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Author(s)	Chandra, Anjan Kumar; Inoue, Jun-ichi
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Similarity in transitions of two distinct number networks

Anjan Kumar Chandra^{1*} and Jun-ichi Inoue^{2†}

¹*Theoretical Condensed Matter Physics Division,
Saha Institute of Nuclear Physics,
1/AF Bidhannagar, Kolkata-700064, India.*

²*Complex Systems Engineering,
Graduate School of Information Science and Technology,
Hokkaido University, N14-W9,
Kita-ku, Sapporo 060-0814, Japan*

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The main objective of our study is to disclose if any network structure is hidden among the pairs of numbers obtained by splitting the consecutive integers. By adopting some preferred rule in placing the links among the pairs obtained, we have a transition from a regular to a random one through a small world one. This type of transition is also similar to the network of primes obtained by splitting the consecutive integers which indicates some general feature in both kinds of networks. We also obtained both some numerical and analytical explanations, that support the transition and also the point of transition.

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I. INTRODUCTION

The study of networks has been quite fruitful in disclosing the statistical properties of a diverse types of complex systems for the last few decades [1–4]. Whether it is a World Wide Web [5] or social networks [6] or complex biological systems [7], its application has become an inevitable tool for the physicists. A network is defined as a graph consisting of some ‘nodes’ connected by some ‘links’ according to some relevant property depending upon the nature of the system. For example, in a traffic network, the ‘stations’ are the nodes and the particular transport that connect two stations is the ‘link’. The fundamental objective to generate a network from a complex system is to identify the nodes and the corresponding links. It is also an important task to find the hidden rule by which the links are established (whether it is according to any preference or just randomly). After having obtained these fundamental informations, we can calculate some basic parameters (e.g. average shortest distance, clustering coefficient, degree distribution) of a network that characterises it and also reflects some gross behaviour (thus avoiding the microscopic details).

The *distance* between two nodes in a network is the number of links along the shortest path connecting them. The *average shortest distance* ($\langle d \rangle$) is the distance averaged over all possible pairs of nodes in the network [1]. The *clustering coefficient* of a node i (C_i) is the ratio between the number of links among the neighbours of that node and the total number of such links possible. This was introduced by Watts and Strogatz [8] to measure the interconnectivity of the network. One pair of linked neighbors form a triangle and thus all such connected triangles increases the clustering coefficient of a network. But in case of bipartite networks, where two types of nodes exist and connections link only nodes of different types, the standard clustering coefficient is zero. In this case, instead of triangles the quotient is measured between the number of observed squares and the total number of possible squares [9]. The average clustering coefficient of a network is the average taken over all the nodes $\langle C \rangle = \sum_i C_i / N$, where N is the total number of nodes in the network. The degree distribution $P(k)$ measures the probability that a node has a degree k where *degree* of a node is the number of links of that node. In case of a random network (i.e. links are established randomly), both $\langle d \rangle$ and $\langle C \rangle$ are very small ($\langle d \rangle \sim \log N$), whereas in a regular network both are very high. In some networks, $\langle d \rangle$ remains low but $\langle C \rangle$ is very high. This kind of network is called *small world network* [10]. This is a very special feature of many real-world networks where apparently the system looks very congested but every node is easily reachable from another by some short route existing within that network. Generally these features are investigated while studying various networks.

*Electronic address: anjan.chandra@saha.ac.in

†Electronic address: j_inoue@complex.eng.hokudai.ac.jp

Apart from its huge applications in real systems (transport, internet, scientific collaborations etc.) networks has also been applied in the world of numbers. Corso considered a network where each node is a natural number and are linked if they share a common prime factor larger than some chosen lower limit [11]. That network appeared to show small-world behaviour unless the lower limit is 1. Some more number network studies has also been done [12, 13]. Another study on number network was done [14] where the nodes were primes and the links were established according to some rule and a tunable parameter α . In that study every even number were split into all possible pairs of primes and one of them were chosen depending on their difference (according to some rule dictated by a parameter α). Near $\alpha = -1.8$, the small world property emerged. In this paper, we have extended the study of this splitting of numbers to more general case (i.e. split any integer into all possible integer pairs) and obtained some striking results (transitions from regular to random network) that shows some similarity to the study of prime numbers. We have also done some extensive studies on the prime network as discussed above and present some features which remain unexplored in the previous study of that network. The transition from regular to random network occurs near the same value of α as it was in case of the prime number network which definitely indicates some general pattern hidden in both kind of splitting.

