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Nonlinear Acoustic Waves Radiated from a Pulsating or Oscillating Sphere and the Shock Formation Problem

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Abstract

The propagation of nonlinear acoustic waves is studied. These waves are radiated from a harmonically pulsating or oscillating sphere in an inviscid perfect gas. In each case a representation of the exact solution is presented for a farfield equation of the first order. This immediately yields the nonlinear distortion process of waveform and the acoustic shock formation distance.

1. Introduction and Summary

The propagation of nonlinear acoustic waves emitted from a pulsating or oscillating sphere is of fundamental importance in nonlinear acoustics.

Let us consider first the nonlinear sound waves emitted by a pulsating sphere in an ideal fluid composed of a perfect gas. The behavior in a farfield of these waves has so far been examined by a number of authors (Heaps[1], Naugol'nykh *et al.*[2], Blackstock[3]). As the result, various representations of the exact solution have been given for a farfield equation in a first approximation. Recently, Ginsberg[4], and Kelly and Nayfeh[5] have analyzed such problems using the method of renormalization. They have succeeded in the following points : (i) The method can precisely estimate the position where a shock emerges ; (ii) It can yield a uniformly valid expansion by means of matching the farfield solution with the nearfield solution. It seems, however, that they did neither ascertain the applicability of the method to this problem, nor could they obtain the correct expression for the axial and azimuthal components of the fluid velocity.

A new representation of the exact solution for the farfield equation is presented here which resembles the approximate solution obtained by the renormalization technique. With the aid of this result we can justify the application of the renormalization technique to this problem. Our solution retains the advantage previously stated, (i) and (ii). Furthermore, we find out a uniformly valid solution in a second approximation by making use of the method of renormalization.

In nonlinear sound waves emitted by an oscillating rigid sphere, a similar representation to the above can only be derived for the radial velocity u_r. On the basis of the expression for u_r we can easily resolve the problem of determining the tangential fluid velocity u_θ. We

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emphasize here that if we start from the expression of u_r obtained by the renormalization technique, it does not lead to the correct result. The result is significant from the point of view of presenting a uniformly valid expansion for a true multidimensional nonlinear acoustic wave.

2. Formulation of the Problem

The flow is irrotational, so that it can described by a velocity potential $\phi^*(r^*, \theta, t^*)$. The basic equation may then be written in a dimensionless form as

$$\Box\phi=\frac{\partial}{\partial t}\left[(\nabla\phi)^2+\frac{\gamma-1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2\right]+(\text{the cubic terms of }\phi), \tag{1}$$

where $r=r^*/R$, $t=c_0t^*/R$, $\phi=\phi^*/c_0R$, R is the (average) radius of the sphere, and c_0 the constant speed of sound at infinity. The asterisk designates dimensional quantities. The solution of Eq. (1) which satisfies an appropriate boundary condition on the surface of the sphere and the radiation condition determines the nonlinear sound wave emitted by the sphere.

3. Nonlinear Waves Radiated from a Pulsating Sphere

3.1 Regular perturbation and nearfield solution

In this case, the boundary condition on the surface of the sphere takes the form

$$\frac{\partial\phi}{\partial r}=\frac{dr}{dt}, \qquad \text{at } r=1+\varepsilon\cos\omega t, \tag{2}$$

where ε is a parameter small compared with unity. The velocity potential may then be expanded in powers of ε as

$$\phi(r,t)=\sum_{n=1}^{\infty}\varepsilon^n\phi_n(r,t). \tag{3}$$

The usual procedure in the regular perturbation problem can yield the solutions of the successive systems of equations for ε as follows. (i) The first order problem for O (ε) (The linear problem):

$$\phi_1=\frac{A\cos\psi}{r}, (\psi=\omega(t-r+1)+\delta), \tag{4}$$

with $A=-\omega/\sqrt{1+\omega^2}$ and $\delta=\arctan(1/\omega)$. (ii) The second order problem for $O(\varepsilon^2)$:

$$\phi_2=\frac{\omega A}{r}\{[-\frac{1}{2r}-\frac{\gamma+1}{8}\omega s(4\omega r)+D_1]\sin 2\psi$$

$$+[\frac{\gamma+1}{8}\omega(\ln r-c(4\omega r)+D_2]\cos 2\psi+\frac{\omega}{2}\},$$

$$s(x)=si(x)\cos(x)-ci(x)\sin(x),\ c(x)=ci(x)\cos(x)-si(x)\sin(x), \tag{5}$$

where D_1 and D_2 are the definite constants. The first two terms on the right-hand side show the generation of the higher harmonics due to the nonlinearity. The last term indicates the existence of steady streaming volume inflow, but it should be noted that the mass flux through a surface surrounding the sphere vanishes.

