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Longitudinal Vibration of a Circular Cylinder under a Thermal Field

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Abstract

Space one dimensional equation of longitudinal vibration of a circular cylinder under a thermal field is obtained from the fundamental equations of an elastic body. Considering that the radius of the cylinder is small, equations in the zero-th, and the first order approximation are derived. The present method enables us to calculate approximate equations in any desired accuracy.

For longitudinal harmonic waves propagating in an infinite long bar under a thermal field, exact and approximate frequency equations can be obtained from equations derived in the present paper. Especially, the approximate frequency equations for a small radius are solved for several thermal conditions at the peripheral surface of the cylinder, and the phase velocity and the damping coefficient are calculated.

1. Introduction

For the longitudinal vibration of a circular cylinder under thermal stress, a method to derive approximate equations in any desired accuracy was presented by Takizawa and Sugiyama[1], and equations for the zero-th, the first, and the second order approximation were derived. In their theory, however, it was assumed that the energy equation is independent of equations of motion, that is, the so-called thermoelastic effect is not taken into account explicitly.

In practice, the variation of the strain in an elastic body is attended by a change in temperature which, in turn, affects the variation of the strain. Thus, it is necessary to take into account the thermoelastic effect appropriately in developing a theory to derive approximate equations for longitudinal vibration of a bar under thermal field.

On the other hand, as for the frequency equation taking into account thermoelastic effect, Suhubi[2] derived a rather complex exact equation for the longitudinal harmonic waves propagating in an infinite long circular cylinder. The exact frequency equation is a transcendental one, and has an infinite number of solutions. Recently, Daimaruya[3] investigated several solutions of the exact frequency equation through a numerical calculation and discussed the thermoelastic effect on the phase velocity and energy dissipation. An approximate solution for a cylinder of a small radius was obtained by Suhubi[2] and Deresiewicz[4] and

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the influence of thermoelastic effect was discussed.

In the present paper, for the longitudinal vibration of a thick circular cylinder under a thermal field, a theory to derive approximate equations in any desired accuracy is presented taking into account the thermoelastic effect, and equations for the zero-th, and the first order approximation are derived. For longitudinal harmonic waves propagating in an infinite long circular cylinder, the exact frequency equation is obtained and approximate frequency equations for small radius are also derived. Especially, the zero-th and the first order approximate frequency equations for small radius are solved under several thermal boundary conditions at the peripheral surface and phase velocity and damping coefficient are calculated. The results obtained in the present paper are compared with those hitherto obtained.

2. Notations and Fundamental equations

Notation

$r,\ \theta,\ z$: cylindrical coordinate,

$u_r,\ u_\theta,\ u_z$: components of displacement,

ρ : density,

t : time,

T : deviation of temperature,

T^* : reference temperature,

α : coefficient of linear thermal expansion,

κ : coefficient of thermal conductivity,

h : heat transfer coefficient,

$\lambda,\ \mu$: Lamé's constants,

$E=\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$: Young's modulus,

$\sigma=\dfrac{\lambda}{2(\lambda+\mu)}$: Poisson's ratio.

Fundamental equations

We shall treat here the longitudinal vibration of a circular cylinder, so we assume that $u_\theta=0$ and $\partial/\partial\theta=0$. In this case, equations of motion under thermal stress and energy equation are expressed as :

$$(\varDelta_{r1}+\mathcal{L}_v^2)u_r+\frac{1}{1-2\sigma}\frac{\partial\Theta}{\partial r}-\frac{2(1+\sigma)}{1-2\sigma}\alpha\frac{\partial T}{\partial r}=0, \tag{2-1}$$

$$(\varDelta_{r0}+\mathcal{L}_v^2)u_z+\frac{1}{1-2\sigma}\frac{\partial\Theta}{\partial z}-\frac{2(1+\sigma)}{1-2\sigma}\alpha\frac{\partial T}{\partial z}=0, \tag{2-2}$$

$$(\varDelta_{r0}+\mathcal{L}_T^2)T-\frac{\gamma-1}{3\alpha}L_T^2\Theta=0, \tag{2-3}$$

with

$$\Theta=\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_z}{\partial z}, \tag{2-4}$$

and

$$\Delta_{r0}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right),\quad \Delta_{r1}=\Delta_{r0}-\frac{1}{r^2},\quad L_T^2=\frac{1}{\nu^2}\frac{\partial}{\partial t},$$

$$\mathcal{L}_T^2=\frac{\partial^2}{\partial z^2}-L_T^2,\quad L_t^2=\rho\frac{\partial^2}{\partial t^2},\quad \mathcal{L}_v^2=\frac{\partial^2}{\partial z^2}-\frac{1}{\mu}L_t^2,$$

$$\nu^2=\varkappa/(\rho c_v),\quad \gamma=c_p/c_v,$$

where α stands for the coefficient of linear thermal expansion, $\varkappa$ thermal conductivity, c_v specific heat at constant volume, and c_p specific heat under constant pressure.