In section II we discuss about the formulation of the network and in section III, we present the results for average distance and clustering coefficient for the case of general splitting as obtained from simulation. We also discuss about the transition of the network properties from those parameters. In section IV, we study the variation of average absolute difference and its fluctuation with α , both analytically and by simulation, to have a deep insight on the possible reasons of the particular transition point ($\alpha = -1.8$) and in the final section we have some general discussions.

II. FORMULATION OF THE NETWORK

Any integer n can be split in a various number of ways ($n = p + q$). Such as the number 8 can be split in 4 unique ways, e.g. $8 = 1 + 7 = 2 + 6 = 3 + 5 = 4 + 4$. The rest are just reverse of these. We can carry on this splitting for all such consecutive integers from 2 to ∞ . We construct a network by a growing process where the nodes are the numbers obtained by splitting the consecutive integers and selecting only a pair of numbers following some rule, which is described below elaborately. We start with the integer 3, which has only one break-up ($2 + 1$) and place the two numbers as two nodes and put a link between them (we have no other choice). Next comes the integer 4, which has two break-ups: ($1 + 3$) and ($2 + 2$). But we will always ignore the pair obtained from equal break-ups, because this gives rise to a self-connected node and again the only choice we have is to take 1 and 3 as two nodes and put a link between them. So now a new node (3) appears in our network and thus the network evolves. Interesting situation arises in case of splitting the next integer 5, for which we have two options : ($1 + 4$) or ($2 + 3$). In order to choose one of these pairs we give priority to the absolute difference of these two pair of numbers and select one of them. We calculate the absolute difference $\Delta = |p - q|$ between the two components for every break-up ($p + q$) and choose one break-up with probability proportional to Δ^α , where α is a tunable parameter. By tuning α we can decide, whether to take pairs of large differences or smaller ones as nodes. So in case of splitting the integer 5, the probability to connect ($1, 4$) is p_1 and ($2, 3$) is p_2 , where $p_1 = |4 - 1|^\alpha / s$ and $p_2 = |3 - 2|^\alpha / s$ with $s = 3^\alpha + 1^\alpha$. If ($1, 4$) is selected, then 4 is incorporated in the network as a new node and gets connected with 1, whereas if ($2, 3$) is selected, no new nodes appear but a link is established between 2 and 3. Like this we go on splitting the integers one after another (continue the growth process) until we obtain a certain number of nodes (say N). Now, for a fixed value of α , we construct networks of various values of N and measure various network properties such as average shortest distance, clustering coefficient etc. We also repeat the same study for different values of α .

So in our study we give preference to pairs according to the value of α . When $\alpha = 0.0$, the choice is random. When the value of α is *high*, higher values of Δ are preferred, whereas when α becomes negative, break-ups with smaller Δ are preferred. This preference gives rise to a transition in the network properties around $\alpha = -1.8$. Near about $\alpha = -1.8$ the network shows small world properties characterised by low average distance ($\langle d \rangle \sim \log N$ shown in Fig. 2) and high clustering coefficient (Fig. 4). As the value of α is gradually decreased the network becomes a regular network ($\langle d \rangle \sim N$ shown in Fig. 3) and high clustering coefficient (Fig. 4), whereas as it is increased it leads to a random one (both $\langle d \rangle$ and $\langle C \rangle$ are small). When the value of α is high, the pair with larger differences are preferred and as a consequence of splitting of each integer one small number is chosen many times, which becomes a hub (highly connected node) and the other one is a large number which changes with change in the integer (cause of randomness) and thus leads to a random network. But when the value of α is low, the pair with smaller differences are preferred and as a result more often we obtain a new pair of nodes and thus leads to a regular one. To have a better image of the network, we have presented six sample networks with $N = 15$ for various values of α (typically $-6.0, -2.0, -1.5, -1.0, 0.0$ and 4.0). The numbers corresponding to each vertices of the network are the numbers obtained by splitting the consecutive integers. It can be clearly seen from the first network ($\alpha = -6.0$), that almost