3.2 A representation of farfield solution

At large distances from the place of the origin, a farfield equation may hold in a first approximation :

$$\frac{\partial w}{\partial z} - w\frac{\partial w}{\partial \psi} = 0, \tag{6}$$

where $w = ru$, $z = (\gamma+1)\omega(\ln r)/2$, and $\psi = \omega(t-r+1)+\delta$.

It can easily be verified that the radial velocity given by

$$u = \varepsilon\frac{\omega A \sin\hat{\psi}}{r}, (\hat{\psi} = \omega(t-\eta+1)+\delta), \tag{7}$$

satisfies exactly Eq. (6) and the given boundary condition, if η is determined by the relation

$$r = \eta + \varepsilon\frac{\gamma+1}{2}A\ln r\sin\hat{\psi}. \tag{8}$$

When we compare the solution (7) with that of the linear problem (4), we recognize that in Eq. (7) r in the phase function of the linear solution in farfield is only replaced by η defined by Eq. (8). Clearly, it shows that the effect of nonlinearity only strains the spatial coordinate in the phase function. As will be seen in the next section, the solution (7) is closely related to the one obtained by the method of renormalization.

3.3 Renormalized expansion and uniformly valid solution in second approximation

Solving the successive systems of equations in the regular perturbation problem, we could have in the farfield

$$u = u^{(1)} + u^{(2)} + O(\varepsilon^3/r),$$

$$u^{(1)} = \varepsilon\omega A\frac{\sin\psi}{r} + \varepsilon^2\frac{\gamma+1}{4}\omega^3 A\frac{\ln r}{r}\sin 2\psi$$

$$-\varepsilon^3\frac{(\gamma+1)^2}{32}\omega^5 A^3\frac{(\ln r)^2}{r}(\sin\psi - \sin 3\psi) + \cdots,$$

$$u^{(2)} = \varepsilon^2\frac{\gamma+1}{4}\omega^3 A^2 B\frac{\sin(2\psi+\beta)}{r} - \varepsilon^3\frac{(\gamma+1)^2}{16}\omega^5 A^3 B$$

$$\times\frac{\ln r}{r}[\sin(\psi+\beta) - \sin(3\psi+\beta)] + \cdots\cdots \tag{9}$$

where

$$\tan\beta = -D_2/D_1, \text{ and } B = 8\sqrt{D_1^2+D_2^2}/\omega(1+\gamma).$$

The first series $u^{(1)}$, which was found by Heaps, does not converge on the range of r satisfying an inequality $\ln r > 1/[\varepsilon(\gamma+1)\omega^2\ |A|\ /4]$. To render $u^{(1)}$ and $u^{(2)}$ uniformly valid at large distances from the sphere, we here employ the method of renormalization. Introducing the coordinate transformation

$$r = \eta + \sum_{n=1}^{\infty}\varepsilon^n S_n^{(1)}(\eta, t) + \sum_{n=2}^{\infty}\varepsilon^n S_n^{(2)}(\eta, t), \tag{10}$$

we attempt to choose $S_n^{(1)}$ and $S_n^{(2)}$ so as to annihilate the nonuniformity in $u^{(1)} + u^{(2)}$ for the new variable η. After some calculations we may infer that

$$S_1^{(1)}=\frac{1}{2}(\gamma+1)\,\omega A\ln\eta\sin\hat{\psi},\ S_n^{(1)}=0\quad(n\geqq 2)\ ;$$

$$S_2^{(2)}=(\gamma+1)\,\omega B\ln\eta\sin(2\hat{\psi}+\beta),\ S_n^{(2)}=0\quad(n\geqq 3)\ . \tag{11}$$

The formulas $S_n^{(1)}=0$ $(n\geqq 2)$ are precisely guaranteed with the aid of Eq. (8). Thus, we have as a second approximation the farfield solution (cf. Eq. (7))

$$u=\varepsilon\frac{\omega A}{\eta}\sin\hat{\psi}+\varepsilon^2\frac{2\omega B}{\eta}\sin(2\hat{\psi}+\beta)+O(\varepsilon^3). \tag{12}$$

A uniformly valid expansion can readily obtained for all r of interest provided that in Eqs. (4) and (5) r is replaced by η.

