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（会場：東京理科大学）

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会場責任者：齋藤　功（東京理科大学）

Seminar on Function Spaces, 2008

CONTENTS

MODULATION SPACES AND APPLICATIONS

MASAHARU KOBAYASHI

ABSTRACT. In this article, we consider the modulation spaces $M_s^{p,q}(\mathbf{R}^d)$. Basic properties of $M_s^{p,q}(\mathbf{R}^d)$ and applications to pseudo-differential operators and partial differential equations are described.

1. 導入

この講演では, modulation 空間と呼ばれる函数空間の基本性質, 及び最近の擬微分作用素や偏微分方程式への応用についての紹介する. Modulation 空間 $M_s^{p,q}$ は Besov 空間とは異なる方法で函数または, 超函数の滑らかさを測るために Feichtinger [4] により導入された函数空間の 1 つであり, 時間周波数解析や擬微分作用素の研究に応用されている.

2. 定義および基本性質

この節では, modulation 空間の定義とその基本性質について紹介する. まず, modulation 空間の定義から始める.

以下では, $\widehat{f}, f^\vee$ で函数 f の Fourier 変換, 逆 Fourier 変換を表わすことにする.

DEFINITION 2.1. $\varphi \in \mathcal{S}(\mathbf{R}^n)$ を次の条件を満たすように取る.

$$\operatorname{supp} \varphi \subset [-1,1]^n, \quad \sum_{k\in\mathbf{Z}^n} \varphi(\xi - k) = 1, \quad \forall \xi \in \mathbf{R}^n.$$

$0 < p, q \le \infty, s \in \mathbf{R}$ とする. このとき, $f \in \mathcal{S}'(\mathbf{R}^n)$ に対して $f \in M_s^{p,q}(\mathbf{R}^n)$ とは,

$$\|f\|_{M_s^{p,q}} = \left(\sum_{k\in\mathbf{Z}^n} \left(\langle k \rangle^s \|\varphi(D-k)f\|_{L^p} \right)^q \right)^{1/q} < \infty$$

が成り立つことであると定義する. ただし $\varphi(D-k)f = (\varphi(\cdot - k)\widehat{f}(\cdot))^\vee$, $\langle k \rangle = (1+|k|^2)^{1/2}$ とする. $s = 0$ の場合は, $M_0^{p,q}$ を $M^{p,q}$ と書くことにする.

この定義からも分るように, modulation 空間はよく知られている Besov 空間と似た擬ノルムで定義される. Besov 空間の定義も復習しておく.

Definition 2.2 ([15]). $\psi_0, \psi \in \mathcal{S}(\mathbf{R}^n)$ を次の条件を満たすように取る:

$$\operatorname{supp} \psi_0 \subset \{\xi \mid |\xi| \le 2\}, \quad \operatorname{supp} \psi \subset \{\xi \mid 1/2 \le |\xi| \le 2\},$$

$$\psi_0(\xi) + \sum_{j=1}^{\infty} \psi(\xi/2^j) = 1, \quad \forall \xi \in \mathbf{R}^n.$$

$0 < p, q \le \infty, s \in \mathbf{R}$ とする. このとき, $f \in \mathcal{S}'(\mathbf{R}^n)$ に対して $f \in B_s^{p,q}(\mathbf{R}^n)$ とは

$$\|f\|_{B_s^{p,q}} = \left(\sum_{j=0}^{\infty} \left(2^{js} \left\| \psi(D/2^j) f \right\|_{L^p} \right)^q \right)^{1/q} < \infty$$

が成り立つことであると定義する. ただし, $\psi(D)f = (\psi_0(\cdot)\widehat{f}(\cdot))^\vee$, $\psi(D/2^j)f = \left(\psi(\cdot/2^j)\widehat{f}(\cdot)\right)^\vee$ $(j \ge 1)$ とする.

次に, modulation 空間の基本性質を紹介する.

Proposition 2.3. $0 < p, q \le \infty, s \in \mathbf{R}$ とする.
(1) $M_s^{p,q}(\mathbf{R}^n)$ は $\varphi \in \mathcal{S}(\mathbf{R}^n)$, $\sum_{k \in \mathbf{R}^n} \varphi(\xi - k) = 1$ の取り方に依らない.
(2) $M_s^{p,q}(\mathbf{R}^n)$ は擬-*Banach* 空間となる.
(3) $0 < p, q < \infty$ ならば, $\mathcal{S}(\mathbf{R}^n)$ は $M_s^{p,q}(\mathbf{R}^n)$ の稠密部分空間となる.
(4) $0 < p_1 \le p_2 \le \infty$, $0 < q_1 \le q_2 \le \infty$, $s_1 \ge s_2$ ならば,

$$M_{s_1}^{p_1,q_1}(\mathbf{R}^n) \hookrightarrow M_{s_2}^{p_2,q_2}(\mathbf{R}^n).$$

(5) ($M^{p,q}$ と L^p の関係)

$$M^{p,p}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n) \hookrightarrow M^{p,p'}(\mathbf{R}^n) \quad (1 \le p \le 2).$$

$$M^{p,p'}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n) \hookrightarrow M^{p,p}(\mathbf{R}^n) \quad (2 \le p \le \infty).$$

(ただし, $1/p + 1/p' = 1$)
(6) (*Dual space*) $0 < p, q < \infty$, $s \in \mathbf{R}$ ならば,

$$(M_s^{p,q}(\mathbf{R}^n))' = M_{-s}^{p',q'}(\mathbf{R}^n).$$

ここで, p' は p の共役指数を表わす : $1 \le p < \infty$ の場合 $\frac{1}{p} + \frac{1}{p'} = 1$. $0 < p < 1$ の場合 $p' = \infty$ (q' も同様に定義する).

3. 擬微分作用素への応用

この節では, modulation 空間の擬微分作用素

$$\sigma(x, D)f(x) = \frac{1}{(2\pi)^n} \iint_{\mathbf{R}^{2n}} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) e^{i(x-y)\cdot\xi} dy d\xi, \quad f \in \mathcal{S}(\mathbf{R}^n)$$

への応用について紹介する.

まず, modulation 空間上での $\sigma(x,D)$ の有界性定理としては次の定理が知られている. $m\in\mathbf{R}$, $0\le\rho\le 1$ に対して

$$S^m_{\rho,0}=\left\{\sigma(x,\xi)\in C^\infty(\mathbf{R}^2)\;\middle|\;|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\beta|}\right\}$$

とおく. このとき次が成り立つ.

THEOREM 3.1 (Tachizawa [13]). $1\le p,q<\infty$, $s\in\mathbf{R}$ とする.

$$\sigma\in S^m_{\rho,0}\Longrightarrow\sigma(x,D)\in\mathcal{L}(M^{p,q}_s(\mathbf{R}),M^{p,q}_{s-m}(\mathbf{R})).$$

他方, modulation 空間をシンボル σ として 用いる $\sigma(x,D)$ の有界性定理としては次の定理が知られている.

THEOREM 3.2 (Sjöstrand [12]). $\sigma(x,\xi)\in M^{\infty,1}(\mathbf{R}^{2n})\Longrightarrow\sigma(x,D)\in\mathcal{L}(L^2(\mathbf{R}^n))$.

これら 2 つの定理は共に, 有名な Calderón-Vaillancourt の定理 (1971)

$$\sigma\in S^0_{0,0}\Longrightarrow\mathcal{L}(L^2(\mathbf{R}^n))$$

の拡張となっている. これらの定理は Gröchenig-Heil によりさらに拡張された.

THEOREM 3.3 (Gröchenig-Heil[7]). $1\le q\le p\le\infty$, $\sigma\in M^{p,q}(\mathbf{R}^{2n})$ として, p_1,p_2,q_1,q_2 を次で定める.

$$p_1,q_1\ge p',\quad p_2,q_2\le p,$$
$$\frac{1}{p_1}-\frac{1}{p_2}=\frac{1}{q_1}-\frac{1}{q_2}=1-\frac{1}{p}-\frac{1}{q}.$$

このとき, $\sigma(x,D)\in\mathcal{L}(M^{p_1,q_1}(\mathbf{R}^n),M^{p_2,q_2}(\mathbf{R}^n))$ が成立する. 特に, $1\le p,q\le\infty$ に対して

$$\sigma\in M^{\infty,1}(\mathbf{R}^{2n})\Longrightarrow\sigma(x,D)\in\mathcal{L}(M^{p,q}(\mathbf{R}^n))$$

が成り立つ.

4. SCHATTEN クラス作用素

まず, Schatten クラス C_p の定義から始める. $1\le p<\infty$, T を $L^2(\mathbf{R}^n)$ 上のコンパクト作用素とする. このとき,

$T\in C_p\Longleftrightarrow T$ の特異値が l^p に入る
$((T^*T)^{1/2}$ の固有値 $)$

で定義する. このとき, 次のことが成り立つ.

THEOREM 4.1 (Gröchenig-Heil [7]). $1\le p\le 2$ とする. このとき

$$\sigma\in M^{p,p}(\mathbf{R}^{2n})\Longrightarrow\sigma(x,D)\in C_p.$$

特に, $p=1$ の時は $\sigma(x,D)$ は Trace 作用素, $p=2$ の時は Hilbert-Schmidt 作用素となる.

5. 偏微分方程式への応用

この節では, modulation 空間の偏微分方程式への応用について紹介する. その際, 重要となる次の定理が知られている.

THEOREM 5.1 ([1]). *$0 \le \alpha \le 2$, $1 \le p, q \le \infty$ とする. このとき, Fourier マルチプライヤー*

$$e^{i|D|^\alpha} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i|\xi|^\alpha} \widehat{f}(\xi) e^{ix\cdot\xi} d\xi$$

は $M^{p,q}(\mathbf{R}^n)$ 上有界となる.

REMARK 5.2. $\alpha = 2$ の場合, $u(t,x) = e^{it|D|^2} u_0(x)$ は Schrödinger 方程式

$$\begin{cases} i\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x), & (t > 0, x \in \mathbf{R}^n) \\ u(0,x) = u_0(x), & (x \in R^n) \end{cases}$$

の解となる. また L^p 空間や Besov 空間の場合, 次のことが知られている.

(1) Hörmander $e^{i|D|^\alpha}$ $(\alpha > 1)$ が $L^p(\mathbf{R}^n)$ 上有界 $\Longleftrightarrow p = 2$.
(2) Mizuhara $e^{i|D|^2}$ が $B^{p,q}_s(\mathbf{R}^n)$ 上有界 $\Longleftrightarrow p = 2$.

Theorem 5.1 から, 次のことが分る.

$$\begin{cases} i\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x), & (t > 0, x \in \mathbf{R}^n) \\ u(0,x) = u_0(x), & (x \in R^n) \end{cases}$$

の解

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{it|\xi|^2} \widehat{u_0}(\xi) e^{ix\cdot\xi} d\xi$$

に対して次が成り立つ: $1 \le p, q \le \infty$, $t \ge 0$ に対して

$$\|u(t,\cdot)\|_{M^{p,q}} \le C(1+t^2)^{n/4} \|u_0\|_{M^{p,q}}.$$

Fourier マルチプライヤーの評価に関して, 次のことも知られている.

THEOREM 5.3 ([16]). *$2 \le p < \infty$, $0 < q < \infty$, $s \in \mathbf{R}$, $t \in \mathbf{R}$, $1/p + 1/p' = 1$ とする. このとき,*

$$\|e^{it|D|^2} f\|_{M^{p,q}_s} \le \frac{1}{(1+|t|)^{n(\frac{1}{2}-\frac{1}{p})}} \|f\|_{M^{p',q}_s}, \quad f \in M^{p',q}_s(\mathbf{R}^n).$$

Modulation 空間は他にも, Korteweg-de Vries 方程式, Benjamin-Ono 方程式, 非線形 Schrödinger 方程式, 非線形波動方程式など現在様々な偏微分方程式にも応用され始めている.

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On BE and BED extensions of some commutative Banach algebras

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We define algebras $C_0^n(\mathbf{R}^d)$ and $\mathrm{Lip}_{n,\alpha}^0(\mathbf{R}^d)$ for $n \in \mathbf{Z}_+$ and $0 < \alpha < 1$, and then investigate their multiplier algebras, BE and BED extensions. The definitions and results contained in this article are generalizations of those for $C_0^1(\mathbf{R})$ and $\mathrm{Lip}_1^0(\mathbf{R})$ which are given in [1].

§1. $C_0^n(\mathbf{R}^d)$ 及び $\mathrm{Lip}_{n,\alpha}^0(\mathbf{R}^d)$ の定義とその性質

この節の目的は可換バナッハ環 $C_0^n(\mathbf{R}^d)$ 及び $\mathrm{Lip}_{n,\alpha}^0(\mathbf{R}^d)$ を定義し、その基本的性質を調べることである。

定義 1 ([3]). **微分環** $(C_b^n(\mathbf{R}^d), \| \ \|_{\infty,n}),\ (C_0^n(\mathbf{R}^d), \| \ \|_{\infty,n}), d, n \in \mathbf{N}$

$$C_b^n(\mathbf{R}^d) := \Bigg\{ f : C^n - \text{function on } \mathbf{R}^d, \\ \frac{\partial^m f}{\partial x_{i_1} \cdots \partial x_{i_m}} \in C_b(\mathbf{R}^d)\ (0 \le m \le n, 1 \le i_k \le d\ (k = 1, ..., m)) \Bigg\}$$

$$C_0^n(\mathbf{R}^d) := \Bigg\{ f \in C_b^n(\mathbf{R}^d) : \quad \frac{\partial^m f}{\partial x_{i_1} \cdots \partial x_{i_m}} \in C_0(\mathbf{R}^d)\ (0 \le m \le n, 1 \le i_k \le d\ (k = 1, ..., m) \Bigg\}$$

$$\|f\|_{\infty,n} := \|f\|_\infty + \sum_{m=1}^{n} \sum_{1 \le i_k \le d\ \ (k=1,\ldots,m)} \frac{1}{m!} \|f_{x_{i_1} \cdots x_{i_m}}\|_\infty$$

スカラー倍、和、積は pointwise とする.

定理 1 (**[3]**). $(C_b^n(\mathbf{R}^d), \| \ \|_{\infty,n})$, **は可環バナッハ環となり、**$C_0^n(\mathbf{R}^d)$ **はその closed ideal である。**
この場合極大イデヤル空間は $\Phi_{C_0^n(\mathbf{R}^d)} = \mathbf{R}^d$ **であり、ゲルファント変換は identity map と一致する。**

定義 2 (**[3]**). **Lipschitz algebras,** $(\mathrm{Lip}_{n,\alpha}(\mathbf{R}^d), \| \ \|_{\mathrm{Lip}_{\alpha,n}}),\ (\mathrm{Lip}_{n,\alpha}^0(\mathbf{R}^d), \| \ \|_{\mathrm{Lip}_{\alpha,n}})$,
$d \in \mathbf{N},\ n \in \{0\} \cup \mathbf{N},\ 0 < \alpha \le 1.$

$n = 0$ **の場合：**

$$\mathrm{Lip}_{0,\alpha}(\mathbf{R}^d) := \left\{ f \in C_b(\mathbf{R}^d) :\ \rho^{(\alpha)}(f) := \sup_{x \ne y,\ x,y \in \mathbf{R}^d} \frac{|\, f(x) - f(y)\, |}{|\, x - y\, |^\alpha} < \infty \right\}.$$

$$\mathrm{Lip}^0_{0,\alpha}(\mathbf{R}^d) := \Big\{ f \in C_0(\mathbf{R}^d) :\ \rho^{(\alpha)}(f) < \infty,$$

$$\rho^{(\alpha)}_M(f) := \sup_{x,y\in\mathbf{R}^d, x\neq y, M\le|x|,|y|} \frac{|\ f(x)-f(y)\ |}{|\ x-y\ |^\alpha} \to 0\ (M\to\infty) \Big\},$$

$$\|f\|_{\mathrm{Lip}_{0,\alpha}} := \|f\|_\infty + \rho^{(\alpha)}(f)\quad (f\in \mathrm{Lip}_{0,\alpha}(\mathbf{R}^d)).$$

$n = 1,2,3,...$ の場合：

$$\mathrm{Lip}_{n,\alpha}(\mathbf{R}^d) := \Big\{ f\in C^n_b(\mathbf{R}^d) :\ \rho^{(\alpha)}(f_{x_{i_1}\cdots x_{i_n}}) < \infty\ (1\le i_k \le d\ (k=1,...,n) \Big\},$$

$$\mathrm{Lip}^0_{n,\alpha}(\mathbf{R}^d) := \Big\{ f\in C^n_0(\mathbf{R}^d) :\ \rho^{(\alpha)}(F) < \infty,\ \lim_{M\to\infty} \rho^{(\alpha)}_M(F) = 0,$$

$$\Big(F = f_{x_{i_1}\cdots x_{i_n}}\ \Big(1\le i_k\le d,\ k=1,...,n\Big)\Big)\Big\},$$

$$\|f\|_{\mathrm{Lip}_{n,\alpha}} := \|f\|_{\infty,n} + \rho^{(\alpha)}(f) + \sum_{m=1}^{n} \sum_{1\le i_k\le d\ (k=1,...,m)} \frac{1}{m!}\rho^{(\alpha)}(f_{x_{i_1}\cdots x_{i_m}}).$$

定理 2 ([3]). $0\le n, 0<\alpha\le 1$ のとき、$\mathrm{Lip}_{n,\alpha}(\mathbf{R}^d)$ は半単純可換バナッハ換で、$\mathrm{Lip}^0_{n,\alpha}(\mathbf{R}^d))$ はその閉イデアルである。この場合極大イデヤル空間は $\Phi_{\mathrm{Lip}^0_{\alpha,n}}(\mathbf{R}^d) = \mathbf{R}^d$, ゲルファント変換は identity map となる。ここで index の間に

$$(n_1,\alpha_1)\le(n_2,\alpha_2) \Longleftrightarrow n_1<n_2 \text{ or } n_1=n_2, \alpha_1\le\alpha_2.$$

