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The Formulation of the Wall Stabilization Problem of an Axisymmetrical Toroidal Sharp-Boundary Plasma with a Horizontally Elongated Noncircular Cross Section

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Abstract

In flat-ring cyclide coordinates, we formulate the wall stabilization problem of an axisymmetric toroidal sharp-boundary plasma with a horizontally elongated noncircular cross section which is considered approximately as an elliptical cross section. As a result of the formulation, the argument of the Wangerin functions, which are solutions of the Laplace equation in the flat-ring cyclide coordinates, is represented by the aspect ratio and the elongation ratio of a toroidal geometry, in such a way that we can treat the energy principle without an expansion in the aspect ratio. The energy principle can be expressed as a symmetric energy matrix which consists of a quadratic form. The problem of stability of a toroidal plasma is reduced to that of examining the eigenvalues of the energy matrix. Therefore, the plasma is marginally stable if the lowest eigenvalue is zero in parameter spaces (the toroidal perturbation number, the poloidal field, the aspect ratio, the elongation ratio, the plasma beta, and the wall shape). Furthermore, critical poloidal beta's in equilibrium are obtained analytically.

1. Introduction

In any magnetically confined fusion device, there is a premium of operation at the highest possible beta because the fusion power output at a fixed magnetic field depends on the square of the plasma beta. Since much of the capital cost of a magnetically confined fusion reactor is associated with the production of a magnetic field, high beta operation is a necessary ingredient in the formulation of a low capital cost system[1]. Generally, the high beta plasma is yielded in such a way that the cross section of plasma can be controlled in a noncircular form. Therefore, it is very important to investigate the stability of a high beta plasma with a noncircular cross section.

The ideal magnetohydrodynamic stability of an axisymmetric toroidal sharp-boundary plasma with a circular cross section has been investigated by many authors[2-6]. Lüst et al[2]., Bateman[3], and Kito and Honma[4,5] treated problems of the stability in toroidal coordinates (σ, ψ, ϕ), while Freidberg and Hass[6] treated

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problems in pseudo-toroidal coordinates (r, θ, ϕ).

On the other hand, the problem of stability of a toroidal sharp-boundary plasma with a noncircular cross section has been solved by Freidberg and Hass[7], and Freidberg and Grossmann[8] limited however to pseudo-toroidal coordinates. However, this requires a very complex method in order to obtain a fast and accurate numerical evaluation of the Laplace equation which appears in the problem. Consequently, flat-ring coordinates (μ, ν, ϕ) are introduced into the plasma in which a solution of the Laplace equation can be expressed analytically in terms of a Wangerin function[9]. The argument of the Wangerin functions is given exactly by both the aspect ratio and elongation ratio of toroidal geometry. Therefore, the only requirement to solve the problem of stability is a numerical evaluation of the Wangerin functions with high accuracy. We can treat the wall stabilization problem of the toroidal sharp-boundary plasma with a horizontally elongated cross section.

In this paper, we assume the cross section of the plasma to be a horizontally elongated noncircular, and the cross section of the wall to be an arbitrary shape. Making use of the flat-ring cyclide coordinates, we formulate the problem of the stability of the axisymmetric toroidal sharp-boundary plasma in a perfectly conducting toroidal wall. This is our aim in the present investigation. The remainder of this paper is arranged as follows. In the next section, the argument of the Wangerin functions is expressed as both the aspect ratio and the elongation ratio. The unit vectors of the flat-ring cyclide coordinates are related to the unit vectors of the rectangular coordinates. In section 3 we discuss an equilibrium of the toroidal sharp-boundary plasma in the flat-ring cyclide coordinates under the assumption of uniform pressure confined only by surface currents. Critical poloidal beta's are obtained. In section 4 we formulate the problem of the wall stabilization of plasma with a horizontally elongated cross section in the flat-ring cyclide coordinates. In the final section we discuss our results.

2. Toroidal Geometry

We consider a toroidal plasma which lies in a perfectly conducting wall. The plasma and wall are assumed to have the same axes of symmetry (X-, Z-axes). The cross section of the plasma is assumed to be a horizontally elongated noncircular, which is considered as an approximate elliptical cross section. However, the cross section of the wall is assumed to have an arbitrary shape as is shown in Fig. 1.

We introduce the flat-ring cyclide coordinates (μ, ν, ϕ) into the plasma geometry. In the coordinates, the curve $\mu=$const. and $\phi=$const. describe the cross section of the plasma, so that the direction of the coordinates ν and ϕ denotes the poloidal and toroidal direction of the plasma, respectively. On the major circumference of the plasma defined by $\mu=\mu_p$, we define the major radius R_p of the plasma as the distance from Z-axis to the midway point between $\nu=K'$ and $\nu=0$ on the horizontal axis, i.e.,

$$R_p=\frac{c}{2k}\frac{1+k\,\mathrm{sn}^2\mu_p}{\mathrm{sn}\,\mu_p},$$

where c is the fundamental length of the flat-ring cyclide coordinates, k is the modulus and K' is the complete elliptic integral of the first kind with the modulus $k'=(1-k^2)^{1/2}$. A horizontal minor radius a_p of the plasma results in

