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# Critical behavior and the limit distribution for long-range oriented percolation. II: Spatial correlation

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## Abstract

We prove that the Fourier transform of the properly-scaled normalized two-point function for sufficiently spread-out long-range oriented percolation with index  $\alpha > 0$  converges to  $e^{-C|k|^{\alpha \wedge 2}}$  for some  $C \in (0, \infty)$  above the upper-critical dimension  $d_c \equiv 2(\alpha \wedge 2)$ . This answers the open question remained in the previous paper [1]. Moreover, we show that the constant  $C$  exhibits crossover at  $\alpha = 2$ , which is a result of interactions among occupied paths. The proof is based on a new method of estimating fractional moments for the spatial variable of the lace-expansion coefficients.

## 1 Introduction and the main result

We consider oriented bond percolation on  $\mathbb{Z}^d \times \mathbb{Z}_+$ , where each time-oriented bond  $((x, n), (y, n + 1))$  is occupied with probability  $pD(y - x)$  and vacant with probability  $1 - pD(y - x)$ , independently of the other bonds. Here,  $D$  is a  $\mathbb{Z}^d$ -symmetric probability distribution on  $\mathbb{Z}^d$ , hence the parameter  $p \in [0, \|D\|_\infty^{-1}]$  can be interpreted as the average number of occupied bonds per vertex. We say that a vertex  $(x, j)$  is connected to  $(y, n)$ , and write  $(x, j) \rightarrow (y, n)$ , if either  $(x, j) = (y, n)$  or there is a time-oriented path of occupied bonds from  $(x, j)$  to  $(y, n)$ . Let  $\mathbb{P}_p$  be the probability distribution of the bond variables, and define the two-point function as

$$\varphi_p(x, n) = \mathbb{P}_p((o, 0) \rightarrow (x, n)),$$

and its Fourier transform as

$$Z_p(k; n) = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} \varphi_p(x, n) \quad (k \in [-\pi, \pi]^d).$$

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Notice that  $Z_p(0; n) \equiv \sum_{x \in \mathbb{Z}^d} \varphi_p(x, n)$  is the expected number of vertices at time  $n$  connected from  $(o, 0)$ . It has been known ([3] and references therein) that there is a  $p_c \geq 1$  such that

$$\chi_p \equiv \sum_{n=0}^{\infty} Z_p(0; n) \begin{cases} < \infty & (p < p_c), \\ = \infty & (p \geq p_c). \end{cases}$$

In the previous paper [1] (often referred to as Part I from now on), we investigated critical behavior of long-range oriented percolation defined by

$$D(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)},$$

where  $h$  is a probability density function on  $\mathbb{R}^d$  satisfying  $h(x) \asymp |x|^{-d-\alpha}$  (i.e.,  $|x|^{d+\alpha}h(x)$  is bounded away from zero and infinity) for large  $x$ . Here,  $\alpha > 0$  is the characteristic index, and  $L \in [1, \infty)$  is the parameter that serves the model to spread out. For example,  $\|D\|_{\infty} = O(\lambda)$ , where

$$\lambda = L^{-d}.$$

See [1, Section 1.1] for the precise definition and other properties of  $D$ . Notice that the variance  $\sigma^2 \equiv \sum_{x \in \mathbb{Z}^d} |x|^2 D(x)$  does not exist if  $\alpha \leq 2$ .

Suppose that there is a positive finite constant  $v_{\alpha}$  ( $= \frac{\sigma^2}{2d}$  if  $\alpha > 2$ ) such that the Fourier transform  $\hat{D}(k) \equiv \sum_{x \in \mathbb{Z}^d} D(x)e^{ik \cdot x}$  obeys the asymptotics

$$1 - \hat{D}(k) \underset{|k| \rightarrow 0}{\sim} \begin{cases} v_{\alpha} |k|^{\alpha \wedge 2} & (\alpha \neq 2), \\ v_2 |k|^2 \log \frac{1}{|k|} & (\alpha = 2). \end{cases} \quad (1.1)$$

The assumption (1.1) with  $v_{\alpha} = O(L^{\alpha \wedge 2})$  indeed holds if, e.g.,  $h(x) \sim c|x|^{-d-\alpha}$  as  $|x| \rightarrow \infty$  for some constant  $c$  (see [7, Section 10.5] for the 1-dimensional case). Let

$$k_n = k \times \begin{cases} (v_{\alpha} n)^{-\frac{1}{\alpha \wedge 2}} & (\alpha \neq 2), \\ (v_2 n \log \sqrt{n})^{-\frac{1}{2}} & (\alpha = 2), \end{cases} \quad (1.2)$$

so that

$$\lim_{n \rightarrow \infty} n(1 - \hat{D}(k_n)) = |k|^{\alpha \wedge 2}. \quad (1.3)$$

Among various results, we proved that, for  $\alpha > 0$ ,  $d > 2(\alpha \wedge 2)$ ,  $L \gg 1$ ,  $p \in (0, p_c]$  and  $k \in \mathbb{R}^d$ , there exists  $c, c' = 1 + O(\lambda)$  such that the normalized two-point function satisfies

$$e^{-c|k|^{\alpha \wedge 2}} \leq \liminf_{n \rightarrow \infty} \frac{Z_p(k_n; n)}{Z_p(0; n)} \leq \limsup_{n \rightarrow \infty} \frac{Z_p(k_n; n)}{Z_p(0; n)} \leq e^{-c'|k|^{\alpha \wedge 2}}. \quad (1.4)$$

Here,  $d_c \equiv 2(\alpha \wedge 2)$  is the upper-critical dimension of this model. We do not expect that (1.4) holds for  $d < d_c$ . Compare this result with the behavior of the two-point function for the branching random walk on  $\mathbb{Z}^d$  whose mean number of offspring per parent is  $p > 0$ :

$$Z_p^{\text{BRW}}(k; n) = p^n \hat{D}(k)^n, \quad \lim_{n \rightarrow \infty} \frac{Z_p^{\text{BRW}}(k_n; n)}{Z_p^{\text{BRW}}(0; n)} = e^{-|k|^{\alpha \wedge 2}}. \quad (1.5)$$

The latter is an immediate consequence of the former and (1.3). We note that  $e^{-|k|^\alpha}$  is the characteristic function of an  $\alpha$ -stable random variable (see, e.g., [10]).

The proof in [1] of (1.4) is based on the lace expansion for the two-point function. To derive information of the sequence  $Z_p(k; n)$  from its sum (= the Fourier-Laplace transform of the two-point function) and prove (1.4), we established optimal control over fractional moments for the *time* variable of the lace-expansion coefficients. However, due to the long-range nature of our  $D$ , we were unable to optimally control fractional moments for the *spatial* variable of the expansion coefficients and squeeze the bounds in (1.4) to identify the limit. We note that, by the standard Taylor-expansion method, the limit has been shown to exist at  $p = p_c$  if  $\alpha > 2$  [6] and for every  $p \in (0, p_c]$  if the model is finite-range [8]. This standard method does not work for  $\alpha < 2$  in the current setting.

In this paper, we develop a new method to estimate fractional moments for the spatial variable of the expansion coefficients and achieve the following result on the normalized two-point function:

**Theorem 1.1.** *Let  $\alpha > 0$ ,  $d > 2(\alpha \wedge 2)$ ,  $L \gg 1$  and  $p \in (0, p_c]$ . There is a  $C = 1 + O(\lambda)$  such that, for any  $k \in \mathbb{R}^d$ ,*

$$\lim_{n \rightarrow \infty} \frac{Z_p(k_n; n)}{Z_p(0; n)} = e^{-C|k|^{\alpha \wedge 2}},$$

where  $k_n$  is defined in (1.2). Moreover,

$$C = \frac{1}{1 + pm_p \sum_{(x,n)} n \pi_p(x, n) m_p^n} \times \begin{cases} 1 + \frac{pm_p}{\sigma^2} \sum_{(x,n)} |x|^2 \pi_p(x, n) m_p^n & (\alpha > 2), \\ 1 & (\alpha \leq 2), \end{cases} \quad (1.6)$$

where  $m_p$  is the radius of convergence for  $\sum_{n=0}^{\infty} Z_p(0; n) m^n$ , and  $\pi_p(x, n)$  is the alternating sum of the lace-expansion coefficients. The sums in (1.6) are absolutely convergent.

See, e.g., [1, Section 3.1] for the precise definition of  $\pi_p(x, n)$ .

The most remarkable observation in the above theorem is that the constant  $C$  exhibits crossover at  $\alpha = 2$ . This phenomenon is observable if  $\pi_p$ , which is model-dependent and contains information about interactions of occupied paths, is nonzero. We recall that, for the branching random walk, occupied paths are independent and  $\pi_p \equiv 0$ , hence  $C$  is always 1 as in (1.5). Therefore, the crossover behavior in (1.6) is a result of interactions among occupied paths.

We should emphasize that our approach developed in this paper and Part I is widely applicable, not only to our long-range oriented percolation, but also to various other (long-range/finite-range) statistical-mechanical models. For example, our methods also apply to show that a similar result to the above limit theorem holds for long-range self-avoiding walk with the characteristic index  $\alpha > 0$ , studied in [5]. Markus Heydenreich is working in this direction [4]. His work will be a generalization of the results in [2, 12], where  $D(x)$  is proportional to  $|x|^{-2}$  if  $x$  is on the coordinate axes, otherwise  $D(x) = 0$ . Since the coordinate axes are 1-dimensional, we should interpret  $\alpha$  for this particular model as 1.