と辞書的に順序を導入すると次の関係が成り立つ。、

$$\mathrm{Lip}^0_{n_2,\alpha_2} \subsetneqq \mathrm{Lip}^0_{n_1,\alpha_1}(\mathbf{R}^d) \Leftrightarrow (n_1,\alpha_1)<(n_2,\alpha_2).$$

§2. BE 拡大、BED 拡大

次に、$(A,\|\ \|_A)$ の BE 拡大、BED 拡大を定義する。$A_c := \{x\in A : \hat{x} \text{ has compact support}\}$ 、$\mathcal{K}(\Phi_A) := \{K :\ K\subseteq \Phi_A,\ \text{compact}\}$ とする。

定義 3 ([1])

$$C_{BE(A)}(\Phi_A) \ := \ \Big\{\sigma\in C_b(\Phi_A) : \|f\|_{BE(A)} < \infty, \text{where}$$

$$\|f\|_{BE(A)} = \sup_{\substack{0\neq p=\sum_{i=1}^m a_i\varphi_i\\ \in \mathrm{span}(\Phi_A)}} \frac{|\ \sum_{i=1}^m a_i\sigma(\varphi_i)\ |}{\|p\|_{A^*}} < \infty\Big\},$$

$$C^0_{BE(A)}(\Phi_A) \ := \ \Big\{\sigma\in C_{BE(A)}(\Phi_A) : \lim_{K\in\mathcal{K}(\Phi_A)} \|f\|_{BE(A),K} := 0\}.$$

$$\text{where } \|f\|_{BE(A),K} := \sup_{\substack{0\neq p=\sum_{i=1}^m a_i\varphi_i\\ \in \mathrm{span}(\Phi_A\setminus K)}} \frac{|\ \sum_{i=1}^m a_i\sigma(\varphi_i)\ |}{\|p\|_{A^*}}.$$

以上の設定より次のことが成立する。

（1）$(C_{BE(A)}(\Phi_A), \| \|_{BE(A)})$ は半単純バナッハ環 ([4]),
（2）$C^0_{BE(A)}(\Phi_A)$ は $C_{BE(A)}(\Phi_A)$ の closed ideal([1]),
（3）$C^0_{BE}(\Phi_A) \supseteq \hat{A}$ if A is Tauberian ([1]), ($\Leftrightarrow A_c$ is dense in A)
（4）$\sigma \in C_b(\Phi_A)$ に対して次が成立 ([4])。

$$\begin{aligned}\sigma \in C_{BE(A)}(\Phi_A) \quad &\Leftrightarrow \quad \exists \text{ bounded net } \{a_\lambda\}_{\lambda\in\Lambda} \text{ in } A \text{ s.t.}\\ &\qquad\qquad \lim_{\lambda\in\Lambda}\widehat{a_\lambda}(\varphi) = \sigma(\varphi) \quad (\varphi \in \Phi_A).\\ &\Leftrightarrow \quad \exists\tau \in A^{**} \text{ s. t.}\sigma(\varphi) = \tau(\varphi) \quad (\varphi \in \Phi_A).\end{aligned}$$

最初の同値関係を使うと、$C_{BE(A)}(\Phi_A) \supseteq \hat{A}$ が導かれる。すなわち, $C_{BE(A)}(\Phi_A)$ は $\hat{A}$ の拡張である。

定義4 (Multiplier). $(A, \| \|_A)$ に対して、($B(A) := \{T :$ bounded linear operator of $A\}$)
$M(A) := \{T \in B(A) : T(ab) = aT(b) \quad (a, b \in A)\}$.
$\hat{M}(A) := \{\sigma \in C_b(\Phi_A) : \sigma\hat{a} \in \hat{A}(a \in A)\}$

注： $\forall T \in M(A), \exists\hat{T} \in \hat{M}(A)$ $s.t.$ $\widehat{Ta} = \hat{T}\hat{a} \quad (a \in A)$.
従って、 $\hat{M}(A)$ は $M(A)$ の Φ_A 上での表現である。
A に対して Φ_A 上の Banach function algebra とその closed ideal という関係についての２組のペア；

$$\hat{M}(A) \underset{\supseteq}{\overset{\text{ideal}}{}} \hat{A}, \qquad C_{BE(A)}(\Phi_A) \underset{\supseteq}{\overset{\text{ideal}}{}} C^0_{BE(A)}(\Phi_A)$$

が存在する。ここで、
”A の **BE**-extension または **BED**-extension ”、”A が **BE** である ”, ”A が **BED** である ” などを次のように定義する。

定義5 （BE, BED）. $(A, \| \|_A)$ に対して、

- $C_{BE(A)}(\Phi_A)$ を A の BE- 拡大 といい、$C^0_{BE(A)}(\Phi_A)$ を A の BED-拡大という。
- A が BE である $\overset{def}{\Leftrightarrow}$ $\hat{M}(A) = C_{BE(A)}(\Phi_A)$
- A が BED である $\overset{def}{\Leftrightarrow}$ $\hat{A} = C^0_{BE(A)}(\Phi_A)$

よく知られた S. Bochner, W. F. Eberlein, J. W. Wendel , R. Doss なのど結果は以下の定理の形に述べることができ、我々の研究の基礎になっている。

Theorem (S. Bochner 1934, W. F. Eberlein, 1955)
$C_{BE(L^1(G))}(\hat{G}) = M(G)^\wedge$. (注：$M(G)$ は G 上の measure algebra,)

Theorem (J. W. Wendel 1952) $\hat{M}(L^1(G)) = M(G)^\wedge$,

Theorem (R. Doss, 1968) $C^0_{BE(L^1(G))}(\hat{G}) = L^1(G)^\wedge$.

Theorem (Bochner, Eberlein, Wendel, Doss)} $L^1(G)$ **は BE かつ BED である。**

我々は $C_{\infty,n}(\mathbf{R}^d)$, $C^0_{\infty,n}(\mathbf{R}^d)$ の BE and BED extensions 及び multiplier algebra について次の結果を得た。

Theorem 3. **([3])** $A = C_0^n(\mathbf{R}^d)$ **のとき;**
$C_{BE(A)}(\mathbf{R}^d) = \mathrm{Lip}_{n-1,1}(\mathbf{R}^d)$, $C_{BE(A)}^0(\mathbf{R}^d) = \mathrm{Lip}_{n-1,1}^0(\mathbf{R}^d)$, $\hat{M}(C_0^n(\mathbf{R}^d)) = C_b^n(\mathbf{R}^d)$.

また、 $\mathrm{Lip}_{n,\alpha}(\mathbf{R}^d)$ **及び** $\mathrm{Lip}_{n,\alpha}(\mathbf{R}^d)$ **については次の結果を得た。**

Theorem 4. **([3])** $\hat{M}(\mathrm{Lip}_{n,\alpha}^0(\mathbf{R}^d)) = \mathrm{Lip}_{n,\alpha}(\mathbf{R}^d)$ **であり、かつ** $\mathrm{Lip}_{n,\alpha}^0(\mathbf{R}^d)$ **は BE かつ BED である。**

註1. $C_0^1(\mathbf{R})$ の **BE** and BED extension がそれぞれ $\mathrm{Lip}_{0,1}(\mathbf{R})$ と $\mathrm{Lip}_{0,1}^0(\mathbf{R})$ であること、また $\mathrm{Lip}_{0,1}^0(\mathbf{R})$ が **BE** かつ **BED** であることは [1] で得られた結果であり、今回の講演の結果はそれらの拡張をもとめて研究した結果である。

註2. 我々は [2] で、$L^1(G)$ で定義されて研究されて来た Segal algebra の概念をより広い可換バナッハ環のクラスに拡張して研究した。今回定義された $\mathrm{Lip}_{n,\alpha}^0(\mathbf{R}^d)$ についても、その Segal algebra は興味深い研究対象だと考えている。

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Three dimensional commutative Banach algebras and Q-algebras

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This is a joint work with Prof. Takahiko Nakazi (Hokusei Gakuen University).
Let $\mathcal{B}$ denote a unital commutative Banach algebra satisfying $\dim \mathcal{B} = 3$. Let $\mathcal{M}(\mathcal{B})$ denote a maximal ideal space of $\mathcal{B}$, that is, the set of all nonzero complex-valued homomorphism. There is a problem of classifying all $\mathcal{B}$ by the number $\sharp\mathcal{M}(\mathcal{B}) = 1, 2, 3$ of elements in $\mathcal{M}(\mathcal{B})$ and establishing the structure of $\mathcal{B}$. If $\mathcal{B}$ is a Q-algebra, then there is a problem of investigating the contractive natural representation of $\mathcal{B}$ to $\mathbf{C}^3$. In this paper, we study the case when $\sharp\mathcal{M}(\mathcal{B}) = 1$, and describe a contractive natural representation of a concrete Q-algebra.

第1章　一般の unital 可換 Banach 環 $\mathcal{B}$ の構造

$\phi \in \mathcal{M}(\mathcal{B})$ は $\phi(fg) = \phi(f)\phi(g)$, $(f, g \in \mathcal{B})$ を満たす．点 ϕ において，$D^1 \in \mathcal{B}^*$ が

$$D^1(fg) = D^1(f)\phi(g) + \phi(f)D^1(g), \quad (f, g \in \mathcal{B})$$

を満たすとき，D^1 を点 ϕ における第1 - 点微分と呼ぶ．また，点 ϕ において，$D^2 \in \mathcal{B}^*$ が

$$D^2(fg) = D^2(f)\phi(g) + 2D^1(f)D^1(g) + \phi(f)D^2(g), \quad (f, g \in \mathcal{B})$$

を満たすとき，D^2 を点 ϕ における第2 - 点微分と呼ぶ．
Nakazi-Osawa [2] は，もし $\sharp\mathcal{M}(\mathcal{B}) = 1$ ならば，$\mathcal{B}^* = \mathrm{span}\{\phi, D^1, D_0\}$ が成り立つことを示した．ただし，$\phi \in \mathcal{M}(\mathcal{B})$ であり，D^1 は 3 次元の unital 可換 Banach 環 $\mathcal{B}$ で定義された点 ϕ における第1 - 点微分であり，D_0 は 2 次元の unital 可換 Banach 環 $\mathcal{B}_0 = \ker D^1$ で定義された点 ϕ における第1 - 点微分である．我々は第2 - 点微分に着目し，以下の定理 1 を得た．
定理 1(1) において，ϕ, D_1^1, D_2^1 は 1 次独立だから $\mathcal{B}^* = \mathrm{span}\{\phi, D_1^1, D_2^1\}$ が成り立つ．
定理 1(2) において，ϕ, D^1, D^2 は 1 次独立だから $\mathcal{B}^* = \mathrm{span}\{\phi, D^1, D^2\}$ が成り立つ．

定理 1.　$\mathcal{B}$ は unital 可換 Banach 環で $\dim \mathcal{B} = 3$ とする．もし $\sharp\mathcal{M}(\mathcal{B}) = 1$ ならば，次の (1) または (2) が成り立つ．ただし $\mathcal{M}(\mathcal{B}) = \{\phi\}$ である．

(1)　$g, h \in \mathcal{B}$ が存在して $\mathcal{B} = \mathrm{span}\{1, g, h\}$，ただし $g^2 = h^2 = gh = 0$ が成り立つ．点 ϕ における第1 - 点微分 D_1^1, D_2^1 が存在して，任意の $f \in \mathcal{B}$ は次のように書ける：

$$f = \phi(f) + D_1^1(f)g + D_2^1(f)h.$$

(2)　$g \in \mathcal{B}$ が存在して $\mathcal{B} = \mathrm{span}\{1, g, g^2\}$，ただし $g^3 = 0$ が成り立つ．点 ϕ における第1 - 点微分 D^1 と第2 - 点微分 D^2 が存在して，任意の $f \in \mathcal{B}$ は次のように書ける：

$$f = \phi(f) + D^1(f)g + \frac{D^2(f)}{2}g^2.$$

定理 1 の証明. (Step 1) $\mathcal{B}_\phi = \ker\phi = \{f \in \mathcal{B} : \phi(f) = 0\}$ と定める．よって $\dim\mathcal{B}_\phi = 2$ が成り立つ．このとき，次の (1) または (2) が成り立つことを示す．
(1) $g, h \in \mathcal{B}$ が存在して $\mathcal{B}_\phi = \mathrm{span}\{g, h\}$, $g^2 = h^2 = gh = 0$.
(2) $g \in \mathcal{B}$ が存在して $\mathcal{B}_\phi = \mathrm{span}\{g, g^2\}$, $g^3 = 0$.

$\dim\mathcal{B}_\phi = 2$ より，$g, h \in \mathcal{B}$ が存在して $\mathcal{B}_\phi = \mathrm{span}\{g, h\}$ と書ける．もし $g^2 = h^2 = gh = 0$ ならば，(1) が成り立つ．もし $g^2 = h^2 = 0$, $gh \neq 0$ ならば，ϕ は homomorphism より，$\mathrm{span}\{g, h\}$ は $\mathrm{span}\{g, gh\}$ または $\mathrm{span}\{h, gh\}$ に等しい．このとき $k = gh$ とおくと，$k^2 = kg = kh = 0$ が成り立ち，$\mathrm{span}\{g, h\}$ は $\mathrm{span}\{g, k\}$ または $\mathrm{span}\{h, k\}$ に等しいから (1) が成り立つ．まとめると，$g^2 = h^2 = 0$ のときは (1) が成り立つ．

次に，$g^2 = h^2 = 0$ でない場合は (2) が成り立つことを示す．$g^2 \neq 0$ として示せばよい．ϕ は homomorphism より $g^2 \in \mathcal{B}_\phi$ だから，ある複素数 α, β が存在して $g^2 = \alpha f + \beta g$ が成り立つ．$\alpha = 0$ と仮定すると $g^2 \neq 0$ だから $\beta \neq 0$ かつ $g^2 = \beta g$. よって，$g(g - \beta) = 0$. $g \in \mathcal{B}_\phi$ より $\phi(g) = 0$ だから $\phi(g - \beta) = -\beta \neq 0$ かつ $\mathcal{M}(\mathcal{B}) = \{\phi\}$. よって $g - \beta$ は $\mathcal{B}$ の可逆元である．よって $g = 0$ となり矛盾．よって $\alpha \neq 0$. よって $\mathcal{B}_\phi = \mathrm{span}\{f, g\} = \mathrm{span}\{g, g^2\}$ である．このとき，$g^3 = 0$ を示す．ϕ は homomorphism より $g^3 \in \mathcal{B}_\phi$ だから，ある複素数 α, β が存在して $g^3 = \alpha g + \beta g^2$. よって $g(g^2 - \alpha - \beta g) = 0$. このとき，$\alpha \neq 0$ と仮定すると，$\phi(g^2 - \alpha - \beta g) = -\alpha \neq 0$ かつ $\mathcal{M}(\mathcal{B}) = \{\phi\}$ だから $g^2 - \alpha - \beta g$ は $\mathcal{B}$ の可逆元である．よって $g = 0$ となり矛盾．よって $\alpha = 0$. よって $g^3 = \beta g^2$. よって $g^2(g - \beta) = 0$. このとき，$\beta \neq 0$ と仮定すると，$\phi(g - \beta) = -\beta \neq 0$ かつ $\mathcal{M}(\mathcal{B}) = \{\phi\}$ だから $g - \beta$ は $\mathcal{B}$ の可逆元である．よって $g^2 = 0$ となり矛盾．よって $\beta = 0$. よって $g^3 = 0$.

(Step 2) $1 \in \mathcal{B}$ だから，Step 1 より，次の (1) または (2) が成り立つ．
(1) $g, h \in \mathcal{B}$ が存在して $\mathcal{B} = \mathrm{span}\{1, g, h\}$, $g^2 = h^2 = gh = 0$.
(2) $g \in \mathcal{B}$ が存在して $\mathcal{B} = \mathrm{span}\{1, g, g^2\}$, $g^3 = 0$.