3.4 The shock formation problem

A shock starts to form at the location where the profile of u attains to a vertical tangent, *i. e.*, $\partial r/\partial\eta\mid\eta=\eta_s,\ t=t_s=0$. Thus the shock formation distance r_s can be obtained using Eq. (8), or more precisely using Eq. (10). To a rough approximation r_s is estimated as

$$r_s\sim\exp\left(\frac{2}{\gamma+1}\frac{\sqrt{1+\omega^2}}{\varepsilon\,\omega^3}\right). \tag{13}$$

We show a typical example of shock formation process in Figure 1.

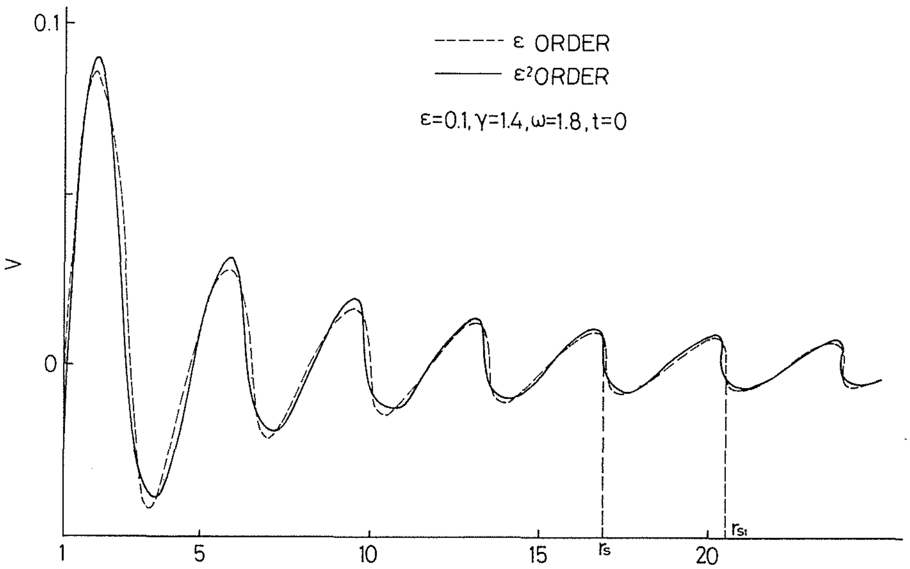

Fig. 1 Velocity profile and the shock formation distance, for a nonlinear acoustic wave emitted from a pulsating sphere.

4. Nonlinear Waves Radiated from an Oscillating Rigid Sphere

In the same manner as in the preceding section we can handle the present problem, except for a matter of determining the tangential fluid velocity u_θ. Similarly, the farfield equation in the first approximation takes the form

$$\frac{\partial w}{\partial z} - w\frac{\partial w}{\partial \psi} = 0\,, \tag{14}$$

where $w = ru_r$, $z = (\gamma+1)\,\omega\,(\ln r)/2$, and $\psi = \omega\,(t-r+1)+\delta'$, $\delta' = -\arctan(\omega^2-2)/(2\omega)$. If the velocity of the sphere in the z-direction is prescribed by $u_z = \varepsilon\omega\cos\omega t$, we obtain from Eq. (14) the exact solution

$$u_r = \varepsilon\omega^2 A'\cos\theta\frac{\sin\hat{\psi}}{r}, \tag{15}$$

where $A' = -\omega/\sqrt{\omega^4+4}$, $\hat{\psi} = \omega\,(t-\eta+1)+\delta'$. Here, η should be determined, as a function of r and t, by the relation

$$r = \eta + \varepsilon\frac{1}{2}(\gamma+1)\,\omega^2 A'\cos\theta\ln r\sin\hat{\psi}. \tag{16}$$

We may note that for the linear problem we have $u_r = \varepsilon\omega^2 A'\cos\theta\sin\psi/r$ in a farfield.

