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Prestressing Analysis of Prestressed Concrete Beams Including Anchorage Slip

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Abstract

In this paper, the techniques of analyzing the tension in the prestressed concrete (PC) beams having curved tendons are discussed. The accuracy is higher than any of the past techniques as the precision elements of two kinds of the finite elements for PC beams developed by the authors before are used, and by showing a number of the examples of numerical calculation, some investigations and examinations are carried out.

1. Introduction

The authors have previously developed two kinds of the finite elements, precision and rough, for the PC beam members which have curved tendons. We reported the techniques of basic analysis for evaluating the tension of tendons and the flexural rigidity of PC beams after completing grouting and their applicability (1). In this paper, the techniques of analyzing the tension in PC beams, for which the precision elements of two kinds of the elements reported already were adapted, and of which the accuracy is higher than any past techniques. Namely, in this paper, first ① the techniques which can evaluate easily and with high accuracy the section affected by the losses of prestress due to the friction between tendons and sheaths and anchorage slip are proposed, and ② two kinds of the techniques for analyzing the effect that the distributions of the tension of tendons and frictional force caused by the exertion on the mechanical behavior of beams proper are shown. Further, ③ in order to examine the validity of a number of the techniques proposed here, the example of the calculation on model PC beams is shown, and the results are investigated and examined.

2. Prestressing Analysis of PC Beams with Curved Tendons

2.1 Finite Element Equation

As reported already, that which becomes the basis for the analysis of the PC beam members with curved (parabolic) tendons is a PC beam element shown in Fig. 1, and the finite element equation in this case is given by the equation below (1).

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$$\begin{bmatrix} K_{uu} & K_{uw} & K_{us} \\ K_{uw}^{\mathrm{T}} & K_{ww} & K_{ws} \\ K_{us}^{\mathrm{T}} & K_{ws}^{\mathrm{T}} & K_{ss} \end{bmatrix} \begin{Bmatrix} \Delta_u \\ \Delta_w \\ \Delta_s \end{Bmatrix} = \begin{Bmatrix} P_u \\ P_w \\ P_{so} \end{Bmatrix} \tag{1}$$

where, $\{\Delta_u\}$, $\{\Delta_w\}$,$\{\Delta_s\}$=displacement vectors at the nodes of PC beam elements related to the axial displacement, bending displacement and slip displacement along the curves of installing all tendons of a beam, respectively; $\{P_u\}$, $\{P_w\}$, $\{P_{so}\}$=external load vectors corresponding to node displacements $\{\Delta_u\}$, $\{\Delta_w\}$ and $\{\Delta_s\}$, respectively. $\{P_{so}\}$ is the vector of the tension introduced at the tension end; $[K_{uu}]$, $[K_{uw}]$, $[K_{us}]$, $[K_{ws}]$, $[K_{ss}]$=partial stiffness matrices.

2.2 Method of taking the Effect of the Tension Losses Due to Friction Exerted on the Mechanical Properties of PC Beams in Calculation

The finite element equation (1) for a PC beam having m (arbitrary) tendon layers can consistently cope with the PC beams from unbonded to perfect bond conditions by adjusting the bond coefficient as reported already. If the unbonded condition without the action of friction between tendons and sheaths can be assumed, also the behavior of a beam when the tension of tendons is introduced can be easily analyzed by equalizing the corresponding tension to the load vector $\{P_{so}\}$ on the right side of the same equation. However actually, by the friction between tendons and sheaths and the wobble of sheaths themselves occurring at the time of execution, the introduced tension gradually reduces as going away from the tension end. Further, when tendons are anchored to a beam proper at the tension end after introducing prestress, additional slip arises, in this way, additional change is to be caused in the tension distribution. Accordingly, only with the finite element equation (1), the effect that this friction, which is completely different from ordinary elastic deformation, exerts itself on the losses of tension cannot be taken into account. In order to take the effect that exerts itself on the mechanical properties of PC beams in cosideration, some method must be derived.

(1) Evaluation of distributions of tension and friction along tendons

For considering the effect that the losses of tension due to the friction between tendons and sheaths exerts on the mechanical properties of PC beams, first the profiles of distributions of tension and friction along tendon curves themselves must be evaluated.

