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The Vacuum Poloidal Flux Functions Satisfying the Grad-Shafranov Equation in the Flat-Ring Cyclide Coordinate System

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Abstract

The Wangerin functions, which are solutions of Laplace's equation in the flat-ring cyclide coordinate system, are derived as series solutions about regular singular points $\mu=0$ and $\mu=K'$ which correspond respectively to the external and internal points of plasmas, where the curve $\mu=$const denotes the flattened cross section of plasmas and K' is the complete elliptic integral of the first kind with the modulus k'. Since the argument of the Wangerin functions is expressed as the aspect and elongation ratios of plasmas, so we can accurately treat the equilibrium problem of axisymmetric toroidal plasmas without an expansion in the aspect ratio. The Grad-Shafranov equation, which governs magnetohydrodynamic plasma equilibria, is analytically solved in the vacuum region by using the flat-ring cyclide coordinates, so that the poloidal flux function is expressed as the Wangerin functions with the toroidal mode number being equal to one.

1. Introduction

Recent investigations in a magnetically confined fusion reactor system suggest that it is important to produce a stable high-beta plasma in order to construct a fusion device of a low capital cost. For this reason, a cross section of the fusion device is designed to be noncircular and the high-beta plasma with noncircular cross section is produced.

We have already studied stability problems of high-beta axisymmetric toroidal plasmas with the circular[1,2] and noncircular[3] cross section from the energy principle which is a stability theory for magnetohydrodynamic plasmas. In the analysis of the stability for axisymmetric toroidal plasma with the circular cross section, we have introduced a toroidal coordinate system into the toroidal plasma and made use of toroidal functions as special functions. We have also solved an equilibrium problem of toroidal plasmas by using the toroidal functions and made properties of toroidal coordinates and functions clear[4,5,6].

On the other hand, for solving the stability problem of axisymmetric toroidal plasma with the horizontally elongated cross section, a flat-ring cyclide coordinate system has been introduced into the the plasma with the noncircular cross section[3].

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In flat-ring cyclide coordinates, we have used the Wangerin functions as solutions of Laplace's equation. However, we have not yet derived a detailed form of the Wangerin functions.

Therefore, we express the Wangerin functions, which play an important part for solving the equilibrium problem, as series solutions about a regular singularity of the Laplace equation in the flat-ring cyclide coordinates. Furthermore, analytical solutions of the Grad-Shafranov equation in the vacuum region are obtained in terms of the Wangerin functions. This is our aim in present paper. The remainder of this paper is arranged as follows. In the next section, the flat-ring cyclide coordinate system is constructed and its properties are summarized. In section 3, from R-separation of the Laplace equation, the Laplace equation is analytically solved and the Wangerin functions expressed as series solutions are obtained. In section 4, we consider the analytical solution of the Grad-Shafranov equation and the discussion is described in the final section.

2. Flat-Ring Cyclide Coordinate System

We develop the flat-ring cyclide coordinate system in accordance with the complex-plane transformation[7], so that we consider the following transformation

$$z = a \,\mathrm{sn}\, w, \tag{2-1}$$

where $w = u + iv$, $z = x + iy$, and a is the fundamental length of the flat-ring coordinates. In this transformation, the rectangular map in the w-plane is transformed into a curvilinear but orthogonal map in the z-plane, so that angles are preserved by the transformation, and squares in the w-plane always map into curvilinear squares in the z-plane.

Separation of eq. (2–1) into real and imaginary parts results in the two equations

$$x = \xi_1(u, v) = \frac{a}{\Lambda} \mathrm{sn}\, u \,\mathrm{dn}\, v \tag{2-2}$$

$$y = \xi_2(u, v) = \frac{a}{\Lambda} \mathrm{cn}\, u \,\mathrm{dn}\, u \,\mathrm{sn}\, v \,\mathrm{cn}\, v \tag{2-3}$$

where $\Lambda = 1 - \mathrm{dn}^2 u \,\mathrm{sn}^2 v$. Since the two families of curves, $u = \text{const}$ and $v = \text{const}$ form the orthogonal map in the z-plane, all intersection are at right angles and all subdivisions are curvilinear squares.