As another nontrivial application of the fractional-moment method of this paper, one of the authors (LCC) will report in his ongoing work that the gyration radius  $\xi_p^{(r)}$  of order

$r \in (0, \alpha)$  for sufficiently spread-out oriented percolation with  $d > 2(\alpha \wedge 2)$  obeys

$$\xi_p^{(r)}(n) \equiv \left( \frac{1}{Z_p(0; n)} \sum_{x \in \mathbb{Z}^d} |x|^r \varphi_p(x, n) \right)^{1/r} \asymp \begin{cases} n^{\frac{1}{\alpha \wedge 2}} & (\alpha \neq 2), \\ (n \log n)^{1/2} & (\alpha = 2), \end{cases}$$

for every  $p \in (0, p_c]$ .

The rest of the paper is organized as follows. In Section 2, we summarize the relevant results from Part I. In Section 3, we prove Theorem 1.1 subject to a key proposition on fractional moments for the spatial variable of the lace-expansion coefficients. We prove that proposition in Section 4 using a certain integral representation for fractional powers of positive reals.

## 2 Summary of the relevant results from Part I

In this section, we summarize the results from Part I that will be used in the rest of the paper.

First, we introduce some notation. Let

$$q_p(x, n) = \mathbb{P}_p \left( ((o, 0), (x, n)) \text{ is occupied} \right) \equiv \begin{cases} pD(x) & (n = 1), \\ 0 & (n \neq 1). \end{cases}$$

We denote the space-time convolution of functions  $f$  and  $g$  on  $\mathbb{Z}^d \times \mathbb{Z}_+$  by

$$(f * g)(x, n) = \sum_{(y, t) \in \mathbb{Z}^d \times \mathbb{Z}_+} f(y, t) g(x - y, n - t),$$

and the Fourier-Laplace transform of  $f$  by

$$\hat{f}(k, z) = \sum_{(x, n) \in \mathbb{Z}^d \times \mathbb{Z}_+} f(x, n) e^{ik \cdot x} z^n \quad (k \in [-\pi, \pi]^d, z \in \mathbb{C}).$$

Notice that  $\frac{-1}{2} \Delta_k \hat{f}(l, z)$ , defined as

$$\begin{aligned} \frac{-1}{2} \Delta_k \hat{f}(l, z) &= \hat{f}(l, z) - \frac{\hat{f}(l + k, z) + \hat{f}(l - k, z)}{2} \\ &\equiv \sum_{(x, n) \in \mathbb{Z}^d \times \mathbb{Z}_+} (1 - \cos(k \cdot x)) f(x, n) e^{il \cdot x} z^n, \end{aligned} \quad (2.1)$$

is the Fourier-Laplace transform of  $(1 - \cos(k \cdot x))f(x, n)$ .

In [1, Section 3.1], we explained the derivation of the convolution equation

$$\varphi_p(x, n) = \pi_p(x, n) + (\pi_p * q_p * \varphi_p)(x, n),$$

where  $\pi_p(x, n)$  is the alternating sum of the  $\mathbb{Z}^d$ -symmetric nonnegative lace-expansion coefficients  $\pi_p^{(N)}(x, n)$  for  $N = 0, 1, 2, \dots$ :

$$\pi_p(x, n) = \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(x, n). \quad (2.2)$$

The precise definition of  $\pi_p^{(N)}$  is unimportant in this paper. However, we will use the following properties of  $\pi_p$  and  $\varphi_p$ :

**Proposition 2.1.** *Let  $\alpha > 0$ ,  $d > 2(\alpha \wedge 2)$  and  $L \gg 1$ . Then,*

$$pm_p \hat{\pi}_p(0, m_p) = 1, \quad (2.3)$$

$$\sum_{(x,n)} n |\pi_p(x, n)| m^n \leq O(\lambda), \quad (2.4)$$

$$\sum_{(x,n)} (1 - \cos(k \cdot x)) |\pi_p(x, n)| m^n \leq O(\lambda)(1 - \hat{D}(k)), \quad (2.5)$$

and

$$|\hat{\varphi}_p(k, me^{i\theta})| \leq \frac{O(1)}{pm_p(1 - \frac{m}{m_p}) + |\theta| + 1 - \hat{D}(k)}, \quad (2.6)$$

$$\begin{aligned} |\Delta_k \hat{\varphi}_p(l, me^{i\theta})| &\leq \sum_{(j,j')=(0,\pm 1), (1,-1)} \frac{1 - \hat{D}(k)}{pm_p(1 - \frac{m}{m_p}) + |\theta| + 1 - \hat{D}(l + jk)} \\ &\quad \times \frac{O(1)}{pm_p(1 - \frac{m}{m_p}) + |\theta| + 1 - \hat{D}(l + j'k)}, \end{aligned} \quad (2.7)$$

uniformly in  $p \in (0, p_c]$ ,  $m \in [0, m_p)$ ,  $k, l \in [-\pi, \pi]^d$  and  $\theta \in [-\pi, \pi]$ .

*Proof.* The identity (2.3) for every  $p \in (0, p_c]$  was proved in [1, (2.17) and (2.22)]. The bounds (2.4) and (2.6) for  $p \in (0, p_c)$  were also proved in Part I, and can be extended up to  $p = p_c$ , as long as  $m$  is strictly less than the radius of convergence,  $m_{p_c} = 1$  (cf., [1, Corollary 1.3]).

The same extension applies to the bounds (2.5) and (2.7), if they hold uniformly in  $p \in (0, p_c)$  and  $m \in [0, m_p)$ . In Part I, we showed that

$$\sum_{(x,n)} (1 - \cos(k \cdot x)) |\pi_p(x, n)| m^n \leq O(\lambda) \left( 1 - \frac{m}{m_p} + 1 - \hat{D}(k) \right),$$

uniformly in  $p \in (0, p_c)$ ,  $m \in [0, m_p)$  and  $k \in [-\pi, \pi]^d$ . However, since the left-hand side is increasing in  $m < m_p$ , we obtain

$$\begin{aligned} \sum_{(x,n)} (1 - \cos(k \cdot x)) |\pi_p(x, n)| m^n &\leq \lim_{m \uparrow m_p} \sum_{(x,n)} (1 - \cos(k \cdot x)) |\pi_p(x, n)| m^n \\ &\leq O(\lambda) \lim_{m \uparrow m_p} \left( 1 - \frac{m}{m_p} + 1 - \hat{D}(k) \right) \\ &\leq O(\lambda)(1 - \hat{D}(k)), \end{aligned}$$

as required. Using this stronger bound and following the steps in [1, Section 4.2], we also obtain (2.7). This completes the proof.  $\blacksquare$

Finally, we summarize the results for the  $n^{\text{th}}$  coefficient  $Z_p(k; n)$  of the series expansion of  $\hat{\varphi}_p(k, m)$  in powers of  $m$ :  $\hat{\varphi}_p(k, m) \equiv \sum_{n=0}^{\infty} Z_p(k; n) m^n$ . Let (cf., [1, (2.33)–(2.34)])

$$\hat{A}_p^{(1)}(k) = \hat{D}(k) + \frac{m_p \partial_m \hat{\pi}_p(k, m_p)}{pm_p \hat{\pi}_p(k, m_p)^2}, \quad (2.8)$$

$$\hat{B}_p(k) = 1 - \hat{D}(k) + \frac{\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)}{\hat{\pi}_p(k, m_p)}, \quad (2.9)$$

where  $\partial_m \hat{\pi}_p(k, m_p) = \lim_{m \uparrow m_p} \partial_m \hat{\pi}_p(k, m)$ . Notice that, by (2.4)–(2.5),

$$|m_p \partial_m \hat{\pi}_p(k, m_p)| \leq \lim_{m \uparrow m_p} \sum_{(x,n)} n |\pi_p(x, n)| m^n \leq O(\lambda), \quad (2.10)$$

$$|\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)| \leq \sum_{(x,n)} (1 - \cos(k \cdot x)) |\pi_p(x, n)| m_p^n \leq O(\lambda)(1 - \hat{D}(k)), \quad (2.11)$$

where the  $O(\lambda)$  terms are uniform in  $p \in (0, p_c]$  and  $k \in [-\pi, \pi]^d$ . Moreover, since  $\pi_p(x, 0)$  equals the Kronecker delta  $\delta_{x,0}$  (cf., [1, (3.2)]), we have  $\hat{\pi}_p(k, m_p) = 1 + O(\lambda)$  and thus  $\hat{A}_p^{(1)}(k) + \hat{B}_p(k) = 1 + O(\lambda)$  uniformly in  $p \in (0, p_c]$  and  $k \in [-\pi, \pi]^d$ .

In [1, Section 2.4], we showed that, for  $\alpha > 0$ ,  $d > 2(\alpha \wedge 2)$ ,  $L \gg 1$  and  $\epsilon \in (0, 1 \wedge \frac{d-2(\alpha \wedge 2)}{\alpha \wedge 2})$ ,

$$m_p^n Z_p(k; n) = \frac{1}{pm_p(\hat{A}_p^{(1)}(k) + \hat{B}_p(k))} \left( \frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)} \right)^n + O(n^{-\epsilon}),$$

hence

$$\frac{Z_p(k; n)}{Z_p(0; n)} = \frac{\hat{A}_p^{(1)}(0)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)} \left( \frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)} \right)^n + O(n^{-\epsilon}), \quad (2.12)$$

uniformly in  $p \in (0, p_c]$  and  $k \in [-\pi, \pi]^d$ . To prove Theorem 1.1, it thus suffices to investigate the first term in (2.12).

