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Hadamard Transform Image Coding

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ABSTRACT

A combination of the Hadamard transform and uniform quantization is investigated. Fundamental properties of the Hadamard transform are discussed in conjunction with image coding applications ; signal energy compaction in the Hadamard domain is reviewed. As for quantization in the transform domain the rate distortion theory plays a major role in the optimum bit allocation. Uniform quantization with entropy coding is used since its performance is close to the rate distortion theoretical limit. The mean-square error measure is inherently used in transform image coding because of its mathematical tractability. The meaning of the mean-square error is discussed. A maximum-square error measure is proposed as a possible way to elude limitations imposed by the mean-square error measure. A few pictorial examples are provided.

INTRODUCTION

The problem of representing an image by binary digits and reconstructing a replica of the original image from the binary digits has recently attracted considerable attention. Transform coding[1)-4)] achieves superior coding performance to differential pulse code modulators (DPCM). Transform coding makes full use of such statistical picture structure as means, covariances, and first order probability density functions. Quantization errors made in the transform domain are less objectionable to the human observer since they tend to spread over the entire picture in the pixel domain.

Quantization is essential in all image coding systems. We can invoke the rate distortion theory[5)] to know the minimum number of bits required in coding a given signal source; however, exact realization of the optimum quantizer, the existence of which is assured by the theory, is not known. Therefore transform coding systems in literature use suboptimal quantizers, mostly without respect to the rate distortion theory.

The mean-square error measure is, in a sense, the basis of transform image coding. But it is widely known that the error measure is not always appropriate for the human viewer. Furthermore, pictorial data may not be modeled by a stationary random process. Thus adaptive coding strategy has been proposed[6)] to solve the problems. A difficulty with adaptive techniques is that evaluation of the overall system performance is not easy.

In this paper a combination of the Hadamard transform[7)] and uniform quantization is

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described. The energy compaction in the Hadamard transform domain is reviewed. Most of the signal energy is packed into low sequency[7] components.

Quantization in the transform domain will be guided by the rate distortion theory. For a Gaussian source, by which our signal source is modeled, a properly designed uniform quantizer with entropy coding achieves a bit rate only slightly greater than the theoretical minimum rate.

The meaning of the mean-square error is discussed. A possible way to elude limitations imposed by the error measure is proposed.

THE HADAMARD TRANSFORM OF A SIGNAL VECTOR

Assume that we are given an analog source modeled by a vector x :

$$x=(x_1\,x_2\cdots x_N)^t, \tag{1}$$

where x_i, i=1.2, ...,N, are random variables with

$$E\{x_i\}=0,$$
$$E\{x_i^2\}=1, \quad i=1, \ 2, \cdots, N, \tag{2}$$

where E $\{\cdot\}$ is the expectation of the term enclosed.

We transform x into another vector

$$y=Tx, \tag{3}$$

where T is an $N\times N$ orthogonal matrix and

$$y \triangleq (y_1\,y_2\cdots y_N)^t.$$

Note that y_i are also zero mean random variables. We can always arrange the rows of the orthogonal matrix so that the variances of y_i satisfy

$$E\{y^2\}\geq E\{y_2^2\}\geq\cdots\geq E\{y_N^2\}. \tag{4}$$

The main idea behind the transformation is that transmitting x is equivalent to transmitting y since x canbe recovered by taking the inverse transform

$$x=T^{-1}y=T^t y \tag{5}$$

at the receiving end. Furthermore, as we shall see, significant improvement on coding efficiency is possibly attained if encoding is done in the transform domain.

Truncation of the transform sequence $\{y_1, y_2,\cdots, y_n\}$ is now considered. Let the first m transform components be retained and the rest be replaced by zeroes. The result is an approximation of y :

$$\hat{y} \triangleq (y_1\,y_2\cdots y_m 00\cdots 0)^t. \tag{6}$$

The vector $\hat{y}$ is sent to the receiver. Note that $(N-m)$ zeroes need not be transmitted. At the receiver a replica of the original signal x is constructed by using (5),

$$\hat{x}=T^t\hat{y}. \tag{7}$$

The portion of the signal energy contained in $\hat{x}$ is given by

$$S_{\hat{x}}=E\{\hat{x}^t\hat{x}\}=E\{\hat{y}^t\,TT^t\,\hat{y}\}=E\{\hat{y}^t\hat{y}\}=\sum_{i=1}^{m} E\{y_i^2\}, \tag{8}$$

where we have used (7) and the relation $TT^t=I$ (identity matrix). It is seen from (4) and (8) that the retained m elements have been selected to maximize the average energy of the reconstruction.

