Lace expansion for the Ising model

Akira Sakai*

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Abstract

The lace expansion has been a powerful tool for investigating mean-field behavior for various stochastic-geometrical models, such as self-avoiding walk and percolation, above their respective upper-critical dimension. In this paper, we prove the lace expansion for the Ising model that is valid for any spin-spin coupling. For the ferromagnetic case, we also prove that the expansion coefficients obey certain diagrammatic bounds that are similar to the diagrammatic bounds on the lace-expansion coefficients for self-avoiding walk. As a result, we obtain Gaussian asymptotics of the critical two-point function for the nearest-neighbor model with $d \gg 4$ and for the spread-out model with $d > 4$ and $L \gg 1$, without assuming reflection positivity.

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*Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK. a.sakai@bath.ac.uk

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1 Introduction and results

1.1 Model and the motivation

The Ising model is a statistical-mechanical model that was first introduced in [22] as a model for magnets. Consider the d-dimensional integer lattice $\mathbb{Z}^d$, and let Λ be a finite subset of $\mathbb{Z}^d$ containing the origin $o \in \mathbb{Z}^d$. For example, Λ is a d-dimensional hypercube centered at the origin. At each site $x \in \Lambda$, there is a spin variable φ_x that takes values either $+1$ or -1. The Hamiltonian represents the energy of the system, and is defined by

$$H_\Lambda^h(\varphi) = -\sum_{\{x,y\}\subset\Lambda} J_{x,y}\varphi_x\varphi_y - h\sum_{x\in\Lambda}\varphi_x, \tag{1.1}$$

where $\varphi \equiv \{\varphi_x\}_{x\in\Lambda}$ is a spin configuration, $\{J_{x,y}\}_{x,y\in\mathbb{Z}^d}$ is a collection of spin-spin couplings, and $h \in \mathbb{R}$ represents the strength of an external magnetic field uniformly imposed on Λ. We say that the model is ferromagnetic if $J_{x,y} \geq 0$ for all pairs $\{x, y\}$; in this case, the Hamiltonian becomes lower as more spins align. The partition function $Z_{p,h;\Lambda}$ at the inverse temperature $p \geq 0$ is the expectation of the Boltzmann factor $e^{-pH_\Lambda^h(\varphi)}$ with respect to the product measure $\prod_{x\in\Lambda}(\frac{1}{2}\mathbb{1}_{\{\varphi_x=+1\}} + \frac{1}{2}\mathbb{1}_{\{\varphi_x=-1\}})$:

$$Z_{p,h;\Lambda} = 2^{-|\Lambda|}\sum_{\varphi\in\{\pm1\}^\Lambda} e^{-pH_\Lambda^h(\varphi)}. \tag{1.2}$$

Then, we denote the thermal average of a function $f = f(\varphi)$ by

$$\langle f\rangle_{p,h;\Lambda} = \frac{2^{-|\Lambda|}}{Z_{p,h;\Lambda}}\sum_{\varphi\in\{\pm1\}^\Lambda} f(\varphi)\, e^{-pH_\Lambda^h(\varphi)}. \tag{1.3}$$

Suppose that the spin-spin coupling is translation-invariant, $\mathbb{Z}^d$-symmetric and finite-range (i.e., there exists an $L < \infty$ such that $J_{o,x} = 0$ if $\|x\|_\infty > L$) and that $J_{o,x} \geq 0$ for any $x \in \mathbb{Z}^d$ and $h \geq 0$. Then, there exist monotone infinite-volume limits of $\langle\varphi_x\rangle_{p,h;\Lambda}$ and $\langle\varphi_x\varphi_y\rangle_{p,h;\Lambda}$. Let

$$M_{p,h} = \lim_{\Lambda\uparrow\mathbb{Z}^d}\langle\varphi_o\rangle_{p,h;\Lambda}, \qquad G_p(x) = \lim_{\Lambda\uparrow\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_{p,h=0;\Lambda}, \qquad \chi_p = \sum_{x\in\mathbb{Z}^d} G_p(x). \tag{1.4}$$

When $d \geq 2$, there exists a unique critical inverse temperature $p_c \in (0, \infty)$ such that the spontaneous magnetization $M_p^+ \equiv \lim_{h\downarrow0} M_{p,h}$ equals zero, $G_p(x)$ decays exponentially as $|x| \uparrow \infty$ (we refer, e.g., to [9] for a sharper Ornstein-Zernike result) and thus the magnetic susceptibility χ_p is finite if $p < p_c$, while $M_p^+ > 0$ and $\chi_p = \infty$ if $p > p_c$ (see [2] and references therein). We should also refer to [7] for recent results on the phase transition for the Ising model.

We are interested in the behavior of these observables around $p = p_c$. The susceptibility χ_p is known to diverge as $p \uparrow p_c$ [1, 4]. It is generally expected that $\lim_{p\downarrow p_c} M_p^+ = \lim_{h\downarrow0} M_{p_c,h} = 0$. We believe that there are so-called critical exponents $\gamma = \gamma(d)$, $\beta = \beta(d)$ and $\delta = \delta(d)$, which are insensitive to the precise definition of $J_{o,x} \geq 0$ (universality), such that (we use below the limit notation "$\approx$" in some appropriate sense)

$$M_p^+ \overset{p\downarrow p_c}{\approx} (p - p_c)^\beta, \qquad \chi_p \overset{p\uparrow p_c}{\approx} (p_c - p)^{-\gamma}, \qquad M_{p_c,h} \overset{h\downarrow0}{\approx} h^{1/\delta}. \tag{1.5}$$

These exponents (if they exist) are known to obey the mean-field bounds: $\beta \le 1/2$, $\gamma \ge 1$ and $\delta \ge 3$. For example, $\beta = 1/8$, $\gamma = 7/4$ and $\delta = 15$ for the nearest-neighbor model on $\mathbb{Z}^2$ [26]. Our ultimate goal is to identify the values of the critical exponents in other dimensions and to understand the universality for the Ising model.

There is a sufficient condition, the so-called bubble condition, for the above critical exponents to take on their respective mean-field values. Namely, the finiteness of $\sum_{x\in\mathbb{Z}^d} G_{p_c}(x)^2$ (or the finiteness of $\sum_{x\in\mathbb{Z}^d} G_p(x)^2$ uniformly in $p < p_c$) implies that $\beta = 1/2$, $\gamma = 1$ and $\delta = 3$ [1, 2, 3, 4]. It is therefore crucial to know how fast $G_{p_c}(x)$ (or $G_p(x)$ near $p = p_c$) decays as $|x| \uparrow \infty$. We note that the bubble condition holds for $d > 4$ if the anomalous dimension η takes on its mean-field value $\eta = 0$, where the anomalous dimension is another critical exponent formally defined as

$$G_{p_c}(x) \overset{|x|\uparrow\infty}{\approx} |x|^{-(d-2+\eta)}. \tag{1.6}$$

Let $\hat{J}_k = \sum_{x\in\mathbb{Z}^d} J_{o,x}\, e^{ik\cdot x}$ and $\hat{G}_p(k) = \sum_{x\in\mathbb{Z}^d} G_p(x)\, e^{ik\cdot x}$ for $p < p_c$. For a class of models that satisfy the so-called reflection positivity [12], the following infrared bound[1] holds:

$$0 \le \hat{G}_p(k) \le \frac{\text{const.}}{\hat{J}_0 - \hat{J}_k} \qquad \text{uniformly in } p < p_c, \tag{1.7}$$

where d is supposed to be large enough to ensure integrability of the upper bound. For finite-range models, d has to be bigger than 2, since $\hat{J}_0 - \hat{J}_k \asymp |k|^2$, where "$f \asymp g$" means that f/g is bounded away from zero and infinity. By Parseval's identity, the infrared bound (1.7) implies the bubble condition for finite-range reflection-positive models above four dimensions, and therefore

$$M_p^+ \overset{p\downarrow p_c}{\asymp} (p - p_c)^{1/2}, \qquad \chi_p \overset{p\uparrow p_c}{\asymp} (p_c - p)^{-1}, \qquad M_{p_c,h} \overset{h\downarrow 0}{\asymp} h^{1/3}. \tag{1.8}$$

The class of reflection-positive models includes the nearest-neighbor model, a variant of the next-nearest-neighbor model, Yukawa potentials, power-law decaying interactions, and their combinations [6]. For the nearest-neighbor model, we further obtain the following x-space Gaussian bound [32]: for $x \ne o$,

$$G_p(x) \le \frac{\text{const.}}{|x|^{d-2}} \qquad \text{uniformly in } p < p_c. \tag{1.9}$$

The problem in this approach to investigate critical behavior is that, since general finite-range models do not always satisfy reflection positivity, their mean-field behavior cannot necessarily be established, even in high dimensions. If we believe in universality, we expect that finite-range models exhibit the same mean-field behavior as soon as $d > 4$. Therefore, it has been desirable to have approaches that do not assume reflection positivity.

The lace expansion has been used successfully to investigate mean-field behavior for self-avoiding walk, percolation, lattice trees/animals and the contact process, above the upper-critical dimension: 4, 6 (4 for oriented percolation), 8 and 4, respectively (see, e.g., [31]). One of the advantages in the application of the lace expansion is that we do not have to require reflection positivity to prove a Gaussian infrared bound and mean-field behavior. Another advantage is the possibility to show an asymptotic result for the decay of correlation. Our goal in this paper is to prove the lace-expansion results for the Ising model.

[1] In (1.7) and (1.9), we also use the fact that, for $p < p_c$, our G_p (i.e., the infinite-volume limit of the two-point function under the free-boundary condition) is equal to the infinite-volume limit of the two-point function under the periodic-boundary condition.

1.2 Main results

From now on, we fix $h = 0$ and abbreviate, e.g., $\langle\varphi_o\varphi_x\rangle_{p,h=0;\Lambda}$ to $\langle\varphi_o\varphi_x\rangle_{p;\Lambda}$. In this paper, we prove the following lace expansion for the two-point function, in which we use the notation

$$\tau_{x,y} = \tanh(pJ_{x,y}). \tag{1.10}$$

Proposition 1.1. *For any $p \geq 0$ and any $\Lambda \subset \mathbb{Z}^d$, there exist $\pi^{(j)}_{p;\Lambda}(x)$ and $R^{(j+1)}_{p;\Lambda}(x)$ for $x \in \Lambda$ and $j \geq 0$ such that*

$$\langle\varphi_o\varphi_x\rangle_{p;\Lambda} = \Pi^{(j)}_{p;\Lambda}(x) + \sum_{u,v} \Pi^{(j)}_{p;\Lambda}(u)\,\tau_{u,v}\langle\varphi_v\varphi_x\rangle_{p;\Lambda} + (-1)^{j+1}R^{(j+1)}_{p;\Lambda}(x), \tag{1.11}$$

where

$$\Pi^{(j)}_{p;\Lambda}(x) = \sum_{i=0}^{j}(-1)^i\,\pi^{(i)}_{p;\Lambda}(x). \tag{1.12}$$

For the ferromagnetic case, we have the bounds

$$\pi^{(j)}_{p;\Lambda}(x) \geq \delta_{j,0}\delta_{o,x}, \qquad\qquad 0 \leq R^{(j+1)}_{p;\Lambda}(x) \leq \sum_{u,v}\pi^{(j)}_{p;\Lambda}(u)\,\tau_{u,v}\langle\varphi_v\varphi_x\rangle_{p;\Lambda}. \tag{1.13}$$

We defer the display of precise expressions of $\pi^{(i)}_{p;\Lambda}(x)$ and $R^{(j+1)}_{p;\Lambda}(x)$ to Section 2.2.3, since we need a certain representation to describe these functions. We introduce this representation in Section 2.1 and complete the proof of Proposition 1.1 in Section 2.2.

It is worth emphasizing that the above proposition holds independently of the properties of the spin-spin coupling: $J_{u,v}$ does not have to be translation-invariant or $\mathbb{Z}^d$-symmetric. In particular, the identity (1.11) holds independently of the sign of the spin-spin coupling. A spin glass, whose spin-spin coupling is randomly negative, is an extreme example for which (1.11) holds.

Whether or not the lace expansion (1.11) is useful depends on the possibility of good control on the expansion coefficients and the remainder. As explained below, it is indeed possible to have optimal bounds on the expansion coefficients for the nearest-neighbor interaction (i.e., $J_{o,x} = \mathbb{1}_{\{\|x\|_1=1\}}$) and for the following spread-out interaction:

$$J_{o,x} = L^{-d}\mu(L^{-1}x) \qquad (1 \leq L < \infty), \tag{1.14}$$

where $\mu : [-1,1]^d \setminus \{o\} \mapsto [0,\infty)$ is a bounded probability distribution, which is symmetric under rotations by $\pi/2$ and reflections in coordinate hyperplanes, and piecewise continuous so that the Riemann sum $L^{-d}\sum_{x\in\mathbb{Z}^d}\mu(L^{-1}x)$ approximates $\int_{\mathbb{R}^d} d^dx\;\mu(x) \equiv 1$. One of the simplest examples would be

$$J_{o,x} = \frac{\mathbb{1}_{\{0<\|x\|_\infty\leq L\}}}{\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{0<\|z\|_\infty\leq L\}}} = O(L^{-d})\,\mathbb{1}_{\{0<\|L^{-1}x\|_\infty\leq 1\}}. \tag{1.15}$$

Proposition 1.2. *Let $\rho = 2(d-4) > 0$. For the nearest-neighbor model with $d \gg 1$ and for the spread-out model with $L \gg 1$, there are finite constants θ and λ such that*

$$|\Pi^{(j)}_{p;\Lambda}(x) - \delta_{o,x}| \leq \theta\delta_{o,x} + \frac{\lambda(1-\delta_{o,x})}{|x|^{d+2+\rho}} \quad (j \geq 0), \qquad\qquad |R^{(j)}_{p;\Lambda}(x)| \to 0 \quad (j \uparrow \infty), \tag{1.16}$$

for any $p \leq p_c$, any $\Lambda \subset \mathbb{Z}^d$ and any $x \in \Lambda$.

The proof of Proposition 1.2 depends on certain bounds on the expansion coefficients in terms of two-point functions. These diagrammatic bounds arise from counting the number of "disjoint connections", corresponding to applications of the BK inequality in percolation (e.g., [5]). We prove these bounds in Section 4, and in anticipation of this, in Section 3 we explain how we use their implication to prove Proposition 1.2, with $\theta = O(d^{-1})$ and $\lambda = O(1)$ for the nearest-neighbor model, and $\theta = O(L^{-2+\epsilon})$ and $\lambda = O(\theta^2)$ with a small $\epsilon > 0$ for the spread-out model.

Let

$$\tau \equiv \tau(p) = \sum_x \tau_{o,x}, \qquad D(x) = \frac{\tau_{o,x}}{\tau}, \qquad \sigma^2 = \sum_x |x|^2 D(x). \tag{1.17}$$

Due to (1.16) uniformly in $\Lambda \subset \mathbb{Z}^d$, there is a limit $\Pi_p(x) \equiv \lim_{\Lambda \uparrow \mathbb{Z}^d} \lim_{j \uparrow \infty} \Pi_{p;\Lambda}^{(j)}(x)$ such that

$$G_p(x) = \Pi_p(x) + (\Pi_p * \tau D * G_p)(x), \qquad |\Pi_p(x) - \delta_{o,x}| \le \theta \delta_{o,x} + \frac{\lambda(1-\delta_{o,x})}{|x|^{d+2+\rho}}, \tag{1.18}$$

for any $p \le p_c$ and any $x \in \mathbb{Z}^d$, where $(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y)\, g(x-y)$. We note that the identity in (1.18) is similar to the recursion equation for the random-walk Green's function:

$$S_r(x) \equiv \sum_{i=0}^{\infty} r^i D^{*i}(x) = \delta_{o,x} + (rD * S_r)(x) \qquad (|r| < 1), \tag{1.19}$$

where $f^{*i}(x) = (f^{*(i-1)} * f)(x)$, with $f^{*0}(x) = \delta_{o,x}$ by convention. The leading asymptotics of $S_1(x)$ for $d > 2$ is known as $\frac{a_d}{\sigma^2}|x|^{-(d-2)}$, where $a_d = \frac{d}{2}\pi^{-d/2}\Gamma(\frac{d}{2}-1)$ (e.g., [14, 15]). Following the model-independent analysis of the lace expansion in [14, 15], we obtain the following asymptotics of the critical two-point function:

Theorem 1.3. *Let $\rho = 2(d-4) > 0$ and fix any small $\epsilon > 0$. For the nearest-neighbor model with $d \gg 1$ and for the spread-out model with $L \gg 1$, we have that, for $x \ne o$,*

$$G_{p_c}(x) = \frac{A}{\tau(p_c)} \frac{a_d}{\sigma^2 |x|^{d-2}} \times \begin{cases} \big(1 + O(|x|^{-\frac{(\rho-\epsilon)\wedge 2}{d}})\big) & (\textit{NN model}), \\ \big(1 + O(|x|^{-\rho\wedge 2+\epsilon})\big) & (\textit{SO model}), \end{cases} \tag{1.20}$$

where constants in the error terms may vary depending on ϵ, and

$$\tau(p_c) = \Big(\sum_x \Pi_{p_c}(x)\Big)^{-1}, \qquad A = \Big(1 + \frac{\tau(p_c)}{\sigma^2} \sum_x |x|^2 \Pi_{p_c}(x)\Big)^{-1}. \tag{1.21}$$

Consequently, (1.8) holds and $\eta = 0$.

In this paper, we restrict ourselves to the nearest-neighbor model for $d \gg 4$ and to the spread-out model for $d > 4$ with $L \gg 1$. However, it is strongly expected that our method can show the same asymptotics of the critical two-point function for *any* translation-invariant, $\mathbb{Z}^d$-symmetric finite-range model above four dimensions, by taking the coordination number sufficiently large.

1.3 Organization

In the rest of this paper, we focus our attention on the model-dependent ingredients: the lace expansion for the Ising model (Proposition 1.1) and the bounds on (the alternating sum of) the expansion coefficients for the ferromagnetic models (Proposition 1.2). In Section 2, we prove Proposition 1.1. In Section 3, we reduce Proposition 1.2 to a few other propositions, which are then results of the aforementioned diagrammatic bounds on the expansion coefficients. We prove these diagrammatic bounds in Section 4. As soon as the composition of the diagrams in terms of two-point functions is understood, it is not so hard to establish key elements of the above reduced propositions. We will prove these elements in Section 5.1 for the spread-out model and in Section 5.2 for the nearest-neighbor model.

2 Lace expansion for the Ising model

The lace expansion was initiated by Brydges and Spencer [8] to investigate weakly self-avoiding walk for $d > 4$. Later, it was developed for various stochastic-geometrical models, such as strictly self-avoiding walk for $d > 4$ (e.g., [18]), lattice trees/animals for $d > 8$ (e.g., [16]), unoriented percolation for $d > 6$ (e.g., [17]), oriented percolation for $d > 4$ (e.g., [25]) and the contact process for $d > 4$ (e.g., [27]). See [31] for an extensive list of references. This is the first lace-expansion paper for the Ising model.

In this section, we prove the lace expansion (1.11) for the Ising model. From now on, we fix $p \geq 0$ and abbreviate, e.g., $\pi_{p;\Lambda}^{(i)}(x)$ to $\pi_{\Lambda}^{(i)}(x)$.

There may be several ways to derive the lace expansion for $\langle\varphi_o\varphi_x\rangle_\Lambda$, using, e.g., the high-temperature expansion, the random-walk representation (e.g., [10]) or the FK random-cluster representation (e.g., [11]). In this paper, we use the random-current representation (Section 2.1), which applies to models in the Griffiths-Simon class (e.g., [1, 4]). This representation is similar in philosophy to the high-temperature expansion, but it turned out to be more efficient in investigating the critical phenomena [1, 2, 3, 4]. The main advantage in this representation is the source-switching lemma (Lemma 2.3 below in Section 2.2.2) by which we have an identity for $\langle\varphi_o\varphi_x\rangle_\Lambda - \langle\varphi_o\varphi_x\rangle_{\mathcal{A}}$ with "$\mathcal{A} \subset \Lambda$" (the meaning will be explained in Section 2.1). We will repeatedly apply this identity to complete the lace expansion for $\langle\varphi_o\varphi_x\rangle_\Lambda$ in Section 2.2.3.

2.1 Random-current representation

In this subsection, we describe the random-current representation and introduce some notation that will be essential in the derivation of the lace expansion.

First we introduce some notions and notation. We call a pair of sites $b = \{u, v\}$ with $J_b \neq 0$ a *bond*. So far we have used the notation $\Lambda \subset \mathbb{Z}^d$ for a site set. However, we will often abuse this notation to describe a *graph* that consists of sites of Λ and are equipped with a certain bond set, which we denote by $\mathbb{B}_\Lambda$. Note that "$\{u, v\} \in \mathbb{B}_\Lambda$" always implies "$u, v \in \Lambda$", but the latter does not necessarily imply the former. If we regard $\mathcal{A}$ and Λ as graphs, then "$\mathcal{A} \subset \Lambda$" means that $\mathcal{A}$ is a subset of Λ as a site set, and that $\mathbb{B}_{\mathcal{A}} \subset \mathbb{B}_\Lambda$.

Now we consider the partition function $Z_{\mathcal{A}}$ on $\mathcal{A} \subset \Lambda$. By expanding the Boltzmann factor in (1.2), we obtain

$$\begin{aligned} Z_{\mathcal{A}} &= 2^{-|\mathcal{A}|} \sum_{\varphi\in\{\pm1\}^{\mathcal{A}}} \prod_{\{u,v\}\in\mathbb{B}_{\mathcal{A}}} \Bigg(\sum_{n_{u,v}\in\mathbb{Z}_+} \frac{(pJ_{u,v})^{n_{u,v}}}{n_{u,v}!} \varphi_u^{n_{u,v}} \varphi_v^{n_{u,v}} \Bigg) \\ &= \sum_{\mathbf{n}\in\mathbb{Z}_+^{\mathbb{B}_{\mathcal{A}}}} \Bigg(\prod_{b\in\mathbb{B}_{\mathcal{A}}} \frac{(pJ_b)^{n_b}}{n_b!} \Bigg) \prod_{v\in\mathcal{A}} \Bigg(\frac{1}{2} \sum_{\varphi_v=\pm1} \varphi_v^{\sum_{b\ni v} n_b} \Bigg), \end{aligned} \tag{2.1}$$

where we call $\mathbf{n} = \{n_b\}_{b\in\mathbb{B}_{\mathcal{A}}}$ a *current configuration*. Note that the single-spin average in the last line equals 1 if $\sum_{b\ni v} n_b$ is an even integer, and 0 otherwise. Denoting by $\partial\mathbf{n}$ the set of *sources* $v \in \Lambda$ at which $\sum_{b\ni v} n_b$ is an *odd* integer, and defining

$$w_{\mathcal{A}}(\mathbf{n}) = \prod_{b\in\mathbb{B}_{\mathcal{A}}} \frac{(pJ_b)^{n_b}}{n_b!} \qquad (\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_{\mathcal{A}}}), \tag{2.2}$$

we obtain

$$Z_{\mathcal{A}} = \sum_{\mathbf{n}\in\mathbb{Z}_+^{\mathbb{B}_{\mathcal{A}}}} w_{\mathcal{A}}(\mathbf{n}) \prod_{v\in\mathcal{A}} \mathbb{1}_{\{\sum_{b\ni v} n_b \text{ even}\}} = \sum_{\partial\mathbf{n}=\varnothing} w_{\mathcal{A}}(\mathbf{n}). \tag{2.3}$$

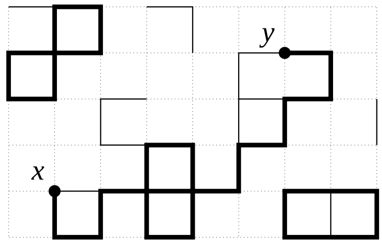


Figure 1: A current configuration with sources at x and y. The thick-solid segments represent bonds with odd currents, while the thin-solid segments represent bonds with positive even currents, which cannot be seen in the high-temperature expansion.

The partition function $Z_{\mathcal{A}}$ equals the partition function on Λ with $J_b = 0$ for all $b \in \mathbb{B}_\Lambda \setminus \mathbb{B}_{\mathcal{A}}$. We can also think of $Z_{\mathcal{A}}$ as the sum of $w_\Lambda(\mathbf{n})$ over $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$ satisfying $\mathbf{n}|_{\mathbb{B}_\Lambda \setminus \mathbb{B}_{\mathcal{A}}} \equiv 0$, where $\mathbf{n}|_{\mathbb{B}}$ is a projection of $\mathbf{n}$ over the bonds in a bond set $\mathbb{B}$, i.e., $\mathbf{n}|_{\mathbb{B}} = \{n_b : b \in \mathbb{B}\}$. By this observation, we can rewrite (2.3) as

$$Z_{\mathcal{A}} = \sum_{\substack{\partial\mathbf{n}=\varnothing \\ \mathbf{n}|_{\mathbb{B}_\Lambda \setminus \mathbb{B}_{\mathcal{A}}} \equiv 0}} w_\Lambda(\mathbf{n}). \tag{2.4}$$

Following the same calculation, we can rewrite $Z_{\mathcal{A}} \langle \varphi_x \varphi_y \rangle_{\mathcal{A}}$ for $x, y \in \mathcal{A}$ as

$$\begin{aligned} Z_{\mathcal{A}} \langle \varphi_x \varphi_y \rangle_{\mathcal{A}} &= \sum_{\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_{\mathcal{A}}}} \bigg(\prod_{b \in \mathbb{B}_{\mathcal{A}}} \frac{(pJ_b)^{n_b}}{n_b!} \bigg) \prod_{v \in \mathcal{A}} \bigg(\frac{1}{2} \sum_{\varphi_v = \pm 1} \varphi_v^{\mathbb{1}_{\{v \in x \triangle y\}} + \sum_{b \ni v} n_b} \bigg) \\ &= \sum_{\partial\mathbf{n} = x \triangle y} w_{\mathcal{A}}(\mathbf{n}) = \sum_{\substack{\partial\mathbf{n} = x \triangle y \\ \mathbf{n}|_{\mathbb{B}_\Lambda \setminus \mathbb{B}_{\mathcal{A}}} \equiv 0}} w_\Lambda(\mathbf{n}), \end{aligned} \tag{2.5}$$

where $x \triangle y$ is an abbreviation for the symmetric difference $\{x\} \triangle \{y\}$:

$$x \triangle y \equiv \{x\} \triangle \{y\} = \begin{cases} \varnothing & \text{if } x = y, \\ \{x, y\} & \text{otherwise.} \end{cases} \tag{2.6}$$

If x or y is in $\mathcal{A}^{\mathrm{c}} \equiv \Lambda \setminus \mathcal{A}$, then we define both sides of (2.5) to be zero. This is consistent with the above representation when $x \neq y$, since, for example, if $x \in \mathcal{A}^{\mathrm{c}}$, then the leftmost expression of (2.5) is a multiple of $\frac{1}{2} \sum_{\varphi_x = \pm 1} \varphi_x = 0$, while the last expression in (2.5) is also zero because there is no way of connecting x and y on a current configuration $\mathbf{n}$ with $\mathbf{n}|_{\mathbb{B}_\Lambda \setminus \mathbb{B}_{\mathcal{A}}} \equiv 0$.

The key observation in the representation (2.5) is that the right-hand side is nonzero only when x and y are connected by a chain of bonds with *odd* currents (see Figure 1). We will exploit this peculiar underlying percolation picture to derive the lace expansion for the two-point function.

2.2 Derivation of the lace expansion

In this subsection, we derive the lace expansion for $\langle \varphi_o \varphi_x \rangle_\Lambda$ using the random-current representation. In Section 2.2.1, we introduce some definitions and perform the first stage of the expansion, namely (1.11) for $j = 0$, simply using inclusion-exclusion. In Section 2.2.2, we perform the second stage of the expansion, where the source-switching lemma (Lemma 2.3) plays a significant role to carry on the expansion indefinitely. Finally, in Section 2.2.3, we complete the proof of Proposition 1.1.

2.2.1 The first stage of the expansion

As mentioned in Section 2.1, the underlying picture in the random-current representation is quite similar to percolation. We exploit this similarity to obtain the lace expansion.

First, we introduce some notions and notation.

Definition 2.1. (i) Given $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$ and $\mathcal{A} \subset \Lambda$, we say that x is $\mathbf{n}$-connected to y in (the graph) $\mathcal{A}$, and simply write $x \underset{\mathbf{n}}{\longleftrightarrow} y$ *in* $\mathcal{A}$, if either $x = y \in \mathcal{A}$ or there is a self-avoiding path (or we simply call it a path) from x to y consisting of bonds $b \in \mathbb{B}_\mathcal{A}$ with $n_b > 0$. If $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\mathcal{A}}$, we omit "in $\mathcal{A}$" and simply write $x \underset{\mathbf{n}}{\longleftrightarrow} y$. We also define

$$\{x \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} y\} = \{x \underset{\mathbf{n}}{\longleftrightarrow} y\} \setminus \{x \underset{\mathbf{n}}{\longleftrightarrow} y \text{ in } \mathcal{A}^{\rm c}\}, \tag{2.7}$$

and say that x is $\mathbf{n}$-connected to y *through* $\mathcal{A}$.

(ii) Given an event E (i.e., a set of current configurations) and a bond b, we define $\{E \text{ off } b\}$ to be the set of current configurations $\mathbf{n} \in E$ such that changing n_b results in a configuration that is also in E. Let $\mathcal{C}^b_\mathbf{n}(x) = \{y : x \underset{\mathbf{n}}{\longleftrightarrow} y \text{ off } b\}$.

(iii) For a *directed* bond $b = (u, v)$, we write $\underline{b} = u$ and $\overline{b} = v$. We say that a directed bond b is *pivotal* for $x \underset{\mathbf{n}}{\longleftrightarrow} y$ from x, if $\{x \underset{\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ off } b\} \cap \{\overline{b} \underset{\mathbf{n}}{\longleftrightarrow} y \text{ in } \mathcal{C}^b_\mathbf{n}(x)^{\rm c}\}$ occurs. If $\{x \underset{\mathbf{n}}{\longleftrightarrow} y\}$ occurs with no pivotal bonds, we say that x is $\mathbf{n}$-*doubly connected to* y, and write $x \underset{\mathbf{n}}{\Longleftrightarrow} y$.

We begin with the first stage of the lace expansion. First, by using the above percolation language, the two-point function can be written as

$$\langle \varphi_o \varphi_x \rangle_\Lambda = \sum_{\partial \mathbf{n} = o \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \equiv \sum_{\partial \mathbf{n} = o \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\longleftrightarrow} x\}}. \tag{2.8}$$

We decompose the indicator on the right-hand side into two parts depending on whether or not there is a pivotal bond for $o \underset{\mathbf{n}}{\longleftrightarrow} x$ from o; if there is, we take the *first* bond among them. Then, we have

$$\mathbb{1}_{\{o \underset{\mathbf{n}}{\longleftrightarrow} x\}} = \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}} + \sum_{b \in \mathbb{B}_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}} \mathbb{1}_{\{n_b > 0\}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{n}}{\longleftrightarrow} x \text{ in } \mathcal{C}^b_\mathbf{n}(o)^{\rm c}\}}. \tag{2.9}$$

Let

$$\pi^{(0)}_\Lambda(x) = \sum_{\partial \mathbf{n} = o \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}. \tag{2.10}$$

Substituting (2.9) into (2.8), we obtain (see Figure 2)

$$\langle \varphi_o \varphi_x \rangle_\Lambda = \pi^{(0)}_\Lambda(x) + \sum_{b \in \mathbb{B}_\Lambda} \sum_{\partial \mathbf{n} = o \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}} \mathbb{1}_{\{n_b > 0\}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{n}}{\longleftrightarrow} x \text{ in } \mathcal{C}^b_\mathbf{n}(o)^{\rm c}\}}. \tag{2.11}$$

Next, we consider the sum over $\mathbf{n}$ in (2.11). Since b is pivotal for $o \underset{\mathbf{n}}{\longleftrightarrow} x$ from $o\,(\neq x$, due to the last indicator) and $\partial \mathbf{n} = o \triangle x$, in fact n_b is an *odd* integer. We alternate the parity of n_b by changing the source constraint into $o \triangle b \triangle x \equiv \{o\} \triangle \{\underline{b}, \overline{b}\} \triangle \{x\}$ and multiplying by

$$\frac{\sum_{n \text{ odd}} (pJ_b)^n/n!}{\sum_{n \text{ even}} (pJ_b)^n/n!} = \tanh(pJ_b) \equiv \tau_b. \tag{2.12}$$

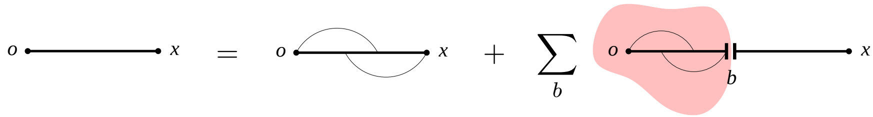


Figure 2: A schematic representation of (2.11). The thick lines are connections consisting of bonds with odd currents, while the thin arcs are connections made of bonds with positive (not necessarily odd) currents. The shaded region represents $\mathcal{C}^b_{\mathbf{n}}(o)$.