A) At the time of introducing tension

All the tendon layers can be treated typically, therefore, only a arbitrary i-layer is considered. As well known, the tension P_{si} of a tension layer i at the position distant from x from the tension end can be expressed by the following equation(2).

$$P_{si}=P_{sio}e^{(-\mu_i\theta_{ix}-\lambda_i\xi_i)} \tag{2}$$

where, P_{sio}=introduced tension of tendon layer i; θ_{ix}=change of the angle of tendon from the tension end to point x; μ_i=coefficient of friction per unit change of angle of tendon; λ_i= coefficient of frictional loss per unit length of tendon.

Since ξ_i on the right side of the above equation is the lenght of the curve from the tension end to point x of a tendon layer i(Fig. 2), by denoting the distance of the tendon at this point from the member axis with h_{si}, it can be determined by the equation below.

$$\xi_i = \int_o^x \sqrt{1+(dh_{si}/dx)^2}\, dx \tag{3}$$

Besides, the frictional force τ_{fi} per unit lenght acting in the tangential direction to a tendon curve i at this position is the rate of change of the tension P_{si} along the curve ξ_i, accordilgly, it is expressed by the following equation (see Fig. 2).

$$\tau_{fi} = dP_{si}/d\xi_i = (dP_{si}/dx) / \sqrt{1+(dh_{si}/dx)^2} \tag{4}$$

Accodingly, the distributions of tension and frictional force along respective tendons over the total length of a beam member are determined by calculating the values of P_{si} and τ_{fi} at every individual node of the total elements.

B) **At the time of anchoring when anchorage slip occurs**

when the desired tension is introduced into respective tendon layers, these are to be anchored to a beam proper at the tension end, but at this time, if anchorage slip dS_{io} occurs at the tension end, and further the effect is exerted on the distributions of tension and frictional force in tendons. Here, the evaluation of the effect of this anchorage slip is considered.

Fig 3 (a) and (b) shows conceptually the profile of an arbitrary tendon layer i, which is composed of the connection of three kinds of curves (curve 1, 2 and 3), and the distribution of the tensile strain (the similar form to the distribution of tension) of the same tendon before and after its anchoring, respectively. The hatched part in Fig. 3 (b) expresses the lowering of tensile strain and the affected section of the tendon caused by the effect of anchorage slip dS_{io}, and as is commonly known, the area of this hatched part becomes equivalent to the anchorage slip dS_{io}. In other word, the amount of slip dS_{io} is to be absorbed by the change of the strain of the tendon from the tednsion end to L_{iR}. Besides, the curves of the strain distribution in the tendon O-P-Q-R and O'-P'-Q'-R before and after anchoring in this affected section L_{iR} become as a matter of course symmetrical in relation to the straight line R_o-R in the figure. Accordingly, in the frictional force after anchoring in this section, only the signs +− change from that before anchoring, and their magnitude is the same. Therefore first, the determination of the value of L_{iR} is the problem.

[Calculation of the affected section L_{iR}]

If the change of tension in the L_{iR} section [the parts O-P, P-Q and Q-R in the example of Fig. 3 (b)] is assumed to be linear, L_{iR} can be determined easily (3), but when we try to determine L_{iR} more accurately as the original curved distribution, it is not so simple. For example, Greunen et al. introduced some approximation by substituting the curve length ξ_i of a tendon, which is given accurately by Equation (3), with the value of x coordinate ($\xi_i \fallingdotseq x$), and presented the method of determining L_{iR} for a parabolic tendon by iterative calculation (4). This technique of Greunen et al. is useful, but when the same tendon layer is formed by the connection of a number of curves, its algorithm becomes considerably complicated.

