第14回 COE研究員連続講演会

開ミラー対称性における最近の進展

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2007.11.19 (月)，20 (火)，21 (水)

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開ミラー対称性における最近の進展
(A REVIEW OF DISC COUNTING ON COMPLETE INTERSECTIONS IN PROJECTIVE SPACES)

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概要

まず，次の古典的な問題を考えよう．

問題 1. 複素 4 次元射影空間内の 5 次超曲面 Q 上に, 次数 d の有理曲線は何本あるか？

ただし，有理曲線の本数は Gromov–Witten 不変量の理論を用いて定式化されるものとする．この問題に対して次のような解答が知られている．有理曲線の本数 (Gromov–Witten 不変量) の次数 d に関する生成母関数を考えると，それはミラー変換という変数変換によって超幾何型の級数で書き表す事が出来る．この事実は, ミラー対称性と呼ばれる位相的弦理論の双対性を通して, 1990 年頃に Candelas 等 [1] によって発見された．最初の数学的証明は 1995 年に Givental [2] によって与えられ, 現在ではミラー定理と呼ばれている．

次に, 問題 1 の '実版' を考えよう．

問題 2. Q を実構造を持つ 5 次超曲面とする時，Q 上に次数 d の実な有理曲線は何本あるか？

この問題は, (次数 d が奇数ならば) Q の実構造から定まる実点の集合 $Q(\mathbb{R})$ に境界を持つ正則円盤を数える問題と等価である．2006 年に Walcher がそのような正則円盤の数え上げ問題に対する開ミラー定理, 即ち正則円盤の数の生成母関数がミラー変換によって超幾何型の級数に写されるということを予想した．その証明は, Pandharipande–Solomon–Walcher [6] によって同じ年の内に直ちに与えられた．彼らの証明は正則円盤の開 Gromov–Witten 不変量の理論と Givental のミラー定理に基づくものである．上の条件を満たす正則円盤の数は実有理曲線の本数の 2 倍とみなせることから, Pandharipande 等の結果は問題 2 に対する 1 つの解答と見なす事もできる．

この連続講演の目標は，Pandharipande–Solomon–Walcher による開ミラー定理の証明ついて概観することである．特にトーラス作用による局所化を用いた具体的計算の部分に重点を置いて説明する．超幾何級数を導くために必要となる Givental の同変 Correlator とそのミラー変換についても概要を説明する． 最後に，5 次超曲面以外の射影空間内の完全交差 Calabi–Yau 多様体であって $h^{1,1}=1$ であるものに対する正則円盤の数え上げ問題について，知られている結果を述べる．

1. Introduction

1.1. **Complex curve counting.** Let us recall the classical mirror symmetry for Calabi–Yau (CY) quintic Q. The genus zero, degree d Gromov–Witten (GW) invariant N_d of Q is given by the following formula:

$$N_d = \int_{\bar{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)} c_{top}(E_d), \tag{1}$$

where E_d is the (orbi-)bundle such that $E_d|_{(f,C)\in\bar{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)} = H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))$. Define the virtual count n_d by the Aspinwall–Morriaon formula [1]

$$N_d = \sum_{k>0,k|d} \frac{1}{k^3} n_{d/k}. \tag{3}$$

It is expected that $n_d \in \mathbb{Z}$.

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[1]This is justified by the following local invariants:

$$\int_{\bar{\mathcal{M}}_{0,0}(\mathbb{P}^1,d)} c_{top}(G_d \oplus G_d) = \frac{1}{d^3}, \tag{2}$$

where $G_d|_{(f,C)\in\bar{\mathcal{M}}_{0,0}(\mathbb{P}^1,d)} = H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1))$.