We now generate a flat-ring cyclide coordinate system (μ, ν, ϕ) which is one of the rotational system. If the z-plane map is rotated about what was originally the y-axis, the flat-ring coordinates is specified by

$$\begin{aligned} x &= \xi_1(\mu, \nu)\cos\phi = \frac{a}{\Lambda} \mathrm{sn}\, \mu \,\mathrm{dn}\, \nu \cos\phi, \\ y &= \xi_1(\mu, \nu)\sin\phi = \frac{a}{\Lambda} \mathrm{sn}\, \mu \,\mathrm{dn}\, \nu \sin\phi, \\ z &= \xi_2(\mu, \nu) = \frac{a}{\Lambda} \mathrm{cn}\, \mu \,\mathrm{dn}\, \mu \,\mathrm{sn}\, \nu \,\mathrm{cn}\, \nu, \end{aligned} \tag{2-2}$$

where the coordinates u and v are replaced by the coordinates μ and ν, respectively. Metric coefficients are

$$\begin{aligned} g_{11} = g_{22} &= \left(\frac{\partial \xi_1}{\partial \mu}\right)^2 + \left(\frac{\partial \xi_2}{\partial \mu}\right)^2 = \left(\frac{a\Omega}{\Lambda}\right)^2, \\ g_{33} &= (\xi_1(\mu, \nu))^2 = \left(\frac{a}{\Lambda} \mathrm{sn}\, \mu \,\mathrm{dn}\, \nu\right)^2, \end{aligned} \tag{2-4}$$

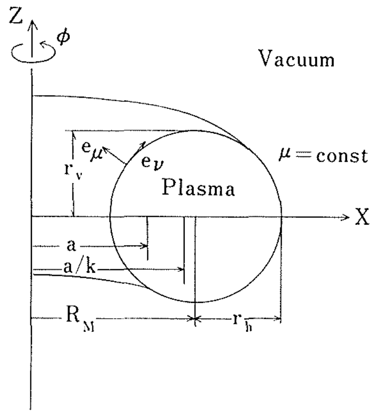

Fig. 1 Toroidal geometry in flat-ring cyclide coordinates (μ, ν, ϕ)

where $\Omega^2=(1-\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu)(\mathrm{dn}^2\nu-k^2\mathrm{sn}^2\mu)$, and k is the modulus.

The flat-ring cyclide coordinate system is shown in Fig. 1. In Fig. 1, the curve $\mu=$const, which denotes the flattened cross section of plasmas, is expressed by

$$(x^2+y^2+z^2)^2+\frac{a^2}{k^4}\frac{(1-k^2)^2-2(1-k^2)\mathrm{dn}^2\mu+(1+k^2)\mathrm{dn}^4\mu}{\mathrm{dn}^2\mu\,\mathrm{cn}^2\mu}z^2$$
$$-a^2\left(\mathrm{sn}^2\mu+\frac{1}{k^2\,\mathrm{sn}^2\mu}\right)(x^2+y^2)+\frac{a^4}{k^2}=0. \tag{2-5}$$

The curve (2-5) is referred to as a flat-ring cyclide. Then a major radius R_M, horizontal and vertical minor radii r_h, r_v of the curve $\mu=$const are defined respectively by

$$R_M=\frac{a}{2k}\frac{1+k\,\mathrm{sn}^2\mu}{\mathrm{sn}\,\mu},$$
$$r_h=\frac{a}{2k}\frac{1-k\,\mathrm{sn}^2\mu}{\mathrm{sn}\,\mu}, \tag{2-6}$$
$$r_v=\frac{a}{2k}\frac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{sn}\,\mu}.$$

A relation between R_M and r_h is given by $a^2/k=R_M^2-r_h^2$. Furthermore, if the aspect and elongation ratios are defined by $A=R_M/r_h$ and $E=r_h/r_v$, respectively, then the modulus k and the function $\mathrm{sn}^2\mu$ can be expressed in terms A of and E, i. e.,

$$k=1+\frac{2}{A^2-1}\left[1-\frac{1}{E^2}-\sqrt{\left(1-\frac{1}{E^2}\right)\left(A^2-\frac{1}{E^2}\right)}\right], \tag{2-7}$$

$$\mathrm{sn}^2\mu=\frac{1}{k}\frac{A-1}{A+1}. \tag{2-8}$$

If we put $E=1$, then the modulus k is equal to one. In this case, the flat-ring cyclide coordinates are reduced to the toroidal coordinates in which the aspect ratio A is given by $\cosh 2\mu$.