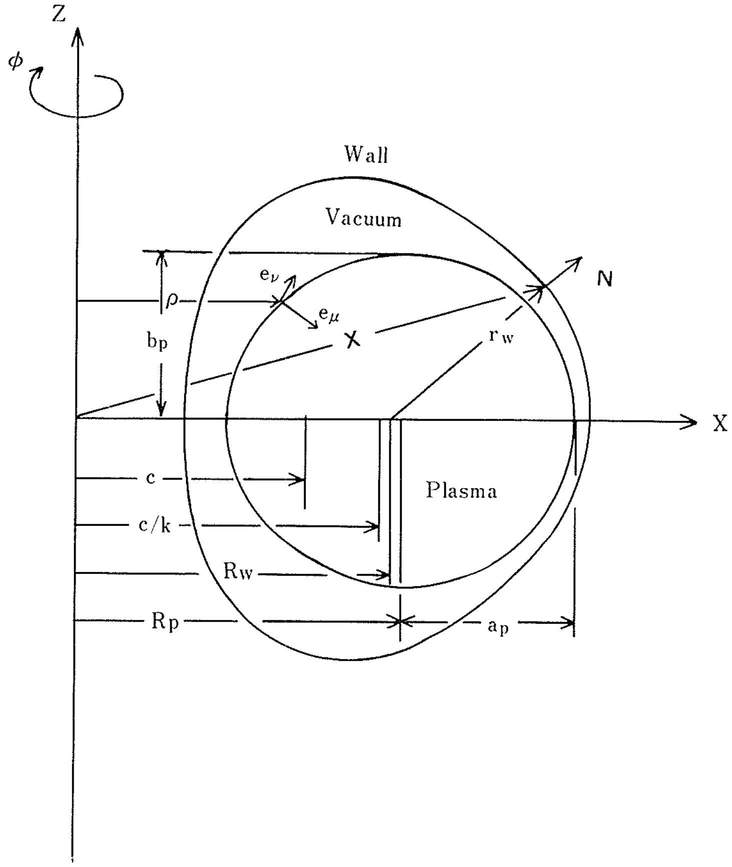

Fig. 1. Toroidal geometry in flat-ring cyclide coordinates (μ, ν, ϕ).

$$a_p=\frac{c}{2k}\frac{1-k\,\mathrm{sn}^2\,\mu_p}{\mathrm{sn}\,\mu_p},$$

and a relation between R_p and a_p is given by $c^2/k=R_p^2-a_p^2$. A vertical minor radius b_p of the plasma is derived from the conditions $\partial Z/\partial\rho=0$ and $\mu=\phi=$ const.,

$$b_p=\frac{c}{2k}\frac{\mathrm{cn}\,\mu_p\,\mathrm{dn}\,\mu_p}{\mathrm{sn}\,\mu_p},$$

where $\rho=(x^2+y^2)^{1/2}$ is the distance from Z-axis.

In accordance with Freidberg and Hass's definition[7], we define an aspect ratio A by R_p/a_p and an elongation ratio E of the horizontal to vertical minor radius by a_p/b_p, so that the modulus k and the function $\mathrm{sn}^2\mu_p$ can be expressed in terms of A and E, i.e.,

$$k=1+\frac{2}{A^2-1}\left[1-\frac{1}{E^2}-\sqrt{\left(1-\frac{1}{E^2}\right)\left(A^2-\frac{1}{E^2}\right)}\;\right], \tag{2-1}$$

$$\mathrm{sn}^2\,\mu_p=\frac{1}{k}\,\frac{A-1}{A+1}. \tag{2-2}$$

In the flat-ring cyclide coordinates, solutions for the Laplace equation can be expressed as the Wangerin functions and the argument of the functions is given by $\mathrm{sn}^2\mu$. Therefore, it is convenient to define the ratio A and E as above. If we put $E=1$, then the modulus k becomes one and the aspect ratio A is $\cosh 2\mu$. For $E=1$, the flat-ring cyclide coordinates agree with the toroidal coordinates.

The flat-ring cyclide coordinate system is one of the orthogonal systems of curvilinear coordinates, so that the metric tensors are given by

$$h_1=h_2=\frac{c}{\Lambda}\Omega, \qquad h_3=\frac{c}{\Lambda}\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu,$$

where

$$\Lambda=1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu, \qquad \Omega^2=(1-\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu)(\mathrm{dn}^2\nu-k^2\mathrm{sn}^2\mu).$$

The flat-ring cyclide coordinates are related to the rectangular ones by the equations

$$\begin{aligned} x&=\frac{c}{\Lambda}\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu\cos\phi,\\ y&=-\frac{c}{\Lambda}\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu\sin\phi, \\ z&=\frac{c}{\Lambda}\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu. \end{aligned} \tag{2-3}$$

The unit vectors $(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu, \boldsymbol{e}_\phi)$ of the flat-ring cyclide coordinates are related to the unit vectors $(\boldsymbol{e}_X, \boldsymbol{e}_Y, \boldsymbol{e}_Z)$ of the rectangular coordinates: $\boldsymbol{e}_\mu=(\partial\boldsymbol{X}/\partial\mu)/h_1$, $\boldsymbol{e}_\nu=(\partial\boldsymbol{X}/\partial\nu)/h_2$, $\boldsymbol{e}_\phi=(\partial\boldsymbol{X}/\partial\phi)/h_3$, where $\boldsymbol{X}=\boldsymbol{e}_X\mathrm{x}+\boldsymbol{e}_Y\mathrm{y}+\boldsymbol{e}_Z\mathrm{z}$. Therefore, we obtain

$$\begin{aligned} \boldsymbol{e}_\mu&=\boldsymbol{e}_X\alpha\cos\phi-\boldsymbol{e}_Y\alpha\sin\phi-\boldsymbol{e}_Z\beta,\\ \boldsymbol{e}_\nu&=\boldsymbol{e}_X\beta\cos\phi-\boldsymbol{e}_Y\beta\sin\phi+\boldsymbol{e}_Z\alpha,\\ \boldsymbol{e}_\phi&=-\boldsymbol{e}_X\sin\phi-\boldsymbol{e}_Y\cos\phi, \end{aligned} \tag{2-4}$$

where

$$\alpha=\frac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{\Lambda\Omega}(\mathrm{cn}^2\nu-k^2\mathrm{sn}^2\mu\,\mathrm{sn}^2\nu)\,\mathrm{dn}\,\nu,$$

$$\beta=\frac{\mathrm{sn}\,\mu}{\Lambda\Omega}(k^2\mathrm{cn}^2\mu+\mathrm{dn}^2\mu\,\mathrm{dn}^2\nu)\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu.$$