From eqs. (2-1) ~ (2-3), we can see that the components of displacement u_r, u_z, and temperature T satisfy the following equations.

$$(\Delta_{r1}+\mathcal{L}_v^2)(\Delta_{r1}+\mathcal{L}_1^2)(\Delta_{r1}+\mathcal{L}_2^2)u_r=0, \tag{2-5}$$

$$(\Delta_{ro}+\mathcal{L}_v^2)(\Delta_{ro}+\mathcal{L}_1^2)(\Delta_{ro}+\mathcal{L}_2^2)u_z=0, \tag{2-6}$$

$$(\Delta_{ro}+\mathcal{L}_1^2)(\Delta_{ro}+\mathcal{L}_2^2)T=0 \tag{2-7}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the operators defined by the relations :

$$\left.\begin{aligned}&\mathcal{L}_1^2+\mathcal{L}_2^2=\mathcal{L}_u^2+\mathcal{L}_T^2-\delta L_T^2,\\&\mathcal{L}_1^2\mathcal{L}_2^2=\mathcal{L}_u^2\mathcal{L}_T^2-\delta L_T^2\frac{\partial^2}{\partial z^2},\end{aligned}\right\} \tag{2-8}$$

with

$$\mathcal{L}_u^2=\frac{\partial^2}{\partial z^2}-\frac{1}{(\lambda+2\mu)}L_t^2,$$

and thermoelastic coupling coefficient :

$$\delta=\frac{1+\sigma}{3(1-\sigma)}(\gamma-1). \tag{2-9}$$

From eqs. (2-5)~(2-7), we obtain u_r, u_z and T, which are finite at the central line ($r=0$) :

$$u_r=\mathcal{L}_1J_1(r\mathcal{L}_1)A_1+\mathcal{L}_2J_1(r\mathcal{L}_2)A_2+\mathcal{L}_v^{-1}J_1(r\mathcal{L}_v)\frac{\partial A_3}{\partial z}, \tag{2-10}$$

$$u_z=J_0(r\mathcal{L}_1)\frac{\partial A_1}{\partial z}+J_0(r\mathcal{L}_2)\frac{\partial A_2}{\partial z}-J_0(r\mathcal{L}_v)A_3, \tag{2-11}$$

$$T=\frac{1-\sigma}{(1+\sigma)\alpha}\{(\mathcal{L}_u^2-\mathcal{L}_1^2)J_0(r\mathcal{L}_1)A_1+(\mathcal{L}_u^2-\mathcal{L}_2^2)J_0(r\mathcal{L}_2)A_2\}, \tag{2-12}$$

where $A_i(i=1, 2, 3)$ are functions of t and z, $J_0(x)$ and $J_1(x)$ represent Bessel functions of the order zero and of the order unity respectively, and operators $J_0(r\mathcal{L}_i)$ and $J_1(r\mathcal{L}_i)$ are defined by :

$$\left.\begin{aligned}&J_0(r\mathcal{L}_i)=\sum_{k=0}^{\infty}(-1)^k\frac{r^{2k}}{2^{2k}(k!)^2}\mathcal{L}_i^{2k},\\&J_1(r\mathcal{L}_i)=\sum_{k=0}^{\infty}(-1)^k\frac{r^{2k+1}}{2^{2k+1}k!(k+1)!}\mathcal{L}_i^{2k+1},\end{aligned}\right\} \tag{2-13}$$

where subscript i means 1, 2, and v. By means of eqs. (2-10)~(2-12) and (2-4), A_i ($i=1, 2, 3$) can be expressed as :

$$\left.\begin{aligned} A_1 &= (\lambda+2\mu)L_t^{-2}(\mathcal{L}_1^2-\mathcal{L}_2^2)^{-1}\left\{(\mathcal{L}_u^2-\mathcal{L}_2^2)\Theta_0-\frac{1+\sigma}{1-\sigma}\alpha\left(\frac{\partial^2}{\partial z^2}-\mathcal{L}_2^2\right)T_0\right\}, \\ A_2 &= (\lambda+2\mu)L_t^{-2}(\mathcal{L}_2^2-\mathcal{L}_1^2)^{-1}\left\{(\mathcal{L}_u^2-\mathcal{L}_1^2)\Theta_0-\frac{1+\sigma}{1-\sigma}\alpha\left(\frac{\partial^2}{\partial z^2}-\mathcal{L}_1^2\right)T_0\right\}, \\ A_3 &= -G_0+(\lambda+2\mu)L_t^{-2}\left\{\frac{\partial\Theta_0}{\partial z}-\frac{1+\sigma}{1-\sigma}\alpha\frac{\partial T_0}{\partial z}\right\}, \end{aligned}\right\} \tag{2-14}$$

where G_o, Θ_o and T_o are the values of u_z, Θ and T at the central line of cylinder ($r=0$) respectively.

From eqs. (2-10)~(2-12) and (2-14), we understand that components of displacement u_r, u_z and temperature T are expressed by three quantities A_i ($i=1, 2, 3$) or G_o, Θ_o and T_o which are functions of t and z.