Finally, we summarize important equations in the flat-ring cyclide coordinate system :

An infinitesimal distance is expressed by

$$(\mathrm{ds})^2=\frac{a^2\Omega^2}{\Lambda^2}[(d\mu)^2+(d\nu)^2]+\frac{a^2}{\Lambda^2}\,\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu\,(d\phi)^2. \tag{2-9}$$

Gradient is

$$\mathrm{grad}\,\varphi=\frac{\Lambda}{a\Omega}\Big[e_\mu\frac{\partial\varphi}{\partial\mu}+e_\nu\frac{\partial\varphi}{\partial\nu}\Big]+\frac{e_\phi\Lambda}{a\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\frac{\partial\varphi}{\partial\phi}. \tag{2-10}$$

Divergence is

$$\mathrm{div}\,E=\frac{\Lambda^3}{a\Omega^2\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\Big[\mathrm{dn}\,\nu\frac{\partial}{\partial\mu}\Big(\frac{\Omega\,\mathrm{sn}\,\mu}{\Lambda^2}E_\mu\Big)+\mathrm{sn}\,\mu\frac{\partial}{\partial\nu}\Big(\frac{\Omega\,\mathrm{dn}\,\nu}{\Lambda^2}E_\nu\Big)\Big]+\frac{\Lambda}{a\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\frac{\partial E_\phi}{\partial\phi}. \tag{2-11}$$

Curl is

$$\mathrm{curl}\,E=\frac{\Lambda^2}{a\Omega^2\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\begin{vmatrix} e_\mu\Omega & e_\nu\Omega & e_\phi\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial\phi} \\ E_\mu\Omega/\Lambda & E_\nu\Omega/\Lambda & E_\phi\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu/\Lambda \end{vmatrix}. \tag{2-12}$$

Since the *scalar Laplacian* of ψ is defined by $\Delta^2\psi=\mathrm{div\ grad}\,\varphi$, so Laplace's equation is expressed by

$$\Delta^2\varphi=\frac{\Lambda^3}{a^2\Omega^2\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\Big[\mathrm{dn}\,\nu\frac{\partial}{\partial\mu}\Big(\frac{\mathrm{sn}\,\mu}{\Lambda}\frac{\partial\varphi}{\partial\mu}\Big)+\mathrm{sn}\,\mu\frac{\partial}{\partial\nu}\Big(\frac{\mathrm{dn}\,\nu}{\Lambda}\frac{\partial\varphi}{\partial\nu}\Big)\Big]+\frac{\Lambda^2}{a^2\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\frac{\partial^2\varphi}{\partial\phi^2}. \tag{2-13}$$

3. Solutions of the Laplace Equation

Since the Laplace equation in the flat-ring cyclide coordinates is R-separable, so we consider the following solution

$$\varphi=\Lambda^{1/2}M(\mu)N(\nu)\Psi(\phi). \tag{3-1}$$

The solution (3-1) permits the separation of the partial differential equation (2-13) into three ordinary differential equations, so that substitution of eq. (3-1) into the Laplace equation (2-13) gives

$$\frac{1}{\mathrm{sn}\,\mu}\frac{d}{d\mu}\Big(\mathrm{sn}\,\mu\frac{dM}{d\mu}\Big)+\Big[k^2\,\mathrm{sn}^2\mu-\alpha_2-\alpha_3\Big(k^2\,\mathrm{sn}^2\mu+\frac{1}{\mathrm{sn}^2\mu}\Big)\Big]M=0, \tag{3-2a}$$

$$\frac{1}{\mathrm{dn}\,\nu}\frac{d}{d\nu}\Big(\mathrm{dn}\,\nu\frac{dN}{d\nu}\Big)+\Big[-\mathrm{dn}^2\nu+\alpha_2+\alpha_3\Big(\mathrm{dn}^2\nu+\frac{k^2}{\mathrm{dn}^2\nu}\Big)\Big]N=0, \tag{3-2b}$$

$$\frac{d^2\Psi}{d\phi^2}+\alpha_3\Psi=0, \tag{3-2c}$$

where α_2 and α_3 are the separation constants. Furthermore, substitution of $\mathrm{sn}^2\mu=z$ in eq. (3-2a) and $\mathrm{dn}^2\nu=z$ in eq. (3-2b) reduces both to the canonical form

$$\frac{d^2Z}{dz^2}+\frac{1}{2}\Big[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{2}{z-a_3}\Big]\frac{dZ}{dz}+\frac{1}{4}\Big[\frac{A_0+A_1z+A_2z^2}{(z-a_1)(z-a_2)(z-a_3)^2}\Big]Z=0, \tag{3-3}$$

where, for M with $\alpha_2=p^2$ and $\alpha_3=q^2$,

$$a_1=b=1,\quad a_2=c=1/k^2,\quad a_3=0,\quad A_0=-q^2c,\quad A_1=-p^2c,\quad A_2=1-q^2.$$

