamma}^M \sum_{\rho=0}^{\alpha-\gamma} \sum_{\mu=0}^{\alpha-\gamma-\rho} \binom{M}{\mu+\rho}^{-1} \binom{\alpha}{\mu+\rho} \binom{\alpha}{\gamma+\mu-\rho} \binom{M-\rho}{M-\mu-\rho} (X_{a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s})^{(\alpha-\gamma-\mu-\rho)} \\
&\quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q})^{(\rho)} T^{a_1 \dots a_r, b_1 \dots b_s(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\rho)} \quad (\text{putting } \beta = \mu+\rho) \\
&= \sum_{\rho=0}^{M-\gamma} \sum_{\mu=0}^{M-\gamma-\rho} \sum_{\omega=\gamma+\mu}^{M-\rho} \binom{M}{\mu+\rho}^{-1} \binom{\alpha}{\mu+\rho} \binom{\omega+\rho}{\gamma+\mu+\rho} \binom{M-\rho}{M-\mu-\rho} (X_{a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s})^{(\omega-\gamma-\mu)} \\
&\quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q})^{(\rho)} T^{a_1 \dots a_r, b_1 \dots b_s(\omega+\rho)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\rho)} \quad (\text{putting } \alpha = \omega+\rho) \\
&= \sum_{\omega=\gamma}^M \sum_{\mu=0}^{\omega-\gamma} \binom{\omega}{\mu} \binom{\omega-\gamma}{\mu} (X_{a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s})^{(\omega-\gamma-\mu)} (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q})^{(\rho)} \\
&\quad \times \sum_{\rho=0}^{M-\omega} \binom{M}{\rho}^{-1} \binom{\omega+\rho}{\rho} T^{a_1 \dots a_r, b_1 \dots b_s(\omega+\rho)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\rho)} \\
&= X_{(\gamma)a_1 \dots a_r, c_1 \dots c_p, j_1 \dots j_s, l_1 \dots l_q}^{i_1 \dots i_r, k_1 \dots k_p, b_1 \dots b_s, d_1 \dots d_q(\omega)} \sum_{\rho=0}^{M-\omega} \binom{M}{\rho}^{-1} \binom{\omega+\rho}{\rho} T^{a_1 \dots a_r, b_1 \dots b_s(\omega+\rho)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\rho)}.
\end{aligned}$$

Q. E. D.

Theorem 3.4. If $T^{i_1 \dots i_r, j_1 \dots j_s}^{(\rho)}$ and $T^{k_1 \dots k_p, l_1 \dots l_q}^{(\nu)}$ are the components of any excontravariant and excovariant hybrid extensors of range M respectively, then the quantities

$$T^{i_1 \dots i_r, k_1 \dots k_p, j_1 \dots j_s, l_1 \dots l_q}^{(\beta)} \equiv \sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \binom{M}{\alpha}^{-1} T^{i_1 \dots i_r, j_1 \dots j_s}^{(\beta-\alpha)} T^{k_1 \dots k_p, l_1 \dots l_q}^{(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\alpha)} \quad (\beta = 0, 1, 2, \dots, M)$$

are the components of excontravariant hybrid extensor of range M .