The average power of the original signal x is given by

$$S_x\triangleq E\{x^t x\}=E\{x_1^2+x_2^2+\cdots+x_N^2\}=N, \tag{9}$$

where (2) has been used. The ratio of $S_{\hat{x}}$ to S_x is called the EPF or energy packing efficiency[8] of the transform T :

$$\eta(m)\triangleq\frac{S_{\hat{x}}}{S_x}=\frac{\sum_{i=1}^{m}\{y_i^2\}}{N}, \tag{10}$$

by using (8) and (9). Obviously $\eta(N)=1$.

For m between 1 and N, $\eta(m)$ is smaller than 1. But by a suitable choice of T and m, it is possible to make $\eta(m), m<N$ very close to 1 ; thus we can pack most of the signal energy into the first m components

The mean-square reconstruction error is evaluated from (6) and (7) by

$$\varepsilon=E\{(x-\hat{x})^t(x-\hat{x})\}=E\{(y-\hat{y})^t(y-\hat{y})\}=\sum_{k=m+1}^{N} E\{y_k^2\} \tag{11}$$

Equation (11) can be rewritten as

$$\varepsilon=\sum_{k=1}^{N} E\{y_k^2\}-\sum_{k=1}^{m} E\{y_k^2\}=S_x\{1-\eta(m)\}, \tag{12}$$

by using (10) and noting

$$\sum_{i=1}^{N} E\{y_i^2\}=S_x.$$

We observe from (12) that the mean-square error is the complement of $S_{\hat{x}}$.

The optimum transform for the role of T is known to be the KLT or Karhunen-Loeve transform. The energy packing efficiency $\eta(m)$ of the KLT is greater than EPF's of any other transforms. The computational cost of the transform is so high that it is often replaced by suboptimum transforms such DFT[9], DCT[10], SCT[11], WHT, etc., usually with tolerable performance degradation. In this work the Walsh-Hadamard transform (WHT) is adopted because of its implementational simplicity. The Hadamard transform is characterized by its elements being all + 1 or − 1 ; therefore no multiplications are involved in computation of the transform. Fast algorithms for the transform are available[4] that have a similar appearance to

FFT routines.

The $N\times N$ Hadamard transform matrix is defined by

$$H_1=[1],$$

$$H_N=\frac{1}{\sqrt{2}}\begin{bmatrix} H_{N/2} & H_{N/2} \\ H_{N/2} & -H_{N/2}\end{bmatrix},\quad N=2^n, n\text{ : positive integer.} \tag{13}$$

It will be appreciated that H_n is symmetric and orthogonal. The 8×8 Hadamard matrix generated by (13) is shown below as an exmple:

$$H_8=\frac{1}{\sqrt{8}}\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix} \qquad \begin{matrix} \text{Sequency}=0 \\ 7 \\ 3 \\ 4 \\ 1 \\ 6 \\ 2 \\ 5 \end{matrix}$$

The Hadamard matrix generated above is in natural order; the rows are not ordered in their sequencies[7] ; the sequency is the number of sign changes along each row.

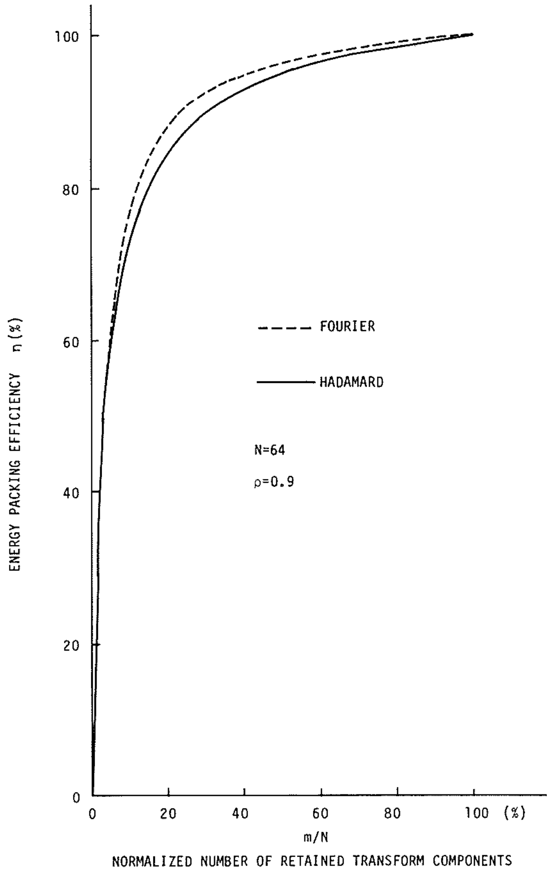

Fig. 1. Comparison effciencies.

