北海道大学工学部研究報告
第 136 号（昭和 62 年）

Bulletin of the Faculty of Engineering,
Hokkaido University. No. 136 (1987)

Theory of the Intermediate Size Distribution in Random Cutting

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(Received March 31, 1987)

ABSTRACT

A homogeneous thread of length L is cut at random subject to the restriction that no thread of a size less than a predetermined length b should be produced. It is shown that the original length L is considerably large as compared with the average size l_0 of the threads generated and the size distribution is characterized by $\rho(x)=(l_0-b)^{-1}\exp[(x-b)/(l_0-b)]$, $x>b$, with $<x> \Sigma \int_b^\infty dx\, x\rho(x)=l_0$. In particular, for $b=0$ (no minimum length), the size distribution is of an exponential decay type: $\rho=l_0^{-1}\exp(-x/l_0)$. A few applications of the theory, including the effect of the radiation damage on polymers and the molecular weight reduction by high speed stirring in polymer solutions, are discussed.

1. Introduction

Let us consider a homogeneous thread of length L. We cut it at random. After M cuts, the average size of the threads will be L/M+1, which is denoted by l_0. It will be shown that if the original length L is very large compared with the average length l_0, the size distribution is of an exponential type:

$$\rho(x)=\frac{1}{l_0}\exp(-x/l_0) \tag{1. 1}$$

with the normalization condition:

$$\int_0^\infty \rho(x)dx=1. \tag{1. 2}$$

When the cuts are made at equal time intervals τ the number of cuts: $M \Sigma t/\tau$ is proportional to the time t. Then, $l_0=L\tau/t$, that is, the average length decreases in proportion to the inverse first power of the time t. The size distribution changes, keeping the exponential form (1. 1).

For various physical reasons, extremely short threads may not be produced in the cut. For example, the scissiors which are used to cut threads, may not have the capacity of

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producing pieces of less than a certain length. Let us now introduce a second kind of the cutting mode.

Consider a random cutting just as before which however does not produce any thread of size smaller than b. In this case, the thread sizes produced will all be greater than b. It will be shown that after a number of cuts the size distribution is characterized by

$$\rho(x)=(l_0-b)^{-1}\exp\left[-(x-b)/(l_0-b)\right],\quad x>b \tag{1. 3}$$

with the normalization condition

$$\int_b^\infty dx\ \rho(x)=1. \tag{1. 4}$$

Eqs. (1. 3) and (1. 4) approach eqs. (1. 1) and (1. 2) as $b\rightarrow 0$.

In section 2, we consider a linear chain of N discrete units (beads), and find an expression (2. 3) for the probability of obtaining chain of r units after M cuts at random. By taking a continuum limit of (2. 3), we obtain, in section 3, the size distribution function given in (1. 1). In section 4, we take the case of a random cutting with a minimum size. The size distribution obtained for the discrete and continuum cases are characterized by (4. 6) and (1. 3) respectively. In section 5, we examine the products of the random cutting when we have initially a set of threads of various sizes. A few applications of the present theory will be discussed in the last section 6.

2. Linear Chain of Discrete Units (Beads)

we consider a linear chain of N discrete units such as beads. This chain will be cut at random at mid-points of successive units. The number of the possible points of cuts is $N-1$. The total number of possibilities in which the chain is cut at M points, is

$$\binom{N-1}{M}=\frac{(N-1)!}{M!\,(N-M-1)} \tag{2. 1}$$

Any member of the resulting set contains $M+1$ chains of one or more beads. Let the size of the first sub-chain, counting from one end, be denoted by r. This r can take any number from 1 to $N-M$. Let us find the number of possible configurations in which the first chain has the size r and the rest is arbitrary. This number will be equal to the number of possibilities in which a chain of $N-r$ beads is cut at $M-1$ points, and it is given by

$$\binom{N-r-1}{M-1}=\frac{(N-r-1)!}{(M-1)!\,(N-M-r)!}. \tag{2. 2}$$

Therefore, the probability p(r) that the first sub-chain has r beads is given by

$$\begin{aligned}P(r)&=\binom{N-r-1}{M-1}\Big/\binom{N-1}{M}\\&=\frac{(N-M-1)!}{(N-1)!}\cdot M\cdot\frac{(N-r-1)!}{(N-M-r)!}.\end{aligned} \tag{2. 3}$$

The probabilities P are normalized:

$$\sum_{r=1}^{N-M}P(r)=1 \tag{2. 4}$$

which can be shown by using the identity.

$$\binom{N-1}{M}=\binom{N-2}{M-1}+\binom{N-3}{M-1}+\cdots\cdots+\binom{M-1}{M-1}. \qquad (2.\ 5)$$

Let us now consider the size distribution of the second (or any other) sub-chains. It is easily verified that this distribution is the same as that of the first sub-chains and therefore can be characterized by P (r). It then follows that the probability of generating a chain of size r is given by P (r).