Proof. Applying successively the transformation for hybrid extensors, we get

$$\begin{aligned}
&\sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \binom{M}{\alpha}^{-1} T^{i_1 \dots i_r, j_1 \dots j_s}^{(\beta-\alpha)} T^{k_1 \dots k_p, l_1 \dots l_q}^{(\alpha)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\alpha)} \\
&= \sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \binom{M}{\alpha}^{-1} X_{(\mu)a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s(\beta-\alpha)} X_{(M-\alpha)c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q(\nu)} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\mu)} T^{c_1 \dots c_p, d_1 \dots d_q(\nu)} \\
&= \sum_{\alpha=0}^{\beta} \sum_{\mu=0}^{\beta-\alpha} \sum_{\nu=M-\alpha}^M \binom{\beta}{\alpha} \binom{M}{\alpha}^{-1} \binom{\beta-\alpha}{\mu} \binom{\nu}{M-\alpha} (X_{a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s})^{(\beta-\alpha-\mu)} \\
&\quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q})^{(\nu-M+\alpha)} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\mu)} T^{c_1 \dots c_p, d_1 \dots d_q(\nu)} \\
&= \sum_{\alpha=0}^{\beta} \sum_{\mu=0}^{\beta-\alpha} \sum_{\rho=0}^{\alpha} \binom{\beta}{\alpha} \binom{M}{\alpha}^{-1} \binom{\beta-\alpha}{\mu} \binom{M-\rho}{M-\alpha} (X_{a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s})^{(\beta-\alpha-\mu)} \\
&\quad \times (X_{c_1 \dots c_p, l_1 \dots l_q}^{k_1 \dots k_p, d_1 \dots d_q})^{(\alpha-\rho)} T^{a_1 \dots a_r, b_1 \dots b_s}^{(\mu)} T^{c_1 \dots c_p, d_1 \dots d_q(M-\rho)} \quad (\text{putting } \nu = M-\rho)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\mu=0}^{\beta} \sum_{\rho=0}^{\beta-\mu} \sum_{\tau=0}^{\beta-\mu-\rho} \binom{\beta}{\tau+\rho} \binom{M}{\tau+\rho}^{-1} \binom{\beta-\tau-\rho}{\mu} \binom{M-\rho}{M-\tau-\rho} (X_{a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s})^{(\beta-\tau-\rho-\mu)} \\
&\quad \times (X_{c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q})^{(\tau)} T^{a_1 \dots a_r b_1 \dots b_s (n)} T^{c_1 \dots c_p d_1 \dots d_q (M-\rho)} \quad (\text{putting } \alpha = \tau + \rho) \\
&= \sum_{\rho=0}^{\beta} \sum_{\tau=0}^{\beta-\rho} \sum_{\omega=0}^{\beta-\tau} \binom{\beta}{\tau+\rho} \binom{M}{\tau+\rho}^{-1} \binom{\beta-\tau-\rho}{\omega-\rho} \binom{M-\rho}{M-\tau-\rho} (X_{a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s})^{(\beta-\tau-\omega)} \\
&\quad \times (X_{c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q})^{(\tau)} T^{a_1 \dots a_r b_1 \dots b_s (\omega-\rho)} T^{c_1 \dots c_p d_1 \dots d_q (M-\rho)} \quad (\text{putting } \mu = \omega - \rho) \\
&= \sum_{\omega=0}^{\beta} \sum_{\tau=0}^{\beta-\omega} \binom{\beta}{\omega} \binom{\beta-\omega}{\tau} (X_{a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s})^{(\beta-\omega-\tau)} (X_{c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q})^{(\tau)} \\
&\quad \times \sum_{\rho=0}^{\omega} \binom{\omega}{\rho} \binom{M}{\rho}^{-1} T^{a_1 \dots a_r b_1 \dots b_s (\omega-\rho)} T^{c_1 \dots c_p d_1 \dots d_q (M-\rho)} \\
&= X_{(\omega) a_1 \dots a_r c_1 \dots c_p j_1 \dots j_s l_1 \dots l_q}^{i_1 \dots i_r k_1 \dots k_p b_1 \dots b_s d_1 \dots d_q (\beta)} \sum_{\rho=0}^{\omega} \binom{\omega}{\rho} \binom{M}{\rho}^{-1} T^{a_1 \dots a_r j_1 \dots j_s (\omega-\rho)} T^{c_1 \dots c_p d_1 \dots d_q (M-\rho)}.
\end{aligned}$$

Q. E. D.

Theorem 3.5. If $T^{i_1 \dots i_r j_1 \dots j_s (\omega)}$ and $T^{k_1 \dots k_p l_1 \dots l_q (\rho)}$ are the components of any excovariant and excontravariant hybrid extensors of range M respectively, then the quantities

$$T^{i_1 \dots i_r k_1 \dots k_p j_1 \dots j_s l_1 \dots l_q (\beta)} \equiv \sum_{\alpha=0}^{M-\beta} \binom{\beta+\alpha}{\alpha} T^{i_1 \dots i_r j_1 \dots j_s (\beta+\alpha)} T^{k_1 \dots k_p l_1 \dots l_q (\alpha)}$$

($\beta = 0, 1, 2, \dots, M$)

are the components of excovariant hybrid extensor of range M .