Then, the sum over $\mathbf{n}$ in (2.11) equals

$$\sum_{\partial\mathbf{n}=o\triangle b\triangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}\text{ off }b\}}\,\tau_b\mathbb{1}_{\{n_b\text{ even}\}}\,\mathbb{1}_{\{\overline{b}\underset{\mathbf{n}}{\longleftrightarrow}x\text{ in }\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}\}}. \tag{2.13}$$

Note that, except for b, there are no positive currents on the boundary bonds of $\mathcal{C}^b_{\mathbf{n}}(o)$.

Now, we condition on $\mathcal{C}^b_{\mathbf{n}}(o)=\mathcal{A}$ and decouple events occurring on $\mathbb{B}_{\mathcal{A}^{\rm c}}$ from events occurring on $\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\rm c}}$, by using the following notation:

$$\tilde{w}_{\Lambda,\mathcal{A}}(\mathbf{k})=\prod_{b\in\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\rm c}}}\frac{(pJ_b)^{k_b}}{k_b!}\qquad(\mathbf{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\rm c}}}). \tag{2.14}$$

Conditioning on $\mathcal{C}^b_{\mathbf{n}}(o)=\mathcal{A}$, multiplying $Z_{\mathcal{A}^{\rm c}}/Z_{\mathcal{A}^{\rm c}}\equiv 1$ (and using the notation $\mathbf{k}=\mathbf{n}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\rm c}}}$ and $\mathbf{m}=\mathbf{n}|_{\mathbb{B}_{\mathcal{A}^{\rm c}}}$) and then summing over $\mathcal{A}\subset\Lambda$, we have

$$\begin{aligned}
(2.13)&=\sum_{\mathcal{A}\subset\Lambda}\sum_{\substack{\partial\mathbf{k}=o\triangle\underline{b}\\ \partial\mathbf{m}=\overline{b}\triangle x}}\frac{\tilde{w}_{\Lambda,\mathcal{A}}(\mathbf{k})\,Z_{\mathcal{A}^{\rm c}}}{Z_\Lambda}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\mathbb{1}_{\{o\underset{\mathbf{k}}{\Longleftrightarrow}\underline{b}\text{ off }b\}\cap\{\mathcal{C}^b_{\mathbf{k}}(o)=\mathcal{A}\}}\,\tau_b\mathbb{1}_{\{k_b\text{ even}\}}\,\mathbb{1}_{\{\overline{b}\underset{\mathbf{m}}{\longleftrightarrow}x\text{ in }\mathcal{A}^{\rm c}\}}\\
&=\sum_{\mathcal{A}\subset\Lambda}\sum_{\partial\mathbf{n}=o\triangle\underline{b}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}\text{ off }b\}\cap\{\mathcal{C}^b_{\mathbf{n}}(o)=\mathcal{A}\}}\,\tau_b\mathbb{1}_{\{n_b\text{ even}\}}\underbrace{\sum_{\partial\mathbf{m}=\overline{b}\triangle x}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\mathbb{1}_{\{\overline{b}\underset{\mathbf{m}}{\longleftrightarrow}x\text{ (in }\mathcal{A}^{\rm c})\}}}_{=\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{A}^{\rm c}}}\\
&=\sum_{\partial\mathbf{n}=o\triangle\underline{b}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}\text{ off }b\}}\,\tau_b\mathbb{1}_{\{n_b\text{ even}\}}\,\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}}.
\end{aligned} \tag{2.15}$$

Furthermore, "off b" and $\mathbb{1}_{\{n_b\text{ even}\}}$ in the last line can be omitted, since $\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}\}\setminus\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}\text{ off }b\}$ and $\{\partial\mathbf{n}=o\triangle\underline{b}\}\cap\{n_b\text{ odd}\}$ are subsets of $\{\overline{b}\in\mathcal{C}^b_{\mathbf{n}}(o)\}$, on which $\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}}=0$. As a result,

$$(2.15)\;=\sum_{\partial\mathbf{n}=o\triangle\underline{b}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}\}}\,\tau_b\,\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}}. \tag{2.16}$$

By (2.11) and (2.16), we arrive at

$$\langle\varphi_o\varphi_x\rangle_\Lambda=\pi^{(0)}_\Lambda(x)+\sum_{b\in\mathbb{B}_\Lambda}\pi^{(0)}_\Lambda(\underline{b})\,\tau_b\,\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda-R^{(1)}_\Lambda(x), \tag{2.17}$$

where

$$R^{(1)}_\Lambda(x)=\sum_{b\in\mathbb{B}_\Lambda}\sum_{\partial\mathbf{n}=o\triangle\underline{b}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}\}}\,\tau_b\Big(\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda-\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}}\Big). \tag{2.18}$$

This completes the proof of (1.11) for $j=0$, with $\pi^{(0)}_\Lambda(x)$ and $R^{(1)}_\Lambda(x)$ being defined in (2.10) and (2.18), respectively.

2.2.2 The second stage of the expansion

In the next stage of the lace expansion, we further expand $R_\Lambda^{(1)}(x)$ in (2.17). To do so, we investigate the difference $\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda - \langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}_{\mathbf{n}}^b(o)^{\mathrm{c}}}$ in (2.18). First, we prove the following key proposition[2]:

Proposition 2.2. *For $v, x \in \Lambda$ and $\mathcal{A} \subset \Lambda$, we have*

$$\langle\varphi_v\varphi_x\rangle_\Lambda - \langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\mathrm{c}}} = \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{v\underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}}x\}}. \tag{2.19}$$

Therefore, $\langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\mathrm{c}}} \le \langle\varphi_v\varphi_x\rangle_\Lambda$ for the ferromagnetic case.

Proof. Since both sides of (2.19) are equal to $\mathbb{1}_{\{x\in\mathcal{A}\}}$ when $v = x$ (see below (2.6)), it suffices to prove (2.19) for $v \ne x$.

First, by using (2.3)–(2.5), we obtain

$$\begin{aligned} Z_\Lambda Z_{\mathcal{A}^{\mathrm{c}}}\Big(\langle\varphi_v\varphi_x\rangle_\Lambda - \langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\mathrm{c}}}\Big) &= \sum_{\partial\mathbf{n}=\{v,x\}} Z_{\mathcal{A}^{\mathrm{c}}}\, w_\Lambda(\mathbf{n}) - \sum_{\partial\mathbf{m}=\{v,x\}} w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})\, Z_\Lambda \\ &= \sum_{\substack{\partial\mathbf{m}=\varnothing,\,\partial\mathbf{n}=\{v,x\}\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} w_\Lambda(\mathbf{m})\, w_\Lambda(\mathbf{n}) - \sum_{\substack{\partial\mathbf{m}=\{v,x\},\,\partial\mathbf{n}=\varnothing\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} w_\Lambda(\mathbf{m})\, w_\Lambda(\mathbf{n}). \end{aligned} \tag{2.20}$$

Note that the second term is equivalent to the first term if the source constraints for $\mathbf{m}$ and $\mathbf{n}$ are exchanged.

Next, we consider the second term of (2.20), whose exact expression is

$$\sum_{\substack{\partial\mathbf{m}=\{v,x\},\,\partial\mathbf{n}=\varnothing\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} \bigg(\prod_{b\in\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}} \frac{(pJ_b)^{n_b}}{n_b!}\bigg) \prod_{b\in\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}} \frac{(pJ_b)^{m_b+n_b}}{m_b!\,n_b!} = \sum_{\partial\mathbf{N}=\{v,x\}} w_\Lambda(\mathbf{N}) \sum_{\substack{\partial\mathbf{m}=\{v,x\}\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}} \binom{N_b}{m_b}. \tag{2.21}$$

The following is a variant of the source-switching lemma [1, 13] and allows us to change the source constraints in (2.21).

Lemma 2.3 (Source-switching lemma).

$$\sum_{\substack{\partial\mathbf{m}=\{v,x\}\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}} \binom{N_b}{m_b} = \mathbb{1}_{\{v\underset{\mathbf{N}}{\longleftrightarrow}x \text{ in } \mathcal{A}^{\mathrm{c}}\}} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}} \binom{N_b}{m_b}. \tag{2.22}$$

The idea of the proof of (2.22) can easily be extended to more general cases, in which the source constraint in the left-hand side of (2.22) is replaced by $\partial\mathbf{m} = \mathcal{V}$ for some $\mathcal{V} \subset \Lambda$ and that in the right-hand side is replaced by $\partial\mathbf{m} = \mathcal{V} \triangle \{v, x\}$ (e.g., [1]). We will explain the proof of (2.22) after completing the proof of Proposition 2.2.

We continue with the proof of Proposition 2.2. Substituting (2.22) into (2.21), we obtain

$$\begin{aligned} (2.21) &= \sum_{\partial\mathbf{N}=\{v,x\}} w_\Lambda(\mathbf{N})\, \mathbb{1}_{\{v\underset{\mathbf{N}}{\longleftrightarrow}x \text{ in } \mathcal{A}^{\mathrm{c}}\}} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}} \binom{N_b}{m_b} \\ &= \sum_{\substack{\partial\mathbf{m}=\varnothing,\,\partial\mathbf{n}=\{v,x\}\\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}}\equiv 0}} w_\Lambda(\mathbf{m})\, w_\Lambda(\mathbf{n})\, \mathbb{1}_{\{v\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}x \text{ in } \mathcal{A}^{\mathrm{c}}\}}. \end{aligned} \tag{2.23}$$

[2]The mean-field results in [1, 2, 3, 4] are based on a couple of differential inequalities for $M_{p,h}$ and χ_p (under the periodic-boundary condition) using a certain random-walk representation. We can simplify the proof of the same differential inequalities (under the free-boundary condition as well) using Proposition 2.2.

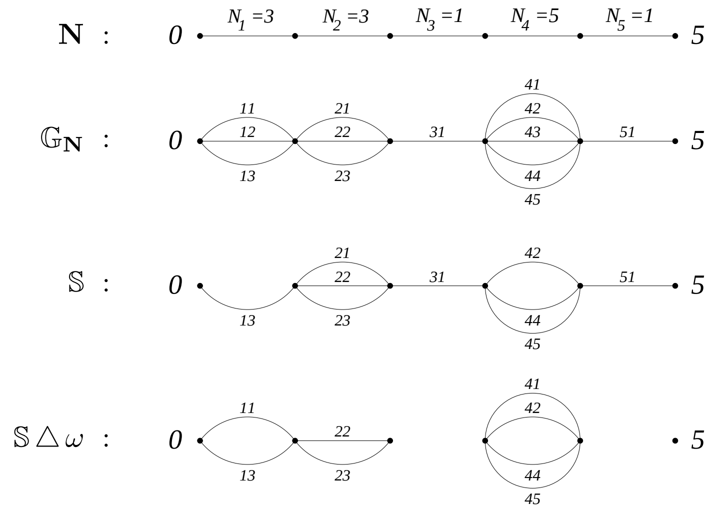


Figure 3: $\mathbf{N} = \{N_b\}_{b=1}^5 = (3,3,1,5,1)$ is an example of a current configuration on $[0,5] \cap \mathbb{Z}_+$ satisfying $\partial\mathbf{N} = \{0,5\}$, and $\mathbb{G}_{\mathbf{N}}$ is the corresponding labeled graph consisting of edges $e = b\ell_b$, where $\ell_b \in \{1,\dots,N_b\}$. The third and fourth pictures show the relation between a subgraph $\mathbb{S}$ with $\partial\mathbb{S} = \{0,5\}$ and its image $\mathbb{S}\,\triangle\,\omega$ of the map defined in (2.27), where ω is a path of edges $(11,21,31,41,51)$.

Note that the source constraints for $\mathbf{m}$ and $\mathbf{n}$ in the last line are identical to those in the first term of (2.20), under which $\mathbb{1}_{\{v \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x\}}$ is always 1. By (2.7), we can rewrite (2.20) as

$$\langle\varphi_v\varphi_x\rangle_\Lambda - \langle\varphi_v\varphi_x\rangle_{\mathcal{A}^c} = \sum_{\substack{\partial\mathbf{m}=\varnothing,\,\partial\mathbf{n}=\{v,x\} \\ \mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^c}}\equiv 0}} \frac{w_\Lambda(\mathbf{m})}{Z_{\mathcal{A}^c}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{v \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}. \tag{2.24}$$

Using (2.3)–(2.4) to omit "$\mathbf{m}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^c}} \equiv 0$" and replace $w_\Lambda(\mathbf{m})$ by $w_{\mathcal{A}^c}(\mathbf{m})$, we arrive at (2.19). This completes the proof of Proposition 2.2. ◻

Sketch proof of Lemma 2.3. We explain the meaning of the identity (2.22) and the idea of its proof. Given $\mathbf{N} = \{N_b\}_{b\in\mathbb{B}_\Lambda}$, we denote by $\mathbb{G}_{\mathbf{N}}$ the graph consisting of N_b *labeled* edges between $\underline{b}$ and $\overline{b}$ for every $b \in \mathbb{B}_\Lambda$ (see Figure 3). For a subgraph $\mathbb{S} \subset \mathbb{G}_{\mathbf{N}}$, we denote by $\partial\mathbb{S}$ the set of vertices at which the number of incident edges in $\mathbb{S}$ is *odd*, and let $\mathbb{S}_{\mathcal{A}} = \mathbb{S} \cap \mathbb{G}_{\mathbf{N}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^c}}}$. Then, the left-hand side of (2.22) equals the cardinality $|\mathfrak{S}|$ of

$$\mathfrak{S} = \{\mathbb{S} \subset \mathbb{G}_{\mathbf{N}} : \partial\mathbb{S} = \{v,x\},\ \mathbb{S}_{\mathcal{A}} = \varnothing\}, \tag{2.25}$$

and the sum in the right-hand side of (2.22) equals the cardinality $|\mathfrak{S}'|$ of

$$\mathfrak{S}' = \{\mathbb{S} \subset \mathbb{G}_{\mathbf{N}} : \partial\mathbb{S} = \varnothing,\ \mathbb{S}_{\mathcal{A}} = \varnothing\}. \tag{2.26}$$

We note that $|\mathfrak{S}|$ is zero when there are no paths on $\mathbb{G}_{\mathbf{N}}$ between v and x consisting of edges whose endvertices are both in $\mathcal{A}^c$, while $|\mathfrak{S}'|$ may not be zero. The identity (2.22) reads that $|\mathfrak{S}|$ equals $|\mathfrak{S}'|$ if we compensate for this discrepancy.

Suppose that there is a path (i.e., a) ω from v to x consisting of edges in $\mathbb{G}_{\mathbf{N}}$ whose endvertices are both in $\mathcal{A}^c$. Then, the map

$$\mathbb{S} \in \mathfrak{S} \ \mapsto\ \mathbb{S}\,\triangle\,\omega \in \mathfrak{S}', \tag{2.27}$$

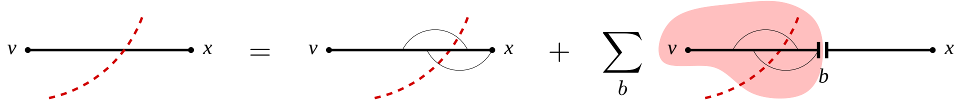


Figure 4: A schematic representation of (2.31). The dashed lines represent $\mathcal{A}$, the thick-solid lines represent connections consisting of bonds b_1 such that $m_{b_1}+n_{b_1}$ is odd, and the thin-solid lines are connections made of bonds b_2 such that $m_{b_2}+n_{b_2}$ is positive (not necessarily odd). The shaded region represents $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)$.

is a bijection [1, 13], and therefore $|\mathfrak{S}|=|\mathfrak{S}'|$. Here and in the rest of the paper, the symmetric difference between graphs is only in terms of *edges*. For example, $\mathbb{S}\,\triangle\,\omega$ is the result of adding or deleting edges (not vertices) contained in ω. This completes the proof of (2.22). $\square$

We now start with the second stage of the expansion by using Proposition 2.2 and applying inclusion-exclusion as in the first stage of the expansion in Section 2.2.1. First, we decompose the indicator in (2.19) into two parts depending on whether or not there is a pivotal bond b for $v \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x$ from v such that $v \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} \underline{b}$. Let

$$E_{\mathbf{N}}(v,x;\mathcal{A}) = \{v \overset{\mathcal{A}}{\underset{\mathbf{N}}{\longleftrightarrow}} x\} \cap \{\nexists \text{ pivotal bond } b \text{ for } v \underset{\mathbf{N}}{\longleftrightarrow} x \text{ from } v \text{ such that } v \overset{\mathcal{A}}{\underset{\mathbf{N}}{\longleftrightarrow}} \underline{b}\}. \tag{2.28}$$

On the event $\{v \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} x\} \setminus E_{\mathbf{m}+\mathbf{n}}(v,x;\mathcal{A})$, we take the *first* pivotal bond b for $v \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x$ from v satisfying $v \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} \underline{b}$. Then, we have (cf., (2.9))

$$\mathbb{1}_{\{v \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} x\}} = \mathbb{1}_{E_{\mathbf{m}+\mathbf{n}}(v,x;\mathcal{A})} + \sum_{b\in\mathbb{B}_\Lambda} \mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A}) \text{ off } b\}}\, \mathbb{1}_{\{m_b+n_b>0\}}\, \mathbb{1}_{\{\overline{b} \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x \text{ in } \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^{\mathrm{c}}\}}. \tag{2.29}$$

Let

$$\Theta_{v,x;\mathcal{A}}[X] = \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{E_{\mathbf{m}+\mathbf{n}}(v,x;\mathcal{A})}\,X(\mathbf{m}+\mathbf{n}), \qquad \Theta_{v,x;\mathcal{A}} = \Theta_{v,x;\mathcal{A}}[1]. \tag{2.30}$$

Substituting (2.29) into (2.19), we obtain (see Figure 4)

$$\begin{aligned} &\langle\varphi_v\varphi_x\rangle_\Lambda - \langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\mathrm{c}}} \\ &= \Theta_{v,x;\mathcal{A}} + \sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A}) \text{ off } b\}}\,\mathbb{1}_{\{m_b \text{ even},\ n_b \text{ odd}\}}\,\mathbb{1}_{\{\overline{b} \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x \text{ in } \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^{\mathrm{c}}\}}, \end{aligned} \tag{2.31}$$

where we have replaced "$m_b+n_b>0$" in (2.29) by "m_b even, n_b odd" that is the only possible combination consistent with the source constraints and the conditions in the indicators. As in (2.13), we alternate the parity of n_b by changing the source constraint from $\partial\mathbf{n}=v\,\triangle\,x$ to $\partial\mathbf{n}=v\,\triangle\,b\,\triangle\,x$ and multiplying by τ_b. Then, the sum over $\mathbf{m}$ and $\mathbf{n}$ in (2.31) equals

$$\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle b\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A}) \text{ off } b\}}\,\tau_b\,\mathbb{1}_{\{m_b,n_b \text{ even}\}}\,\mathbb{1}_{\{\overline{b} \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x \text{ in } \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^{\mathrm{c}}\}}. \tag{2.32}$$

Then, as in (2.15), we condition on $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v) = \mathcal{B}$ and decouple events occurring on $\mathbb{B}_{\mathcal{B}^c}$ from events occurring on $\mathbb{B}_\Lambda \setminus \mathbb{B}_{\mathcal{B}^c}$. Let $\mathbf{m}' = \mathbf{m}|_{\mathbb{B}_{\mathcal{A}^c}\setminus\mathbb{B}_{\mathcal{A}^c\cap\mathcal{B}^c}}$, $\mathbf{m}'' = \mathbf{m}|_{\mathbb{B}_{\mathcal{A}^c\cap\mathcal{B}^c}}$, $\mathbf{n}' = \mathbf{n}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{B}^c}}$ and $\mathbf{n}'' = \mathbf{n}|_{\mathbb{B}_{\mathcal{B}^c}}$. Note that $\partial\mathbf{m}' = \partial\mathbf{m}'' = \varnothing$, $\partial\mathbf{n}' = v \,\triangle\, \underline{b}$ and $\partial\mathbf{n}'' = \overline{b} \,\triangle\, x$. Multiplying (2.32) by $(Z_{\mathcal{A}^c\cap\mathcal{B}^c}/Z_{\mathcal{A}^c\cap\mathcal{B}^c})(Z_{\mathcal{B}^c}/Z_{\mathcal{B}^c}) \equiv 1$ and using the notation (2.14), we obtain

$$\begin{aligned}
(2.32) &= \sum_{\mathcal{B}\subset\Lambda} \sum_{\substack{\partial\mathbf{m}'=\varnothing\\ \partial\mathbf{n}'=v\triangle\underline{b}}} \frac{\tilde{w}_{\mathcal{A}^c,\mathcal{B}}(\mathbf{m}')\, Z_{\mathcal{A}^c\cap\mathcal{B}^c}}{Z_{\mathcal{A}^c}} \frac{\tilde{w}_{\Lambda,\mathcal{B}}(\mathbf{n}')\, Z_{\mathcal{B}^c}}{Z_\Lambda} \mathbb{1}_{\{E_{\mathbf{m}'+\mathbf{n}'}(v,\underline{b};\mathcal{A}) \text{ off } b\}\cap\{\mathcal{C}^b_{\mathbf{m}'+\mathbf{n}'}(v)=\mathcal{B}\}} \\
&\qquad \times \tau_b \mathbb{1}_{\{m'_b,n'_b \text{ even}\}} \sum_{\substack{\partial\mathbf{m}''=\varnothing\\ \partial\mathbf{n}''=\overline{b}\triangle x}} \frac{w_{\mathcal{A}^c\cap\mathcal{B}^c}(\mathbf{m}'')}{Z_{\mathcal{A}^c\cap\mathcal{B}^c}} \frac{w_{\mathcal{B}^c}(\mathbf{n}'')}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{m}''+\mathbf{n}''}{\longleftrightarrow} x \text{ in } \mathcal{B}^c\}} \\
&= \sum_{\mathcal{B}\subset\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle\underline{b}}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A}) \text{ off } b\}\cap\{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)=\mathcal{B}\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}} \langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{B}^c} \\
&= \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle\underline{b}}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A}) \text{ off } b\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}} \langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^c},
\end{aligned} \tag{2.33}$$

where we have been able to perform the sum over $\mathbf{m}''$ and $\mathbf{n}''$ independently, due to the fact that $\mathbb{1}_{\{\overline{b} \underset{\mathbf{m}''+\mathbf{n}''}{\longleftrightarrow} x \text{ in } \mathcal{B}^c\}} \equiv 1$ for any $\mathbf{n}'' \in \mathbb{Z}_+^{\mathbb{B}_{\mathcal{B}^c}}$ with $\partial\mathbf{n}'' = \overline{b} \,\triangle\, x$. As in the derivation of (2.16) from (2.15), we can omit "off b" and $\mathbb{1}_{\{m_b,n_b \text{ even}\}}$ in (2.33) using the source constraints and the fact that $\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^c} = 0$ whenever $\overline{b} \in \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)$. Therefore,

$$(2.33) \;=\; \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle\underline{b}}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A})} \tau_b \langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^c}. \tag{2.34}$$

By (2.30)–(2.34), we arrive at

$$\begin{aligned}
\langle\varphi_v\varphi_x\rangle_\Lambda - \langle\varphi_v\varphi_x\rangle_{\mathcal{A}^c} = \Theta_{v,x;\mathcal{A}} &+ \sum_{b\in\mathbb{B}_\Lambda} \Theta_{v,\underline{b};\mathcal{A}}\, \tau_b \langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda \\
&- \sum_{b\in\mathbb{B}_\Lambda} \Theta_{v,\underline{b};\mathcal{A}} \Big[\tau_b\Big(\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda - \langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b(v)^c}\Big)\Big],
\end{aligned} \tag{2.35}$$

where $\mathcal{C}^b(v) \equiv \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)$ is a variable for the operation $\Theta_{v,\underline{b};\mathcal{A}}$. This completes the second stage of the expansion.

2.2.3 Completion of the lace expansion

For notational convenience, we define $w_\varnothing(\mathbf{m})/Z_\varnothing = \mathbb{1}_{\{\mathbf{m}\equiv 0\}}$. Since $E_\mathbf{n}(o,x;\Lambda) = \{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}$ (cf., (2.28)), we can write

$$\pi^{(0)}_\Lambda(x) = \Theta_{o,x;\Lambda}. \tag{2.36}$$

Also, we can write $R^{(1)}_\Lambda(x)$ in (2.18) as

$$R^{(1)}_\Lambda(x) = \sum_b \Theta_{o,\underline{b};\Lambda} \Big[\tau_b\Big(\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda - \langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b(o)^c}\Big)\Big]. \tag{2.37}$$

Using (2.35), we obtain

$$R_\Lambda^{(1)}(x)=\sum_b\bigg(\Theta_{o,\underline{b};\Lambda}\Big[\tau_b\,\Theta_{\overline{b},x;\mathcal{C}^b(o)}\Big]+\sum_{b'}\Theta_{o,\underline{b};\Lambda}\Big[\tau_b\,\Theta_{\overline{b},\underline{b}';\mathcal{C}^b(o)}\Big]\,\tau_{b'}\langle\varphi_{\overline{b}'}\varphi_x\rangle_\Lambda$$
$$-\sum_{b'}\Theta_{o,\underline{b};\Lambda}\Big[\tau_b\,\Theta_{\overline{b},\underline{b}';\mathcal{C}^b(o)}\Big[\tau_{b'}\Big(\langle\varphi_{\overline{b}'}\varphi_x\rangle_\Lambda-\langle\varphi_{\overline{b}'}\varphi_x\rangle_{\mathcal{C}^{b'}(\overline{b})^{\rm c}}\Big)\Big]\Big]\bigg),\tag{2.38}$$

where $\mathcal{C}^b(o)\equiv\mathcal{C}^b_{\mathbf{n}}(o)$ is a variable for the outer operation $\Theta_{o,\underline{b};\Lambda}$, and $\mathcal{C}^{b'}(\overline{b})\equiv\mathcal{C}^{b'}_{\mathbf{m}'+\mathbf{n}'}(\overline{b})$ is a variable for the inner operation $\Theta_{\overline{b},\underline{b}';\mathcal{C}^b(o)}$. For $j\ge1$, we define

$$\pi_\Lambda^{(j)}(x)=\sum_{b_1,\dots,b_j}\Theta^{(0)}_{o,\underline{b}_1;\Lambda}\Big[\tau_{b_1}\Theta^{(1)}_{\overline{b}_1,\underline{b}_2;\tilde{\mathcal{C}}_0}\Big[\cdots\tau_{b_{j-1}}\Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Big[\tau_{b_j}\Theta^{(j)}_{\overline{b}_j,x;\tilde{\mathcal{C}}_{j-1}}\Big]\cdots\Big]\Big],\tag{2.39}$$

$$R_\Lambda^{(j)}(x)=\sum_{b_1,\dots,b_j}\Theta^{(0)}_{o,\underline{b}_1;\Lambda}\Big[\tau_{b_1}\Theta^{(1)}_{\overline{b}_1,\underline{b}_2;\tilde{\mathcal{C}}_0}\Big[\cdots\tau_{b_{j-1}}\Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Big[\tau_{b_j}\Big(\langle\varphi_{\overline{b}_j}\varphi_x\rangle_\Lambda-\langle\varphi_{\overline{b}_j}\varphi_x\rangle_{\tilde{\mathcal{C}}^{\rm c}_{j-1}}\Big)\Big]\cdots\Big]\Big],\tag{2.40}$$

where the operation $\Theta^{(i)}$ determines the variable $\tilde{\mathcal{C}}_i=\mathcal{C}^{b_{i+1}}_{\mathbf{m}_i+\mathbf{n}_i}(\overline{b}_i)$ (provided that $\overline{b}_0=o$). Then, we can rewrite (2.38) as

$$R_\Lambda^{(1)}(x)=\pi_\Lambda^{(1)}(x)+\sum_{b'}\pi_\Lambda^{(1)}(\underline{b}')\,\tau_{b'}\langle\varphi_{\overline{b}'}\varphi_x\rangle_\Lambda-R_\Lambda^{(2)}(x).\tag{2.41}$$

As a result,

$$\langle\varphi_o\varphi_x\rangle_\Lambda=\big(\pi_\Lambda^{(0)}(x)-\pi_\Lambda^{(1)}(x)\big)+\sum_b\big(\pi_\Lambda^{(0)}(\underline{b})-\pi_\Lambda^{(1)}(\underline{b})\big)\,\tau_b\,\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda+R_\Lambda^{(2)}(x).\tag{2.42}$$

By repeated applications of (2.35) to the remainder $R_\Lambda^{(j)}(x)$, we obtain (1.11)–(1.12) in Proposition 1.1.

For the ferromagnetic case, τ_b and $w_{\mathcal{A}}(\mathbf{n})$ for any $\mathcal{A}\subset\Lambda$ and $\mathbf{n}\in\mathbb{Z}_+^{\mathbb{B}_{\mathcal{A}}}$ are nonnegative. This proves the first inequality in (1.13) and, with the help of Proposition 2.2, the nonnegativity of $R_\Lambda^{(j+1)}(x)$. To prove the upper bound on $R_\Lambda^{(j+1)}(x)$, we simply ignore $\langle\varphi_{\overline{b}_j}\varphi_x\rangle_{\tilde{\mathcal{C}}^{\rm c}_{j-1}}$ in (2.40) and replace j by $j+1$, where $b_{j+1}=\{u,v\}$. This completes the proof of Proposition 1.1. □

2.3 Comparison to percolation

Since we have exploited the underlying percolation picture to derive the lace expansion (1.11) for the Ising model, it is not so surprising that the expansion coefficients (2.36) and (2.39) (also recall (2.30)) are quite similar to the lace-expansion coefficients for unoriented bond-percolation (cf., [17]):

$$\pi_p^{(j)}(x)=\begin{cases}\mathbb{E}_p^{(0)}\big[\mathbb{1}_{\{o\underset{\mathbf{n}_0}{\Longleftrightarrow}x\}}\big]\equiv\mathbb{P}_p(o\Longleftrightarrow x) & (j=0),\\ \displaystyle\sum_{b_1,\dots,b_j}\mathbb{E}_p^{(0)}\Big[\mathbb{1}_{\{o\underset{\mathbf{n}_0}{\Longleftrightarrow}\underline{b}_1\}}\,p_{b_1}\mathbb{E}_p^{(1)}\Big[\mathbb{1}_{E_{\mathbf{n}_1}(\overline{b}_1,\underline{b}_2;\tilde{\mathcal{C}}_0)}\cdots p_{b_j}\mathbb{E}_p^{(j)}\Big[\mathbb{1}_{E_{\mathbf{n}_j}(\overline{b}_j,x;\tilde{\mathcal{C}}_{j-1})}\Big]\cdots\Big]\Big] & (j\ge1),\end{cases}\tag{2.43}$$

where $p\equiv\sum_x p_{o,x}$ is the bond-occupation parameter, and each $\mathbb{E}_p^{(i)}$ denotes the expectation with respect to the product measure $\prod_b(p_b\mathbb{1}_{\{\mathbf{n}_i|_b=1\}}+(1-p_b)\mathbb{1}_{\{\mathbf{n}_i|_b=0\}})$. In particular, the events involved in (2.36) and (2.39) are identical to those in (2.43).