(1) の場合，任意の $f \in \mathcal{B}$ に対し，複素数 $c_0(f), c_1(f), c_2(f)$ が一意的に存在して

$$f = c_0(f) + c_1(f)g + c_2(f)h$$

が成り立つ．よって $\phi(f) = c_0(f)$ が成り立ち，$c_1, c_2 \in \mathcal{B}^*$ である．このとき，$c_1 = D_1^1, c_2 = D_2^1$ を示せば証明が終了する．$F = \alpha + \beta g + \gamma h$, $G = a + bg + ch$ に対して，

$$FG = \alpha a + (\beta a + \alpha b)g + (\gamma a + \alpha c)h$$

が成り立つ．このとき，

$$\begin{pmatrix} \phi(F) & c_1(F) & c_2(F) \\ \phi(G) & c_1(G) & c_2(G) \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \end{pmatrix}$$

だから，

$$c_1(FG) = \beta a + \alpha b = c_1(F)\phi(G) + \phi(F)c_1(G),$$
$$c_2(FG) = \gamma a + \alpha c = c_2(F)\phi(G) + \phi(F)c_2(G)$$

が成り立つ．よって $c_1 = D_1^1$, $c_2 = D_2^1$ と書ける．

(2) の場合，任意の $f \in \mathcal{B}$ に対し，複素数 $\hat{f}(0), \hat{f}(1), \hat{f}(2)$ が一意的に存在して

$$f = \hat{f}(0) + \hat{f}(1)g + \hat{f}(2)g^2$$

が成り立つ．よって $\phi(f) = \hat{f}(0)$ が成り立つ．このとき，$\delta_1(f) = \hat{f}(1)$, $\delta_2(f) = 2\hat{f}(2)$ と定めると，$\delta_1, \delta_2 \in \mathcal{B}^*$ である．このとき，$\delta_1 = D^1, \delta_2 = D^2$ を示せば証明が終了する．$F = \alpha + \beta g + \gamma g^2$, $G = a + bg + cg^2$ に対して，

$$FG = \alpha a + (\beta a + \alpha b)g + (\alpha c + \beta b + \gamma a)g^2$$

が成り立つ．このとき，

$$\begin{pmatrix} \phi(F) & \delta_1(F) & \delta_2(F) \\ \phi(G) & \delta_1(G) & \delta_2(G) \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 2\gamma \\ a & b & 2c \end{pmatrix}$$

だから，

$$\begin{aligned} \delta_1(FG) &= \beta a + \alpha b = \delta_1(F)\phi(G) + \phi(F)\delta_1(G), \\ \delta_2(FG) &= 2(\alpha c + \beta b + \gamma a) = (2\gamma)a + 2\beta b + \alpha(2c) \\ &= \delta_2(F)\phi(G) + 2\delta_1(F)\delta_1(G) + \phi(F)\delta_2(G) \end{aligned}$$

が成り立つ．よって $\delta_1 = D^1$, $\delta_2 = D^2$ である．□

任意の可換な正方行列は unitary 行列により同時三角化可能だから，定理 1 より次の系を得る．

系 1. $\mathcal{B}$ は unital 可換 3×3 行列環で $\dim \mathcal{B} = 3$ とする．もし $\sharp\mathcal{M}(\mathcal{B}) = 1$ ならば，次の (1) または (2) が成り立つ．ただし，$\mathcal{M}(\mathcal{B}) = \{\phi\}$ である．

(1) 点 ϕ における第 1 - 点微分 D_1^1, D_2^1 が存在して，任意の $f \in \mathcal{B}$ に対し，unitary 同値の意味で次の等式が成り立つ：

$$f = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + D_1^1(f)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + D_2^1(f)\begin{pmatrix} 0 & 0 & 0 \\ z & 0 & 0 \\ w & 0 & 0 \end{pmatrix}.$$

ただし x, y, z, w は任意の複素数である．

(2) 点 ϕ における第 1 - 点微分 D^1 と第 2 - 点微分 D^2 が存在して，任意の $f \in \mathcal{B}$ に対し，unitary 同値の意味で次の等式が成り立つ：

$$f = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + D^1(f)\begin{pmatrix} 0 & 0 & 0 \\ z & 0 & 0 \\ y & x & 0 \end{pmatrix} + \frac{D^2(f)}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ zx & 0 & 0 \end{pmatrix}.$$

ただし x, y, z は任意の複素数である．

第 2 章　$\mathcal{B}$ が Q - 環の場合の自然な contractive 表現

A はコンパクト Hausdorff 空間 X 上の関数環とし，I は A の閉 ideal とする．このとき，商環 A/I は unital 可換 Banach 環である．本報告では，もし Banach 環 $\mathcal{B}$ が A/I に等距離 isomorphic ならば，$\mathcal{B}$ を Q - 環 と呼ぶ. (Bonsall-Duncan は IQ - 環 と呼んでいる.) また，確率測度 μ に対し，A の $L^2(\mu)$ ノルム 閉包を $H^2(\mu)$ で表し，抽象的 Hardy 空間と呼ぶ.
このとき $\mathcal{H}=H^2(\mu)\cap I^{\perp}$ と定め，$P:H^2(\mu)\to\mathcal{H}$ を直交射影とし，Q-環 A/I の元 $f+I$ に対し，作用素 $S^{\mu}_f:\mathcal{H}\to\mathcal{H}$ を $S^{\mu}_f g=P(fg)$, $(f\in A)$ と定める．もし $\dim\mathcal{H}=3$ ならば，3 次元 Q-環 A/I と $B(\mathcal{H})$ の 3 次元部分環 $\{S^{\mu}_f\ :\ f\in A\}$ は 1 対 1 に対応し $\|S^{\mu}_f\|\leq\|f+I\|$. ただし $B(\mathcal{H})$ は $\mathcal{H}$ 上の有界線形作用素の全体を表し，$\|S^{\mu}_f\|$ は作用素 norm $\|S^{\mu}_f\|_{B(\mathcal{H})}$ を表し，$\|f+I\|$ は商 norm $\|f+I\|_{A/I}$ を表す.

第 1 章の系 1 における行列の成分 x,y,z,w は（その比が）任意であるが，次の定理 2 における行列の成分の比は任意であるか否かが問題である.

定理 2. A はコンパクト Hausdorff 空間 X 上の関数環とし，I は A の閉 ideal とする．μ は X 上の確率測度とし，$\mathcal{H}=H^2(\mu)\cap I^{\perp}$ と定める．このとき，次の (1) と (2) が成り立つ.

(1) $I=\{f\in A\ :\ \phi(f)=D^1_1(f)=D^1_2(f)=0\}$ と定める．このとき $k_1,k_2,k_3\in\mathcal{H}$ が存在して $\phi(f)=\langle f,k_1\rangle$, $D^1_1(f)=\langle f,k_2\rangle$, $D^1_2(f)=\langle f,k_3\rangle$, $(f\in A)$ を満たす．このとき $\mathcal{H}=\mathrm{span}\{\psi_1,\psi_2,\psi_3\}$ が成り立つ．ただし $\{\psi_1,\psi_2,\psi_3\}$ は $\{k_1,k_2,k_3\}$ から Gram-Schmidt の方法で作られた $\mathcal{H}$ の正規直交基底である．この基底に関して,

$$S^{\mu}_f=\phi(f)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+D^1_1(f)C\begin{pmatrix}0&0&0\\1&0&0\\y&0&0\end{pmatrix}+D^1_2(f)\begin{pmatrix}0&0&0\\0&0&0\\x&0&0\end{pmatrix},$$

$$C=\frac{\|k_1\|}{\sqrt{\|k_2\|^2-|\langle k_2,\psi_1\rangle|^2}},$$

$$x=\frac{\|k_1\|}{\sqrt{\|k_3\|^2-|\langle k_3,\psi_1\rangle|^2-|\langle k_3,\psi_2\rangle|^2}},\quad y=\frac{-\langle\psi_2,k_3\rangle}{\sqrt{\|k_3\|^2-|\langle k_3,\psi_1\rangle|^2-|\langle k_3,\psi_2\rangle|^2}}.$$

(2) $I=\{f\in A\ :\ \phi(f)=D^1(f)=D^2(f)=0\}$ と定める．このとき $k_1,k_2,k_3\in\mathcal{H}$ が存在して $\phi(f)=\langle f,k_1\rangle$, $D^1(f)=\langle f,k_2\rangle$, $D^2(f)=\langle f,k_3\rangle$, $(f\in A)$ を満たす．このとき $\mathcal{H}=\mathrm{span}\{\psi_1,\psi_2,\psi_3\}$ が成り立つ．ただし $\{\psi_1,\psi_2,\psi_3\}$ は $\{k_1,k_2,k_3\}$ から Gram-Schmidt の方法で作られた $\mathcal{H}$ の正規直交基底である．この基底に関して,

$$S^{\mu}_f=\phi(f)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+D^1(f)C\begin{pmatrix}0&0&0\\1&0&0\\y&x&0\end{pmatrix}+\frac{D^2(f)}{2}C^2\begin{pmatrix}0&0&0\\0&0&0\\x&0&0\end{pmatrix},$$

$$C=\frac{\|k_1\|}{\sqrt{\|k_2\|^2-|\langle k_2,\psi_1\rangle|^2}},$$

$$x=\frac{\frac{2}{C}\sqrt{\|k_2\|^2-|\langle k_2,\psi_1\rangle|^2}}{\sqrt{\|k_3\|^2-|\langle k_3,\psi_1\rangle|^2-|\langle k_3,\psi_2\rangle|^2}},\quad y=\frac{\frac{2}{C}\langle\psi_1,k_2\rangle-\langle\psi_2,k_3\rangle}{\sqrt{\|k_3\|^2-|\langle k_3,\psi_1\rangle|^2-|\langle k_3,\psi_2\rangle|^2}}.$$

ただし $\langle\cdot,\cdot\rangle$ は $\mathcal{H}$ における内積 $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ を表し，$\|\cdot\|$ は $\mathcal{H}$ における norm $\|\cdot\|_{\mathcal{H}}$ を表す.

定理 2 の証明の方針. $\{\psi_1,\psi_2,\psi_3\}$ は $\{k_1,k_2,k_3\}$ から Gram-Schmidt の方法で作られた $\mathcal{H}$ の正規直交基底だから,

$$\psi_1=\frac{k_1}{\|k_1\|},\quad \psi_2=\frac{k_2-\langle k_2,\psi_1\rangle\psi_1}{\|k_2-\langle k_2,\psi_1\rangle\psi_1\|},\quad \psi_3=\frac{k_3-\langle k_3,\psi_1\rangle\psi_1-\langle k_3,\psi_2\rangle\psi_2}{\|k_3-\langle k_3,\psi_1\rangle\psi_1-\langle k_3,\psi_2\rangle\psi_2\|}.$$

$a_{ij}=\langle S_f^\mu\psi_j,\psi_i\rangle$ と定める. $P:H^2(\mu)\to\mathcal{H}$ は直交射影だから, $a_{ij}=\langle f\psi_j,\psi_i\rangle$ が成り立つ. 各 i,j $(i,j=1,2,3)$ について, 右辺の内積を 点 ϕ における第 1 - 点微分 D^1,D_1^1,D_2^2, 第 2 - 点微分 D^2 に関する Leibniz の公式を用いて計算すれば良い. □

定理 2 における対応: $\mathcal{B}=A/I \quad\longleftrightarrow\quad \{S_f^{\mathcal{H}}\in B(\mathcal{H})\ :\ f\in A\}=\mathcal{B}'$ において, 右側を左側の自然な表現と呼ぶ. 特に $\mathcal{H}$ が抽象的 Hardy 空間 $H^2(\mu)$ を用いて $\mathcal{H}=H^2(\mu)\cap I^\perp$ と書けるとき, $S_f^\mu=S_f^{\mathcal{H}}$ と書く. 目標は, 3 次元 Q-環 $\mathcal{B}_1=A/I$ と 3 次元 unital 可換 3×3 行列環 $\mathcal{B}_2$ について, もし $\sharp\mathcal{M}(\mathcal{B}_1)=\sharp\mathcal{M}(\mathcal{B}_2)$ ならば, unitary 同値の意味で等式 $\mathcal{B}_1=\mathcal{B}_2$ が成り立つか否かを調べることである. 我々は [3] において, $\mathbf{T}=\{|z|=1\}$ 上の円板環 $A(\mathbf{T})$ に関する 3 次元 Q-環 $\mathcal{B}_1=A(\mathbf{T})/I$ と 3 次元 unital 可換 3×3 行列環 $\mathcal{B}_2$ について, もし $\sharp\mathcal{M}(\mathcal{B}_1)=\sharp\mathcal{M}(\mathcal{B}_2)=3$ ならば, unitary 同値の意味で $\mathcal{B}_1\neq\mathcal{B}_2$ を示した.

第 3 章　$\mathcal{B}$ が円板環から作られる Q - 環の場合

例 1. $A(\mathbf{T}^2)$ は 2 重円板環とし, $I=\{f\in A(\mathbf{T}^2)\ :\ f(0,0)=f_z(0,0)=f_w(0,0)=0\}$ とする. $z=e^{i\theta},w=e^{i\phi},d\mu=d\theta d\phi/(2\pi)^2$ として定理 2(1) を適用すると, $\{1,z,w\}$ は $\mathcal{H}=H^2(\mu)\cap I^\perp$ の正規直交基底であり,

$$S_f^\mu=f(0,0)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+f_z(0,0)\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}+f_w(0,0)\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}.$$

S_f^μ は Q - 環 $A(\mathbf{T}^2)/I$ の元 $f+I$ の自然な contractive 表現である.

例 2. $A(\overline{\mathbf{D}})$ を $\overline{\mathbf{D}}=\{|z|\le 1\}$ 上の円板環とし, $I=z^3A=\{f\in A\ :\ f(0)=f'(0)=f''(0)=0\}$ とする. 区間 $[0,1]$ 上の正の測度 $\nu(r)$ は $\int_0^1 d\nu(r)=1$ を満たすとする. μ を $\mu(E)=\int\int_E d\nu(r)d\theta/2\pi$ と定めると, μ は $\overline{\mathbf{D}}$ 上の radial 確率測度だから k_1,k_2,k_3 は互いに直交する. $f(z)=z$ について, $(f(0),f'(0),f''(0))=(0,1,0)$ だから, 定理 2(2) より,

$$S_z^\mu=C\begin{pmatrix}0&0&0\\1&0&0\\0&x&0\end{pmatrix}=\begin{pmatrix}0&0&0\\C&0&0\\0&Cx&0\end{pmatrix},$$

$$C=\frac{\|k_1\|_{H^2(\mu)}}{\|k_2\|_{H^2(\mu)}}=\left\{\int_0^1 r^2d\nu(r)\right\}^{1/2}\le 1,\quad x=\frac{2\|k_2\|_{H^2(\mu)}}{C\|k_3\|_{H^2(\mu)}}=\frac{\{\int_0^1 r^4d\nu(r)\}^{1/2}}{\int_0^1 r^2d\nu(r)}\ge 1,$$

$$k_1=1,\quad k_2=\frac{z}{\int_0^1 r^2d\nu(r)},\quad k_3=\frac{2z^2}{\int_0^1 r^4d\nu(r)},$$

$$\psi_1=1,\quad \psi_2=\frac{z}{\{\int_0^1 r^2d\nu(r)\}^{1/2}},\quad \psi_3=\frac{z^2}{\{\int_0^1 r^4d\nu(r)\}^{1/2}}.$$

よって $Cx = C\max\{|x|, 1\} = \|S_z^\mu\| \leq 1$. よって $C \leq Cx \leq 1$. S_f^μ は Q - 環 $A(\overline{\mathbf{D}})$ の元 $f+I$ の自然な contractive 表現である.

例 3. Hardy 空間 $H^2 = H^2(\mathbf{D})$ と Bergman 空間 $L_a^2 = L_a^2(\mathbf{D})$ の norm を

$$\|f\|_{H^2}^2 = \sup_{0\leq r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta/2\pi, \quad \|f\|_{L_a^2}^2 = \int_{\mathbf{D}} |f(z)|^2 dA(z) = \int_{\mathbf{D}} |f(re^{i\theta})|^2 rdrd\theta/\pi$$

と定める. ただし $\mathbf{D} = \{|z| < 1\}$, $z = x+iy = re^{i\theta}$, $dA(z) = dxdy/\pi = rdrd\theta/\pi$ である. Dirichlet 空間 $\mathcal{D}^2$ の norm を

$$\|f\|_{\mathcal{D}^2}^2 = \|f\|_{H^2}^2 + \int_{\mathbf{D}} |f'(re^{i\theta})|^2 rdrd\theta/\pi$$

と定める. このとき, $\|f\|_{\mathcal{D}^2} = (\int_{|z|\leq 1} |f|^2 d\mu)^{1/2}$ となる確率測度 μ は存在しないから, $\mathcal{D}^2$ は抽象的 Hardy 空間 $H^2(\mu)$ にならない.

$A(\overline{\mathbf{D}})$ は円板環とし, $I = z^3 A(\overline{\mathbf{D}})$ とする. H^2 は測度 $\mu_1 = d\delta_{r=1} d\theta/2\pi$ について $H^2 = H^2(\mu_1)$ を満たし, L_a^2 は測度 $\mu_2 = rdrd\theta/\pi$ について $L_a^2 = H^2(\mu_2)$ を満たすから, 定理 2 を $X = \overline{\mathbf{D}}$ に関して適用できる. このとき, 3 次元 Hilbert 空間をそれぞれ,

$$\mathcal{H}_1 = H^2 \cap I^\perp, \ \mathcal{H}_2 = L_a^2 \cap I^\perp, \ \mathcal{H}_3 = \mathcal{D}^2 \cap I^\perp$$

と定めると, shift 作用素 S_z の $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ への制限作用素 $S_z^{\mathcal{H}_1}, S_z^{\mathcal{H}_2}, S_z^{\mathcal{H}_3}$ は, それぞれ正規直交基底 $\{1, z, z^2\}$, $\{1, \sqrt{2}z, \sqrt{3}z^2\}$, $\{1, z/\sqrt{2}, z^2/\sqrt{3}\}$ に関して,

$$S_z^{\mu_1} = S_z^{\mathcal{H}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_z^{\mu_2} = S_z^{\mathcal{H}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2}/\sqrt{3} & 0 \end{pmatrix}, \quad S_z^{\mathcal{H}_3} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \end{pmatrix}$$

と書ける. $S_z^{\mathcal{H}_1}$ と $S_z^{\mathcal{H}_2}$ の行列は, 例 2 の特殊な場合である. 一方, $S_z^{\mathcal{H}_3}$ の行列を求めるとき, $\mathcal{D}^2$ は抽象的 Hardy 空間 $H^2(\mu)$ にならないから, 定理 2 は使えないが, $\mathcal{D}^2$ の内積 :

$$\langle f, g\rangle_{\mathcal{D}^2} = \langle f, g\rangle_{H^2} + \int_{\mathbf{D}} f'(re^{i\theta})\overline{g'(re^{i\theta})} rdrd\theta/\pi$$

により, (i, j) 成分 $\langle S_z^{\mathcal{H}_3}\psi_j, \psi_i\rangle_{\mathcal{D}^2} = \langle z\psi_j, \psi_i\rangle_{\mathcal{D}^2}$ を計算すると上のようになる. $S_z^{\mathcal{H}_3}$ の成分比は $S_z^{\mathcal{H}_2}$ とは逆になっている. $S_z^{\mathcal{H}_1}$ と $S_z^{\mathcal{H}_2}$ は Q - 環 $A(\overline{\mathbf{D}})/I$ の元 $z+I$ の自然な contractive 表現である. 一方, $S_z^{\mathcal{H}_3}$ は Q - 環 $A(\overline{\mathbf{D}})/I$ の元 $z+I$ の自然な expansive 表現である.