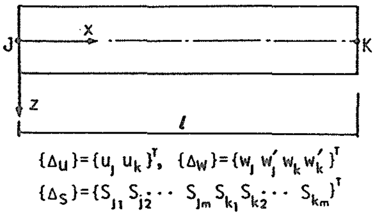

Fig. 1 PC beam element including tendon bond slip effect

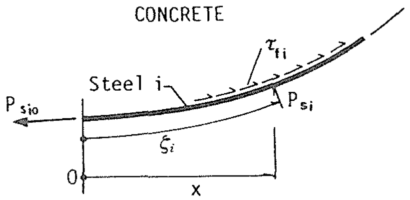

Fig. 2 Frictional force in tangential direction acting on a tendon

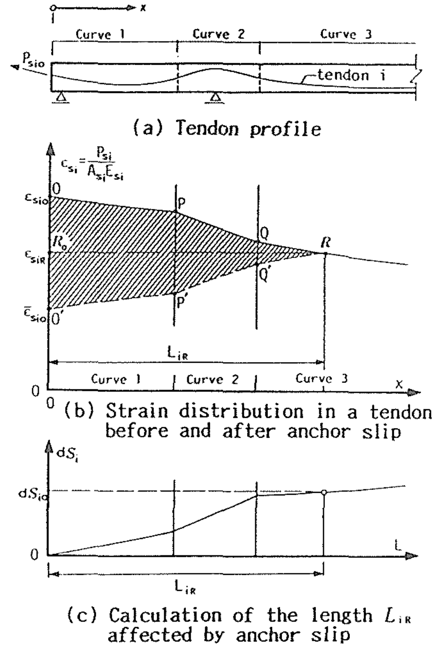

Fig. 3 Conceptual chart for calculating the length L_{iR} of the affected section due to anchor slip at the time of anchoring tendons

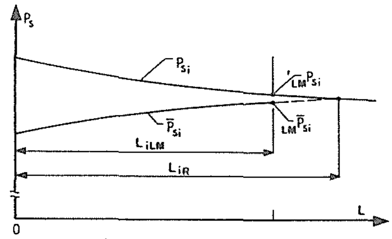

Fig. 4 Distribution of tension in a tendon before and after anchor slip when the length of the affected section L_{iR} larger than the length of the limit section L_{iLM}

Accordingly in this paper, it was decided to determine L_{iR} by using the simple method explained herein after.

Namely, the node at the position distant by L_J from the tension end is assumed to be J. Now, when it is assumed that this position is the affected section corresponding to a certain anchorage slip, the amount of slip dS_{iJ} at this time is determined by the following equation.

$$dS_{iJ} = 2\int_{o}^{Lj} (\varepsilon_{si} - \varepsilon_{sJ})\, dx \tag{5}$$

where, ε_{si} = strain of a tendon layer i at an arbitrary position x ; ε_{siJ} = strain of a tendon layer

i at the position of a node J.

Since ε_{si} and ε_{siJ} in the above equation are known, the integration is easy. As shown in Fig. 3 (c), dS_{iJ} corresponding to total node positions are calculated beforehand, and L_{iR} corresponding to the actual anchorage slip dS_{io} that finally arises at the tension end is determined and taken as its solution. The method of Greunen et al. (4) mentioned before is limited to the application to only parabolic tendons, but in contrast, the algorithm for L_{iR} used here is very simple, and can be applied to whatever forms of tendon curves arise.

[Correction of tension distribution in tendons in section L_{iR}]

As mentioned before, the distribution of frictional force in the section L_{iR} becomes the same form as the case without anchorage slip as only positive and negative signs change, therefore, only the correction of the distribution of tension in tendons becomes the problem. In the case of L_{iR} being smaller than the limit section length L_{iLM} of a member (in the case of one side pulling, the distance from the tension end to the fixed end of corresponding tendons, and in the case of symmetrical both-side pulling, the distance to the symmetry axis), as observed in Fig. 3 (b) before, the form of distribution can be determined easily, therefore, the case of L_{iR} which is larger than L_{iLM} becomes the problem.

Fig. 4 shows an example of the change of tension distribution in a tendon layer i at this time. When the amount of slip absorbed in the section L_{iLM} is assumed to be dS_{iLM}, the value can be obtained by Equation (5). The amount equivalent to the difference from the actual amount of slip dS_{io} occurring at the tension end is the amount remaining unabsorbed. This is uniformly distributed in the section L_{iLM} as the lowering of tension in this whole tendon. Accordingly, when the tension P_{si} of a tendon after anchorage slip in this section is expressed for both cases of $L_{iR}<L_{iLM}$ and $L_{iR}>L_{iLM}$, it becomes as follows.

① $L_{iR}<L_{iLM}$:

$$\bar{P}_{si}=2P_{siR}-P_{si} \tag{6}$$

② $L_{iR}>L_{iLM}$:

$$\bar{P}_{si}=(2\,{}_{LM}P_{si}-P_{si})-(dS_{io}-dS_{iLM})/L_{iLM} \tag{7}$$

where, P_{si}=tension before anchorage slip ; P_{siR}=tension before anchorage slip at L_{iR} position ; ${}_{LM}P_{si}$=tension before anchorage slip at L_{iLM} position.