Consider the generating series

$$\mathcal{F}(T) := \frac{5}{6}T^3 + \sum_{d>0} N_d e^{dT}. \tag{4}$$

Let $I_i(t)$ be the functions defined by

$$\sum_{i=0}^{3} I_i(t)H^i = \sum_{d=0}^{\infty} e^{(H+d)t} \frac{\prod_{r=1}^{5d}(5H+r)}{\prod_{r=1}^{d}(H+r)^5} \quad \mathrm{mod} H^4. \tag{5}$$

These are a basis of the Picard–Fuchs (PF) equation

$$[(\frac{d}{dt})^4 - 5e^t(5\frac{d}{dt}+1)(5\frac{d}{dt}+2)(5\frac{d}{dt}+3)(5\frac{d}{dt}+4)]I(t) = 0. \tag{6}$$

Under the mirror transform $T(t) = \frac{I_1}{I_0}(t)$, we have

$$\mathcal{F}(T(t)) = \frac{5}{2}(\frac{I_1}{I_0}(t)\frac{I_2}{I_0}(t) - \frac{I_3}{I_0}(t)). \tag{7}$$

This is the mirror formula predicted by Candelas et al [1] and proven by Givental [2], Lian–Liu–Yau [3].

1.2. **Disk counting.** Consider the symplectic manifold $(\mathbb{P}^4, \omega_{FS})$ and the anti-symplectic involution on it :

$$c : [x_0, x_1, x_2, x_3, x_4] \mapsto [\overline{x_0}, \overline{x_2}, \overline{x_1}, \overline{x_4}, \overline{x_3}]. \tag{8}$$

Let $s \in H^0(\mathbb{P}^4, \mathcal{O}(5))$ be a degree 5 homogeneous polynomial which is real w.r.t c, i.e. satisfies

$$\overline{s(x)} = s(c(x)). \tag{9}$$

Let Q be defined by $s = 0$. Such a Q is called a real quintic. Let $Q_{\mathbb{R}}$ be the c-fixed points of Q. This is a Lagrangian submanifold of $(Q, \iota^*\omega_{FS})$ $(\iota : Q \hookrightarrow \mathbb{P}^4)$.

We consider the disk invariants N_d^{disk} of odd degree d which counts the number of holomorphic discs inside Q whose boundaries lie in $Q_{\mathbb{R}}$. (Definition will be given later.)

Define

$$\mathcal{F}^{disk}(T) := \sum_{d:\mathrm{odd}} N_d^{disk} e^{\frac{dT}{2}}. \tag{10}$$

The mirror formula for $\mathcal{F}^{disk}$ was predicted by Walcher, and proven by Pandharipande–Solomon–Walcher. Let

$$J(t) = 2\sum_{d:\mathrm{odd}} e^{\frac{dt}{2}} \frac{(5d)!!}{(d!!)^5} = 30e^{t/2} + \cdots. \tag{11}$$

This is a solution to the in-homogeneous PF equation:

$$[(\frac{d}{dt})^4 - 5e^t(5\frac{d}{dt}+1)(5\frac{d}{dt}+2)(5\frac{d}{dt}+3)(5\frac{d}{dt}+4)]J(t) = \frac{15}{8}e^{\frac{t}{2}}. \tag{12}$$

Theorem 1 ([6, 8])**.** *Under the mirror transform $T(t) = \frac{I_1}{I_0}(t)$,*

$$\mathcal{F}^{disk}(T(t)) = \frac{J(t)}{I_0(t)}. \tag{13}$$

The aim of this talk is to outline the proof of this theorem.

2. Givental's correlators and Mirror transforms

Before explaining the definition of disk invariants, we quickly review Givental's correlators, which will be important tool. See [5] for details.

2.1. **Equivariant Cohomology of $\mathbb{P}^4$.** Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Consider the action of $\mathbb{T}^5$ on $\mathbb{P}^4$:

$$[x_0, \cdots, x_4] \mapsto [t_0 x_0, \cdots, t_4 x_4], \tag{14}$$

where $(t_0, \cdots, t_4) \in \mathbb{T}^5$. Let $\lambda_0, \cdots, \lambda_4$ be the weights of the induced representation on $H^0(\mathbb{P}^4, \mathcal{O}(1))$. Let $H := c_{\text{top}}(\mathcal{O}(1)) \in H^*_{\mathbb{T}^5}(\mathbb{P}^4)$. Then $H^*_{\mathbb{T}^5}(\mathbb{P}^4) \cong \mathbb{C}[H, \lambda_0, \cdots, \lambda_4]/(H-\lambda_0)\cdots(H-\lambda_4)$. Let ζ_i $(i = 0, \cdots, 4)$ be $\mathbb{T}^5$-fixed points on $\mathbb{P}^4$. An element $\eta \in H^*_{\mathbb{T}^5}(\mathbb{P}^4)$ is determined by its restrictions $\eta|_{\zeta_i}$. Let $\phi_i \in H^*_{\mathbb{T}^5}(\mathbb{P}^4)$ be the P.D. of ζ_i. Explicitly, it is given by

