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Numerical Studies for the Nonlinear Behavior of the One-Dimensional Vlasov Plasma by the Power Transform Method

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Abstract

The nonlinear electrostatic wave behavior of a Vlasov plasma is studied numerically for perturbation applied to a Maxwellian plasma with two beams. The solutions for large perturbations are carried out beyond the time points of minimum electrostatic amplitudes. After reaching their minimum values, some modes which are linearly stable grow, due to the interaction with particles and the mode coupling effect. The effect of a weak beam of electrons with shifted Maxwellian distribution on a Maxwellian plasma is studied. After the initial conditions die out, mode behavior as would be expected according to quasilinear theory was seen. The nonlinear response of Vlasov plasma is calculated for the perturbations applied to a spacially inhomogenous equilibrium. The period of a fundamental spacial mode closely agrees with the second and third modes. Each mode approaches a new equilibrium after $t \approx 18$.

1. Introduction

Two fundamentally different formulations for the investigation of the nonlinear effect in plasma physics exist in the numerical simulation. One method consists of numerically solving the Vlasov-Poisson set of equations, while the other computes the dynamics of a large number of charged particles as their self-consistent electric field.

In this paper, a numerical simulation of Vlasov-Pisson set of equations was performed using the Power Transform method developed by Joyce and others[1,2]. This method is amenable to an extrapolation procedure which avoids the difficulty of cutoff of higher terms and yields an appreciable savings in computer time compared with the Fourier Hermite expansion method.

We consider the problems for the plasma oscillations regarding spacially homogeneous and inhomogeneous equilibriums. The problems are one dimensional, with periodic boundary conitions. The plasma is considered to be a collisionless electron gas; i.e. only the motions of electrons will be considered and the system is macroscopically neutral with a uniform immobile positive background. Only electrostatic forces between the charges are considered. The Vlasov-Poisson set of equation is given by

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{e}{m}E\frac{\partial f}{\partial v}=0, \tag{1}$$

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$$\frac{\partial E}{\partial x}=4\pi ne\left(1-\int_{-\infty}^{+\infty}fdv\right)+4\pi\rho_{\text{ext}}, \tag{2}$$

where v is the velocity, e and m are the electronic charge and mass respectively, $f(x, v, t)$ is the distribution, $E(x, t)$ is the electric field and ρ_{ext} (x) is an external source charge density which is regarded as generating the external part of the total electric field. In § 2, eqs. (1), (2) are written in dimensionless form for the convenience of calculations. The equations are expanded in the infinite system of ordinary differential equations by using the Power Transform method. A self-consistent system is obtained by evaluating the coefficients $a_{n,\nu_{\max}+1}$ from $a_{n,0}$, $a_{n,1}\cdots$ $a_{n,\nu_{\max}}$ by a polynomial extrapolation.

Emery and Joyce[2] have solved the problemes of weak Landau damping in a Maxwellian plasma and the two beam instability with electron beams of equal intensity by the Power Transform method. Joyce and others[3] have studied the effect of a weak beam of electrons on a Maxwellian plasma. In § 3, we study two problems ; first, strong nonlinear Landau damping in a Maxwellian plasma with two beams, second, the effect of a weak beam of shifted Maxwellian electrons on a Maxwellian plasma. The nonlinear response of two Fourier modes for a short time has been studied using the Fourier Hermite expansion method for a spacially inhomogeneous equilibrated plasma by Harding[4,5]. In § 4, we study the nonlinear response of three Fourier modes for a long time by Power Transform method.

2. Computational Method

To write eqs. (1), (2) in dimensionless form, we introduce the dimensionless quantities $fv_{th}\to f$, $t\omega_p\to t$, $v/v_{th}\to v$, $x/\lambda_D\to x$, $\rho_{\text{ext}}/ne\to\rho_{\text{ext}}$, $Ee\lambda_D/mv_{th}^2\to E$, where v_{th} is the thermal velocity, ω_p is the plasma frequency, and λ_D is the Deby length. In terms of the dimensionless quantities, eqs. (1), (2) become