### 3 Proof of Theorem 1.1 subject to a key proposition

In this section, we first prove Theorem 1.1 assuming convergence of fractional moments for the spatial variable of  $\pi_p$ , as stated in the following proposition:

**Proposition 3.1.** *Let  $\alpha > 0$ ,  $d > 2(\alpha \wedge 2)$ ,  $L \gg 1$  and*

$$\delta \begin{cases} \in (0, \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))) & (\alpha \neq 2), \\ = 0 & (\alpha = 2). \end{cases} \quad (3.1)$$

*Then, for any  $p \in (0, p_c]$ ,*

$$\sum_{(x,n)} |x|^{\alpha \wedge 2 + \delta} |\pi_p(x, n)| m_p^n < \infty. \quad (3.2)$$

We will roughly explain why  $\delta$  is chosen as in (3.1), after the proof of Theorem 1.1 is completed. The proof of Proposition 3.1 is deferred to Section 4.

*Proof of Theorem 1.1 subject to Proposition 3.1.* As explained at the end of Section 2, it suffices to investigate the term

$$\left( \frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)} \right)^n \equiv \left( \left( 1 + \frac{\hat{B}_p(k)}{\hat{A}_p^{(1)}(k)} \right)^{-\frac{\hat{A}_p^{(1)}(k)}{\hat{B}_p(k)}} \right)^{\frac{n(1-\hat{D}(k))}{\hat{A}_p^{(1)}(k)} \frac{\hat{B}_p(k)}{1-\hat{D}(k)}}.$$

Notice that, by (2.8)–(2.11) and (1.3),

$$\left(1 + \frac{\hat{B}_p(k)}{\hat{A}_p^{(1)}(k)}\right)^{-\frac{\hat{A}_p^{(1)}(k)}{\hat{B}_p(k)}} \xrightarrow{|k| \rightarrow 0} e^{-1}, \quad \frac{n(1 - \hat{D}(k_n))}{\hat{A}_p^{(1)}(k_n)} \xrightarrow{n \rightarrow \infty} \frac{|k|^{\alpha \wedge 2}}{\hat{A}_p^{(1)}(0)},$$

where

$$\hat{A}_p^{(1)}(0) = 1 + pm_p \sum_{(x,n)} n \pi_p(x, n) m_p^n.$$

Moreover,

$$\begin{aligned} \frac{\hat{B}_p(k)}{1 - \hat{D}(k)} &= 1 + pm_p \frac{\hat{\pi}_p(0, m_p)}{\hat{\pi}_p(k, m_p)} \frac{\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)}{1 - \hat{D}(k)} \\ &\xrightarrow{|k| \rightarrow 0} 1 + pm_p \lim_{|k| \rightarrow 0} \frac{\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)}{1 - \hat{D}(k)}, \end{aligned}$$

if the limit exists. To complete the proof of Theorem 1.1, it remains to show

$$\lim_{|k| \rightarrow 0} \frac{\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)}{1 - \hat{D}(k)} = \begin{cases} \frac{1}{\sigma^2} \sum_{(x,n)} |x|^2 \pi_p(x, n) m_p^n & (\alpha > 2), \\ 0 & (\alpha \leq 2). \end{cases} \quad (3.3)$$

Now we choose  $\delta$  as in (3.1) and use Proposition 3.1 to prove (3.3) for (i)  $\alpha \leq 2$  and (ii)  $\alpha > 2$ , separately.

(i) Let  $\alpha \leq 2$  and  $\alpha + \delta \leq 2$ . Then, we have

$$0 \leq 1 - \cos(k \cdot x) \leq O(|k \cdot x|^{\alpha + \delta}).$$

By the spatial symmetry of the model and using (3.2) with  $\delta$  satisfying  $\alpha + \delta \leq 2$  and (3.1),

$$\begin{aligned} |\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)| &= \left| \sum_{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+} (1 - \cos(k \cdot x)) \pi_p(x, n) m_p^n \right| \\ &\leq O(|k|^{\alpha + \delta}) \sum_{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+} |x|^{\alpha + \delta} |\pi_p(x, n)| m_p^n \\ &= O(|k|^{\alpha + \delta}). \end{aligned}$$

By (1.1), we thus obtain that, for small  $|k|$ ,

$$\frac{|\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)|}{1 - \hat{D}(k)} \leq \begin{cases} O(|k|^\delta) & (\alpha < 2), \\ O(1/\log \frac{1}{|k|}) & (\alpha = 2). \end{cases}$$

This yields (3.3) for  $\alpha \leq 2$ .

(ii) Let  $\alpha > 2$  and  $\delta \leq 2$ . By the Taylor expansion,

$$1 - \cos(k \cdot x) = \frac{(k \cdot x)^2}{2} + O(|k \cdot x|^{2 + \delta}).$$

Then, by the spatial symmetry of the model and using (3.2) with  $\delta$  satisfying (3.1),

$$\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p) = \frac{|k|^2}{2d} \sum_{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+} |x|^2 \pi_p(x, n) m_p^n + O(|k|^{2+\delta}). \quad (3.4)$$

The limit (3.3) for  $\alpha > 2$  follows from (3.4) and the asymptotics (1.1) with  $v_\alpha = \frac{\sigma^2}{2d}$ .

This completes the proof of Theorem 1.1 subject to Proposition 3.1.  $\blacksquare$

Before closing this section, we roughly explain why  $\delta < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$  for  $\alpha \neq 2$  (the necessity of  $\delta = 0$  for  $\alpha = 2$  and  $\delta > 0$  for  $\alpha \neq 2$  is obvious from the above proof of Theorem 1.1). This is a sort of preview of Section 4.

In Section 4, we will use diagrammatic bounds on the expansion coefficients  $\pi_p^{(N)}$  in (2.2). In each bound (cf., (4.1)–(4.3) below), there are *two* sequences of two-point functions from  $(o, 0)$  to  $(x, n)$ . To bound  $\sum_{(x,n)} |x|^{\alpha \wedge 2 + \delta} \pi_p^{(N)}(x, n) m^n$ , we will split the power  $\alpha \wedge 2 + \delta$  into  $\delta_1$  and  $\delta_2$ , and multiply one of the aforementioned two sequences of two-point functions by  $|x|^{\delta_1}$  and the other by  $|x|^{\delta_2}$ . Here, we choose  $\delta_1$  and  $\delta_2$  both less than  $\alpha \wedge 2$ , so as to potentially control the weighted two-point functions, like  $|y|^{\delta_1} \varphi_p(y, s)$ .

Then,  $\sum_{(x,n)} |x|^{\alpha \wedge 2 + \delta} \pi_p^{(N)}(x, n) m^n$  will be bounded by the product of diagram functions (cf., Lemma 4.3 below). Those diagram functions are the “triangle”  $T_{p,m}$ , which is independent of  $\delta_1$  and  $\delta_2$ , its weighted version  $T'_{p,m}(\delta_1)$  and the weighted “bubbles”  $W'_{p,m}(\delta_2)$  and  $W''_{p,m}(\delta_1, \delta_2)$  (cf., (4.4)–(4.7) below). As shown in Section 4.3, it is not hard to bound  $W'_{p,m}(\delta_2)$  uniformly in  $p$  and  $m$  for  $d > 2(\alpha \wedge 2)$  and  $L \gg 1$  as long as  $\delta_2 < \alpha \wedge 2$ . However, to bound  $T'_{p,m}(\delta_1)$  and  $W''_{p,m}(\delta_1, \delta_2)$  uniformly in  $p$  and  $m$ , we will have to choose  $\delta_1$  to be small depending on how close  $d$  is to the upper-critical dimension  $2(\alpha \wedge 2)$ . As described in Lemma 4.5 below, we will choose  $\delta_1$  less than  $d - 2(\alpha \wedge 2)$ .

To summarize the above, we have

$$0 < \delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2)), \quad 0 < \delta_2 < \alpha \wedge 2, \quad \delta_1 + \delta_2 = \alpha \wedge 2 + \delta.$$

To satisfy all, it suffices to choose  $\delta_1$  “slightly” larger than  $\delta$  and let  $\delta_2 = \alpha \wedge 2 - (\delta_1 - \delta)$ . This is why we choose  $\delta < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$  when  $\alpha \neq 2$ .

## 4 Proof of Proposition 3.1

Finally, in this section, we prove Proposition 3.1. First, in Section 4.1, we bound fractional moments for the spatial variable of the expansion coefficients  $\pi_p^{(N)}$  in (2.2) in terms of certain diagram functions. In Section 4.2, we use an integral representation of  $a^\delta$  for  $a > 0$  and  $\delta \in (0, 2)$ , which is the key to the proof of Proposition 3.1. In Section 4.3, we show that the aforementioned diagram functions are convergent, and complete the proof of Proposition 3.1.

### 4.1 Diagrammatic bounds on the expansion coefficients

In this subsection, we bound  $\sum_{(x,n)} |x|^r |\pi_p(x, n)| m^n$  for  $r > 0$  in terms of the diagram functions  $T_{p,m}$ ,  $T'_{p,m}$ ,  $W'_{p,m}$  and  $W''_{p,m}$  defined in Lemma 4.3 below.