Proof. With the same process as the proof of the preceding theorem, we have the following equalities:

$$\begin{aligned}
&\sum_{\alpha=0}^{M-\beta} \binom{\beta+\alpha}{\alpha} T^{i_1 \dots i_r j_1 \dots j_s (\beta+\alpha)} T^{k_1 \dots k_p l_1 \dots l_q (\alpha)} \\
&= \sum_{\alpha=0}^{M-\beta} \binom{\beta+\alpha}{\alpha} X_{(\beta+\alpha) a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s (n)} X_{(\nu) c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q (\alpha)} T^{a_1 \dots a_r b_1 \dots b_s (n)} T^{c_1 \dots c_p d_1 \dots d_q (\nu)} \\
&= \sum_{\alpha=0}^{M-\beta} \sum_{\mu=\beta+\alpha}^M \sum_{\nu=0}^{\alpha} \binom{\beta+\alpha}{\alpha} \binom{\mu}{\beta+\alpha} \binom{\alpha}{\nu} (X_{a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s})^{(\mu-\alpha-\beta)} \\
&\quad \times (X_{c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q})^{(\alpha-\nu)} T^{a_1 \dots a_r b_1 \dots b_s (n)} T^{c_1 \dots c_p d_1 \dots d_q (\nu)} \\
&= \sum_{\mu=\beta}^M \sum_{\nu=0}^{\mu-\beta} \sum_{\rho=0}^{\mu-\beta-\nu} \binom{\beta+\rho+\nu}{\rho+\nu} \binom{\mu}{\beta+\rho+\nu} \binom{\rho+\nu}{\nu} (X_{a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s})^{(\mu-\rho-\nu-\beta)} \\
&\quad \times (X_{c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q})^{(\rho)} T^{a_1 \dots a_r b_1 \dots b_s (n)} T^{c_1 \dots c_p d_1 \dots d_q (\nu)} \quad (\text{putting } \alpha = \rho + \nu) \\
&= \sum_{\nu=0}^{M-\beta} \sum_{\tau=\beta}^{M-\nu} \sum_{\rho=0}^{\tau-\beta} \binom{\beta+\rho+\nu}{\rho+\nu} \binom{\tau+\nu}{\beta+\rho+\nu} \binom{\rho+\nu}{\nu} (X_{a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s})^{(\tau-\beta-\rho)} \\
&\quad \times (X_{c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q})^{(\rho)} T^{a_1 \dots a_r b_1 \dots b_s (\tau+\nu)} T^{c_1 \dots c_p d_1 \dots d_q (\nu)} \quad (\text{putting } \mu = \tau + \nu) \\
&= \sum_{\tau=\beta}^M \sum_{\rho=0}^{\tau-\beta} \binom{\tau}{\beta} \binom{\tau-\beta}{\rho} (X_{a_1 \dots a_r j_1 \dots j_s}^{i_1 \dots i_r b_1 \dots b_s})^{(\tau-\beta-\rho)} (X_{c_1 \dots c_p l_1 \dots l_q}^{k_1 \dots k_p d_1 \dots d_q})^{(\rho)} \\
&\quad \times \sum_{\nu=0}^{M-\nu} \binom{\tau+\nu}{\nu} T^{a_1 \dots a_r b_1 \dots b_s (\tau+\nu)} T^{c_1 \dots c_p d_1 \dots d_q (\nu)} \\
&= X_{(\beta) a_1 \dots a_r c_1 \dots c_p j_1 \dots j_s l_1 \dots l_q}^{i_1 \dots i_r k_1 \dots k_p b_1 \dots b_s d_1 \dots d_q (\tau)} \sum_{\nu=0}^{M-\tau} \binom{\tau+\nu}{\nu} T^{a_1 \dots a_r b_1 \dots b_s (\tau+\nu)} T^{c_1 \dots c_p d_1 \dots d_q (\nu)}.
\end{aligned}$$