On the wall surface, we obtain

$$x=(R_W+r_W\cos\theta)\cos\phi, \qquad y=-(R_W+r_W\cos\theta)\sin\phi, \qquad z=r_W\sin\theta, \tag{2-5}$$

where R_W is the major radius of the wall and the wall cross section is given by $r_W=ag(\theta)$. Note that the center of the wall is defined to be midway between the inner and outer edge of the wall. The function $g(\theta)$ represents the cross section of the wall and is arbitrary. It is normalized so that $g(0)=g(\pi)=1$; thus a represents the horizontal radius of the wall. The normal unit vector $\boldsymbol{N}$ on the wall surface is expressed as $\boldsymbol{N}=(\boldsymbol{X}_1\times\boldsymbol{X}_2)/(g_{11}g_{22}-g_{12}^2)^{1/2}$, where $\boldsymbol{X}_1=\partial\boldsymbol{X}/\partial\theta$, $\boldsymbol{X}_2=\partial\boldsymbol{X}/\partial\phi$, $g_{ij}=\mathrm{X}_i\mathrm{X}_j$ $(i,j=1,2)$. Therefore, we obtain[5)]

$$\boldsymbol{N}=\frac{-\boldsymbol{e}_X(g\sin\theta)'\cos\phi+\boldsymbol{e}_Y(g\sin\theta)'\sin\phi+\boldsymbol{e}_Z(g\cos\theta)'}{[(g')^2+g^2]^{1/2}}, \tag{2-6}$$

where the prime represents the derivative with respect to θ.

3. Equilibrium

We consider an equilibrium of the plasma. The plasma is assumed to have a sharp boundary with uniform pressure confined by currents flowing only at the plasma surface. The aspect ratio A and the elongation ration E of the plasma are assumed arbitrarily.

In the sharp-boundary constant pressure model, the plasma has only a toroidal field. On the plasma surface, this field is given by $B_{\phi i}=cB_i/\rho$, where $\rho=h_3$ is the distance from Z-axis, and B_i is the internal toroidal field at $\nu=\nu_0$ which satisfies $\rho=c$, *i. e.*,

$$\mathrm{dn}\,\nu_0=\frac{1}{2\,\mathrm{dn}^2\,\mu}\left[(1-k^2)\mathrm{sn}\,\mu+\sqrt{(1-k^2)^2\,\mathrm{sn}^2\,\mu+4k^2\,\mathrm{dn}^2\,\mu\,\mathrm{cn}^2\,\mu}\ \right].$$

In the vacuum region between the plasma and wall, there are both a toroidal field and a poloidal field induced by toroidal currents flowing on the plasma surface. The toroidal field on the plasma surface is expressed as $\hat{B}_{\phi e}=c\hat{B}_0/\rho$, where $\hat{B}_0$ is the external toroidal field at $\nu=\nu_0$.

On the plasma surface $\mu=\mu_p$, we obtain the pressure balance relation

$$\hat{B}_\nu^2+\hat{B}_{\phi e}^2-B_{\phi i}^2=2\,P, \tag{3-1}$$

where P is the plasma pressure which is constant. If we define the plasma β_p by $\beta_p=\langle 2P\rangle/\langle\hat{B}_\nu^2\rangle$, where $\langle\ \rangle$ means the surface average, we obtain

$$2\,P=\frac{\beta_p(\hat{B}_0^2-B_i^2)I_1}{(\beta_p-1)\ I_2\,\mathrm{sn}^2\ \mu_p}, \tag{3-2}$$

where

$$I_1=\int_0^{K'}\frac{\Omega}{\mathrm{dn}\,\nu}d\nu,\qquad I_2=\int_0^{K'}\frac{\Omega}{\Lambda^2}\,\mathrm{dn}\,\nu d\nu.$$

The relation (3-1) at $\rho=c$ satisfies

$$\hat{B}_{\nu0}^2+\hat{B}_0^2-B_i^2=2P, \tag{3-3}$$

where $\hat{B}_{\nu0}=\hat{B}_\nu(\mu_p,\nu_0)$. Eliminating $2P$, $\hat{B}_0^2-B_i^2$ from Eqs. (3-1), (3-2) and (3-3), we obtain the poloidal field on the plasma surface

$$\frac{\hat{B}_\nu^2}{\hat{B}_{\nu0}^2}=\frac{\beta_pI_1-(\Lambda/\mathrm{dn}\,\nu)^2(\beta_p-1)\,I_2}{\beta_pI_1-(\beta_p-1)\,I_2\,\mathrm{sn}^2\,\mu_p}. \tag{3-4}$$