Hoever, there are significant differences between these two models. The major differences are the following:

(a) Each current configuration must satisfy not only the conditions in the indicators, but also its source constraint that is absent in percolation.

(b) An operation Θ is not an expectation, since the source constraints in the numerator and denominator of Θ in (2.30) are different.

(c) In each $\Theta^{(i)}$ for $i \geq 1$, the sum $\mathbf{m}_i + \mathbf{n}_i$ of two current configurations is coupled with $\mathbf{m}_{i-1} + \mathbf{n}_{i-1}$ via the cluster $\tilde{\mathcal{C}}_{i-1}$ determined by $\mathbf{m}_{i-1} + \mathbf{n}_{i-1}$. By contrast, in each $\mathbb{E}_p^{(i)}$ in (2.43), a single percolation configuration $\mathbf{n}_i$ is coupled with $\mathbf{n}_{i-1}$ via $\tilde{\mathcal{C}}_{i-1} = \mathcal{C}^{b_i}_{\mathbf{n}_{i-1}}(\overline{b}_{i-1})$. In addition, $\mathbf{m}_i$ is nonzero only on bonds in $\mathbb{B}_{\tilde{\mathcal{C}}^{\rm c}_{i-1}}$, while the current configuration $\mathbf{n}_i$ has no such restriction.

These elements are responsible for the difference in the method of bounding diagrams for the expansion coefficients. Take the $0^{\rm th}$-expansion coefficient for example. For percolation, the BK inequality simply tells us that

$$\pi_p^{(0)}(x) \leq \mathbb{P}_p(o \longleftrightarrow x)^2. \tag{2.44}$$

For the ferromagnetic Ising model, on the other hand, we first recall (2.10), i.e.,

$$\pi_\Lambda^{(0)}(x) = \sum_{\partial \mathbf{n} = o \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}, \tag{2.45}$$

where $w_\Lambda(\mathbf{n})/Z_\Lambda \geq 0$. Due to the indicator, every current configuration $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$ that gives nonzero contribution has at least *two bond-disjoint* paths ζ_1, ζ_2 from o to x such that $n_b > 0$ for all $b \in \zeta_1 \dot{\cup} \zeta_2$. Also, due to the source constraint, there should be at least one path ζ from o to x such that n_b is odd for all $b \in \zeta$. Suppose, for example, that $\zeta = \zeta_1$ and that n_b for $b \in \zeta_2$ are all positive-even. Since a positive-even integer can split into two odd integers, on the labeled graph $\mathbb{G}_\mathbf{n}$ with $\partial \mathbb{G}_\mathbf{n} = o \triangle x$ (recall the notation introduced above (2.25)) there are at least *three edge-disjoint* paths from o to x. This observation naturally leads us to expect that

$$\pi_\Lambda^{(0)}(x) \leq \langle \varphi_o \varphi_x \rangle_\Lambda^3 \tag{2.46}$$

holds for the ferromagnetic Ising model. This naive argument to justify (2.46) will be made rigorous in Section 4 by taking account of partition functions.

The higher-order expansion coefficients are more involved, due to the above item (c). This will also be explained in detail in Section 4.

3 Bounds on $\Pi_\Lambda^{(j)}(x)$ for the ferromagnetic models

From now on, we restrict ourselves to the ferromagnetic models. In this section, we explain how to prove Proposition 1.2 assuming a few other propositions (Propositions 3.1–3.3 below). These propositions are results of diagrammatic bounds on the expansion coefficients in terms of two-point functions. We will show these diagrammatic bounds in Section 4.

The strategy to prove Proposition 1.2 is model-independent, and we follow the strategy in [14] for the nearest-neighbor model and that in [15] for the spread-out model. Since the latter is simpler, we first explain the strategy for the spread-out model. In the rest of this paper, we will frequently use the notation

$$|\!|\!| x |\!|\!| = |x| \vee 1. \tag{3.1}$$

We also emphasize that constants in the O-notation used below (e.g., $O(\theta_0)$ in (3.3)) are independent of $\Lambda \subset \mathbb{Z}^d$.

3.1 Strategy for the spread-out model

Using the diagrammatic bounds below in Section 4, we will prove in detail in Section 5.1 that the following proposition holds for the spread-out model:

Proposition 3.1. *Let $J_{o,x}$ be the spread-out interaction. Suppose that*

$$\tau \le 2, \qquad\qquad G(x) \le \delta_{o,x} + \theta_0 \|\!|x|\!\|^{-q} \tag{3.2}$$

hold for some $\theta_0 \in (0,\infty)$ and $q \in (\frac{d}{2}, d)$. Then, for sufficiently small θ_0 (with $\theta_0 L^{d-q}$ being bounded away from zero) and any $\Lambda \subset \mathbb{Z}^d$, we have

$$\pi_\Lambda^{(i)}(x) \le \begin{cases} O(\theta_0)^i \delta_{o,x} + O(\theta_0^3)\|\!|x|\!\|^{-3q} & (i = 0,1), \\ O(\theta_0)^i \|\!|x|\!\|^{-3q} & (i \ge 2). \end{cases} \tag{3.3}$$

The exact value of the assumed upper bound on τ in (3.2) is unimportant and can be any finite number, as long as it is independent of θ_0 and bigger than the mean-field critical point 1. We note that the exponent $3q$ in (3.3) is due to (2.46) (and diagrammatic bounds on the higher-expansion coefficients), and is replaced by $2q$ with $q \in (\frac{2d}{3}, d)$ for percolation, due to, e.g., (2.44).

Sketch proof of Proposition 1.2 for the spread-out model. We will show below that, at $p = p_c$,

$$\tau \le 2, \qquad\qquad G(x) \le \delta_{o,x} + O(L^{-2+\epsilon})\|\!|x|\!\|^{-(d-2)}, \tag{3.4}$$

for some small $\epsilon > 0$. Since τ and $G(x)$ are nondecreasing and continuous in $p \le p_c$ for the ferromagnetic models, these bounds imply (3.2) for all $p \le p_c$, with $\theta_0 = cL^{-2+\epsilon} > 0$ and $q = d-2$, where $q \in (\frac{d}{2}, d)$ if $d > 4$ and $\theta_0 L^{d-q} = cL^\epsilon > 0$. Then, by Proposition 3.1, the bound (3.3) with $\theta_0 = O(L^{-2+\epsilon})$ and $q = d-2$ holds for $d > 4$ and $\theta_0 \ll 1$ (thus $L \gg 1$). Therefore, by (1.13) with $\langle \varphi_v \varphi_x \rangle_\Lambda \le 1$,

$$0 \le R_\Lambda^{(j+1)}(x) \le \tau \sum_u \pi_\Lambda^{(j)}(u) = O(\theta_0)^j \to 0 \qquad (j \uparrow \infty), \tag{3.5}$$

and by (1.12) for $j \ge 0$,

$$|\Pi_\Lambda^{(j)}(x) - \delta_{o,x}| \le O(\theta_0)\delta_{o,x} + \frac{O(\theta_0^2)}{\|\!|x|\!\|^{3(d-2)}} = O(\theta_0)\delta_{o,x} + \frac{O(\theta_0^2)(1-\delta_{o,x})}{|x|^{d+2+\rho}}, \tag{3.6}$$

where $\rho = 2(d-4)$. This completes the proof of Proposition 1.2 for the spread-out model, assuming (3.4) at $p = p_c$.

It thus remains to show the bounds in (3.4) at $p = p_c$. These bounds are proved by adapting the model-independent bootstrapping argument in [15] (see the proof of [15, Proposition 2.2] for self-avoiding walk and percolation), together with the fact that $G(x)$ decays exponentially as $|x| \uparrow \infty$ for every $p < p_c$ [23, 30] so that $\sup_x G(x)$ is continuous in $p < p_c$ [28]. We complete the proof. □

3.2 Strategy for the nearest-neighbor model

Since $\sigma^2 = O(1)$ for short-range models, we cannot expect that θ_0 in (3.2) is small, or that Proposition 3.1 is applicable to bound the expansion coefficients in this setting.

Under this circumstance, we follow the strategy in [14]. The following is the key proposition, whose proof will be explained in Section 5.2:

Proposition 3.2. *Let $J_{o,x}$ be the nearest-neighbor or spread-out interaction, and suppose that*

$$\tau - 1 \le \theta_0, \qquad \sup_x (D * G^{*2})(x) \le \theta_0, \qquad \sup_{\substack{x\equiv(x_1,\dots,x_d)\neq o\\ l=1,\dots,d}} \bigg(\frac{x_l^2}{\sigma^2} \vee 1\bigg) G(x) \le \theta_0 \tag{3.7}$$

hold for some $\theta_0 \in (0,\infty)$. Then, for sufficiently small θ_0 and any $\Lambda \subset \mathbb{Z}^d$, we have

$$\sum_x \pi_\Lambda^{(i)}(x) \le \begin{cases} 1+O(\theta_0^2) & (i=0),\\ O(\theta_0)^i & (i\ge 1),\end{cases} \qquad \sum_x |x|^2 \pi_\Lambda^{(i)}(x) \le d\sigma^2 (i+1)^2 O(\theta_0)^{i\vee 2}. \tag{3.8}$$

Furthermore, in addition to (3.7) with $\theta_0 \ll 1$, if

$$G(x) \le \lambda_0 \| x \|^{-q} \tag{3.9}$$

holds for some $\lambda_0 \in [1,\infty)$ and $q \in (0,d)$, then we have for $i \ge 0$

$$\pi_\Lambda^{(i)}(x) \le O(\theta_0)^i \delta_{o,x} + \frac{\lambda_0^3 (i+1)^{3q+2} O(\theta_0)^{(i-2)\vee 0}}{|x|^{3q}} (1-\delta_{o,x}). \tag{3.10}$$

Sketch proof of Proposition 1.2 (primarily) for the nearest-neighbor model. First we claim that the assumed bounds in (3.7) indeed hold for any $p \le p_c$ if $d > 4$ and $\theta_0 \ll 1$, where $\theta_0 = O(d^{-1})$ for the nearest-neighbor model and $\theta_0 = O(L^{-d})$ for the spread-out model. The proof is based on the orthodox model-independent bootstrapping argument in, e.g., [24] (see also [21] for improved random-walk estimates; bootstrapping assumptions that are different from, but philosophically similar to, (3.7) are used in [20]). Therefore, (3.8) holds for $p \le p_c$ and hence ensures the existence of an infinite-volume limit $\Pi(x) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \lim_{j\uparrow\infty} \Pi_\Lambda^{(j)}(x)$ that satisfies

$$\sum_x |\Pi(x)| = 1 + O(\theta_0), \qquad \sum_x |x|^2 |\Pi(x)| = d\sigma^2 O(\theta_0^2). \tag{3.11}$$

As a byproduct, we obtain the identity in (1.21) for $\tau(p_c)$ for both models. Suppose that

$$G(x) \le \lambda_0 \| x \|^{-(d-2)} \tag{3.12}$$

holds at $p = p_c$. Then, by Proposition 3.2, we obtain (3.10) with $q = d-2$. Using this in (3.5)–(3.6), we can prove Proposition 1.2.

To complete the proof, it thus remains to show (3.12) at $p = p_c$. To show this, we use the following proposition:

Proposition 3.3. *Let*

$$\bar G^{(s)} = \sup_x |x|^s G(x), \qquad \bar W^{(t)} = \sup_x \sum_y |y|^t G(y)\, G(x-y), \tag{3.13}$$

and suppose that the bounds in (3.7) hold with $\theta_0 \ll 1$.

(i) If $\sum_x \Pi(x) = \tau^{-1}$ and $|\Pi(x)| \le O(\| x \|^{-(d+2)})$, then we have

$$G(x) \sim \frac{\sum_x \Pi(x)}{\tau \sum_x |x|^2 (D * \Pi)(x)} \frac{a_d}{|x|^{d-2}} \qquad \text{as } |x| \uparrow \infty. \tag{3.14}$$

(ii) If $\sum_x |x|^r |\Pi(x)| < \infty$ for some $r > 0$, then, for $s,t > 0$ which are not odd integers, we have

$$\begin{cases} \bar G^{(s)} < \infty & \text{if} \quad s \le r \quad \text{and} \quad s < d-2,\\ \bar W^{(t)} < \infty & \text{if} \quad t \le \lfloor r \rfloor \quad \text{and} \quad t < d-4.\end{cases} \tag{3.15}$$

(iii) If $\bar{W}^{(t)} < \infty$ for some $t \geq 0$, then $\sum_x |x|^{t+2}|\Pi(x)| < \infty$.

The above proposition is a summary of key elements in [14, Proposition 1.3 and Lemmas 1.5–1.6] that are sufficient to prove (3.12) in the current setting. The proofs of Propositions 3.3(i) and 3.3(ii) are model-independent and can be found in [14, Sections 2 and 4], respectively. The proof of Proposition 3.3(iii) is similar to that of the first statement of Proposition 3.2: (3.7) implies (3.8). We will explain this in Section 5.2.

Now we continue with the proof of (3.12). Fix $p = p_c$. Since the asymptotic behavior (3.14) is good enough for the bound (3.12), it suffices to check the assumptions of Proposition 3.3(i). The first assumption on the sum of $\Pi(x)$ is satisfied at $p = p_c$, as mentioned below (3.11). The second assumption is also satisfied if $\bar{G}^{(\frac{d+2}{3})} < \infty$, because of the second statement of Proposition 3.2: (3.9) implies (3.10). By Proposition 3.3(ii), it thus suffices to show that $\sum_x |x|^{\frac{d+2}{3}}|\Pi(x)|$ is finite if $d > 4$.

To show this, we let

$$r_0 = 2, \qquad\qquad r_{i+1} = \Big((d-2) \wedge \big(\lfloor r_i \rfloor + 2\big)\Big) - \epsilon, \tag{3.16}$$

where $0 < \epsilon \leq \frac{2}{3}(d-4)$. Note that, by this definition, r_i for $i \geq 1$ equals $((d-2) \wedge (i+3)) - \epsilon$ and increases until it reaches $d - 2 - \epsilon$. We prove below by induction that $\sum_x |x|^{r_i}|\Pi(x)|$ is finite for all $i \geq 0$. This is sufficient for the finiteness of $\sum_x |x|^{\frac{d+2}{3}}|\Pi(x)|$, since

$$\lim_{i\uparrow\infty} r_i = d - 2 - \epsilon \geq d - 2 - \tfrac{2}{3}(d-4) = \tfrac{d+2}{3}. \tag{3.17}$$

Note that, by (3.11), $\sum_x |x|^{r_0}|\Pi(x)| < \infty$. Suppose $\sum_x |x|^{r_i}|\Pi(x)| < \infty$ for some $i \geq 0$. Then, by Proposition 3.3(ii), $\bar{W}^{(t)}$ is finite for $t \in (0, \lfloor r_i \rfloor] \cap (0, d-4)$. Since $\lfloor r_0 \rfloor = 2$ and $\lfloor r_i \rfloor = (d-3) \wedge (i+2)$ for $i \geq 1$, $\bar{W}^{(T)}$ with $T = (i+2) \wedge (d-4-\epsilon)$ is finite. Then, by Proposition 3.3(iii), $\sum_x |x|^{T+2}|\Pi(x)|$ is finite. Since

$$T + 2 = (i+4) \wedge (d-2-\epsilon) \geq \big((d-2) \wedge (i+4)\big) - \epsilon = r_{i+1}, \tag{3.18}$$

we obtain that $\sum_x |x|^{r_{i+1}}|\Pi(x)| < \infty$. This completes the induction and the proof of (3.12). The proof of Proposition 1.2 is now completed. □

4 Diagrammatic bounds on $\pi_\Lambda^{(j)}(x)$

In this section, we prove diagrammatic bounds on the expansion coefficients. In Section 4.1, we construct diagrams in terms of two-point functions and state the bounds. In Section 4.2, we prove a key lemma for the diagrammatic bounds and show how to apply this lemma to prove the bound on $\pi_\Lambda^{(0)}(x)$. In Section 4.3, we prove the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \geq 1$.

4.1 Construction of diagrams

To state bounds on the expansion coefficients (as in Proposition 4.1 below), we first define diagrammatic functions consisting of two-point functions. Let

$$\tilde{G}_\Lambda(y, x) = \sum_{b:\bar{b}=x} \langle \varphi_y \varphi_{\underline{b}} \rangle_\Lambda \tau_b, \tag{4.1}$$

which satisfies[3]

$$\langle \varphi_y \varphi_x \rangle_\Lambda \leq \delta_{y,x} + \sum_{b:\bar{b}=x} \sum_{\substack{\partial\mathbf{n}=y\triangle x \\ n_b \text{ odd}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} = \delta_{y,x} + \sum_{b:\bar{b}=x} \tau_b \sum_{\substack{\partial\mathbf{n}=y\triangle \underline{b} \\ n_b \text{ even}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \leq \delta_{y,x} + \tilde{G}_\Lambda(y, x). \tag{4.2}$$

[3]Repeated applications of (4.2) to the translation-invariant models result in the random-walk bound: $\langle \varphi_o \varphi_x \rangle_\Lambda \leq S_\tau(x)$ for $\Lambda \subset \mathbb{Z}^d$ and $\tau \leq 1$.

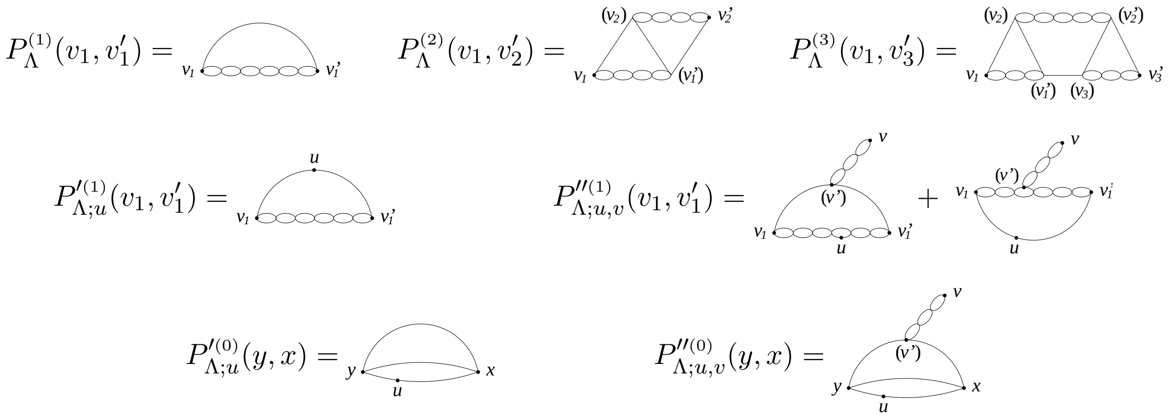


Figure 5: Schematic representations of $P_\Lambda^{(j)}(v_1,v_j')$ for $j=1,2,3$, $P'^{(1)}_{\Lambda;u}(v_1,v_1')$, $P''^{(1)}_{\Lambda;u,v}(v_1,v_1')$, $P'^{(0)}_{\Lambda;u}(y,x)$ and $P''^{(0)}_{\Lambda;u,v}(y,x)$. The labels in the parentheses represent vertices that are summed over, each sequence of bubbles from v_i and v_i' represents $\psi_\Lambda(v_i,v_i')-\delta_{v_i,v_i'}$, and the sequence of bubbles from v' to v represents $\psi_\Lambda(v',v)$.

Let

$$\psi_\Lambda(y,x)=\sum_{j=0}^\infty \big(\tilde G_\Lambda^2\big)^{*j}(y,x)\equiv\delta_{y,x}+\sum_{j=1}^\infty\sum_{\substack{u_0,\dots,u_j\\ u_0=y,\,u_j=x}}\prod_{l=1}^j\tilde G_\Lambda(u_{l-1},u_l)^2, \tag{4.3}$$

and define (see the first line in Figure 5)

$$P_\Lambda^{(1)}(v_1,v_1')=2\big(\psi_\Lambda(v_1,v_1')-\delta_{v_1,v_1'}\big)\,\langle\varphi_{v_1}\varphi_{v_1'}\rangle_\Lambda, \tag{4.4}$$

$$\begin{aligned}P_\Lambda^{(j)}(v_1,v_j')=\sum_{\substack{v_2,\dots,v_j\\ v_1',\dots,v_{j-1}'}}&\bigg(\prod_{i=1}^j\big(\psi_\Lambda(v_i,v_i')-\delta_{v_i,v_i'}\big)\bigg)\langle\varphi_{v_1}\varphi_{v_2}\rangle_\Lambda\langle\varphi_{v_2}\varphi_{v_1'}\rangle_\Lambda\\ &\times\bigg(\prod_{i=2}^{j-1}\langle\varphi_{v_{i-1}'}\varphi_{v_{i+1}}\rangle_\Lambda\langle\varphi_{v_{i+1}}\varphi_{v_i'}\rangle_\Lambda\bigg)\langle\varphi_{v_{j-1}'}\varphi_{v_j'}\rangle_\Lambda\qquad(j\ge2),\end{aligned} \tag{4.5}$$

where the empty product for $j=2$ is regarded as 1.

Next, we define $P'^{(j)}_{\Lambda;u}(v_1,v_j')$ by replacing one of the $2j-1$ two-point functions on the right-hand side of (4.4)–(4.5) by the product of *two* two-point functions, such as replacing $\langle\varphi_z\varphi_{z'}\rangle_\Lambda$ by $\langle\varphi_z\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{z'}\rangle_\Lambda$, and then summing over all $2j-1$ choices of this replacement. For example, we define (see the second line in Figure 5)

$$P'^{(1)}_{\Lambda;u}(v_1,v_1')=2\big(\psi_\Lambda(v_1,v_1')-\delta_{v_1,v_1'}\big)\langle\varphi_{v_1}\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{v_1'}\rangle_\Lambda, \tag{4.6}$$

and

$$\begin{aligned}P'^{(2)}_{\Lambda;u}(v_1,v_2')=\sum_{v_2,v_1'}\bigg(\prod_{i=1}^2\big(\psi_\Lambda(v_i,v_i')-\delta_{v_i,v_i'}\big)\bigg)\Big(&\langle\varphi_{v_1}\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{v_2}\rangle_\Lambda\langle\varphi_{v_2}\varphi_{v_1'}\rangle_\Lambda\langle\varphi_{v_1'}\varphi_{v_2'}\rangle_\Lambda\\ &+\langle\varphi_{v_1}\varphi_{v_2}\rangle_\Lambda\langle\varphi_{v_2}\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{v_1'}\rangle_\Lambda\langle\varphi_{v_1'}\varphi_{v_2'}\rangle_\Lambda\\ &+\langle\varphi_{v_1}\varphi_{v_2}\rangle_\Lambda\langle\varphi_{v_2}\varphi_{v_1'}\rangle_\Lambda\langle\varphi_{v_1'}\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{v_2'}\rangle_\Lambda\Big).\end{aligned} \tag{4.7}$$

We define $P''^{(j)}_{\Lambda;u,v}(v_1,v'_j)$ similarly as follows. First we take *two* two-point functions in $P^{(j)}_{\Lambda}(v_1,v'_j)$, one of which (say, $\langle\varphi_{z_1}\varphi_{z'_1}\rangle_\Lambda$ for some z_1,z'_1) is among the aforementioned $2j-1$ two-point functions, and the other (say, $\tilde G_\Lambda(z_2,z'_2)$ for some z_2,z'_2) is among those of which $\psi_\Lambda(v_i,v'_i)-\delta_{v_i,v'_i}$ for $i=1,\dots,j$ are composed. The product $\langle\varphi_{z_1}\varphi_{z'_1}\rangle_\Lambda\tilde G_\Lambda(z_2,z'_2)$ is then replaced by

$$\bigg(\sum_{v'}\langle\varphi_{z_1}\varphi_{v'}\rangle_\Lambda\langle\varphi_{v'}\varphi_{z'_1}\rangle_\Lambda\,\psi_\Lambda(v',v)\bigg)\Big(\langle\varphi_{z_2}\varphi_u\rangle_\Lambda\tilde G_\Lambda(u,z'_2)+\tilde G_\Lambda(z_2,z'_2)\,\delta_{u,z'_2}\Big)$$
$$+\langle\varphi_{z_1}\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{z'_1}\rangle_\Lambda\sum_{v'}\Big(\langle\varphi_{z_2}\varphi_{v'}\rangle_\Lambda\tilde G_\Lambda(v',z'_2)+\tilde G_\Lambda(z_2,z'_2)\,\delta_{v',z'_2}\Big)\,\psi_\Lambda(v',v). \tag{4.8}$$

Finally, we define $P''^{(j)}_{\Lambda;u,v}(v_1,v'_j)$ by taking account of all possible combinations of $\langle\varphi_{z_1}\varphi_{z'_1}\rangle_\Lambda$ and $\tilde G_\Lambda(z_2,z'_2)$. For example, we define $P''^{(1)}_{\Lambda;u,v}(v_1,v'_1)$ as (see Figure 5)

$$P''^{(1)}_{\Lambda;u,v}(v_1,v'_1)$$
$$=\sum_{u',u'',v'}\bigg(2\psi_\Lambda(v_1,u')\,\tilde G_\Lambda(u',u'')\Big(\langle\varphi_{u'}\varphi_u\rangle_\Lambda\tilde G_\Lambda(u,u'')+\tilde G_\Lambda(u',u'')\,\delta_{u,u''}\Big)\,\psi_\Lambda(u'',v'_1)$$
$$\times\langle\varphi_{v_1}\varphi_{v'}\rangle_\Lambda\langle\varphi_{v'}\varphi_{v'_1}\rangle_\Lambda\psi_\Lambda(v',v)+(\text{permutation of }u\text{ and }v')\bigg), \tag{4.9}$$

where the permutation term corresponds to the second term for $P''^{(1)}_{\Lambda;u,v}(v_1,v'_1)$ in Figure 5.

In addition to the above quantities, we define (see the third line in Figure 5)

$$P'^{(0)}_{\Lambda;u}(y,x)=\langle\varphi_y\varphi_x\rangle_\Lambda^2\langle\varphi_y\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_x\rangle_\Lambda, \tag{4.10}$$

$$P''^{(0)}_{\Lambda;u,v}(y,x)=\langle\varphi_y\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_x\rangle_\Lambda\sum_{v'}\langle\varphi_y\varphi_{v'}\rangle_\Lambda\langle\varphi_{v'}\varphi_x\rangle_\Lambda\,\psi_\Lambda(v',v), \tag{4.11}$$

and let

$$P'_{\Lambda;u}(y,x)=\sum_{j\ge0}P'^{(j)}_{\Lambda;u}(y,x),\qquad P''_{\Lambda;u,v}(y,x)=\sum_{j\ge0}P''^{(j)}_{\Lambda;u,v}(y,x), \tag{4.12}$$

where $P'^{(0)}_{\Lambda;u}(y,x)$ and $P''^{(0)}_{\Lambda;u,v}(y,x)$ are the leading contributions to $P'_{\Lambda;u}(y,x)$ and $P''_{\Lambda;u,v}(y,x)$, respectively.

Finally, we define

$$Q'_{\Lambda;u}(y,x)=\sum_z\big(\delta_{y,z}+\tilde G_\Lambda(y,z)\big)P'_{\Lambda;u}(z,x), \tag{4.13}$$

$$Q''_{\Lambda;u,v}(y,x)=\sum_z\big(\delta_{y,z}+\tilde G_\Lambda(y,z)\big)P''_{\Lambda;u,v}(z,x)$$
$$+\sum_{v',z}\big(\delta_{y,v'}+\tilde G_\Lambda(y,v')\big)\,\tilde G_\Lambda(v',z)\,P'_{\Lambda;u}(z,x)\,\psi_\Lambda(v',v). \tag{4.14}$$

The following are the diagrammatic bounds on the expansion coefficients (see Figure 6):

Proposition 4.1 (Diagrammatic bounds). *For the ferromagnetic Ising model, we have*

$$\pi^{(j)}_\Lambda(x)\le\begin{cases}P'^{(0)}_{\Lambda;o}(o,x)\equiv\langle\varphi_o\varphi_x\rangle_\Lambda^3 & (j=0),\\[1ex] \displaystyle\sum_{\substack{b_1,\dots,b_j\\ v_1,\dots,v_j}}P'^{(0)}_{\Lambda;v_1}(o,\underline{b}_1)\bigg(\prod_{i=1}^{j-1}\tau_{b_i}Q''_{\Lambda;v_i,v_{i+1}}(\overline{b}_i,\underline{b}_{i+1})\bigg)\,\tau_{b_j}Q'_{\Lambda;v_j}(\overline{b}_j,x) & (j\ge1),\end{cases} \tag{4.15}$$

where, as well as in the rest of the paper, the empty product is regarded as 1 by convention.

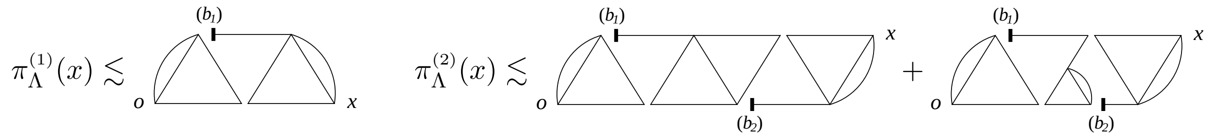


Figure 6: The leading diagrams for $\pi_\Lambda^{(1)}(x)$ and $\pi_\Lambda^{(2)}(x)$. The segments that terminate with b_i for $i = 1, 2$ represent $\delta + \tilde{G}_\Lambda$ (cf., (4.13)–(4.14)). The labels in the parentheses represent bonds that are summed over. There are artificial gaps in the figures to distinguish different building blocks.

4.2 Bound on $\pi_\Lambda^{(0)}(x)$

The key ingredient of the proof of Proposition 4.1 is Lemma 4.2 below, which is an extension of the GHS idea used in the proof of Lemma 2.3. In this subsection, we demonstrate how this extension works to prove the bound on $\pi_\Lambda^{(0)}(x)$ and the inequality

$$\sum_{\partial\mathbf{n}=o\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{o \underset{\mathbf{n}}{\longleftrightarrow} y\}} \le P'^{(0)}_{\Lambda;y}(o, x), \tag{4.16}$$

which will be used in Section 4.3 to obtain the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \ge 1$.

Proof of (4.15) for $j = 0$. Since the inequality is trivial if $x = o$, we restrict our attention to the case of $x \ne o$.

First we note that, for each current configuration $\mathbf{n}$ with $\partial\mathbf{n} = \{o, x\}$ and $\mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}} = 1$, there are at least *three edge-disjoint* paths on $\mathbb{G}_\mathbf{n}$ between o and x. See, for example, the first term on the right-hand side in Figure 2. Suppose that the thick line in that picture, referred to as ζ_1 and split into $\zeta_{11} \mathbin{\dot\cup} \zeta_{12} \mathbin{\dot\cup} \zeta_{13}$ from o to x, consists of bonds b with $n_b = 1$, and that the thin lines, referred to as ζ_2 and ζ_3 that terminate at o and x respectively, consist of bonds b' with $n_{b'} = 2$. Let ζ_i', for $i = 2, 3$, be the duplication of ζ_i. Then, the three paths $\zeta_2 \mathbin{\dot\cup} \zeta_{13}$, $\zeta_2' \mathbin{\dot\cup} \zeta_{12} \mathbin{\dot\cup} \zeta_3$ and $\zeta_{11} \mathbin{\dot\cup} \zeta_3'$ are edge-disjoint.