3. Boundary Conditions at the Peripheral Surface and Equation of Longitudinal Vibration

As for boundary conditions at the peripheral surface of the cylinder, *i. e.* at $r=a$, we shall take the view that no surface traction exists, and that heat flow obeys Newton' s law :

$$A_{rz}=2\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)=0, \tag{3-1}$$

$$A_{rr}=\lambda\Theta+2\mu\frac{\partial u_r}{\partial r}-(3\lambda+2\mu)\alpha T=0, \tag{3-2}$$

and

$$\frac{\partial T}{\partial r}+\frac{h}{\chi}T=0, \tag{3-3}$$

where h is heat transfer coefficient. Substituting (2-10)~(2-12) into (3-1)~(3-3), we obtain the governing equations of longitudinal vibration of a circular cylinder under a thermal field.

$$2\sum_{k=1}^{2}\mathcal{L}_k J_1(a\mathcal{L}_k)\frac{\partial A_k}{\partial z}-\left(\mathcal{L}_v^2+\frac{\partial^2}{\partial z^2}\right)\mathcal{L}_v^{-1}J_1(a\mathcal{L}_v)A_3=0, \tag{3-4}$$

$$\sum_{k=1}^{2}\left\{\left(\mathcal{L}_v^2+\frac{\partial^2}{\partial z^2}\right)J_0(a\mathcal{L}_k)-\frac{2\mathcal{L}_k}{a}J_1(a\mathcal{L}_k)\right\}A_k- -2\left\{J_0(a\mathcal{L}_v)-\frac{\mathcal{L}_v^{-1}}{a}J_1(a\mathcal{L}_v)\right\}\frac{\partial A_3}{\partial z}=0, \tag{3-5}$$

$$\sum_{k=1}^{2}(\mathcal{L}_u^2-\mathcal{L}_k^2)\left\{\mathcal{L}_k J_1(a\mathcal{L}_k)-\frac{h}{\chi}J_0(a\mathcal{L}_k)\right\}A_k=0. \tag{3-6}$$

Eq. (3-3), *i. e.* eq. (3-6), contains the following two thermal conditions as special cases.

a) Peripheral surface of the cylinder is impervious to heat, that is :

$$\frac{\partial T}{\partial r}=0. \quad (at\ r=a) \tag{3-7}$$

Eq. (3-7) can be obtained, if we put $h=0$ into eq. (3-3). Putting $h=0$ into eq. (3-6), we get :

$$\sum_{k=1}^{2}(\mathcal{L}_u^2-\mathcal{L}_k^2)\mathcal{L}_k J_1(a\mathcal{L}_k)A_k=0. \tag{3-8}$$

b) Peripheral surface is kept at the ambient temperature. In this case the thermal boundary condition is :

$$T=0. \qquad (at\ r=a) \tag{3-9}$$

We can obtain eq. (3-9), if we put $h\to\infty$ into eq. (3-3). Putting $h\to\infty$ into eq. (3-6), we have :

$$\sum_{k=1}^{2}(\mathcal{L}_u^2-\mathcal{L}_k^2)J_0(a\mathcal{L}_k)A_k=0. \tag{3-10}$$

We shall refer to conditions a), b), simply as a) adiabatic case, b) isothermal case hereafter.

4. Approximate Equations

When the radius of cylinder a is small compared with the characteristic length (wave length or length of cylinder), we can expand the left-hand sides of eqs. (3-4)~(3-6) into power series of radius a by using (2-13). If we truncate the series at terms of a^{2n}, we can obtain the equation of the longitudinal vibration in the n-th order approximation in our theory. Now, we shall show equations in the zero-th, and the first order approximation.

A) Zero-th Order Approximation

If we retain terms of the order a^0 in (3-4)~(3-6), and use (2-14), we obtain :

$$\left(L_t^2-2\mu\frac{\partial^2}{\partial z^2}\right)G_0-\lambda\frac{\partial\Theta_0}{\partial z}+(3\lambda+2\mu)\alpha\frac{\partial T_0}{\partial z}=0, \tag{4-1}$$

$$\Theta_0=(1-2\sigma)\frac{\partial G_0}{\partial z}-2(1+\sigma)\alpha T_0, \tag{4-2}$$

and

$$\left(L_T^2-\frac{\partial^2}{\partial z^2}+\frac{2h}{\varkappa a}\right)T_0+\frac{1-\sigma}{1+\sigma}\ \frac{\delta}{\alpha}\ L_T^2\Theta_0=0. \tag{4-3}$$

Eliminating Θ_o from (4-1)~(4-3), we have :

$$\left(L_t^2-E\frac{\partial^2}{\partial z^2}\right)G_0+E\alpha\frac{\partial T_0}{\partial z}=0, \tag{4-4}$$

and

$$\left\{L_T^2-\frac{\partial^2}{\partial z^2}+\frac{2h}{\varkappa a}+2(1-\sigma)\delta L_T^2\right\}T_0+\frac{(1-\sigma)(1-2\sigma)}{1+\sigma}\ \frac{\delta}{\alpha}L_T^2\frac{\partial G_0}{\partial z}=0. \tag{4-5}$$