First, we show the following elementary inequality:

**Lemma 4.1.** For any  $r > 0$  and  $m \geq 0$ ,

$$\sum_{(x,n)} |x|^r |\pi_p(x, n)| m^n \leq d^{\frac{r}{2}+1} \sum_{N=0}^{\infty} \sum_{(x,n)} |x_1|^r \pi_p^{(N)}(x, n) m^n,$$

where  $x_1$  is the first coordinate of  $x \equiv (x_1, \dots, x_d)$ .

*Proof.* For any  $r > 0$ , we have

$$|x|^r = \left( \sum_{j=1}^d |x_j|^2 \right)^{r/2} \leq \left( \sum_{j=1}^d \|x\|_r^2 \right)^{r/2} = d^{r/2} \|x\|_r^r \equiv d^{r/2} \sum_{j=1}^d |x_j|^r.$$

By this inequality and using the nonnegativity and the spatial symmetry of  $\pi_p^{(N)}$ , we obtain

$$\begin{aligned} \sum_{(x,n)} |x|^r |\pi_p(x, n)| m^n &\leq d^{r/2} \sum_{j=1}^d \sum_{(x,n)} |x_j|^r \left| \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(x, n) \right| m^n \\ &\leq d^{r/2} \sum_{j=1}^d \sum_{N=0}^{\infty} \sum_{(x,n)} |x_j|^r \pi_p^{(N)}(x, n) m^n \\ &= d^{\frac{r}{2}+1} \sum_{N=0}^{\infty} \sum_{(x,n)} |x_1|^r \pi_p^{(N)}(x, n) m^n, \end{aligned}$$

as required. ■

Next, we use [9, Lemma 1] to investigate  $\sum_{(x,n)} |x_1|^r \pi_p^{(N)}(x, n) m^n$ . For notational convenience, we denote vertices in  $\mathbb{Z}^{d+1}$  by bold letters, e.g.,  $\mathbf{o} \equiv (o, 0)$  and  $\mathbf{x} = (x, t_x)$ , where  $t_x$  is the temporal part of  $\mathbf{x}$ . Let

$$\psi_p(\mathbf{x}) = (q_p * \varphi_p)(\mathbf{x}).$$

Given a sequence of vertices  $\mathbf{y}_1, \dots, \mathbf{y}_j \in \mathbb{Z}^{d+1}$ , we write

$$\vec{\mathbf{y}}_j = \sum_{i=1}^j \mathbf{y}_i.$$

For  $\mathbf{y}_1, \mathbf{z}_1, \mathbf{y}_2, \mathbf{z}_2, \dots \in \mathbb{Z}^{d+1}$ , we define

$$\begin{aligned} \Lambda_p(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) &= \psi_p(\mathbf{y}_i) \psi_p(\mathbf{z}_i) \frac{\varphi_p(\vec{\mathbf{y}}_i - \vec{\mathbf{z}}_i) + \varphi_p(\vec{\mathbf{z}}_i - \vec{\mathbf{y}}_i)}{2^{\delta_{\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i}}}, \\ \tilde{\Lambda}_p(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) &= \frac{\varphi_p(\vec{\mathbf{y}}_i - \vec{\mathbf{z}}_i) + \varphi_p(\vec{\mathbf{z}}_i - \vec{\mathbf{y}}_i)}{2^{\delta_{\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i}}} \psi_p(\mathbf{y}_{i+1}) \psi_p(\mathbf{z}_{i+1}). \end{aligned}$$

**Lemma 4.2** (Equivalent to Lemma 1 [9]). For  $N = 0$ ,

$$0 \leq \pi_p^{(0)}(\mathbf{x}) - \delta_{\mathbf{x}, \mathbf{o}} \leq \psi_p(\mathbf{x})^2. \quad (4.1)$$

For  $N \geq 1$ ,

$$\pi_p^{(N)}(\mathbf{x}) \leq \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N+1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1} = \vec{\mathbf{z}}_{N+1} = \mathbf{x}) \\ (t_{\mathbf{y}_1} \geq t_{\mathbf{z}_1})}} \varphi_p(\mathbf{y}_1) \varphi_p(\mathbf{z}_1) \prod_{i=1}^N \tilde{\Lambda}_p(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}), \quad (4.2)$$

and, for any  $j \in \{2, \dots, N+1\}$ ,

$$\begin{aligned} \pi_p^{(N)}(\mathbf{x}) \leq & \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N+1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1} = \vec{\mathbf{z}}_{N+1} = \mathbf{x})}} \varphi_p(\mathbf{y}_1) \varphi_p(\mathbf{z}_1) \varphi_p(\mathbf{y}_1 - \mathbf{z}_1) \left( \prod_{i=2}^{j-1} \Lambda_p(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) \right) \\ & \times \psi_p(\mathbf{y}_j) \psi_p(\mathbf{z}_j) \left( \prod_{i=j}^N \tilde{\Lambda}_p(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \right), \end{aligned} \quad (4.3)$$

where an empty product is regarded as 1.

For further notational convenience, we let

$$\begin{aligned} \varphi_p^{(m)}(\mathbf{x}) &= \varphi_p(\mathbf{x}) m^{t_{\mathbf{x}}}, \\ \psi_p^{(m)}(\mathbf{x}) &= \psi_p(\mathbf{x}) m^{t_{\mathbf{x}}}, \\ \Lambda_p^{(m)}(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) &= \Lambda_p(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) m^{t_{\mathbf{z}}_i}, \\ \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) &= \tilde{\Lambda}_p(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) m^{t_{\mathbf{z}}_{i+1}}. \end{aligned}$$

Given arbitrary  $\delta_1, \delta_2 > 0$ , we define  $T_{p,m}, T'_{p,m} \equiv T'_{p,m}(\delta_1), W'_{p,m} \equiv W'_{p,m}(\delta_2)$  and  $W''_{p,m} \equiv W''_{p,m}(\delta_1, \delta_2)$  as

$$T_{p,m} = \sup_{\mathbf{x}} \sum_{\mathbf{y}} \psi_p(\mathbf{y}) \left( (\psi_p^{(m)} * \varphi_p)(\mathbf{y} - \mathbf{x}) + \sum_{\mathbf{z}} \varphi_p(\mathbf{z} - \mathbf{y}) \psi_p^{(m)}(\mathbf{z} - \mathbf{x}) \right), \quad (4.4)$$

$$T'_{p,m} = \sup_{\mathbf{x}} \sum_{\mathbf{y}} |y_1|^{\delta_1} \psi_p(\mathbf{y}) \left( (\psi_p^{(m)} * \varphi_p)(\mathbf{y} - \mathbf{x}) + \sum_{\mathbf{z}} \varphi_p(\mathbf{z} - \mathbf{y}) \psi_p^{(m)}(\mathbf{z} - \mathbf{x}) \right), \quad (4.5)$$

$$W'_{p,m} = \sup_{\mathbf{x}} \sum_{\mathbf{y}} \psi_p(\mathbf{y}) |y_1 - x_1|^{\delta_2} \psi_p^{(m)}(\mathbf{y} - \mathbf{x}), \quad (4.6)$$

$$W''_{p,m} = \sup_{\mathbf{x}} \sum_{\mathbf{y}} |y_1|^{\delta_1} \psi_p(\mathbf{y}) |y_1 - x_1|^{\delta_2} \psi_p^{(m)}(\mathbf{y} - \mathbf{x}). \quad (4.7)$$

Using the above diagram functions and Lemma 4.2, we obtain the following:

**Lemma 4.3.** For any  $N \geq 0$  and  $m \geq 0$ ,

$$\begin{aligned} \sum_{(x,n)} |x_1|^{\delta_1 + \delta_2} \pi_p^{(N)}(x, n) m^n &\leq (N+1)^{\delta_1 + \delta_2} (T_{p,m})^{N-2} \left( \left( N(1 + T_{p,m}) + T_{p,m} \right) T_{p,m} W''_{p,m} \right. \\ &\quad \left. + N \left( (N-1)(1 + T_{p,m}) + 3T_{p,m} \right) T'_{p,m} W'_{p,m} \right). \end{aligned} \quad (4.8)$$

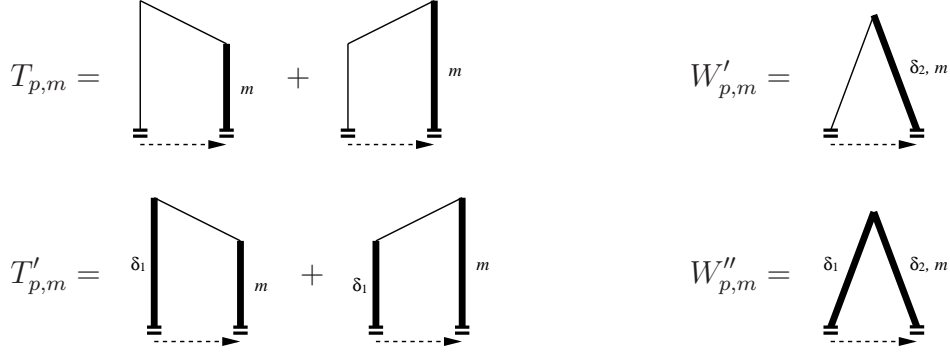


Figure 1: Schematic representations of the diagram functions. Each pair of horizontal short line segments represents  $q_p$ , and the other longer line segments represent  $\varphi_p$ . A bold line segment representing  $\varphi_p(x, n)$  is weighted by the factor  $m^n$  if the line segment is indexed by  $m$ , and by the factor  $|x_1|^\delta$  if the line segment is indexed by  $\delta$ . A dashed arrow represents the supremum over its terminal point  $\mathbf{x} \in \mathbb{Z}^{d+1}$ , with its initial point fixed at the origin  $\mathbf{o}$ .