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Simple proofs of some L^∞ inequalities for Sobolev functions

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Abstract

L^∞ estimates of for Sobolev functions u in terms of $\Delta^m u$ and $\nabla(\Delta^m u)$ have been recently obtained by N. Okazawa. The purpose of this note is to show that his results can be, *qualitatively speaking*, readily obtained from textbook-level fundamental Sobolev embedding theorem. Some generalizations and estimations of constants are considered.

1 Introduction

As a part of so-called Sobolev embedding theorems, the following result concerning the embedding into the L^∞ space is well known as Morrey's theorem:

- Let m, $N \in \mathbb{N}$ and $1 \leq p \leq \infty$ satisfy $m > N/p$. Then $W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$. In other words, there exists a constant $C > 0$ such that

$$\forall u \in W^{m,p}(\mathbb{R}^N) \quad \|u\|_{L^\infty(\mathbb{R}^N)} \leq C\,\|u\|_{W^{m,p}(\mathbb{R}^N)}. \tag{1}$$

As for the inequality (1), there remains a possibility of improving the estimate of $\|u\|_{L^\infty}$ by quantities other than $\|u\|_{W^{m,p}}$. Recently, N. Okazawa [4, Theorem 1] obtained the following results on L^∞ estimates in terms of $\|\Delta^k u\|_{L^p}$ or $\|\nabla\Delta^k u\|_{L^p}$ for some k.

Theorem 1 (N. Okazawa)**. Part I.** *Let $1 < p < \infty$ and $N \in \mathbb{N}$, with $N/p \notin \mathbb{N}$. Moreover suppose that Let $2m-2 < N/p < 2m-1$ hold for $m \in \mathbb{N}$ (i.e., $\lfloor N/p \rfloor = 2m-2$). Then there exists a constant C_{2m-1} for which every $u \in C_c^{2m-1}(\mathbb{R}^N)$ satisfies the inequality*

$$\|u\|_{L^\infty}^p \leq C_{2m-1}\|\Delta^{m-1}u\|_{L^p}^{(2m-1)p-N}\|\nabla(\Delta^{m-1}u)\|_{L^p}^{N-2(m-1)p}. \tag{2}$$

Part II. *The case where $\lfloor N/p \rfloor$ is odd: Let $2m-1 < N/p < 2m$ hold for $m \in \mathbb{N}$ (i.e., $\lfloor N/p \rfloor = 2m-1$). Then there exists a constant C_{2m} for which every $u \in C_c^{2m-1}(\mathbb{R}^N)$ satisfies the inequality*

$$\|u\|_{L^\infty}^p \leq C_{2m}\|\nabla(\Delta^{m-1}u)\|_{L^p}^{2mp-N}\|\Delta^m u\|_{L^p}^{N-(2m-1)p}. \tag{3}$$

For the proof of this theorem, Okazawa used integral representations of $u(x)$ via a series of kernel functions, one of which being the fundamental solution for the Laplace operator (*cf.* [4, Lemma 3]). In addition, an elaborate use of cut-off functions with the help of some inequalities including Hardy's inequality and Rellich's inequality ([4, Lemma 6]) yields the proof of Theorem 1. His proof is very lengthy and his estimate of constants for some cases of N or p is very complicated. For example, In the case of odd $N \in \mathbb{N}$ and $p = 2$, the constant C_{2m-1} in (2) is given by $C_{2m-1} = 2c_m K_{43}/\omega_N$, where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$ and

$$c_k = c_k(N) := \prod_{j=1}^{k-1}\frac{1}{2j(N-2j)} \quad (2 \leq k \leq m+1), \quad c_1 := 1,$$

$$K(k,2,N) := 2^{2k}\prod_{j=1}^{k}\frac{1}{(N-4j)(N+4j-4)},$$

$$K_{43} := c_m + \sum_{k=1}^{m-1} c_{m-k}\left[K(k,2,N) + \frac{2^{2k-1}}{2(m+k)-3}\prod_{j=1}^{2k-1}\frac{1}{N-2j}\right].$$

Therefore, the following questions naturally arise:

- Does there exist a shorter proof if qualitative?
- Are the explicit expression for constants in (2) or (3) best possible?

In the present note, we show that, qualitatively speaking, Theorem 1 is derived by an exercise-level simple argument from standard Sobolev embedding theorems and the L^p-boundedness of Riesz transforms. In addition, our argument readily yields a concrete estimates of the constants in the inequalities. Some generalizations of Theorem 1 are also obtained.

2 Results

2.1 Main idea and the starting point

The idea of our proof is very simple. It can be summarized as follows.

1. Instead of sticking to the form $\Delta^m u$ or $\nabla\Delta^m u$, we pay attention to the order of differentiation and use the following quantity: For every $k \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$, we set
$$|u|_{k,p} := \max_{\alpha\colon |\alpha|=k} \|\partial^\alpha u\|_{L^p(\mathbb{R}^N)}.$$
2. To recover $\|\Delta^m u\|_{L^p}$ or $\|\nabla(\Delta^m u)\|_{L^p}$ from $|u|_{k,p}$, we make use of the boundedness of Riesz Transform in L^p norm for $1 < p < \infty$.

Although we lose much in the second step as for the estimation of C_{2m-1} or C_{2m} in (2) or (3), respectively, this method has simplicity and uniformity as advantages.

The starting point of our proof for Theorem 1 is the following two basic inequalities that are explained in any textbook on Sobolev spaces (e.g., [2]):

- (Gagliardo–Nirenberg) Let $N \in \mathbb{N}$ and $1 \le p < N$. Then there exists a constant C_1 that satisfies
$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \le C_1\|\nabla u\|_{L^p(\mathbb{R}^N)} \quad (\forall u \in W^{1,p}(\mathbb{R}^N)), \tag{4}$$
where $p^* := p/(p-1)$.
- (Morrey) For every $p > N \in \mathbb{N}$, there exists a constant C_2 that satisfies
$$\|u\|_{L^\infty} \le C_2\, \|u\|_{L^p}^{1-N/p}\|\nabla u\|_{L^p}^{N/p} \quad (\forall u \in W^{1,p}(\mathbb{R}^N)). \tag{5}$$

2.2 Unified theorem

Although the statement of Theorem 1 is divided in two parts according as $\lfloor N/p \rfloor$ is even or odd, we can obtain a unified form of result by using $|u|_{k,p}$.

Theorem 2. *For every $m \in \mathbb{N}$, the following assertion* $(\mathrm{INEQ})_m$ *holds:*

$(\mathrm{INEQ})_m$ *For every $N \in \mathbb{N}$ and p with $1 < p < \infty$ and $m - 1 < N/p < m$, there exists a constant $C > 0$ such that*
$$\|u\|_{L^\infty} \le C\, |u|_{m-1,p}^{m-N/p} |u|_{m,p}^{N/p-m+1} \quad (\forall u \in W^{m,p}(\mathbb{R}^N)) \tag{6}$$

Proof. The proof is done by an induction on m.

1. When $m = 1$, (6) follows immediately from (5). (Note that $|u|_{1,p} \sim \|\nabla u\|_{L^p}$.)
2. Suppose that $(\mathrm{INEQ})_m$ holds for $m = k$. Moreover, let $1 < p < \infty$ satisfy $k < N/p < k+1$ and $u \in W^{k+1,p}(\mathbb{R}^N)$. Then, for every multi-index α with $|\alpha| \le k$, $\partial^\alpha u \in W^{1,p}(\mathbb{R}^N)$. Hence, by setting $p^* := Np/(N-p) < \infty$, we obtain from the inequality (4) $\|\partial^\alpha u\|_{L^{p^*}} \le C\,\|\nabla\partial^\alpha u\|_{L^p}$ for some constant C independent of u. Therefore, we obtain
$$0 \le \ell \le k \Longrightarrow |u|_{\ell,p^*} \le C\,|u|_{\ell+1,p}. \tag{7}$$
for some another constant C independent of u.

3. Now we know that $u \in W^{k,p^*}(\mathbb{R}^N)$, hence by the assumption in Step 2 $\|u\|_{L^\infty}$ is estimated as
$$\|u\|_{L^\infty} \leq C\,|u|_{k-1,p^*}^{k-N/p^*}\,|u|_{k,p^*}^{N/p-k+1},$$
where C denotes another constant independent of u. Rewriting the right-hand side of the inequality above by using (7), we can see that the assertion of the theorem holds for $m = k+1$.

$\square$

2.3 Riesz transform

To connect $|u|_{k,p}$ with $\|\Delta^m u\|_{L^p}$ or $\|\nabla\Delta^m u\|_{L^p}$, we should recall the Riesz transform. Riesz transform R_j $(1 \leq j \leq N)$ in $\mathbb{R}^N$ is the pseudo-differential operator with symbol $\sqrt{-1}\xi_i/|\xi|$:

$$R_j u(x) := \frac{1}{\sqrt{(2\pi)^{N/2}}}\int_{\mathbb{R}^N} \sqrt{-1}\frac{\xi_j}{|\xi|}\widehat{u}(\xi)\,e^{ix\xi}\,d\xi.$$

The following theorem is a standard fact in real analysis (see, e.g., Stein [6]).

Theorem 3. *R_j has a unique bounded extension onto $L^p(\mathbb{R}^N)$ for every $1 < p < \infty$.*

As an immediate consequence of this theorem, we obtain the following

Lemma 4. *Let $N \in \mathbb{N}$. Then for every $p \in (1,\infty)$ and $m \in \mathbb{N}$, there exists a positive constant C such that*

$$C^{-1}\|\Delta^m u\|_{L^p} \leq |u|_{2m,p} \leq C\|\Delta^m u\|_{L^p} \quad (\forall u \in W^{2m,p}(\mathbb{R}^N)), \tag{8}$$
$$C^{-1}\|\nabla(\Delta^{m-1}u)\|_{L^p} \leq |u|_{2m-1,p} \leq C\|\nabla(\Delta^{m-1}u)\|_{L^p} \ (\forall u \in W^{2m-1,p}(\mathbb{R}^N)). \tag{9}$$

Proof. Note that for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ with $|\alpha| = 2m$ $(m \in \mathbb{N})$,

$$\|D^\alpha u\|_{L^p} = \|R_1^{\alpha_1}R_2^{\alpha_2}\ldots R_N^{\alpha_N}\Delta^m u\|_{L^p} \leq \Big(\prod_{j=1}^{N}\|R_j\|_p^{\alpha_j}\Big)\,\|\Delta^m u\|_{L^p},$$

which proves the second inequality in (8). The second inequality in (9) is similarly verified. Other inequalities in the lemma are trivial. $\square$

2.4 Proof of Theorems 1

Now, Okazawa's results are immediately obtained since Lemma 4 allows us to replace $|u|_{k,p}$'s in the inequality (INEQ)$_m$ (Theorem 2) with $\|\Delta^\ell u\|_{L^p}$ or $\|\nabla\Delta^\ell u\|_{L^p}$.

2.5 Case of integral N/p

The following result treats the case of integral N/p, which is excluded from Theorem 1.

Theorem 5. *For every $m \in \mathbb{N}$, the following assertion* (INEQ2)$_m$ *holds:*
(INEQ2)$_m$ *If $N \in \mathbb{N}$ and p satisfy $1 \leq p \leq N$ and $N/p = m$, then there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty} \leq C\,|u|_{m-1,p}^{1/2}|u|_{m+1,p}^{1/2} \quad (\forall u \in W^{m+1,p}(\mathbb{R}^N)). \tag{10}$$

We can prove this result by an induction on m as in the argument of the proof of Theorem 2. However, the proof of the case $m = 1$ requires some argument other than the fundamental inequality (4) or (5). See [5] for details.

Theorem 5 immediately yields the following estimate in terms of $\|\Delta^m u\|_{L^p}$ or $\|\nabla\Delta^m u\|_{L^p}$ through Lemma 4.

Theorem 6. *Let $1 < p < \infty$ and $N \in \mathbb{N}$, with $N/p \in \mathbb{N}$. Then the following assertions hold.*

Part III. *The case where N/p is odd: In this case, there exists an $m \in \mathbb{N}$ such that $N/p = 2m - 1$, and there exists a constant C_{2m} for which the following inequality holds for every $u \in C_c^{2m}(\mathbb{R}^N)$:*

$$\|u\|_{L^\infty} \leq C_{2m} \|\Delta^{m-1} u\|_{L^p}^{1/2} \|\Delta^m u\|_{L^p}^{1/2}.$$

Part IV. *The case where N/p is even: In this case, there exists an $m \in \mathbb{N}$ such that $N/p = 2m$, and there exists a constant C_{2m-1} for which the following inequality holds for every $u \in C_c^{2m+1}(\mathbb{R}^N)$:*

$$\|u\|_{L^\infty} \leq C_{2m+1} \|\nabla(\Delta^{m-1} u)\|_{L^p}^{1/2} \|\nabla(\Delta^m u)\|_{L^p}^{1/2}.$$

2.6 Generalization

It is easy to generalize the powers of Laplacian in Theorem 1 by 2nd order elliptic differential operators of constant coefficients.

Proposition 7. *Let $L = -\sum_{j,k=1}^N a_{jk} \partial_j \partial_k$ be a uniformly elliptic second order differential operator with constant coefficients (i.e., the matrix $(a_{jk})_{j,k}$ is positive definite). Then the following assertions hold.*

(1) *Let $1 < p < \infty$, $N \in \mathbb{N}$ satisfy $2m - 2 < N/p < 2m - 1$ for some $m \in \mathbb{N}$. Then there exists a constant C_{2m-1} such that*

$$\|u\|_{L^\infty} \leq C_{2m-1} \|L^{m-1} u\|_{L^p}^{(2m-1)-N/p} \|\nabla(L^{m-1} u)\|_{L^p}^{N/p-2(m-1)}$$

holds for every $u \in C_c^{2m-1}(\mathbb{R}^N)$.

(2) *Let $1 < p < \infty$, $N \in \mathbb{N}$ satisfy $2m - 1 < N/p < 2m$ for some $m \in \mathbb{N}$. Then there exists a constant C_{2m} such that*

$$\|u\|_{L^\infty} \leq C_{2m} \|\nabla(L^{m-1} u)\|_{L^p}^{2m-N/p} \|L^m u\|_{L^p}^{N/p-2m+1}$$

holds for every $u \in C_c^{2m}(\mathbb{R}^N)$.

As for the 2nd order elliptic differential operator with variable coefficients, only a very limited result is obtained.

Proposition 8. *Suppose that the coefficients $a_{ij}(x)$ of a uniformly elliptic differential operator $L = -\sum_{j,k=1}^N a_{jk}(x) \partial_j \partial_k$ or $L = -\sum_{j,k=1}^N \partial_j (a_{jk}(x) \partial_k)$ are bounded up to second derivatives. Then,*

$$\|u\|_{L^\infty} \leq C \, \|\nabla u\|_{L^2}^{1/2} \|(L + \lambda) u\|_{L^2}^{1/2} \quad (\forall u \in W^{2,2}(\mathbb{R}^3))$$

holds for some $\lambda > 0$ and $C > 0$.

2.7 Application

As shown in Okazawa [4], estimates like Proposition 7 or Proposition 8 leads to an L^∞-estimate of eigenfunctions for L. Since Proposition 8 is a weak assertion, we can obtain only the following result for the case of variable coefficients.

Corollary 9. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary. Moreover, suppose that the coefficients $a_{ij}(x)$ of a uniformly elliptic self-adjoint differential operator $L = -\sum_{j,k=1}^3 \partial_j (a_{jk}(x) \partial_k)$ in $L^2(\Omega)$ are bounded up to second derivatives. Then, normalized eigenfunction $u_\mu \in H^2(\Omega) \cap H_0^1(\Omega)$ with eigenvalue μ (i.e., $\|u_\mu\|_{L^2} = 1$ and $L u_\mu = \mu u_\mu$) satisfies $\|u_\mu\|_{L^\infty} = O(\mu^{3/4})$ as $\mu \to \infty$.*

3 Estimates of constants

In this section, we discuss on an estimate of the constants appearing in the inequalities treated in this note.

3.1 Basic constants

In the first place, let us consider some basic constants appearing in foundational inequalities for the proofs of the results in this note.

- C_1 in (4): The following best constant has been found by Aubin [1] and Talenti [7]:

$$\frac{p-1}{N-p}\left[\frac{N-p}{N(p-1)}\right]^{1/p}\left[\frac{p\Gamma(N+1)}{(N-p)\Gamma(N/p-1)\Gamma(N+1-N/p)\omega_N}\right]^{1/N} \tag{11}$$

- As C_2 in (5), one may take

$$\left(\frac{p}{N}\right)^{N/p}\left(\frac{N}{\omega_N}\right)^{1/p}\cdot\frac{1}{1-N/p}\cdot\left(\frac{N}{p'+N}\right)^{N/pp'}, \tag{12}$$

  where p' is the conjugate exponent for p. This result could be proved by inspecting the proof of Théorème IX.12 in [2]. See [5] for details. As a by-product, this estimate shows that the explicit estimate given in [4] for constants in Theorem 1 is *not best possible.*

- $\|R_j\|$ in $L^p(\mathbb{R}^N)$:(Iwaniek and Martin [3])

$$\|R_j\| = H_p := \begin{cases} \tan\left(\frac{\pi}{2p}\right) & (1 < p \le 2), \\ \cot\left(\frac{\pi}{2p}\right) & (2 \le p < \infty). \end{cases} \tag{13}$$

3.2 Estimates of constants in theorems

We need another constant linking $|u|_{1,p}$ and $\|\nabla u\|$

Lemma 10. *For $N \in \mathbb{N}$ and $p \in [1,\infty)$, set*

$$E(N,p) := \begin{cases} N^{1/2} & (2 \le p), \\ N^{1/p} & (1 \le p < 2). \end{cases}$$

Then, the following inequalities hold for every $u \in W^{1,p}(\mathbb{R}^N)$.

$$|u|_{1,p} \le \|\nabla u\|_{L^p} \le E(N,p)\,|u|_{1,p} \tag{14}$$

It is easy to see that our method of proof gives the following estimate of the constant in (6).