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-E\frac{\partial f}{\partial v}=0, \tag{3}$$

$$\frac{\partial E}{\partial x}=1-\int fdv+\rho_{\text{ext}}. \tag{4}$$

Furthermore, using a Fourier expansion in x and Fourier transform in v for f, E, ρ_{ext} and eqs. (3), (4), we obtain

$$f(x,\ v,\ t)=\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_n(y,\ t)\exp(-ivy)\frac{dy}{2\pi}\exp(ink_0x), \tag{5}$$

$$E(x,\ t)=\sum_{n=-\infty}^{+\infty}E_n(t)\exp(ink_0x), \tag{6}$$

$$\rho_{\text{ext}}(x)=\sum_{n=-\infty}^{+\infty}\rho_n\exp(ink_0x), \tag{7}$$

$$\frac{\partial F_n}{\partial t}+nk_0\frac{\partial F_n}{\partial y}+\sum_{m=-\infty}^{+\infty}iyF_mF_{n-m}=0, \tag{8}$$

$$ink_0E_n(t)=-F_n(0,t)+\rho_n \qquad (n\neq 0), \tag{9}$$

$$n=0,\ \pm1,\ \pm2,\ \pm3,\cdots,$$

where k_0 is the fundamental wave number, and we require $E_0(t)=0$.

According to Joyce[1], F_n (y, t) is written as an expansion in powers of y,

$$F_n(y,\ t)=\sum_{\nu=0}^{\infty}a_{n,\nu}(t)g_\nu y^\nu\exp\left(-\frac{1}{2}y^2\right), \tag{10}$$

$$g_\nu = 2^{\nu/2}\Gamma(\nu/2+1)/\Gamma(\nu+1),$$
$$\nu = 0,\ 1,\ 2,\ 3, \cdots.$$

When series (10) is inserted into eqs. (8), (9) and equal powers of y are calculated, we obtain a system

$$\left.\begin{aligned} &\dot{a}_{n,\nu} + nk_0\gamma_\nu[(\nu+1)a_{n,\nu+1} - \nu a_{n,\nu-1}] + i\nu\gamma_\nu \sum_{m=-\infty}^{+\infty} E_m a_{n-m,\nu-1} = 0, \\ &E_n = -(a_{n,0} - \rho_n)/ink_0, \\ &\gamma_\nu = g_{\nu+1}/g_\nu. \end{aligned}\right\} \qquad (11)$$

For the actual computations, g_ν was chosen in such a manner that $a_{n,\nu}$ a tendency to be of the same order of magnitude was obtained. Equation (10) is truncated in $\nu = \nu_{max}$. As eq. (11) is the infinite recursive system of ordinary differential equations, it must be terminated by evaluating $a_{n,\nu max+1}$. There is no reason to assume any regularity between the $a_{n,\nu}$, however, when a version of eq. (11) corresponding to linearized Vlasov equation is numerically integrated, $a_{n,\nu}$ lies on a smooth curve as a function of ν for even small ν_{max}[1]. Therefore, for the nonlinear system, we used the polynomial extrapolation scheme for large ν_{max}, so that $a_{n,\nu max+1}$ are determined by a fourth order polynomial fit to the last five of the coefficients.

To evaluate to what extent of accuracy the computer program solves the truncated system, the conservation law[5] which is obtained from eqs. (7), (8),

$$-\frac{1}{2}\frac{\partial^2 F_0(y,t)}{\partial t^2}\bigg|_{y=0} + \frac{1}{2}\sum_{n=-\infty}^{+\infty} |E_n(t)|^2 = \text{const.} \qquad (12)$$

is used. Equation (11) is solved by using the Runge-Kutta method.

3. Numerical Results of Plasma Oscillations about Homogeneous Equilibrium

We consider the no external source charge density ρ_{ext}, in this section is present.

3.1 Initial condition with two beams

The equilibrium distribution of particles f_0 and the initial perturbation f_1 are given by

$$f_0(v) = \frac{A}{\sqrt{2\pi}}(1+Bv^2)e^{-\frac{1}{2}v^2}, \qquad (13)$$

$$f_1(x,v) = \sum_{n=-m}^{m} \varepsilon|n|k_0 f_0(v)\exp(ink_0x), \qquad (14)$$

For large m, the initial electric field for the perturbation is

$$E(x,0) = \varepsilon\sum_{n=-m}^{m} i\exp(ink_0x), \qquad (15)$$

$$\left.\begin{aligned} &\approx 2m\varepsilon i \qquad \text{for } x=0, \\ &\approx 0 \qquad \text{for } x \neq 0. \end{aligned}\right\} \qquad (16)$$

From eqs. (5), (10) and $\int(f_0+f_1)dvdx = 1$, initial values of $a_{n,\nu}$ are obtained as follows;

$$\left.\begin{aligned} &a_{0,0}=1, \qquad a_{0,1}=0, \qquad a_{0,2}=-B/(B+1); \\ &a_{0,\nu}=0 \qquad\qquad \text{for } \nu \geq 3; \\ &a_{n,0}=\varepsilon|n|k_0, \qquad a_{n,1}=0, \qquad a_{n,2}=-\varepsilon|n|k_0B/(B+1) \text{ for } n\neq 0; \end{aligned}\right\} \qquad (17)$$

$$a_{n,\nu}=0 \qquad \text{for } n \neq 0, \ \nu \geq 3. \tag{17}$$

Under the initial condition (17), the system (11) is solved for $B=0.4$, $k_0=0.075$, $m=8$, $\nu_{max}=50$ with $\varepsilon=10^{-3}$, 3×10^{-2} and 7×10^{-2}. In Figs. 1, 2, 3, the amplitudes