Eq. (4-4) coincides with the result obtained by Takizawa and Sugiyama[1], and is nothing but the usual equation of the longitudinal vibration for a thin bar under thermal stress. Eliminating T_0 from eqs. (4-4) and (4-5), we get :

$$\left(L_T^2-\frac{\partial^2}{\partial z^2}+\frac{2h}{\varkappa a}\right)\left(L_t^2-E\frac{\partial^2}{\partial z^2}\right)G_0+2(1-\sigma)\delta L_T^2\left(L_t^2-3\mu\frac{\partial^2}{\partial z^2}\right)G_0=0. \tag{4-6}$$

Eq. (4-6) is the equation for longitudinal vibration of a cylinder in the zero-th order approximation in our theory.

If we put $h=0$ into the equations obtained above, we can readily get the zero-th order approximate equations for an adiabatic case.

In an isothermal case, putting $h\to\infty$ into the equations obtained above, we have :

$$T_0=0, \tag{4-7}$$

$$\Theta_0=(1-2\sigma)\frac{\partial G_0}{\partial z}, \tag{4-8}$$

and

$$\left(L_t^2-E\frac{\partial^2}{\partial z^2}\right)G_0=0. \tag{4-9}$$

Eq. (4-9) is nothing but the usual equation of longitudinal vibration of a thin bar without thermal stress. This result shows that in an isothermal case there is no thermal effect in the zero-th order approximation.

B) First Order Approximation

When we retain terms of order a^2 and neglect terms of $O(a^4)$ in (3-4)~(3-6), and then use eq. (2-14) and equations of the zero-th order approximation in terms of order a^2, we obtain the following equations.

$$\Theta_0=(1-2\sigma)\frac{\partial G_0}{\partial z}+2(1+\sigma)\alpha T_0+\frac{K^2}{4}\left[\frac{2\sigma(1-2\sigma)(3-2\sigma^2)}{1-\sigma}\frac{\partial^3 G_0}{\partial z^3}-\right.$$
$$\left.-2(1+\sigma)\alpha\left\{\frac{3-2\sigma}{\lambda+2\mu}L_t^2+2\sigma(1-2\sigma)\frac{\partial^2}{\partial z^2}+\frac{1-2\sigma}{1-\sigma}\frac{h}{\varkappa a}\right\}T_0\right], \tag{4-10}$$

$$\left(L_t^2-E\frac{\partial^2}{\partial z^2}\right)G_0+E\alpha\frac{\partial T_0}{\partial z}-K^2E\left[\sigma^2\frac{\partial^4 G_0}{\partial z^4}+\right.$$
$$\left.+\frac{\alpha}{4}\left\{\frac{L_t^2}{\lambda+\mu}-2(1+2\sigma^2)\frac{\partial^2}{\partial z^2}+\frac{2h}{\varkappa a}\right\}\frac{\partial T_0}{\partial z}\right]=0, \tag{4-11}$$

and

$$\left\{L_T^2-\frac{\partial^2}{\partial z^2}+\frac{2h}{\varkappa a}+2(1-\sigma)\delta L_T^2\right\}T_0+\frac{(1-\sigma)(1-2\sigma)}{1+\sigma}\frac{\delta}{\alpha}L_T^2\frac{\partial G_0}{\partial z}-$$
$$-\frac{K^2}{4}\left[2\sigma(3+2\sigma)\left(L_T^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial z^2}+2(1-\sigma)\delta L_T^2\left\{\frac{L_t^2}{\lambda+\mu}+(1+6\sigma)\frac{\partial^2}{\partial z^2}\right\}+\right.$$
$$\left.+\frac{2h}{\varkappa a}\left\{L_T^2-(1-6\sigma-4\sigma^2)\frac{\partial^2}{\partial z^2}+2(1-\sigma)\delta L_T^2+\frac{4h}{\varkappa a}\right\}T_0=0, \tag{4-12}$$

where $K=a/\sqrt{2}$ is the radius of gyration of the cross-section of the cylinder around its central line. Eq. (4-11) is slightly different from Takizawa and Sugiyama' s result[1] in terms of order a^2. When no thermal stress is present *i. e.* $\alpha=0$, eq. (4-11) coincides with the equation taking into account the lateral deformation in the cross-sectional plane of the cylinder[5].