$$\phi_i = \prod_{k \neq i} (H - \lambda_k). \tag{15}$$

Denote the equivariant intersection pairing by $\langle\, , \rangle$. Then for example, one has

$$\langle H, \phi_i \rangle = \lambda_i. \tag{16}$$

In general

$$\eta|_{\zeta_i} = \langle \eta, \phi_i \rangle, \tag{17}$$

for any $\eta \in H^*_{\mathbb{T}^5}(\mathbb{P}^4)$. This follows from the localization formula.

2.2. **Equivariant Givental's correlators.** The action of $\mathbb{T}^5$ on $\mathbb{P}^4$ induces that on $\bar{\mathcal{M}}_{0,n}(\mathbb{P}^4, d)$. Define Givental's equivariant correlator $\mathcal{S}_{\mathbb{T}^5}(T, \hbar)$ as

$$\mathcal{S}_{\mathbb{T}^5}(T, \hbar) := \sum_{d \geq 0} e^{(\frac{H}{\hbar} + d)T} \frac{1}{5H} \mathrm{ev}_{1*}\left(\frac{c_{\text{top}}(E_d)}{\hbar - \psi_1}\right) \in H^*_{\mathbb{T}^5}(\mathbb{P}^4)[[\hbar^{-1}, T, e^T]]. \tag{18}$$

This makes sense because $\mathrm{ev}_{1*}(\frac{c_{\text{top}}(E_d)}{\hbar - \psi_1})$ is divisible by $5H$ for $d > 0$ and $\frac{1}{5H} ev_{1*}(\frac{c_{\text{top}}(E_0)}{\hbar - \psi_1})$ is defined to be 1. Here $\mathrm{ev}_1 : \bar{\mathcal{M}}_{0,2}(\mathbb{P}^4, d) \to \mathbb{P}^4$, ψ_1 is the ψ-class at the first marked point, and $E_{d,2} \to \bar{\mathcal{M}}_{0,2}(\mathbb{P}^4, d)$.

Let

$$\mathcal{Z}_i(Q, \hbar) = \frac{1}{5\lambda_i} \sum_{d \geq 0} Q^d \int_{\bar{\mathcal{M}}_{0,2}(\mathbb{P}^4, d)} \frac{c_{\text{top}}(E_d)}{\hbar - \psi_1} \mathrm{ev}_1^*(\phi_i), \tag{19}$$

where the degree zero term is defined to be 1. Then

$$\langle \mathcal{S}_{\mathbb{T}^5}(T, \hbar), \phi_i \rangle = e^{\frac{\lambda_i T}{\hbar}} \mathcal{Z}_i(e^T, \hbar), \tag{20}$$

where the pairing is taken to be linear in the auxiliary parameters $\hbar^{-1}, T, e^T$. Note that, by string equation, we have

$$\mathcal{Z}_i(Q, \hbar) = 1 + \hbar^{-1} \frac{1}{5\lambda_i} \sum_{d=0}^{\infty} Q^d \int_{\bar{\mathcal{M}}_{0,1}(\mathbb{P}^4, d)} \frac{c_{\text{top}}(E_{d,1})}{\hbar - \psi_1} \mathrm{ev}_1^*(\phi_i). \tag{21}$$

2.3. **Mirror transformations.** Define

$$\mathcal{S}^*(t, \hbar) = \frac{1}{5H} \sum_{r \geq 0} e^{(\frac{H}{\hbar} + r)t} \frac{\prod_{s=0}^{5r} (5H + s\hbar)}{\prod_{i=0}^{4} \prod_{s=1}^{r} (H - \lambda_i + s\hbar)}. \tag{22}$$

Givental's mirror theorem states that, under the mirror transformation $T(t) = \frac{I_1}{I_0}(t)$, we have

$$\mathcal{S}_{\mathbb{T}^5}(T(t), \hbar) = \frac{1}{I_0(t)} \mathcal{S}^*(t, \hbar). \tag{23}$$

It follows that

$$\langle \mathcal{S}_{\mathbb{T}^5}(T(t), \hbar), \phi_i \rangle = \frac{1}{I_0(t)} \langle \mathcal{S}^*(t, \hbar), \phi_i \rangle. \tag{24}$$

The mirror formula (7) can be derived form this.