(2) Method of analyzing tension in PC beams taking losses of tension into account

In the previous paragraph (1), the distributions of tension and frictional force in tendons were determined on the basis of the theory of friction. In this paragraph, the method of calculating the mechanical effect that these distributed forces induce in PC beams is considered. As the method of carrying out the analysis of the tension in PC beams by using the finite elements employed in this paper, two techniques are conceivable. Namely A) the method of treating the introduced tension as an external load acting at the tension end (5), and B) the method of treating tension and the distribution of friction as the initial stress (6). Herein after, these techniques are outlined.

A) **Method of treating the introduced tension as an external load acting at the Tension end**

In the case of treating tension as an external load, if perfectly unbonded condition can be assumed between tendons and sheaths, as already mentioned before in Section 2.2, the analysis of tension is very easy. For taking the action of friction between both into account by this technique, a number of the methods are conceivable. In the present case, such method is used that while tension is applied as external load at the tension end, the distribution of friction which is evaluated by Equation (4) is forced on a structural system as the initial stress. Namely, first the frictional force at the total element nodes in respective tendon layers is calculated by Equation (4). Next by assuming that those node frictional forces change linearly in elements, they are converted to the equivalent node loads for total tendon layers. Further by applying these equivalent node loads and tension load to the finite element equation (1), the iterative calculation is carried out till the distribution of frictional force in respective tendons, which was calculated as the results, coincides with the distribution of frictional force evaluated by Equation (4). As the result, the distribution of tension in tendons which is nearly equal to Equation (2) is realized, and the analysis is finished. By this method, the procedure of analysis itself becomes considerably simpler than the technique of B) mentioned next, but it becomes difficult to cope with the effect of anchorage slip (though it is theoretically possible), therefore, this effect is not considered in this case.

B) **Method of treating the distribution curves of tension and friction as the initial stress**

This method regards the tension and the frictional force in tendon determined in the previous paragraph (1) as the internal force, and by forcing the forms of these distributions to a structural system using initial stress method, the analysis of the tension in PC beams taking the action of friction into account is executed.

Namely, first the tensions at the total element nodes in respective tendon layers are determined by Equation (2) [for occasional demands, Equations (6) and (7)], and by assuming that these tensions at nodes change linearly within elements, they are converted to the equivalent nodal loads for all tendon layers. Next, the frictional forces at the nodes in respective tendon layers are converted to the equivalent nodal load similarly to the case A), and these and respective equivalent nodal loads determined here are applied to the finite element equation (1) both as the load terms. This initial stress method is repeated until the distributions of tension and frictional force obtained as a result coincide with the forms of distributions corresponding to these, which were calculated beforehand, and the desired solution is to be obtained.

Further, here space is restricted, therefore, the basic procedure of two method A) and B) is only outlined, and the details could not be explained, but as clearly seen in Equations (2) ~(4) in the previous paragraph (1), the distributions of tension and friction in respective tendons depend largely on their curved forms. In order to more accurately determine these distributed forces, it becomes necessary to consider also the effect that the deformation of PC beams themselves accompanying the introduction of tension exerts on the curved form of respective tendons. By this method of analysis, it is easy to take this effect into account (We

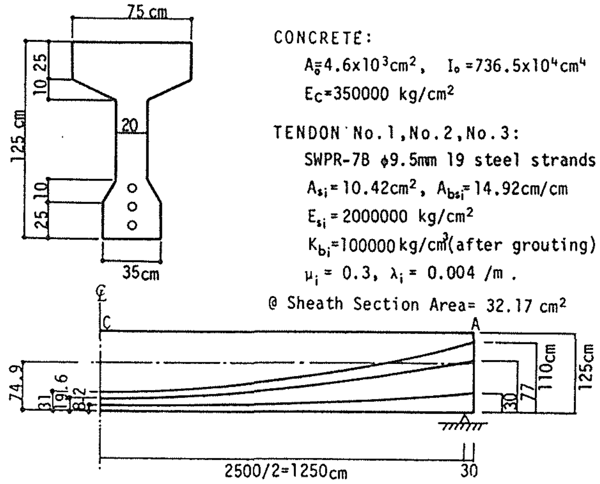

Fig. 5 General description of a model PC beam

merely add the deflection curve of a beam to the original curves of respective tendons and to simulate by these newly developed curves) and in both techniques of A) and B) mentioned above, its effect is considered.