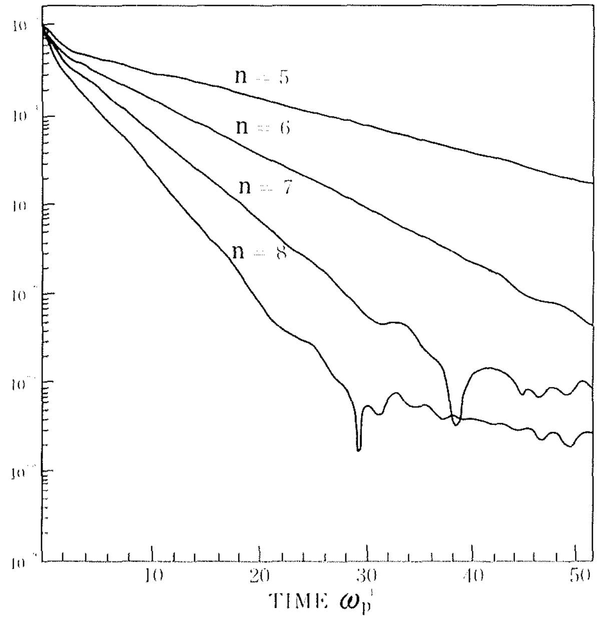

Fig. 1 Oscilations about homogeneous eqilibrium plasma with two beams; the envelopes of maxima of electric fields for $k_0=0.075$, $\varepsilon=10^{-3}$ with $n=5, 6, 7, 8$.

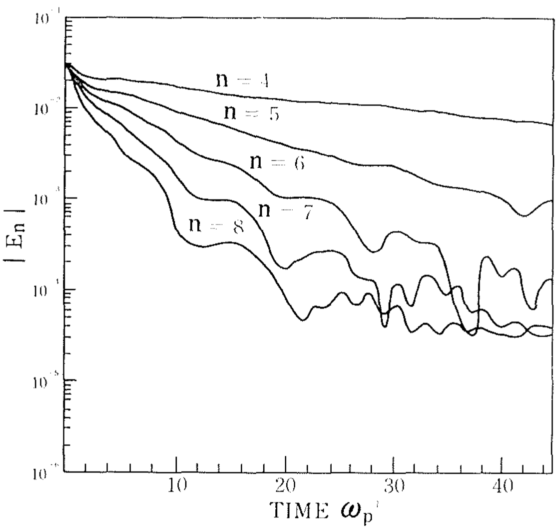

Fig. 2 Oscillations about homogeneous equilibrium plasma with two beams; the envelopes of maxima of electric fields for $k_0=0.075$, $\varepsilon=3\times10^{-2}$ with $n=4, 5, 6, 7, 8$.

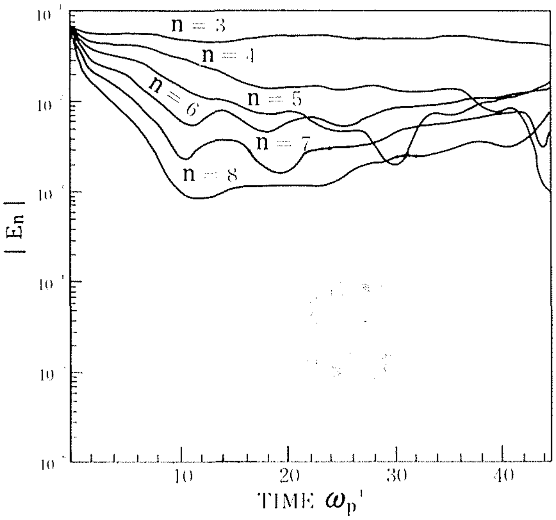

Fig. 3 Oscillations about homogeneous equilibrium plasma with two beams; the envelopes of maxima of electric fields for $k_0=0.075$, $\varepsilon=7\times10^{-2}$ with $n=3, 4, 5, 6, 7, 8$.

of electric fields $E_3 \sim E_8$ are plotted on a logarithmic scale versus time so that each curve exhibits an envelope of the maxima of $|E_n|$. Each solution departs from linear behavior; the severity of the departure and the time at which the departure occurs depends on ε and n. As ε or n is increased, the time of departure value decreases. For example, when $\varepsilon=10^{-3}$, 3×10^{-2}, the times of that are $t\approx40$, 10 or $t\approx25$, 8 according to $n=6$ or 7. When the mode coupling terms are neglected, the electric field does not fluctuate after the nonlinear effect appears[3]. Therefore, we find that the fluctuations of each mode indicate mode coupling with other modes. When $\varepsilon=7\times10^{-2}$, the nonlinear effects due to the interaction with particles and mode coupling are sufficiently large so that modes with $n\geq6$ begin growing at the time point $t\approx10$.