Then, by multiplying $\pi_\Lambda^{(0)}(x)$ by *two* dummies $(Z_\Lambda/Z_\Lambda)^2$ $(\equiv 1)$, we obtain

$$\begin{aligned}\pi_\Lambda^{(0)}(x) &= \sum_{\substack{\partial\mathbf{n}=\{o,x\}\\ \partial\mathbf{m}'=\partial\mathbf{m}''=\varnothing}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \frac{w_\Lambda(\mathbf{m}')}{Z_\Lambda} \frac{w_\Lambda(\mathbf{m}'')}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}} \\ &= \sum_{\partial\mathbf{N}=\{o,x\}} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^3} \sum_{\substack{\partial\mathbf{n}=\{o,x\}\\ \partial\mathbf{m}'=\partial\mathbf{m}''=\varnothing\\ \mathbf{N}\equiv\mathbf{n}+\mathbf{m}'+\mathbf{m}''}} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}} \prod_b \frac{N_b!}{n_b!\, m_b'!\, m_b''!},\end{aligned} \tag{4.17}$$

where the sum over $\mathbf{n}, \mathbf{m}', \mathbf{m}''$ in the second line equals the cardinality of the following set of partitions:

$$\mathfrak{S}_0 = \bigg\{(\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2) : \mathbb{G}_\mathbf{N} = \dot{\bigcup_{i=0,1,2}} \mathbb{S}_i,\ \partial\mathbb{S}_0 = \{o, x\},\ \partial\mathbb{S}_1 = \partial\mathbb{S}_2 = \varnothing,\ o \Longleftrightarrow x \text{ in } \mathbb{S}_0\bigg\}, \tag{4.18}$$

where "$o \Longleftrightarrow x$ in $\mathbb{S}_0$" means that there are at least two *bond*-disjoint paths in $\mathbb{S}_0$. We will show $|\mathfrak{S}_0| \le |\mathfrak{S}_0'|$, where

$$\mathfrak{S}_0' = \bigg\{(\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2) : \mathbb{G}_\mathbf{N} = \dot{\bigcup_{i=0,1,2}} \mathbb{S}_i,\ \partial\mathbb{S}_0 = \partial\mathbb{S}_1 = \partial\mathbb{S}_2 = \{o, x\}\bigg\}. \tag{4.19}$$

This implies (4.15) for $j=0$, because

$$|\mathfrak{S}_0'| = \sum_{\substack{\partial\mathbf{n}=\partial\mathbf{m}'=\partial\mathbf{m}''=\{o,x\}\\ \mathbf{N}\equiv\mathbf{n}+\mathbf{m}'+\mathbf{m}''}} \prod_b \frac{N_b!}{n_b!\,m_b'!\,m_b''!}, \tag{4.20}$$

and

$$\sum_{\partial\mathbf{N}=\{o,x\}} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^3} \sum_{\substack{\partial\mathbf{n}=\partial\mathbf{m}'=\partial\mathbf{m}''=\{o,x\}\\ \mathbf{N}\equiv\mathbf{n}+\mathbf{m}'+\mathbf{m}''}} \prod_b \frac{N_b!}{n_b!\,m_b'!\,m_b''!} = \bigg(\sum_{\partial\mathbf{n}=\{o,x\}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\bigg)^3. \tag{4.21}$$

It remains to show $|\mathfrak{S}_0| \le |\mathfrak{S}_0'|$. To do so, we use the following lemma, in which we denote by $\Omega^{\mathbf{N}}_{z\to z'}$ the set of paths on $\mathbb{G}_{\mathbf{N}}$ from z to z' and write $\omega\cap\omega'=\varnothing$ to mean that ω and ω' are *edge*-disjoint (not necessarily *bond*-disjoint).

Lemma 4.2. *Given a current configuration* $\mathbf{N}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}$, $k\ge1$, $\mathcal{V}\subset\Lambda$ *and* $z_i\ne z_i'\in\Lambda$ *for* $i=1,\dots,k$, *we let*

$$\mathfrak{S} = \left\{ (\mathbb{S}_0,\mathbb{S}_1,\dots,\mathbb{S}_k) : \begin{array}{r} \mathbb{G}_{\mathbf{N}} = \dot{\bigcup}_{i=0}^k \mathbb{S}_i,\ \partial\mathbb{S}_0=\mathcal{V},\ \partial\mathbb{S}_i=\varnothing\ (i=1,\dots,k),\\ {}^\exists\omega_i\in\Omega^{\mathbf{N}}_{z_i\to z_i'}\ (i=1,\dots,k) \text{ such that } \omega_i\subset\mathbb{S}_0\,\dot\cup\,\mathbb{S}_i\\ \text{and } \omega_i\cap\omega_j=\varnothing\ (i\ne j) \end{array} \right\}, \tag{4.22}$$

and define $\mathfrak{S}'$ *to be the right-hand side of (4.22) with "*$\partial\mathbb{S}_0=\mathcal{V}$, $\partial\mathbb{S}_i=\varnothing$*" being replaced by "*$\partial\mathbb{S}_0=\mathcal{V}\,\triangle\,\{z_1,z_1'\}\,\triangle\cdots\triangle\,\{z_k,z_k'\}$, $\partial\mathbb{S}_i=\{z_i,z_i'\}$*". Then,* $|\mathfrak{S}|=|\mathfrak{S}'|$.

We will prove this lemma at the end of this subsection.

Now we use Lemma 4.2 with $k=2$ and $\mathcal{V}=\{z_1,z_1'\}=\{z_2,z_2'\}=\{o,x\}$. Note that $\mathfrak{S}_0$ in (4.18) is a subset of $\mathfrak{S}$, since $\mathfrak{S}$ includes partitions $(\mathbb{S}_0,\mathbb{S}_1,\mathbb{S}_2)$ in which there does not exist two *bond*-disjoint paths on $\mathbb{S}_0$. In addition, $\mathfrak{S}'$ is trivially a subset of $\mathfrak{S}_0'$ in (4.19). Therefore, we have $|\mathfrak{S}_0|\le|\mathfrak{S}_0'|$. This completes the proof of (4.15) for $j=0$. ☐

Here, we summarize the basic steps that we have followed to bound $\pi_\Lambda^{(0)}(x)$ and which we generalize to prove (4.16) below and the bounds on $\pi_\Lambda^{(j)}(x)$ for $j\ge1$ in Section 4.3.2.

(i) Count the (minimum) number, say, $k+1$, of *edge-disjoint* paths on $\mathbb{G}_{\mathbf{n}}$ that satisfy the source constraint (as well as other additional conditions, if there are) of the considered function $f(x)$. For example, $k=2$ for $\pi_\Lambda^{(0)}(x)\equiv\frac{1}{Z_\Lambda}\sum_{\partial\mathbf{n}=\{o,x\}} w_\Lambda(\mathbf{n})\,\mathbb{1}_{\{o\overset{}{\underset{\mathbf{n}}{\Longleftrightarrow}}x\}}$.

(ii) Multiply $f(x)$ by $(\frac{Z_\Lambda}{Z_\Lambda})^k=\prod_{i=1}^k(\frac{1}{Z_\Lambda}\sum_{\partial\mathbf{m}_i=\varnothing} w_\Lambda(\mathbf{m}_i))\ (\equiv1)$ and then overlap the k dummies $\mathbf{m}_1,\dots,\mathbf{m}_k$ on the original current configuration $\mathbf{n}$. Choose k paths $\omega_1,\dots,\omega_k$ among $k+1$ edge-disjoint paths on $\mathbb{G}_{\mathbf{n}+\sum_{i=1}^k\mathbf{m}_i}$.

(iii) Use Lemma 4.2 to exchange the occupation status of edges on ω_i between $\mathbb{G}_{\mathbf{n}}$ and $\mathbb{G}_{\mathbf{m}_i}$ for every $i=1,\dots,k$. The current configurations after the mapping, denoted by $\tilde{\mathbf{n}},\tilde{\mathbf{m}}_1,\dots,\tilde{\mathbf{m}}_k$, satisfy $\partial\tilde{\mathbf{n}}=\partial\mathbf{n}\,\triangle\,\partial\omega_1\,\triangle\cdots\triangle\,\partial\omega_k$ and $\partial\tilde{\mathbf{m}}_i=\partial\omega_i$ for $i=1,\dots,k$.

Proof of (4.16). If $y=o$ or x, then (4.16) is reduced to the inequality for $\pi_\Lambda^{(0)}(x)$. Also, if $y\ne o=x$, then the left-hand side of (4.16) multiplied by $Z_\Lambda/Z_\Lambda=\sum_{\partial\mathbf{m}=\varnothing} w_\Lambda(\mathbf{m})/Z_\Lambda\equiv1$ equals

$$\begin{aligned}\sum_{\partial\mathbf{n}=\partial\mathbf{m}=\varnothing} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\frac{w_\Lambda(\mathbf{m})}{Z_\Lambda}\,\mathbb{1}_{\{o\underset{\mathbf{n}}{\longleftrightarrow}y\}} &\le \sum_{\partial\mathbf{n}=\partial\mathbf{m}=\varnothing} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\frac{w_\Lambda(\mathbf{m})}{Z_\Lambda}\,\mathbb{1}_{\{o\underset{\mathbf{n}+\mathbf{m}}{\longleftrightarrow}y\}}\\ &= \sum_{\partial\mathbf{n}=\partial\mathbf{m}=\{o,y\}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\frac{w_\Lambda(\mathbf{m})}{Z_\Lambda} = \langle\varphi_o\varphi_y\rangle_\Lambda^2, \end{aligned}\tag{4.23}$$

where the first equality is due to Lemma 2.3. Therefore, we can assume $o \neq x \neq y \neq o$.

We follow the three steps described above.

(i) Since $y \notin \partial\mathbf{n} = \{o, x\}$ and $\mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{o \underset{\mathbf{n}}{\longleftrightarrow} y\}} = 1$, it is not hard to see that there is an *edge*-disjoint cycle (closed path) $o \to y \to x \to o$. Since a cycle does not have a source, there must be another edge-disjoint connection from o to x, due to the source constraint $\partial\mathbf{n} = \{o, x\}$. Therefore, there are at least $4\,(= k+1)$ edge-disjoint paths on $\mathbb{G}_{\mathbf{n}}$: one is between o and y, another is between y and x, and the other two are between o and x.

(ii) Multiplying both sides of (4.16) by $(Z_\Lambda/Z_\Lambda)^3$ is equivalent to

$$\begin{aligned}
&\sum_{\partial\mathbf{N}=\{o,x\}} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^4} \sum_{\substack{\partial\mathbf{n}=\{o,x\} \\ \partial\mathbf{m}_i=\varnothing\ \forall i=1,2,3 \\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^3 \mathbf{m}_i}} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{o \underset{\mathbf{n}}{\longleftrightarrow} y\}} \prod_b \frac{N_b!}{n_b!\, m_b^{(1)}!\, m_b^{(2)}!\, m_b^{(3)}!} \\
&\leq \sum_{\partial\mathbf{N}=\{o,x\}} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^4} \sum_{\substack{\partial\mathbf{n}=\partial\mathbf{m}_3=\{o,x\} \\ \partial\mathbf{m}_1=\{o,y\},\ \partial\mathbf{m}_2=\{y,x\} \\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^3 \mathbf{m}_i}} \prod_b \frac{N_b!}{n_b!\, m_b^{(1)}!\, m_b^{(2)}!\, m_b^{(3)}!},
\end{aligned} \tag{4.24}$$

where we have used the notation $m_b^{(i)} = \mathbf{m}_i|_b$. Note that the second sum on the left-hand side equals the cardinality of

$$\left\{ (\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3) : \begin{array}{c} \mathbb{G}_{\mathbf{N}} = \dot{\bigcup}_{i=0}^3 \mathbb{S}_i,\ \partial\mathbb{S}_0 = \{o, x\},\ \partial\mathbb{S}_1 = \partial\mathbb{S}_2 = \partial\mathbb{S}_3 = \varnothing \\ o \Longleftrightarrow x \text{ in } \mathbb{S}_0,\ o \longleftrightarrow y \text{ in } \mathbb{S}_0 \end{array} \right\}, \tag{4.25}$$

and the second sum on the right-hand side of (4.24) equals the cardinality of

$$\Big\{ (\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3) : \mathbb{G}_{\mathbf{N}} = \dot{\bigcup}_{i=0}^3 \mathbb{S}_i,\ \partial\mathbb{S}_0 = \partial\mathbb{S}_3 = \{o, x\},\ \partial\mathbb{S}_1 = \{o, y\},\ \partial\mathbb{S}_2 = \{y, x\} \Big\}. \tag{4.26}$$

Therefore, to prove (4.24), it is sufficient to show that the cardinality of (4.25) is not bigger than that of (4.26).

(iii) Now we use Lemma 4.2 with $k = 3$ and $\mathcal{V} = \{z_3, z_3'\} = \{o, x\}$, $\{z_1, z_1'\} = \{o, y\}$ and $\{z_2, z_2'\} = \{y, x\}$. Since (4.25) is a subset of $\mathfrak{S}$ in the current setting, while $\mathfrak{S}'$ is a subset of (4.26), we obtain (4.24). This completes the proof of (4.16). ☐

Proof of Lemma 4.2. We prove Lemma 4.2 by decomposing $\mathfrak{S}^{(\prime)}$ into $\dot{\bigcup}_{\vec{\omega}_k} \mathfrak{S}_{\vec{\omega}_k}^{(\prime)}$ (described in detail below) and then constructing a bijection from $\mathfrak{S}_{\vec{\omega}_k}$ to $\mathfrak{S}'_{\vec{\omega}_k}$ for every $\vec{\omega}_k$. To do so, we first introduce some notation.

1. For every $i = 1, \ldots, k$, we introduce an arbitrarily fixed order among elements in $\Omega^{\mathbf{N}}_{z_i \to z_i'}$. For $\omega, \omega' \in \Omega^{\mathbf{N}}_{z_i \to z_i'}$, we write $\omega \prec \omega'$ if ω is earlier than ω' in this order. Let $\tilde{\Omega}^{\mathbf{N}}_{z_1 \to z_1'}$ be the set of paths $\zeta \in \Omega^{\mathbf{N}}_{z_1 \to z_1'}$ such that there are $k-1$ edge-disjoint paths on $\mathbb{G}_{\mathbf{N}} \setminus \zeta$ (= the resulting graph by removing the edges in ζ) each of which connects z_i and z_i' for every $i = 2, \ldots, k$.

2. Then, for $\omega_1 \in \tilde{\Omega}^{\mathbf{N}}_{z_1 \to z_1'}$, we define $\Xi^{\mathbf{N};\omega_1}_{z_2 \to z_2'}$ to be the set of paths $\zeta \in \Omega^{\mathbf{N}}_{z_2 \to z_2'}$ on $\mathbb{G}_{\mathbf{N}} \setminus \omega_1$ such that $\zeta \not\supset \xi$ for any $\xi \in \tilde{\Omega}^{\mathbf{N}}_{z_1 \to z_1'}$ earlier than ω_1. Then, we define $\tilde{\Omega}^{\mathbf{N};\omega_1}_{z_2 \to z_2'}$ to be the set of paths $\zeta \in \Xi^{\mathbf{N};\omega_1}_{z_2 \to z_2'}$ such that there are $k-2$ edge-disjoint paths on $\mathbb{G}_{\mathbf{N}} \setminus (\omega_1 \,\dot{\cup}\, \zeta)$ each of which is from z_i to z_i' for $i = 3, \ldots, k$.

3. More generally, for $l < k$ and $\vec{\omega}_l = (\omega_1, \dots, \omega_l)$ with $\omega_1 \in \tilde{\Omega}^{\mathbf{N}}_{z_1 \to z'_1}$, $\omega_2 \in \tilde{\Omega}^{\mathbf{N};\omega_1}_{z_2 \to z'_2}, \dots, \omega_l \in \tilde{\Omega}^{\mathbf{N};\vec{\omega}_{l-1}}_{z_l \to z'_l}$, we define $\Xi^{\mathbf{N};\vec{\omega}_l}_{z_{l+1} \to z'_{l+1}}$ to be the set of paths $\zeta \in \Omega^{\mathbf{N}}_{z_{l+1} \to z'_{l+1}}$ on $\mathbb{G}_{\mathbf{N}} \setminus \dot{\bigcup}_{i=1}^l \omega_i$ such that $\zeta \not\supset \xi$ for any $\xi \in \tilde{\Omega}^{\mathbf{N};\vec{\omega}_{i-1}}_{z_i \to z'_i}$ earlier than ω_i, for every $i = 1, \dots, l$. Then, we define $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_l}_{z_{l+1} \to z'_{l+1}}$ to be the set of paths $\zeta \in \Xi^{\mathbf{N};\vec{\omega}_l}_{z_{l+1} \to z'_{l+1}}$ such that there are $k - (l+1)$ edge-disjoint paths on $\mathbb{G}_{\mathbf{N}} \setminus (\dot{\bigcup}_{i=1}^l \omega_i \dot\cup \zeta)$ each of which is from z_i to z'_i for $i = l+2, \dots, k$.

4. If $l = k-1$, then we simply define $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_{k-1}}_{z_k \to z'_k} = \Xi^{\mathbf{N};\vec{\omega}_{k-1}}_{z_k \to z'_k}$. We will also abuse the notation to denote $\tilde{\Omega}^{\mathbf{N}}_{z_1 \to z'_1}$ by $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_0}_{z_1 \to z'_1}$.

Using the above notation, we can decompose $\mathfrak{S}^{(\prime)}$ disjointly as follows. For a collection $\omega_i \in \tilde{\Omega}^{\mathbf{N};\vec{\omega}_{i-1}}_{z_i \to z'_i}$ for $i = 1, \dots, k$, we denote by $\mathfrak{S}^{(\prime)}_{\vec{\omega}_k}$ the set of partitions $\vec{\mathbb{S}}_k \equiv (\mathbb{S}_0, \mathbb{S}_1, \dots, \mathbb{S}_k) \in \mathfrak{S}^{(\prime)}$ such that, for every $i = 1, \dots, k$, the earliest element of $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_{i-1}}_{z_i \to z'_i}$ contained in $\mathbb{S}_0 \dot\cup \mathbb{S}_i$ is ω_i. Then, $\mathfrak{S}^{(\prime)}$ is decomposed as

$$\mathfrak{S}^{(\prime)} = \dot{\bigcup_{\omega_1 \in \tilde{\Omega}^{\mathbf{N}}_{z_1 \to z'_1}}} \; \dot{\bigcup_{\omega_2 \in \tilde{\Omega}^{\mathbf{N};\omega_1}_{z_2 \to z'_2}}} \cdots \dot{\bigcup_{\omega_k \in \tilde{\Omega}^{\mathbf{N};\vec{\omega}_{k-1}}_{z_k \to z'_k}}} \mathfrak{S}^{(\prime)}_{\vec{\omega}_k}. \tag{4.27}$$

To complete the proof of Lemma 4.2, it suffices to construct a bijection from $\mathfrak{S}_{\vec{\omega}_k}$ to $\mathfrak{S}'_{\vec{\omega}_k}$ for every $\vec{\omega}_k$. For $\vec{\mathbb{S}}_k \in \mathfrak{S}_{\vec{\omega}_k}$, we define

$$\vec{F}_{\vec{\omega}_k}(\vec{\mathbb{S}}_k) \equiv \big(F^{(0)}_{\vec{\omega}_k}(\mathbb{S}_0), \dots, F^{(k)}_{\vec{\omega}_k}(\mathbb{S}_k)\big) = \Big(\mathbb{S}_0 \triangle \dot{\bigcup}_{i=1}^k \omega_i,\, \mathbb{S}_1 \triangle \omega_1, \dots, \mathbb{S}_k \triangle \omega_k\Big), \tag{4.28}$$

where $\partial F^{(0)}_{\vec{\omega}_k}(\mathbb{S}_0) = \mathcal{V} \triangle \{z_1, z'_1\} \triangle \cdots \triangle \{z_k, z'_k\}$ and $\partial F^{(i)}_{\vec{\omega}_k}(\mathbb{S}_i) = \{z_i, z'_i\}$ for $i = 1, \dots, k$. Note that, by definition using symmetric difference, we have $\vec{F}_{\vec{\omega}_k}(\vec{F}_{\vec{\omega}_k}(\vec{\mathbb{S}}_k)) = \vec{\mathbb{S}}_k$. Also, by simple combinatorics using $\omega_i \cap \omega_j = \mathbb{S}_i \cap \mathbb{S}_j = \varnothing$ and $\omega_j \subset \mathbb{S}_0 \dot\cup \mathbb{S}_j$ for $1 \le j \le k$ and $i \ne j$, we have

$$F^{(i)}_{\vec{\omega}_k}(\mathbb{S}_i) \cap F^{(j)}_{\vec{\omega}_k}(\mathbb{S}_j) = \varnothing, \qquad F^{(0)}_{\vec{\omega}_k}(\mathbb{S}_0) \dot\cup F^{(j)}_{\vec{\omega}_k}(\mathbb{S}_j) = \Big(\mathbb{S}_0 \triangle \dot{\bigcup}_{i \ne j} \omega_i\Big) \dot\cup \mathbb{S}_j. \tag{4.29}$$

Since $\omega_j \subset \mathbb{S}_0 \dot\cup \mathbb{S}_j$ and $\omega_j \cap \dot{\bigcup}_{i \ne j} \omega_i = \varnothing$, we have $\omega_j \subset F^{(0)}_{\vec{\omega}_k}(\mathbb{S}_0) \dot\cup F^{(j)}_{\vec{\omega}_k}(\mathbb{S}_j)$.

It remains to show that ω_j is the earliest element of $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_{j-1}}_{z_j \to z'_j}$ in $F^{(0)}_{\vec{\omega}_k}(\mathbb{S}_0) \dot\cup F^{(j)}_{\vec{\omega}_k}(\mathbb{S}_j)$. To see this, we first recall that $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_{j-1}}_{z_j \to z'_j}$ is a set of paths on $\mathbb{G}_{\mathbf{N}} \setminus \dot{\bigcup}_{i<j} \omega_i$, so that its earliest element contained in $(\mathbb{S}_0 \triangle \dot{\bigcup}_{i<j} \omega_i) \dot\cup \mathbb{S}_j$ is still ω_j. Furthermore, since each $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_{i-1}}_{z_i \to z'_i}$ for $i > j$ is a set of paths that do not fully contain ω_j or any earlier element of $\tilde{\Omega}^{\mathbf{N};\vec{\omega}_{j-1}}_{z_j \to z'_j}$ as a subset, ω_j is still the earliest element of

$$\bigg(\Big(\mathbb{S}_0 \triangle \dot{\bigcup}_{i<j} \omega_i\Big) \dot\cup \mathbb{S}_j\bigg) \triangle \Big(\dot{\bigcup}_{i>j} \omega_i\Big) = \Big(\mathbb{S}_0 \triangle \dot{\bigcup}_{i \ne j} \omega_i\Big) \dot\cup \mathbb{S}_j \equiv F^{(0)}_{\vec{\omega}_k}(\mathbb{S}_0) \dot\cup F^{(j)}_{\vec{\omega}_k}(\mathbb{S}_j). \tag{4.30}$$

Therefore, $\vec{F}_{\vec{\omega}_k}$ is a bijection from $\mathfrak{S}_{\vec{\omega}_k}$ to $\mathfrak{S}'_{\vec{\omega}_k}$. This completes the proof of Lemma 4.2. ∎

4.3 Bounds on $\pi_\Lambda^{(j)}(x)$ for $j \geq 1$

First we prove (4.15) for $j \geq 1$ assuming the following two lemmas, in which we recall (2.30) and use

$$E'_{\mathbf{N}}(z,x;\mathcal{A}) = \{z \overset{\mathcal{A}}{\underset{\mathbf{N}}{\longleftrightarrow}} x\} \cap \{z \underset{\mathbf{N}}{\Longleftrightarrow} x\}, \qquad E''_{\mathbf{N}}(z,x,v;\mathcal{A}) = E'_{\mathbf{N}}(z,x;\mathcal{A}) \cap \{z \underset{\mathbf{N}}{\longleftrightarrow} v\}, \tag{4.31}$$

$$\Theta'_{z,x;\mathcal{A}} = \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=z\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E'_{\mathbf{m}+\mathbf{n}}(z,x;\mathcal{A})}, \qquad \Theta''_{z,x,v;\mathcal{A}} = \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=z\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E''_{\mathbf{m}+\mathbf{n}}(z,x,v;\mathcal{A})}. \tag{4.32}$$

Lemma 4.3. *For the ferromagnetic Ising model, we have*

$$\Theta_{y,x;\mathcal{A}} \leq \sum_z \big(\delta_{y,z} + \tilde{G}_\Lambda(y,z)\big)\,\Theta'_{z,x;\mathcal{A}}, \tag{4.33}$$

$$\begin{aligned} \Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{\{y\longleftrightarrow v\}}\big] \leq{}& \sum_z \big(\delta_{y,z} + \tilde{G}_\Lambda(y,z)\big)\,\Theta''_{z,x,v;\mathcal{A}} \\ &+ \sum_{v',z} \big(\delta_{y,v'} + \tilde{G}_\Lambda(y,v')\big)\,\tilde{G}_\Lambda(v',z)\,\Theta'_{z,x;\mathcal{A}}\,\psi_\Lambda(v',v). \end{aligned} \tag{4.34}$$

Lemma 4.4. *For the ferromagnetic Ising model, we have*

$$\Theta'_{y,x;\mathcal{A}} \leq \sum_{u\in\mathcal{A}} P'_{\Lambda;u}(y,x), \qquad \Theta''_{y,x,v;\mathcal{A}} \leq \sum_{u\in\mathcal{A}} P''_{\Lambda;u,v}(y,x). \tag{4.35}$$

We prove Lemma 4.3 in Section 4.3.1, and Lemma 4.4 in Section 4.3.2.

Proof of (4.15) for $j \geq 1$ assuming Lemmas 4.3–4.4. Recall (2.39). By (4.33), (4.35) and (4.13), we obtain

$$\begin{aligned} \Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Big[\tau_{b_j}\Theta^{(j)}_{\overline{b}_j,x;\tilde{\mathcal{C}}_{j-1}}\Big] &\leq \Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Bigg[\sum_z \tau_{b_j}\big(\delta_{\overline{b}_j,z} + \tilde{G}_\Lambda(\overline{b}_j,z)\big)\sum_{v_j\in\tilde{\mathcal{C}}_{j-1}} P'_{\Lambda;v_j}(z,x)\Bigg] \\ &\leq \sum_{v_j} \Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\big[\mathbb{1}_{\{\overline{b}_{j-1}\longleftrightarrow v_j\}}\big]\,\tau_{b_j} Q'_{\Lambda;v_j}(\overline{b}_j,x). \end{aligned} \tag{4.36}$$

For $j = 1$, we use (4.16) and (4.36) to obtain

$$\begin{aligned} \pi_\Lambda^{(1)}(x) \equiv \sum_{b_1} \Theta^{(0)}_{o,\underline{b}_1;\Lambda}\Big[\tau_{b_1}\,\Theta^{(1)}_{\overline{b}_1,x;\tilde{\mathcal{C}}_0}\Big] &\leq \sum_{b_1,v_1} \Theta^{(0)}_{o,\underline{b}_1;\Lambda}\big[\mathbb{1}_{\{o\longleftrightarrow v_1\}}\big]\,\tau_{b_1} Q'_{\Lambda;v_1}(\overline{b}_1,x) \qquad (4.37) \\ = \sum_{b_1,v_1}\Bigg(\sum_{\partial\mathbf{n}=o\triangle\underline{b}_1} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{o\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}_1\}\cap\{o\underset{\mathbf{n}}{\longleftrightarrow}v_1\}}\Bigg)\tau_{b_1} Q'_{\Lambda;v_1}(\overline{b}_1,x) &\leq \sum_{b_1,v_1} P'^{(0)}_{\Lambda;v_1}(o,\underline{b}_1)\,\tau_{b_1} Q'_{\Lambda;v_1}(\overline{b}_1,x). \end{aligned}$$

For $j \geq 2$, we use (4.34)–(4.35) and then (4.13)–(4.14) to obtain

$$\begin{aligned}
\Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Big[\tau_{b_j}\Theta^{(j)}_{\overline{b}_j,x;\tilde{\mathcal{C}}_{j-1}}\Big] &\leq \sum_{v_j}\Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\big[\mathbb{1}_{\{\overline{b}_{j-1}\longleftrightarrow v_j\}}\big]\,\tau_{b_j}Q'_{\Lambda;v_j}(\overline{b}_j,x)\\
&\leq \sum_{v_j}\tau_{b_j}Q'_{\Lambda;v_j}(\overline{b}_j,x)\Bigg(\sum_z\big(\delta_{\overline{b}_{j-1},z}+\tilde{G}_\Lambda(\overline{b}_{j-1},z)\big)\sum_{v_{j-1}\in\tilde{\mathcal{C}}_{j-2}}P''_{\Lambda;v_{j-1},v_j}(z,\underline{b}_j)\\
&\qquad+\sum_{v',z}\big(\delta_{\overline{b}_{j-1},v'}+\tilde{G}_\Lambda(\overline{b}_{j-1},v')\big)\,\tilde{G}_\Lambda(v',z)\sum_{v_{j-1}\in\tilde{\mathcal{C}}_{j-2}}P'_{\Lambda;v_{j-1}}(z,\underline{b}_j)\,\psi_\Lambda(v',v_j)\Bigg)\\
&\leq \sum_{v_{j-1},v_j}\mathbb{1}_{\{v_{j-1}\in\tilde{\mathcal{C}}_{j-2}\}}\,Q''_{\Lambda;v_{j-1},v_j}(\overline{b}_{j-1},\underline{b}_j)\,\tau_{b_j}Q'_{\Lambda;v_j}(\overline{b}_j,x).
\end{aligned}\tag{4.38}$$

We repeatedly use (4.34)–(4.35) to bound $\Theta^{(i)}_{\overline{b}_i,\underline{b}_{i+1};\tilde{\mathcal{C}}_{i-1}}[\mathbb{1}_{\{\overline{b}_i\longleftrightarrow v_{i+1}\}}]$ for $i=j-2,\dots,1$ as in (4.38), and then at the end we apply (4.16) as in (4.37) to obtain (4.15). This completes the proof. $\square$

4.3.1 Proof of Lemma 4.3

Proof of (4.33). Recall (2.30) and (4.32). Then, to prove (4.33), it suffices to bound the contribution from $\mathbb{1}_{E_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})\setminus E'_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})}$ by $\sum_z \tilde{G}_\Lambda(y,z)\,\Theta'_{z,x;\mathcal{A}}$.