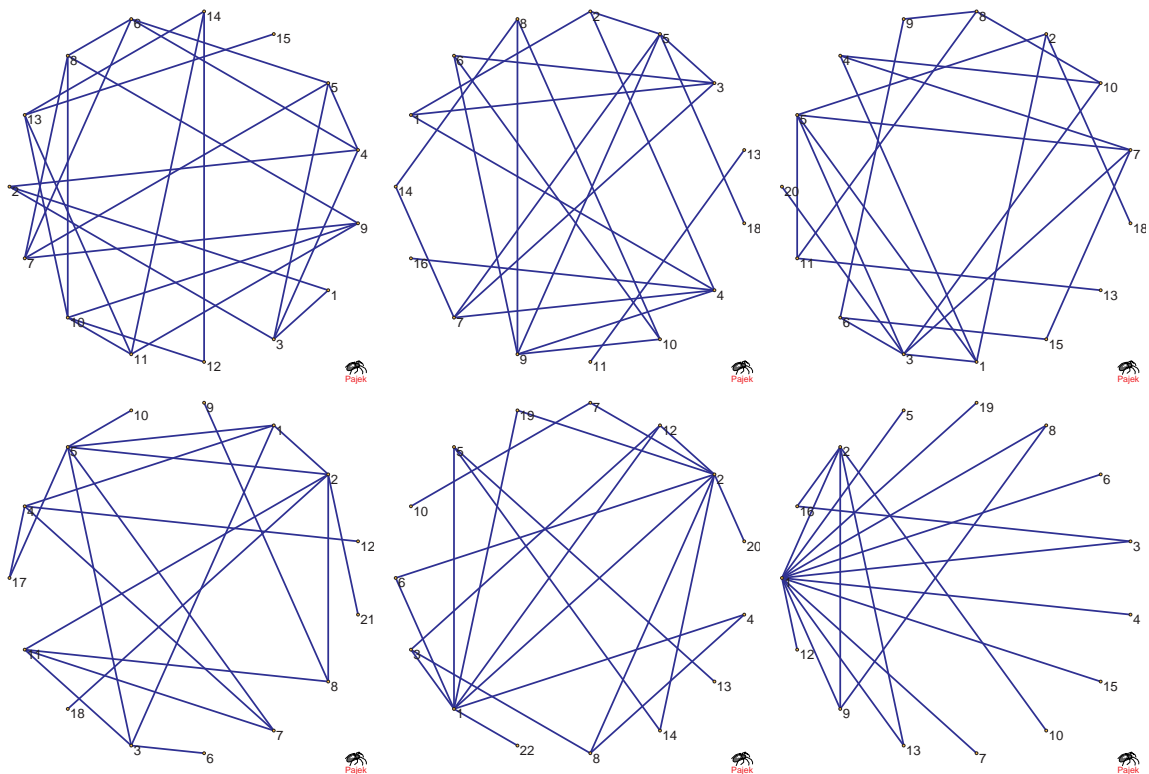


FIG. 1: Sample networks for $\alpha = -6.0, -2.0, -1.5, -1.0, 0.0$ and 4.0 for $N = 15$ (from left to right). The numbers corresponding to each vertices of the networks are the split numbers.

all the nodes have equal degrees indicating regular network, whereas in the last one the smaller integers are highly connected and the larger ones only one degree leading to a random network. Previously we obtained a similar kind of transition in case of prime number networks [14] as discussed above. In that study, the nodes were primes obtained by splitting consecutive even numbers according to the same rule followed here. There also the transition occurred around $\alpha = -1.8$, which shows a similarity between the two networks. We also justify this value of $\alpha = -1.8$ by some analytic arguments.