3. Continuous Chain (Thread)

Consider a linear chain of beads. Let Δl be the length of a bead. We introduce

$$L \Sigma N \Delta l \Sigma \text{total length} \qquad (3.\ 1)$$

$$L/(M+1)=(N/M+1)\Delta l=\text{average length}=l_0 \qquad (3.\ 2)$$

$$r\Delta l \Sigma x \Sigma \text{the length of r-bead chain.} \qquad (3.\ 3)$$

we now consider the limits

$$\Delta l \to 0$$

$$T\to\infty,\ M\to\infty,\ r\to\infty, \qquad (3.\ 4)$$

so that L, l_0 and x all will be finite. Let us introduce the size *distribution function* $\rho(x)$ by

$\rho(x)dx$

=the probability of finding a thread (continuous chain) of size in (x, x+dx). (3. 5)

This function ρ can be identified as the limit

$$\rho(x)=\lim_{\Delta l\to 0}\frac{P(r)}{\Delta l}. \qquad (3.\ 6)$$

By means of Stirling's formula:

$$\ln N!=N(\ln N-1) \text{ for large N}, \qquad (3.\ 7)$$

we obtain from (2. 3)

$$\ln P(r)=N\ln\left(\frac{(N-M)(N-r)}{N(N-M-r)}\right)+M\ln\left(\frac{N-M-r}{N-M}\right)$$
$$+r\ln\left(\frac{N-M-r}{N-r}\right)-\ln\left(\frac{N-r}{M}\right). \qquad (3.\ 8)$$

It is apparent that $\ln$ P is a complicated function of $x=r\Delta l$ for a general N. If N is made much greater than M, then $\ln$ P takes a simple form:

$$\ln P=-\frac{Mr}{N}-\ln\frac{N}{M} \text{ for } N\oplus M,\ r \qquad (3.\ 9)$$

or

$$P=\frac{M}{N}\exp\left(-\frac{Mr}{N}\right)=\frac{\Delta l}{l_0}\exp\ (-x/l_0), \qquad (3.\ 10)$$

where (3. 1)−(3. 3) are used.

Substitution of this expression into (3. 6) yields

$$\rho(x)=\frac{1}{l_0}\exp\ (-x/l_0). \qquad (3.\ 11)$$

This shows that the distribution is of an exponential decay type. We note that ρ is normalized in a standard form:

$$\int_0^\infty \rho(x)\ dx = 1. \tag{3. 12}$$

The average size with the distribution $\rho(x)$:

$$< x > \Sigma \int_0^\infty x\ \rho(x) dx = \frac{1}{l_0} \int_0^\infty x e^{-x/l_0}\ dx = l_0 \tag{3. 13}$$

is equal to l_0 as expected.

4. Random Cutting at a Minimum Size

Let us take a chain of N beads. We make M cuts subject to the restriction that all the sub-chains produced have a predetermined number B of beads or more. In any final distribution there will be M+1 subchains, each of which contains B or more beads.

Let us remove B beads from each and every subchain and construct a "contracted" configuration. Clearly the one-to-one correspondence can be maintained between the original and contracted configurations.

A contracted configuration has M+1 chains of zero or more beads; the total number of beads in it is equal to $N-(M+f)B$. The total number of the contracted configurations is equal to the number of possible ways in which $N-(N+1)B$ identical beads are distributed over M+1 boxes. The latter is given by

$$\binom{N-(M+1)B+M}{M} = \frac{(N-MB+M-B)\,!}{M\,!\,(N-MB-B)\,!}. \tag{4. 1}$$

Let the size of the first sub-chain in a contracted configuration be denoted by s. This s can take on any integer from zero to $N-(M+1)B$. Let us now find the number of possible configurations in which the first sub-chain has s beads and the rest is arbitrary. This number is equal to the number of ways in which $N-(M+1)B-s$ beads are distributed over M boxes, and is given by

$$\binom{N-MB-B+M-s-1}{M-1} = \frac{(N-MB-B+M-s-1)\,!}{(M-1)\,!\,(N-MB-B-s)\,!}. \tag{4. 2}$$

Thus, the probability P_s that the first sub-chain has s beads, is given by

$$P_s = \binom{N-MB-B+M-s-1}{M-1} \Big/ \binom{N-MB-B+M}{M}. \tag{4. 3}$$

The probabilities P_s are normalized:

$$\sum_{s=0}^{N-(M+1)B} P_s = 1. \tag{4. 4}$$

Recalling the one-to-one correspondence between the original and contracted configurations we can deduce that the probability P_s represents the probability $P(B+s)$ in which the first sub-chain in the original configuration contains $B+s$ beads. We now put

$$r = B + s. \tag{4. 5}$$

As in the case of unrestricted random cutting, the probability

$$P(r)=P_{r-B}=M\frac{(N-MB-B)!}{(N-MB-B+M)!}\frac{(N-MB+M-r-1)!}{(N-MB-r)!} \tag{4. 6}$$

can be regarded as the probability of generating a chain of size $r \gtrsim B$. we note that for $B=1$ expression (4. 6) reduces to expression (2. 3).