Eliminating T_0 from (4-11) and (-4-12), we obtain :

$$\left(L_T^2-\frac{\partial^2}{\partial z^2}+\frac{2h}{\varkappa a}\right)\left(L_t^2-E\frac{\partial^2}{\partial z^2}\right)G_0+2(1-\sigma)\delta L_T^2\left(L_t^2-3\mu\frac{\partial^2}{\partial z^2}\right)G_0-$$
$$-K^2\left[E\sigma^2\left(L_T^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial^4}{\partial z^4}+\frac{1-\sigma}{2}\delta L_T^2\left\{\frac{L_t^4}{\lambda+\mu}-2(1-\sigma)(1-2\sigma)L_t^2\frac{\partial^2}{\partial z^2}+6\mu\sigma\frac{\partial^4}{\partial z^4}\right\}+\right.$$
$$\left.+\frac{2h}{\varkappa a}\left\{E\sigma^2\frac{\partial^4}{\partial z^4}+\frac{h}{2\varkappa a}\left(L_t^2-E\frac{\partial^2}{\partial z^2}\right)\right\}\right]G_0=0, \tag{4-13}$$

which is the equation of longitudinal vibration under a thermal field in the first order approximation in our theory.

If we put $h=0$ into eqs. (4-10)~(4-13), we promptly get equations in the first order approximation for an adiabatic case.

If we use eq. (3-10) instead of (3-6), and apply the same procedure as we used in deriving (4-10)~(4-13), we have :

$$T_0=-\frac{K^2}{2}\frac{(1-\sigma)(1-2\sigma)}{1+\sigma}\frac{\delta}{\alpha}L_T{}^2\frac{\partial G_0}{\partial z}, \tag{4-14}$$

$$\Theta_0=(1-2\sigma)\frac{\partial G_0}{\partial z}+\frac{1-2\sigma}{4}K^2\left\{\frac{2\sigma(3-2\sigma^2)}{1-\sigma}\frac{\partial^2}{\partial z^2}-(3-2\sigma)\delta L_T{}^2\right\}\frac{\partial G_0}{\partial z}, \tag{4-15}$$

and

$$\left(L_t{}^2-E\frac{\partial^2}{\partial z^2}\right)G_0-K^2E\left\{\sigma^2\frac{\partial^2}{\partial z^2}+\frac{(1-\sigma)(1-2\sigma)}{4(1+\sigma)}\delta L_T{}^2\right\}\frac{\partial^2 G_0}{\partial z^2}=0. \tag{4-16}$$

Eq. (4-16) is the equation of longitudinal vibration for an isothermal case in the first order approximation.

In a similar manner, we are able to continue the process of approximation and can obtain the approximate equations for longitudinal vibration of a cylinder under thermal field with any desired accuracy.

5. Frequency Equations

Let us consider harmonic waves propagating in the z direction. In this case A_k ($k=1, 2, 3$) are represented by :

$$A_k(z,t)=a_k\exp\{i(qz+\omega t)\},\quad (k=1,2,3) \tag{5-1}$$

where a_k ($k=1, 2, 3$) are costants, q is wave number, and ω is frequency. Since operators $\partial/\partial t$ and $\partial/\partial z$ can be replaced by $i\omega$ and iq respectively, we put :

$$\left.\begin{aligned}&L_T{}^2=i\frac{\omega}{\nu^2},\ \mathcal{L}_T{}^2=-q^2-i\frac{\omega}{\nu^2},\ L_t{}^2=-\rho\omega^2,\ \mathcal{L}_1=\lambda_1,\\&\mathcal{L}_2=\lambda_2,\ \mathcal{L}_v{}^2=\lambda_3{}^2=\frac{\omega^2}{c_2{}^2}-q^2,\ \mathcal{L}_u{}^2=\frac{\omega^2}{c_1{}^2}-q^2,\\&\text{with } c_1{}^2=(\lambda+2\mu)/\rho,\ c_2{}^2=\mu/\rho.\end{aligned}\right\} \tag{5-2}$$

From eqs. (2-8) and (5-2), λ_1 and λ_2 satisfy the relations :

$$\left.\begin{aligned}&\lambda_1{}^2+\lambda_2{}^2=\frac{\omega^2}{c_1{}^2}-2q^2-i(1+\delta)\frac{\omega}{\nu^2},\\&\lambda_1{}^2\lambda_2{}^2=q^2\left(q^2-\frac{\omega^2}{c_1{}^2}\right)+i\frac{\omega}{\nu^2}\left\{(1+\delta)q^2-\frac{\omega^2}{c_1{}^2}\right\}.\end{aligned}\right\} \tag{5-3}$$

Substituting (5-1) into (3-4)~(3-6), and using (5-2) and (5-3), we obtain the following equations.

$$\left.\begin{aligned}&2q\sum_{k=1}^{2}\lambda_k J_1(a\lambda_k)a_k+i(\lambda_3^2-q^2)J_1(a\lambda_3)\frac{a_3}{\lambda_3}=0,\\&\sum_{k=1}^{2}\left\{(\lambda_3^2-q^2)J_0(a\lambda_k)-2\frac{\lambda_k}{a}J_1(a\lambda_k)\right\}a_k-2iq\Big\{J_0(a\lambda_3)-\\&\qquad-\frac{1}{a\lambda_3}J_1(a\lambda_3)\Big\}a_3=0,\\&\sum_{k=1}^{2}\left(\frac{\omega^2}{c_1^2}-q^2-\lambda_k^2\right)\left\{\lambda_k J_1(a\lambda_k)-\frac{h}{\chi}J_0(a\lambda_k)\right\}a_k=0.\end{aligned}\right\}\qquad(5\text{-}4)$$