*Proof.* First of all, by (4.1), we immediately obtain

$$\sum_{(x,n)} |x_1|^{\delta_1 + \delta_2} \pi_p^{(0)}(x, n) m^n \leq \sum_{(x,n)} |x_1|^{\delta_1} \psi_p(x, n) |x_1|^{\delta_2} \psi_p^{(m)}(x, n) \leq W''_{p,m},$$

as required.

Let  $N \geq 1$ . We denote the first coordinate of the spatial part of  $\mathbf{y}_i$  by  $y_{i,1}$ :  $\mathbf{y}_i = ((y_{i,1}, \dots, y_{i,d}), t_{\mathbf{y}_i})$ . Similarly, we write, e.g.,  $\vec{\mathbf{y}}_i = ((\vec{y}_{i,1}, \dots, \vec{y}_{i,d}), t_{\vec{\mathbf{y}}_i})$ . Notice that, since

$$|\vec{y}_{N+1,1}|^{\delta_1} = \left| \sum_{j=1}^{N+1} y_{j,1} \right|^{\delta_1} \leq (N+1)^{\delta_1} \max_j |y_{j,1}|^{\delta_1} \leq (N+1)^{\delta_1} \sum_{j=1}^{N+1} |y_{j,1}|^{\delta_1},$$

we have that, for  $\vec{\mathbf{y}}_{N+1} = \vec{\mathbf{z}}_{N+1} = \mathbf{x}$ ,

$$|x_1|^{\delta_1 + \delta_2} = |\vec{y}_{N+1,1}|^{\delta_1} |\vec{z}_{N+1,1}|^{\delta_2} \leq (N+1)^{\delta_1 + \delta_2} \sum_{j,j'=1}^{N+1} |y_{j,1}|^{\delta_1} |z_{j',1}|^{\delta_2}.$$

By this inequality and using (4.2)–(4.3), we obtain

$$\sum_{(x,n)} |x_1|^{\delta_1 + \delta_2} \pi_p^{(N)}(x, n) m^n \leq (N+1)^{\delta_1 + \delta_2} \sum_{j'=1}^{N+1} S_{j'}, \quad (4.9)$$

where

$$S_1 = \sum_{j=1}^{N+1} \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N+1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1} = \vec{\mathbf{z}}_{N+1}) \\ (t_{\mathbf{y}_1} \geq t_{\mathbf{z}_1})}} |y_{j,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^N \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}),$$

and, for  $j' > 1$ ,

$$S_{j'} = \sum_{j=1}^{N+1} \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N+1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1} = \vec{\mathbf{z}}_{N+1})}} |y_{j,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) \varphi_p^{(m)}(\mathbf{z}_1) \varphi_p(\mathbf{y}_1 - \mathbf{z}_1) \left( \prod_{i=2}^{j'-1} \Lambda_p^{(m)}(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) \right) \\ \times \psi_p(\mathbf{y}_{j'}) |z_{j',1}|^{\delta_2} \psi_p^{(m)}(\mathbf{z}_{j'}) \left( \prod_{i=j'}^N \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \right).$$

It remains to estimate each  $S_{j'}$ . To do so, we follow the same line of argument in [9, Section 2]. Here, we explain in detail how to estimate  $S_1$ . First we note that, by translation-invariance,

$$\sup_{\mathbf{y}} \sum_{\mathbf{w}, \mathbf{x}} \tilde{\Lambda}_p^{(m)}(\mathbf{o}, \mathbf{w}; \mathbf{x}, \mathbf{x} + \mathbf{y}) \leq T_{p,m}, \quad (4.10)$$

$$\sup_{\mathbf{y}} \sum_{\mathbf{w}, \mathbf{x}} |y_1|^{\delta_1} \tilde{\Lambda}_p^{(m)}(\mathbf{o}, \mathbf{w}; \mathbf{x}, \mathbf{x} + \mathbf{y}) \leq T'_{p,m}. \quad (4.11)$$

Then, by repeated use of translation-invariance, the contribution to  $S_1$  from  $j = 1$  is bounded as

$$\sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N+1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1} = \vec{\mathbf{z}}_{N+1}) \\ (t_{y_1} \geq t_{z_1})}} |y_{1,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^N \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \\ = \sum_{\mathbf{w}, \mathbf{x}} \tilde{\Lambda}_p^{(m)}(\mathbf{o}, \mathbf{w}; \mathbf{x}, \mathbf{x}) \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_N \\ \mathbf{z}_1, \dots, \mathbf{z}_N \\ (\vec{\mathbf{z}}_N = \vec{\mathbf{y}}_N + \mathbf{w}) \\ (t_{y_1} \geq t_{z_1})}} |y_{1,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^{N-1} \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \\ \leq T_{p,m} \sup_{\mathbf{w}} \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_N \\ \mathbf{z}_1, \dots, \mathbf{z}_N \\ (\vec{\mathbf{z}}_N = \vec{\mathbf{y}}_N + \mathbf{w}) \\ (t_{y_1} \geq t_{z_1})}} |y_{1,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^{N-1} \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \\ \leq (T_{p,m})^2 \sup_{\mathbf{w}} \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N-1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N-1} \\ (\vec{\mathbf{z}}_{N-1} = \vec{\mathbf{y}}_{N-1} + \mathbf{w}) \\ (t_{y_1} \geq t_{z_1})}} |y_{1,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^{N-2} \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \\ \vdots \\ \leq (T_{p,m})^N \sup_{\mathbf{w}: t_{\mathbf{w}} \geq 0} \sum_{\substack{\mathbf{y}_1, \mathbf{z}_1 \\ (\mathbf{y}_1 = \mathbf{z}_1 + \mathbf{w})}} |y_{1,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \\ \leq (T_{p,m})^N W''_{p,m}, \quad (4.12)$$

where we have used  $\varphi_p(\mathbf{x}) \leq \delta_{\mathbf{x}, \mathbf{o}} + \psi_p(\mathbf{x})$ . Similarly, the contribution to  $S_1$  from  $j > 1$  is bounded as

$$\begin{aligned}
& \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N+1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N+1} \\ (\tilde{\mathbf{y}}_{N+1} = \tilde{\mathbf{z}}_{N+1}) \\ (t_{\mathbf{y}_1} \geq t_{\mathbf{z}_1})}} |y_{j,1}|^{\delta_1} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^N \tilde{\Lambda}_p^{(m)}(\tilde{\mathbf{y}}_i, \tilde{\mathbf{z}}_i; \tilde{\mathbf{y}}_{i+1}, \tilde{\mathbf{z}}_{i+1}) \\
& \leq (T_{p,m})^{N-1} T'_{p,m} \sup_{\mathbf{w}: t_{\mathbf{w}} \geq 0} \sum_{\substack{\mathbf{y}_1, \mathbf{z}_1 \\ (\mathbf{y}_1 = \mathbf{z}_1 + \mathbf{w})}} \varphi_p(\mathbf{y}_1) |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \\
& \leq (T_{p,m})^{N-1} T'_{p,m} W'_{p,m}.
\end{aligned}$$

Therefore,

$$S_1 \leq N(T_{p,m})^{N-1} T'_{p,m} W'_{p,m} + (T_{p,m})^N W''_{p,m}. \quad (4.13)$$

To estimate  $S_{j'}$  for  $j' > 1$ , we first use (4.10)–(4.11). For example, the contribution from  $j = j'$  is bounded, similarly to (4.12), as

$$\begin{aligned}
& \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{N+1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{N+1} \\ (\tilde{\mathbf{y}}_{N+1} = \tilde{\mathbf{z}}_{N+1})}} \varphi_p(\mathbf{y}_1) \varphi_p^{(m)}(\mathbf{z}_1) \varphi_p(\mathbf{y}_1 - \mathbf{z}_1) \left( \prod_{i=2}^{j'-1} \Lambda_p^{(m)}(\tilde{\mathbf{y}}_{i-1}, \tilde{\mathbf{z}}_{i-1}; \tilde{\mathbf{y}}_i, \tilde{\mathbf{z}}_i) \right) \\
& \quad \times |y_{j',1}|^{\delta_1} \psi_p(\mathbf{y}_{j'}) |z_{j',1}|^{\delta_2} \psi_p^{(m)}(\mathbf{z}_{j'}) \left( \prod_{i=j'}^N \tilde{\Lambda}_p^{(m)}(\tilde{\mathbf{y}}_i, \tilde{\mathbf{z}}_i; \tilde{\mathbf{y}}_{i+1}, \tilde{\mathbf{z}}_{i+1}) \right) \\
& \leq (T_{p,m})^{N+1-j'} \sup_{\mathbf{x}} \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{j'} \\ \mathbf{z}_1, \dots, \mathbf{z}_{j'} \\ (\tilde{\mathbf{z}}_{j'} = \tilde{\mathbf{y}}_{j'} + \mathbf{x})}} \varphi_p(\mathbf{y}_1) \varphi_p^{(m)}(\mathbf{z}_1) \varphi_p(\mathbf{y}_1 - \mathbf{z}_1) \\
& \quad \times \left( \prod_{i=2}^{j'-1} \Lambda_p^{(m)}(\tilde{\mathbf{y}}_{i-1}, \tilde{\mathbf{z}}_{i-1}; \tilde{\mathbf{y}}_i, \tilde{\mathbf{z}}_i) \right) |y_{j',1}|^{\delta_1} \psi_p(\mathbf{y}_{j'}) |z_{j',1}|^{\delta_2} \psi_p^{(m)}(\mathbf{z}_{j'}). \\
& \leq (T_{p,m})^{N+1-j'} W''_{p,m} \sup_{\mathbf{x}} \sum_{\substack{\mathbf{y}_1, \dots, \mathbf{y}_{j'-1} \\ \mathbf{z}_1, \dots, \mathbf{z}_{j'-1} \\ (\tilde{\mathbf{z}}_{j'-1} = \tilde{\mathbf{y}}_{j'-1} + \mathbf{x})}} \varphi_p(\mathbf{y}_1) \varphi_p^{(m)}(\mathbf{z}_1) \varphi_p(\mathbf{y}_1 - \mathbf{z}_1) \\
& \quad \times \left( \prod_{i=2}^{j'-1} \Lambda_p^{(m)}(\tilde{\mathbf{y}}_{i-1}, \tilde{\mathbf{z}}_{i-1}; \tilde{\mathbf{y}}_i, \tilde{\mathbf{z}}_i) \right). \quad (4.14)
\end{aligned}$$