Furthermore, if we define the plasma β by $\beta=\langle 2\,P\rangle/\langle\hat{B}_\nu^2+\hat{B}_{\phi e}^2\rangle$, we obtain

$$\beta_p=\beta\frac{\gamma_a^2+1}{\gamma_a^2+\beta-\beta I_1/I_2\,\mathrm{sn}^2\,\mu_p},$$

where

$$\gamma_a=\hat{B}_{\nu0}/\hat{B}_0.$$

Another parameter of the plasma is a safety factor q which is defined by $q=\oint\mathrm{d}\,\phi/2\pi$ on the plasma surface. In the flat-ring cyclide coordinates, we obtain $\hat{B}_{\phi e}/\hat{B}_\nu=\mathrm{d}s_3/\mathrm{d}s_2=(\mathrm{sn}\,\mu_p\,\mathrm{dn}\,\nu/\Omega)\ (\mathrm{d}\,\phi/\,\mathrm{d}\nu)$, so that the factor is expressed as

$$q=\frac{B_0}{\pi\,\mathrm{sn}^2\,\mu_p}\int_0^{K'}\frac{\Lambda\Omega}{\mathrm{dn}^2\,\nu\hat{B}_\nu}\,d\nu. \tag{3-5}$$

A condition $\hat{B}_\nu(\nu=0)=0$ gives $q=\infty$, so that the condition represents an equilibrium limit. From eq. (3-4), the condition leads to

$$Max.\,\beta_p=\frac{I_2}{I_2-I_1}. \tag{3-6}$$

If we put $I_1/I_2\,\mathrm{sn}^2\,\mu_p=1$ in eq. (3-4), we obtain the other expression for critical poloidal beta, i.e.,

$$Max.\,\beta_p=\frac{1}{2}\left[E^2A+1+\sqrt{(E^2-1)(E^2A^2-1)}\right]. \tag{3-7}$$

For the elongation ratio equal to one, eq. (3-7) agrees with critical β_p's expression in the toroidal coordinates[4]. Critical poloidal beta's for the aspect ratio A and the ratio E is shown in Fig. 2, in which the critical β_p's increase in keeping with an elongation ratio at a fixed aspect ratio.

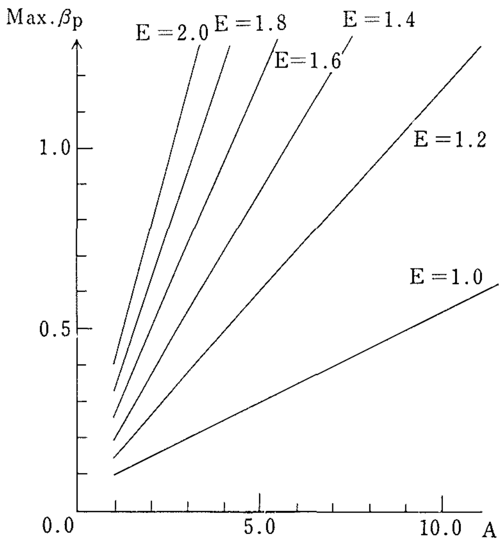

Fig. 2. Critical β_p's for the aspect ratio A and elongation ratio E.

4. Energy Principle

A stability of the plasma is examined by an analysis of the energy principle[10]. The principle for the sharp-boundary model is expressed as

$$\delta W=\delta W_p+\delta W_v+\delta W_s,$$

$$\delta W_p=\frac{1}{2}\int_p |B_1|^2 dV, \qquad \delta W_v=\frac{1}{2}\int_v |\hat{B}_1|^2 dV,$$

$$\delta W_s=\frac{1}{2}\int_s |\xi|^2 \boldsymbol{n}\cdot\langle \nabla B^2/2\rangle dS,$$

where the subscripts p, v, s represent the plasma, vacuum and surface terms respectively, $\boldsymbol{n}=-\boldsymbol{e}_\mu$ is the unit normal vector from the plasma surface to vacuum, and $\langle\ \rangle$ denotes the increment of a quantity across the perturbed magnetic fields.

In the sharp-boundary model, we only require the normal component of the perturbations $\boldsymbol{\xi}$ at the plasma surface. The normal component ξ_μ can be Fourier analyzed in the form

$$\xi_\mu=e^{-in\phi}\sum_{m=-\infty}^{\infty}\xi_m e^{im(\pi/K')\nu}, \tag{4-1}$$

where the ξ_m are undetermined coefficients.

We consider first the surface term. In the flat-ring cyclide coordinates, we obtain

$$\boldsymbol{n}\cdot\langle \nabla B^2/2\rangle=f/h_1h_3, \tag{4-2}$$

at $\mu=\mu_p$, where

$$f=\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{\Omega}\frac{\partial h_2}{\partial\mu}\hat{B}_\mu^2+\frac{\partial h_3}{\partial\mu}(\hat{B}_{\phi e}^2-B_{\phi i}^2).$$

In this derivation, we use the relations $\partial(\mathrm{h}_2\hat{B}_\nu)/\partial\mu=0$, $\partial(\mathrm{h}_3\hat{B}_{\phi e})/\partial\mu=0$, $\partial(\mathrm{h}_3 B_{\phi i})/\partial\mu=0$, which are obtained from $\nabla\times\hat{\boldsymbol{B}}_e=0$ and $\nabla\times\boldsymbol{B}_i=0$, where $\hat{\boldsymbol{B}}_e=\boldsymbol{e}_\mu\hat{B}_\mu+\boldsymbol{e}_\phi\hat{B}_{\phi e}$, $\boldsymbol{B}_i=\boldsymbol{e}_\phi B_{\phi i}$ at $\mu=\mu_p$.