From eq. (5-4), we have the frequency equation as follows.

$$\begin{aligned}&\left[2\eta^2\left\{\Lambda_3\frac{J_0(\Lambda_3)}{J_1(\Lambda_3)}-1\right\}+(\Lambda_3^2-\eta^2)\left\{\frac{\Lambda_3^2-\eta^2}{2\Lambda_2}\frac{J_0(\Lambda_2)}{J_1(\Lambda_2)}-1\right\}\right](\Lambda_1^2+\eta^2-\Omega_1^2)\times\\&\times\left\{\frac{H}{\Lambda_1}\frac{J_0(\Lambda_1)}{J_1(\Lambda_1)}-1\right\}-\left[2\eta^2\left\{\Lambda_3\frac{J_0(\Lambda_3)}{J_1(\Lambda_3)}-1\right\}+(\Lambda_3^2-\eta^2)\times\right.\\&\left.\times\left\{\frac{\Lambda_3^2-\eta^2}{2\Lambda_1}\frac{J_0(\Lambda_1)}{J_1(\Lambda_1)}-1\right\}\right](\Lambda_2^2+\eta^2-\Omega_2^2)\left\{\frac{H}{\Lambda_2}\frac{J_0(\Lambda_2)}{J_1(\Lambda_2)}-1\right\}=0,\end{aligned}\qquad(5\text{-}5)$$

where we introduced dimensionless quantities :

$$\eta=qa,\quad H=\frac{ha}{\chi},\quad \Omega_i=\frac{\omega a}{c_i},\quad \Lambda_k=\lambda_k a.\ (i=1,2\ ;\ k=1,2,3)\qquad(5\text{-}6)$$

From eqs. (5-3) and (5-6), Λ_1 and Λ_2 satisfy the relations :

$$\left.\begin{aligned}&\Lambda_1^2+\Lambda_2^2=\Omega_1^2-2\eta^2-i(1+\delta)N\Omega_1,\\&\Lambda_1^2\Lambda_2^2=\eta^2(\eta^2-\Omega_1^2)+iN\Omega_1\{(1+\delta)\eta^2-\Omega_1^2\},\end{aligned}\right\}\qquad(5\text{-}7)$$

with $N=ac_1/\nu^2$.

Eq. (5-5) coincides with the equation derived by Daimaruya[3].

If we put $H=0$, or $H\to\infty$ into eq. (5-5), we can get at once the frequency equation for an adiabatic case, or for an isothermal case, respectively. The results coincide with the frequency equations presented by Suhubi[2].

If we neglect the thermoelastic effect, *i. e.* when $\delta=0$, we have Λ_1 and Λ_2 from eq. (5-7) as follows :

$$\Lambda_1^2=\Omega_1^2-\eta^2,\quad \Lambda_2^2=-\eta^2-iN\Omega_1.\qquad(5\text{-}8)$$

In this case, frequency equations (5-5) is reduced to :

$$2\eta^2\left\{\Lambda_3\frac{J_0(\Lambda_3)}{J_1(\Lambda_3)}-1\right\}+(\Lambda_3^2-\eta^2)\left\{\frac{\Lambda_3^2-\eta^2}{2\Lambda_1}\frac{J_0(\Lambda_1)}{J_1(\Lambda_1)}-1\right\}=0,\qquad(5\text{-}9)$$

which is the frequency equation obtained by Pochhammer and Chree[6].

A) Frequency equations for a large radius

When the radius of cylinder a is sufficiently large compared with wave length, we can take $|\eta|, |\Omega_1|, |\Lambda_k|$, $H\to\infty$, and $J_0(\Lambda_k)/J_1(\Lambda_k)\simeq\pm i$ ($k=1, 2, 3$) in (5-5) and (5-9), and can obtain the frequency equations of thermoelastic Rayleigh waves.[7),8)]

B) Approximation for a small radius or a long wave length

When the radius of cylinder a is sufficiently small compared with the wave length, we can obtain the approximate frequency equations for longitudinal waves from eqs. (4-6) and (4-13). We treat predominantly elastic space harmonic waves travelling in the negative z direction, and put :

$$G_o = g_o \exp\{i(qz+\omega t)\}, \tag{5-10}$$

where g_o is constant. We presume that q is a real positive quantity and that ω is expressed as :

$$\omega = \omega_R + i\,\omega_I, \tag{5-11}$$

with real positive quantities ω_R and ω_I.

B-1) Zero-th order approximation

Introducing (5-10) into (4-6), and using (5-6), we have the frequency equation in the zero - th order approximation.

$$\left\{\eta^2 - \left(\frac{c_1}{c_0}\right)^2 \Omega_1^2\right\}(\eta^2 + 2H + iN\Omega_1) + i\frac{1-\sigma}{1+\sigma} N\Omega_1\delta\left\{3\eta^2 - \left(\frac{c_1}{c_2}\right)^2 \Omega_1^2\right\} = 0, \tag{5-12}$$

with $c_0^2 = E/\rho$.