Proposition 11. *Suppose that for every $N \in \mathbb{N}$ and $p \in [1, N)$, a constant $C_1(N,p)$ is given that satisfies inequality (4). Suppose also that a constant $C_2(N,p)$ is given for every $N \in \mathbb{N}$ and $p \in (N,\infty)$ that satisfies inequality (5). Then, if N, $m \in \mathbb{N}$ and $p \in (1,\infty)$ satisfy $m-1 < N/p < m$, one can take the following value as the constant C in (6) in Theorem 2:*

$$E(N,p)^{N/p+m-1}C_2(N,p^{(*(m-1))})\prod_{j=0}^{m-2}C_1(N,p^{(*j)}). \tag{15}$$

Here, $p^{(*j)}$ is defined by $1/p^{(*j)} = 1/p - j/N$ for every $j = 0,1,\dots,m-1$, and $\prod_{j=0}^{m-2}C_1(N,p^{(*j)})$ is construed as 1 when $m = 1$.

Combining the result of Proposition 11, the proof of Lemma 4 and the precise value for $\|R_j\|$, we can obtain estimates of the constants appearing in Theorem 1, Theorem 2, Theorem 5 and Theorem 6. For example, the following result holds.

Proposition 12. *Suppose $1 < p < \infty$ and $N \in \mathbb{N}$, with $N/p \notin \mathbb{N}$. Let $C_m(N,p)$ be a constant for which inequality* (6) *in our Theorem is satisfied when $C = C_m(N,p)$ and $m = \lfloor N/p \rfloor + 1$. Moreover, let H_p be as defined in* (13). *Then the following assertions hold.*

(Part I): *Suppose $\lfloor N/p \rfloor$ is even and $2m-2 < N/p < 2m-1$ for $m \in \mathbb{N}$. Then we obtain*

$$\|u\|_{L^\infty} \leq C_{2m-1}(N,p) H_p^{2m-2} \|\Delta^{m-1} u\|_{L^p}^{2m-1-N/p} \|\nabla(\Delta^{m-1} u)\|_{L^p}^{N/p-2m+2}$$

for every $u \in W^{2m-1,p}(\mathbb{R}^N)$.

As a consequences of Proposition 11 and Proposition 12, the following inequalities with explicit constants are obtained if we adopt (11) for $C_1(N,p)$ and (12) for $C_2(N,p)$, respectively.

$$\|u\|_{L^\infty} \leq 3.27 \, \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2} \quad (\forall u \in W^{2,2}(\mathbb{R}^3)),$$
$$\|u\|_{L^\infty} \leq 14.11 \, \|\Delta u\|_{L^2}^{1/2} \|\nabla \Delta u\|_{L^2}^{1/2} \quad (\forall u \in W^{3,2}(\mathbb{R}^5)),$$
$$\|u\|_{L^\infty} \leq 18.92 \, \|\nabla u\|_{L^4}^{3/4} \|\Delta u\|_{L^4}^{1/4} \quad (\forall u \in W^{2,4}(\mathbb{R}^5)).$$

Here, we should note that in the case of inequalities above, [4] gives much better constants.

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Further extension of an order preserving operator inequality

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An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

The celebrated Löwner-Heinz inequality asserts that *if $A \geq B \geq 0$, then $A^{\alpha} \geq B^{\alpha}$ holds for any $\alpha \in [0,1]$, but $A^p \geq B^p$ does not always hold for $p > 1$.* From this point of view, we shall show *an order preserving operator inequality* in Theorem 2.3 and we shall state an implication among Theorem 2.3 and the previous known results.

Theorem 2.3. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geq 1$ for natural number n. Then the following inequality holds for $r \geq t$:*

$$A^{1-t+r} \geq$$

$$\left\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \ldots\ldots\ldots [A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} \; B^{p_1} \; \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{t}{2}} \ldots\ldots\ldots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} \Big]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{q[2n]+r-t}}$$

where $q[2n] = \underbrace{\Big\{ \ldots\ldots [\{[(p_1 - t)p_2 + t]p_3 - t\}p_4 + t]p_5 - \ldots\ldots - t \Big\} p_{2n} + t}_{\text{-t and t alternately } n \text{ times appear}}.$

Remark 1. Consider the following operator function $f(\square)$ defined by

$$f(\square) = \left\{ A^{\frac{r}{2}} \Big[\underbrace{A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \ldots\ldots\ldots [A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} (A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} \; \square \; \underbrace{A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{t}{2}} \ldots\ldots\ldots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} \Big]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{q[2n]+r-t}}.$$

Theorem 2.3 asserts that $f(A^{p_1}) \geq f(B^{p_1})$ holds for any $p_1 \geq 1$ whenever $A \geq B \geq 0$ with $A > 0$ under some suitable conditions, that is, $f(\square)$ can be considered as order preserving operator function. Therorem 2.3 is further extension of the following previous one: *if $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$, $A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ holds for $r \geq t$ and $s \geq 1$, in particular, $A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$.* Finally an application of Theorem 2.3 is shown.

§1 Introduction

An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

Theorem LH (Löwner-Heinz inequality, denoted by (LH) briefly).

$$\text{If } A \geq B \geq 0 \text{ holds, then } A^{\alpha} \geq B^{\alpha} \text{ for any } \alpha \in [0,1]. \tag{LH}$$

Although (LH) asserts that $A \geq B \geq 0$ *ensures* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0,1]$, unfortunately $A^p \geq B^p$ *does not always hold for* $p > 1$. The following result has been obtained from this point of view.

Theorem F.
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

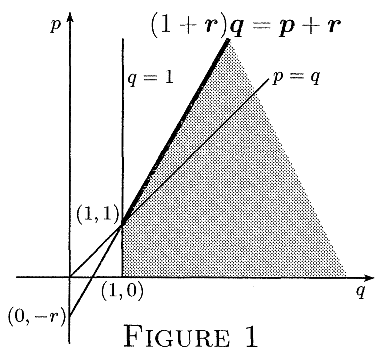

FIGURE 1

The original proof of Theorem F is shown in [F1], an elementary one-page proof is in [F2] and alternative ones are in [MF],[K1]. It is shown in [T1] that the conditions p, q and r in FIGURE 1 are best possible.

Theorem G. *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for $r \geq t$ and $s \geq 1$.

The original proof of Theorem G is in [F3], and an alternative one is in [MF-K]. An elementary one-page proof of (1.1) is in [F4]. We mention that further extensions of Theorem G and related results to Theorem F are in [MF-K-N],[F5],[F-H-I],[F-Y-Y], [K2] and etc. It is shown in [T2] that the exponent value $\dfrac{1-t+r}{(p-t)s+r}$ of the right hand of (1.1) is best possible and alternative ones are in [MF-M-N],[Y]. It is known that the operator inequality (1.1) interpolates Theorem F and an inequality (see Theorem AH under below at

4 page) equivalent to the main result of Ando-Hiai log majorization [A-H] by the parameter $t \in [0,1]$.

§2 Further extension of Theorem G

Theorem 2.1 [F7]. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geq 1$. Then the following inequality holds,*

$$A \geq \left[\underbrace{A^{\frac{t}{2}}\{A^{\frac{-t}{2}}[A^{\frac{t}{2}}\ldots\ldots\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}\ldots A^{\frac{t}{2}}]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} \right]^{\frac{1}{q[2n]}} \tag{2.1}$$

where $q[2n] = q[2n; p_1, p_2, .., p_n, .., p_{2(n-1)}, p_{2n-1}, p_{2n}]$

$$= \underbrace{\Big\{\ldots..[\{[(p_1 - t)p_2 + t]p_3 - t\}p_4 + t]p_5 - \ldots\ldots - t\Big\}}_{\text{-t and t alternately } n \text{ times appear}} p_{2n} + t. \tag{2.2}$$

Corollary 2.2 [F6]. *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$, then the following inequality holds,*

$$A \geq \left\{A^{\frac{t}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{[\{(p_1-t)p_2+t\}p_3-t]p_4+t}}.$$

Theorem 2.3 [F7].*Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, ..., p_{2n} \geq 1$ for natural number n. Then the following inequality holds for $r \geq t$,*

$$A^{1-t+r} \geq$$

$$\left\{A^{\frac{r}{2}}\Big[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots\ldots\ldots\ldots\ldots\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\ldots\ldots\ldots A^{\frac{-t}{2}}}_{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and } A^{\frac{t}{2}}\ n-1 \text{ times by turns}}\Big]^{p_{2n}} A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{q[2n]+r-t}} \tag{2.3}$$

where $q[2n]$ is defined in (2.2).

Corollary 2.4 [F6]. *If $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geq 1$,*

$$A^{1-t+r} \geq \left\{A^{\frac{r}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $r \geq t$.

Corollary 2.4 obtained by Theorem 2.3 and previous known results

Corollary 2.4. *If* $A \geq B \geq 0$ *with* $A > 0$, $t \in [0,1]$ *and* $p_1, p_2, p_3, p_4 \geq 1$,

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $r \geq t$.

$p_2 = p_3 = 1$ $\downarrow$

Theorem G. *If* $A \geq B \geq 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geq 1$,

$$A^{1-t+r} \geq \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ *and* $s \geq 1$.

$t = 0$ and $s = 1$ $\swarrow$ $\qquad$ $t = 1$ and $r = s$ $\searrow$

Theorem F. $A \geq B \geq 0 \Longrightarrow$

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

for $p \geq 1$ *and* $r \geq 0$.

Theorem AH. *If* $A \geq B \geq 0$ *with* $A > 0 \Longrightarrow$

$$A^r \geq \{ A^{\frac{r}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}} \}^{\frac{1}{p}}$$

for $r, p \geq 1$.

$\Updownarrow$

Theorem AH-L. *For every* $A, B \geq 0$ *and* $0 \leq \alpha \leq 1$,

$$(A \#_\alpha B)^r \underset{(\log)}{\succ} A^r \#_\alpha B^r \qquad \text{for } r \geq 1.$$

$\Updownarrow$

Theorem F.
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $\qquad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\qquad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p + r$.

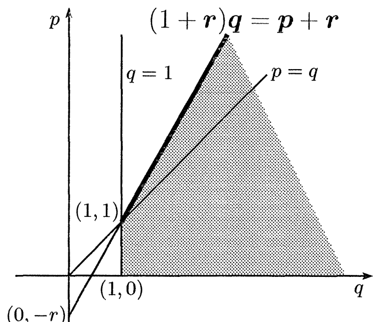

FIGURE 1

@

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Majorization and some operator monotone functions

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1 Introduction

$\mathbf{P}(I)$: the set of all operator monotone functions on I
$\mathbf{P}_+(I) := \{f|f(t) \geqq 0, f \in \mathbf{P}(I)\}$.
$\mathbf{LP}_+(I) := \{f|f(t) > 0, \log f \in \mathbf{P}(I^\circ)\}$.
$\mathbf{P}_+^{-1}(I) := \{h|$ h is increasing on (I) and its range is $(0,\infty)$, $h^{-1} \in \mathbf{P}(0,\infty)\}$.

Example: $t^a \in \mathbf{LP}_+(0,\infty) \cap \mathbf{P}_+^{-1}(0,\infty)$ $(a \geq 1)$.
$e^t \in \mathbf{LP}_+(I) \cap \mathbf{P}_+^{-1}(I)$ for $I = (-\infty,\infty)$.

Definition h: non-decreasing function on I
k: increasing function on J.
h is **majorized** by k, in symbols
$h \preceq k \quad (J)$,
if $J \subset I$ and $h \circ k^{-1} \in \mathbf{P}(k(J))$.

This definition is equivalent with

$$\sigma(A), \sigma(B) \subset J,\ k(A) \leqq k(B) \Longrightarrow h(A) \leqq h(B).$$

Simple properties:

(i)(**Löwner - Heinz**) $k^\alpha \preceq k^\beta$ for any increasing function $k(t) \geqq 0$ and $0 < \alpha \leqq \beta$;

(ii) $g \preceq h, \quad h \preceq k \Longrightarrow g \preceq k$;

(iii) if τ is an increasing function whose range is contained in the domain of k, then

$h \preceq k \Longrightarrow h \circ \tau \preceq k \circ \tau$;
moreover, if the range of τ coincides with the domain of k, then
$h \preceq k \Longleftrightarrow h \circ \tau \preceq k \circ \tau$;

(iv) $h_1 \preceq k,\ h_2 \preceq k \quad \Longrightarrow h_1 + h_2 \preceq k$;

(v) if the range of k is $[0, \infty)$ and $h \geqq 0$, then
$h \preceq k \Longrightarrow h^r \preceq k^r$ for $r \geqq 1$;
indeed, $\phi(t) := h \circ k^{-1}(t) \in \mathbf{P}_+[0, \infty)$ implies $\phi(t^{1/r})^r \in \mathbf{P}_+[0, \infty)$.

(vi) if h and k are increasing and unbounded to the above, then
$h \preceq k, \quad k \preceq h \Longleftrightarrow h = ck + d$ for real numbers $c > 0,\ d$.

Product Lemma $h, g \geqq 0$ on $[a, b)$. Suppose $hg \nearrow$, $(hg)(a) = 0$, $(hg)(b - 0) = \infty$.
Then
$g \preceq hg \Longrightarrow h \preceq hg$.
Moreover for every $\psi_1,\ \psi_2$ in $\mathbf{P}_+[0, \infty)$
$g \preceq hg \Longrightarrow \psi_1(h)\psi_2(g) \preceq hg$.

Product theorem For I:right open interval,

$$\mathbf{P}_+^{-1}(I) \cdot \mathbf{P}_+^{-1}(I) \subset \mathbf{P}_+^{-1}(I), \quad \mathbf{LP}_+(I) \cdot \mathbf{P}_+^{-1}(I) \subset \mathbf{P}_+^{-1}(I).$$

Further, let $g_i(t) \in \mathbf{LP}_+(I)$ for $1 \leqq i \leqq m$ and $h_j(t) \in \mathbf{P}_+^{-1}(I)$ for $1 \leqq j \leqq n$. Then for $\psi_i, \phi_j \in \mathbf{P}_+[0, \infty)$

$$\prod_{i=1}^m \psi_i(g_i) \prod_{j=1}^n \phi_j(h_j) \preceq \prod_{i=1}^m g_i \prod_{j=1}^n h_j.$$

2 A function and its derivative

Polynomial For
$\{a_i\}_{i=1}^n$ $(a_i \geqq a_{i+1})$ and for $\{b_i\}_{i=1}^m$ $(b_i \geqq b_{i+1})$,

$$u(t) := \prod_{i=1}^n (t - a_i)\ (t \geqq a_1), \quad v(t) := \prod_{i=1}^m (t - b_i)\ (t \geqq b_1).$$

Then
$u(t) \in \mathbf{P}_+^{-1}[a_1, \infty)$, and

$$m \leqq n, \quad \sum_{i=1}^k b_i \leqq \sum_{i=1}^k a_i\ (1 \leqq k \leqq m)$$
$$\Longrightarrow v \preceq u \quad ([a_1, \infty)).$$

This implies

$$u(t) \in \mathbf{P}_+^{-1}[a_1, \infty), \quad u' \preceq u. \tag{1}$$

Recall a simple fact: $-\frac{1}{t} \in \mathbf{P}(0,\infty)$, $\int \frac{1}{t}dt = \log t \in \mathbf{P}(0,\infty)$.
We have generalized this as follows:
Proposition Let $f(t)$ be an increasing and differentiable function defined on $I = (a, \infty)$.

$$-f'(t) \in \mathbf{P}(I) \Longrightarrow f(t) \in \mathbf{P}(I). \tag{2}$$

The converse assertion of (2) does not hold; indeed,
$f(t) := \dfrac{t}{t+1} \in \mathbf{P}(0,\infty)$, but $-f'(t) = \dfrac{-1}{(1+t)^2} \notin \mathbf{P}(0,\infty)$.

About (1) and the converse of (2) we have

Theorem 2.1

(i) $f(t) \in \mathbf{P}(0,\infty) \Rightarrow -f'(\sqrt{t}) \in \mathbf{P}(0,\infty)$.

(ii) $h(t) \in \mathbf{P}_+^{-1}[a,b) \Rightarrow h'(t) \preceq h(t)^2 \quad (a < t < b)$.

(iii) Suppose $h \in \mathbf{P}_+^{-1}[a,b)$ and $f := h^{-1} \in \mathbf{P}(0,\infty)$. Then $-f' \in \mathbf{P}(0,\infty) \Longleftrightarrow h' \preceq h$.

3 Operator monotone function

Petz-Hasegawa[1] showed that

$$\frac{(t-1)^2}{(t^r - 1)(t^{1-r} - 1)} \in \mathbf{P}_+[0,\infty) \tag{3}$$

for $0 \leqq r \leqq 1$.

In this section we will make use of the majorization to extend (3). It is well-known that
$\dfrac{t}{\log(1+t)} \in \mathbf{P}_+[0,\infty)$, $\dfrac{t-1}{\log t} \in \mathbf{P}_+(0,\infty)$.