On the basis of Eq. (15) the velocity potential $\phi\,(\eta,\,\theta,\,t)$ can easily be evaluated, so that from it we succeed in deriving the tangential fluid velocity u_θ :

$$\begin{aligned} ru_\theta &= \frac{\partial\phi}{\partial\theta} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial\theta} \\ &= -\varepsilon\omega A'\sin\theta\frac{\cos\hat{\psi}}{\eta} - \varepsilon^2\frac{\gamma+1}{8}\omega^2\sin 2\theta\frac{\ln\eta}{\eta}. \end{aligned} \tag{17}$$

Figure 2 shows the profiles of the radial and tangential fluid velocity in the instant of forming a shock wave. A shock formation process is illustrated in Figure 3.

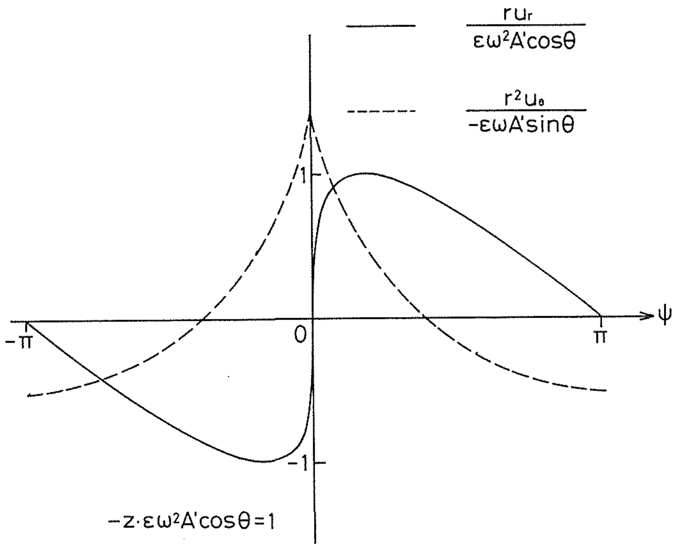

Fig. 2 Profiles of the radial and tangential fluid velocity in the instant of forming a shock, for a nonlinear acoustic wave emitted from an oscillating sphere.

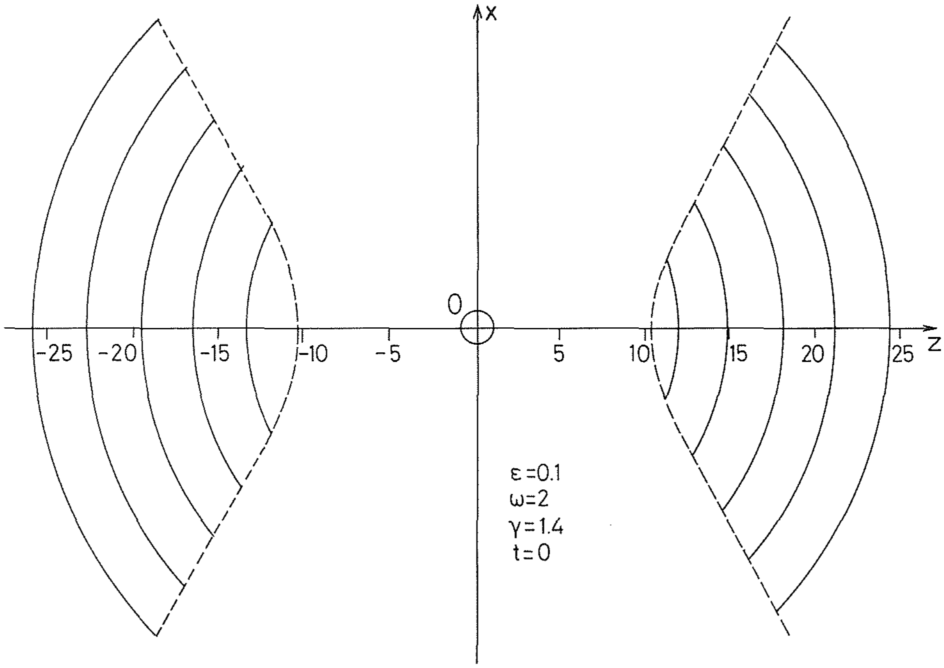

Fig. 3 Propagation of shock waves designated by the solid lines. The shock waves cannot exist beyond the broken lines.

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