From eqs. (4–1) and (4–2), we obtain

$$\delta W_s=\pi c\hat{B}_0^2\boldsymbol{\xi}^T\boldsymbol{H}\boldsymbol{\xi}^*.$$

Here, $\boldsymbol{\xi}$ is the column vector, $\boldsymbol{\xi}^T$ and $\boldsymbol{\xi}^*$ are the transpose and complex conjugated vectors of $\boldsymbol{\xi}$. The matrix $\boldsymbol{H}$ has the following elements at $\mu=\mu_p$

$$H_{mp}=\frac{2}{c\hat{B}_0^2}\int_0^{K'}f\cos\left(\frac{m-p}{K'}\pi\nu\right)d\nu=2\frac{r_a^2}{c}\int_0^{K'}d\nu\cos\left(\frac{m-p}{K'}\pi\nu\right)$$
$$\times\left[\frac{\Lambda^2}{\mathrm{sn}^2\,\mu\,\mathrm{dn}^2\,\nu}(\beta_p-1)\left(\frac{\partial h_3}{\partial\mu}-\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Omega}\frac{\partial h_2}{\partial\mu}\right)+\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Omega}\frac{\partial h_2}{\partial\mu}\beta_p\right],$$

where

$$\frac{\partial h_3}{\partial\mu}=\frac{c}{\Lambda^2}\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{dn}\,\nu(2\,\mathrm{cn}^2\,\nu-\Lambda),$$

$$\frac{\partial h_2}{\partial\nu}=\frac{2c}{\Lambda^2}\mathrm{sn}\,\nu\;\mathrm{cn}\,\nu\;\mathrm{dn}\,\nu\;[-k^{*2}(1-\mathrm{dn}^2\,\mu\,\mathrm{sn}^2\,\nu)(k^2\,\mathrm{sn}^4\,\mu-2\mathrm{sn}^2\,\mu\,\mathrm{dn}^2\,\nu+1)$$
$$+(1-\mathrm{sn}^2\,\mu\,\mathrm{dn}^2\,\nu)(\mathrm{dn}^2\,\nu-k^2\,\mathrm{sn}^2\,\mu)\mathrm{dn}^2\,\mu].$$

We now consider the plasma term. Under the minimizing perturbations, there exists the scalar function V, which we can write as $\boldsymbol{B}_1=\nabla V$ satisfying $\nabla^2V=0$. Therefore, δW_p is expressed as

$$\delta W_p=\frac{1}{2}\int_p\nabla V\cdot\nabla V^*dV.$$

If we apply Green's theorem

$$\int_v(f\nabla^2g+\nabla f\cdot\nabla g)dV=\int_s f\frac{\mathrm{d}g}{\mathrm{dn}}dS,$$

to the above integral, we obtain

$$\delta W_p=-\frac{\pi}{2}\int_{-K'}^{K'}d\nu h_3\left(V\frac{\partial V^*}{\partial\mu}+V^*\frac{\partial V}{\partial\mu}\right),\tag{4–3}$$

at $\mu=\mu_p$, where we use the directional derivative $\mathrm{d}\varphi/\mathrm{dn}=\boldsymbol{n}\cdot\nabla\varphi$.

Since we require V regular at $\mu=K$, the solution V of Laplace's equation is expressed as

$$V=i\Lambda^{1/2}e^{-in\phi}\sum_{m=-\infty}^{\infty}B_mT_m^ne^{im(\pi/K')\nu},\tag{4–4}$$

where T_m^n is the abbreviation of the Wangerin function $T_m^n(\mathrm{sn}^2\,\mu)$, and B_m is an arbitrary constant. Substituting eq. (4–4) into eq. (4–3), we obtain

$$\delta W_p=-\pi c\boldsymbol{B}^T\boldsymbol{G}\boldsymbol{B}^*,$$

where $\boldsymbol{B}$ is the column vector and $\boldsymbol{G}$ is the matrix whose elements are

$$G_{mp}=\mathrm{sn}\,\mu\int_0^{K'}d\nu\cos\left(\frac{m-p}{K'}\pi\nu\right)\frac{\mathrm{dn}\,\nu}{\Lambda^{1/2}}\left\{T_m^n\frac{\partial}{\partial\mu}(\Lambda^{1/2}T_p^n)+T_p^n\frac{\partial}{\partial\mu}(\Lambda^{1/2}T_m^n)\right\},$$

at $\mu=\mu_p$.

A relation between $\boldsymbol{B}$ and $\boldsymbol{\xi}$ is obtained from the boundary condition $\boldsymbol{n}\cdot\boldsymbol{B}_1=\boldsymbol{n}\cdot\nabla V=\boldsymbol{n}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_1)$ at $\mu=\mu_p$. After a straightforward calculation we find

$$\sum_{p=-\infty}^{\infty}\xi_p e^{ip(\pi/K')\nu}=-\frac{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}{\Lambda\Omega}\frac{1}{nB_i}\sum_{m=-\infty}^{\infty}e^{im(\pi/K')\nu}\frac{\partial}{\partial\mu}(\Lambda^{1/2}T_m^n)B_m,$$

so that the Fourier coefficients ξ_p are expressed as

$$\xi_p=-\sum_{m=-\infty}^{\infty}C_{pm}B_m,$$

where

$$C_{pm}=\frac{\mathrm{sn}^2\mu}{K'}\frac{1}{n\gamma_b}\int_0^{K'}d\nu\frac{\mathrm{dn}^2\nu}{\Omega\Lambda}\frac{\partial}{\partial\mu}(\Lambda^{1/2}T_m^n)\cos\Big(\frac{m-p}{K'}\pi\nu\Big),\qquad \gamma_b=\frac{B_i}{\hat{B}_0},$$

at $\mu=\mu_p$. Therefore, we obtain

$$\delta W_p=-\pi c\hat{B}_0^2\boldsymbol{\xi}^T(\boldsymbol{C}^{-1})^T\boldsymbol{G}\boldsymbol{C}^{-1}\boldsymbol{\xi}^*,$$

where $\boldsymbol{C}$ is the matrix whose elements are C_{pm} and $\boldsymbol{C}^{-1}$ represents the inverse matrix of $\boldsymbol{C}$.