3. Disc invariants

In this section, we always assume that d is a positive odd integer.

3.1. **Disk invariants.** Denote by $\bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)$ the moduli space of maps $f : (\Sigma, \partial\Sigma) \to (\mathbb{P}^4, \mathbb{P}^4_{\mathbb{R}})$, where Σ is a nodal bordered Riemann surface [2] , such that the reflected genus zero map $(\tilde{f}, C)$ belongs to $\bar{\mathcal{M}}_{0,0}(\mathbb{P}^4, d)$. This is an orbifold with boundary (more precisely, with corners). The real dimension of this space is $5d + 1$.

Integration of cohomology classes on $\bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}_{\mathbb{R}}, d)$ is not well-defined. Therefore, we define certain equivalence relation $\sim$ on corners of $\bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)$ to close corners. Let

$$\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d) := \bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)/\sim . \tag{25}$$

Then $\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)$ is a closed orbifold.

Let $\hat{F}_d$ be a real vector bundle over $\bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)$ such that $\hat{F}_d|_{(f,D)} = H^0(C, \tilde{f}^*\mathcal{O}(5))_{\mathbb{R}}$, where $H^0(C, \tilde{f}^*\mathcal{O}(5))_{\mathbb{R}}$ denotes the real sections. The real rank of this bundle is $5d + 1$. $\hat{F}_d$ naturally descends to a vector bundle F_d over $\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)$. (Although F_d is not orientable,) its Euler class $e(F_d)$ is well-defined.

For d odd, we have

$$N_d^{disk} = \int_{\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)} e(F_d). \tag{26}$$

Actually, this is a theorem (cf. [**?**, Theorem 3]), but we take this as a working definition of the disk invariant N_d^{disk}.

3.2. **Multiple cover formula.** Define the virtual count n_d^{disk} by the Ooguri–Vafa formula

$$N_d^{\mathrm{disk}} = \sum_{k:\mathrm{odd}, k|d} \frac{1}{k^2} n_{d/k}^{\mathrm{disk}}. \tag{27}$$

It is conjectured that $n_d^{\mathrm{disk}} \in 2\mathbb{Z}$. The reason for evenness is that one counts the disk and its complex conjugate separately.

The Ooguri–Vafa foumula is justified by the following result on local invariants [6, Proposition 19].

$$\int_{\tilde{\mathcal{M}}_D(\mathbb{P}^1/\mathbb{P}^1_{\mathbb{R}}, d)} e((G_d)_{\mathbb{R}} \oplus (G_d)_{\mathbb{R}}) = \frac{2}{d^2}. \tag{28}$$

Here $\mathbb{P}^1_{\mathbb{R}}$ is defined by the following anti-holomorphic involution:

$$[x_0, x_1] \mapsto [\overline{x_1}, \overline{x_0}], \tag{29}$$

and $(G_d)_{\mathbb{R}}$ is a real vector bundle given by $(G_d)_{\mathbb{R}}|_{(f,\Sigma)\in\tilde{\mathcal{M}}_D(\mathbb{P}^1/\mathbb{P}^1_{\mathbb{R}}, d)} = H^1(C, \tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1))_{\mathbb{R}}$.

4. Localization calculation

We shall calculate N_d^{disk} via localization on $\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)$ w.r.t. a $\mathbb{T}^2$-action.

4.1. **$\mathbb{T}^2$-action and equivariant weights.** Consider the rank 2 subtorus $\mathbb{T}^2 \subset \mathbb{T}^5$ acting on $\mathbb{P}^4$ by

$$(t_1, t_2) \cdot [x_0, \cdots, x_4] = [x_0, t_1 x_1, t_1^{-1} x_2, t_2 x_3, t_2^{-1} x_4]. \tag{30}$$

Again, ζ_i are the $\mathbb{T}^2$-fixed points. Note that only ζ_0 is the REAL fixed point.

Since this action preserves $\mathbb{P}^4_{\mathbb{R}}$, $\mathbb{T}^2$ acts on $\bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}_{\mathbb{R}}, d)$ and $\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)$. By complexifyng the action of $\mathbb{T}^2$, an algebraic torus $(\mathbb{C}^*)^2$ acts on $\bar{\mathcal{M}}_{0,0}(\mathbb{P}^4, d)$.