3. Example of Numerical Calculation and Examination

In order to show the example of the calculation by the method of analyzing the tension in PC beam members proposed in this paper, the analysis was carried out on a model T-shape beam having three parabolic tendons as shown in Fig. 5, and some investigation and examination were attempted. However, it is assumed that the tendons in this beam are jacked up by both end pulling, and the tension of 144 t is applied to each tendon at the tendon ends.

(1) Case without anchorage slip

Fig. 6 is the result of an inverstigation of the difference in values of the present analysis A) and B) developed in the previous section, using respective distribution curves of deflection, bending moment, tendon stress and frictional force. Both analytical values in the figure show good agreement in all the distribution curves, and the equivalence of both analysis methods are clearly shown.

As the points to which attention must be paid in the case of carrying out the analysis of the tension in PC beams, in addition to the effect that the deformation of a beam itself exerts on the curved forms of tendons mentioned before, there is the problem of the effect that the loss of concrete cross section due to the provision of sheaths exerts itself on the rigidity of beams. By the present analytical method, these points can be easily taken into account, and in both analytical values by A) and B) shown in Fig. 6, the effects were considered. Fig. 7 and Fig. 8 show the results of investigating the difference that arises in the analytical values when these were neglected. Namely, Fig. 7 are the results of observing the effect of the deflection of beams (the loss of concrete cross section due to sheaths was teken into account), and Fig.

8 is the results of observing the effect of the loss of concrete cross section due to sheaths (the effect of the deflection of beams was taken into account). The results of both figures show that the effect of the deflection of beams themselves exerted on the curved forms of tendons was almost negligible, and that when the loss of concrete cross section due to provision of sheaths was neglected, the deflection curves of beams were underestimated (the deflection at the middle point of a beam becomes small by slightly less than 6%), but in the curves of bending moment and other stress, such a difference did not arise.

Besides, when the deflection at the middle point is computed by the customary method in which the effect of bending by the tension of tendons is treated as the upward uniformly distributed load (2) by using the tensions of respective tendons at the middle point of a beam (the loss of cross section due to sheaths connot be taken into account), the deflection of 5.92 cm is given and its value is close to 5.83 cm calculated by this analysis developed in this paper as shown in Fig. 8 in which the loss of concrete cross section due to sheaths is neglected (the value in the case of taking this effect into account is 6.18 cm.).

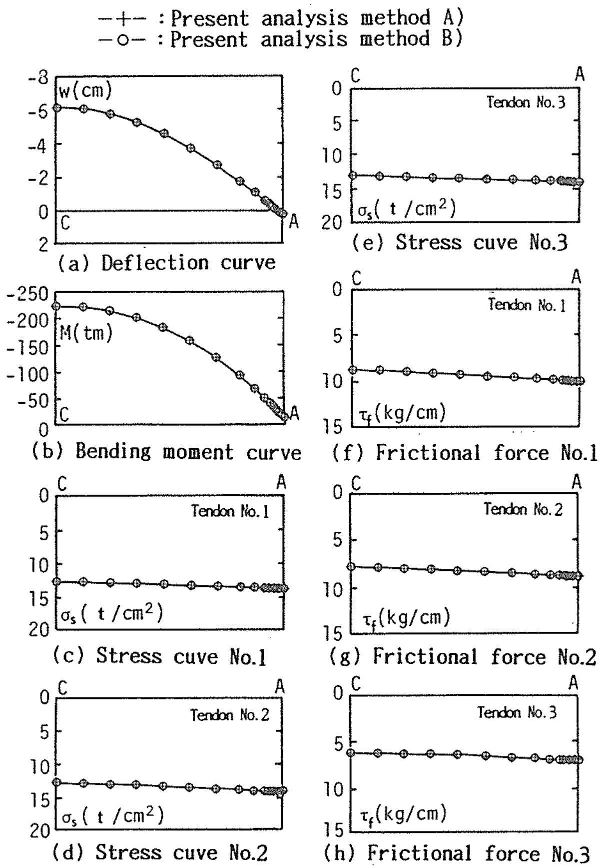

Fig. 6 Comparison of the solutions by the present analysis method A) and B)