For N, with $\alpha_2=p'^2c$ and $\alpha_3=q^2$,

$$a_1=b=1,\quad a_2=c=k^2,\quad a_3=0,\quad A_0=-q^2c,\quad A_1=-p'^2c,\quad A_2=1-q^2,\quad p'^2=p^2/k^2.$$

The equation (3-3) represents one of the Bôcher equation with four singularities, which is referred to as the Wangerin equation. The solutions of the Wangerin equation are called the Wangerin functions and denoted by $S_p^q(k,z)$ and $T_p^q(k,z)$.

We consider an application of the Wangerin function to both the equilibrium and stability problems of axisymmetric toroidal plasmas in the flat-ring cyclide coordinates. The direction of the coordinate ν denotes the poloidal direction with the period $2K'$, where K' is the complete elliptic integral of the first kind with the modulus $k'=(1-k^2)^{1/2}$. Therefore, in order to solve the eq. (3-2b), we assume that the solution $N(\nu)$ is Fourier analyzed in the following form

$$N(\nu)=\sum_{m=-\infty}^{\infty} C_m e^{im(\pi/K')\nu}, \tag{3-4}$$

where C_m are the Fourier coefficients. Furthermore, we can consider the differential equation (3-2b) as the Sturm-Liouville system with the boundary condition $[\mathrm{dn}\,\nu N(\nu)N'(\nu)]_{-K'}^{K'}=0$, so that we can formulate the variational problem which is equivalent to eq. (3-2b). Therefore, the separation constants α_2 can be determined by the following proper equation

$$C^T(\boldsymbol{A}\boldsymbol{B}^{-1}-\lambda\boldsymbol{I})=0, \tag{3-5}$$

where $\lambda=\alpha_2$, C is the column vector, and the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the following elements

$$A_{mp}=2\int_0^{K'} d\nu\,\mathrm{dn}\,\nu\Big[\frac{\pi^2}{K'^2}mp+\Big\{\mathrm{dn}^2\nu-n^2\Big(\mathrm{dn}^2\nu+\frac{k^2}{\mathrm{dn}^2\nu}\Big)\Big\}\Big]\cos\Big(\frac{m-p}{K'}\pi\nu\Big),$$

$$B_{mp}=2\int_0^{K'} d\nu\,\mathrm{dn}\,\nu\cos\Big(\frac{m-p}{K'}\pi\nu\Big),$$

$n^2=\alpha_3$ is the toroidal mode number determined by eq. (3-2c). We can obtain the eigenvalues λ and eigenvector $\boldsymbol{C}$ from eq. (3-5), so that the value of the solution $N(\nu)$ is determined by these results.

The remainder of the problem is to solve the eq. (3-2a) and derive the Wangerin function regular at $\mu=0$ or $\mu=K$. First of all, we consider the series solution of eq. (3-2a) about $\mu=0$. For $\mu=0$, the variable z becomes zero, so that we find that $z=0$ means the regular singular point. In this case, the roots of the indicial equation differs by an integer, so that two distinct series solutions of the Wangerin equation (3-2a) are given by

$$S_p^q(k,z)=z^{q/2}\sum_{m=0}^{\infty}\frac{(-)^m\Delta_m(q/2)}{\prod_{l=1}^{m}\Big\{\Big(l+\frac{q}{2}\Big)^2-\frac{q^2}{4}\Big\}}z^m,$$