Finally, we consider the vacuum term. Since the vacuum region lies between the wall and the plasma surface, δW_v is expressed as

$$\delta W_v=I_w-I_p,\qquad I_w=\frac{1}{2}\int_{wall}|\hat{\boldsymbol{B}}_1|^2dV,\qquad I_p=\frac{1}{2}\int_{plasma}|\hat{\boldsymbol{B}}_1|^2dV,$$

where the integrals of I_w and I_p are integrated over the volumes of the wall and plasma respectively.

Under the minimizing perturbations, there exists a scalar function $\hat{V}$ which we can write as $\hat{\boldsymbol{B}}_1=\nabla\hat{V}$ satisfying $\nabla^2\hat{V}=0$. The solution $\hat{V}$ of Laplace's equation is expressed as

$$\hat{V}=i\Lambda^{1/2}e^{-in\phi}\sum_{m=-\infty}^{\infty}(\hat{A}_mS_m^n+\hat{B}_mT_m^n)e^{im(\pi/K')\nu}, \tag{4-5}$$

where S_m^n is the abbreviation of the Wangerin function $S_m^n(\mathrm{sn}^2\mu)$, and $\hat{A}_m$, $\hat{B}_m$ are arbitrary constants. Substituting eq. (4-5) into I_p, we obtain

$$I_p=-c\pi\sum_{m,p=-\infty}^{\infty}[\hat{A}_m\hat{F}_{mp}^{(1)}\hat{A}_p^*+\hat{A}_m\hat{G}_{mp}^{(1)}\hat{B}_p^*+\hat{B}_m\hat{G}_{pm}^{(1)}\hat{A}_p^*+\hat{B}_m\hat{H}_{mp}^{(1)}\hat{B}_p^*], \tag{4-6}$$

at $\mu=\mu_p$, where

$$\hat{F}_{mp}^{(1)}=\int_0^{K'}d\nu\,\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Lambda^{1/2}}\cos\Big(\frac{m-p}{K'}\pi\nu\Big)(S_m^nS_p+S_p^nS_m),$$

$$\hat{G}_{mp}^{(1)}=\int_0^{K'}d\nu\,\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Lambda^{1/2}}\cos\Big(\frac{m-p}{K'}\pi\nu\Big)(S_m^nT_p+T_p^nS_m),$$

$$\hat{H}_{mp}^{(1)}=\int_0^{K'}d\nu\,\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Lambda^{1/2}}\cos\Big(\frac{m-p}{K'}\pi\nu\Big)(T_m^nT_p+T_p^nT_m),$$

S_p and T_p are the abbreviations of $\partial(\Lambda^{1/2}S_p^n)/\partial\mu$ and $\partial(\Lambda^{1/2}T_p^n)/\partial\mu$, respectively, and if we replace S_p and S_p^n by T_p and T_p^n in $\hat{F}_{mp}^{(1)}$ respectively, then we obtain the formula for $\hat{G}_{mp}^{(1)}$. Moreover, if we replace S_m^n and S_m by T_m^n and T_m in $\hat{G}_{mp}^{(1)}$ respectively, we obtain the formula for $\hat{H}_{mp}^{(1)}$. This derivation is similar to that of δW_p.

In accordance with Green's theorem, the integral I_w is replaced by the following surface integral on the wall surface

$$I_w=\frac{1}{2}\int_S\hat{V}\frac{d\hat{V}^*}{dN}dS.$$

The directional derivative at the wall surface is expressed as

$$\frac{d\hat{V}}{dN}=-\boldsymbol{N}\cdot\nabla\hat{V}=-(\boldsymbol{N}\cdot\boldsymbol{e}_\mu)\frac{1}{h_1}\frac{\partial\hat{V}}{\partial\mu}-(\boldsymbol{N}\cdot\boldsymbol{e}_\nu)\frac{1}{h_2}\frac{\partial\hat{V}}{\partial\nu}.$$

Therefore, we obtain

$$I_w=-c\pi\sum_{m,p=-\infty}^{\infty}[\hat{A}_m\hat{F}^{(2)}_{mp}\hat{A}^*_p+\hat{A}_m\hat{G}^{(2)}_{mp}\hat{B}^*_p+\hat{B}_m\hat{G}^{(2)}_{pm}\hat{A}^*_p+\hat{B}_m\hat{H}^{(2)}_{mp}\hat{B}^*_p], \tag{4-7}$$

where

$$\hat{F}^{(2)}_{mp}=\int_0^{K'}d\nu\,\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Lambda^{1/2}}[a_{mp}(S^n_mS_p+S^n_pS_m)+b_{mp}S^n_mS^n_p],$$

$$\hat{G}^{(2)}_{mp}=\int_0^{K'}d\nu\,\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Lambda^{1/2}}[a_{mp}(S^n_mT_p+T^n_pS_m)+b_{mp}S^n_mT^n_p],$$

$$\hat{H}^{(2)}_{mp}=\int_0^{K'}d\nu\,\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Lambda^{1/2}}[a_{mp}(T^n_mT_p+T^n_pT_m)+b_{mp}T^n_mT^n_p],$$

$$a_{mp}=(\boldsymbol{e}_\mu\cdot\boldsymbol{N})\cos\left(\frac{m-p}{K'}\pi\nu\right),$$

$$b_{mp}=(\boldsymbol{e}_\nu\cdot\boldsymbol{N})\left\{\frac{1}{\Lambda^{1/2}}\frac{\partial\Lambda}{\partial\nu}\cos\left(\frac{m-p}{K'}\pi\nu\right)-\frac{m-p}{K'}\pi\Lambda^{1/2}\sin\left(\frac{m-p}{K'}\pi\nu\right)\right\}.$$