We first generalize this as:

Theorem 3.1 *Suppose $f(t) \in \mathbf{P}(I)$, where $I = (0, \infty)$, or $I = [0, \infty)$. Then*

$$\frac{t-a}{f(t)-f(a)} \in \mathbf{P}_+[0, \infty) \quad (a \in I).$$

In particular, if $f(t) > 0$, then

$$\frac{t-a}{\log f(t) - \log f(a)} \in \mathbf{P}_+[0, \infty).$$

Example 3.1 *Suppose $a_i > 0$. Then*

$$\frac{t-1}{\sum_{i=1}^{n}\{\log(t+a_i) - \log(1+a_i)\}} \in \mathbf{P}_+[0, \infty),$$

$$\frac{t}{t-1}\sum_{i=1}^{n}\{\log(t+a_i) - \log(1+a_i)\} \in \mathbf{P}_+[0, \infty).$$

Corollary 3.2 *Suppose $f(t) \in \mathbf{P}(I)$, where $I = (0, \infty)$ or $[0, \infty)$, and non-constant $h(t) \in \mathbf{P}_+[0, \infty)$. Then for $a \in I$*

$$\frac{h(t)-h(a)}{f \circ h(t) - f \circ h(a)} \in \mathbf{P}_+[0, \infty).$$

Proof. Put $f_a(t) = \dfrac{t-a}{f(t)-f(a)}$. Then by Theorem3.1

$f_{h(a)}(t) \preceq t \quad ([0, \infty))$ and $f_{h(a)}(t) \geqq 0$.

The property (iii) of the first section yields

$f_{h(a)} \circ h(t) \preceq h(t) \preceq t.$

Hence $f_{h(a)} \circ h(t) \in \mathbf{P}_+[0, \infty)$. □

The following result was well-known:

$$\frac{t-a}{t^p - a^p} \preceq t.$$

We now give a more accurate fact:

Lemma 3.3 *If $0 < p \leqq \frac{1}{2}$, then*

$$\frac{t-a}{t^p - a^p} \preceq t^{1-p}.$$

Proof. For $0 < r \leqq 1$ and for $b : 0 \leqq b < \infty$,
$\dfrac{t^{r+1} - b^{r+1}}{t^r - b^r} = t + b^r \dfrac{t - b}{t^r - b^r} \preceq t.$
Replace r with $\frac{p}{1-p}$ to get
$\dfrac{t^{\frac{1}{1-p}} - b^{\frac{1}{1-p}}}{t^{\frac{p}{1-p}} - b^{\frac{p}{1-p}}} \preceq t.$
Since t^{1-p} is increasing and bijective, by property (iii),
$\dfrac{t - b^{\frac{1}{1-p}}}{t^p - b^{\frac{p}{1-p}}} \preceq t^{1-p}$. Replace $b^{\frac{1}{1-p}}$ with a. □

Theorem 3.4 *For* $0 < p < 1,\ 0 < a, b < \infty$

$$\frac{(t-a)(t-b)}{(t^p - a^p)(t^{1-p} - b^{1-p})} \in \mathbf{P}_+[0, \infty).$$

Proof. We may assume $p \leqq 1/2$. By Lemma3.3
$\dfrac{t-a}{t^p - a^p} \preceq t^{1-p} \preceq t.$
Define h by $h(t^{1-p}) = \dfrac{t-a}{t^p - a^p}$. Then
$h(t) \in \mathbf{P}_+[0, \infty)$.
So $h(z)$ is holomorphic in the open upper half plane Π_+; moreover
$0 < \arg h(z) \leqq \arg z$ for $z \in \Pi_+$.
Hence for $z \in \Pi_+$

$$0 < \arg \frac{z-a}{z^p - a^p} \leqq \arg z^{1-p}. \tag{4}$$

On the other hand, since
$t^{1-p} \dfrac{t-b}{t^{1-p} - b^{1-p}} = t - b + b^{1-p} \dfrac{t-b}{t^{1-p} - b^{1-p}} \in \mathbf{P}_+[0, \infty)$,
we get
$0 < \arg z^{1-p} \dfrac{z-b}{z^{1-p} - b^{1-p}} < \pi$ for z in Π_+.
Since
$0 < \arg \dfrac{z-b}{z^{1-p} - b^{1-p}} < \pi$,
by (4)
$0 < \arg \dfrac{(z-a)(z-b)}{(z^p - a^p)(z^{1-p} - b^{1-p})} < \pi.$
$\dfrac{(t-a)(t-b)}{(t^p - a^p)(t^{1-p} - b^{1-p})} \in \mathbf{P}_+[0, \infty).$ □

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Antisymmetricity Of Some Binary Relations On Self-adjoint Operators

自己共役作用素の間のある二項関係の反対称性

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Abstract

We introduce an "order-like" binary relation $\ll$ among self-adjoint operators on a separable Hilbert space defined by: $A \ll B$ if there exists an operator T on $\mathcal{H}$ such that $E_A(\Delta) \leq T^* E_B(\Delta) T$ for any Borel set Δ on the real line $\boldsymbol{R}$. The relation $\ll$ has reflexivity and transitivity. To consider the antisymmetricity for $\ll$ on the unitary equivalence classes of self-adjoint operators, we will show that $A \ll B$, $B \ll A$ imply that A and B are unitarily equivalent.

1. 序論

$\mathcal{H}$ を可分 Hilbert 空間, $\mathcal{B}(\mathcal{H})$ を $\mathcal{H}$ 上の有界線型作用素の全体とし, 自己共役作用素の全体を $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ とする. 二つの正規作用素 A, $B \in \mathcal{B}(\mathcal{H})$ について, 任意の $\boldsymbol{C}$ 上の Borel 集合 Δ に対し

$$E_A(\Delta) \leq T^* E_B(\Delta) T$$

が成り立つとき, $A \ll_T B$ と表し, ある $T \in \mathcal{B}(\mathcal{H})$ が存在して $A \ll_T B$ となるとき, $A \ll B$ と表すことにより, 正規作用素のユニタリ同値類の間における二項関係 $\ll$ を定義する. 上の不等式を単に $E_A \leq T^* E_B T$ で表すことにする. $A \geq O$ のとき, $A \ll_T B$ ならば $A \leq T^* B T$ である. 他方, $A \leq T^* B T$ を満たし $A \ll_T B$ とならないような正の対角作用素 A, B と weighted shift 作用素 T の例を作ることが出来るため, 逆は成り立たない.

一方, 任意の $\lambda \in$R に対して $E_A((-\infty,\ \lambda]) \geq E_B((-\infty,\ \lambda])$ のとき $A \preceq B$ として定義されるスペクトル順序 $\preceq$ (Cf.[1]) との関連については,

$$A \ll_T B \quad \Longrightarrow \quad E_A((-\infty,\ \lambda]) \geq T^* E_B((-\infty,\ \lambda]) T$$

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1

であり, 逆は不成立であることが判る.

さて, 二項関係 $\ll$ が反射律・推移律を満たすことは直ちにわかる. そこで $\ll$ の反対称律の真偽を考察したい. すなわち命題 "$A \ll B$ かつ $B \ll A$ ならば, A, B はユニタリ同値となるか？" (Cf.[2]) が成り立つか否かを解明したい.

2. 二項関係 $\ll$ の特徴づけ

二項関係 $\ll$ は幾つかの興味深い性質を持っている. 初めにそれらを述べる.

Proposition 1. $T \in \mathcal{B}(\mathcal{H})$ とし, A, $B \in \mathcal{B}(\mathcal{H})$ を $A \ll_T B$ なる正規作用素とするとき,

(i) $\sigma(A) \subseteq \sigma(B)$.
(ii) 任意の $\boldsymbol{C}$ 上の Borel 集合 Δ に対し, $\mathrm{rank}E_A(\Delta) \leq \mathrm{rank}E_A(\Delta)$.
(iii) 任意の $\sigma(B)$ 上有界・非負である Borel 可測関数 ϕ に対して, $\phi(A) \leq T^*\phi(B)T$.
(iv) 任意の $\sigma(B)$ 上有界な Borel 可測関数 ϕ に対して, $\phi(A) \ll_T \phi(B)$.

上の命題 (i), (ii) より, 以下が直ちに得られる:

Corollary 2. 正規作用素 A, $B \in \mathcal{B}(\mathcal{H})$ について, $A \ll B$ かつ $B \ll A$ ならば, $\sigma(A) = \sigma(B)$ かつ 任意の $\boldsymbol{C}$ 上の Borel 集合 Δ に対して $\mathrm{rank}E_A(\Delta) = \mathrm{rank}E_A(\Delta)$ である.

Corollary 3. 正規作用素 A, $B \in \mathcal{B}(\mathcal{H})$ の少なくとも一方がコンパクトであり, $A \ll B$ かつ $B \ll A$ であるとき, A, B はユニタリ同値である.

$A \ll_T B$ のとき, $I \leq T^*T$ である. 故に次が成り立つ.

Proposition 4. 正規作用素 A, $B \in \mathcal{B}(\mathcal{H})$ について, $A \ll_T B$ かつ $||T|| \leq 1$ であるとき, 任意の $\sigma(B)$ 上有界な Borel 可測関数 ϕ に対して, $\phi(A) = T^*\phi(B)T$.

上の Proposition 4 より, $A \ll_T B$ かつ $||T|| \leq 1$ ならば, $|A|^n = T^*|B|^nT$ $(n = 1, 2, \ldots)$ である. 逆に, 次の命題が成り立つ:

Proposition 5. 正規作用素 A, $B \in \mathcal{B}(\mathcal{H})$ と $T \in \mathcal{B}(\mathcal{H})$ について, $A^n = T^*B^nT$, $A^{*n} = T^*B^{*n}T$ $(n = 1, 2, \ldots)$ のとき, 任意の $\boldsymbol{C}$ 上の Borel 集合 Δ に対して,

$$E_A(\Delta) = T^*E_B(\Delta)T + \chi_\Delta(0)(I - T^*T).$$

更に, 以下が成り立つ:
(i) T が等長ならば, $E_A = T^*E_BT$.
(ii) $I \leq T^*T$ ならば, $A \ll_T B$.
(iii) $I \neq T^*T$ ならば, $0 \notin \sigma(A) \cup \sigma(B)$.

以下, A, $B \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ について考える. 二項関係 $\ll$ の反対称性を考察するにあたり, 次の定理が議論の主軸を為す. これは, 二項関係 $\ll$ に対する特徴づけともいえる結果である.

Theorem 6. $A,\ B \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ に対し, 以下の条件は互いに同値である:

(i) $A \ll B$.

(ii) ある全射縮小作用素 V が存在して, $VB = AV$.

(iii) ある等長作用素 W が存在して, $A = W^*BW$ かつ $(WW^*)\mathcal{H}$ は B を約す.

(iv) ある等長作用素 W が存在して, $E_A = W^*E_BW$ かつ $(WW^*)\mathcal{H}$ は B を約す.

[証明] (i)$\Longrightarrow$(ii): 仮定より, ある $T \in \mathcal{B}(\mathcal{H})$ が存在し, 任意の $\boldsymbol{R}$ 上の Borel 集合 Δ に対して $E_A(\Delta) \leq T^*E_B(\Delta)T$. 有限個の $\boldsymbol{R}$ 上の Borel 集合の列 $\Delta_1,\ \Delta_2,\ \ldots,\ \Delta_n$ 及び $\mathcal{H}$ の 0 でないベクトルの列 $\xi_1,\ \xi_2,\ \ldots,\ \xi_n$ をとり, $\bigcup_{k=1}^{n} \Delta_k$ の互いに素な Borel 分割を $\{\ \Delta'_j\ :\ 1 \leq j \leq N\ \}$ とする. すなわち, $\bigcup_{k=1}^{n} \Delta_k = \bigcup_{j=1}^{N} \Delta'_j$ であり, 各 $k = 1,\ 2,\ \ldots,\ n$ に対し部分集合 N_k がとれて $\Delta_k = \bigcup_{j\in N_k} \Delta'_j$ となるとする.

このとき,

$$\begin{aligned}
\left\| \sum_{k=1}^{n} E_A(\Delta_k)\xi_k \right\|^2 &= \sum_{j=1}^{N} \|E_A(\Delta'_j)\xi'_j\|^2 \qquad \text{where} \quad \xi'_j = \sum_{k=1}^{n} \chi_{N_k}(j)\xi_k \\
&\leq \sum_{j=1}^{N} \|E_B(\Delta'_j)T\xi'_j\|^2 \\
&= \sum_{j=1}^{N} \left\langle\ E_B(\Delta'_j)T\left(\sum_{k=1}^{n} \chi_{N_k}(j)\xi_k\right),\ \left(\sum_{k=1}^{n} \chi_{N_k}(j)\xi_k\right)\ \right\rangle \\
&= \left\| \sum_{k=1}^{n} E_B(\Delta_k)T\xi_k \right\|^2.
\end{aligned}$$

そこで, $\boldsymbol{R}$ 上の Borel 集合 Δ およびベクトル $\xi \in \mathcal{H}$ に対し, $V_0(E_B(\Delta)T\xi) := E_A(\Delta)\xi$ により作用素 V_0 を定めると上の計算より $x \in \mathrm{span}\,[\,W^*(B)T\mathcal{H}\,]$ に対し $\|V_0x\| \leq \|x\|$. 故に V_0 は $\mathcal{M} = \overline{\mathrm{span}\,[\,W^*(B)T\mathcal{H}\,]}$ 上の縮小作用素に拡張できる. ここで $W^*(B)$ は B の生成する von Neumann 環である. 次に,

$$V(x \oplus y) := V_0x \oplus 0, \quad x \oplus y \in \mathcal{M} \oplus \mathcal{M}^{\perp}$$

により作用素 V を定める. $\boldsymbol{R}$ 上の Borel 集合 $\Delta,\ \Delta'$ とベクトル $\xi \in \mathcal{H}$ に対し, $x = E_B(\Delta)T\xi$ とおくと,

$$\begin{aligned}
VE_B(\Delta')x &= VE_B(\Delta')E_B(\Delta)T\xi = V_0(\ E_B(\Delta \cap \Delta')T\xi\ ) = E_A(\Delta \cap \Delta')\xi \\
&= E_A(\Delta')E_A(\Delta)\xi = E_A(\Delta')Vx
\end{aligned}$$

となるので, 任意の $\sigma(B)$ 上の有界 Borel 可測関数 ϕ に対して $V\phi(B) = \phi(A)V$. また, $V_0(E_B(\boldsymbol{R})T\xi) = E_A(\boldsymbol{R})\xi$ より $VT = I$. よって $\mathcal{H} = VT\mathcal{H} \subseteq V\mathcal{H}$ となるので V は全射である. 故に上で構成した V が所要のものとなる.

3

(ii)$\Longrightarrow$(iii): 全射縮小作用素 V が存在して, $VB = AV$ であるとする. $V = U|V|$ を V の極分解とすると, $VB = AV$ および $V^*A = BV^*$ より, $|V|B = B|V|$. 故に $U^*U\mathcal{H} = \overline{|V|\mathcal{H}}$ は B を約す. また $\mathcal{H} = V\mathcal{H}$ より $UU^*\mathcal{H} = \mathcal{H}$; すなわち $UU^* = I$.

次に, $x = |V|\xi \in \overline{|V|\mathcal{H}}$ に対し,

$$UBx = UB|V|\xi = U|V|B\xi = VB\xi = AV\xi = AU|V|\xi = AUx.$$

故に $x \in \overline{|V|\mathcal{H}}$ に対し $UBx = AUx$. 更に $\eta \in \mathcal{H}$ に対し, $U\mathcal{H} = \mathcal{H}$ より $\eta = Ux$ となる $x \in \mathcal{H}$ をとれば, $U^*Ux \in \overline{|V|\mathcal{H}}$ より,

$$UBU^*\eta = UBU^*Ux = AUU^*Ux = AUx = A\eta.$$

故に $W = U^*$ とすれば $W^*BW = A$.

(iii)$\Longrightarrow$(iv): 仮定 (iii) より $A^n = W^*B^nW$ $(n = 1,\ 2,\ \ldots)$. 故に $\boldsymbol{R}$ 上の Borel 集合 Δ に対し χ_Δ を連続関数により近似することにより (iv) を得る.

(iv)$\Longrightarrow$(i) は明らかである. □

上の定理に於いて, 特に $A \ll_T B$ であるとき, (ii) における V は $VT = I$ を満たす. このことから T の可逆性に注目すれば, 次が得られる.

Corollary 7. 稠密な値域をもつ T に対して $A \ll_T B$ ならば, $A,\ B$ はユニタリ同値である. また, $\mathcal{H}$ が有限次元のとき, $A \ll_T B$ ならば $A,\ B$ はユニタリ同値である.

なお, $\mathcal{H}$ が無限次元で T が可逆でないとき, $A \ll_T B$ であるが $A,\ B$ がユニタリ同値とならないような $A,\ B,\ T$ の例を作ることができる.

また, Theorem 6 の条件 (iii) より, 以下の結果が得られる. これは巡回的な自己共役作用素に対する解決を与える.

Corollary 8. $A \ll B$ かつ B が巡回的ならば, A も巡回的である.

Theorem 9. $A,\ B \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ の少なくとも一方が巡回的であり, $A \ll B$ かつ $B \ll A$ であるとき, $A,\ B$ はユニタリ同値である.

3. 主定理

Theorem 6 の条件 (iii), (iv) に注目することにより, 次の主定理が得られる:

Theorem 10. $A,\ B \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ が, $A \ll B$ かつ $B \ll A$ であるとき, $A,\ B$ はユニタリ同値である.

[証明] Theorem 6 の条件 (iii), (iv) における W について, $\phi(A) = W^*\phi(B)W$ が任意の $\sigma(B)$ 上有界な Borel 可測関数 ϕ に対して成り立つことがわかる.