Q. E. D.

According to the results of these theorems, we can establish the relations among blocked extensors, generalized crossed extensors and hybrid extensors. Assuming that $T^{\hat{i}_1 \dots \hat{i}_r, (\nu)}$ and $T_{j_1 \dots j_s, (\nu)}$ are the components of any excontravariant and excovariant blocked extensors respectively, we construct the components of excontravariant and excovariant hybrid extensors as follows:

$$\begin{aligned} T^{\hat{i}_1 \dots \hat{i}_r, j_1 \dots j_s, (\omega)} &\equiv \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \binom{M}{\nu}^{-1} T^{\hat{i}_1 \dots \hat{i}_r, (\omega-\nu)} T_{j_1 \dots j_s, (M-\nu)}, \\ T^{\hat{i}_1 \dots \hat{i}_r, j_1 \dots j_s, (\nu)} &\equiv \sum_{\mu=0}^{M-\nu} \binom{\nu+\mu}{\mu} T^{\hat{i}_1 \dots \hat{i}_r, (\mu)} T_{j_1 \dots j_s, (\nu+\mu)}. \end{aligned}$$

Similarly, from the components of any excontravariant generalized crossed extensor and the components of any excovariant one, we get the following hybrid extensors,

$$\begin{aligned} T^{k_1 \dots k_s, \hat{i}_1 \dots \hat{i}_r, (\omega)} &\equiv \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \binom{M}{\nu}^{-1} T_{\hat{i}_1 \dots \hat{i}_r, (\omega-\nu)} T^{k_1 \dots k_s, (M-\nu)}, \\ T^{\hat{i}_1 \dots \hat{i}_r, k_1 \dots k_s, (\nu)} &\equiv \sum_{\mu=0}^{M-\nu} \binom{\nu+\mu}{\mu} T_{k_1 \dots k_s, (\mu)} T^{\hat{i}_1 \dots \hat{i}_r, (\nu+\mu)}. \end{aligned}$$

Also from the components of any blocked extensor and of any generalized crossed extensor, we construct hybrid extensors as follows:

$$\begin{aligned} T^{\hat{i}_1 \dots \hat{i}_r, k_1 \dots k_s, (\omega)} &\equiv \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} T^{\hat{i}_1 \dots \hat{i}_r, (\omega-\nu)} T_{k_1 \dots k_s, (\nu)}, \\ T^{\hat{i}_1 \dots \hat{i}_r, k_1 \dots k_s, (\nu)} &\equiv \sum_{\mu=0}^{M-\nu} \binom{M}{\nu}^{-1} \binom{\nu+\mu}{\mu} T_{k_1 \dots k_s, (\mu+\nu)} T^{\hat{i}_1 \dots \hat{i}_r, (M-\mu)}. \end{aligned}$$

4. Contractions and Reduced Range

In the hybrid extensor, the operation of contraction is the same as that of an ordinary extensor. Then, we obtain the following contraction of hybrid extensors.

Theorem 4.1. For an arbitrary given hybrid extensor of range M , the following $M+1$ kinds of quantities

$$\sum_{\rho=A}^M \binom{\rho}{A} T^{\hat{i}_1 \dots \hat{i}_r, j_1 \dots j_s, (\rho-A)} T^{j_1 \dots j_s, \hat{i}_1 \dots \hat{i}_r, (\rho)} \quad (A=0, 1, 2, \dots, M)$$

are invariant.

Proof. Applying the transformation for hybrid extensor, we know that these quantities are invariant:

$$\begin{aligned} &\sum_{\rho=A}^M \binom{\rho}{A} T^{\hat{i}_1 \dots \hat{i}_r, j_1 \dots j_s, (\rho-A)} T^{j_1 \dots j_s, \hat{i}_1 \dots \hat{i}_r, (\rho)} \\ &= \sum_{\rho=A}^M \binom{\rho}{A} X_{(\nu) \hat{a}_1 \dots \hat{a}_r, \hat{j}_1 \dots \hat{j}_s}^{\hat{i}_1 \dots \hat{i}_r, b_1 \dots b_s, (\rho-A)} X_{(\rho) c_1 \dots c_s, \hat{i}_1 \dots \hat{i}_r}^{j_1 \dots j_s, d_1 \dots d_r, (\omega)} T^{a_1 \dots a_r, b_1 \dots b_s, (\nu)} T^{c_1 \dots c_s, d_1 \dots d_r, (\omega)} \\ &= \sum_{\rho=A}^M \sum_{\nu=0}^{\rho-A} \sum_{\omega=\rho}^M \binom{\rho}{A} X_{(\nu) \hat{a}_1 \dots \hat{a}_r, \hat{j}_1 \dots \hat{j}_s}^{\hat{i}_1 \dots \hat{i}_r, b_1 \dots b_s, (\rho-A)} X_{(\rho) c_1 \dots c_s, \hat{i}_1 \dots \hat{i}_r}^{j_1 \dots j_s, d_1 \dots d_r, (\omega)} T^{a_1 \dots a_r, b_1 \dots b_s, (\nu)} T^{c_1 \dots c_s, d_1 \dots d_r, (\omega)} \\ &= \sum_{\beta=0}^{M-A} \sum_{\nu=0}^{\beta} \sum_{\omega=\beta+A}^M \binom{\beta+A}{A} X_{(\nu) \hat{a}_1 \dots \hat{a}_r, \hat{j}_1 \dots \hat{j}_s}^{\hat{i}_1 \dots \hat{i}_r, b_1 \dots b_s, (\beta)} X_{(\beta+A) c_1 \dots c_s, \hat{i}_1 \dots \hat{i}_r}^{b_1 \dots b_s, d_1 \dots d_r, (\omega)} T^{a_1 \dots a_r, b_1 \dots b_s, (\nu)} T^{c_1 \dots c_s, d_1 \dots d_r, (\omega)} \\ &\hspace{15em} (\text{putting } \rho = \beta + A) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\beta=0}^{M-A} \sum_{\nu=0}^{\beta} \sum_{\rho=\beta}^{M-A} \binom{\beta+A}{A} X_{(\nu) a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s(\beta)} X_{(\beta+A) c_1 \dots c_s, b_1 \dots b_s}^{j_1 \dots j_s, d_1 \dots d_r(\rho+A)} T^{a_1 \dots a_r, b_1 \dots b_s}{}^{(\nu)} T^{c_1 \dots c_s, d_1 \dots d_r(\rho+A)} \\
&\hspace{15em} (\text{putting } \omega = A + \rho) \\
&= \sum_{\beta=0}^{M-A} \sum_{\nu=0}^{\beta} \sum_{\rho=\beta}^{M-A} \binom{\beta+A}{A} \binom{\rho+A}{\beta+A} \binom{\rho}{\beta}^{-1} X_{(\nu) a_1 \dots a_r, j_1 \dots j_s}^{i_1 \dots i_r, b_1 \dots b_s(\beta)} \\
&\quad \times X_{(\beta) c_1 \dots c_s, b_1 \dots b_s}^{j_1 \dots j_s, d_1 \dots d_r(\rho)} T^{a_1 \dots a_r, b_1 \dots b_s}{}^{(\nu)} T^{c_1 \dots c_s, d_1 \dots d_r(\rho+A)} \\
&= \sum_{\nu=0}^{M-A} \sum_{\rho=\nu}^{M-A} \binom{\rho+A}{A} \delta_{\nu}^{\rho} \delta_{a_1 \dots a_r, c_1 \dots c_s}^{d_1 \dots d_r, b_1 \dots b_s} T^{a_1 \dots a_r, b_1 \dots b_s}{}^{(\nu)} T^{c_1 \dots c_s, d_1 \dots d_r(\rho+A)} \\
&= \sum_{\rho=0}^{M-A} \binom{\rho+A}{A} T^{a_1 \dots a_r, b_1 \dots b_s}{}^{(\rho)} T^{b_1 \dots b_s, a_1 \dots a_r(\rho+A)} \\
&= \sum_{\delta=A}^M \binom{\delta}{A} T^{a_1 \dots a_r, b_1 \dots b_s}{}^{(\delta-A)} T^{b_1 \dots b_s, a_1 \dots a_r(\delta)}. \quad (\text{putting } \rho = \delta - A) \quad \text{Q. E. D.}
\end{aligned}$$

From the results of Theorem 3.2, 3.3, 3.4 and 3.5, we obtain the following other kind of contraction.