Notice that μ on the wall depends on ν. The scalar products $\boldsymbol{e}_\mu\cdot\boldsymbol{N}$ and $\boldsymbol{e}_\nu\cdot\boldsymbol{N}$ can be calculated in accordance with eqs. (2–4) and (2–6).

To obtain a relation between $\hat{A}_m$ and $\hat{B}_m$, we impose the boundary condition $\boldsymbol{N}\cdot\hat{\boldsymbol{B}}_1=0$ on the wall surface. Substituting $\hat{\boldsymbol{B}}_1=\nabla\hat{V}$ into the above condition, we obtain

$$e^{ip(\pi/K')\nu}\hat{A}_pS_p+\sum_{m=-\infty}^{\infty}e^{im(\pi/K')\nu}\{\hat{A}_mS_m\varepsilon_{pm}+\hat{B}_mT_m+(d_1+id_2)(\hat{A}_mS^n_m+\hat{B}_mT^n_m)\}=0, \tag{4-8}$$

where

$$d_1=e\frac{1}{2\Lambda^{1/2}}\frac{\partial\Lambda}{\partial\nu},\qquad d_2=e\frac{m\pi}{K'}\Lambda^{1/2},\qquad e=\frac{\boldsymbol{N}\cdot\boldsymbol{e}_\nu}{\boldsymbol{N}\cdot\boldsymbol{e}_\mu},$$

and we define ε_{pm} as $\varepsilon_{pm}=0$ for $m=p$ and $\varepsilon_{pm}=1$ for $m\neq p$.

Dividing eq. (4–8) by S_p and summing over p, we obtain the Fourier coefficient $\hat{A}_p$

$$\hat{A}_p=-\sum_{m=-\infty}^{\infty}(D^{(1)}_{pm}\hat{A}_m+D^{(2)}_{pm}\hat{B}_m),$$

where

$$D^{(1)}_{pm}=I_d(S_m,S^n_m)-\delta_{m,p},\qquad D^{(2)}_{pm}=I_d(T_m,T^n_m),$$

and we define

$$I_d(A,B)=\frac{1}{K'}\int_0^{K'}d\nu\left(\sum_{l=-\infty}^{\infty}\frac{1}{S_l}\right)\left[(A+d_1B)\cos\left(\frac{m-p}{K'}\pi\nu\right)-d_2B\sin\left(\frac{m-p}{K'}\pi\nu\right)\right].$$

In the vector form, we obtain

$$\hat{\boldsymbol{B}}=-\boldsymbol{L}\hat{\boldsymbol{A}},\qquad \boldsymbol{L}=(\boldsymbol{D}^{(2)})^{-1}(\boldsymbol{I}+\boldsymbol{D}^{(2)}),$$

wherc $\boldsymbol{I}$ is the unit matrix, and $\boldsymbol{D}^{(1)}$, $\boldsymbol{D}^{(2)}$ are the matrices whose elements are $D^{(1)}_{pm}$, $D^{(2)}_{pm}$.

A relation between $\boldsymbol{\xi}$ and $\hat{\boldsymbol{A}}$ is obtained from the boundary condition $\boldsymbol{n}\cdot\hat{\boldsymbol{B}}_1=\boldsymbol{n}\cdot\nabla\hat{V}=\boldsymbol{n}\cdot\nabla\times(\boldsymbol{\xi}\times\hat{\boldsymbol{B}}_e)$ at $\mu=\mu_p$. After a straightforward calculation, we find

$$\sum_{m=-\infty}^{\infty}e^{im(\pi/K')\nu}(\hat{A}_mS_m+\hat{B}_mT_m)=\sum_{m=-\infty}^{\infty}\xi_me^{im(\pi/K')\nu}\left[\left(\frac{m}{K'}\pi+i\frac{1}{h_1}\frac{\partial h_1}{\partial\nu}\right)\hat{\boldsymbol{B}}_{e\nu}-\frac{n\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Omega}\hat{\boldsymbol{B}}_{e\phi}\right].$$

In this derivation, we use the relation $\partial(h_3\hat{\boldsymbol{B}}_{e\nu})/\partial\nu=-(\hat{\boldsymbol{B}}_{e\nu}\mathrm{sn}\mu\ \mathrm{dn}\nu/\Omega)\ (\partial h_1/\partial\nu)$ which is obtained from $\nabla\cdot\hat{\boldsymbol{B}}_e=0$. Therefore, the Fourier coefficient ξ_p is expressed as

$$\xi_p=\frac{1}{B_0}\sum_{m=-\infty}^{\infty}\left\{V^{(3)}_{pm}(B_0\xi_m)-V^{(1)}_{pm}\hat{A}_m-V^{(2)}_{pm}\hat{B}_m\right\}, \tag{4-9}$$

where

$$V_{pm}^{(1)}=V_1\int_0^{K'}d\nu V_2 S_m \cos\left(\frac{m-p}{K'}\pi\nu\right),$$

$$V_{pm}^{(2)}=V_1\int_0^{K'}d\nu V_2 T_m \cos\left(\frac{m-p}{K'}\pi\nu\right),$$

$$V_{pm}^{(3)}=V_1\int_0^{K'}d\nu V_2 b_\nu \left\{\frac{m}{K'}\pi\cos\left(\frac{m-p}{K'}\pi\nu\right)-\frac{1}{h_1}\frac{\partial h_1}{\partial\nu}\sin\left(\frac{m-p}{K'}\pi\nu\right)\right\}.$$