Let λ and λ' be the generators of $H^*_{\mathbb{T}^2}(\mathrm{pt})$ defined by the pull-back $\rho^* : H^*_{\mathbb{T}^2}(\mathrm{pt}) \to H^*_{\mathbb{T}^5}(\mathrm{pt})$ and equations

$$\rho^*(\lambda_1) = -\rho^*(\lambda_2) = \lambda, \quad \rho^*(\lambda_3) = -\rho^*(\lambda_4) = \lambda'. \tag{31}$$

The pull-back $\rho^*(\lambda_0)$ is zero. For convenience, we omit the map ρ^* and write

$$\lambda_0 = 0, \quad \lambda_1 = -\lambda_2 = \lambda, \quad \lambda_3 = -\lambda_4 = -\lambda'. \tag{32}$$

[2]For a nodal bordered Riemann surface Σ, there exists a closed nodal Riemann sueface C together with an anti-holomorphic involution σ and a double covering $\pi : C \to \Sigma$ such that $\pi \circ \sigma = \pi$. There is a embeding $i : \Sigma \to C$ such that $\pi \circ i = \mathrm{id}$. The triple (C, σ, i) is unique up to isomorphism and called the complex double of Σ. A holomorphic map from a borderd Riemann surface to a complex manifold with anti-holomorphic involution can be reflected (or doubled). See [4].

The $\mathbb{T}^2$-equivariant Givental correlator $\mathcal{S}_{\mathbb{T}^2}(T,\hbar)$ is obtained from $\mathcal{S}_{\mathbb{T}^2}(T,\hbar)$ by the specialization of the weights.

4.2. **$\mathbb{T}^2$-fixed disk maps.** Let

$$[f:(D,\partial D)\to(\mathbb{P}^4,\mathbb{P}^4_{\mathbb{R}})]\in\bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}_{\mathbb{R}},d)^{\mathbb{T}^2} \tag{33}$$

be a $\mathbb{T}^2$-fixed disk map. The boundary distinguishes a minimal, c-invariant, central curve $P\subset C$ of the domain curve of the reflected map $[\tilde{f};C]\in\bar{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)$. satisfying $\partial D=P_{\mathbb{R}}$. Define the central degree p of f by the degree of $\tilde{f}|_P:P\to\mathbb{P}^4$. The central degree p is odd number $\le d$. The moduli point $[\tilde{f}_P]\in\bar{\mathcal{M}}_{0,0}(\mathbb{P}^4,p)$ is fixed by the $(\mathbb{C}^*)^2$-action.

Lemma 2 (Lemma 4 in [6])**.** *The two lines*

$$L=\{[x_0,\cdots,x_4]\in\mathbb{P}^4\mid x_0=x_3=x_4=0\},\quad L'=\{[x_0,\cdots,x_4]\in\mathbb{P}^4\mid x_0=x_1=x_2=0\}, \tag{34}$$

are the only real $(\mathbb{C}^*)^2$-invariant irreducible curves of odd degree.

If $P_{\mathbb{R}}$ contains a node, it must be mapped to the unique real fixed point ζ_0. However, the above lemma implies that $\tilde{f}(P)=L\quad\text{or}\quad L'$ (since the central degree p is odd). Hence $P_{\mathbb{R}}$ cannot contain a node. Note also that the oddness of d implies that the central curve P cannot be a contracted component. These imply the following

Lemma 3 (Lemma 5 in [6])**.** *(i) The central curve P is $\mathbb{P}^1$ and $\tilde{f}|_P:P\to L\mathrm{or}L'$ is the Galois covering of odd degree p. Explicitly, $\tilde{f}_P$ is given by*

$$[z_1,z_2]\mapsto[0,z_1^p,z_2^p,0,0]\quad\text{or}\quad[0,0,0,z_1^p,z_2^p]. \tag{35}$$

(ii) $\bar{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}_{\mathbb{R}},d)^{\mathbb{T}^2}\cong\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}},d)^{\mathbb{T}^2}$.