For the Hadamard transform, the EPF $\eta\ (m)$, $m=N/2^j$, is given by[8]

$$\eta(n/2^j)=\frac{1}{2^j}\{1+2c(0)^{-1}\sum_{t=1}^{2^j-1}c(t)(1-\frac{t}{2^j})\},\quad j=1,2,\cdots,n \tag{14}$$

where we have assumed that the signal covariances are given by

$$E\{x_i x_j\}=\sigma^2\ c(|i-j|),\quad 1\leq i,\quad j\leq N. \tag{15}$$

In Fig.1 the EPF's of the Hadamard trnsform and DFT are compared for the case

$$E\{x_i x_j\}=\rho^{|i-j|},\quad 0\leq\rho<1. \tag{16}$$

The DFT is known to be close to the KLT[9]. Although the DFT and WHT are close to each other, the small difference that does exist may have significant impacts on coding performance.

Equation (14) indicates that the EPF of the Hadamard transform is not improved by increasing N. This is in sharp contrast with trigonometric transforms such as DFT, DCT...... They become ever closer to the KLT as N increases.

In concluding this section we remark that the energy compaction is mathematically equivalent to decorrelation in the transform domain. If T is the KLT, y is completely decorrelated, i.e.,

$$E\{y_i y_j\}=0 \text{ for all } i,\ j.$$

For other transforms this is satisfied only approximately.

THE TWO-DIMENSIONAL HADAMARD TRANSFORM OF IMAGES

In the foregoing section we have seen that the one-dimensional Hadamard transform is fairly close to the DFT, which is close to the KLT, for the signal covariance of the type in (16). Suppose a one-dimensional transform T is efficient in energy packing performance for a one-dimensional signal with the covariance (15). Then it is known that T is also usable for energy compaction in the form

$$Y=TXT^t \tag{17}$$

for the two-dimensional signal matrix

$$X=[x_{ij}]$$

with x_{ij} being zero mean random variables satisfying

$$E\{x_{ij}x_{i'j'}\}=\sigma^2\ c(|i-i'|)\cdot c(|j-j'|). \tag{18}$$

The covariance defined by (18) is said to be separable.

We shall model our images by the matrix introduced above and further assume that (18) takes the form

$$E\{x_{ij}x_{i'j'}\}=\rho^{|i-i'|+|j-j'|}. \tag{19}$$

Then from the foregoing arguments the two-dimensional transform

$$Y=HXH^t=HXH \tag{20}$$

works well in energy compaction applications; y_{ij}, elements of y, are approximately decorrelated.

QUANTIZATION

Quantization is the process of approximating analog signals by a finite set of representative values. The finite set may be identified with an alphabet ; thus the quantized signal source will render itself to subsequent efficient coding. The truncation discussed in the previous section is a trivial example of quatization ; all values are represented by a single constant equal to zero.

Quantization of a source modeled by a Gaussian random variable with zero mean and variance σ^2 is considered. From the rate distortion theory[12] the minimum bit rate required in coding the source is given by

$$R(D)=\frac{1}{2}\log_2\frac{\sigma^2}{D}, \quad \sigma^2>D \tag{21}$$

where D is the mean-sguare quantization distortion. If the signal is smaller than or equal to the allowed mean-square error, we need not transmit the signal. At the receiving end, it will be simply replaced by its expected value ; the resulting error will be σ^2. Unfortunately no simple practical way to achieve the theoretical rate in (21) is knovn ; however, uniform quantization with entropy coding of the output levels can achieve a rate just above the limit[13],[14]:

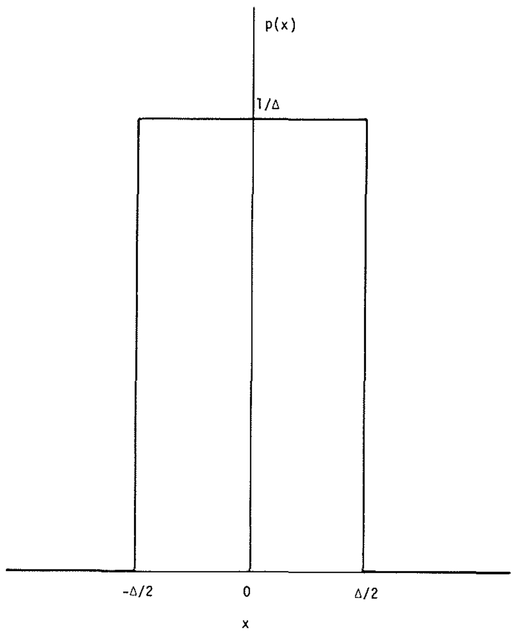

Fig. 2. Probability density of quantization errors.

$$R^*(D)=\frac{1}{4}+\frac{1}{2}\log_2\frac{\sigma^2}{D} \tag{22}$$

Design parameters of a uniform quantizer with its output rate given in (22) will be found in 14).

For a uniform quantizer with the step size Δ, the mean-square quantization error is given by

$$D=\frac{\Delta^2}{12}, \tag{23}$$

which follows from the fact that the probability density function of quantization noise can be approximated by a rectagular function of Fig. 2.

Now quantization of N Gaussian variables $y_1, y_2, \ldots, y_n$, with respective variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma^2$, is considered. If the mean-square quantization errors $D_1, D_2, \ldots, D_n$ of $y_1, y_2, \ldots, y_n$ are specified, the rate in coding each y_i will be known from (21) :

$$R_i(D_i)=\frac{1}{2}\log_2\frac{\sigma_i}{D}, \quad \sigma_i^2 \geq D_i, \quad i=1,2,\cdots,N. \tag{24}$$

The average mean-square error is written as

$$D=\frac{1}{N}\sum_{i=1}^{N} D_i, \tag{25}$$

and the average transmission rate is

$$R(D)=\frac{1}{N}\sum_{i=1}^{N} R_i(D_i)=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\log_2\frac{\sigma_i^2}{D_i}, \quad \sigma_i^2 \geq D_i \text{ for all } i. \tag{26}$$

The quantity in (26) can be minimized subject to the constraint (25) by setting

$$D_i = min\{\theta, \ \sigma_i^2\}, \quad i=1,2,\cdots,N, \tag{27}$$

where the parameter θ is chosen so that

$$D=\frac{1}{N}\sum_{i=1}^{N} D_i=\frac{1}{N}\sum_{i=1}^{N} min\{\theta, \ \sigma_i^2\}. \tag{28}$$

The minimum rate

$$R(D)=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\log_2\frac{\sigma_i^2}{min\{\theta, \sigma_i^2\}} \tag{29}$$

is the N-block rate distortion function.

From (27) the quantization principle is established:

Quantize y_i with the mean-square distortion θ lf $\theta < \sigma^2$.

Discard y_i if $\theta > \sigma_i^2$.

Given a sequence of correlated Gaussian varialbes $x_1, x_2, \ldots, x_n$, we can always transform it into a sequence of statistically independent Gaussian random variables $y_1, y_2, \ldots, y_n$ by means of the KLT. Hence quatization of the original sequence is accomplished through quantization of the transformed sequence in the manner just described. Extension of the foregoing quantization principle to the two-dimensional signal is obvious. Recall that the two-dimensional transform defined by (20) generates statistically (almost) independent Gaussian random variables with variances

$$\sigma_{ij}^2 \triangleq E\{y_{ij}^2\}, \tag{30}$$

if x is Gaussian. The elements of Y, y_{ij}, i, $j=1, 2, \ldots, N$, can be arranged one-dimensionally in the descending order of variances. Thus the quantization principle is directly applicable to images.

We opt for using uniform quantizers for their implementational simplicity and fairly good performance. (See (22).)