First we recall (2.28) and (4.31). Then, we have

$$E_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})\setminus E'_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A}) = E_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})\cap\big\{\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}x\}\setminus\{y\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x\}\big\}. \tag{4.39}$$

On $\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}x\}\setminus\{y\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x\}$, there is at least one pivotal bond for $y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}x$ from y. Let b be the last pivotal bond among them. Then, we have $\overline{b}\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x$ off b, $m_b+n_b>0$, and $y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}\underline{b}$ in $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\mathrm{c}}$. Moreover, on the event $E_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})$, we have that $y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}\underline{b}$ in $\mathcal{A}^{\mathrm{c}}$ and $\overline{b}\overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}}x$. Since $\{\overline{b}\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x \text{ off } b\}\cap\{\overline{b}\overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}}x\}=\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}$ on the event that b is pivotal for $y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}x$ from y, we have

$$\begin{aligned}
&E_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})\setminus E'_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})\\
&=\dot{\bigcup_b}\Big\{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}\cap\{m_b+n_b>0\}\cap\big\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}\underline{b} \text{ in } \mathcal{A}^{\mathrm{c}}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\mathrm{c}}\big\}\Big\}.
\end{aligned}\tag{4.40}$$

Therefore, we obtain

$$\begin{aligned}
&\Theta_{y,x;\mathcal{A}}-\Theta'_{y,x;\mathcal{A}}\\
&=\sum_b\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\triangle x}}\frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}}\mathbb{1}_{\{m_b+n_b>0\}}\mathbb{1}_{\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}\underline{b} \text{ in } \mathcal{A}^{\mathrm{c}}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\mathrm{c}}\}}.
\end{aligned}\tag{4.41}$$

It remains to bound the right-hand side of (4.41), which is nonzero only if m_b is even and n_b is odd, due to the source constraints and the conditions in the indicators. First, as in (2.31), we alternate the parity of n_b by changing the source constraint into $\partial\mathbf{n}=y\triangle b\triangle x$ and multiplying by τ_b. Then, by conditioning on $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)$ as in (2.33) (i.e., conditioning on $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)=\mathcal{B}$, letting $\mathbf{m}'=\mathbf{m}|_{\mathbb{B}_{\mathcal{A}^{\mathrm{c}}}\setminus\mathbb{B}_{\mathcal{A}^{\mathrm{c}}\cap\mathcal{B}^{\mathrm{c}}}}$, $\mathbf{m}''=\mathbf{m}|_{\mathbb{B}_{\mathcal{A}^{\mathrm{c}}\cap\mathcal{B}^{\mathrm{c}}}}$, $\mathbf{n}'=\mathbf{n}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{B}^{\mathrm{c}}}}$ and $\mathbf{n}''=\mathbf{n}|_{\mathbb{B}_{\mathcal{B}^{\mathrm{c}}}}$, and then summing over

$\mathcal{B} \subset \Lambda$), we obtain

$$\sum_{\mathcal{B}\subset\Lambda} \sum_{\substack{\partial\mathbf{m}'=\varnothing \\ \partial\mathbf{n}'=\overline{b}\triangle x}} \frac{\tilde{w}_{\mathcal{A}^{\rm c},\mathcal{B}}(\mathbf{m}')\, Z_{\mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}}}{Z_{\mathcal{A}^{\rm c}}} \frac{\tilde{w}_{\Lambda,\mathcal{B}}(\mathbf{n}')\, Z_{\mathcal{B}^{\rm c}}}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}'+\mathbf{n}'}(\overline{b},x;\mathcal{A}) \text{ off } b\}\cap\{\mathcal{C}^b_{\mathbf{m}'+\mathbf{n}'}(x)=\mathcal{B}\}}$$
$$\times \tau_b \mathbb{1}_{\{m'_b,n'_b \text{ even}\}} \underbrace{\sum_{\substack{\partial\mathbf{m}''=\varnothing \\ \partial\mathbf{n}''=y\triangle\underline{b}}} \frac{w_{\mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}}(\mathbf{m}'')}{Z_{\mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}}} \frac{w_{\mathcal{B}^{\rm c}}(\mathbf{n}'')}{Z_{\mathcal{B}^{\rm c}}} \mathbb{1}_{\{y \underset{\mathbf{m}''+\mathbf{n}''}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}\}}}_{\because (2.23)\ \equiv\ \langle\varphi_y\varphi_{\underline{b}}\rangle_{\mathcal{A}^{\rm c}\cap\,\mathcal{B}^{\rm c}}}$$
$$= \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=\overline{b}\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}} \langle\varphi_y\varphi_{\underline{b}}\rangle_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}}. \tag{4.42}$$

Since $\langle\varphi_y\varphi_{\underline{b}}\rangle_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}} = 0$ on $E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \setminus \{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\} \subset \{\underline{b} \in \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)\}$ and on the event that m_b or n_b is odd (see below (2.15) or above (2.34)), we can omit "off b" and $\mathbb{1}_{\{m_b,n_b \text{ even}\}}$ in (4.42). Since $\langle\varphi_y\varphi_{\underline{b}}\rangle_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}} \le \langle\varphi_y\varphi_{\underline{b}}\rangle_\Lambda$ due to Proposition 2.2, we have

$$(4.42) \le \langle\varphi_y\varphi_{\underline{b}}\rangle_\Lambda \tau_b \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=\overline{b}\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A})} = \langle\varphi_y\varphi_{\underline{b}}\rangle_\Lambda \tau_b\, \Theta'_{\overline{b},x;\mathcal{A}}. \tag{4.43}$$

Therefore, (4.41) is bounded by $\sum_b \langle\varphi_y\varphi_{\underline{b}}\rangle_\Lambda \tau_b\, \Theta'_{\overline{b},x;\mathcal{A}} \equiv \sum_z \tilde{G}_\Lambda(y,z)\, \Theta'_{z,x;\mathcal{A}}$. This completes the proof of (4.33). □

Proof of (4.34). Recall (2.30) and (4.32). To prove (4.34), we investigate

$$L \equiv \big\{E_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A}) \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}\big\} \setminus E''_{\mathbf{m}+\mathbf{n}}(y,x,v;\mathcal{A})$$
$$= \{E_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A}) \setminus E'_{\mathbf{m}+\mathbf{n}}(y,x;\mathcal{A})\} \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}, \tag{4.44}$$

where $\Theta_{y,x;\mathcal{A}}[\mathbb{1}_L] = \Theta_{y,x;\mathcal{A}}[\mathbb{1}_{\{y\longleftrightarrow v\}}] - \Theta''_{y,x,v;\mathcal{A}}$.

First we recall (4.40), in which b is the last pivotal bond for $y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x$ from y, and define

$$R_1(b) = \{E''_{\mathbf{m}+\mathbf{n}}(\overline{b},x,v;\mathcal{A}) \text{ off } b\} \cap \{m_b+n_b>0\} \cap \big\{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}\big\}, \tag{4.45}$$

$$R_2(b) = \{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\} \cap \{m_b+n_b>0\} \cap \big\{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c},\ y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\big\}, \tag{4.46}$$

where $v \in \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)$ on $R_1(b)$, while $v \in \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(y)$ on $R_2(b)$. Since

$$L = \dot{\bigcup_b} \{R_1(b) \,\dot\cup\, R_2(b)\}, \tag{4.47}$$

we have

$$\Theta_{y,x;\mathcal{A}}[\mathbb{1}_{\{y\longleftrightarrow v\}}] - \Theta''_{y,x,v;\mathcal{A}} = \sum_b \Big(\Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{R_1(b)}\big] + \Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{R_2(b)}\big]\Big). \tag{4.48}$$

Following the same argument as in (4.42)–(4.43), we easily obtain

$$\Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{R_1(b)}\big] = \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\Delta x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E''_{\mathbf{m}+\mathbf{n}}(\overline{b},x,v;\mathcal{A}) \text{ off } b\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}} \langle \varphi_y \varphi_{\underline{b}} \rangle_{\mathcal{A}^{\rm c}\cap \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}}$$

$$\le \langle \varphi_y \varphi_{\underline{b}} \rangle_\Lambda \tau_b \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\Delta x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E''_{\mathbf{m}+\mathbf{n}}(\overline{b},x,v;\mathcal{A})} = \langle \varphi_y \varphi_{\underline{b}} \rangle_\Lambda \tau_b\, \Theta''_{\overline{b},x,v;\mathcal{A}}. \tag{4.49}$$

Similarly, we have

$$\Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{R_2(b)}\big] = \sum_{\mathcal{B}\subset\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\Delta x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}\cap\{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)=\mathcal{B}\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}}$$

$$\times \sum_{\substack{\partial\mathbf{h}=\varnothing\\ \partial\mathbf{k}=y\Delta\underline{b}}} \frac{w_{\mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}}(\mathbf{h})}{Z_{\mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}}} \frac{w_{\mathcal{B}^{\rm c}}(\mathbf{k})}{Z_{\mathcal{B}^{\rm c}}} \mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c},\ y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v \text{ (in } \mathcal{B}^{\rm c})\}}$$

$$\le \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\Delta x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}} \Psi_{y,\underline{b},v;\mathcal{A},\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)}, \tag{4.50}$$

where

$$\Psi_{y,z,v;\mathcal{A},\mathcal{B}} = \sum_{\substack{\partial\mathbf{h}=\varnothing\\ \partial\mathbf{k}=y\Delta z}} \frac{w_{\mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}}(\mathbf{h})}{Z_{\mathcal{A}^{\rm c}\cap\mathcal{B}^{\rm c}}} \frac{w_{\mathcal{B}^{\rm c}}(\mathbf{k})}{Z_{\mathcal{B}^{\rm c}}} \mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\}}. \tag{4.51}$$

We note that, by ignoring the indicator in (4.51), we have $0 \le \Psi_{y,z,v;\mathcal{A},\mathcal{B}} \le \langle \varphi_y\varphi_z\rangle_{\mathcal{B}^{\rm c}}$, which is zero whenever $z\in\mathcal{B}$. Therefore, we can omit "off b" and $\mathbb{1}_{\{m_b,n_b \text{ even}\}}$ in (4.50) to obtain

$$\Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{R_2(b)}\big] \le \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\Delta x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A})} \tau_b\, \Psi_{y,\underline{b},v;\mathcal{A},\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)}. \tag{4.52}$$

Substituting (4.49) and (4.52) to (4.48), we arrive at

$$\Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{\{y\longleftrightarrow v\}}\big] \le \sum_z \big(\delta_{y,z} + \tilde G_\Lambda(y,z)\big)\, \Theta''_{z,x,v;\mathcal{A}}$$

$$+ \sum_b \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\Delta x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A})} \tau_b\, \Psi_{y,\underline{b},v;\mathcal{A},\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)}. \tag{4.53}$$

The proof of (4.34) is completed by using

$$\Psi_{y,z,v;\mathcal{A},\mathcal{B}} \le \sum_{v'} \langle \varphi_y\varphi_{v'}\rangle_\Lambda \langle \varphi_{v'}\varphi_z\rangle_\Lambda\, \psi_\Lambda(v',v), \tag{4.54}$$

and replacing $\langle \varphi_y\varphi_{v'}\rangle_\Lambda$ in (4.54) by $\delta_{y,v'} + \tilde G_\Lambda(y,v')$, due to (4.2).

To complete the proof of (4.34), it thus remains to show (4.54). First we note that, if $\mathcal{A}\subset\mathcal{B}$, then by Lemma 2.3 we have

$$\Psi_{y,z,v;\mathcal{A},\mathcal{B}} = \sum_{\substack{\partial\mathbf{h}=\varnothing\\ \partial\mathbf{k}=y\Delta z}} \frac{w_{\mathcal{B}^{\rm c}}(\mathbf{h})}{Z_{\mathcal{B}^{\rm c}}} \frac{w_{\mathcal{B}^{\rm c}}(\mathbf{k})}{Z_{\mathcal{B}^{\rm c}}} \mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\}} = \langle \varphi_y\varphi_v\rangle_{\mathcal{B}^{\rm c}} \langle \varphi_v\varphi_z\rangle_{\mathcal{B}^{\rm c}} \le \langle \varphi_y\varphi_v\rangle_\Lambda \langle \varphi_v\varphi_z\rangle_\Lambda. \tag{4.55}$$

However, to prove (4.54) for a general $\mathcal{A}$ that does not necessarily satisfy $\mathcal{A} \subset \mathcal{B}$, we use

$$\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\} = \{y \underset{\mathbf{k}}{\longleftrightarrow} v\} \mathbin{\dot\cup} \big\{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\} \setminus \{y \underset{\mathbf{k}}{\longleftrightarrow} v\}\big\}, \tag{4.56}$$

and consider the two events on the right-hand side separately. The contribution to $\Psi_{y,z,v;\mathcal{A},\mathcal{B}}$ from $\{y \underset{\mathbf{k}}{\longleftrightarrow} v\}$ is easily bounded, similarly to (4.23), as

$$\begin{aligned}\sum_{\partial\mathbf{k}=y\Delta z} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} v\}} &\le \sum_{\substack{\partial\mathbf{k}=y\Delta z\\ \partial\mathbf{k}'=\varnothing}} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \frac{w_{\mathcal{B}^c}(\mathbf{k}')}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{y \underset{\mathbf{k}+\mathbf{k}'}{\longleftrightarrow} v\}} = \langle\varphi_y\varphi_v\rangle_{\mathcal{B}^c}\langle\varphi_v\varphi_z\rangle_{\mathcal{B}^c}\\ &\le \langle\varphi_y\varphi_v\rangle_\Lambda\langle\varphi_v\varphi_z\rangle_\Lambda.\end{aligned} \tag{4.57}$$

Next we consider the contribution to $\Psi_{y,z,v;\mathcal{A},\mathcal{B}}$ from $\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\}\setminus\{y \underset{\mathbf{k}}{\longleftrightarrow} v\}$ in (4.56). We denote by $\mathcal{C}_{\mathbf{k}}(y)$ the set of sites $\mathbf{k}$-connected from y. Since $v \in \mathcal{C}_{\mathbf{h}+\mathbf{k}}(y)\setminus\mathcal{C}_{\mathbf{k}}(y)$, there is a *nonzero* alternating chain of mutually-disjoint $\mathbf{h}$-connected clusters and mutually-disjoint $\mathbf{k}$-connected clusters, from some $u_0 \in \mathcal{C}_{\mathbf{k}}(y)$ to v. Therefore, we have

$$\begin{aligned}\mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\}\setminus\{y \underset{\mathbf{k}}{\longleftrightarrow} v\}} \le \sum_{j=1}^{\infty} \sum_{\substack{u_0,\dots,u_j\\ u_l\ne u_{l'}\ \forall l\ne l'\\ u_j=v}} \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} u_0\}} \bigg(\prod_{l\ge0} \mathbb{1}_{\{u_{2l} \underset{\mathbf{h}}{\longleftrightarrow} u_{2l+1}\}}\bigg)\bigg(\prod_{l\ge1} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}}\bigg)\\ \times\bigg(\prod_{\substack{l,l'\ge0\\ l\ne l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{h}}(u_{2l})\cap\mathcal{C}_{\mathbf{h}}(u_{2l'})=\varnothing\}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l})\cap\mathcal{C}_{\mathbf{k}}(u_{2l'})=\varnothing\}}\bigg),\end{aligned} \tag{4.58}$$

where we regard an empty product as 1. Using this bound, we can perform the sums over $\mathbf{h}$ and $\mathbf{k}$ in (4.51) independently.

For $j = 1$ and given $u_0 \ne u_1 = v$, the summand of (4.58) equals $\mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} u_0\}}\mathbb{1}_{\{u_0 \underset{\mathbf{h}}{\longleftrightarrow} v\}}$, which is simply equal to $\mathbb{1}_{\{y \underset{\mathbf{h}}{\longleftrightarrow} v\}}$ if $u_0 = y$. Then, by (4.57) and (4.2), the contribution from this to $\Psi_{y,z,v;\mathcal{A},\mathcal{B}}$ is

$$\sum_{\partial\mathbf{k}=y\Delta z} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} u_0\}} \sum_{\partial\mathbf{h}=\varnothing} \frac{w_{\mathcal{A}^c\cap\mathcal{B}^c}(\mathbf{h})}{Z_{\mathcal{A}^c\cap\mathcal{B}^c}} \mathbb{1}_{\{u_0 \underset{\mathbf{h}}{\longleftrightarrow} v\}} \le \langle\varphi_y\varphi_{u_0}\rangle_\Lambda\langle\varphi_{u_0}\varphi_z\rangle_\Lambda\, \tilde G_\Lambda(u_0,v)^2. \tag{4.59}$$

Fix $j \ge 2$ and a sequence of distinct sites $u_0,\dots,u_j\,(=v)$, and first consider the contribution to the sum over $\mathbf{k}$ in (4.51) from the relevant indicators in the right-hand side of (4.58), which is

$$\begin{aligned}&\sum_{\partial\mathbf{k}=y\Delta z} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} u_0\}} \bigg(\prod_{l\ge1} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}}\bigg) \prod_{\substack{l,l'\ge0\\ l\ne l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l})\cap\mathcal{C}_{\mathbf{k}}(u_{2l'})=\varnothing\}} \qquad (4.60)\\ &= \sum_{\partial\mathbf{k}=y\Delta z} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \bigg(\prod_{l\ge1} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}}\bigg)\bigg(\prod_{\substack{l,l'\ge1\\ l\ne l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l})\cap\mathcal{C}_{\mathbf{k}}(u_{2l'})=\varnothing\}}\bigg) \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} u_0\}\cap\{\mathcal{C}_{\mathbf{k}}(u_0)\cap\mathcal{U}_{\mathbf{k};1}=\varnothing\}},\end{aligned}$$

where $\mathcal{U}_{\mathbf{k};1}=\dot{\bigcup}_{l\geq 1}\mathcal{C}_{\mathbf{k}}(u_{2l})$. Conditioning on $\mathcal{U}_{\mathbf{k};1}$, we obtain that

$$(4.60)=\sum_{\partial\mathbf{k}=\varnothing}\frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}}\bigg(\prod_{l\geq 1}\mathbb{1}_{\{u_{2l-1}\underset{\mathbf{k}}{\longleftrightarrow}u_{2l}\}}\bigg)\bigg(\prod_{\substack{l,l'\geq 1\\ l\neq l'}}\mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l})\cap\mathcal{C}_{\mathbf{k}}(u_{2l'})=\varnothing\}}\bigg)$$

$$\times\underbrace{\sum_{\partial\mathbf{k}'=y\vartriangle z}\frac{w_{\mathcal{B}^{\mathrm{c}}\cap\mathcal{U}^{\mathrm{c}}_{\mathbf{k};1}}(\mathbf{k}')}{Z_{\mathcal{B}^{\mathrm{c}}\cap\mathcal{U}^{\mathrm{c}}_{\mathbf{k};1}}}\mathbb{1}_{\{y\underset{\mathbf{k}'}{\longleftrightarrow}u_0\}}}_{\overset{\because(4.57)}{\leq}\langle\varphi_y\varphi_{u_0}\rangle_\Lambda\langle\varphi_{u_0}\varphi_z\rangle_\Lambda}. \tag{4.61}$$

Then, by conditioning on $\mathcal{U}_{\mathbf{k};2}\equiv\dot{\bigcup}_{l\geq 2}\mathcal{C}_{\mathbf{k}}(u_{2l})$, following the same computation as above and using (4.2), we further obtain that

$$(4.60)\leq\langle\varphi_y\varphi_{u_0}\rangle_\Lambda\langle\varphi_{u_0}\varphi_z\rangle_\Lambda\sum_{\partial\mathbf{k}=\varnothing}\frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}}\bigg(\prod_{l\geq 2}\mathbb{1}_{\{u_{2l-1}\underset{\mathbf{k}}{\longleftrightarrow}u_{2l}\}}\bigg)\bigg(\prod_{\substack{l,l'\geq 2\\ l\neq l'}}\mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l})\cap\mathcal{C}_{\mathbf{k}}(u_{2l'})=\varnothing\}}\bigg)$$

$$\times\underbrace{\sum_{\partial\mathbf{k}'=\varnothing}\frac{w_{\mathcal{B}^{\mathrm{c}}\cap\mathcal{U}^{\mathrm{c}}_{\mathbf{k};2}}(\mathbf{k}')}{Z_{\mathcal{B}^{\mathrm{c}}\cap\mathcal{U}^{\mathrm{c}}_{\mathbf{k};2}}}\mathbb{1}_{\{u_1\underset{\mathbf{k}'}{\longleftrightarrow}u_2\}}}_{\leq\tilde{G}_\Lambda(u_1,u_2)^2}. \tag{4.62}$$

We repeat this computation until all indicators for $\mathbf{k}$ are used up. We also apply the same argument to the sum over $\mathbf{h}$ in (4.51). Summarizing these bounds with (4.57) and (4.59), and replacing u_0 in (4.58)–(4.61) by v', we obtain (4.54). This completes the proof of (4.34). □

4.3.2 Proof of Lemma 4.4

We note that the common factor $\mathbb{1}_{\{y\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x\}}$ in $\Theta'_{y,x;\mathcal{A}}$ and $\Theta''_{y,x,v;\mathcal{A}}$ can be decomposed as

$$\mathbb{1}_{\{y\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x\}}=\mathbb{1}_{\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}+\mathbb{1}_{\{y\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x\}\setminus\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}. \tag{4.63}$$

We estimate the contributions from $\mathbb{1}_{\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}$ to $\Theta'_{y,x;\mathcal{A}}$ and $\Theta''_{y,x,v;\mathcal{A}}$ in the following paragraphs (a) and (b), respectively. Then, in the paragraphs (c) and (d) below, we will estimate the contributions from $\mathbb{1}_{\{y\underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow}x\}\setminus\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}$ in (4.63) to $\Theta'_{y,x;\mathcal{A}}$ and $\Theta''_{y,x,v;\mathcal{A}}$, respectively.

(a) First we investigate the contribution to $\Theta'_{y,x;\mathcal{A}}$ from $\mathbb{1}_{\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}$:

$$\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\vartriangle x}}\frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}}x\}\cap\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}. \tag{4.64}$$

For a set of events $E_1,\dots,E_N$, we define $E_1\circ\cdots\circ E_N$ to be the event that $E_1,\dots,E_N$ occur *bond*-disjointly. Then, we have

$$\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}}x\}\cap\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}\leq\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}}x\}\cap\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}\leq\sum_{u\in\mathcal{A}}\mathbb{1}_{\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x\}}, \tag{4.65}$$

where the right-hand side does not depend on $\mathbf{m}$. Therefore, the contribution to $\Theta'_{y,x;\mathcal{A}}$ is bounded by

$$(4.64)\leq\sum_{u\in\mathcal{A}}\sum_{\partial\mathbf{n}=y\vartriangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x\}}\leq\sum_{u\in\mathcal{A}}P'^{(0)}_{\Lambda;u}(y,x), \tag{4.66}$$

where we have applied the same argument as in the proof of (4.16), which is around (4.23)–(4.26). ☐

(b) Next we investigate the contribution to $\Theta''_{y,x,v;\mathcal{A}}$ from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ in (4.63):

$$\sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=y\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}}. \tag{4.67}$$

Note that, by using (4.56) and $\mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} x\}} \leq \mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\}}$, we have

$$\mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}} \leq \mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}} \Big(\mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} v\}} + \mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\} \setminus \{y \underset{\mathbf{n}}{\longleftrightarrow} v\}} \Big). \tag{4.68}$$

We investigate the contributions from the two indicators in the parentheses separately.

We begin with the contribution from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} v\}}$, which is independent of $\mathbf{m}$. Since

$$\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{y \underset{\mathbf{n}}{\longleftrightarrow} v\} \subset \{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\} \circ \{y \underset{\mathbf{n}}{\longleftrightarrow} x,\ y \underset{\mathbf{n}}{\longleftrightarrow} v\}, \tag{4.69}$$

$$\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\} \subset \bigcup_{u\in\mathcal{A}} \{y \underset{\mathbf{n}}{\longleftrightarrow} u\} \circ \{u \underset{\mathbf{n}}{\longleftrightarrow} x\}, \tag{4.70}$$

the contribution to (4.67) from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} v\}}$ in (4.68) is bounded by

$$\sum_{u\in\mathcal{A}} \sum_{\partial\mathbf{n}=y\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} u\} \circ \{u \underset{\mathbf{n}}{\longleftrightarrow} x\} \circ \{y \underset{\mathbf{n}}{\longleftrightarrow} x,\ y \underset{\mathbf{n}}{\longleftrightarrow} v\}}. \tag{4.71}$$

We follow Steps (i)–(iii) described above (4.23) in Section 4.2. Without loss of generality, we can assume that y, u, x and v are all different; otherwise, the following argument can be simplified. (i) Since y and x are sources, but u and v are not, there is an edge-disjoint cycle $y \to u \to x \to v \to y$, with an extra edge-disjoint path from y to x. Therefore, we have in total at least $5\,(= 4+1)$ edge-disjoint paths. (ii) Multiplying by $(Z_\Lambda/Z_\Lambda)^4$, we have

$$(4.71) = \sum_{u\in\mathcal{A}} \sum_{\partial\mathbf{N}=y\triangle x} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^5} \sum_{\substack{\partial\mathbf{n}=y\triangle x \\ \partial\mathbf{m}_i=\varnothing\ \forall i=1,\dots,4 \\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^4 \mathbf{m}_i}} \mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} u\} \circ \{u \underset{\mathbf{n}}{\longleftrightarrow} x\} \circ \{y \underset{\mathbf{n}}{\longleftrightarrow} x,\ y \underset{\mathbf{n}}{\longleftrightarrow} v\}} \prod_b \frac{N_b!}{n_b! \prod_{i=1}^4 m_b^{(i)}!}, \tag{4.72}$$

where we have used the notation $m_b^{(i)} = \mathbf{m}_i|_b$. (iii) The sum over $\mathbf{n}, \mathbf{m}_1, \dots, \mathbf{m}_4$ in (4.72) is bounded by the cardinality of $\mathfrak{S}$ in Lemma 4.2 with $k=4$, $\mathcal{V}=\{y,x\}$, $\{z_1,z_1'\}=\{y,u\}$, $\{z_2,z_2'\}=\{u,x\}$, $\{z_3,z_3'\}=\{y,v\}$ and $\{z_4,z_4'\}=\{v,x\}$. Bounding the cardinality of $\mathfrak{S}'$ in Lemma 4.2 for this setting, we obtain

$$\begin{aligned}
(4.72) &\leq \sum_{u\in\mathcal{A}} \sum_{\partial\mathbf{N}=y\triangle x} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^5} \sum_{\substack{\partial\mathbf{n}=y\triangle x \\ \partial\mathbf{m}_1=y\triangle u,\ \partial\mathbf{m}_2=u\triangle x \\ \partial\mathbf{m}_3=y\triangle v,\ \partial\mathbf{m}_4=v\triangle x \\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^4 \mathbf{m}_i}} \prod_b \frac{N_b!}{n_b! \prod_{i=1}^4 m_b^{(i)}!} \\
&\leq \sum_{u\in\mathcal{A}} \langle\varphi_y\varphi_x\rangle_\Lambda \langle\varphi_y\varphi_u\rangle_\Lambda \langle\varphi_u\varphi_x\rangle_\Lambda \langle\varphi_y\varphi_v\rangle_\Lambda \langle\varphi_v\varphi_x\rangle_\Lambda.
\end{aligned} \tag{4.73}$$

Next we investigate the contribution to (4.67) from $\mathbb{1}_{\{y\underset{\mathbf{m+n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}}$ in (4.68). On the event $\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}\cap\{\{y\underset{\mathbf{m+n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}\}$, there exists a $v_0\neq v$ such that $\{y\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x,\ y\underset{\mathbf{n}}{\longleftrightarrow}v_0\}$ occurs and that v_0 and v are connected via a nonzero alternating chain of mutually-disjoint **m**-connected clusters and mutually-disjoint **n**-connected clusters. Therefore, by (4.58) and (4.70) (see also (4.71)), we obtain

$$\begin{aligned}
&\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}}x\}\cap\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}\cap\{\{y\underset{\mathbf{m+n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}\}}\\
&\leq\sum_{u\in\mathcal{A}}\sum_{j\geq1}\sum_{\substack{v_0,\dots,v_j\\ v_l\neq v_{l'}\ \forall l\neq l'\\ v_j=v}}\mathbb{1}_{\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x,\ y\underset{\mathbf{n}}{\longleftrightarrow}v_0\}}\bigg(\prod_{l\geq0}\mathbb{1}_{\{v_{2l}\underset{\mathbf{m}}{\longleftrightarrow}v_{2l+1}\}}\bigg)\\
&\qquad\times\bigg(\prod_{l\geq1}\mathbb{1}_{\{v_{2l-1}\underset{\mathbf{n}}{\longleftrightarrow}v_{2l}\}}\bigg)\bigg(\prod_{\substack{l,l'\geq0\\ l\neq l'}}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}}(v_{2l})\cap\mathcal{C}_{\mathbf{m}}(v_{2l'})=\varnothing\}}\mathbb{1}_{\{\mathcal{C}_{\mathbf{n}}(v_{2l})\cap\mathcal{C}_{\mathbf{n}}(v_{2l'})=\varnothing\}}\bigg). \qquad (4.74)
\end{aligned}$$

For the three products of indicators, we repeate the same argument as in (4.59)–(4.62) to derive the factor $\psi_\Lambda(v_0,v)-\delta_{v_0,v}$. As a result, we have

$$\begin{aligned}
&\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\Delta x}}\frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}}x\}\cap\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}\cap\{\{y\underset{\mathbf{m+n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}\}}\\
&\leq\sum_{v_0}\big(\psi_\Lambda(v_0,v)-\delta_{v_0,v}\big)\sum_{u\in\mathcal{A}}\sum_{\partial\mathbf{n}=y\Delta x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x,\ y\underset{\mathbf{n}}{\longleftrightarrow}v_0\}}. \qquad (4.75)
\end{aligned}$$

Following the same argument as in (4.71)–(4.73), we obtain

$$\begin{aligned}
(4.75)&\leq\sum_{u\in\mathcal{A},\ v_0}\big(\psi_\Lambda(v_0,v)-\delta_{v_0,v}\big)\,\langle\varphi_y\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_{v_0}\rangle_\Lambda\langle\varphi_{v_0}\varphi_x\rangle_\Lambda\\
&\leq\sum_{u\in\mathcal{A}}\Big(P''^{(0)}_{\Lambda;u,v}(y,x)-\langle\varphi_y\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_v\rangle_\Lambda\langle\varphi_v\varphi_x\rangle_\Lambda\Big). \qquad (4.76)
\end{aligned}$$

Summarizing (4.68), (4.73) and (4.76), we arrive at

$$(4.67)\leq\sum_{u\in\mathcal{A}}P''^{(0)}_{\Lambda;u,v}(y,x). \qquad (4.77)$$

This completes the bound on the contribution to $\Theta''_{y,x,v;\mathcal{A}}$ from $\mathbb{1}_{\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}$ in (4.63). □

(c) The contribution to $\Theta'_{y,x;\mathcal{A}}$ from $\mathbb{1}_{\{y\underset{\mathbf{m+n}}{\Longleftrightarrow}x\}\setminus\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}$ in (4.63) equals

$$\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\Delta x}}\frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{m+n}}{\longleftrightarrow}}x\}\cap\{\{y\underset{\mathbf{m+n}}{\Longleftrightarrow}x\}\setminus\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}\}}. \qquad (4.78)$$

Note that, if $\mathbb{1}_{\{\partial\mathbf{n}=y\Delta x\}\setminus\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}}=1$, then y is **n**-connected, but not **n**-doubly connected, to x, and therefore there exists at least one pivotal bond for $y\underset{\mathbf{n}}{\longleftrightarrow}x$. Given an ordered set of bonds $\vec{b}_T=(b_1,\dots,b_T)$, we define

$$H_{\mathbf{n};\vec{b}_T}(y,x)=\{y\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}_1\}\cap\bigcap_{i=1}^{T}\Big\{\{\overline{b}_i\underset{\mathbf{n}}{\Longleftrightarrow}\underline{b}_{i+1}\}\cap\{n_{b_i}>0,\ b_i\text{ is pivotal for }y\underset{\mathbf{n}}{\longleftrightarrow}x\}\Big\}, \qquad (4.79)$$

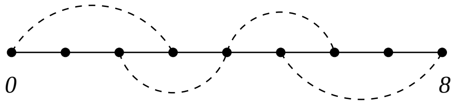


Figure 7: An element in $\mathcal{L}^{(4)}_{[0,8]}$, which consists of $s_1t_1 = \{0,3\}$, $s_2t_2 = \{2,4\}$, $s_3t_3 = \{4,6\}$ and $s_4t_4 = \{5,8\}$.

where, by convention, $\underline{b}_{T+1} = x$. Then, by $\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}} \le \mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}$, we obtain

$$
\begin{aligned}
(4.78) &= \sum_{T\ge1}\sum_{\vec{b}_T}\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}\cap H_{\mathbf{n};\vec{b}_T}(y,x)\cap\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}} \\
&\le \sum_{T\ge1}\sum_{\vec{b}_T}\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}\cap H_{\mathbf{n};\vec{b}_T}(y,x)\cap\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}}.
\end{aligned}
\tag{4.80}
$$

On the event $H_{\mathbf{n};\vec{b}_T}(y,x)$, we denote the $\mathbf{n}$-double connections between the pivotal bonds $b_1,\dots,b_T$ by

$$
\mathcal{D}_{\mathbf{n};i} = \begin{cases} \mathcal{C}^{b_1}_{\mathbf{n}}(y) & (i=0),\\ \mathcal{C}^{b_{i+1}}_{\mathbf{n}}(y)\setminus\mathcal{C}^{b_i}_{\mathbf{n}}(y) & (i=1,\dots,T-1),\\ \mathcal{C}_{\mathbf{n}}(y)\setminus\mathcal{C}^{b_T}_{\mathbf{n}}(y) & (i=T). \end{cases}
\tag{4.81}
$$

As in Figure 7, we can think of $\mathcal{C}_{\mathbf{n}}(y)$ as the interval $[0,T]$, where each integer $i\in[0,T]$ corresponds to $\mathcal{D}_{\mathbf{n};i}$ and the unit interval $(i-1,i)\subset[0,T]$ corresponds to the pivotal bond b_i. Since $y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x$, we see that, for every b_i, there must be an $(\mathbf{m}+\mathbf{n})$-bypath (i.e., an $(\mathbf{m}+\mathbf{n})$-connection that does not go through b_i) from some $z\in\mathcal{D}_{\mathbf{n};s}$ with $s<i$ to some $z'\in\mathcal{D}_{\mathbf{n};t}$ with $t\ge i$. We abbreviate $\{s,t\}$ to st if there is no confusion. Let $\mathcal{L}^{(1)}_{[0,T]} = \{\{0T\}\}$, $\mathcal{L}^{(2)}_{[0,T]} = \{\{0t_1, s_2T\} : 0<s_2\le t_1<T\}$ and generally for $j\le T$ (see Figure 7),

$$
\mathcal{L}^{(j)}_{[0,T]} = \big\{\{s_it_i\}_{i=1}^j : 0=s_1<s_2\le t_1<s_3\le\cdots\le t_{j-2}<s_j\le t_{j-1}<t_j=T\big\}.
\tag{4.82}
$$

For every $j\in\{1,\dots,T\}$, we have $\bigcup_{st\in\Gamma}[s,t] = [0,T]$ for any $\Gamma\in\mathcal{L}^{(j)}_{[0,T]}$, which implies double connection. Conditioning on $\mathcal{C}_{\mathbf{n}}(y)\equiv\bigcup_{i=0}^T\mathcal{D}_{\mathbf{n};i} = \mathcal{B}$ (and denoting $\mathbf{k}=\mathbf{n}|_{\mathbb{B}_{\mathcal{B}^{\mathrm{c}}}}$, $\mathbf{h}=\mathbf{n}|_{\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{B}^{\mathrm{c}}}}$ and $\mathcal{D}_{\mathbf{n};i}\equiv\mathcal{D}_{\mathbf{h};i}=\mathcal{B}_i$) and multiplying by $Z_{\mathcal{B}^{\mathrm{c}}}/Z_{\mathcal{B}^{\mathrm{c}}}$, we obtain

$$
\begin{aligned}
(4.80) &= \sum_{\mathcal{B}\subset\Lambda}\sum_{T\ge1}\sum_{\vec{b}_T}\sum_{\substack{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing\\ \partial\mathbf{h}=y\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}}\,\frac{\tilde{w}_{\Lambda,\mathcal{B}}(\mathbf{h})\,Z_{\mathcal{B}^{\mathrm{c}}}}{Z_\Lambda}\,\frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}}\,\mathbb{1}_{\{y \underset{\mathbf{h}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}\cap H_{\mathbf{h};\vec{b}_T}(y,x)\cap\{\mathcal{C}_{\mathbf{h}}(y)=\mathcal{B}\}} \\
&\quad\times\sum_{j=1}^T\sum_{\{s_it_i\}_{i=1}^j\in\mathcal{L}^{(j)}_{[0,T]}}\sum_{\substack{z_1,\dots,z_j\\ z'_1,\dots,z'_j}}\Bigg(\prod_{i=1}^j\mathbb{1}_{\{z_i\in\mathcal{B}_{s_i},\ z'_i\in\mathcal{B}_{t_i}\}\cap\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_i\}}\Bigg)\prod_{i\ne l}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}}.
\end{aligned}
\tag{4.83}
$$

Reorganizing this expression and then summing over $\mathcal{B} \subset \mathcal{A}$, we obtain

$$\begin{aligned}(4.83) = \sum_{T\geq 1}\sum_{\vec{b}_T}\sum_{\partial\mathbf{n}=y\Delta x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\}\cap H_{\mathbf{n};\vec{b}_T}(y,x)} \\ \times \sum_{j=1}^{T} \sum_{\{s_i t_i\}_{i=1}^j \in \mathcal{L}_{[0,T]}^{(j)}} \sum_{\substack{z_1,\dots,z_j \\ z'_1,\dots,z'_j}} \bigg(\prod_{i=1}^j \mathbb{1}_{\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\ z'_i\in\mathcal{D}_{\mathbf{n};t_i}\}}\bigg) \\ \times \sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing} \frac{w_{\mathcal{A}^\mathrm{c}}(\mathbf{m})}{Z_{\mathcal{A}^\mathrm{c}}} \frac{w_{\tilde{\mathcal{D}}^\mathrm{c}}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^\mathrm{c}}} \bigg(\prod_{i=1}^j \mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_i\}}\bigg) \prod_{i\neq l} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}},\end{aligned} \tag{4.84}$$

where we have denoted $\mathcal{C}_\mathbf{n}(y)$ by $\tilde{\mathcal{D}}$. In the rightmost expression, the first line determines $\tilde{\mathcal{D}}$ that contains vertices z_i, z'_i for all $i = 1, \dots, j$ in a specific manner, while the second line determines the bypaths $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ joining z_i and z'_i for every $i = 1, \dots, j$. We first derive $\mathbf{n}$-independent bounds on these bypaths in the following paragraph (c-1). Then, in (c-2) below, we will bound the first two lines of the rightmost expression in (4.84).