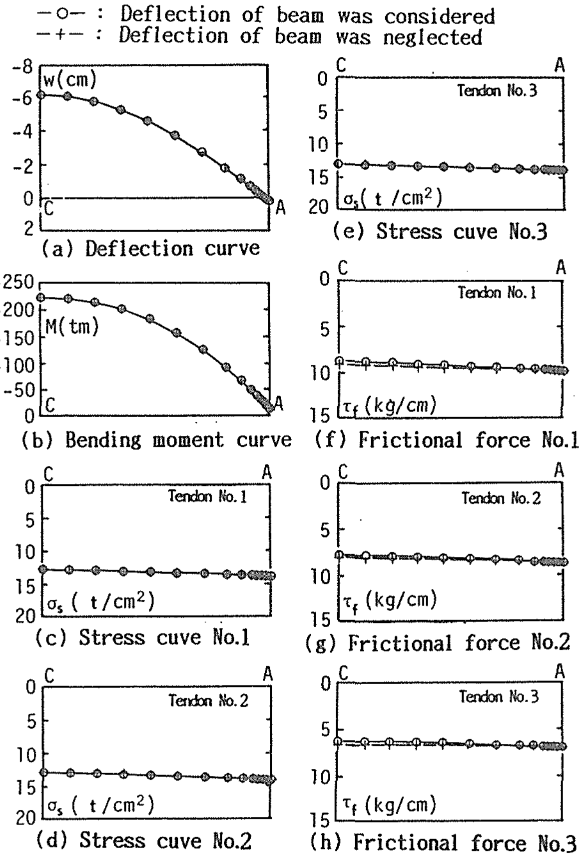

Fig. 7 Investifation into the effect that the deflection of beam exerts on the curved forms of tendons

(2) Case with anchorage slip

Fig. 9 shows the values by the present analysis B) when the anchorage slip in respective tendons was assumed to be 7 mm and the results of calculation in Fig. 6 shown before when this was neglected in comparison. In Fig. 9 (a)~(h), the effect that anchorage slip exerted on the mechanical properties of the PC beam is clearly illustrated. When attention was paid to the distribution of frictional force in Fig. (f), (g) and (h), the section where tension was reduced due to anchorage slip was 12.472 m from the tension end and did not reach the middle point of the beam in the tendon No. 1, but the sections in other No. 2 and No. 3 layers reached the middle point of the beam. When this section of tension loss was determined by the past customary method (2), in the case of No. 1 layer in which the error was the largest, it become 12.123 m, and it is known that the difference from the value by the more precise present analysis was only 2.3% at most.

Besides, the loss of tension at the main points in tendons caused by anchorage slip was $\Delta P_{s1o}=23.03$t, $\Delta P_{s2o}=21.62$t, $\Delta P_{s3o}=19.30$t at the tension ends in the tendon layers, 1, 2 and 3, respectively. $\Delta P_{s2c}=0.51$t and $\Delta P_{s3c}=2.6$t at the middle points in the tendon layers 2 and

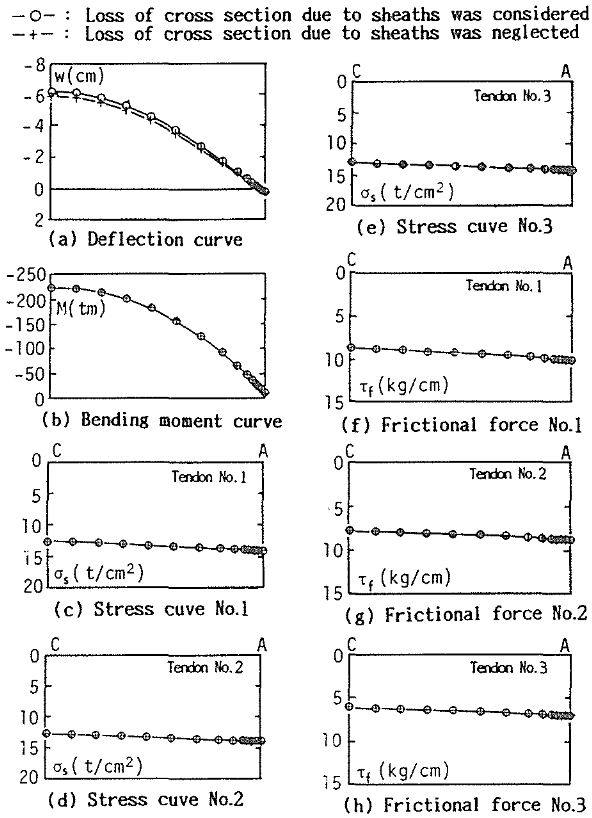

Fig. 8 Investigation into the effect of the beam cross section due to the provision of sheaths