Notice that

$$\sup_{\mathbf{x}} \sum_{\mathbf{y}, \mathbf{z}} \Lambda_p^{(m)}(\mathbf{o}, \mathbf{x}; \mathbf{y}, \mathbf{z}) \leq T_{p,m}, \quad \sup_{\mathbf{x}} \sum_{\mathbf{y}, \mathbf{z}} |y_1|^{\delta_1} \Lambda_p^{(m)}(\mathbf{o}, \mathbf{x}; \mathbf{y}, \mathbf{z}) \leq T'_{p,m}.$$

By repeated use of translation-invariance, we obtain

$$(4.14) \leq (1 + T_{p,m})(T_{p,m})^{N-1} W''_{p,m}.$$

It is not hard to see that the contribution from  $j$  not being either  $j'$  or 1, which is possible only if  $N \geq 2$ , is bounded by  $(1 + T_{p,m})(T_{p,m})^{N-2}T'_{p,m}W'_{p,m}$ , and the contribution from  $j = 1$  is bounded by  $2(T_{p,m})^{N-1}T'_{p,m}W'_{p,m}$ . Therefore, for  $j' > 1$ ,

$$S_{j'} \leq ((N-1)(1 + T_{p,m}) + 2T_{p,m})(T_{p,m})^{N-2}T'_{p,m}W'_{p,m} + (1 + T_{p,m})(T_{p,m})^{N-1}W''_{p,m}. \quad (4.15)$$

The proof of (4.8) is completed by assembling (4.9), (4.13) and (4.15).  $\blacksquare$

## 4.2 Integral representation of fractional-power functions

In this subsection, we use an integral representation of  $a^\delta$  for  $a > 0$  and  $\delta \in (0, 2)$  to bound the diagram functions  $T'_{p,m}$ ,  $W'_{p,m}$  and  $W''_{p,m}$ .

First we note that, for  $\delta \in (0, 2)$ ,

$$K_\delta = \int_0^\infty \frac{1 - \cos t}{t^{1+\delta}} dt$$

is a positive finite constant. Replacing  $t$  by  $u = t/a$  with  $a > 0$ , we obtain

$$a^\delta = \frac{1}{K_\delta} \int_0^\infty \frac{1 - \cos(ua)}{u^{1+\delta}} du \leq \frac{1}{K_\delta} \left( \frac{2}{\delta} + \int_0^1 \frac{1 - \cos(ua)}{u^{1+\delta}} du \right), \quad (4.16)$$

which is the key inequality.

To describe bounds on  $T'_{p,m}$ ,  $W'_{p,m}$  and  $W''_{p,m}$  below, we define

$$\hat{Y}_k(l, z) = |\Delta_k \hat{D}(l)| |\hat{\varphi}_p(l, z)| + |\Delta_k \hat{\varphi}_p(l, z)|,$$

and, by denoting  $\vec{u} = (u, 0, \dots, 0) \in [-\pi, \pi]^d$ ,

$$\hat{I}_1(u) = \int_{[-\pi, \pi]^d} \frac{d^d l}{(2\pi)^d} \int_{-\pi}^\pi \frac{d\theta}{2\pi} \hat{Y}_{\vec{u}}(l, e^{i\theta}) |\hat{\varphi}_p(l, e^{i\theta})| |\hat{\varphi}_p(l, me^{i\theta})|, \quad (4.17)$$

$$\hat{I}_2(v) = \int_{[-\pi, \pi]^d} \frac{d^d l}{(2\pi)^d} \int_{-\pi}^\pi \frac{d\theta}{2\pi} |\hat{\varphi}_p(l, e^{i\theta})| \hat{Y}_{\vec{v}}(l, me^{i\theta}), \quad (4.18)$$

$$\hat{I}_3(u) = \int_{[-\pi, \pi]^d} \frac{d^d l}{(2\pi)^d} \int_{-\pi}^\pi \frac{d\theta}{2\pi} \hat{Y}_{\vec{u}}(l, e^{i\theta}) |\hat{\varphi}_p(l, me^{i\theta})|, \quad (4.19)$$

$$\hat{I}_4(u, v) = \int_{[-\pi, \pi]^d} \frac{d^d l}{(2\pi)^d} \int_{-\pi}^\pi \frac{d\theta}{2\pi} \hat{Y}_{\vec{u}}(l, e^{i\theta}) \hat{Y}_{\vec{v}}(l, me^{i\theta}). \quad (4.20)$$

Taking the Fourier-Laplace transform of (4.5)–(4.7) (also recalling (2.1)) and using (4.16), we obtain the following:

**Lemma 4.4.** *For any  $p \in (0, p_c)$  and  $m \in [0, m_p)$ ,*

$$T'_{p,m} \leq \frac{1}{K_{\delta_1}} \left( \frac{2}{\delta_1} T_{p,m} + 5p^2 m \int_0^1 \frac{du}{u^{1+\delta_1}} \hat{I}_1(u) \right), \quad (4.21)$$

$$W'_{p,m} \leq \frac{1}{K_{\delta_2}} \left( \frac{1}{\delta_2} T_{p,m} + \frac{5p^2 m}{2} \int_0^1 \frac{dv}{v^{1+\delta_2}} \hat{I}_2(v) \right), \quad (4.22)$$

$$W''_{p,m} \leq \frac{1}{K_{\delta_1}} \left( \frac{2}{\delta_1} W'_{p,m} + \frac{5p^2 m}{K_{\delta_2} \delta_2} \int_0^1 \frac{du}{u^{1+\delta_1}} \hat{I}_3(u) + \frac{25p^2 m}{4K_{\delta_2}} \int_0^1 \frac{du}{u^{1+\delta_1}} \int_0^1 \frac{dv}{v^{1+\delta_2}} \hat{I}_4(u, v) \right). \quad (4.23)$$

*Proof.* We only prove (4.22), since the other two inequalities can be proved in the same way.

First we use (4.16) to bound  $|y_1 - x_1|^{\delta_2}$  in (4.6). The first term in (4.22) is due to the first term in (4.16) and the trivial inequality

$$\sum_{\mathbf{y}} \psi_p(\mathbf{y}) \psi_p^{(m)}(\mathbf{y} - \mathbf{x}) \leq \frac{1}{2} T_{p,m}.$$

To complete the proof of (4.22), it thus remains to show

$$\sum_{\mathbf{y}} \psi_p(\mathbf{y} + \mathbf{x}) (1 - \cos(vy_1)) \psi_p^{(m)}(\mathbf{y}) \leq \frac{5p^2 m}{2} \hat{I}_2(v). \quad (4.24)$$

However, since  $1 - \cos \sum_{j=1}^J t_j \leq (2J+1) \sum_{j=1}^J (1 - \cos t_j)$  (cf., [11, (4.50)]), we have

$$\begin{aligned} (1 - \cos(vy_1)) \psi_p(\mathbf{y}) &\equiv (1 - \cos(vy_1)) (q_p * \varphi_p)(\mathbf{y}) \\ &\leq 5p \sum_{\mathbf{w}} (1 - \cos(vw_1)) \left( D(w) \varphi_p(\mathbf{y} - \mathbf{w}) + \varphi_p(\mathbf{w}) D(\mathbf{y} - \mathbf{w}) \right). \end{aligned}$$

Applying this to the left-hand side of (4.24), then taking the Fourier-Laplace transform and using  $|\hat{\varphi}_p(l, e^{i\theta})| = |\hat{\varphi}_p(l, e^{-i\theta})|$ , we obtain (4.24). This completes the proof of (4.22).  $\blacksquare$

### 4.3 Bounds on the diagram functions

In this subsection, we complete the proof of Proposition 3.1 using the following lemma:

**Lemma 4.5.** *Let  $\alpha > 0$  and  $d > 2(\alpha \wedge 2)$ , and choose  $\delta$  as in (3.1) and  $\delta_1, \delta_2 \in (0, 2)$  as*

$$\delta < \delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2)), \quad \delta_2 = \alpha \wedge 2 + \delta - \delta_1.$$

*Then,*

$$T_{p,m} = O(\lambda), \quad \left. \begin{array}{l} T'_{p,m} \\ W'_{p,m} \\ W''_{p,m} \end{array} \right\} = O(1), \quad (4.25)$$