III. AVERAGE DISTANCE AND CLUSTERING COEFFICIENT

Now let us discuss about the variations in average distance $\langle d \rangle$ and clustering coefficient $\langle C \rangle$ with the number of nodes N for different values of α . This has been shown in Fig. 2, 3 and 4. For $\alpha \geq -1.8$, $\langle d \rangle \sim \log N$ (Fig. 2) whereas as α is decreased gradually it varies linearly with the number of nodes (Fig. 3). The nature of the clustering coefficient also shows a transition near $\alpha = -1.8$. When the value of α is very high than -1.8 , the value of $\langle C \rangle$ is very low, but as α decreases, $\langle C \rangle$ increases and at very small value of α (e.g. $\alpha = -6.0$) it becomes very high (0.5) (Fig. 4). The clustering coefficient is almost independent of N for all values of α . So for values of α much greater than -1.8 , both the average distance and clustering coefficient is low which indicates random nature, whereas for values of α much smaller than -1.8 , both of the parameters ($\langle d \rangle$ and $\langle C \rangle$) are high leading to a regular network. Around $\alpha = -1.8$ for a finite region the average distance of the network is low but simultaneously the clustering coefficient remains high which is the signature of *small world network*. Also upto Just as we decrease the value of α below -1.8 , $\langle d \rangle$ does not increase linearly with $\log N$ (Fig. 2), indicating the transition point at $\alpha = -1.8$.

In order to have a more clear insight to the transitions we present a comparative picture of the change in $\langle d \rangle$ and $\langle C \rangle$ with that of the distance and clustering coefficient obtained for the regular network with the same number of nodes (i.e., the values of $\langle d \rangle$ and $\langle C \rangle$ for very low values of α which we say $\langle d_{reg} \rangle$ and $\langle C_{reg} \rangle$ respectively). So $\langle d \rangle / \langle d_{reg} \rangle$ and $\langle C \rangle / \langle C_{reg} \rangle$ gives the normalised average distance and clustering coefficient respectively, and when our network becomes regular, both the values will approach 1.0. We plot $\langle d \rangle / \langle d_{reg} \rangle$ and $\langle C \rangle / \langle C_{reg} \rangle$ as a function of α for $N = 10000$ (Fig. 5), where d_{reg} and C_{reg} denote the average distance and clustering coefficient for the regular network. Here we have assumed that the network becomes regular for values of $\alpha \ll -0.2$ (we went up to $\alpha = -6.0$)

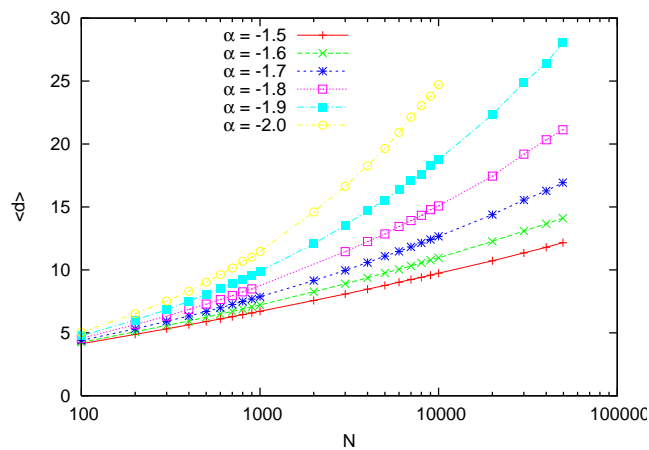


FIG. 2: Plot of average shortest distance $\langle d \rangle$ as a function of the number of nodes (N) for $\alpha = -1.5, -1.6, -1.7, -1.8, -1.9$ and -2.0 . This is a log-linear plot. It is clearly shown that as α reaches -1.8 the linearity destroys. Up to $\alpha = -1.8$, $\langle d \rangle \sim \log N$ indicating small world property.

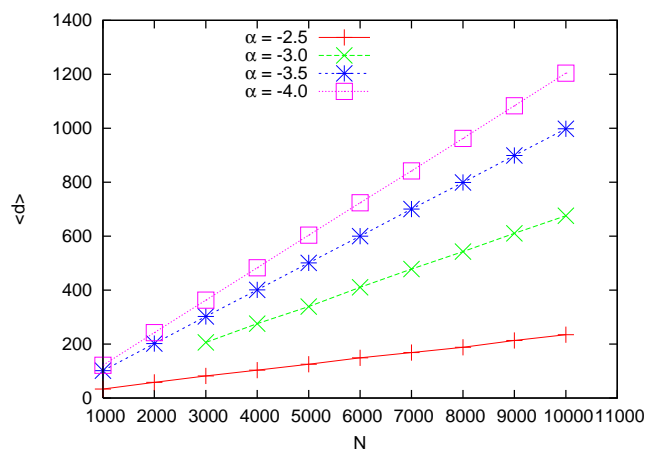


FIG. 3: Plot of average shortest distance $\langle d \rangle$ as a function of the number of nodes (N) for $\alpha < -1.8$. For all values of α , $\langle d \rangle \sim N$ indicating the absence of small world property.