3, respectively, by the present analysis method. Respective values obtained by a customary method corresponding with these become $\Delta P_{s1o}=24.07$t, $\Delta P_{s2o}=22.64$t, $\Delta P_{s3o}=20.24$t and $\Delta P_{s2c}=0.15$t, $\Delta P_{s3c}=2.55$t, and it was found that the customary method which is adopted in design standards and others has the sufficient accuracy that in satisfactory.

4. Summary and Conclusion

In this paper, two kinds of the techniques for analyzing the tension in PC beam members using the PC beam elements having curved tendons, which were reported by the authors before, and taking also the losses of tension due to the friction between tendons and sheaths into account are described, and by showing the example of numerical calculation, some investigation and examination are attempted. The results obtained by this study are summarized as follows.

(1) In the case without anchorage slip in tendons, two kinds of the techniques A) and B) presented in this paper gave the nearly same values, however, in view of the adaptability of the techniques, in such point that also show the effect of anchorage slip can be easily taken into account, the latter technique B) that treats the distributions of tension and frictional

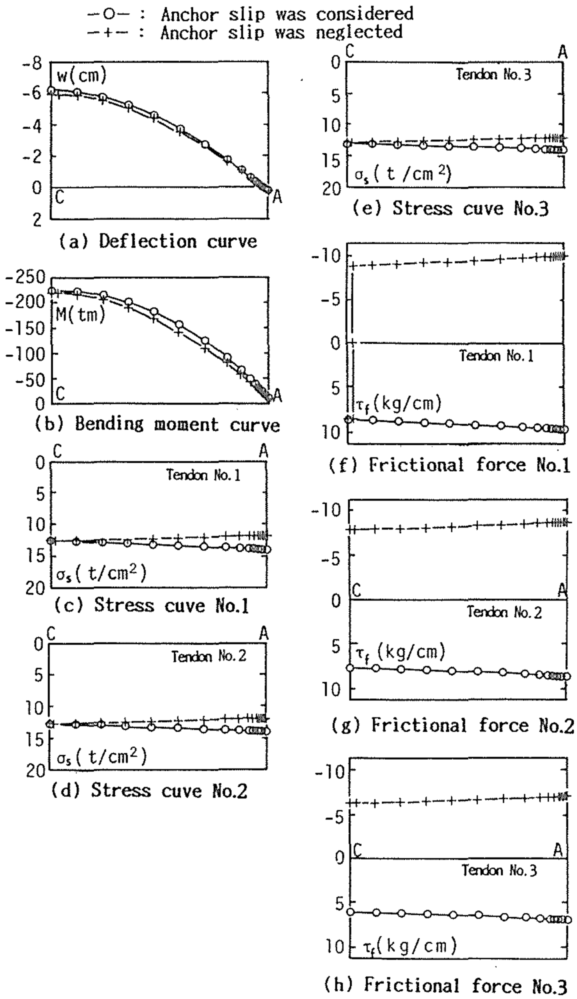

Fig. 9 Rsults of analysis in the case of taking anchor slip into account

force both as internal forces is superior.

(2) Even if the effect that the deformation of a beam exerts on the curved forms of tendons was neglected, the change was scarcely observed in the results of analysis, but when the loss of concrete cross section due to sheaths was neglected, deflection was underestimated (in the example of calculation in this paper, slightly less than 6% at the middle point of a beam), therefore, for ensuring the accuracy, it is advisable to take this into account.

(3) The past customary method such as the method of calculating the lenght of the section where the effect of anchorage slip exists was somewhat inferior in the accuracy as compared with the more precise present analysis method, but from the viewpoint of practical design, it has the sufficient accuracy.

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