Theorem 4.2. If $T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\nu)}$ and $T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(\mu)}$ are the components of arbitrary excontravariant and excovariant hybrid extensor of range M respectively, then the quantities

$$T^{i_1 \dots i_r, l_1 \dots l_q}{}^{(\beta+\alpha)} \equiv \sum_{\alpha=0}^{M-\beta} \binom{\beta+\alpha}{\alpha} T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\nu)} T^{j_1 \dots j_s, l_1 \dots l_q}{}^{(\beta+\alpha)}$$

are also the components of excovariant hybrid extensor of range M .

The existence of the hybrid extensor of reduced range is shown by the following theorem.

Theorem 4.3. If $T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\mu)}$ and $T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(\nu)}$ are the components of excontravariant and excovariant hybrid extensor of range M respectively, then the quantities

$$\begin{aligned}
&T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\omega)}, \quad (0 \leq \omega \leq K \leq M) \\
&\binom{M-K+\nu}{\nu} T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(M-K+\nu)} \equiv \mathfrak{B}_{[M-K]} T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(\nu)*} \quad (0 \leq \nu \leq K \leq M)
\end{aligned}$$

are the components of hybrid extensor of reduced range $0 \leq \omega \leq K$ and $0 \leq \nu \leq K$ respectively.

Applying these quantities to Theorem 3.2, 3.3, 3.4 and 3.5, we have

Theorem 4.4. If $T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\mu)}$ are the components of reduced range $0 \leq \mu \leq K$ and $T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(\nu)}$ are of range $0 \leq \nu \leq L$ ($K \leq L \leq M$), then the quantities

$$\sum_{\nu=0}^{\omega} \binom{\omega}{\nu} T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\omega-\nu)} T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(\nu)}$$

are the components of excontravariant hybrid extensor of reduced range $0 \leq \omega \leq K$.

Theorem 4.5. If $\mathfrak{B}_{[M-K]} T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\delta)}$ are the components of reduced range $0 \leq \delta \leq K$ and $T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(\sigma)}$ are of range M , then the quantities

$$\sum_{\beta=0}^{K-\gamma} \binom{M}{\beta}^{-1} \binom{\gamma+\beta}{\beta} [\mathfrak{B}_{[M-K]} T^{i_1 \dots i_r, j_1 \dots j_s}{}^{(\gamma+\beta)}] T^{k_1 \dots k_s, l_1 \dots l_q}{}^{(M-\beta)}$$

are the components of excovariant hybrid extensor of reduced range $0 \leq \gamma \leq K$.

* This notation is due to A. Kawaguchi's paper 4).

Theorem 4.6. If $\mathfrak{B}_{[M-K]} T^{i_1 \dots i_r}_{j_1 \dots j_s(r)}$ are the components of reduced range $0 \leq r \leq K$ and $T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\alpha)}$ are of range $0 \leq \alpha \leq L$ ($K \leq L \leq M$), then the quantities

$$\sum_{\alpha=0}^{K-\beta} \binom{\beta+\alpha}{\alpha} [\mathfrak{B}_{[M-K]} T^{i_1 \dots i_r}_{j_1 \dots j_s(\beta+\alpha)}] T^{k_1 \dots k_p}_{l_1 \dots l_q}^{(\alpha)}$$

are the components of excovariant hybrid extensor of reduced range $0 \leq \beta \leq K$.

5. Conclusions

In order to consolidate the extensor analysis, an investigation of the existence of various unknown extensors is important at present.

The hybrid extensor analysis, proposed in this paper, is one of the available means to treat numerous varieties of quantities concerned with higher order differentials. And we have studied its properties in detail. Moreover, we have proved that the hybrid extensor contains the extensor, the crossed extensor and the generalized crossed extensor as a special case.

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