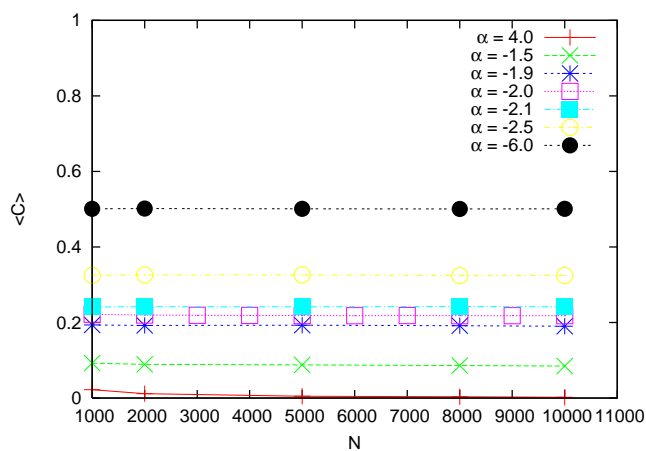


FIG. 4: Plot of average clustering coefficient $\langle C \rangle$ as a function of the number of nodes (N) for different values of α .

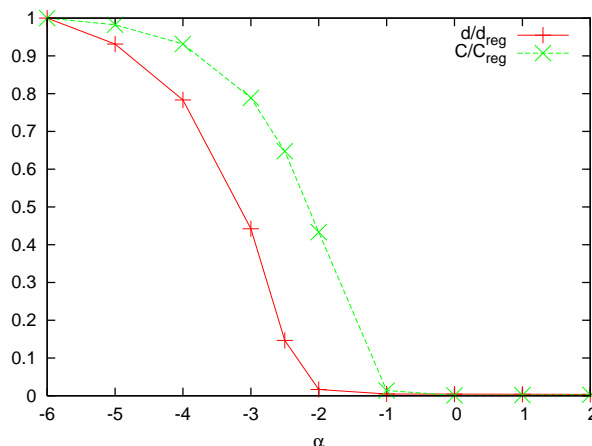


FIG. 5: The plot of normalised average shortest distance $\langle d \rangle / \langle d_{reg} \rangle$ and normalised average clustering coefficient $\langle C \rangle / \langle C_{reg} \rangle$ as a function of α for general splitting.

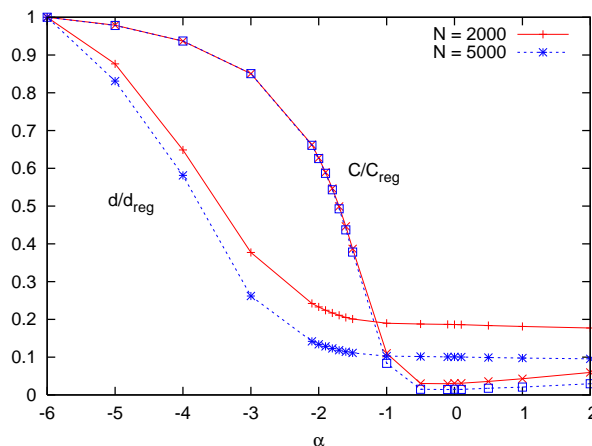


FIG. 6: The plot of normalised average shortest distance $\langle d \rangle / \langle d_{reg} \rangle$ and normalised average clustering coefficient $\langle C \rangle / \langle C_{reg} \rangle$ as a function of α for splitting evens into prime numbers.

and taken the corresponding average distance as d_{reg} and clustering coefficient as C_{reg} . From Fig. 5 it is very clear that for $\alpha \gg -1.8$, both the d/d_{reg} and C/C_{reg} are low which indicates random nature, whereas for $\alpha \ll -1.8$, both of the parameters are high which is a signature of a regular network. Near around $\alpha = -1.8$ there is a transition from this random to a regular one where for a large region of α , $\langle d \rangle / \langle d_{reg} \rangle$ remains low but $\langle C \rangle / \langle C_{reg} \rangle$ is high which leads to a small world network.