4.3. **Intersection disk term.** The original disk map $f\in\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}},d)^{\mathbb{T}^2}$ is obtained from a half of $\tilde{f}$. A half of P determines a pair (ζ,p) where $\zeta\in\{\zeta_1,\cdots,\zeta_4\}$ is a non real fixed point and p is the central degree. The data (ζ,p) is called the intersection disk type of f with the real Lagrangian $\mathbb{P}^4_{\mathbb{R}}$. The half of P is called the intersection disk.

Let $[f:D]\in\tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}},p)^{\mathbb{T}^2}$ be the unique $\mathbb{T}^2$-fixed point incident to ζ_i with the central degree p and domain consinsing only of the intersection disk. The intersection disk term $I(\zeta_i,p)$ is defined to be

$$I(\zeta_i,p)=\frac{e(F_p|_{[f,D]})}{e(Norm_{[f,D]})}\in\mathbb{Q}(\lambda,\lambda'). \tag{36}$$

Explicit evaluation of $I(\zeta_i,p)$ is not difficult. See [6, §4] for a proof. Let

$$C_p(\lambda,\lambda')=\frac{(-1)^{\frac{p-1}{2}}}{p}\frac{2\lambda}{p}\frac{\frac{(5p)!!}{p!p!!}(\frac{\lambda}{2p})^p}{\prod_{i=0}^{(p-1)/2}((1-\frac{2i}{p})\lambda-\lambda')((1-\frac{2i}{p})\lambda+\lambda')}. \tag{37}$$

Then we have

Lemma 4 (Lemma 6 in [6])**.** *$I(\zeta_1,p)=I(\zeta_2,p)=C_p(\lambda,\lambda')$, $I(\zeta_3,p)=I(\zeta_4,p)=C_p(\lambda',\lambda)$.*

For example, one obtains the number of degree 1 disk on the real quintic $(Q,Q_{\mathbb{R}})$ as

$$N_1^{\mathrm{disk}}=n_1^{\mathrm{disk}}=\sum_{i=1}^4 I(\zeta_i,1)=30, \tag{38}$$

which was first computed by Solomon [7] (using another method).

4.4. **Contribution of type (ζ_i,p).** We may separate the contribution to N_d^{disk} form $\mathbb{T}^2$-fixed maps into those from intersection disk type:

$$N_d^{disk}=\sum_{i=1}^4\sum_{p:\mathrm{odd},0<p\le d}\mathrm{Cont}_{(\zeta_i,p)}(N_d^{\mathrm{disk}}). \tag{39}$$

Let

$$\mathrm{Cont}_{(\zeta_i,p)}(\mathcal{F}^{\mathrm{disk}})=\sum_{d:\mathrm{odd}}e^{\frac{dT}{2}}\mathrm{Cont}_{(\zeta_i,p)}(N_d^{\mathrm{disk}}). \tag{40}$$

It turns out that the intersection disk term can be factored out of $\mathrm{Cont}_{(\zeta_i,p)}(\mathcal{F}^{\mathrm{disk}})$ by removing the intersection disk. What remains after intersection disk of type (ζ_1, p) is removed from $[f] \in \tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, d)^{\mathbb{T}^2}$ is a genus zero stable map with one marked point;

$$[f'] \in \mathrm{ev}_1^{-1}(\zeta_i) \subset \bar{\mathcal{M}}_{0,1}(\mathbb{P}^4, \frac{d-p}{2})^{(\mathbb{C}^*)^2}. \tag{41}$$

Contributions from these maps can be identified with Givental's equivariant correlator.

Proposition 5 (Lemma 7 in [6])**.**

$$\mathrm{Cont}_{(\zeta_i,p)}(\mathcal{F}^{\mathrm{disk}}) = \langle \mathcal{S}_{\mathbb{T}^2}(T, \frac{2p}{\lambda_i}), \phi_i \rangle \cdot I(\zeta_i, p). \tag{42}$$

Proof. Rewrite $\mathrm{Cont}_{(\zeta_i,p)}(\mathcal{F}^{\mathrm{disk}})$ as

$$\mathrm{Cont}_{(\zeta_i,p)}(\mathcal{F}^{\mathrm{disk}}) = \sum_{r>0} e^{(\frac{p}{2}+r)T} \mathrm{Cont}_{(\zeta_i,p)}(N^{\mathrm{disk}}_{p+2r}). \tag{43}$$