(c-1) For $j = 1$, the last line of the rightmost expression in (4.84) simply equals

$$\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing} \frac{w_{\mathcal{A}^\mathrm{c}}(\mathbf{m})}{Z_{\mathcal{A}^\mathrm{c}}} \frac{w_{\tilde{\mathcal{D}}^\mathrm{c}}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^\mathrm{c}}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_1\}}. \tag{4.85}$$

Since $z_1, z'_1 \in \tilde{\mathcal{D}}$ and $z_1 \neq z'_1$, these two vertices are connected via a nonzero alternating chain of mutually-disjoint $\mathbf{m}$-connected clusters and mutually-disjoint $\mathbf{k}$-connected clusters. Moreover, since $z_1, z'_1 \in \tilde{\mathcal{D}}$ and $\mathbf{k} \in \mathbb{Z}_+^{\mathbb{B}_{\tilde{\mathcal{D}}^\mathrm{c}}}$, this chain of bubbles starts and ends with $\mathbf{m}$-connected clusters (possibly with a single $\mathbf{m}$-connected cluster), not with $\mathbf{k}$-connected clusters. Therefore, by following the argument around (4.58)–(4.62), we can easily show

$$(4.85) \leq \sum_{l\geq 1} (\tilde{G}_\Lambda^2)^{*(2l-1)}(z_1, z'_1). \tag{4.86}$$

For $j \geq 2$, since $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ for $i = 1, \dots, j$ are mutually-disjoint due to the last product of the indicators in (4.84), we can treat each bypath separately by the conditioning-on-clusters argument. By conditioning on $\mathcal{V}_{\mathbf{m}+\mathbf{k}} \equiv \dot{\bigcup}_{i\geq 2} \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$, the last line in the rightmost expression of (4.84) equals

$$\begin{aligned}\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing} \frac{w_{\mathcal{A}^\mathrm{c}}(\mathbf{m})}{Z_{\mathcal{A}^\mathrm{c}}} \frac{w_{\tilde{\mathcal{D}}^\mathrm{c}}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^\mathrm{c}}} \bigg(\prod_{i=2}^j \mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_i\}}\bigg) \bigg(\prod_{\substack{i,l\geq 2 \\ i\neq l}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}}\bigg) \\ \times \sum_{\partial\mathbf{m}'=\partial\mathbf{k}'=\varnothing} \frac{w_{\mathcal{A}^\mathrm{c}\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^\mathrm{c}}(\mathbf{m}')}{Z_{\mathcal{A}^\mathrm{c}\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^\mathrm{c}}} \frac{w_{\tilde{\mathcal{D}}^\mathrm{c}\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^\mathrm{c}}(\mathbf{k}')}{Z_{\tilde{\mathcal{D}}^\mathrm{c}\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^\mathrm{c}}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow} z'_1\}}.\end{aligned} \tag{4.87}$$

By using (4.86) (and replacing $\mathcal{A}^\mathrm{c}$ and $\tilde{\mathcal{D}}^\mathrm{c}$ in (4.85) by $\mathcal{A}^\mathrm{c} \cap \mathcal{V}_{\mathbf{m}+\mathbf{k}}^\mathrm{c}$ and $\tilde{\mathcal{D}}^\mathrm{c} \cap \mathcal{V}_{\mathbf{m}+\mathbf{k}}^\mathrm{c}$, respectively), the second line of (4.87) is bounded by $\sum_{l\geq 1}(\tilde{G}_\Lambda^2)^{*(2l-1)}(z_1, z'_1)$. Repeating the same argument until the remaining products of the indicators are used up, we obtain

$$(4.87) \leq \prod_{i=1}^j \sum_{l\geq 1} (\tilde{G}_\Lambda^2)^{*(2l-1)}(z_i, z'_i). \tag{4.88}$$

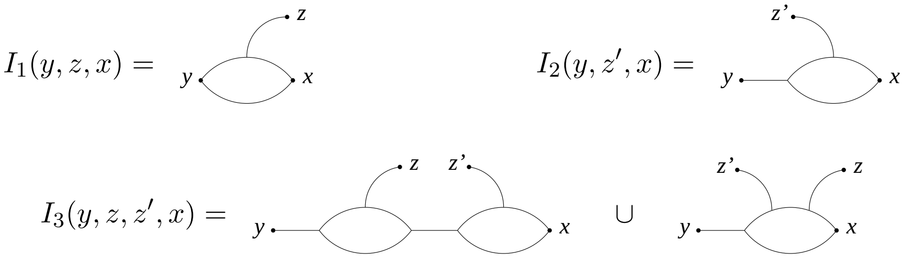


Figure 8: Schematic representations of $I_1(y,z,x)$, $I_2(y,z',x)$ and $I_3(y,z,z',x)$.

We have proved that

$$(4.84) \le \sum_{j\ge1}\sum_{\substack{z_1,\dots,z_j\\ z'_1,\dots,z'_j}} \Big(\prod_{i=1}^{j}\sum_{l\ge1}(\tilde{G}_\Lambda^2)^{*(2l-1)}(z_i,z'_i)\Big)\sum_{\partial\mathbf{n}=y\Delta x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{y\xleftrightarrow[\mathbf{n}]{\mathcal{A}}x\}}$$

$$\times\sum_{T\ge j}\sum_{\vec{b}_T}\sum_{\{s_it_i\}_{i=1}^{j}\in\mathcal{L}^{(j)}_{[0,T]}}\mathbb{1}_{H_{\mathbf{n};\vec{b}_T}(y,x)}\prod_{i=1}^{j}\mathbb{1}_{\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\,z'_i\in\mathcal{D}_{\mathbf{n};t_i}\}}. \tag{4.89}$$

(c-2) Since (4.89) depends only on a single current configuration, we may use Lemma 4.2 to obtain an upper bound. To do so, we first simplify the second line of (4.89), which is, by definition, equal to the indicator of the disjoint union

$$\dot{\bigcup_{T\ge j}}\;\dot{\bigcup_{\vec{b}_T}}\;\dot{\bigcup_{\{s_it_i\}_{i=1}^{j}\in\mathcal{L}^{(j)}_{[0,T]}}}\Big\{H_{\mathbf{n};\vec{b}_T}(y,x)\cap\bigcap_{i=1}^{j}\big\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\;z'_i\in\mathcal{D}_{\mathbf{n};t_i}\big\}\Big\} \tag{4.90}$$

$$=\dot{\bigcup_{e_1,\dots,e_j}}\Bigg\{\dot{\bigcup_{T\ge j}}\;\dot{\bigcup_{\vec{b}_T}}\;\dot{\bigcup_{\substack{\{s_it_i\}_{i=1}^{j}\in\mathcal{L}^{(j)}_{[0,T]}\\ b_{t_i+1}=e_{i+1}\ \forall i=0,\dots,j-1}}}\Big\{H_{\mathbf{n};\vec{b}_T}(y,x)\cap\bigcap_{i=1}^{j}\big\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\;z'_i\in\mathcal{D}_{\mathbf{n};t_i}\big\}\Big\}\Bigg\},$$

where $t_0=0$ by convention. On the left-hand side of (4.90), the first two unions identify the number and location of the pivotal bonds for $y\xleftrightarrow[\mathbf{n}]{}x$, and the third union identifies the indices of double connections associated with the bypaths between z_i and z'_i, for every $i=1,\dots,j$. The union over $e_1,\dots,e_j$ on the right-hand side identifies some of the pivotal bonds $b_1,\dots,b_T$ that are essential to decompose the chain of double connections $H_{\mathbf{n};\vec{b}_T}(y,x)$ into the following building blocks (see Figure 8):

$$I_1(y,z,x)=\{y\overset{}{\underset{\mathbf{n}}{\Longleftrightarrow}}x,\;y\xleftrightarrow[\mathbf{n}]{}z\},\qquad I_2(y,z',x)=\bigcup_u\big\{\{y\xleftrightarrow[\mathbf{n}]{}u\}\circ I_1(u,z',x)\big\}, \tag{4.91}$$

$$I_3(y,z,z',x)=\bigcup_u\Big\{\{I_2(y,z,u)\circ I_2(u,z',x)\}\cup\big\{\{y\xleftrightarrow[\mathbf{n}]{}u\}\circ\{I_1(u,z,x)\cap I_1(u,z',x)\}\big\}\Big\}. \tag{4.92}$$

For example, since $\mathcal{L}^{(1)}_{[0,T]} = \{\{0T\}\}$, we have

$$((4.90)\text{ for } j=1) = \dot{\bigcup_{e_1}}\dot{\bigcup_{T\geq 1}}\dot{\bigcup_{\vec{b}_T: b_1=e_1}} \Big\{H_{\mathbf{n};\vec{b}_T}(y,x) \cap \{z_1 \in \mathcal{D}_{\mathbf{n};0},\ z'_1 \in \mathcal{D}_{\mathbf{n};T}\}\Big\}$$

$$\subset \dot{\bigcup_{e_1}} \Big\{\{I_1(y,z_1,\underline{e}_1) \circ I_2(\overline{e}_1,z'_1,x)\} \cap \{n_{e_1} > 0,\ e_1 \text{ is pivotal for } y \underset{\mathbf{n}}{\longleftrightarrow} x\}\Big\}. \tag{4.93}$$

It is not hard to see in general that

$$((4.90)\text{ for } j\geq 2)$$
$$\subset \dot{\bigcup_{e_1,\dots,e_j}} \bigg\{\Big\{I_1(y,z_1,\underline{e}_1) \circ I_3(\overline{e}_1,z_2,z'_1,\underline{e}_2) \circ \cdots \circ I_3(\overline{e}_{j-1},z_j,z'_{j-1},\underline{e}_j) \circ I_2(\overline{e}_j,z'_j,x)\Big\}$$
$$\cap \bigcap_{i=1}^{j} \{n_{e_i} > 0,\ e_i \text{ is pivotal for } y \underset{\mathbf{n}}{\longleftrightarrow} x\}\bigg\}. \tag{4.94}$$

To bound (4.89) using Lemma 4.2, we further consider an event that includes (4.93)–(4.94) as subsets. Without losing generality, we can assume that $y \neq \underline{e}_1$, $\overline{e}_{i-1} \neq \underline{e}_i$ for $i = 2,\dots,j$, and $\overline{e}_j \neq x$; otherwise, the following argument can be simplified. We consider each event I_i in (4.93)–(4.94) individually, and to do so, we assume that y and $\underline{e}_1$ are the only sources for $I_1(y,z_1,\underline{e}_1)$, that $\overline{e}_{i-1}$ and $\underline{e}_i$ are the only sources for $I_3(\overline{e}_{i-1},z_i,z'_{i-1},\underline{e}_i)$ for every $i = 2,\dots,j$, and that $\overline{e}_j$ and x are the only sources for $I_2(\overline{e}_j,z'_j,x)$. This is because y and x are the only sources for the entire event (4.94), and every e_i is pivotal for $y \underset{\mathbf{n}}{\longleftrightarrow} x$.

On $I_1(y,z,x)$ with y,x being the only sources, according to the observation in Step (i) described below (4.23), we have two edge-disjoint connections from y to z, one of which may go through x, and another edge-disjoint connection from y to x (cf., $I_1(y,z,x)$ in Figure 8). Therefore,

$$I_1(y,z,x) \subset \big\{{}^\exists\omega_1,\omega_2 \in \Omega^{\mathbf{n}}_{y\to z}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{y\to x} \text{ such that } \omega_i \cap \omega_l = \varnothing\ (i\neq l)\big\}. \tag{4.95}$$

Similarly, for $I_2(y,z',x)$ with y,x being the only sources (cf., $I_2(y,z',x)$ in Figure 8),

$$I_2(y,z',x) \subset \big\{{}^\exists\omega_1,\omega_2 \in \Omega^{\mathbf{n}}_{x\to z'}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{y\to x} \text{ such that } \omega_i \cap \omega_l = \varnothing\ (i\neq l)\big\}. \tag{4.96}$$

On $I_3(y,z,z',x)$ with y,x being the only sources, there are at least three edge-disjoint paths, one from y to z, another one from z to z', and another one from z' to x. It is not hard to see this from $\bigcup_u\{I_2(y,z,u) \circ I_2(u,z',x)\}$ in (4.92), which corresponds to the first event depicted in Figure 8. It is also possible to extract such three edge-disjoint paths from the remaining event in (4.92). See the second event depicted in Figure 8 for one of the worst topological situations. Since there are at least three edge-disjoint paths between u and x, say, ζ_1,ζ_2 and ζ_3, we can go from y to z via ζ_1 and a part of ζ_2, and go from z to z' via the middle part of ζ_2, and then go from z' to x via the remaining part of ζ_2 and ζ_3. The other cases can be dealt with similarly. As a result, we have

$$I_3(y,z,z',x) \subset \big\{{}^\exists\omega_1 \in \Omega^{\mathbf{n}}_{y\to z}\ {}^\exists\omega_2 \in \Omega^{\mathbf{n}}_{z\to z'}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{z'\to x} \text{ such that } \omega_i \cap \omega_l = \varnothing\ (i\neq l)\big\}. \tag{4.97}$$

Since

$$\bigcup_e \Big\{\{\{{}^\exists\omega \in \Omega^{\mathbf{n}}_{z\to\underline{e}}\} \circ \{{}^\exists\omega \in \Omega^{\mathbf{n}}_{\overline{e}\to z'}\}\} \cap \{n_e > 0\}\Big\} \subset \{{}^\exists\omega \in \Omega^{\mathbf{n}}_{z\to z'}\}, \tag{4.98}$$

we see that (4.93) is a subset of

$$\tilde{I}^{(1)}_{z_1,z'_1}(y,x) = \left\{\begin{array}{c} {}^\exists\omega_1,\omega_2 \in \Omega^{\mathbf{n}}_{z_1\to y}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{y\to x}\ {}^\exists\omega_4,\omega_5 \in \Omega^{\mathbf{n}}_{x\to z'_1} \\ \text{such that } \omega_i \cap \omega_l = \varnothing\ (i\neq l) \end{array}\right\}, \tag{4.99}$$

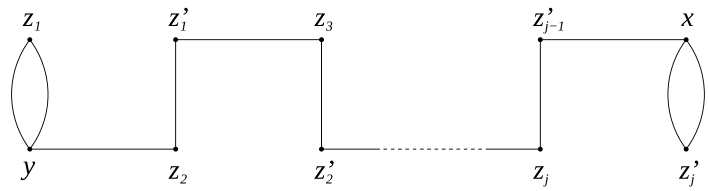


Figure 9: A schematic representation of $\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x)$ for $j \geq 2$ consisting of $2j+3$ edge-disjoint paths on $\mathbb{G}_{\mathbf{n}}$.

and that (4.94) is a subset of (see Figure 9)

$$\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x) = \left\{ \begin{array}{c} {}^\exists\omega_1,\omega_2 \in \Omega^{\mathbf{n}}_{z_1\to y} \; {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{y\to z_2} \; {}^\exists\omega_4 \in \Omega^{\mathbf{n}}_{z_2\to z'_1} \; {}^\exists\omega_5 \in \Omega^{\mathbf{n}}_{z'_1\to z_3} \cdots \\ \cdots {}^\exists\omega_{2j} \in \Omega^{\mathbf{n}}_{z_j\to z'_{j-1}} \; {}^\exists\omega_{2j+1} \in \Omega^{\mathbf{n}}_{z'_{j-1}\to x} \; {}^\exists\omega_{2j+2},\omega_{2j+3} \in \Omega^{\mathbf{n}}_{x\to z'_j} \\ \text{such that } \; \omega_i \cap \omega_l = \varnothing \; (i \neq l) \end{array} \right\}, \tag{4.100}$$

where $\vec{z}^{(\prime)}_j = (z^{(\prime)}_1,\dots,z^{(\prime)}_j)$. Therefore,

$$(4.89) \leq \sum_{j\geq 1} \sum_{\substack{z_1,\dots,z_j \\ z'_1,\dots,z'_j}} \left(\prod_{i=1}^{j} \sum_{l\geq 1} \big(\tilde{G}^2_\Lambda\big)^{*(2l-1)}(z_i,z'_i) \right) \sum_{\partial\mathbf{n}=y\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}} \mathbb{1}_{\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x)}. \tag{4.101}$$

Now we apply Lemma 4.2 to bound (4.101). To clearly understand how it is applied, for now we ignore $\mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}$ in (4.101) and only consider the contribution from $\mathbb{1}_{\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x)}$. Without losing generality, we assume that y, x, z_i, z'_i for $i = 1,\dots,j$ are all different. Since there are $2j+3$ edge-disjoint paths on $\mathbb{G}_{\mathbf{n}}$ as in (4.99)–(4.100) (see also Figure 9), we multiply (4.101) by $(Z_\Lambda/Z_\Lambda)^{2j+2}$, following Step (ii) of the strategy described in Section 4.2. Overlapping the $2j+3$ current configurations and using Lemma 4.2 with $\mathcal{V} = \{y,x\}$ and $k = 2j+2$, we obtain

$$\begin{aligned} \sum_{\partial\mathbf{n}=y\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x)} &\leq \langle\varphi_{z_1}\varphi_y\rangle^2_\Lambda \langle\varphi_x\varphi_{z'_j}\rangle^2_\Lambda \qquad (4.102) \\ &\quad\times \begin{cases} \langle\varphi_y\varphi_x\rangle_\Lambda & (j=1), \\ \langle\varphi_y\varphi_{z_2}\rangle_\Lambda \langle\varphi_{z_2}\varphi_{z'_1}\rangle_\Lambda \left(\prod_{i=2}^{j-1} \langle\varphi_{z'_{i-1}}\varphi_{z_{i+1}}\rangle_\Lambda \langle\varphi_{z_{i+1}}\varphi_{z'_i}\rangle_\Lambda \right) \langle\varphi_{z'_{j-1}}\varphi_x\rangle_\Lambda & (j\geq 2). \end{cases} \end{aligned}$$

Note that, by (4.2), we have

$$\left. \begin{array}{r} \sum_{l\geq 1} (\tilde{G}^2_\Lambda)^{*(2l-1)}(y,x) \\ \sum_z \langle\varphi_z\varphi_y\rangle^2_\Lambda \sum_{l\geq 1} (\tilde{G}^2_\Lambda)^{*(2l-1)}(z,x) \\ \sum_{z'} \langle\varphi_x\varphi_{z'}\rangle^2_\Lambda \sum_{l\geq 1} (\tilde{G}^2_\Lambda)^{*(2l-1)}(y,z') \end{array} \right\} \leq \psi_\Lambda(y,x) - \delta_{y,x}, \tag{4.103}$$

$$\sum_{z,z'} \langle\varphi_z\varphi_y\rangle^2_\Lambda \langle\varphi_x\varphi_{z'}\rangle^2_\Lambda \sum_{l\geq 1} \big(\tilde{G}^2_\Lambda\big)^{*(2l-1)}(z,z') \leq 2\big(\psi_\Lambda(y,x) - \delta_{y,x}\big). \tag{4.104}$$

Therefore, (4.101) without $\mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}$ is bounded by

$$\begin{aligned}
&\langle\varphi_y\varphi_x\rangle_\Lambda \sum_{z_1,z_1'} \langle\varphi_{z_1}\varphi_y\rangle_\Lambda^2 \langle\varphi_x\varphi_{z_1'}\rangle_\Lambda^2 \sum_{l\geq 1} \big(\tilde{G}_\Lambda^2\big)^{*(2l-1)}(z_1,z_1') \\
&+\sum_{j\geq 2} \sum_{\substack{z_2,\dots,z_j \\ z_1',\dots,z_{j-1}'}} \bigg(\prod_{i=2}^{j-1} \big(\psi_\Lambda(z_i,z_i') - \delta_{z_i,z_i'}\big)\bigg) \bigg(\sum_{z_1} \langle\varphi_y\varphi_{z_1}\rangle_\Lambda^2 \sum_{l\geq 1} \big(\tilde{G}_\Lambda^2\big)^{*(2l-1)}(z_1,z_1')\bigg) \\
&\qquad \times \bigg(\sum_{z_j'} \langle\varphi_x\varphi_{z_j'}\rangle_\Lambda^2 \sum_{l\geq 1} \big(\tilde{G}_\Lambda^2\big)^{*(2l-1)}(z_j,z_j')\bigg) \langle\varphi_y\varphi_{z_2}\rangle_\Lambda \langle\varphi_{z_2}\varphi_{z_1'}\rangle_\Lambda \\
&\qquad \times \bigg(\prod_{i=2}^{j-1} \langle\varphi_{z_{i-1}'}\varphi_{z_{i+1}}\rangle_\Lambda \langle\varphi_{z_{i+1}}\varphi_{z_i'}\rangle_\Lambda\bigg) \langle\varphi_{z_{j-1}'}\varphi_x\rangle_\Lambda \leq \sum_{j\geq 1} P_\Lambda^{(j)}(y,x).
\end{aligned} \tag{4.105}$$

If $\mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}$ is present in the above argument, then at least one of the paths ω_i for $i = 3,\dots,2j+1$ has to go through $\mathcal{A}$. For example, if ω_3 ($\in \Omega_{y\to z_2}^{\mathbf{n}}$) goes through $\mathcal{A}$, then we can split it into two edge-disjoint paths at some $u \in \mathcal{A}$, such as $\omega_3' \in \Omega_{y\to u}^{\mathbf{n}}$ and $\omega_3'' \in \Omega_{u\to z_2}^{\mathbf{n}}$. The contribution from this case is bounded, by following the same argument as above, by (4.102) with $\langle\varphi_y\varphi_{z_2}\rangle_\Lambda$ being replaced by $\sum_{u\in\mathcal{A}} \langle\varphi_y\varphi_u\rangle_\Lambda \langle\varphi_u\varphi_{z_2}\rangle_\Lambda$. Bounding the other $2j-2$ cases similarly and summing these bounds over $j \geq 1$, we obtain

$$(4.101) \leq \sum_{u\in\mathcal{A}} \sum_{j\geq 1} P_{\Lambda;u}'^{(j)}(y,x). \tag{4.106}$$

This together with (4.66) in the above paragraph (a) complete the proof of the bound on $\Theta_{y,x;\mathcal{A}}'$ in (4.35). ☐

(d) Finally, we investigate the contribution to $\Theta_{y,x,v;\mathcal{A}}''$ from $\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}\setminus\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ in (4.63):

$$\sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=y\vartriangle x}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\} \cap \{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}\setminus\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}\} \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}}. \tag{4.107}$$

Using $H_{\mathbf{n};\vec{b}_T}(y,x)$ defined in (4.79), we can write (4.107) as (cf., (4.80))

$$(4.107) = \sum_{T\geq 1} \sum_{\vec{b}_T} \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=y\vartriangle x}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\} \cap H_{\mathbf{n};\vec{b}_T}(y,x) \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\} \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}}. \tag{4.108}$$

To bound this, we will also use a similar expression to (4.84), in which $\mathbf{k} = \mathbf{n}|_{\mathbb{B}_{\tilde{\mathcal{D}}^c}}$ with $\tilde{\mathcal{D}} = \mathcal{C}_{\mathbf{n}}^b(y)$. We investigate (4.108) separately (in the following paragraphs (d-1) and (d-2)) depending on whether or not there is a bypath $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ for some $i \in \{1,\dots,j\}$ containing v.

(d-1) If there is such a bypath, then we use $\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}} \leq \mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}$ as in (4.80) to bound the contribution from this case to (4.108) by

$$\begin{aligned}
&\sum_{T\geq 1} \sum_{\vec{b}_T} \sum_{\partial\mathbf{n}=y\vartriangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\} \cap H_{\mathbf{n};\vec{b}_T}(y,x)} \sum_{j=1}^{T} \sum_{\{s_it_i\}_{i=1}^j \in \mathcal{L}_{[0,T]}^{(j)}} \sum_{\substack{z_1,\dots,z_j \\ z_1',\dots,z_j'}} \bigg(\prod_{i=1}^{j} \mathbb{1}_{\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\ z_i'\in\mathcal{D}_{\mathbf{n};t_i}\}}\bigg) \\
&\quad \times \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{k}=\varnothing}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_{\tilde{\mathcal{D}}^c}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^c}} \bigg(\prod_{i=1}^{j} \mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z_i'\}}\bigg) \bigg(\prod_{i\neq l} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}}\bigg) \sum_{i=1}^{j} \mathbb{1}_{\{v\in\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\}}.
\end{aligned} \tag{4.109}$$

Note that the last sum of the indicators is the only difference from (4.84).

When $j = 1$, the second line of (4.109) equals

$$\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_{\tilde{\mathcal{D}}^c}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^c}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z_1'\}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v\}}. \tag{4.110}$$

As described in (4.85)–(4.86), we can bound (4.110) without $\mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v\}}$ by a chain of bubbles $\sum_{l\geq1}(\tilde{G}_\Lambda^2)^{*(2l-1)}(z_1, z_1')$. If $\mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v\}} = 1$, then, by the argument around (4.58)–(4.62), one of the bubbles has an extra vertex v' that is further connected to v with another chain of bubbles $\psi_\Lambda(v', v)$. That is, the effect of $\mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v\}}$ is to replace one of the $\tilde{G}_\Lambda$'s in the chain of bubbles, say, $\tilde{G}_\Lambda(a, a')$, by $\sum_{v'}(\langle\varphi_a\varphi_{v'}\rangle_\Lambda\tilde{G}_\Lambda(v', a') + \tilde{G}_\Lambda(a, a')\delta_{v',a'})\,\psi_\Lambda(v', v)$. Let

$$g_{\Lambda;y}(z, z') = \sum_{l\geq1}\sum_{i=1}^{2l-1}\sum_{a,a'} \big(\tilde{G}_\Lambda^2\big)^{*(i-1)}(z, a)\,\tilde{G}_\Lambda(a, a')\,\big(\tilde{G}_\Lambda^2\big)^{*(2l-1-i)}(a', z') \\ \times\Big(\langle\varphi_a\varphi_y\rangle_\Lambda\tilde{G}_\Lambda(y, a') + \tilde{G}_\Lambda(a, a')\,\delta_{y,a'}\Big). \tag{4.111}$$

Then, we have

$$(4.110) \leq \sum_{v'} g_{\Lambda;v'}(z_1, z_1')\,\psi_\Lambda(v', v). \tag{4.112}$$

Let $j \geq 2$ and consider the contribution to (4.109) from $\mathbb{1}_{\{v\in\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_1)\}}$; the contribution from $\mathbb{1}_{\{v\in\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\}}$ with $i \neq 1$ can be estimated in the same way. By conditioning on $\mathcal{V}_{\mathbf{m}+\mathbf{k}} \equiv \dot{\bigcup}_{i\geq2}\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ as in (4.87), the contribution to the second line of (4.109) from $\mathbb{1}_{\{v\in\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_1)\}} \equiv \mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v\}}$ equals

$$\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_{\tilde{\mathcal{D}}^c}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^c}} \bigg(\prod_{i=2}^{j}\mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z_i'\}}\bigg)\bigg(\prod_{\substack{i,i'\geq2\\ i\neq i'}}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i'})=\varnothing\}}\bigg) \\ \times\sum_{\partial\mathbf{m}'=\partial\mathbf{k}'=\varnothing} \frac{w_{\mathcal{A}^c\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^c}(\mathbf{m}')}{Z_{\mathcal{A}^c\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^c}} \frac{w_{\tilde{\mathcal{D}}^c\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^c}(\mathbf{k}')}{Z_{\tilde{\mathcal{D}}^c\cap\mathcal{V}_{\mathbf{m}+\mathbf{k}}^c}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow} z_1'\}}\mathbb{1}_{\{z_1 \underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow} v\}}, \tag{4.113}$$

where the second line is bounded by (4.112) for $j = 1$, and then the first line is bounded by $\prod_{i=2}^{j}\sum_{l\geq1}(\tilde{G}_\Lambda^2)^{*(2l-1)}(z_i, z_i')$, due to (4.87)–(4.88).