*uniformly in  $p \in (0, p_c)$  and  $m \in [0, m_p]$ .*

*Proof of Proposition 3.1.* First, by Lemmas 4.1 and 4.3 with  $r = \delta_1 + \delta_2 \equiv \alpha \wedge 2 + \delta < 4$ , we obtain that, for any  $p \in (0, p_c]$ ,

$$\begin{aligned} &\sum_{(x,n)} |x|^{\alpha \wedge 2 + \delta} |\pi_p(x, n)| m_p^n \\ &\leq d^3 \sum_{N=0}^{\infty} (N+1)^{\delta_1 + \delta_2} (T_{p,m_p})^{N-2} \left( \left( N(1 + T_{p,m_p}) + T_{p,m_p} \right) T_{p,m_p} W''_{p,m_p} \right. \\ &\quad \left. + N \left( (N-1)(1 + T_{p,m_p}) + 3T_{p,m_p} \right) T'_{p,m_p} W'_{p,m_p} \right). \end{aligned} \quad (4.26)$$

Since the diagram functions (4.4)–(4.7) are increasing in  $m \geq 0$  for every  $p \geq 0$  and in  $p \geq 0$  for every  $m \geq 0$ , the uniform bounds in (4.25) imply that these diagram functions at  $m = m_p$  obey the same bounds uniformly in  $p \in (0, p_c]$ . Therefore, the right-hand side of (4.26) is convergent, if  $\lambda$  is sufficiently small. This completes the proof of Proposition 3.1.  $\blacksquare$

*Proof of Lemma 4.5.* It is not hard to extend [1, Lemma 4.1] to show that  $T_{p,m} = O(\lambda)$  uniformly in  $p \in (0, p_c)$  and  $m \in [0, m_p)$ . Recall Lemma 4.4. To complete the proof of Lemma 4.5, it thus suffices to show that the integrals in (4.21)–(4.23) of  $\hat{I}_1, \dots, \hat{I}_4$  are bounded uniformly in  $p \in (0, p_c)$  and  $m \in [0, m_p)$ .

The integrals of  $\hat{I}_2$  and  $\hat{I}_3$  are easy and can be estimated similarly. For example, by (2.6)–(2.7) and  $|\Delta_{\vec{v}}\hat{D}(l)| \leq 2(1 - \hat{D}(\vec{v}))$  (cf., (2.1)),

$$\begin{aligned} \hat{I}_2(v) &= \int \frac{d^d l}{(2\pi)^d} \int \frac{d\theta}{2\pi} |\hat{\varphi}_p(l, e^{i\theta})| \left( |\Delta_{\vec{v}}\hat{D}(l)| |\hat{\varphi}_p(l, me^{i\theta})| + |\Delta_{\vec{v}}\hat{\varphi}_p(l, me^{i\theta})| \right) \\ &\leq O(1 - \hat{D}(\vec{v})) \int \frac{d^d l}{(2\pi)^d} \int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \left( \frac{1}{|\theta| + 1 - \hat{D}(l)} \right. \\ &\quad \left. + \sum_{(j,j')=(0,\pm 1), (1,-1)} \frac{1}{(|\theta| + 1 - \hat{D}(l + j\vec{v}))(|\theta| + 1 - \hat{D}(l + j'\vec{v}))} \right) \end{aligned}$$

holds uniformly in  $p \in (0, p_c)$  and  $m \in [0, m_p)$ . Using the Hölder inequality twice and the translation-invariance of  $D$ , we have

$$\begin{aligned} &\int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \frac{1}{(|\theta| + 1 - \hat{D}(l + j\vec{v}))(|\theta| + 1 - \hat{D}(l + j'\vec{v}))} \\ &\leq \left( \int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \left( \frac{1}{|\theta| + 1 - \hat{D}(l + j\vec{v})} \right)^2 \right)^{1/2} \\ &\quad \times \left( \int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \left( \frac{1}{|\theta| + 1 - \hat{D}(l + j'\vec{v})} \right)^2 \right)^{1/2} \\ &\leq \left( \int \frac{d\theta}{2\pi} \left( \frac{1}{|\theta| + 1 - \hat{D}(l)} \right)^3 \right)^{1/6} \left( \int \frac{d\theta}{2\pi} \left( \frac{1}{|\theta| + 1 - \hat{D}(l + j\vec{v})} \right)^3 \right)^{1/3} \\ &\quad \times \left( \int \frac{d\theta}{2\pi} \left( \frac{1}{|\theta| + 1 - \hat{D}(l)} \right)^3 \right)^{1/6} \left( \int \frac{d\theta}{2\pi} \left( \frac{1}{|\theta| + 1 - \hat{D}(l + j'\vec{v})} \right)^3 \right)^{1/3} \\ &= \int \frac{d\theta}{2\pi} \left( \frac{1}{|\theta| + 1 - \hat{D}(l)} \right)^3. \end{aligned}$$

Since, by (1.1),

$$\int \frac{d^d l}{(2\pi)^d} \int \frac{d\theta}{2\pi} \left( \frac{1}{|\theta| + 1 - \hat{D}(l)} \right)^3 \leq \int \frac{d^d l}{(2\pi)^d} \frac{O(1)}{(1 - \hat{D}(l))^2} < \infty$$

holds for  $d > 2(\alpha \wedge 2)$ , we conclude that, for  $\delta_2 = \alpha \wedge 2 - (\delta_1 - \delta) < \alpha \wedge 2$ ,

$$\int_0^1 \frac{dv}{v^{1+\delta_2}} \hat{I}_2(v) \leq \int_0^1 \frac{dv}{v^{1+\delta_2}} O(1 - \hat{D}(\vec{v})) < \infty,$$

as required.

Next, we consider the integral of  $\hat{I}_1$ . In fact, we only need consider the contribution from  $|\Delta_{\vec{u}} \hat{\varphi}_p(l, e^{i\theta})|$  in  $\hat{Y}_{\vec{u}}(l, e^{i\theta})$  of (4.17), because the contribution from the other term in  $\hat{Y}_{\vec{u}}(l, e^{i\theta})$  can be estimated similarly to the integral of  $\hat{I}_2$ , as explained above. Using (2.6)–(2.7) and ignoring some factors of  $|\theta|$ , we obtain

$$\begin{aligned} & \int \frac{d^d l}{(2\pi)^d} \int \frac{d\theta}{2\pi} |\Delta_{\vec{u}} \hat{\varphi}_p(l, e^{i\theta})| |\hat{\varphi}_p(l, e^{i\theta})| |\hat{\varphi}_p(l, me^{i\theta})| \\ & \leq \sum_{(j, j')} \int \frac{d^d l}{(2\pi)^d} \frac{O(1 - \hat{D}(\vec{u}))}{(1 - \hat{D}(l + j\vec{u}))(1 - \hat{D}(l + j'\vec{u}))} \int \frac{d\theta}{2\pi} \left( \frac{1}{|\theta| + 1 - \hat{D}(l)} \right)^2 \\ & \leq O(1 - \hat{D}(\vec{u})) \sum_{(j, j')} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(1 - \hat{D}(l + j\vec{u}))(1 - \hat{D}(l + j'\vec{u}))(1 - \hat{D}(l))}, \end{aligned} \quad (4.27)$$

where  $\sum_{(j, j')}$  is the sum over  $(j, j') = (0, \pm 1), (1, -1)$ . By the translation-invariance and  $\mathbb{Z}^d$ -symmetry of  $D$ , the integral for  $(j, j') = (0, \pm 1)$  equals

$$\hat{J}(u) = \int_{[-\pi, \pi]^d} \frac{d^d l}{(2\pi)^d} \frac{1}{(1 - \hat{D}(l))^2 (1 - \hat{D}(l - \vec{u}))}. \quad (4.28)$$

Moreover, by the Schwarz inequality, the integral for  $(j, j') = (1, -1)$  is bounded by

$$\begin{aligned} & \left( \int \frac{d^d l}{(2\pi)^d} \frac{1}{(1 - \hat{D}(l + \vec{u}))^2 (1 - \hat{D}(l))} \right)^{1/2} \\ & \times \left( \int \frac{d^d l}{(2\pi)^d} \frac{1}{(1 - \hat{D}(l - \vec{u}))^2 (1 - \hat{D}(l))} \right)^{1/2} = \hat{J}(u). \end{aligned}$$

Therefore,

$$(4.27) \leq O(1 - \hat{D}(\vec{u})) \hat{J}(u). \quad (4.29)$$

Now we show

$$\hat{J}(u) \leq O(u^{(d-3(\alpha \wedge 2)) \wedge 0}), \quad (4.30)$$

which is sufficient for the integral of  $\hat{I}_1$  to be convergent for  $\delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$ .  
Let

$$R_1 = \{l \in [-\pi, \pi]^d : |l| \geq \frac{3}{2}u\}, \quad (4.31)$$

$$R_2 = \{l \in [-\pi, \pi]^d : |l| \leq \frac{3}{2}u, |l| \leq |l - \vec{u}|\}, \quad (4.32)$$

$$R_3 = \{l \in [-\pi, \pi]^d : |l| \leq \frac{3}{2}u, |l - \vec{u}| \leq |l|\}. \quad (4.33)$$