If the coupling coefficient δ is small, we can obtain an approximate solution of eq. (5-12) as follows. When $\delta = 0$, the solution of eq. (5-12) is :

$$\omega = \omega_R = \omega_0 = c_0 q, \tag{5-13}$$

and phase velocity is :

$$c = \frac{\omega_R}{q} = c_0. \tag{5-14}$$

For small δ, we put :

$$\omega_R \fallingdotseq \omega_0(1 + x_1\delta), \quad \omega_I \fallingdotseq \omega_0 y_1 \delta. \tag{5-15}$$

Introducing (5-15) into (5-12), we get a phase velocity c and a damping coefficient ω_I as follows.

$$c = c_0\left\{1 + \frac{1}{2}\ \frac{c_0^4 q^2}{c_0^2 q^2 + \nu^4\{q^2 + 2h/(\varkappa a)\}^2}\ \frac{\alpha^2 T_0^*}{c_v}\right\}, \tag{5-16}$$

$$\omega_I = \frac{1}{2} c_0^4 q^2 \frac{\nu^2\{q^2 + 2h/(\varkappa a)\}}{c_0^2 q^2 + \nu^4\{q^2 + 2h/(\varkappa a)\}^2}\ \frac{\alpha^2 T_0^*}{c_v}, \tag{5-17}$$

where we used the thermodynamical relation :

$$\rho c_v\ (\gamma - 1) = 3\ (3\lambda + 2\mu)\ \alpha^2\ T_o^*, \tag{5-18}$$

with the reference temperature T_o^*, and eq. (2-9) was used.

In an adiabatic case $(h=0)$, eqs. (5-16) and (5-17) coincide with the results derived by Suhubi[2] if we put $1/2\,(1-2\sigma)$ instead of $1/2$ in both equations, (5-16) and (5-17).

In an isothermal case, there is no thermal effect in the zero-th order approximation. Thus we shall examine the first order approximation.

B-2) First order approximation

Introducing (5-10) into (4-13), and using (5-6), we get the frequency equation in the first order approximation.

$$\left\{\eta^2-\left(\frac{c_1}{c_0}\right)^2\Omega_1^2\right\}(\eta^2+2H+iN\Omega_1)-\frac{\eta^2}{2}\left[(\eta^2+2H+iN\Omega_1)\sigma^2\eta^2+\left\{\eta^2-\left(\frac{c_1}{c_0}\right)^2\Omega_1^2\right\}\frac{H^2}{\eta^2}\right]+$$
$$+i\frac{1-\sigma}{1+\sigma}N\Omega_1\delta\left[3\eta^2-\left(\frac{c_1}{c_2}\right)^2\Omega_1^2-\frac{1-2\sigma}{8}\left\{\frac{12\sigma^2}{1-2\sigma}\eta^4+2(1-\sigma)\left(\frac{c_1}{c_2}\right)^2\Omega_1^2\eta^2-\left(\frac{c_1}{c_2}\right)^4\Omega_1^4\right\}\right]=0. \tag{5-19}$$

When we put $\delta=0$ and use the zero-th order approximation in terms of a^2, we have :

$$\omega_R=c_0q\left(1-\frac{\sigma^2}{2}K^2q^2\right),\qquad \omega_I=0. \tag{5-20}$$

For small δ, we introduce (5-15) into (5-19) and obtain :

$$c=c_0\left[1-\frac{\sigma^2}{2}K^2q^2+\frac{1}{2}\frac{c_0^4q^2}{c_0^2q^2+\nu^4\{q^2+2h/(\varkappa a)\}^2}\frac{\alpha^2T_0^*}{c_v}\left\{1+4\sigma(1+2\sigma)K^2q^2-\right.\right.$$
$$\left.\left.-K^2\frac{\sigma^2c_0^2q^4-\nu^4[q^2+2h/(\varkappa a)][\sigma^2q^4+4h^2/(\varkappa a)^2]}{c_0^2q^2+\nu^4[q^2+2h/(\varkappa a)]^2}\right\}\right], \tag{5-21}$$

and

$$\omega_I=\frac{1}{2}\frac{c_0^4\nu^2q^2}{c_0^2q^2+\nu^4\{q^2+2h/(\varkappa a)\}^2}\frac{\alpha^2T_0^*}{c_v}\left[\{q^2+2h/(\varkappa a)\}\{1+4\sigma(1+2\sigma)K^2q^2\}-\right.$$
$$\left.-2K^2\frac{\sigma^2c_0^2q^4\{q^2+2h/(\varkappa a)\}+h^2/(\varkappa a)^2\{c_0^2q^2-\nu^2[q^2+2h/(\varkappa a)]^2\}}{c_0^2q^2+\nu^4\{q^2+2h/(\varkappa a)\}^2}\right]. \tag{5-22}$$

If we put $h=0$ into eqs. (5-21) and (5-22), we can obtain at once the phase velocity and the damping coefficient for an adiabatic case.