Let us go over to the limits of a continuous chain(thread), the limits defined through (3. 1)−(3. 4). The size distribution function $\rho(x)$ is a complicated function of x for a general initial size. If, however, the initial length is much greater than the average size, we can readily obtain from (4. 6)

$$\rho(x)=\lim_{\Delta\to 0}\frac{P(r)}{\Delta l}=\frac{1}{l_0-b}\exp\ [-(x-b)/(l_0-b)],\ x>b \tag{4. 7}$$

where

$$b=B\Delta l \tag{4. 8}$$

is the minimum size of the thread.

The normalization condition of ρ is given by

$$\int_b^\infty dx\rho(x)=1. \tag{4. 9}$$

The average size with the distribution $\rho(x)$:

$$\langle x\rangle\Sigma\int_b^\infty dx\ x\ \rho(x)=\frac{1}{l_0-b}\int_b^\infty dx\ x\ \exp$$
$$[-(x-b)/(l_0-b)]=l_0 \tag{4. 10}$$

is equal to l_0 as expected.

5. Random Cutting for a Set of Threads

We have seen that the random cutting with (and without) a minimum length of a single thread generates exponential size distributions characterized by (1. 3) [and (1. 1)]. This behavior is interesting in itself. However, we often required to know the results of the random cutting when we have initially a set of threads of various sizes. In this section we study this general case separately for each cutting mode.

a. Random cutting with no minimum length.

In some cutting processes we may assume that (a) the rate of cutting is proportional to the size of each thread and (b) the rate is constant in time. After a number of cuts, the average size of the threads produced from a single thread will become the same and will be independent of its original size (since longer threads are cut proportionately more often). Then, regardless of the form of the original size distribution the size distribution for all the threads is characterized by the same distribution function $l_0^{-1}\exp(-x/l_0)$. That is, any size distribution (of Gaussian double-peaked and other forms) *converges* to the " exponential-decay " type after a sufficient time. From the assumption (b), the average size l_0 decreases with the time t like $l\approx t^{-1}$. Thus, the size distribution changes with time, maintaining the exponential-decay form and the " decay-constant " l_0^{-1} increasing linearly with time : $l_0^{-1} \approx t$.

This behavior does not change much when the rate of cutting depends on time. After a sufficiently long time, the size distribution becomes of an exponential form with an increasing " decay constant " l_0^{-1}.

b. Random cutting with a minimum length b

If all of the initial sizes are much greater than the minimum length b, the same arguments apply. That is, after a number of cuts, the size distribution changes with time, maintaining the exponential-decay form : $(l_0-b)^{-1} \exp[-(x-b)/(l_0-b)]$ and the decay -constant $(l_0-b)^{-1}$ growing with time. If, however, the initial size distribution contains threads whose sizes are close to or smaller than the length b, then these threads are less likely or unlikely to be cut in the process. This set of threads therefore should be treated separately. Out theory for a chain of beads [formula (4. 6) in particular] applied in general, and may be used to discuss the behavior of this set.

6. Discussion

Among many potential applications, we may consider the radiation damage of polymer molecules. It is known that only certain specific sites of the molecule are susceptible to the damage[1]. These sites are periodically located along the polymer backbone and this period may be chosen as the discrete unit length. In most experimental conditions these sites will be damaged at random, and the number of damaged sites will be prportional to the exposure time. The distribution of distances separating successive damage sites will then follow an exponential distribution law: (1. 3).

Recently one of the present authors (J. S.) and his coworkers investigated the molecular weight changes induced by high speed stirring (14,000 rpm) of polymer solutions[2]. The main-chain scissions were observed for polymers longer than certain critical sizes (e. g. $M_C = 3 \times 10^5$ for high density polyethylene) possibly because the chemical bond is broken by a concentration of shearing stress in molecules longer than the critical value. Such scissions should occur at random for longer polymers, and scission products with certain minimum lengths should exhibit an exponential decay-type distribution. The average molecular size was observed to decrease and become stationary in about two hours of high speed stirring presumably because most of the polymers were reduced in size to the critical size or smaller. The stationary size distribution after the random cutting with minimum and maximum (critical) sizes in the products is quite different from the intermediate size distribution dicussed in the present work. The former case will be discussed in a separate publication[3].

References

1. e. g. R. B. Setlow and E. C. Pollard, *Molecular Biophysics*, Addison-Wesley, Reading, Mass., 1962
2. O. Watanabe, M. Tabata, T. Kudo, J. Sohma and T. Ogiwara, Rept. Prog. Polym. Phys. Jpn., **28**, (1985) 285

3. A. Vilenkin, B. L. Scipioni, H. Hara, S. Fujita and J. Sohma, Polym. Deg. and Stab. **17**, No. 2, 173(1987)