3.2 Initial condition with one beam

The unstable equilibrium of particles f_0 is given by Maxwellian plasma with shifted Maxwellian electrons. The initial perturbation f_1 is given by a beam of shifted Maxwellian electrons.

$$f_0(v)=\frac{A}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}+\frac{B}{\sqrt{2\pi}}e^{-\frac{1}{2}(v-v_0)^2} \tag{18}$$

$$f_1(x, v)=\sum_{n=-m}^{m}\varepsilon|n|k_0\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(v-v_0)^2}\exp(ink_0x). \tag{19}$$

From eqs. (5), (10) and $\int(f_0+f_1)dvdx=1$, initial values of $a_{n,\nu}$ are obtained as follows;

$$a_{0,0}=1, \qquad a_{0,\nu}=\frac{B}{g\nu}\frac{i^\nu v_0^\nu}{\nu!} \quad (\nu\neq0), \tag{20}$$

$$a_{n,\nu}=\frac{\varepsilon|n|k_0}{g^\nu}\frac{i^\nu v_0^\nu}{\nu!} \quad (n\neq0), \tag{21}$$

$$E_n(0)=i\varepsilon \tag{22}$$

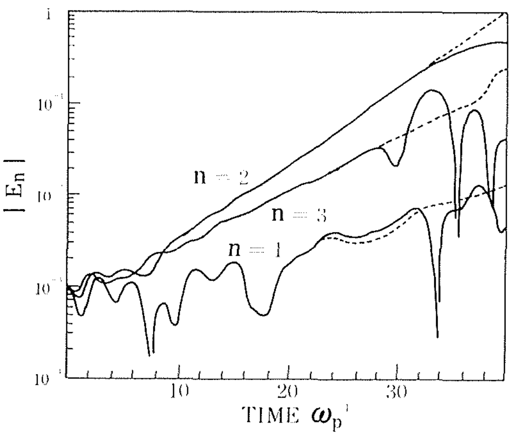

Fig. 4 Oscillations about homogeneous equilibrium plasma with one beam; absolute values of the unstable modes $n=1, 2, 3$, for $k_0=0.0795$, $\varepsilon=10^{-3}$. The solid and dotted lines represent the solutions for the nonlinear Vlasov equation and the linearized Vlasov equation.

Since eqs. (20), (21) are imaginary, we consider the absolute values of $a_{n,\nu}$ are the function of ν and extrapolate $a_{n,\nu_{max}+1}$.

Under the initial conditions (20), (21), the system (11) is solved for $A=0.092$, $B=0.08$, $v_0=7$, $k_0=0.0795$, $m=3$, $\nu_{max}=80$, with $\varepsilon=10^{-3}$. In Fig. 4, the absolute values of three unstable modes are plotted on a logarithmic scale versus time. We note that each curve does not exhibit an envelope of the maxima of $|E_n|$. The solution departs from linear behavior at $t\approx 25$, and after $t\approx 35$ the mode coupling effect appears, and three modes behave as may be expected according to the quasilinear theory[6].

4. Numerical Results of Plasma Oscillations in the Vicinity of Inhomogeneous Equilibrium

We evaluate a spacially inhomogeneous equilibrium of a trigonometrical type and study nonlinear plasma oscillation in the vicinity of the equilibrium given by Harding[4].

The spacially inhomogeneous equilibrium is constructed from

$$\rho_{ext}=\varepsilon \cos k_0 x, \qquad \varepsilon=\text{const}, \tag{23}$$

$$f=\frac{A}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}v^2+\phi(x)\right), \tag{24}$$

where $\phi(x)$ is the electrostatic potential.