故に $A \ll B$ かつ $B \ll A$ のとき, 等長作用素 V, W が存在して, $X = \sigma(A) = \sigma(B)$ 上有界な任意の Borel 可測関数 ϕ に対して

$$\phi(A) = W^*\phi(B)W \quad \text{かつ} \quad \phi(B) = V^*\phi(A)V$$

が成り立ち, $(WW^*)\mathcal{H}$ は B を, $(VV^*)\mathcal{H}$ は A をそれぞれ約す. そこで

$$\pi_1(\phi(B)) = W^*\phi(B)W, \qquad \pi_2(\phi(A)) = V^*\phi(A)V$$

により写像 π_1, π_2 を定義すると, π_1 は $W^*(B)$ から $W^*(A)$ への, π_2 は $W^*(A)$ から $W^*(B)$ へのそれぞれ $*$-準同型となり, $\pi_1(\phi(B)) = \phi(A)$, $\pi_2(\phi(A)) = \phi(B)$ を満たす. ここで $W^*(A)$, $W^*(B)$ は各々 A, B が生成する von Neumann 環を表す. 一方, $W^*(A)$, $W^*(B)$ は以下のように分解される:

$$W^*(A) = \left(\bigoplus_{n\geq 1} \mathcal{A}_n\right) \oplus \mathcal{A}_\infty, \quad W^*(B) = \left(\bigoplus_{n\geq 1} \mathcal{B}_n\right) \oplus \mathcal{B}_\infty.$$

ここで, $\mathcal{A}_n$, $\mathcal{B}_n$ $(1 \leq n \leq \infty)$ は, それぞれ $W^*(A)$, $W^*(B)$ の重複度 (uniform multiplicity) n をもつ極大な斉次部分 (maximal homogeneous part) である. すなわち, $\mathcal{A}_n$, $\mathcal{B}_n$ が $\{0\}$ にならないような各 n に対して, ある正則 Borel 測度 μ_n, ν_n が存在して $\mathcal{A}_n$, $\mathcal{B}_n$ はそれぞれ $L^\infty(\mu_n) \quad I_n$, $L^\infty(\nu_n) \quad I_n$ とユニタリ同値となる. このとき, $\pi_1(\mathcal{B}_n) \subseteq \mathcal{A}_n$, $\pi_2(\mathcal{A}_n) \subseteq \mathcal{B}_n$ が各 n に対して成り立つ. 故に, π_1, π_2 から自然に導かれる中への $*$-準同型写像 $\pi_1^n : L^\infty(\nu_n) \longrightarrow L^\infty(\mu_n)$ および $\pi_2^n : L^\infty(\mu_n) \longrightarrow L^\infty(\nu_n)$ を各 n に対して考えることが出来る. これらの写像に対し, 以下に述べる Theorem 11 を用いることにより証明が完結する. □

Theorem 11. X, Y を compact Hausdorff 空間とし, それぞれ X から Y への, Y から X への中への Borel 同型写像が存在すれば, X から Y への上への Borel 同型写像が存在する.

上の定理は集合論におけるベルンシュタインの定理の証明と同様の議論を用いることにより証明できる.

Theorem 10 は Corollary 3, Theorem 9 の結果を含み, この定理より二項関係 $\ll$ が自己共役作用素のユニタリ同値類全体の上において順序となることがわかった.

二項関係 $\ll$ に関連して, 条件 "正規作用素 A, B に対して, ある $S \in \mathcal{B}(\mathcal{H})$ が存在して $S^*E_AS \leq E_B$" によって定まる二項関係を考えることができる. もし S が可逆であれば, $S^*E_AS \leq E_B$ から $A \ll_{S^{-1}} B$ が従うので $\ll$ の結果に帰着する. S が可逆でない場合には $\ll$ と一致するとは限らない. この二項関係については次が成り立つ:

Theorem 12. A, $B \in \mathcal{B}(\mathcal{H})_{\rm sa}$ および $\overline{S\mathcal{H}} = \mathcal{H}$ である $S \in \mathcal{B}(\mathcal{H})$ に対し, 以下の条件は互いに同値である:

(i) $S^*E_AS \leq E_B$.
(ii) $SB = AS$.
(iii) ある等長作用素 W が存在して, $A = W^*BW$ かつ $(WW^*)\mathcal{H}$ は B を約す.
(iv) ある等長作用素 W が存在して, $E_A = W^*E_BW$ かつ $(WW^*)\mathcal{H}$ は B を約す.

証明は Theorem 6 とほぼ同様の議論による. 上の結果より, S が稠密な値域を持つ場合にも, $S^*E_AS \leq E_B$ は $A \ll B$ と同値であることがわかった. 従って次の系を得る:

Corollary 13. $A,\ B\in\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ および 稠密な値域を持つ $S,\ T\in\mathcal{B}(\mathcal{H})$ に対し, $S^*E_AS\leq E_B$ かつ $T^*E_BT\leq E_A$ ならば, $A,\ B$ はユニタリ同値である.

最後に, $S,\ T$ が可逆でない場合について, 対角作用素 $A,\ B$ および weighted shift 作用素 (故に, 非可逆である) $S,\ T$ で $S^*E_BS\leq E_A\leq T^*E_BT$ かつ $A,\ B$ がユニタリ同値ではないような例を紹介する. 有界な数列 $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ を

$$k\neq l\Rightarrow \alpha_k\neq\alpha_l,\quad \alpha_1\notin\overline{\{\ \alpha_n : n\geq 2\ \}}\quad\text{and}\quad \sup_{n\geq 1}|\lambda_n|\leq 1\leq\inf_{n\geq 1}|\mu_n|$$

を満たすようにとり, $A,\ B,\ S$ and T を

$$A=\begin{pmatrix}\alpha_2 & & \\ & \alpha_3 & \\ & & \ddots\end{pmatrix},\qquad B=\begin{pmatrix}\alpha_1 & & \\ & \alpha_2 & \\ & & \ddots\end{pmatrix},$$

$$S=\begin{pmatrix}0 & & & \\ \lambda_1 & 0 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & \end{pmatrix}\quad\text{and}\quad T=\begin{pmatrix}0 & & & \\ \mu_1 & 0 & & \\ & \mu_2 & \ddots & \\ & & \ddots & \end{pmatrix}.$$

と定めれば, これらが所要の反例を為す. また上の $A,\ B$ and T は $A\ll_T B$ かつ $A,\ B$ がユニタリ同値ではない例となっている.

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SEMILINEAR DEGENERATE ELLIPTIC BOUNDARY VALUE PROBLEMS
（半線形退化楕円型境界値問題の L^p 理論）

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Abstract. The purpose of this talk is to study a class of semilinear elliptic boundary value problems with *degenerate* boundary conditions which include as particular cases the Dirichlet problem and the Robin problem. The approach here is based on the super-sub-solution method in the degenerate case, and is distinguished by the extensive use of an L^p Schauder theory elaborated for second-order, elliptic differential operators with *discontinuous* zero-th order term. By using Schauder's fixed point theorem, we prove that the existence of an ordered pair of sub- and supersolutions of our problem implies the existence of a solution of the problem. The results extend an earlier theorem due to Kazdan and Warner to the degenerate case.

はじめに

本講演は、半線形楕円型境界値問題の一般的研究について、関数空間論の立場から考察したものである。筆者は、主として、無限次元版の陰関数定理に基づく非自明解の分岐理論を援用して、実際の応用上からも重要であるパラメータに関する非自明解の分岐の様子、正値解の個数、定常解の安定性の問題について解析学の立場から多角的な研究を行ってきた。例えば、化学反応速度論では、アレニウスの法則とニュートンの冷却の法則をみたす「化学反応物に対する燃焼問題」を考察し、活性化エネルギーが十分小さい場合には、化学反応は連続的であって爆発現象が起こることはないが、活性化エネルギーが十分大きい場合には、酸素と化学反応物の比率に応じて、不連続に爆発が起きたり、起こらなかったりするという化学実験データの数学的な証明を与えた。その証明においては、確率論における多次元マルコフ過程の境界値問題を詳しく考察した研究成果が本質的な役割を果たした。

それらの研究成果によって、半線形楕円型方程式に対する退化型 Robin 境界条件の場合に、Leray–Schauder の写像度理論及び Schauder の不動点定理に代表される位相的な手法を用いることが初めて可能となった。この手法は、古典的な変分法と組み合わせることで極めて強力な武器を提供し、個々の問題を、Dirichlet 境界条件及び Robin 境界条件を含む退化型 Robin 境界値問題として統一的に取り扱うことができる。変分法を適用するに当たっての本質的な着眼点は、藤田宏・加藤敏夫にその源を持つ解析的半群の分数巾を考える点にある。この着想によって、Amann–Hess [AH], Ambrosetti–Prodi [AP1], [AP2], Ambrosetti–Mancini [AM], Berger–Podolak [BP], Kazdan–Warner [KW], Landesman–Laser [LL] 等による従来の様々な非線型問題に対する先行研究を一般化することができた。

本講演で考察する非線形項の形は、大きく分けて、次の 3 つの場合に分類される：

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(I) 可逆な線形項を持つ、漸近的に線形な場合：Perron にその源を持つ優解・劣解の方法を利用するが、その際、Schauder の不動点定理が重要な役割を果たす。また、応用上からも重要であるパラメータに関する解の個数を調べるには、Leray–Schauder の写像度の理論を使う。

(II) 固有値と一致する可逆ではない線形項（共鳴項）を持つ、漸近的に線形な場合：Lyapunov–Schmidt の直交分解及び無限次元版の大局的逆関数定理を使う。

(III) 非対称な非線形項を持つ場合：特異点を持つ場合に拡張された無限次元版の大局的逆関数定理を使う。この場合は、パラメータに関する解の個数について、完全に分類できる。

紙数の関係で、研究テーマ (I) について詳しく解説し、残りの研究テーマについては、講演の中で詳しく述べる。

問題の定式化及び主結果

Ω を、ユークリッド空間 $\mathbf{R}^N$ ($N \geq 2$) 内の滑らかな境界 $\partial\Omega$ を持つ有界領域とする。その閉包 $\overline{\Omega} = \Omega \cup \partial\Omega$ は、N 次元の滑らかなコンパクト多様体である。まず、主要部の線形作用素

$$Au := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u \tag{1.1}$$

は、滑らかな実係数を持つ、一様に楕円型な 2 階の偏微分作用素とする。より正確には、次の 2 条件を仮定する：

(1) $a^{ij}(x) \in C^\infty(\overline{\Omega})$, $a^{ij}(x) = a^{ji}(x)$, $x \in \overline{\Omega}$,

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq \exists a_0|\xi|^2, \quad (x,\xi) \in \overline{\Omega} \times \mathbf{R}^N.$$

(2) $c(x) \in C^\infty(\overline{\Omega})$, $c(x) \geq 0$ in Ω.

次に、考える境界条件

$$Bu = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u \tag{1.2}$$

は、滑らかな実係数を持つ、一般の Robin 型境界作用素とする。より正確には、次の 3 条件を仮定する：

(1) $a(x') \in C^\infty(\partial\Omega)$, $a(x') \geq 0$ on $\partial\Omega$.
(2) $b(x') \in C^\infty(\partial\Omega)$, $b(x') \geq 0$ on $\partial\Omega$.
(3) $\partial/\partial\boldsymbol{\nu}$ は、次式で与えられる余法線微分である。

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i=1}^{N} a^{ij}(x')\, n_j \frac{\partial}{\partial x_i}.$$

ここで $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ は単位長さを持つ外向き法線ベクトルである。

本講演では、次の半線形楕円型境界値問題を考察する：与えられた非線形項 $f(x,\xi)$ に対して, 境界値問題

$$
(1.3) \qquad \begin{cases} Au = f(x,u) & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega \end{cases}
$$

を満たす解 $u(x)$ を求めよ。

一般 Robin 型境界条件 B に関する基本的な仮定は、次の 2 つである：

(H.1) $a(x') + b(x') > 0$ on $\partial\Omega$.

(H.2) $b(x') \not\equiv 0$ on $\partial\Omega$.

注意 *1.1.* 境界条件 B について、重要な注意を述べる。

(i) もし $a(x') \equiv 0$, $b(x') \equiv 1$ ならば、B は Dirichlet 条件となり、もし $a(x') \equiv 1$, $b(x') \geq 0$ ならば、通常の Robin 条件となる。さらに、境界値問題 (1.3) が非退化であるためには、(a) $a(x') > 0$ on $\partial\Omega$ であるか (b) $a(x') \equiv 0$, $b(x') > 0$ on $\partial\Omega$ の場合のどちらかに限ることが分かる。

(ii) 仮定 (H.1) の下では、関数 $a(x')$ のゼロ点において、Lopatinskii–Shapiro 条件が成り立たない。従って、我々の境界条件 B は退化型 Robin 境界条件となる。

(iii) 仮定 (H.2) は、我々の境界条件 B は Neumann 条件と一致しないことを意味する。以下の定理 1.1 で述べるように、我々の場合は、付随する第 1 固有値 λ_1 は正かつ代数的に単純である。しかしながら、Neumann 条件の場合は、第 1 固有値 λ_1 がゼロとなるので、改めて考察する必要があるので、本講演では除外する。

まず、境界値問題 (1.3) に付随して、線形作用素

$$
\mathfrak{A}\colon L^2(\Omega) \longrightarrow L^2(\Omega)
$$

を次式で定義する：

(1) $\mathcal{D}(\mathfrak{A}) := \{u \in H^2(\Omega) = W^{2,2}(\Omega) : Bu = 0\}$.

(2) $\mathfrak{A}u := Au$, $u \in \mathcal{D}(\mathfrak{A})$.

このとき、我々の出発点となる基本的な結果は、次の定理である：

定理 1.1. 作用素 $\mathfrak{A}$ は、*Hilbert* 空間 $L^2(\Omega)$ において自己共役かつ正定値である。さらに、その第 *1* 固有値 λ_1 は正かつ代数的に単純であり、対応する固有関数 $\varphi_1(x)$ は Ω で正の値をとる。すなわち、

$$
\begin{cases} \mathfrak{A}\varphi_1 = \lambda_1 \varphi_1 & \textit{in } L^2(\Omega), \\ \varphi_1(x) > 0 & \textit{in } \Omega. \end{cases}
$$

本講演では、パラメーター t に依存する非線形項 $f(x,\xi,t)$ について次の 4 条件を課す：

(F1) 関数 $f(x,\xi,t)$ は、$\overline{\Omega} \times \mathbf{R} \times \mathbf{R}$ 上で定義された滑らかな実数値関数である。

(F2) 任意の $m \in \mathbf{R}$ について、連続関数 $h(x) \in C(\overline{\Omega})$ が存在して、全ての $x \in \Omega$, $\xi \geq m$, $t \in \mathbf{R}$ に対して、不等式

$$
\frac{\partial f}{\partial t}(x,\xi,t) \geq h(x) > 0
$$

が成り立つ。

(F3) 全ての $x \in \overline{\Omega}$ と有界区間内を動く実数 t について、不等式

$$\begin{cases} \limsup_{\xi \to -\infty} \dfrac{f(x,\xi,t)}{\xi} < \lambda_1, \\ \liminf_{\xi \to +\infty} \dfrac{f(x,\xi,t)}{\xi} > \lambda_1 \end{cases}$$

が成り立つ。

(F4) 全ての $x \in \overline{\Omega}$ と有界区間内を動く実数 t について、不等式

$$\limsup_{\xi \to +\infty} \frac{f(x,\xi,t)}{\xi} < +\infty$$

が成り立つ。

本講演の最初の目的は、L^p 版の Sobolev 空間の枠組みで、非線形境界値問題 (1.3) の解の個数を下から評価することである。次の定理は、先行する研究 Kazdan–Warner [KW] 及び Amann–Hess [AH] を退化型 Robin 境界条件の場合へ拡張した結果である：

定理 1.2. 次の *3* 条件を仮定する：

(a) 任意の $m \in \mathbf{R}$ について、関数 $h(x) \in C(\overline{\Omega})$ が存在し、全ての $x \in \Omega$, $\xi > m$, $t \in \mathbf{R}$ に対して、不等式

$$\frac{\partial f}{\partial t}(x,\xi,t) > h(x) > 0, \quad x \in \overline{\Omega}$$

が成り立つ。

(b) 全ての $t \in \mathbf{R}$ について、不等式

$$\limsup_{\xi \to -\infty} \frac{f(x,\xi,t)}{\xi} < \lambda_1$$

が成り立つ。ただし、上極限は $x \in \overline{\Omega}$ について一様とする。

(c) 定数 s_1, t_1, c が存在し、全ての $x \in \overline{\Omega}$, $s > s_1$ に対して、不等式

$$f(x,s,t_1) - \lambda_1 s > c$$

が成り立つ。

このとき、実数 t_0 が存在して、非線形境界値問題 *(1.3)* の解 $u(x) \in W^{2,p}(\Omega)$ *(*$N < p < \infty$*)* は、$t > t_0$ ならば非存在、$t < t_0$ ならば少なくとも一つ存在する。

系 1.3. 非線形項 $f(x,\xi,t)$ は次の形とする：

$$f(x,\xi,t) = f_0(x) + t\,r(x) + g(x,\xi).$$

ここで、$f_0(x) \in C^\infty(\overline{\Omega})$, $r(x) \in C^\infty(\overline{\Omega})$, $r(x) > 0$ *in* Ω, $g(x,\xi) \in C^\infty(\overline{\Omega} \times \mathbf{R})$. さらに、関数 $g(x,\xi)$ について、次の *2* 条件を満たす定数 $\gamma < \lambda_1$, k, $s_\pm$ が存在するものとする：

(a) $s < s_-$ ならば $g(x,s) - \gamma s > k$.

(b) $s > s_+$ ならｂ $g(x,s) - \lambda_1 s > k$.