Then, by eq.(21), eq.(42) is equivalent to

$$\mathrm{Cont}_{(\zeta_i,p)}(N^{\mathrm{disk}}_{p+2r}) = \int_{\bar{\mathcal{M}}_{0,1}(\mathbb{P}^4,r)} \frac{c_{\mathrm{top}}(E_{r,1})}{\frac{2p}{\lambda_i} - \psi_1} \mathrm{ev}_1^*(\phi_i) \cdot \frac{I(\zeta_i, p)}{(5\lambda_i)(\frac{2p}{\lambda_i})}. \tag{44}$$

We apply localization formula to the LHS and compare with the RHS. Note that the $\mathbb{T}^2$-fixed loci of the both sides are isomorphic. Let $[f : D = C \cup_x \Delta] \in \tilde{\mathcal{M}}_D(\mathbb{P}^4/\mathbb{P}^4_{\mathbb{R}}, p+2r)^{\mathbb{T}^2}$, where Δ is the intersection disc of type (ζ_i, p). Let

$$[f', (C, x)] \in \mathrm{ev}_1^{-1}(\zeta_1) \subset \bar{\mathcal{M}}_{0,1}(\mathbb{P}^4, r)^{(\mathbb{C}^*)^2}. \tag{45}$$

be the stable map obtained from f by removing Δ. On the numerators, we have

$$e(F_{p+2r}) = c_{\mathrm{top}}(E_r) \cdot \frac{e(F_p)}{5\lambda_i}. \tag{46}$$

The factor $5\lambda_i$ comes from the nomalization sequence:

$$0 \to F_{p+2r}|_{[f]} \to E_r|_{[f']} \oplus F_p|_{[\Delta]} \to \mathcal{O}_{\mathbb{P}^4}(5)|_{\zeta_i} \to 0. \tag{47}$$

On the denominator of the LHS, we have

$$e(Norm_{[f]}) = \frac{e(Def(f)) \cdot e(Def(D))}{e(Aut(D))}. \tag{48}$$

For automomorphisms, we have

$$e(Aut(D)) = c_{\mathrm{top}}(Aut(C, x)) \cdot \frac{e(Aut(\Delta))}{\frac{2\lambda_i}{p}}. \tag{49}$$

The factor $\frac{2\lambda_i}{p}$ is the automorphism of Δ which moves the origin x. For deformation of maps, we have

$$e(Def(f)) = \frac{c_{\mathrm{top}}(Def(f')) \cdot e(Def(f|_\Delta))}{c_{\mathrm{top}}(T_{\zeta_i}\mathbb{P}^4)}. \tag{50}$$

The factor $c_{\mathrm{top}}(T_{\zeta_i}\mathbb{P}^4)$ comes from the normalization sequence

$$0 \to f^*T\mathbb{P}^4 \to f'^*T\mathbb{P}^4 \oplus (f|_\Delta)^*T\mathbb{P}^4 \to T_{\zeta_i}\mathbb{P}^4 \to 0. \tag{51}$$

For deformation of the domain, we have

$$e(Def(D)) = c_{\mathrm{top}}(Def(C, x)) \cdot (\frac{2\lambda_i}{p} - \psi_1). \tag{52}$$

The last factor comes from smoothing of the node x. In total, we have

$$\frac{1}{e(Norm_{[f]})} = \frac{1}{c_{top}(Nor_{[f']})} \frac{c_{top}(T_{\zeta_i}\mathbb{P}^4)}{(\frac{2\lambda_i}{p} - \psi_1)\frac{2\lambda_i}{p}} \frac{1}{e(Norm_{f|_\Delta})}. \tag{53}$$

This complete the proof of eq.(44). □

4.5. **Proof of Theorem 1.** Under the mirror map $T(t)$, we have

$$\mathcal{F}^{disk}(T(t)) = \sum_{i=1}^{4}\sum_{p:odd} \mathrm{Cont}_{(\zeta_i,p)}(\mathcal{F}^{\mathrm{disk}}(T(t))) \tag{54}$$

$$= \sum_{i=1}^{4}\sum_{p:odd} \langle \mathcal{S}_{\mathbb{T}^2}(T(t),\frac{2p}{\lambda_i}),\phi_i\rangle \cdot I(\zeta_i,p) \tag{55}$$

$$= \sum_{i=1}^{4}\sum_{p:odd} \frac{1}{I_0(t)}\langle \mathcal{S}^*_{\mathbb{T}^2}(t,\frac{2p}{\lambda_i}),\phi_i\rangle \cdot I(\zeta_i,p). \tag{56}$$