In an isothermal case, introducing (5-10) into (4-16), and using (5-6), we get :

$$\left\{\eta^2-\left(\frac{c_1}{c_0}\right)^2\Omega_1^2\right\}-\frac{1}{2}\eta^2\left\{\sigma^2\eta^2-i\frac{(1-\sigma)(1-2\sigma)}{4(1+\sigma)}N\Omega_1\delta\right\}=0. \tag{5-23}$$

If we put $\delta=0$ into eq. (5-23), we have :

$$\omega_R=c_0q\left(1-\frac{\sigma^2}{2}K^2q^2\right),\quad \omega_I=0. \tag{5-24}$$

Substituting (5-15) into (5-23), we obtain the following results for small δ.

$$c=c_0\left(1-\frac{\sigma^2}{2}K^2q^2\right), \tag{5-25}$$

and

$$\omega_I=\frac{K^2}{8}\frac{c_0^4q^2}{\nu^2}\frac{\alpha^2T_0^*}{c_v}. \tag{5-26}$$

6. Discussion

In the previous section, we treated space harmonic waves and obtained an approximate solution of frequency equation for small radius assuming that thermoelastic coupling coefficient (δ) is small. In a similar manner, we can treat time harmonic waves. However, we treat the time harmonic waves in a somewhat different manner in this section. We assume in eq. (5-10), that ω is a real positive quantity and that q is complex quantity :

$$q = q_R - i q_I, \tag{6-1}$$

with real positive quantities q_R and q_I. If we introduce the quantity $\Omega = \nu^2 \omega / c_1{}^2$, the frequency equation in the zero-th order approximation, (5-12) can be written as :

$$\left\{\eta^2 - \left(\frac{c_1}{c_0}\right)^2 \Omega_1^2\right\}\left(\eta^2 + 2H + i\frac{\Omega_1^2}{\Omega}\right) + i\frac{1-\sigma}{1+\sigma}\frac{\Omega_1^2}{\Omega}\delta\left\{3\eta^2 - \left(\frac{c_1}{c_2}\right)^2 \Omega_1{}^2\right\} = 0. \tag{6-2}$$

The quantity Ω is sufficiently small in most engineering materials even for extremely high frequency[4]. So, we expand q_R and q_I as :

$$\left.\begin{aligned} q_R &= x_0 + x_1\Omega + x_2\Omega^2 + \cdots, \\ q_I &= y_0 + y_1\Omega + y_2\Omega^2 + \cdots. \end{aligned}\right\} \tag{6-3}$$

Introducing (6-3) into (6-2), we obtain the phase velocity $c = \omega / q_R$ and the space damping coefficient q_I for small Ω as follows.

$$c = \sqrt{\frac{\alpha}{\beta}}\, c_0\left[1 - \frac{1}{8\alpha^2}\left(1 - \frac{\beta}{\alpha}\right)\left(1 + 2\frac{\alpha}{\beta}\frac{H}{\Omega_0^2}\right)\left\{7\frac{\beta}{\alpha} - 3 + \frac{2H}{\Omega_0^2}\left(3 + \frac{\alpha}{\beta}\right)\right\}\Omega^2\right], \tag{6-4}$$

$$q_I = \sqrt{\frac{\beta}{\alpha}}\,\frac{\omega}{c_0}\,\frac{1}{2\alpha}\left(1 - \frac{\beta}{\alpha}\right)\left(1 + 2\frac{\alpha}{\beta}\frac{H}{\Omega_0^2}\right)\Omega, \tag{6-5}$$

with

$$\alpha = \left(\frac{c_0}{c_1}\right)^2\left(1 + 3\frac{1-\sigma}{1+\sigma}\delta\right), \quad \beta = \left(\frac{c_0}{c_1}\right)^2\left\{1 + 2(1-\sigma)\delta\right\}. \tag{6-6}$$

If we assume that δ is small and neglect $O(\delta^2)$ in the bracket of eq. (6-4), we obtain :

$$c = \sqrt{\frac{\alpha}{\beta}}\, c_0\left\{1 - \frac{(1-\sigma)^3}{2(1+\sigma)^3(1-2\sigma)}\left(1 + \frac{2H}{\Omega_0^2}\right)\delta\,\Omega^2\right\}. \tag{6-7}$$

This result is different from the phase velocity derived by Deresiewicz[4] for an adiabatic case ($H=0$) :

$$c = \sqrt{\frac{\alpha}{\beta}}\, c_0\left\{1 - \frac{(1-\sigma)^2}{8(1+\sigma)^2(1-2\sigma)^2}\Omega^2\right\}. \tag{6-8}$$

From eq.(6-8), the phase velocity does not contain the thermoelastic effect in terms of order Ω^2 but our present result, eq. (6-7) shows the thermoelastic effect in terms of order Ω^2.

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