A similar plot for the prime number network (discussed above and in [14]) has also been given in Fig. 6. This plot also shows a similar kind of transition near $\alpha = -1.8$ which definitely indicates some universality of the value of this particular α .

IV. AVERAGE ABSOLUTE DIFFERENCE OF THE SPLIT NUMBERS

Now let us look after some other parameters that also reflects such kind of transitions near $\alpha = -1.8$. One such parameter is the average absolute difference of the pair of numbers $\langle \Delta \rangle$ that are selected for the splitting upto a certain integer. For the splitting of a certain integer I which has say m break ups with absolute values of differences j_1, j_2, \dots and j_m , $\langle \Delta \rangle_I = \sum_{i=1}^m j_i^{\alpha+1} / j_i^\alpha$. If we split upto n integers, $\langle \Delta \rangle = \sum_{I=1}^n \langle \Delta \rangle_I / n$. We have calculated the value of $\langle \Delta \rangle$ and also the relative fluctuation of Δ i.e. $\sigma / \langle \Delta \rangle$ (where σ is given by Eq.2) as a function of α for different values of n (for $n = 1000, 2000, 3000$ and 4000) from simulation. The results are given in Fig. 7 and Fig. 8. There is a sharp rise of $\langle \Delta \rangle$ at $\alpha = -1.9$ and it becomes sharper with increasing n (Fig. 7). Also the relative fluctuation

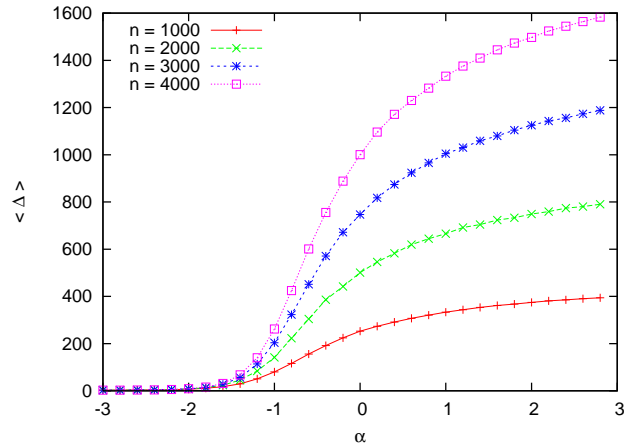


FIG. 7: Plot of average absolute difference $\langle \Delta \rangle$ of the split numbers for splitting up to a certain number of integers n . Typical error bar for $\alpha = -1.8$ is ± 0.10511627 .

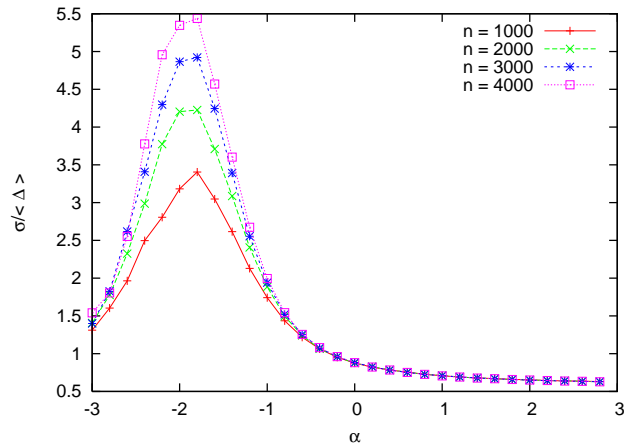


FIG. 8: Plot of $\sigma / \langle \Delta \rangle$ of the split numbers with α for splitting up to a certain number of integers n . Typical error bar for $\alpha = -1.8$ is ± 0.0396026149 .

shows a peak at $\alpha = -1.9$, which again becomes sharper with increasing n (Fig. 8). From both the figures there is a clear indication of a transition near $\alpha = -1.9$. Although this point does not exactly matches with the transition point obtained earlier i.e. $\alpha = -1.8$, but it gives strong indication of the presence of the original transition point near it. The anomaly comes from the fact that in the original network we measure the average shortest distance between the nodes, whereas here we measure the average absolute difference between the numbers obtained by splitting the consecutive integers following the rule adopted by us to create our network. So it gives us a clue about the presence of the transition point which is very close to that we obtained here.