ERROR ANALYSIS

Quantization noise for uncorrelated, Gaussian random variables $y_1, y_2, \ldots, y_n$ with

$$\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_N^2 \tag{31}$$

is considered. Let θ be a positive constant that satisfies

$$\sigma_m^2 > \theta > \sigma_{m+1}^2 \tag{32}$$

for some m, $1 < m < N$. We quantize $y_1, y_2, \ldots, y_m$ with the mean-square quantization error θ each and discard the rest $y_{m+1}, y_{m+2}, \ldots, y_n$, the variances of which are smaller than θ. Recall that this is the optimum allocation of quantization errors of y_i according to the quantization principle in the previous section. The resultant average mean-square distortion is

$$D=\frac{1}{N}\left(m\theta+\sum_{i=m+1}^{N}\sigma_i^2\right) \tag{33}$$

from (28), (31) and (32). If the sequence $\{y_i, i=1, 2, \ldots, N\}$ is an orthogonal transform of a correlated sequence $\{x_i, i=1, 2, \ldots, N\}$, the distortion D in (33) is invariant under the transformation ; thus (33) can be used to evaluate the average distortion in $\{x_i\}$.

The mean-square error measure is widely used in image processing because of its mathematical tractability. But its effectiveness is known to be limited. A system with a moderate mean-square error may produce a low quality picture.

Suppose the sequence $\{x_i\}$ represents images and $\{y_i\}$ is an orthogonal transform of it. If the average mean-square error is extremely small, we can expect that the probability of a sample quantization error of each x_i being objectionably large is negligibly small: this is from Tchebycheff's Theorem[15]. Low average mean-squarbe distortion can only be achieved at a high transmission rate. At a relatively low transmission rate,which is of particular interest in most situations, the mean-square error of each x_i being sizable magnitude and therefore large errors will occasionally occur.

As a supplemental error measure, a maximum error is proposed here. It is the opposite of the mean-square error ; it shows how large squared errors may be in the worst case. Will the worst case actually occur ? Not likely, in the sense that the probability of the squared error assuming the maximum value is very small. Yet it serves as a conceptual system performance bound. Two different image coding systems can be compared in terms of the maximum squared errors.

Let the average maximum-square error be defined by

$$\varepsilon_{max}=\frac{1}{N}\,max\{\xi_1^2+\xi_2^2+\cdots+\xi_N^2\}, \tag{34}$$

where ξ_i, $i=1, 2, \ldots, N$, are random variables representing errors in the reconstruction of the sequence $\{x_i\}$, and max $\{\cdot\}$ denotes the maximum value of the random variable enclosed. Since

$$\sum_{i=1}^{N} \xi_i^2=\sum_{i=1}^{N} \zeta_i^2, \tag{35}$$

where ζ_i, $i=1, 2, \ldots, N$, are quantization errors in the transform $\{y_i\}$, we have

$$\varepsilon_{max}=max\{\zeta_1^2+\zeta_2^2+\cdots+\zeta_N^2\}. \tag{36}$$

Assuming $\zeta_1, \zeta_2, \ldots, \zeta_n$ are statistically independent, we can rewrite (36) as

$$\varepsilon_{max}=max\{\zeta_1^2\}+max\{\zeta_2^2\}+\cdots+max\{\zeta_N^2\}. \tag{37}$$

In the quantization strategy, $y_1, y_2, \ldots, y_m$ are quantized with the mean-sqrare error θ each and $y_{m+1}, \ldots, y_n$ are disregarded. The m. s. e. θ can be obtained by uniform quantization with the step size Δ chosen in such a way that

$$\theta=\frac{\Delta^2}{12}, \tag{38}$$

from (23). Since $\zeta_1, \ldots, \zeta_m$ statistically distribute with the density function in Fig. 2,

$$max\{\zeta_i^2\}=\frac{\Delta^2}{4}, \quad i=1, 2, \cdots, m \tag{39}$$

Note that

$$\zeta_k=y_k, \quad k=m+1, \cdots, N.$$

Hence ζ_k, $k=m+1, \ldots, N$, are Gaussian. For a Gaussian random variable with variance σ^2, the maximum of its squared value does not exist; however, we shall use

$$(2.5\sigma)^2=6.25\sigma^2 \tag{40}$$

as the practical maximum squared value of the variable. The probability of the variable lying outside the ± 2.5 region is 1.242 〔%〕. Thus we set

$$max\{\zeta_i^2\}=6.25\sigma_i^2, \quad i=m+1, \cdots, N. \tag{41}$$

Substitution of (39) and (41) into (37) yields

$$\varepsilon_{max}=\frac{1}{N}\left(\frac{\Delta^2}{4}m+6.25\sum_{i=m+1}^{N}\sigma_i^2\right) \tag{42}$$