$$T_p^q(k,z)=-\frac{\Delta_q(-q/2)}{q!(q-1)!}S_p^q(k,z)\ln z+\frac{z^{-q/2}}{(q-1)!}\sum_{m=0}^{q-1}\frac{(q-m-1)!\Delta_m(-q/2)}{m!}z^m$$
$$+\frac{z^{-q/2}}{(q-1)!}\sum_{m=q}^{\infty}\frac{(-)^{m-q+1}}{(m-q)!m!}\Big\{\Big[\frac{\partial\Delta_m}{\partial\beta}\Big]_{\beta\to-q/2}-\Delta_m\Big(-\frac{q}{2}\Big)\Big[\sum_{l=1}^{m-q}\frac{1}{l}+\sum_{l=1}^{m}\frac{1}{l}-\sum_{l=1}^{q-1}\frac{1}{l}\Big]\Big\}, \tag{3-6}$$

where

$$A_0=1,\ A_1=-\frac{1+c}{2c},\ A_2=-\frac{1+c^2}{2c^2},\ A_j=\frac{1}{c}\{A_{j-1}(1+c)-A_{j-2}\},\ j>2$$

$$B_0=-\frac{q^2}{4},\ B_1=-\frac{1}{4c}\{(1+c)q^2+p^2c\},$$

$$B_2=-\frac{1}{4c^2}\{(1+c+c^2)q^2+(1+c)p^2c+c(1-q^2)\},$$

$$B_j=\frac{1}{c}\{B_{j-1}(1+c)-B_{j-2}\},\ j>2$$

$$f_0(\beta)=\beta^2+(A_0-1)\beta+B_0,\quad f_j(\beta)=\beta A_j+B_j,\quad j=1,2,3,\ldots$$

$$\Delta_0=1,$$

$$\Delta_j(\beta)=\begin{vmatrix} f_1(\beta) & f_0(\beta+1) & 0 & 0 & 0\cdots & 0 & 0 \\ f_2(\beta) & f_1(\beta+1) & f_0(\beta+2) & 0 & 0\cdots & 0 & 0 \\ f_3(\beta) & f_2(\beta+1) & f_1(\beta+2) & f_0(\beta+3) & 0\cdots & 0 & 0 \\ \cdots & & & & & & \\ f_{j-1}(\beta) & f_{j-2}(\beta+1) & f_{j-3}(\beta+2) & \cdots & & f_1(\beta+j-2) & f_0(\beta+j-1) \\ f_j(\beta) & \cdots & & & & f_2(\beta+j-2) & f_1(\beta+j-1) \end{vmatrix}, \quad j\geqq 1.$$

On the other hand, we consider the series solution about $\mu=K$. For $\mu=K$, the variable z becomes one, so that $z=1$ means a regular singular point as well as $z=0$. We find that the roots of the indicial equation for $z=1$ are distinct and do not differ by an integer, so that two distinct series solutions of the Wangerin equation (3–2a) are given by

$$S_p^q(k,z)=(z-1)^{1/2}\sum_{m=0}^{\infty}\frac{(-)^m\Delta_m(1/2)}{\prod_{l=1}^{m}(l+1)(l+1/2)}(z-1)^m,$$

$$T_p^q(k,z)=\sum_{m=0}^{\infty}\frac{(-)^m\Delta_m(0)}{\prod_{l=1}^{m}(l+1)(l+1/2)}(z-1)^m \tag{3-7}$$

where

$$A_0=\frac{1}{2},\qquad A_1=\frac{1}{2}\frac{2c-3}{c-1},\qquad A_2=-\frac{1}{2}\frac{2c^2-4c+3}{(c-1)^2},$$

$$A_j=-\frac{1}{c-1}\{A_{j-1}(c-2)-A_{j-2}\},\qquad j>2,$$

$$B_0=0,\ B_1=-\frac{1}{4}\frac{1}{c-1}\{1-q^2-(p^2+q^2)c\},$$

$$B_2=\frac{1}{4}\frac{1}{(c-1)^2}\{-(1-q^2)-(c-2)p^2c-(2c-3)q^2c\},$$

$$B_j=-\frac{1}{c-1}\{B_{j-1}(c-2)-B_{j-2}\}+\frac{q^2c}{4(-)^{j+1}(c-1)},\qquad j>2.$$

The series solutions above converge within the circle whose center is at a regular singular point, so that the solutions (3–6) and (3–7) are appropriate for the external and internal solutions of the equilibrium or stability problems of plasmas, respectively. That is the reason why $\mu=0$ and $\mu=K$ mean the external and internal points of plasmas, respectively.