Summarizing the above bounds, we have (cf., (4.101))

$$(4.109) \leq \sum_{j\geq1}\sum_{\substack{z_1,\dots,z_j\\ z_1',\dots,z_j'}}\bigg(\sum_{h=1}^{j}\sum_{v'} g_{\Lambda;v'}(z_h, z_h')\,\psi_\Lambda(v', v)\prod_{i\neq h}\sum_{l\geq1}\big(\tilde{G}_\Lambda^2\big)^{*(2l-1)}(z_i, z_i')\bigg) \\ \times\sum_{\partial\mathbf{n}=y\triangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\}}\mathbb{1}_{\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x)}, \tag{4.114}$$

to which we can apply the bound discussed between (4.80) and (4.106).

(d-2) If $v \notin \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ for any $i = 1, \dots, j$, then there exists a $v' \in \mathcal{D}_{\mathbf{n};l}$ for some $l \in \{0, \dots, T\}$ such that $v' \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v$ and $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(v')\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i) = \varnothing$ for any i. In addition, since all connections from

y to x on the graph $\tilde{\mathcal{D}} \cup \dot{\bigcup}_{i=1}^{j} \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ have to go through $\mathcal{A}$, there is an $h \in \{1, \dots, j\}$ such that $z_h \underset{\mathbf{m}+\mathbf{k}}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_h$. Therefore, the contribution from this case to (4.108) is bounded by

$$\sum_{T \geq 1} \sum_{\vec{b}_T} \sum_{\partial \mathbf{n} = y \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{H_{\mathbf{n};\vec{b}_T}(y,x)} \sum_{j=1}^{T} \sum_{\{s_i t_i\}_{i=1}^{j} \in \mathcal{L}_{[0,T]}^{(j)}} \sum_{\substack{v', z_1, \dots, z_j \\ z'_1, \dots, z'_j}} \bigg(\prod_{i=1}^{j} \mathbb{1}_{\{z_i \in \mathcal{D}_{\mathbf{n};s_i},\, z'_i \in \mathcal{D}_{\mathbf{n};t_i}\}} \bigg) \sum_{l=0}^{T} \mathbb{1}_{\{v' \in \mathcal{D}_{\mathbf{n};l}\}}$$
$$\times \sum_{\substack{\partial \mathbf{m} = \varnothing \\ \partial \mathbf{k} = \varnothing}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}} \frac{w_{\tilde{\mathcal{D}}^{\mathrm{c}}}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^{\mathrm{c}}}} \bigg(\sum_{h=1}^{j} \mathbb{1}_{\{z_h \underset{\mathbf{m}+\mathbf{k}}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_h\}} \prod_{i=1}^{j} \mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_i\}} \bigg) \bigg(\prod_{i \neq i'} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i) \cap \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i'}) = \varnothing\}} \bigg)$$
$$\times \mathbb{1}_{\{v' \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v\}} \prod_{i=1}^{j} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(v') \cap \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i) = \varnothing\}}, \tag{4.115}$$

where, by conditioning on $\mathcal{S}_{\mathbf{m}+\mathbf{k}} \equiv \dot{\bigcup}_{i=1}^{j} \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$, the last two lines are (see below (4.87))

$$\sum_{\partial \mathbf{m} = \partial \mathbf{k} = \varnothing} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}} \frac{w_{\tilde{\mathcal{D}}^{\mathrm{c}}}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^{\mathrm{c}}}} \bigg(\sum_{h=1}^{j} \mathbb{1}_{\{z_h \underset{\mathbf{m}+\mathbf{k}}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_h\}} \prod_{i=1}^{j} \mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_i\}} \bigg) \bigg(\prod_{i \neq i'} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i) \cap \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i'}) = \varnothing\}} \bigg)$$
$$\times \underbrace{\sum_{\partial \mathbf{m}'' = \partial \mathbf{k}'' = \varnothing} \frac{w_{\mathcal{A}^{\mathrm{c}} \cap \mathcal{S}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}(\mathbf{m}'')}{Z_{\mathcal{A}^{\mathrm{c}} \cap \mathcal{S}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}} \frac{w_{\tilde{\mathcal{D}}^{\mathrm{c}} \cap \mathcal{S}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}(\mathbf{k}'')}{Z_{\tilde{\mathcal{D}}^{\mathrm{c}} \cap \mathcal{S}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}} \mathbb{1}_{\{v' \underset{\mathbf{m}''+\mathbf{k}''}{\longleftrightarrow} v\}}}_{\leq \psi_\Lambda(v', v)}. \tag{4.116}$$

When $j = 1$, we have

$$((4.116) \text{ for } j = 1) \leq \psi_\Lambda(v', v) \sum_{\partial \mathbf{m} = \partial \mathbf{k} = \varnothing} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}} \frac{w_{\tilde{\mathcal{D}}^{\mathrm{c}}}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^{\mathrm{c}}}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_1\}}. \tag{4.117}$$

If we ignore the "through $\mathcal{A}$"-condition in the last indicator, then the sum is bounded, as in (4.86), by a chain of bubbles $\sum_{l \geq 1} (\tilde{G}_\Lambda^2)^{*(2l-1)}(z_1, z'_1)$. However, because of this condition, one of the $\tilde{G}_\Lambda$'s in the bound, say, $\tilde{G}_\Lambda(a, a')$, is replaced by $\sum_{u \in \mathcal{A}} (\langle \varphi_a \varphi_u \rangle_\Lambda \tilde{G}_\Lambda(u, a') + \tilde{G}_\Lambda(a, a') \delta_{u,a'})$. Using (4.111), we have

$$(4.117) \leq \psi_\Lambda(v', v) \sum_{y \in \mathcal{A}} g_{\Lambda;y}(z_1, z'_1). \tag{4.118}$$

Let $j \geq 2$ and consider the contribution to (4.116) from $\mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_1\}}$; the contributions from $\mathbb{1}_{\{z_h \underset{\mathbf{m}+\mathbf{k}}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_h\}}$ with $h \neq 1$ can be estimated similarly. By conditioning on $\mathcal{V}_{\mathbf{m}+\mathbf{k}} \equiv \dot{\bigcup}_{i \geq 2} \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$, the contribution to (4.116) from $\mathbb{1}_{\{z_1 \underset{\mathbf{m}+\mathbf{k}}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_1\}}$ equals

$$\sum_{\partial \mathbf{m} = \partial \mathbf{k} = \varnothing} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}} \frac{w_{\tilde{\mathcal{D}}^{\mathrm{c}}}(\mathbf{k})}{Z_{\tilde{\mathcal{D}}^{\mathrm{c}}}} \bigg(\prod_{i=2}^{j} \mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_i\}} \bigg) \bigg(\prod_{\substack{i, i' \geq 2 \\ i \neq i'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i) \cap \mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i'}) = \varnothing\}} \bigg)$$
$$\times \psi_\Lambda(v', v) \sum_{\partial \mathbf{m}' = \partial \mathbf{k}' = \varnothing} \frac{w_{\mathcal{A}^{\mathrm{c}} \cap \mathcal{V}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}(\mathbf{m}')}{Z_{\mathcal{A}^{\mathrm{c}} \cap \mathcal{V}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}} \frac{w_{\tilde{\mathcal{D}}^{\mathrm{c}} \cap \mathcal{V}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}(\mathbf{k}')}{Z_{\tilde{\mathcal{D}}^{\mathrm{c}} \cap \mathcal{V}_{\mathbf{m}+\mathbf{k}}^{\mathrm{c}}}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}'+\mathbf{k}'}{\overset{\mathcal{A}}{\longleftrightarrow}} z'_1\}}, \tag{4.119}$$

where the second line is bounded by (4.118) for $j = 1$, and then the first line is bounded by $\prod_{i=2}^{j} \sum_{l \geq 1} (\tilde{G}_\Lambda^2)^{*(2l-1)}(z_i, z'_i)$, as described below (4.113).

As a result, (4.115) is bounded by

$$\sum_{j\ge1}\sum_{\substack{v',z_1,\dots,z_j\\ z'_1,\dots,z'_j}}\psi_\Lambda(v',v)\bigg(\sum_{h=1}^{j}\sum_{y\in\mathcal{A}}g_{\Lambda;y}(z_h,z'_h)\prod_{i\ne h}\sum_{l\ge1}\big(\tilde G_\Lambda^2\big)^{*(2l-1)}(z_i,z'_i)\bigg)$$

$$\times\sum_{\partial\mathbf{n}=y\triangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\sum_{T\ge j}\sum_{\vec b_T}\sum_{\{s_it_i\}_{i=1}^{j}\in\mathcal{L}_{[0,T]}^{(j)}}\mathbb{1}_{H_{\mathbf{n};\vec b_T}(y,x)}\bigg(\prod_{i=1}^{j}\mathbb{1}_{\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\,z'_i\in\mathcal{D}_{\mathbf{n};t_i}\}}\bigg)\sum_{l=0}^{T}\mathbb{1}_{\{v'\in\mathcal{D}_{\mathbf{n};l}\}}. \quad (4.120)$$

The second line can be bounded by following the argument between (4.89) and (4.105); note that the sum of the indicators in (4.120), except for the last factor $\sum_{l=0}^{T}\mathbb{1}_{\{v'\in\mathcal{D}_{\mathbf{n};l}\}}$, is identical to that in (4.89). First, we rewrite the sum of the indicators in (4.120) as a single indicator of an event $\mathcal{E}$ similar to (4.90). Then, we construct another event similar to $\tilde I^{(j)}_{\vec z_j,\vec z'_j}(y,x)$ in (4.99)–(4.100), of which $\mathcal{E}$ is a subset. Due to $\sum_{l=0}^{T}\mathbb{1}_{\{v'\in\mathcal{D}_{\mathbf{n};l}\}}$ in (4.120), one of the paths in the definition of $\tilde I^{(j)}_{\vec z_j,\vec z'_j}(y,x)$, say, $\omega_i\in\Omega^{\mathbf{n}}_{a\to a'}$ for some a,a' (depending on i) is split into two edge-disjoint paths $\omega'_i\in\Omega^{\mathbf{n}}_{a\to v'}$ and $\omega''_i\in\Omega^{\mathbf{n}}_{v'\to a'}$, followed by the summation over $i=3,\dots,2j+1$ (cf., Figure 9). Finally, we apply Lemma 4.2 to obtain the desired bound on the last line of (4.120).

Summarizing the above (d-1) and (d-2), we obtain

$$(4.108)\le\sum_{j\ge1}\sum_{u\in\mathcal{A}}P''^{(j)}_{\Lambda;u,v}(y,x). \quad (4.121)$$

This together with (4.77) in the above paragraph (b) complete the proof of the bound on $\Theta''_{y,x,v;\mathcal{A}}$ in (4.35). ☐

5 Bounds on $\pi_\Lambda^{(j)}(x)$ assuming the decay of $G(x)$

Using the diagrammatic bounds proved in the previous section, we prove Proposition 3.1 in Section 5.1, and Propositions 3.2 and 3.3(iii) in Section 5.2.

5.1 Bounds for the spread-out model

We prove Proposition 3.1 for the spread-out model using the following convolution bounds:

Proposition 5.1. *(i) Let $a\ge b>0$ and $a+b>d$. There is a $C=C(a,b,d)$ such that*

$$\sum_y\frac{1}{|||y-v|||^a}\,\frac{1}{|||x-y|||^b}\le\frac{C}{|||x-v|||^{(a\wedge d+b)-d}}. \quad (5.1)$$

(ii) Let $q\in(\frac{d}{2},d)$. There is a $C'=C'(d,q)$ such that

$$\sum_z\frac{1}{|||x-z|||^q}\,\frac{1}{|||x'-z|||^q}\,\frac{1}{|||z-y|||^q}\,\frac{1}{|||z-y'|||^q}\le\frac{C'}{|||x-y|||^q|||x'-y'|||^q}. \quad (5.2)$$

Proof. The inequality (5.1) is identical to [15, Proposition 1.7(i)]. We use this to prove (5.2). By the triangle inequality, we have $\frac12|||x-y|||\le|||x-z|||\vee|||z-y|||$ and $\frac12|||x'-y'|||\le|||x'-z|||\vee|||z-y'|||$. Suppose that $|||x-z|||\le|||z-y|||$ and $|||x'-z|||\le|||z-y'|||$. Then, by (5.1) with $a=b=q$, the contribution from this case is bounded by

$$\frac{2^{2q}}{|||x-y|||^q|||x'-y'|||^q}\sum_z\frac{1}{|||x-z|||^q}\,\frac{1}{|||x'-z|||^q}\le\frac{2^{2q}c|||x-x'|||^{d-2q}}{|||x-y|||^q|||x'-y'|||^q}, \quad (5.3)$$

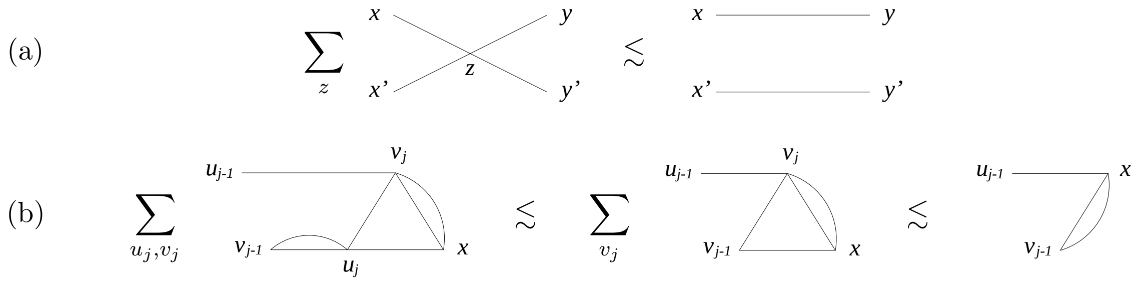


Figure 10: (a) A schematic representation of Proposition 5.1(i), where each segment, say, from x to y represent $\|x-y\|^{-q}$. (b) A schematic representation of (5.19), which is a result of successive applications of Proposition 5.1(ii) with $x=x'$ or $y=y'$.

for some $c<\infty$, where we note that $\|x-x'\|^{d-2q}\le 1$ because of $\frac{1}{2}d<q$. The other three possible cases can be estimated similarly (see Figure 10(a)). This completes the proof of Proposition 5.1. $\square$

Before going into the proof of Proposition 3.1, we summarize prerequisites. Recall that (4.13)–(4.14) involve $\tilde G_\Lambda$, and note that, by (4.2),

$$\langle\varphi_o\varphi_x\rangle_\Lambda^3\le\delta_{o,x}+\tilde G_\Lambda(o,x)^3. \tag{5.4}$$

We first show that

$$\tilde G_\Lambda(o,x)\le\frac{O(\theta_0)}{\|x\|^q},\qquad\qquad \sum_{b:\underline{b}=o}\tau_b\big(\delta_{\overline{b},x}+\tilde G_\Lambda(\overline{b},x)\big)\le\frac{O(\theta_0)}{\|x\|^q} \tag{5.5}$$

hold assuming the bounds in (3.2).

Proof. By the assumed bound $\tau\le 2$ in (3.2), we have

$$\tilde G_\Lambda(o,x)=\tau D(x)+\sum_{y\ne x}\tau D(y)\,\langle\varphi_y\varphi_x\rangle_\Lambda\le 2D(x)+\sum_{y\ne x}2D(y)\,G(x-y), \tag{5.6}$$

where, and from now on without stating explicitly, we use the translation invariance of $G(x)$ and the fact that $G(x-y)$ is an increasing limit of $\langle\varphi_y\varphi_x\rangle_\Lambda$ as $\Lambda\uparrow\mathbb{Z}^d$. By (1.14) and the assumption in Proposition 3.1 that θ_0L^{d-q}, with $q<d$, is bounded away from zero, we obtain

$$D(x)\le O(L^{-d})\mathbb{1}_{\{0<\|x\|_\infty\le L\}}\le\frac{O(L^{-d+q})}{\|x\|^q}\le\frac{O(\theta_0)}{\|x\|^q}. \tag{5.7}$$

For the last term in (5.6), we consider the cases for $|x|\le 2\sqrt{d}L$ and $|x|\ge 2\sqrt{d}L$ separately.

When $|x|\le 2\sqrt{d}L$, we use (5.7), (3.2) and (5.1) with $\frac{1}{2}d<q<d$ to obtain

$$\sum_{y\ne x}D(y)\,G(x-y)\le\sum_y\frac{O(L^{-d+q})}{\|y\|^q}\,\frac{\theta_0}{\|x-y\|^q}\le\frac{O(\theta_0L^{-d+q})}{\|x\|^{2q-d}}\le\frac{O(\theta_0)}{\|x\|^q}. \tag{5.8}$$

When $|x|\ge 2\sqrt{d}L$, we use the triangle inequality $|x-y|\ge|x|-|y|$ and the fact that $D(y)$ is nonzero only when $0<\|y\|_\infty\le L$ (so that $|y|\le\sqrt{d}\|y\|_\infty\le\sqrt{d}L\le\frac{1}{2}|x|$). Then, we obtain

$$\sum_{y\ne x}D(y)\,G(x-y)\le\sum_y D(y)\,\frac{2^q\theta_0}{\|x\|^q}=\frac{2^q\theta_0}{\|x\|^q}. \tag{5.9}$$

This completes the proof of the first inequality in (5.5). The second inequality can be proved similarly. $\square$

By repeated use of (5.5) and Proposition 5.1(i) with $a = b = 2q$ (or Proposition 5.1(ii) with $x = x'$ and $y = y'$), we obtain

$$\psi_\Lambda(v', v) \le \delta_{v',v} + \frac{O(\theta_0^2)}{|||v - v'|||^{2q}}. \tag{5.10}$$

Together with the naive bound $G(x) \le O(1)|||x|||^{-q}$ (cf., (3.2)) as well as Proposition 5.1(ii) (with $x = x'$ or $y = y'$), we also obtain

$$\begin{aligned}\sum_{v'} G(v'-y)\, G(z-v')\, \psi_\Lambda(v', v) &\le G(v-y)\, G(z-v) + \sum_{v'} \frac{O(\theta_0^2)}{|||v'-y|||^q |||z-v'|||^q |||v-v'|||^{2q}} \\ &\le \frac{O(1)}{|||v-y|||^q |||z-v|||^q}. \end{aligned} \tag{5.11}$$

The $O(1)$ term in the right-hand side is replaced by $O(\theta_0)$ or $O(\theta_0^2)$ depending on the number of G's on the left being replaced by $\tilde{G}_\Lambda$'s.

Proof of Proposition 3.1. Since (5.4)–(5.5) immediately imply the bound on $\pi_\Lambda^{(0)}(x)$, it suffices to prove the bounds on $\pi_\Lambda^{(i)}(x)$ for $i \ge 1$. To do so, we first estimate the building blocks of the diagrammatic bound (4.15): $\sum_{b:\underline{b}=y} \tau_b\, Q'_{\Lambda;u}(\overline{b}, x)$ and $\sum_{b:\underline{b}=y} \tau_b\, Q''_{\Lambda;u,v}(\overline{b}, x)$.

Recall (4.10)–(4.14). First, by using $G(x) \le O(1)|||x|||^{-q}$ and (5.11), we obtain

$$P'^{(0)}_{\Lambda;u}(y, x) \le \frac{O(1)}{|||x-y|||^{2q} |||u-y|||^q |||x-u|||^q}, \tag{5.12}$$

$$P''^{(0)}_{\Lambda;u,v}(y, x) \le \frac{O(1)}{|||x-y|||^q |||u-y|||^q |||x-u|||^q |||v-y|||^q |||x-v|||^q}. \tag{5.13}$$

We will show at the end of this subsection that, for $j \ge 1$,

$$P'^{(j)}_{\Lambda;u}(y, x) \le \frac{O(j)\, O(\theta_0^2)^j}{|||x-y|||^{2q} |||u-y|||^q |||x-u|||^q}, \tag{5.14}$$

$$P''^{(j)}_{\Lambda;u,v}(y, x) \le \frac{O(j^2)\, O(\theta_0^2)^j}{|||x-y|||^q |||u-y|||^q |||x-u|||^q |||v-y|||^q |||x-v|||^q}. \tag{5.15}$$

As a result, $P'^{(0)}_{\Lambda;u}(y, x)$ (resp., $P''^{(0)}_{\Lambda;u,v}(y, x)$) is the leading term of $P'_{\Lambda;u}(y, x)$ (resp., $P''_{\Lambda;u,v}(y, x)$), which thus obeys the same bound as in (5.12) (resp., (5.13)), with a different constant in $O(1)$. Combining these bounds with (5.5) and (5.11) (with both G in the left-hand side being replace by $\tilde{G}_\Lambda$) and then using Proposition 5.1(ii), we obtain

$$\sum_{b:\underline{b}=y} \tau_b\, Q'_{\Lambda;u}(\overline{b}, x) \le \sum_z \frac{O(\theta_0)}{|||z-y|||^q} \frac{1}{|||x-z|||^{2q} |||u-z|||^q |||x-u|||^q} \le \frac{O(\theta_0)}{|||x-y|||^q |||x-u|||^{2q}}, \tag{5.16}$$

and

$$\begin{aligned}\sum_{b:\underline{b}=y} \tau_b\, Q''_{\Lambda;u,v}(\overline{b}, x) &\le \sum_z \frac{O(\theta_0)}{|||z-y|||^q} \frac{1}{|||x-z|||^q |||u-z|||^q |||x-u|||^q |||v-z|||^q |||x-v|||^q} \\ &\qquad + \sum_z \frac{O(\theta_0)}{|||v-y|||^q} \frac{O(\theta_0)}{|||z-v|||^q} \frac{1}{|||x-z|||^{2q} |||u-z|||^q |||x-u|||^q} \\ &\le \frac{O(\theta_0)}{|||v-y|||^q |||x-v|||^q |||x-u|||^{2q}}. \end{aligned} \tag{5.17}$$

This completes bounding the building blocks.

Now we prove the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \geq 1$. For the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \geq 2$, we simply apply (5.12) and (5.16)–(5.17) to the diagrammatic bound (4.15). Then, we obtain

$$\pi_\Lambda^{(j)}(x) \leq \sum_{\substack{u_1,\dots,u_j \\ v_1,\dots,v_j}} \frac{O(1)}{\|u_1\|^{2q}\|v_1\|^q\|u_1-v_1\|^q} \bigg(\prod_{i=1}^{j-1} \frac{O(\theta_0)}{\|v_{i+1}-u_i\|^q\|u_{i+1}-v_{i+1}\|^q\|u_{i+1}-v_i\|^{2q}} \bigg) \times \frac{O(\theta_0)}{\|x-u_j\|^q\|x-v_j\|^{2q}} \qquad (j \geq 2). \tag{5.18}$$

First, we consider the sum over u_j and v_j. By successive applications of Proposition 5.1(ii) (with $x = x'$ or $y = y'$), we obtain (see Figure 10(b))

$$\sum_{v_j}\sum_{u_j} \frac{O(\theta_0)}{\|v_j-u_{j-1}\|^q\|u_j-v_j\|^q\|u_j-v_{j-1}\|^{2q}} \frac{O(\theta_0)}{\|x-u_j\|^q\|x-v_j\|^{2q}} \tag{5.19}$$
$$\leq \sum_{v_j} \frac{O(\theta_0)^2}{\|v_j-u_{j-1}\|^q\|v_{j-1}-v_j\|^q\|x-v_{j-1}\|^q\|x-v_j\|^{2q}} \leq \frac{O(\theta_0)^2}{\|x-u_{j-1}\|^q\|x-v_{j-1}\|^{2q}},$$

and thus

$$\pi_\Lambda^{(j)}(x) \leq \sum_{\substack{u_1,\dots,u_{j-1} \\ v_1,\dots,v_{j-1}}} \frac{O(1)}{\|u_1\|^{2q}\|v_1\|^q\|u_1-v_1\|^q} \bigg(\prod_{i=1}^{j-2} \frac{O(\theta_0)}{\|v_{i+1}-u_i\|^q\|u_{i+1}-v_{i+1}\|^q\|u_{i+1}-v_i\|^{2q}} \bigg) \times \frac{O(\theta_0)^2}{\|x-u_{j-1}\|^q\|x-v_{j-1}\|^{2q}}. \tag{5.20}$$

Repeating the application of Proposition 5.1(ii) as in (5.19), we end up with

$$\pi_\Lambda^{(j)}(x) \leq \sum_{u_1,v_1} \frac{O(1)}{\|u_1\|^{2q}\|v_1\|^q\|u_1-v_1\|^q} \frac{O(\theta_0)^j}{\|x-u_1\|^q\|x-v_1\|^{2q}} \leq \frac{O(\theta_0)^j}{\|x\|^{3q}}. \tag{5.21}$$

For the bound on $\pi_\Lambda^{(1)}(x)$, we use the following bound, instead of (5.12):

$$P'^{(0)}_{\Lambda;v}(o,u) = \delta_{o,u}\delta_{o,v} + (1-\delta_{o,u}\delta_{o,v})\, P'^{(0)}_{\Lambda;v}(o,u) \leq \delta_{o,u}\delta_{o,v} + \frac{O(\theta_0^2)}{\|u\|^{2q}\|v\|^q\|u-v\|^q}. \tag{5.22}$$

In addition, instead of using (5.16), we use

$$\sum_{b:\underline{b}=u} \tau_b\, Q'_{\Lambda;v}(\overline{b},x) \leq \sum_z \frac{O(\theta_0)}{\|z-u\|^q} \bigg(\delta_{z,v}\delta_{z,x} + (1-\delta_{z,x}\delta_{z,v})\, P'^{(0)}_{\Lambda;v}(z,x) + \sum_{j\geq 1} P'^{(j)}_{\Lambda;v}(z,x) \bigg)$$
$$\leq \frac{O(\theta_0)}{\|x-u\|^q}\,\delta_{v,x} + \sum_z \frac{O(\theta_0^3)}{\|z-u\|^q\|x-z\|^{2q}\|v-z\|^q\|x-v\|^q}$$
$$\leq \frac{O(\theta_0)}{\|x-u\|^q}\,\delta_{v,x} + \frac{O(\theta_0^3)}{\|x-u\|^q\|x-v\|^{2q}}, \tag{5.23}$$

due to (5.5), (5.14) and (5.22). Applying (5.22)–(5.23) to (4.15) for $j = 1$ and then using Proposition 5.1(ii), we end up with

$$\pi_\Lambda^{(1)}(x) \leq O(\theta_0)\,\delta_{o,x} + \frac{O(\theta_0^3)}{\|x\|^{3q}} + \sum_{u,v} \frac{O(\theta_0^2)}{\|u\|^{2q}\|v\|^q\|u-v\|^q} \bigg(\frac{O(\theta_0)\,\delta_{v,x}}{\|x-u\|^q} + \frac{O(\theta_0^3)}{\|x-u\|^q\|x-v\|^{2q}} \bigg)$$
$$\leq O(\theta_0)\,\delta_{o,x} + \frac{O(\theta_0^3)}{\|x\|^{3q}}. \tag{5.24}$$

To complete the proof of Proposition 3.1, it thus remains to show (5.14)–(5.15). The inequality (5.14) for $j = 1$ immediately follows from the definition (4.6) of $P'^{(1)}_{\Lambda;u}$ (see also Figure 5) and the bound (5.10) on $\psi_\Lambda - \delta$. To prove (5.15) for $j = 1$, we first recall the definition (4.9) of $P''^{(1)}_{\Lambda;u,v}$ (and Figure 5). Note that, by (5.11), $\sum_{v'} G(v' - y)\, G(z - v')\, \psi_\Lambda(v', v)$ obeys the same bound on $\sum_{v'} G(v' - y)\, G(z - v')$ (with a different $O(1)$ term). That is, the effect of an additional ψ_Λ is not significant. Therefore, the bound on $P''^{(1)}_{\Lambda;u,v}$ is identical, with a possible modification of the $O(1)$ multiple, to the bound on $P'^{(1)}_{\Lambda;u}$ (or $P'^{(1)}_{\Lambda;v}$) with v (resp., u) "being embedded" in one of the bubbles consisting of $\psi_\Lambda - \delta$. By (5.10), $\psi_\Lambda(y, x) - \delta_{y,x}$ with v being embedded in one of its bubbles is bounded as

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{l=1}^{k}\sum_{y',x'} (\tilde G^2_\Lambda)^{*(l-1)}(y, y')\, \tilde G_\Lambda(y', x')\Big(\langle\varphi_{y'}\varphi_v\rangle_\Lambda \tilde G_\Lambda(v, x') + \tilde G_\Lambda(y', x')\, \delta_{v,x'}\Big)(\tilde G^2_\Lambda)^{*(k-l)}(x', x)\\
&= \sum_{y',x'} \psi_\Lambda(y, y')\, \tilde G_\Lambda(y', x')\Big(\langle\varphi_{y'}\varphi_v\rangle_\Lambda \tilde G_\Lambda(v, x') + \tilde G_\Lambda(y', x')\, \delta_{v,x'}\Big)\psi_\Lambda(x', x)\\
&\le \sum_{y',x'} \frac{O(1)}{\|y' - y\|^{2q}}\, \frac{O(\theta_0)}{\|x' - y'\|^q}\, \frac{O(\theta_0)}{\|v - y'\|^q\|x' - v\|^q}\, \frac{O(1)}{\|x - x'\|^{2q}} \le \frac{O(\theta_0^2)}{\|x - y\|^q\|v - y\|^q\|x - v\|^q}. \qquad (5.25)
\end{aligned}$$

By this observation and using (3.2) to bound the remaining two two-point functions consisting of $P''^{(1)}_{\Lambda;u,v}$ (recall (4.9)), we obtain (5.15) for $j = 1$.

For (5.14)–(5.15) with $j \ge 2$, we first note that, by applying (3.2) and (5.10) to the definition (4.5) of $P^{(j)}_\Lambda(y, x)$, we have

$$\begin{aligned}
P^{(j)}_\Lambda(y, x) \le \sum_{\substack{v_2,\dots,v_j\\ v'_1,\dots,v'_{j-1}}} &\frac{O(\theta_0^2)}{\|v'_1 - y\|^{2q}\|v_2 - y\|^q\|v'_1 - v_2\|^q} \prod_{i=2}^{j-1} \frac{O(\theta_0^2)}{\|v'_i - v_i\|^{2q}\|v_{i+1} - v'_{i-1}\|^q\|v'_i - v_{i+1}\|^q}\\
&\times \frac{O(\theta_0^2)}{\|x - v_j\|^{2q}\|x - v'_{j-1}\|^q}. \qquad (5.26)
\end{aligned}$$

By definition, the bound on $P'^{(j)}_{\Lambda;u}(y, x)$ is obtained by "embedding u" in one of the $2j - 1$ factors of $\|\cdots\|^q$ (not $\|\cdots\|^{2q}$) and then summing over all these $2j-1$ choices. For example, the contribution from the case in which $\|v_2 - y\|^q$ is replaced by $\|u - y\|^q\|v_2 - u\|^q$ is bounded, similarly to (5.21), by

$$\begin{aligned}
&\sum_{v_2,v'_1} \frac{O(\theta_0^2)}{\|v'_1 - y\|^{2q}\|u - y\|^q\|v_2 - u\|^q\|v'_1 - v_2\|^q}\, \frac{O(\theta_0^2)^{j-1}}{\|x - v'_1\|^q\|x - v_2\|^{2q}}\\
&\le \sum_{v'_1} \frac{O(\theta_0^2)^j}{\|v'_1 - y\|^{2q}\|u - y\|^q\|x - u\|^q\|x - v'_1\|^{2q}} \le \frac{O(\theta_0^2)^j}{\|x - y\|^{2q}\|u - y\|^q\|x - u\|^q}. \qquad (5.27)
\end{aligned}$$

The other $2j-2$ contributions can be estimated in a similar way, with the same form of the bound. This completes the proof of (5.14).