Observations

From (31), (32) and (33) it will be appreciated that the contribution of each discarded term to D is smaller than the contribution of each transmitted term. On the other hand, in (42),

$$\text{Contribution from each transmitted term}=\frac{\Delta^2}{4}=3\theta, \tag{43}$$

$$\text{Contribution of discarded term}=6.25\sigma_i^2, \quad i=m+1, \ldots, N. \tag{44}$$

Thus some of the errors due to the truncation may have a more significant contribution to ε_{max} than the quantization error of each transmitted term does.

We propose to modify the truncation threshold to correct the unnatural situation:

$$\text{Discard} \quad y_k \text{ if } \frac{\Delta^2}{4}>6.25\sigma_k^2, \tag{45}$$

or

$$\sigma_k^2 < \frac{1}{25}\Delta^2. \tag{46}$$

(See (43) and (44).) Recall that in the rate distortion theory y_k is discarded if

$$\sigma_k^2 < \theta = \frac{\Delta^2}{12}.$$

With the use of (46), more terms will be retained. The average maximum-square error will now be witten in the form

$$\varepsilon'_{max} = \frac{1}{N}\left(\frac{\Delta^2}{4}n + 6.25\sum_{i=n+1}^{N}\sigma_i^2\right), \tag{47}$$

where $n > m$. The first n transform components are retained with the maximum squared error $\Delta^2/4$ each and the rest are discarded with maximum-squared errors satisfying (45), i.e.,

$$6.25\sigma_k^2 < \frac{\Delta^2}{4}, \quad k = n+1, \quad n+2, \cdots, N \tag{48}$$

From (47) and (48), we have

$$\varepsilon'_{max} < \frac{\Delta^2}{4} \tag{49}$$

Thus the modified threshold puts the average maximum-squared error under control. Numerical results indicate that the modified threshold is accompanied with a small increase in the transmission rate.

The final remark in this section is that a properly designed uniform quantizer does not only achieve a bit rate close to the theoretical limit but is also optimum in suppressing the maximum squared error.

CODING EXPERIMENTS

We have tested the Hadamard domain quantization with the modified truncation threshold described in the previous section. The coding system used in the experiments is illustrated in Fig. 3. We have modeled the image source by a zero mean Gaussian process with the separable covariance given by (19), $\rho = 0.9$.

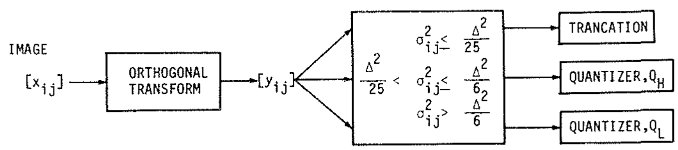

Fig. 3. Block diagram of the coding system.

The retained transform components are divided into two groups according to their variances and each group is quantized by a uniform quantizer designed for the transform component with the maximum variance in the group. Two separate Huffman codes will then be used. Instead of actually generating Huffman codes we have computed entropies of the outputs of the two quantizers as they give the limits of transmission rates attainable. The number of bits accompanying each reconstructed image in Fig. 4 is the sum of the two entropies. Therefore the number does not represent the actual number of bits/pixel used in coding the particular image

Fig. 4. Results of the coding experiments.
(a) Original picture, GIRL.
(b) Reconstruction of GIRL, 2.1 bits/pixel, $\Delta=0.36$.
(c) Reconstruction of GIRL, 0.7 bits/pixel, $\Delta=1.15$.

but the expected number of bits/pixel used in the coding system.

We observe that the use of the Hadamard transform with the quantization strategy yields fairly good results.

CONCLUDING REMARKS

Signal energy compaction in the Hadamard transform domain has been reviewed. Effects of truncation and quantization errors have been analyzed. It has been argued that more transform components should be retained to suppress the maximum-square error.

Coding experiments were conducted. It has been disclosed that Hadamard domain image coding is practical if proper quantization strategy is employed. A possible increase in the bit rate required by the use of the Hadamard transform will be well compensated by computational advantages of the transform.

A remaining problem to be solved is a search for efficient fast decodable variable length codes. Similar coding experiments in the SCT[11] domain are under way ; the results will be reported in a future paper.

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