4. Vacuum Flux Functions

Magnetohydrodynamic (MHD) plasma equilibria are governed by an equilibrium equation grad $P=\boldsymbol{J}\times\boldsymbol{B}$ and Maxwell's equations $\boldsymbol{J}=\mathrm{curl}\,\boldsymbol{B}$ and $\mathrm{div}\,\boldsymbol{B}=0$, where P is the plasma pressure, $\boldsymbol{J}$ the current density and $\boldsymbol{B}$ the magnetic field. The equilibrium equation of axisymmetric plasmas in the cylindrical coordinates (R, Z, ϕ) is expressed by the Grad-Shafranov equation

$$R^2\,\mathrm{div}\,(R^{-2}\,\mathrm{grad}\,\Psi)=-R^2P'-II'. \tag{4-1}$$

Here $\Psi(R,Z)$ is the poloidal magnetic flux, the plasma pressure $P(\Psi)$ and the toroidal magnetic flux $I(\Psi)$ are functions of Ψ and the primes indicate derivatives with respect to Ψ. MHD plasma equilibria is determined by solving eq. (4–1) with expressions of $P'(\Psi)$ and $II'(\Psi)$ which are given as linear or nonlinear functions in Ψ. On the other hand, in the vacuum region, eq. (4–1) is solved with $P'(\Psi)=0$ and $II'(\Psi)=0$ so that we obtain analytical solutions of eq. (4–1) in the vacuum region in which the solutions are expressed in the form of the Wangerin functions.

Equation (4–1) in the vacuum region is rewritten in the following form

$$\frac{\partial^2\Psi}{\partial R^2}-\frac{1}{R}\frac{\partial\Psi}{\partial R}+\frac{\partial^2\Psi}{\partial Z^2}=0. \tag{4–2}$$

In order to eliminate the differential $\partial\Psi/\partial R$ in eq. (4–2), we replace the poloidal flux $\Psi(R, Z)$ by $\Psi_0(R/a)^{1/2}F^*(R, Z)$, where Ψ_0 is constant, so that eq. (4–2) reduces to

$$\frac{\partial^2 F^*}{\partial R^2}+\frac{\partial^2 F^*}{\partial Z^2}=\frac{3}{4}\frac{1}{R^2}F^*. \tag{4–3}$$

Since eq. (4–3) with the rigth-hand side being equal to zero means the Laplace equation in 2-space, so we consider the conformal transformation from the coordinates (R, Z) into (μ, ν) and obtain the following relation

$$\frac{\partial^2 F^*}{\partial \mu^2}+\frac{\partial^2 F^*}{\partial \nu^2}=\frac{\partial(\mu, \nu)}{\partial(R, Z)}\left(\frac{\partial^2 F^*}{\partial R^2}+\frac{\partial^2 F^*}{\partial Z^2}\right), \tag{4–4}$$

where the Jacobian determinant $\partial(\mu, \nu)/\partial(R, Z)$ is given by

$$\frac{\partial(\mu, \nu)}{\partial(R, Z)}=\left(\frac{\Lambda}{a\Omega}\right)^2.$$

Application of eq. (4–4) to eq. (4–3) results in

$$\frac{\partial^2 F^*}{\partial \mu^2}+\frac{\partial^2 F^*}{\partial \nu^2}=\frac{3}{4}\frac{\Omega^2}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}F^*. \tag{4–5}$$

Furthermore, for eq. (4–5) we consider the transformation $F^*=(2\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu)^{-1/2}F$, which implies the transformation $\Psi=\Psi_0(2\Lambda)^{-1/2}F$, so that eq. (4–5) can be expressed as follows

$$\frac{\partial^2 F}{\partial \mu^2}+\frac{\partial^2 F}{\partial \nu^2}-\frac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{sn}\,\mu}\frac{\partial F}{\partial \mu}+k'^2\frac{\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}{\mathrm{dn}\,\nu}\frac{\partial F}{\partial \nu}-k^2\mathrm{sn}^2\mu F+\mathrm{dn}^2\nu F=0. \tag{4–6}$$

Namely, we can show that the Grad-Safranov equation (4–2) under the transformation $\Psi=\Psi_0(2\Lambda)^{-1/2}F$ reduces to the partial differential equation (4–6) written in the form of the flat-ring cyclide coordinates.