By (5.25), the bound on $P''^{(j)}_{\Lambda;u,v}(y, x)$ is also obtained by "embedding u and v" in one of the $2j - 1$ factors of $\|\cdots\|^q$ and one of the j factors of $\|\cdots\|^{2q}$ in (5.26), and then summing over all these combinations. For example, the contribution from the case in which $\|v_2-y\|^q$ and $\|v'_1-y\|^{2q}$ in (5.26) are replaced, respectively, by $\|u-y\|^q\|v_2-u\|^q$ and $\|v'_1-y\|^q\|v-y\|^q\|v'_1-v\|^q$, is bounded by

$$\begin{aligned}
&\sum_{v_2,v_1'} \frac{O(\theta_0^2)}{\|v_1'-y\|^q\|v-y\|^q\|v_1'-v\|^q\|u-y\|^q\|v_2-u\|^q\|v_1'-v_2\|^q} \frac{O(\theta_0^2)^{j-1}}{\|x-v_1'\|^q\|x-v_2\|^{2q}}\\
&\le \sum_{v_1'} \frac{O(\theta_0^2)^j}{\|v_1'-y\|^q\|v-y\|^q\|v_1'-v\|^q\|u-y\|^q\|x-u\|^q\|x-v_1'\|^{2q}}\\
&\le \frac{O(\theta_0^2)^j}{\|x-y\|^q\|u-y\|^q\|x-u\|^q\|v-y\|^q\|x-v\|^q}.
\end{aligned} \tag{5.28}$$

The other $(2j-1)j-1$ contributions can be estimated similarly, with the same form of the bound. This completes the proof of (5.15) and thus Proposition 3.1. ☐

5.2 Bounds for finite-range models

First, we prove (3.8) and Proposition 3.3(iii) assuming (3.7). Then, we prove (3.10) assuming (3.7) and (3.9) to complete the proof of Propositions 3.2.

Proof of (3.8) assuming (3.7). By applying (4.2) to the bound (4.15) on $\pi_\Lambda^{(0)}(x)$, it is easy to show that, for $r=0,2$,

$$\begin{aligned}
\sum_x |x|^r \pi_\Lambda^{(0)}(x) \le \delta_{r,0} + \sum_{x\ne o} |x|^r \langle\varphi_o\varphi_x\rangle_\Lambda^3 &\le \delta_{r,0} + \Big(\sup_{x\ne o} |x|^r G(x)\Big) \sum_{x\ne o} (\tau D * G)(x)\, G(x)\\
&\le \delta_{r,0} + (d\sigma^2)^{\delta_{r,2}} O(\theta_0)^2.
\end{aligned} \tag{5.29}$$

For $i \ge 1$, by using the diagrammatic bound (4.15) and translation invariance, we have

$$\sum_x \pi_\Lambda^{(i)}(x) \le \Big(\sum_{v,x} P'^{(0)}_{\Lambda;v}(o,x)\Big)\Big(\sup_y \sum_{z,v,x} \tau_{y,z} Q''_{\Lambda;o,v}(z,x)\Big)^{i-1}\Big(\sup_y \sum_{z,x} \tau_{y,z} Q'_{\Lambda;o}(z,x)\Big). \tag{5.30}$$

The proof of the bound on $\sum_x \pi_\Lambda^{(i)}(x)$ for $i \ge 1$ is completed by showing that

$$\Big(\sum_{v,x} P'^{(0)}_{\Lambda;v}(o,x) - 1\Big) \vee \Big(\sup_y \sum_{z,v,x} \tau_{y,z} Q''_{\Lambda;o,v}(z,x)\Big) \vee \Big(\sup_y \sum_{z,x} \tau_{y,z} Q'_{\Lambda;o}(z,x)\Big) = O(\theta_0). \tag{5.31}$$

The key idea to obtain this estimate is that the bounding diagrams for the Ising model are similar to those for self-avoiding walk (cf., Figure 6). The diagrams for self-avoiding walk are known to be bounded by products of bubble diagrams (see, e.g., [24]), and we can apply the same method to bound the diagrams for the Ising model by products of bubbles.

For example, consider

$$\sum_{z,x} \tau_{y,z} Q'_{\Lambda;o}(z,x) = \sum_{z',x} \Big(\sum_z \tau_{y,z}\big(\delta_{z,z'} + \tilde G_\Lambda(z,z')\big)\Big) P'_{\Lambda;o}(z',x). \tag{5.32}$$

The factor of θ_0 is due to the nonzero line segment $\sum_z \tau_{y,z}(\delta_{z,z'} + \tilde G_\Lambda(z,z'))$, because

$$\sum_z \tau_{o,z}\big(\delta_{z,x} + \tilde G_\Lambda(z,x)\big) = \tau D(x) + \tau \sum_z D(z)\, \tilde G_\Lambda(z,x) \le O(\theta_0) + \tau \sup_x \tilde G_\Lambda(o,x), \tag{5.33}$$

$$\tilde G_\Lambda(o,x) \le \tau D(x) + \tau \sum_{y\ne o} G(y)\, D(x-y) \le O(\theta_0) + \tau \sup_{y\ne o} G(y) = O(\theta_0), \tag{5.34}$$

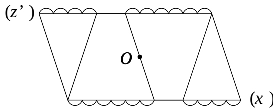
where we have used translation invariance, (3.7) and $\sup_x D(x) = O(\theta_0)$. By (4.12),

$$(5.32) \le O(\theta_0) \sum_{z',x} P'_{\Lambda;o}(z',x) = O(\theta_0) \sum_{z',x} \bigg(P'^{(0)}_{\Lambda;o}(z',x) + \sum_{j\ge 1} P'^{(j)}_{\Lambda;o}(z',x) \bigg). \tag{5.35}$$

Similarly to (5.29) for $r = 0$, the sum of $P'^{(0)}_{\Lambda;v}(z',x)$ is easily estimated as $1 + O(\theta_0)$. We claim that the sum of $P'^{(j)}_{\Lambda;o}(z',x)$ for $j \ge 1$ is $(2j-1)\,O(\theta_0)^j$, since $P'^{(j)}_{\Lambda;o}(z',x)$ is a sum of $2j-1$ terms, each of which contains j chains of nonzero bubbles; each chain is $\psi_\Lambda(v,v') - \delta_{v,v'}$ for some v, v' and satisfies

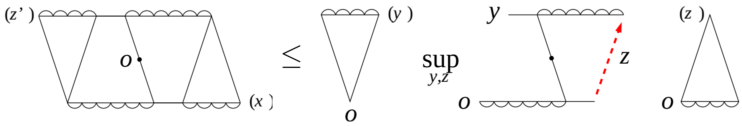
$$\sum_{v'} \big(\psi_\Lambda(v,v') - \delta_{v,v'}\big) \le \sum_{l\ge 1} \Big(\tau^2 \big(D * (D * G^{*2})\big)(o)\Big)^l = \sum_{l \ge 1} O(\theta_0)^l = O(\theta_0). \tag{5.36}$$

For example,

$$\sum_{z',x} P'^{(4)}_{\Lambda;o}(z',x) =$$ 

$$+\,6 \text{ other possibilities,} \tag{5.37}$$

which can be estimated, by translation invariance, as



$$\le \bigg(\sum_y \big(\psi_\Lambda(o,y) - \delta_{o,y}\big)\bigg)^4 \big(\bar{W}^{(0)}\big)^4 = O(\theta_0)^4, \tag{5.38}$$

where $\bar{W}^{(t)}$ is given by (3.13).

The sum of $\tau_{y,z} Q''_{\Lambda;o,v}(z,x)$ in (5.31) is estimated similarly [29]. We complete the proof of the bound on $\sum_x \pi^{(j)}_\Lambda(x)$ for $j \ge 1$.

To estimate $\sum_x |x|^2 \pi^{(j)}_\Lambda(x)$ for $j \ge 1$, we recall that, in each bounding diagram, there are at least three distinct paths between o and x: the uppermost path (i.e., $o \to b_1 \to v_2 \to b_3 \to \cdots \to x$ in (4.15); see also Figure 6), the lowermost path (i.e., $o \to v_1 \to b_2 \to v_3 \to \cdots \to x$) and a middle zigzag path. We use the lowermost path to bound $|x|^2$ as

$$|x|^2 = \sum_{n=0}^{j} |a_n|^2 + 2 \sum_{0 \le m < n \le j} a_m \cdot a_n \le (j+1) \sum_{n=0}^{j} |a_n|^2, \tag{5.39}$$

where $a_0 = v_1$, $a_1 = \underline{b}_2 - v_1$,$a_2 = v_3 - \underline{b}_2, \dots,$ and $a_j = x - v_j$ or $x - \underline{b}_j$ depending on the parity of j. We discuss the contributions to $\sum_x |x|^2 \pi^{(j)}_\Lambda(x)$ from (i) $|a_j|^2$, (ii) $|a_0|^2$ and (iii) $|a_n|^2$ for $n \ne 0, j$, separately (cf., Figure 11).

(i) The contribution from $|a_j|^2$ is bounded by

$$\bigg(\sum_{v,y} P'^{(0)}_{\Lambda;v}(o,y) \bigg) \bigg(\sup_y \sum_{\substack{b,v,z \\ \underline{b}=y}} \tau_b Q''_{\Lambda;o,v}(\overline{b}, z) \bigg)^{j-1}$$
$$\times \bigg(\sup_y \sum_{\substack{b,x \\ \underline{b}=y}} \Big(|x|^2 \mathbb{1}_{\{j \text{ odd}\}} + |x - \underline{b}|^2 \mathbb{1}_{\{j \text{ even}\}} \Big) \tau_b Q'_{\Lambda;o}(\overline{b}, x) \bigg)$$
$$\le O(\theta_0)^{j-1} \sup_y \sum_{z,z',x} \Big(|x|^2 \mathbb{1}_{\{j \text{ odd}\}} + |x-y|^2 \mathbb{1}_{\{j \text{ even}\}} \Big) \tau_{y,z} \big(\delta_{z,z'} + \tilde{G}_\Lambda(z,z')\big) P'_{\Lambda;o}(z',x). \tag{5.40}$$

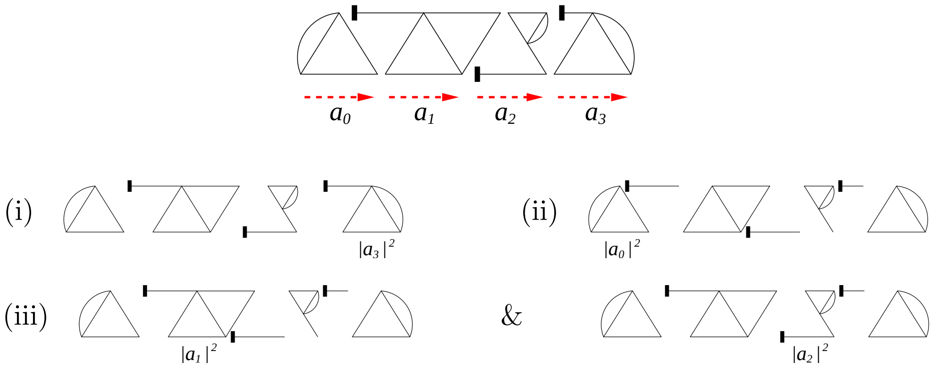


Figure 11: One of the leading diagrams for $\sum_x |x|^2\pi_\Lambda^{(3)}(x)$ and its decompositions depending on whether the assigned weight is (i) $|a_3|^2$, (ii) $|a_0|^2$ and (iii) $|a_n|^2$ for $n = 1, 2$, respectively.

By (4.10), the leading contribution from $P'^{(0)}_{\Lambda;o}(z', x)$ for an odd j can be estimated as

$$\begin{aligned}
&\sup_y \sum_{z,z',x} |x|^2 \tau_{y,z}\big(\delta_{z,z'} + \tilde{G}_\Lambda(z,z')\big) P'^{(0)}_{\Lambda;o}(z',x) \\
&= \sup_y \sum_{z,z',x} \tau_{y,z}\big(\delta_{z,z'} + \tilde{G}_\Lambda(z,z')\big)\, \langle \varphi_{z'}\varphi_o\rangle_\Lambda\, \langle \varphi_{z'}\varphi_x\rangle_\Lambda^2\, |x|^2 \langle \varphi_o\varphi_x\rangle_\Lambda \\
&\le \sup_y \Big((\tau D * G)(y) + (\tau D * G)^{*2}(y) \Big)\, G^{*2}(o)\, \bar{G}^{(2)} = d\sigma^2 O(\theta_0)^2, \qquad (5.41)
\end{aligned}$$

where $\bar{G}^{(s)}$ is given by (3.13). The other contributions from $P'^{(i)}_{\Lambda;o}(z', x)$ for $i \ge 1$ and from the even-j case can be estimated similarly; if j is even, then, by using $|x - y|^2 \le 2|z' - y|^2 + 2|x - z'|^2$ and estimating the contributions from $|z' - y|^2$ and $|x - z'|^2$ separately, we obtain that the supremum in (5.40) is $d\sigma^2 O(\theta_0)$. Consequently, (5.40) is $d\sigma^2 O(\theta_0)^{2\lfloor \frac{j+1}{2} \rfloor}$.

(ii) To bound the contributions to $\sum_x |x|^2\pi_\Lambda^{(j)}(x)$ from $|a_n|^2$ for $n < j$, we define (cf., Figure 12)

$$\tilde{Q}''_{\Lambda;u,v}(y,x) = \sum_b \bigg(P''_{\Lambda;u,v}(y,\underline{b}) + \sum_{y'} \tilde{G}_\Lambda(y,y')\, P'_{\Lambda;u}(y',\underline{b})\, \psi_\Lambda(y,v) \bigg)\, \tau_b\big(\delta_{\overline{b},x} + \tilde{G}_\Lambda(\overline{b},x)\big). \qquad (5.42)$$

By translation invariance and a similar argument to show (5.31), we can easily prove

$$\sup_z \sum_{y,v} \tilde{Q}''_{\Lambda;o,v}(y, v+z) = \sum_{y,v} \tilde{Q}''_{\Lambda;v,o}(y,z) = O(\theta_0). \qquad (5.43)$$

Therefore, the contribution from $|a_0|^2$ to $\sum_x |x|^2\pi_\Lambda^{(j)}(x)$ is bounded by

$$\begin{aligned}
&\bigg(\sup_y \sum_{v,b} |v|^2 P'^{(0)}_{\Lambda;o}(v,\underline{b})\, \tau_b\big(\delta_{\overline{b},y} + \tilde{G}_\Lambda(\overline{b},y)\big) \bigg) \bigg(\sup_z \sum_{y,v} \tilde{Q}''_{\Lambda;v,o}(y,z) \bigg)^{j-1} \bigg(\sum_{z,x} P'_{\Lambda;o}(z,x) \bigg) \\
&\le d\sigma^2 O(\theta_0)^{j+1}. \qquad (5.44)
\end{aligned}$$

(iii) By translation invariance and (5.42)–(5.43), the contribution from $|a_n|^2$ for an $n \ne 0, j$ is

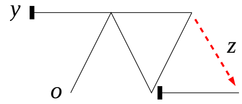
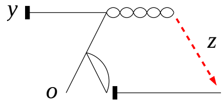
Figure 12: The leading diagrams of $\tilde{Q}''_{\Lambda;u,v}(y,x)$, due to $P''^{(0)}_{\Lambda;u,v}(y,\underline{b})$ and $P'^{(0)}_{\Lambda;u}(y',\underline{b})$ in (5.42), respectively.

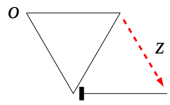
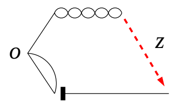
bounded by

$$\bigg(\sum_{v,y} P'^{(0)}_{\Lambda;v}(o,y)\bigg)\bigg(\sup_y \sum_{\substack{b,v,z\\ \underline{b}=y}} \tau_b Q''_{\Lambda;o,v}(\overline{b},z)\bigg)^{n-1}\bigg(\sup_z \sum_{y,v} \tilde{Q}''_{\Lambda;v,o}(y,z)\bigg)^{j-1-n}\bigg(\sum_{z,x} P'_{\Lambda;o}(z,x)\bigg)$$

$$\times\Bigg(\sup_{y,z}\sum_{\substack{b,b',v\\ \underline{b}=y}}\Big(|\underline{b}'|^2\mathbb{1}_{\{n\text{ odd}\}}+|v-\underline{b}|^2\mathbb{1}_{\{n\text{ even}\}}\Big)\,\tau_b Q''_{\Lambda;o,v}(\overline{b},\underline{b}')\,\tau_{b'}\big(\delta_{\overline{b}',v+z}+\tilde{G}_\Lambda(\overline{b}',v+z)\big)\Bigg), \tag{5.45}$$

where the first line is $O(\theta_0)^{j-2}$. The leading contribution to the second line from $P''^{(0)}_{\Lambda;o,v}$ and $P'^{(0)}_{\Lambda;o}$ in $Q''_{\Lambda;o,v}$ for an odd n is bounded, due to translation invariance, by

$$\bar{G}^{(2)}\sup_{y,z}\Bigg(\quad$$  $$+$$  $$\quad\Bigg)$$

$$\le d\sigma^2 O(\theta_0)^2 \sup_z\Bigg(\quad$$  $$+$$  $$\quad\Bigg) \le d\sigma^2 O(\theta_0)^3. \tag{5.46}$$

The other contributions from $P''^{(i)}_{\Lambda;o,v}$ and $P'^{(i)}_{\Lambda;o}$ for $i \ge 1$ and from the even-n case can be estimated similarly; if n is even, then the second supremum in (5.46) is $O(\theta_0)$. Therefore, (5.45) is $d\sigma^2 O(\theta_0)^{2\lfloor\frac{j+1}{2}\rfloor}$.

Summarizing the above (i)–(iii) and using $2\lfloor\frac{j+1}{2}\rfloor \ge j\vee 2$ for $j\ge 1$, we have

$$\frac{1}{j+1}\sum_x |x|^2\pi^{(j)}_\Lambda(x) \le d\sigma^2\Big(jO(\theta_0)^{2\lfloor\frac{j+1}{2}\rfloor}+O(\theta_0)^{j+1}\Big) \le d\sigma^2(j+1)\,O(\theta_0)^{j\vee 2}. \tag{5.47}$$

This together with (5.29) complete the proof of (3.8). □

Proof of Proposition 3.3(iii) assuming (3.7). It is easy to see that

$$\sum_x |x|^{t+2}\pi^{(0)}_\Lambda(x) \le \sum_x |x|^{t+2}G(x)^3 \le \bar{G}^{(2)}\sum_x |x|^t G(x)^2 \le d\sigma^2\theta_0\bar{W}^{(t)}. \tag{5.48}$$

We show below that, for $j\ge 1$,

$$\sum_x |x|^{t+2}\pi^{(j)}_\Lambda(x) \le d\sigma^2\bar{W}^{(t)}(j+1)^{t+3}O(\theta_0)^{j\vee 2-1}, \tag{5.49}$$

where the bound is independent of Λ. Due to these uniform bounds, we conclude that the sum of $|x|^{t+2}|\Pi(x)|$ is finite if $\theta_0 \ll 1$.

Now we explain the main idea of the proof of (5.49). First we recall that, in the proof of the bound on $\sum_x |x|^2\pi^{(j)}_\Lambda(x)$, we distribute $|x|^2$ along the lowermost path of each bounding diagram.

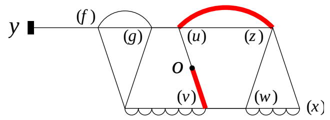
To bound $\sum_x |x|^{t+2}\pi_\Lambda^{(j)}(x)$, we again use the lowermost path in the same way to distribute $|x|^2$, and use the uppermost path to distribute the remaining $|x|^t$. More precisely, we use

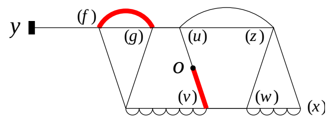
$$|x| \le (j+1) \max_{n=0,1,\dots,j} |a'_n|, \tag{5.50}$$

where $a'_0, a'_1, \dots, a'_j$ are the displacements along the uppermost path: $a'_0 = \underline{b}_1$, $a'_1 = v_2 - \underline{b}_1$, $a'_2 = \underline{b}_3 - v_2, \dots,$ and $a'_j = x - v_j$ or $x - \underline{b}_j$ depending on the parity of j. Let m be such that $|a'_m| = \max_n |a'_n|$.

For the contribution to $\sum_x |x|^{t+2}\pi_\Lambda^{(j)}(x)$ from $|a_n|^2$ in (5.39) for $n \ne m$, we simply follow the same strategy as explained above in the paragraphs (i)–(iii) to prove the bound on $\sum_x |x|^2\pi_\Lambda^{(j)}(x)$. The only difference is that one of the bubbles $\bar{W}^{(0)}$ contained in the bound on the m^{th} block is now replaced by $\bar{W}^{(t)}$.

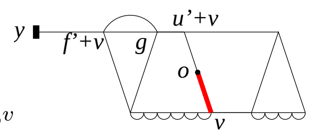
The contribution from $|a_m|^2$ in (5.39) can be estimated in a similar way, except for a few complicated cases, due to $P'^{(i)}_{\Lambda;u}$ and $P''^{(i)}_{\Lambda;u,v}$ for $i \ge 1$ contained in the m^{th} block. For example, let j be even and let $m = j$ (cf., the second line of (5.40)). The following are two possibile diagrams in the contribution from $P'^{(4)}_{\Lambda;o}(f,x)$ to $\sum_{z,x} |x-y|^2 |x|^t \tau_{y,z} Q'_{\Lambda;o}(z,x)$:

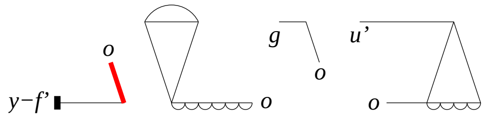
(i) 

(ii) 

(5.51)

where, for simplicity, $\psi_\Lambda(f,g) - \delta_{f,g}$ and $\psi_\Lambda(u,z) - \delta_{u,z}$ are reduced to $\tilde{G}_\Lambda(f,g)^2$ and $\tilde{G}_\Lambda(f,g)^2$, respectively. We suppose that $|v|$ is bigger than $|w-v|$ and $|x-w|$ along the lowermost path from o to x through v and w, so that $|x|^t$ is bounded by $3^t|v|^t$. We also suppose that $|z-u|$ in (5.51.i) (resp., $|g-f|$ in (5.51.ii)) is bigger than the end-to-end distance of any of the other four segments along the uppermost path from y to x through f, g, u and z. Therefore, we can bound $|x-y|^2$ by $5^2|z-u|^2$ in (5.51.i) (resp., $5^2|g-f|^2$ in (5.51.ii)) and bound the weighted arc between u and z (resp., between f and g) by $5^2\bar{G}^{(2)}$. By translation invariance, the remaining diagram of (5.51.i) is easily bounded as

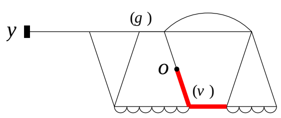
$\sum_{f',g,u',v}$ 

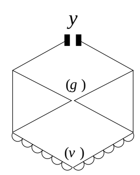
$= \sup_{f',g,u'}$ 

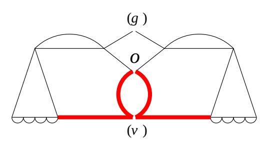
$\le \bar{W}^{(t)}\, O(\theta_0)^4,$ (5.52)

where the power 4 (not 3) is due to the fact that the segment from u' in the last block is nonzero.

To bound the remaining diagram of (5.51.ii) is a little trickier. We note that at least one of $|u|, |z-u|, |w-z|$ and $|v-w|$ along the path from o to v through u, z, w is bigger than $\frac{1}{4}|v|$. Suppose $|v-w| \ge \frac{1}{4}|v|$, so that $|v|^t \le 2^t|v-w|^{t/2}|v|^{t/2}$. Then, by using the Schwarz inequality, we obtain

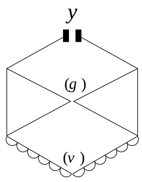


$\le \Bigg($ 

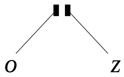
$\Bigg)^{1/2} \Bigg($ 

$\Bigg)^{1/2},$ (5.53)

where the two weighted arcs between o and v in the second term is $|v|^t G(v)^2 \equiv (|v|^{t/2}G(v))^2$. By translation invariance and the fact that the north-east and north-west segments from g in the first term are nonzero, we obtain



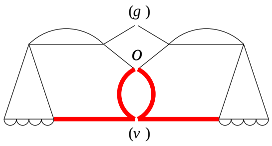
$$\leq \Big(\sup_z$$



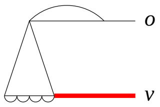
$$\Big) \Big(\sup_{g'} \tau(D * G^{*2})(g') \Big)^2 \bar{W}^{(0)} \Big(\sum_v \big(\psi_\lambda(o,v) - \delta_{o,v}\big) \Big)^2$$

$$\leq O(\theta_0)^5. \tag{5.54}$$

With the help of $(\bar{W}^{(t/2)})^2 \leq \bar{W}^{(0)}\bar{W}^{(t)}$ (due to the Schwarz inequality), we also obtain



$$\leq \bar{W}^{(0)}\,\bar{W}^{(t)} \Big(\sup_v$$



$$\Big)^2 \leq (\bar{W}^{(t)})^2 O(\theta_0)^4. \tag{5.55}$$

Therefore, (5.53) is bounded by $\bar{W}^{(t)} O(\theta_0)^{9/2}$.

The other cases can be estimated similarly [29]. As a result, we obtain

$$\sum_x |x|^{t+2} \pi_\Lambda^{(j)}(x) \leq \sum_{m=0}^{j} d\sigma^2 \bar{W}^{(t)} (j+1)^{t+2} O(\theta_0)^{j \vee 2 - 1}, \tag{5.56}$$

which implies (5.49). This completes the proof of Proposition 3.3(iii). □

Proof of (3.10) assuming (3.7) and (3.9). If $x = o$, then we simply use the bound on the sum in (3.8) to obtain $\pi_\Lambda^{(i)}(o) \leq O(\theta_0)^i$ for any $i \geq 0$. It is also easy to see that $\pi_\Lambda^{(0)}(x)$ with $x \neq o$ obeys (3.10), due to (3.9) and the diagrammatic bound (4.15). It thus remains to show (3.10) for $\pi_\Lambda^{(j)}(x)$ with $x \neq o$ and $j \geq 1$.

The idea of the proof is somewhat similar to that of Proposition 3.3(iii) explained above. First, we take $|a_m| \equiv \max_n |a_n|$ from the lowermost path and $|a'_l| \equiv \max_n |a'_n|$ from the uppermost path of a bounding diagram. Note that, by (5.50), $|a_m|$ and $|a'_l|$ are both bigger than $\frac{1}{j+1}|x|$. That is, $|a_m|^{-q}$ and $|a'_l|^{-q}$ are both bounded from above by $(j+1)^q |x|^{-q}$. If the path corresponding to a_m in the m^{th} block consists of N segments, we take the "longest" segment whose end-to-end distance is therefore bigger than $\frac{1}{N(j+1)}|x|$. That is, the corresponding two-point function is bounded by $\lambda_0 N^q (j+1)^q |x|^{-q}$. Here, N depends on the parity of m, as well as on $i \geq 0$ for $P''^{(i)}_{\Lambda;u,v}$ (or $P'^{(i)}_{\Lambda;u}$ if $m = 0$ or j) and the location of u, v in each diagram, and is at most $N \leq O(i+1)$. However, the number of nonzero chains of bubbles contained in each diagram of $P'^{(i)}_{\Lambda;u}$ and $P''^{(i)}_{\Lambda;u,v}$ is $O(i)$, and hence their contribution would be $O(\theta_0)^{O(i)}$. This compensates the growing factor of N^q, and therefore we will not have to take the effect of N seriously. The same is true for a'_l, and we refrain from repeating the same argument.

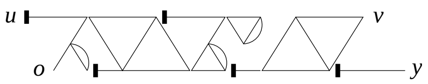
Next, we take the "longest" segment, denoted a'', among those which together with a'_l (or a part of it) form a "loop"; a similar observation was used to obtain (5.53). The loop consists of segments contained in the l^{th} block and possibly in the $(l-1)^{\text{st}}$ block, and hence the number of choices for a'' is at most $O(i_{l-1} + i_l + 1)$, where i_l is the index of $P'^{(i_l)}_\Lambda$ or $P''^{(i_l)}_\Lambda$ in the l^{th} block ($i_{-1} = 0$ by convention). By (5.50), we have $|a''| \geq O(i_{l-1} + i_l + 1)^{-1} |a'_l|$, and the corresponding two-point function is bounded by $\lambda_0 O(i_{l-1} + i_l + 1)^q (j+1)^q |x|^{-q}$. As explained above, the effect of $O(i_{l-1} + i_l + 1)^q$ would not be significant after summing over i_{l-1} and i_l.

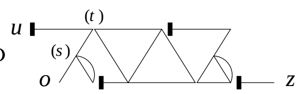
We have explained how to extract three "long" segments from each bounding diagram, which provide the factor $\lambda_0^3 (j+1)^{3q} |x|^{-3q}$ in (3.10); the extra factor of $(j+1)^2$ in (3.10) is due to the number of choices of $m, l \in \{0, 1, \ldots, j\}$. Therefore, the remaining task is to control the rest of the diagram.

Suppose, for example, $0 < m < l < j$ (so that $j \geq 3$). Using $\tilde{Q}''_\Lambda$ defined in (5.42), we can reorganize the diagrammatic bound (4.15) on $\pi_\Lambda^{(j)}(x)$ as (cf., (5.45))

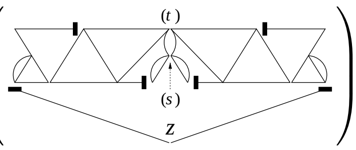
$$\begin{aligned}\pi_\Lambda^{(j)}(x) \leq \sum_{\substack{b_m, v_m \\ y_{l+1}, v_{l+1}}} & \bigg(\sum_{\substack{b_1,\dots,b_{m-1} \\ v_1,\dots,v_{m-1}}} P'^{(0)}_{\Lambda;v_1}(o, \underline{b}_1) \prod_{i=1}^{m-1} \tau_{b_i} Q''_{\Lambda;v_i,v_{i+1}}(\overline{b}_i, \underline{b}_{i+1}) \bigg) \\ & \times \sum_{b_{l+1}} \bigg(\sum_{\substack{b_{m+1},\dots,b_l \\ v_{m+1},\dots,v_l}} \prod_{i=m}^{l} \tau_{b_i} Q''_{\Lambda;v_i,v_{i+1}}(\overline{b}_i, \underline{b}_{i+1}) \bigg) \tau_{b_{l+1}} \big(\delta_{\overline{b}_{l+1}, y_{l+1}} + \tilde{G}_\Lambda(\overline{b}_{l+1}, y_{l+1}) \big) \\ & \times \bigg(\sum_{\substack{y_{l+2},\dots,y_j \\ v_{l+2},\dots,v_j}} \bigg(\prod_{i=l+1}^{j-1} \tilde{Q}''_{\Lambda;v_i,v_{i+1}}(y_i, y_{i+1}) \bigg) P'_{\Lambda;v_j}(y_j, x) \bigg). \end{aligned} \tag{5.57}$$

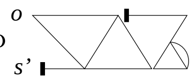
As explained above, we bound three "long" two-point functions contained in the second line of (5.57); let $Y_{m,l}$ be the supremum of what remains in the second line over $b_m, v_m, y_{l+1}, v_{l+1}$. Then we can perform the sum of the first line over b_m, v_m and the sum of the third line over y_{l+1}, v_{l+1} independently; the former is $O(\theta_0)^{m-1}$ and the latter is $O(\theta_0)^{j-1-l}$, due to (5.31) and (5.43), respectively. Finally, we can bound $Y_{m,l}$ using the Schwarz inequality by $O(\theta_0)^{l-m}$, where $l - m$ is the number of nonzero segments in the second line of (5.57) (i.e., $\sum_{b_i} \tau_{b_i}(\delta_{\overline{b}_i, y_i} + \tilde{G}_\Lambda(\overline{b}_i, y_i))$ for some $y_m, \dots, y_{l+1}$) minus 2 (= the maximum number of those along the uppermost and lowermost paths that are extracted to obtain the aforementioned $|x|$-decaying term). For example, one of the leading contributions to $Y_{m,m+4}$ is bounded, by using translation invariance and the Schwarz inequality, as

$\sup_{u,v,y}$ 

$\leq O(\theta_0) \sup_{u,z}$ 

(5.58)

$\leq O(\theta_0)^{3/2} \Bigg($ 

$\Bigg)^{1/2} \leq O(\theta_0)^2 \sup_{s'}$ 

$\leq O(\theta_0)^4.$

The other cases can be estimated similarly [29]. This completes the proof of (3.10). ☐

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