After explicit calculation (some combinatorial identity arising from multiple cover formula is necessary, see [6, Lemma 8]), we obtain the desired result:

$$\mathcal{F}^{disk}(T(t)) = \frac{2}{I_0(t)}\sum_{d:\mathrm{odd}} e^{\frac{dt}{2}}\frac{(5d)!!}{(d!!)^5}. \tag{57}$$

5. Other complete intersection CY's in projective spaces

It is possible to generalization to other complete intersection CY's in projective spaces with $h^{1,1}=1$. In the following, the choice of anti-holomorphic involution c on $\mathbb{P}^N$ is similar to the case of $\mathbb{P}^4$ (resp. $\mathbb{P}^1$) if N is even (resp. odd).

5.1. Compact examples.

5.1.1. $X_{3,3}\subset\mathbb{P}^5$ *(Studied in* [9]*.)* This is the case of $(\mathbb{P}^5,\mathcal{O}(3)^{\oplus 2})$. Let

$$J(t) = 2\sum_{d:odd} e^{dt/2}\frac{(3d)!!^2}{(d!!)^6}. \tag{58}$$

This is a solution to

$$[(\frac{d}{dt})^4 - 9e^t(3\frac{d}{dt}+1)^2(3\frac{d}{dt}+2)^2]J(t) = \frac{9}{8}e^{\frac{t}{2}}. \tag{59}$$

It gives the disk invariants under the mirror map.

5.1.2. $X_{4,2}\subset\mathbb{P}^5$. This is the case of $(\mathbb{P}^5,\mathcal{O}(4)\oplus\mathcal{O}(2))$. In this case, $N_d=0$ for any d. (Reason : Euler class of real vector bundle of odd rank vanishes.) In fact, it seems thhat the in-homogeneous PF

$$[(\frac{d}{dt})^4 - 16e^t(2\frac{d}{dt}+1)^2(4\frac{d}{dt}+1)(4\frac{d}{dt}+3)]J(t) = Ce^{\frac{t}{2}}. \tag{60}$$

does not give integer n_d^{disk} unless $C=0$.

5.1.3. $X_{3,2,2}\subset\mathbb{P}^6$. This is the case of $(\mathbb{P}^6,\mathcal{O}(3)\oplus\mathcal{O}(2)^{\oplus 2})$. Same as above. PF operator is:

$$[(\frac{d}{dt})^4 - 12e^t(2\frac{d}{dt}+1)^2(3\frac{d}{dt}+1)(3\frac{d}{dt}+2)]J(t) = Ce^{\frac{t}{2}}. \tag{61}$$

5.1.4. $X_{2,2,2,2}\subset\mathbb{P}^7$. This is the case of $(\mathbb{P}^7,\mathcal{O}(2)^{\oplus 4})$. Same as above. PF operator is:

$$[(\frac{d}{dt})^4 - 16e^t(2\frac{d}{dt}+1)^4]J(t) = Ce^{\frac{t}{2}}. \tag{62}$$

5.2. Local examples.

5.2.1. *Local* $\mathbb{P}^1$. The case of $(\mathbb{P}^1,\mathcal{O}(-1)^{\oplus 2})$. This is nothing but multiple cover formula [6]. See also [4].

5.2.2. *Local* $\mathbb{P}^2$ *(Studied in* [9].*)* The case of $(\mathbb{P}^2, \mathcal{O}(-3))$. Let

$$J(t) = 2\sum_{d:odd} e^{dt/2}(-1)^{\frac{3(d-1)}{2}}\frac{(3d-2)!!}{(d!!)^3}. \tag{63}$$

This is a solution to

$$[(\frac{d}{dt})^3 - 3e^t(3\frac{d}{dt}+1)(3\frac{d}{dt}+2)]J(t) = \frac{1}{4}e^{\frac{t}{2}}. \tag{64}$$

In this case, $I_0 = 1$ and the mirror map is given by $T(t) = I_1(t)$. Under these, $J(t)$ gives the disk invariants.

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