We need to solve eq. (4–6) in order to obtain the poloidal flux function. However, since we can show that the following equation

$$(\mathrm{curl\ curl}\ \boldsymbol{A})_\phi=0, \tag{4–7}$$

is identical with eq. (4–6) under the transformation $A_\phi=\Psi_0(2\Lambda g_{33})^{-1/2}F$ in which the poloidal flux function is given by $\Psi=g_{33}^{1/2}A_\phi$, so we consider eq. (4–7) in place of eq. (4–6) and express eq. (4–7) in the flat-ring cyclide coordinates as follows

$$\frac{\partial}{\partial\mu}\left[\frac{\partial A_\phi}{\partial\mu}+\left(-\frac{1}{\Lambda}\frac{\partial\Lambda}{\partial\mu}+\frac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{sn}\,\mu}\right)A_\phi\right]+\frac{\partial}{\partial\nu}\left[\frac{\partial A_\phi}{\partial\nu}+\left(-\frac{1}{\Lambda}\frac{\partial\Lambda}{\partial\nu}-\frac{k'^2\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}{\mathrm{dn}\,\nu}\right)A_\phi\right]=0. \tag{4–8}$$

We apply the following R-separable solution

$$A_\phi=\Lambda^{1/2}M(\mu)N(\nu),$$

to eq. (4–7), so that we have, two second order differential equation

$$\frac{1}{\mathrm{sn}\,\mu}\frac{d}{d\mu}\left(\mathrm{sn}\,\mu\frac{dM}{d\mu}\right)-\left(\alpha_2+\frac{1}{\mathrm{sn}^2\mu}\right)M=0, \tag{4–9a}$$

$$\frac{1}{\mathrm{dn}\,\nu}\frac{d}{d\nu}\left(\mathrm{dn}\,\nu\frac{dN}{d\nu}\right)+\left(\alpha_2+\frac{k^2}{\mathrm{dn}^2\nu}\right)N=0. \tag{4–9b}$$

We find that eqs. (4–9a) and (4–9b) are identical respectively with eqs. (3–2a) and (3–2b) with $\alpha_3=1$, so that solutions of eqs. (4–9a) and (4–9b) are determined by the Wangerin functions with the toroidal mode number $\alpha_3=1$ and the poloidal flux function can be expressed by

$$\Psi(\mu, \nu)=a\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}{\Lambda^{1/2}}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}C_{ml}e^{im(\pi/K')\nu}\times[A_l S^1_{\lambda l 1/2}(k, \mathrm{sn}^2\mu)+B_l T^1_{\lambda l 1/2}(k, \mathrm{sn}^2\mu)], \quad (4\text{–}10)$$

where A_l and B_l are any constants determined by boundary conditions.

Therefore, we conclude that the analytical solution of the Grad-Shafranov equation in the vacuum region is obtained by using the Wangerin functions which are solutions of Laplace's equation in the flat-ring cyclide system. Furthermore, for the modulus $k=1$, the metric coefficient g_{33} in the flat-ring cyclide coordinates becomes

$$g_{33}=\left(\frac{a\sinh\sigma}{\cosh\sigma-\cos\phi}\right)^2 \quad (4\text{–}10)$$

where $\sigma=2\mu$, $\phi=2\nu-\pi$, and identical with the metric one in the toroidal coordinates. Therefore, we find that the analytical solution of the Grad-Shafranov equation in the toroidal coordinates is derived from the same procedure that is used in the flat-ring cyclide coordinates and the poloidal flux function can be also expressed by $\Psi=g_{33}^{1/2}A_\phi$.

5. Discussion

We obtain the series representation of the Wangerin functions which are the series solution of the Laplace equation in the flat-ring cyclide coordinate system. The Wangerin functions are represented as the internal and external solutions of plasmas with the flattened cross section. The argument of the Wangerin functions is expressed as the aspect ratio and elongation ratio of toroidal plasma, so that we can treat the equilibrium and stability problems of plasmas with noncircular cross sections without using the expansion of the aspect ratio for the physical quantities. We apply the Wangerin functions to the equilibrium problem of axisymmetric toroidal plasmas with a horizontally elongated noncircular cross section and show that the the analytical solution of the Grad-Shafranov equation in the vacuum region is obtained by using the Wangerin functiion in the flat-ring cyclide coordinates.

However, we have not yet developed the programming code from which the values of the Wangerin functions are evaluated, so that the equilibrium magnetic structure consisting of the poloidal flux function is not shown but will be investigated in the near future.

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