Here $V_1=(\mathrm{sn}^2\,\mu/nK')$, $V_2=(\mathrm{dn}^2\,\nu/\Omega A)$, and if we replace S_m by T_m in $V_{pm}^{(1)}$, we obtain the formula for $V_{pm}^{(2)}$. The vector form for eq. (4–9) is $\boldsymbol{V}^{(1)}\boldsymbol{A}+\boldsymbol{V}^{(2)}\boldsymbol{B}=(\boldsymbol{V}^{(3)}-\boldsymbol{I})B_0\boldsymbol{\xi}$, where $\boldsymbol{V}^{(1)}$, $\boldsymbol{V}^{(2)}$ and $\boldsymbol{V}^{(3)}$ are the matrices whose elements are $V_{pm}^{(1)}$, $V_{pm}^{(2)}$ and $V_{pm}^{(3)}$. Therefore, we obtain

$$\hat{\boldsymbol{A}}=\boldsymbol{M}\boldsymbol{\xi},\qquad \hat{\boldsymbol{B}}=-\boldsymbol{L}\boldsymbol{M}\boldsymbol{\xi},\tag{4–10}$$

where $\boldsymbol{M}=(\boldsymbol{V}^{(1)}-\boldsymbol{V}^{(2)}\boldsymbol{L})^{-1}(\boldsymbol{V}^{(3)}-\boldsymbol{I})$.

Substituting eq. (4–10) into eqs. (4–6) and (4–7), we obtain

$$\delta W_v=\pi c\hat{B}_0^2\boldsymbol{\xi}^T\boldsymbol{M}^T(\hat{\boldsymbol{F}}-\hat{\boldsymbol{G}}\boldsymbol{L}-\boldsymbol{L}^T\hat{\boldsymbol{G}}^T+\boldsymbol{L}^T\hat{\boldsymbol{H}}\boldsymbol{L})\boldsymbol{M}\boldsymbol{\xi}^*.$$

Here, the elements of the matrices $\hat{\boldsymbol{F}}$, $\hat{\boldsymbol{G}}$, $\hat{\boldsymbol{H}}$ are $\hat{F}_{mp}^{(1)}-\hat{F}_{mp}^{(2)}$, $\hat{G}_{mp}^{(1)}-\hat{G}_{mp}^{(2)}$, $\hat{H}_{mp}^{(1)}-\hat{H}_{mp}^{(2)}$ respectively. Therefore, we obtain the final result

$$\delta W=\pi c\hat{B}_0^2\boldsymbol{\xi}^T\boldsymbol{W}\boldsymbol{\xi}^*,$$

where $\boldsymbol{W}=\boldsymbol{H}-(\boldsymbol{C}^{-1})^T\boldsymbol{G}\boldsymbol{C}^{-1}+\boldsymbol{M}^T(\hat{\boldsymbol{F}}-\hat{\boldsymbol{G}}\boldsymbol{L}-\boldsymbol{L}^T\hat{\boldsymbol{G}}^T+\boldsymbol{L}^T\hat{\boldsymbol{H}}\boldsymbol{L})\boldsymbol{M}$. If we put $\boldsymbol{L}=\boldsymbol{0}$, the energy matrix $\boldsymbol{W}$ corresponds to the energy matrix in the case of no wall.

5. Discussion

A wall stabilization problem of an axisymmetric toroidal sharp boundary plasma is formulated in flat-ring cyclide coordinates. The cross section of plasma is assumed to be a horizontally elongated noncircular, and the cross section of the wall is assumed to be of an arbitrary shape. The flat-ring cyclide coordinates (μ, ν, ϕ) are more useful than the pseudo-toroidal coordinates (r, θ, ϕ), because a solution of Laplace's equation can be analytically represented in terms of the Wangerin functions which are calculated numerically in accordance with standard work[9].

The argument of the Wangerin functions represents the aspect ratio and elongation ratio of a toroidal geometry, so that we can treat the energy principle without an expansion of the aspect ratio. Also, expansions in the plasma beta, such as the conventional and high beta orderings, are not necessary.

The energy principle is reduced to the energy matrix which consists of a quadratic form; therefore, the stability of the plasma is determined by examining the eigenvalues λ_i of the energy matrix $\boldsymbol{W}$. Since the matrix $\boldsymbol{W}$ is symmetrical, the eigenvalues are always real. If all the $\lambda_i>0$, the plasma is stable and if any $\lambda_i<0$, the plasma is unstable. Therefore, if the lowest eigenvalue is zero, the plasma is marginally stable, so that our problem is to find the lowest eigenvalue of the energy matrix which is zero in the parameter spaces. In practice, we give the toroidal perturbation number n, the aspect ratio A, the elongation ratio E, and wall shape $g\,(\theta)$. Then, we find such a plasma beta in which the lowest eigenvalue is zero for variation of the parameter γ_a.

The remainder of the wall stabilization problem is to evaluate the Wangerin functions numerically and to find the critical plasma beta for several parameters. In the near future, this will be investigated.

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