We have also obtained an analytical expression of $\langle \Delta \rangle$ and $\sigma / \langle \Delta \rangle$. It can be easily noticed that as we consider the splitting of an odd(even) integer I , we get odd(even) absolute differences starting from 1 up to $I - 3$ at an interval of 2. The analytical expression obtained is given by

$$\langle \Delta \rangle = \frac{1}{n-2} \sum_{I=3}^n \frac{\sum_{k=1}^{I-2} k^{\alpha+1}}{\sum_{k=1}^{I-2} k^{\alpha}} \quad [\text{k-odd(even) for I-odd(even)}] \quad (1)$$

and

$$\sigma = \sqrt{\langle \Delta^2 \rangle - \langle \Delta \rangle^2} \quad (2)$$

The corresponding curves are shown in Fig. 9 which qualitatively same as we found in Fig. 7 and Fig. 8 and also confirms the transition point.

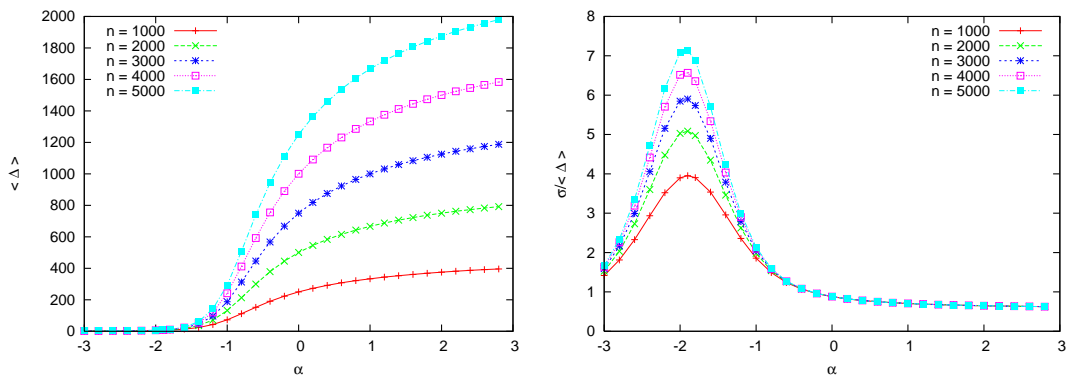


FIG. 9: The plots of $\langle \Delta \rangle$ and $\sigma/\langle \Delta \rangle$ for discrete sum.

V. DISCUSSION

The entire study shows that the network formed by general splitting of integers into pairs of integers, shows a transition depending upon our choice of splitting. The choice of splitting the integers is controlled by a single parameter α , which determines whether we prefer a pair having large difference or a small one. Above the value $\alpha = -1.8$, pairs having large differences are favoured and consequently the smaller integers are selected a large number of times leading to the creation of hubs and small $\langle d \rangle$. For very high values of α the network becomes random. As α is decreased below -1.8 , the linear nature of $\langle d \rangle$ with $\ln N$ breaks (N is the number of nodes) showing a clear signature of transition. At and just below $\alpha = -1.8$ both $\langle d \rangle \sim \ln N$ and the clustering coefficient is high indicating small world nature. Although as α is increased further, $\langle d \rangle$ increases linearly with N which along with high clustering coefficient suggests regular one. So given a tunable parameter, the network formed by the integers displays a transition from a random to a regular one through an intermediate small world region, the transition occurring near $\alpha = -1.8$. This transition along with the point of transition ($\alpha = -1.8$) is almost similar as the one we obtained while splitting the consecutive even integers into pairs of primes. This similarity discloses some common pattern in both kinds of splitting. Moreover, it is also quite interesting to have such presence of network structures in the world of numbers.

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