Notice that  $1 - \hat{D}(l - \vec{u}) \geq O(|l - \vec{u}|^{\alpha \wedge 2})$  for any  $l \in [-\pi, \pi]^d$  and  $u \in [0, 1]$  (cf., [1, Proposition 1.1]). Since  $|l - \vec{u}| \geq |l| - u \geq \frac{1}{3}|l|$  for  $l \in R_1$ , we have

$$\int_{R_1} \frac{d^d l}{(2\pi)^d} \frac{1}{(1 - \hat{D}(l))^2 (1 - \hat{D}(l - \vec{u}))} \leq O(1) \int_{R_1} \frac{d^d l}{|l|^{3(\alpha \wedge 2)}} \leq O(u^{(d-3(\alpha \wedge 2)) \wedge 0}).$$

Moreover, since  $|l - \vec{u}| \geq \frac{u}{2}$  for  $l \in R_2$  and  $|l| \geq \frac{u}{2}$  for  $l \in R_3$ , we have

$$\int_{R_2} \frac{d^d l}{(2\pi)^d} \frac{1}{(1 - \hat{D}(l))^2 (1 - \hat{D}(l - \vec{u}))} \leq O(u^{-\alpha \wedge 2}) \int_{|l| \leq \frac{3}{2}u} \frac{d^d l}{|l|^{2(\alpha \wedge 2)}} \leq O(u^{d-3(\alpha \wedge 2)}),$$

and

$$\int_{R_3} \frac{d^d l}{(2\pi)^d} \frac{1}{(1 - \hat{D}(l))^2 (1 - \hat{D}(l - \vec{u}))} \leq O(u^{-2(\alpha \wedge 2)}) \int_{|l| \leq \frac{3}{2}u} \frac{d^d l}{|l|^{\alpha \wedge 2}} \leq O(u^{d-3(\alpha \wedge 2)}).$$

This completes the proof of (4.30), as required.

Finally, we discuss the integral of  $\hat{I}_4$ . We only need consider the contribution from  $|\Delta_{\vec{u}} \hat{\varphi}_p(l, e^{i\theta})| |\Delta_{\vec{v}} \hat{\varphi}_p(l, me^{i\theta})|$  in  $\hat{Y}_{\vec{u}}(l, e^{i\theta}) \hat{Y}_{\vec{v}}(l, me^{i\theta})$  of (4.20), since the contributions from the other combinations are bounded similarly to the integrals of  $\hat{I}_1, \hat{I}_2, \hat{I}_3$  as long as  $d > 2(\alpha \wedge 2)$ ,  $\delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$  and  $\delta_2 < \alpha \wedge 2$ . Using (2.7) and ignoring some factors of  $|\theta|$ , we have

$$\begin{aligned} & \int \frac{d^d l}{(2\pi)^d} \int \frac{d\theta}{2\pi} |\Delta_{\vec{u}} \hat{\varphi}_p(l, e^{i\theta})| |\Delta_{\vec{v}} \hat{\varphi}_p(l, me^{i\theta})| \\ & \leq \sum_{(j_1, j'_1), (j_2, j'_2)} \int \frac{d^d l}{(2\pi)^d} \frac{O(1 - \hat{D}(\vec{u}))}{(1 - \hat{D}(l + j_1 \vec{u}))(1 - \hat{D}(l + j'_1 \vec{u}))} \\ & \quad \times \int \frac{d\theta}{2\pi} \frac{1 - \hat{D}(\vec{v})}{(|\theta| + 1 - \hat{D}(l + j_2 \vec{v}))(|\theta| + 1 - \hat{D}(l + j'_2 \vec{v}))}. \end{aligned} \quad (4.34)$$

Notice that

$$\begin{aligned} & \sum_{(j_2, j'_2)} \int \frac{d\theta}{2\pi} \frac{1}{(|\theta| + 1 - \hat{D}(l + j_2 \vec{v}))(|\theta| + 1 - \hat{D}(l + j'_2 \vec{v}))} \\ & \leq \sum_{(j_2, j'_2)} \frac{1}{1 - \hat{D}(l + j_2 \vec{v}) \vee \hat{D}(l + j'_2 \vec{v})} \leq \sum_{j=0, \pm 1} \frac{2}{1 - \hat{D}(l + j \vec{v})}. \end{aligned}$$

The contribution from  $j = 0$  is bounded, similarly to (4.29), by  $O(1 - \hat{D}(\vec{u})) \hat{J}(u) (1 - \hat{D}(\vec{v}))$ , where  $(1 - \hat{D}(\vec{u})) \hat{J}(u) / u^{1+\delta_1}$  is integrable if  $\delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$  and  $(1 - \hat{D}(\vec{v})) / v^{1+\delta_2}$  is integrable if  $\delta_2 < \alpha \wedge 2$  (see around (4.30)). On the other hand, the contribution from  $j = \pm 1$  is bounded, due to the Schwarz inequality and the  $\mathbb{Z}^d$ -symmetry and translation-

invariance of  $D$ , by

$$\begin{aligned}
& \sum_{(j_1, j'_1)} \int \frac{d^d l}{(2\pi)^d} \frac{O(1 - \hat{D}(\vec{u}))}{(1 - \hat{D}(l + j_1 \vec{u}))(1 - \hat{D}(l + j'_1 \vec{u}))} \frac{1 - \hat{D}(\vec{v})}{1 - \hat{D}(l + j \vec{v})} \\
& \leq \sum_{(j_1, j'_1)} \left( \int \frac{d^d l}{(2\pi)^d} \frac{O(1 - \hat{D}(\vec{u}))^2}{(1 - \hat{D}(l + j_1 \vec{u}))^2 (1 - \hat{D}(l + j'_1 \vec{u}))} \right)^{1/2} \\
& \quad \times \left( \int \frac{d^d l}{(2\pi)^d} \frac{(1 - \hat{D}(\vec{v}))^2}{(1 - \hat{D}(l + j'_1 \vec{u}))(1 - \hat{D}(l + j \vec{v}))} \right)^{1/2} \\
& \leq O(1 - \hat{D}(\vec{u}))(1 - \hat{D}(\vec{v})) \sum_{(j_1, j'_1)} \hat{J}((1 - j_1 j'_1)u)^{1/2} \hat{J}(|v - j j'_1 u|)^{1/2} \\
& = O(1 - \hat{D}(\vec{u}))(1 - \hat{D}(\vec{v})) \left( (\hat{J}(u)^{1/2} + \hat{J}(2u)^{1/2}) \hat{J}(|v + ju|)^{1/2} \right. \\
& \quad \left. + \hat{J}(u)^{1/2} \hat{J}(|v - ju|)^{1/2} \right). \tag{4.35}
\end{aligned}$$

It is not hard to show that  $\hat{J}(2u)$  and  $\hat{J}(v+u)$  obey the same bound as  $\hat{J}(u)$  for  $u, v \in [0, 1]$ . Therefore, the contribution to (4.35) from  $\hat{J}(v+u)$  is bounded by  $O(1 - \hat{D}(\vec{u}))(1 - \hat{D}(\vec{v}))\hat{J}(u)$ , which divided by  $u^{1+\delta_1}v^{1+\delta_2}$  is integrable if  $d > 2(\alpha \wedge 2)$ ,  $\delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$  and  $\delta_2 < \alpha \wedge 2$ , as explained above. Moreover, since

$$\begin{aligned}
\int_{\frac{u}{2}}^{\frac{3u}{2}} \frac{dv}{v^{1+\delta_2}} (1 - \hat{D}(\vec{v})) \hat{J}(|v - u|)^{1/2} & \leq \underbrace{\int_0^{\frac{u}{2}} dr \hat{J}(r)^{1/2}}_{O(u^{\frac{d-3(\alpha \wedge 2)}{2} \wedge 0+1})} \times \begin{cases} O(u^{(\alpha \wedge 2 - \delta_2 - 1) \wedge 0}) & (\alpha \neq 2) \\ O(u^{(1 - \delta_2) \wedge 0} \log \frac{1}{u}) & (\alpha = 2) \end{cases} \\
& \leq O\left(u^{\frac{d-3(\alpha \wedge 2)}{2} \wedge 0}\right),
\end{aligned}$$

and

$$\begin{aligned}
\int_{[0,1] \setminus [\frac{u}{2}, \frac{3u}{2}]} \frac{dv}{v^{1+\delta_2}} (1 - \hat{D}(\vec{v})) \hat{J}(|v - u|)^{1/2} & \leq O\left(u^{\frac{d-3(\alpha \wedge 2)}{2} \wedge 0}\right) \int_0^1 \frac{dv}{v^{1+\delta_2}} (1 - \hat{D}(\vec{v})) \\
& \leq O\left(u^{\frac{d-3(\alpha \wedge 2)}{2} \wedge 0}\right),
\end{aligned}$$

we have

$$\int_0^1 \frac{dv}{v^{1+\delta_2}} (1 - \hat{D}(\vec{v})) \hat{J}(|v - u|)^{1/2} \leq O\left(u^{\frac{d-3(\alpha \wedge 2)}{2} \wedge 0}\right), \tag{4.36}$$

i.e., the left-hand side of (4.36) obeys the same bound as  $\hat{J}(u)^{1/2}$ . Therefore, the contribution to (4.35) from  $\hat{J}(|v - u|)$ , divided by  $u^{1+\delta_1}v^{1+\delta_2}$ , is also integrable if  $d > 2(\alpha \wedge 2)$ ,  $\delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$  and  $\delta_2 < \alpha \wedge 2$ , as required. This completes the proof of Lemma 4.5.  $\blacksquare$

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