We consider the solution for $\phi(x)$ which can be written as a rapidly converging Fourier series $\phi(x)=A_1 \cos k_0x+A_2 \cos k_0x+A_3 \cos k_0x+\ldots$ with $|A_1|\gg|A_2|\gg|A_3|$, etc. and A_1 sufficiently small so that exponential series $\exp(\phi(x))=1+\phi+\phi^2/2!+\ldots$ also converges rapidly. Using these expansions and eq. (24) in Poisson's eq. (4) and relating terms through order A_1^3, we obtain a set of relations involving A_1 and k_0

$$A_1=\frac{1}{1+A_1^2/4},\qquad A_2=\frac{-AA_1^2/4}{A+4k_0^2},\qquad A_3=-\frac{AA_1}{24}\left(\frac{A_1^2+12A_2}{A+9k_0^2}\right),$$
$$\epsilon=A_1(A+k_0^2)+AA_1(A_1^2+4A_2)/8. \tag{26}$$

Expanding the eq. (24) by eqs. (5), (10), we obtain the equilibrium coefficients $a_{n,\nu}$.

$$a_{0,0}=A(1+A_1^2/4),\qquad a_{1,0}=A(A_1^2+A_1A_2/2+A_1^3/8)/2,$$
$$a_{2,0}=A(A_2+A_1^2/4)/2,\qquad a_{3,0}=(A_3+A_1A_2/2+A_1^3/24)/2,$$
$$a_{n,\nu}=0\ (n>3). \tag{26}$$

The coefficients $a_{n,\nu}$ for $A_1=0.1$ and $k_0=0.5$ are used as initial values of eq. (11) for the first three modes and $\nu_{max}=60$ to see how well they represent a time independent solution, the coefficients $a_{1,0}$, $a_{2,0}$, $a_{3,0}$ from $t=0$ to 40 are found to oscillate with amplitudes smaller than 0.1% for $t\lesssim 2$ and 0.001% for $t>2$, in the vicinity of the predicted equilibrium values. Furthermore, after three or four trials it is possible to reduce the relative amplitudes of the oscillations of the $a_{1,0}$, $a_{2,0}$, $a_{3,0}$ to nearly 10^{-6}%, 5×10^{-6}%, 3×10^{-5}% by adjusting the starting values, respectively. The amplitudes are much smaller than that calculated by the Fourier Hermite expansion method[4,5].

We study the nonlinear electrostatic wave behavior for the following initial perturbation f_1 applied to the inhomogeneous equilibrium;

$$f_1(x,v)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}\cdot\varepsilon k_0\cos k_0x. \tag{27}$$

The perturbed electric field is evaluated from eq. (11) for $A_1=0.1$, $k_0=0.5$, $\varepsilon=0.01$ with $\nu_{max}=60$ and 100. In Fig. 5, the absolute values of the perturbed electric fields are plotted on a logarithmic scale versus time. The solutions for each ν_{max} disagree with other ν_{max} after $t\approx 20$, but does not disagree qualitatively.

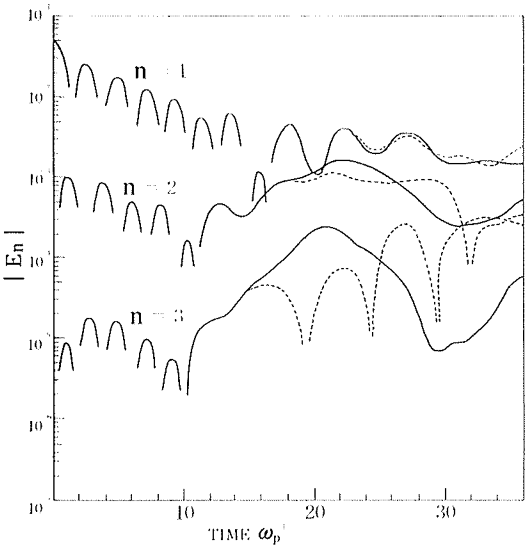

Fig. 5 Plasma oscillations about inhomogeneous equilibrium; the absolute values of the perturbed electric fields for $A_1=0.1$, $\epsilon=0.1248$, $k_0=0.5$, $\varepsilon=0.1$ with $n=1,2,3$ and $\nu_{max}=60$ (—), $\nu_{max}=100$ (···)

Each solution departs from linear behavior; the time of departure from linear behavior, i.e. $t \approx 10$ is much smaller than that of the solution for a homogeneous equilibrium of Maxwellian plasma, and the plasma oscillation disappears after $t \approx 18$. Thus the nonlinear plasma oscillation approches the equilibrium in a different manner from the initial one. It shuld be noted that before $t \approx 12$ or 10 the period of the fundamental mode nearly agrees with that of the second or third mode due to the effect of the trapped electrons.

Harding calculated the nonlinear response of two Fourier modes from $t=0$ to 14, and concluded that the damping of the second mode can be considered to be essentially linear. Our results for the fudamental mode agrees with the results of Harding, but the second mode is slightly different.

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