このとき、実数 t_0 が存在して、非線形境界値問題

$$\begin{cases} Au = f_0(x) + t\,r(x) + g(x,u) & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega \end{cases}$$

の解 $u(x) \in W^{2,p}(\Omega)$ *(*$N < p < \infty$*)* は、$t > t_0$ ならば非存在、$t < t_0$ ならば少なくとも一つ存在する。

系 1.4. 非線形項 $f(x,\xi,t)$ は次の特別な形とする：

$$f(x,\xi,t) = f_0(x) + t\,\varphi_1(x) + g(\xi).$$

ここで $f_0(x) \in C^\infty(\overline{\Omega})$ であって、$g(\xi) \in C^\infty(\mathbf{R})$ は強凸関数とする。さらに、*2* つの極限値 $g'(-\infty)$, $g'(+\infty)$ が存在し、条件

$$0 < g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$$

を満たすとする。

このとき、実数 t_0 が存在して、非線形境界値問題

$$\begin{cases} Au = f_0(x) + t\varphi_1(x) + g(u) & \mathit{in}\ \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \mathit{on}\ \partial\Omega \end{cases}$$

の解 $u(x) \in W^{2,p}(\Omega)$ *(*$N < p < \infty$*)* は、$t > t_0$ ならば非存在、$t = t_0$ ならば唯一つ存在し、$t < t_0$ ならば丁度 *2* 個存在する。

今後の問題

(1) 境界条件が非斉次の場合は、Taira-Palagachev–Popivanov [TPP] におけるように、いわゆる Continuity method を利用して、Schaefer の不動点定理を適用すればよい。

(2) 将来的には、弾性体における Von Kármán 方程式に対して、応用面で重要なパラメーターの臨界値の数値解析を実際に行うことを目指している。そのためには、Ambrosetti–Mancini [AM] におけるように、第 1 固有値 λ_1 を超えて第 2 固有値 λ_2 の近くでの考察が不可欠になる。

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Interpolation sequence and closed range integration operators

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Let D be the open unit disk in complex plane C. For $z, w \in D$, $0 < r < 1$, let $\varphi_w(z) = \dfrac{w - z}{1 - \overline{w}z}$ and let $\rho(z, w) = \left|\dfrac{w - z}{1 - \overline{w}z}\right|$ and $D(w, r) = \left\{z \in D, \rho(w, z) < r\right\}$. Let $H(D)$ be the space of all analytic functions on D.

For $\alpha \geq 0$, the space L^∞_α is defined to be the Banach space of Lebesgue measurable functions f on the open unit disk D with

$$\| f \|_{L^\infty_\alpha} = \sup_{z \in D} \left(1 - |z|^2\right)^\alpha |f(z)| < +\infty.$$

Note that $L^\infty_0 = L^\infty$ is the classical Banach space of Lebesgue measurable functions f on the open unit disk D with

$$\| f \|_{L^\infty} = \sup_{z \in D} |f(z)| < +\infty.$$

For $\alpha \geq 0$, the Bloch-type space $\mathcal{B}^\alpha$ of D is defined by

$$\mathcal{B}^\alpha = H(D) \cap L^\infty_\alpha.$$

The space of bounded analytic functions on D will be denoted by H^∞.

Let $\alpha > 0$. Then the space $\mathcal{B}_\alpha$ of D is defined to be the space of analytic functions f on D such that

$$\| f \|_{\mathcal{B}_\alpha} = |f(0)| + \| f \|_{\mathcal{B}_\alpha} < +\infty ,$$

where $\| f \|_{\mathcal{B}_\alpha} = \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)|$. Note that $\mathcal{B}_1 = \mathcal{B}$ is the Bloch space .

The little Bloch space $\mathcal{B}_0$ is defined to be the space of analytic functions f on D such that

$$(1 - |z|^2) |f'(z)| \to 0 \qquad (|z| \to 1^-).$$

For $\alpha > -1$ and $p > 0$, the space $L^p\left((1 - |z|^2)^\alpha dA(z)\right)$ is defined to be the space of Lebesgue measurable functions f on D such that

$$\| f \|_{L^p((1-|z|^2)^\alpha dA(z))} = \left\{\int_D |f(z)|^p (\alpha + 1)(1 - |z|^2)^\alpha dA(z)\right\}^{\frac{1}{p}} < +\infty,$$

where $dA(z)$ denote the area measure on D. The weighted Bergman space is defined by

$$L^p_a\left((1 - |z|^2)^\alpha dA(z)\right) = H(D) \cap L^p\left((1 - |z|^2)^\alpha dA(z)\right).$$

For $\alpha > -1$ and $p > 0$, the weighted Dirichret space D^α_p is defined to be the space of analytic functions f on D such that

$$\int_D (1 - |z|^2)^\alpha |f'(z)|^p (\alpha + 1) dA(z) < +\infty.$$

In the case of $\alpha = 1$ and $p = 2$, then $D_2^1 = H^2$ is the Hardy space. In the case of $\alpha = 2$ and $p = 2$, then $D_2^2 = L_a^2$ is the Bergman space.

Let X, Y be Banach spaces and let T be a linear operator from X into Y. Then T is called to be bounded below from X to Y if there exists a positive constant $C > 0$ such that $\| Tf \|_Y \geq C \| f \|_X$ for all $f \in X$, where $\| * \|_X$, $\| * \|_Y$ be the norm of X, Y, respectively. And we define the operator infimum norm by the following:

$$\| T \|_{\inf, X \to Y} = \inf_{\|x\|_X = 1} \| Tx \|_Y .$$

For g analytic on D, the generalized integration operators J_g is defined by the following:

$$J_g(f)(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta.$$

Theorem.1.([1]) *Let $\alpha > -1$ and $p \geq 1$. Then for g analytic on D, J_g is bounded on the weighted Bergman space $L_a^p\left((1-|z|^2)^\alpha dA(z)\right)$ if and only if $g \in \mathcal{B}$. And J_g is compact on $L_a^p\left((1-|z|^2)^\alpha dA(z)\right)$ if and only if*

$$\lim_{|z| \to 1^-} (1-|z|^2)|g'(z)| = 0 \qquad (i.e. \quad g \in \mathcal{B}_0).$$

A sequence $\Gamma = \{z_k\}$ is called an *uniformly dense sequence* if

(1) $\Gamma = \{z_k\}$ is separated, i.e. , $\inf_{k \neq n} \rho(z_k, z_n) > 0$,

(2) There exists a $r < 1$ such that $D = \cup_k D(z_k, r)$.

A sequence $\{z_k\}$ is called an *uiversal interpolation sequence* if for each $\{w_k\} \in l^\infty$, there exists a function $f \in H^\infty$ with $f(z_k) = w_k$ for $n = 1, 2, \cdots$.

For $p > 0$, a sequence $\{z_k\}$ is called an *interpolation sequence for H^p* if for each $\{w_k\}$ with $\sum_k (1-|z_k|^2)|w_k|^p < \infty$, there exists a function $f \in H^p$ with $f(z_k) = w_k$ for $n = 1, 2, \cdots$.

For $\alpha > 0$, a sequence $\{z_k\}$ is called an *interpolation sequence for $\mathcal{B}^\alpha$* if for each $\{w_k\}$ with $\sup_k (1-|z_k|^2)^\alpha |w_k| < \infty$, there exists a function $f \in \mathcal{B}^\alpha$ with $f(z_k) = w_k$ for $n = 1, 2, \cdots$.

For $p > 0$ and $\alpha > 0$, a sequence $\{z_k\}$ is called an *interpolation sequence for $L_a^p\left((1-|z|^2)^\alpha dA(z)\right)$* if for each $\{w_k\}$ with $\sum_k (1-|z_k|^2)^{2+\alpha}|w_k|^p < \infty$, there exists a function $f \in L_a^p\left((1-|z|^2)^\alpha dA(z)\right)$ with $f(z_k) = w_k$ for $n = 1, 2, \cdots$.

If z_n is a zero of a Blaschke product $B(z)$ $(\in H^\infty)$, then

$$\prod_{j \neq n} \rho(z_j, z_n) = (1-|z_n|^2)\,|B'(z_n)|\,.$$

A Blaschke product $B(z) \in H^\infty$ is called interpolating if the left hand side is bounded away from zero independently of n. So if $B(z)$ belongs to the little Bloch space $\mathcal{B}_0$, then it is definitely not interpolating, since the right hand side goes to zero uniformly as $z_n \to 1$.

In [3], C.J.Bishop studied the bounded function in the little Bloch space $\mathcal{B}_0$. By Theorem 1.1, $g \in \mathcal{B}_0$ is the equivalent condition with the compactness of J_g on $L_a^2\left((1-|z|^2)^\alpha dA(z)\right)$.

In this paper, we study when J_g is bounded below on $L_a^2\left((1-|z|^2)^\alpha dA(z)\right)$ (J_g is not compact on $L_a^2\left((1-|z|^2)^\alpha dA(z)\right)$). And we characterize the interpolation sequences for the

Bergman space, the weighted Bergman space and the Bloch-type space $\mathcal{B}^{\alpha}$ $(\alpha > 0)$.

The interpolation sequences for H^{∞} (the space of the bounded analytic functions) is determined by L.Carleson using uniformly separared sequence. And the interpolation sequences for the Hardy space H^p is also determined by Shapiro and Shields using uniformly separared sequence. And the interpolation sequences for the Bergman space L^p_a is also determined by Kristian Seip using Seip's density.

Definition 2. *Let $\alpha > 0$. A set Γ of the open unit disk D is called a sampling set for $\mathcal{B}^{\alpha}$ if there exists a positive constant $C > 0$ such that*

$$\sup_{z\in D}(1-|z|^2)^{\alpha}|f(z)| \leq C \sup_{z\in\Gamma}(1-|z|^2)^{\alpha}|f(z)|,$$

for all $f \in \mathcal{B}^{\alpha}$.

Theorem 3. ([17]) *For any inner function φ, the following are equivalent:*
(i) *The operator $M_{\varphi} : L^2_a \to L^2_a$ is bounded below,*
(ii) *φ is a finite product of interpolating Blaschke products.*

Theorem 4. ([23]) *Let $\beta \geq \alpha > 0$. Suppose $h \in H(D)$ with $\sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|h(z)| < +\infty$. Then the following are equivalent :*
(1) *$M_h : L^2_a\left((1-|z|^2)^{2\alpha}dA(z)\right) \to L^2_a\left((1-|z|^2)^{2\beta}dA(z)\right)$ is bounded below i.e. there is a constant $k > 0$ such that*

$$\left\{\int_D |h(z)f(z)|^2(1-|z|^2)^{2\beta}dA(z)\right\}^{\frac{1}{2}} \geq k\left\{\int_D |f(z)|^2(1-|z|^2)^{2\alpha}dA(z)\right\}^{\frac{1}{2}}$$

for all $f \in L^2_a\left((1-|z|^2)^{2\alpha}dA(z)\right)$,
(2) *$M_h : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is bounded below,*
(3) *There exists a positive constant $\epsilon > 0$ such that $\left\{z \in D, (1-|z|^2)^{\beta-\alpha}|h(z)| \geq \epsilon\right\}$ is a sampling set for $\mathcal{B}^{\alpha}$.*

Moreover $\| M_h \|_{\inf, L^2_a((1-|z|^2)^{2\alpha}dA(z))\to L^2_a((1-|z|^2)^{2\beta}dA(z))} \approx \| M_h \|_{\inf,\mathcal{B}^{\alpha}\to\mathcal{B}^{\beta}}$

$$\approx \inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^{\alpha}(1-|z|^2)^{\beta-\alpha}|h(z)|\right\}.$$

Theorem 5. ([23]) *Let $\alpha > 0$. For any inner function φ, the following are equivalent.*
(1) *M_{φ} is bounded below on $L^2_a\left((1-|z|^2)^{2\alpha}dA(z)\right)$,*
(2) *There exists a positive constant $\epsilon > 0$ such that $\left\{z \in D, |\varphi(z)| \geq \epsilon\right\}$ is a sampling set for $\mathcal{B}^{\alpha}$,*
(3) $\inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^{\alpha}|\varphi(z)|\right\} > 0$,
(4) *φ is a finite product of interpolating Blaschke products,*
(5) *For some $\epsilon > 0$ and $0 < r < 1$ the area of the subset of the disc $D_{z,r} = \left\{w : \rho(z,w) < r\right\}$ where $|\varphi(w)| > \epsilon$ is comparable to the area of the whole disc $D_{z,r}$, $z \in D$.*

Theorem 6.([24]) *Let $\beta \geq \alpha > 0$ and $g \in H(D)$. Suppose that $J_g : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is bounded. Then the operator $J_g : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is bounded below if and only if there exists a* *positive constant* $\epsilon > 0$ *such that* $\left\{z \in D, (1-|z|^2)^{\beta-\alpha+1}|g'(z)| \geq \epsilon\right\}$ *is a sampling set for* $\mathcal{B}^\alpha$

Theorem 7. ([24]) *Let* $\alpha > -1$. *There is a constant* $C > 0$ *such that*

$$\int_D |f(z)|^2(1-|z|^2)^\alpha dA(z) \leq C \int_G |f(z)|^2(1-|z|^2)^\alpha dA(z)$$

for all $f \in L_a^2\left((1-|z|^2)^\alpha dA(z)\right)$ *if and only if a subset* G *of* D *satisfy the condition that there exist* $\delta > 0$ *and* $r > 0$ *such that* $\delta|D(a,r)| \leq |D(a,r) \cap G|$, *where* $|D(a,r)|$ *is the (normalized) area of* $D(a,r)$.

Theorem 8.([24]) *Let* $\alpha > 0$. *Suppose that* $g \in \mathcal{B}$. *If* $J_g : L_a^2\left((1-|z|^2)^{2\alpha} dA(z)\right) \to L_a^2\left((1-|z|^2)^{2\alpha} dA(z)\right)$ *is bounded below, then* $J_g : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ *is bounded below.*

Theorem 9.([24]) *Let* $\alpha > 0$. *Suppose that* $g \in \mathcal{B}$. *Then the following are equivalent:*

(1) J_g *is bounded below on* $L_a^2\left((1-|z|^2)^{2\alpha} dA(z)\right)$

i.e. *there is a constant* $k > 0$ *such that*

$$\left\{\int_D |J_g f(z)|^2(1-|z|^2)^{2\alpha} dA(z)\right\}^{\frac{1}{2}} \geq k\left\{\int_D |f(z)|^2(1-|z|^2)^{2\alpha} dA(z)\right\}^{\frac{1}{2}}$$

for all $f \in L_a^2\left((1-|z|^2)^{2\alpha} dA(z)\right)$,

(2) *There exists a positive constant* $\epsilon > 0$ *such that* $\left\{z \in D, (1-|z|^2)|g'(z)| \geq \epsilon\right\}$ *is a sampling set for* $\mathcal{B}^\alpha$,

(3) $\inf_{w \in D} \sup_{z \in D} (1-|\varphi_w(z)|^2)^\alpha (1-|z|^2)|g'(z)| > 0$.

Theorem 10.([24]) *Let* $\alpha > 0$. *Let* $B(z)$ *be a Blaschke product (a* H^∞ *function) of the form* $B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j}\left(\frac{z_j - z}{1-\overline{z_j}z}\right)$ *and let* $\{z_n\}$ *be its zeros, counted with their multiplicities. Then the following are equivalent:*

(1) $B(z)$ *is a finite product of interpolating Blaschke products,*

(2) $\inf_{w \in D}\left\{\sup_{z \in D}(1-|\varphi_w(z)|^2)^\alpha |B(z)|\right\} > 0$,

(3) $\inf_{w \in D}\left\{\sup_{z \in D}(1-|\varphi_w(z)|^2)^\alpha (1-|z|^2)|B'(z)|\right\} > 0$,

(4) M_B *is bounded below on* $L_a^2\left((1-|z|^2)^{2\alpha} dA(z)\right)$,

(5) M_B *is bounded below on* L_a^2,

(6) J_B *is bounded below on* $L_a^2\left((1-|z|^2)^{2\alpha} dA(z)\right)$,

(7) *There exists a positive constant* $\epsilon > 0$ *such that* $\left\{z \in D, (1-|z|^2)|B'(z)| \geq \epsilon\right\}$ *is a sampling set for* $\mathcal{B}^\alpha$.

Corollary 11.([24]) *If an infinite Blaschke product* $B(z) \in H^\infty$ *belongs to the little Bloch space* $\mathcal{B}_0$, *then* M_B *is not bounded below on* L_a^2.

Corollary 12.([24]) *Let $\alpha > 0$. Let $B(z)$ be an infinite Blaschke product (a H^∞ function) of the form* $B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j}\left(\frac{z_j - z}{1-\overline{z_j}z}\right)$ *and let $\{z_n\}$ be its zeros,* $\inf_{n\neq m} \rho(z_n, z_m) > 0$.
Then the following are equivalent:
(1) *$B(z)$ is an interpolating Blaschke product,*
(2) $\inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^\alpha |B(z)|\right\} > 0$,
(3) $\inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^\alpha (1-|z|^2)|B'(z)|\right\} > 0$.
Moreover

$$\inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^\alpha |B(z)|\right\} \sim \inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^\alpha (1-|z|^2)|B'(z)|\right\}$$

$$\sim \inf_n \prod_{j\neq n} \rho(z_j, z_n) = \inf_n (1-|z_n|^2)\left|B'(z_n)\right|.$$

Theorem 13.([24]) *Let $\alpha > 0$ and $0 < p < \infty$. Let $H(z)$ be an infinite product of the form* $H(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j}\left(\frac{z_j - z}{1-\overline{z_j}z}\right)\left(2 - \frac{|z_j|}{z_j}\left(\frac{z_j - z}{1-\overline{z_j}z}\right)\right)$ *that belongs to the Bloch space $\mathcal{B}$ and let $\{z_n\}$ be its zeros. Then the following are equivalent:*
(1) *$\Gamma = \{z_n\}$ is an interpolation sequence for $\mathcal{B}^\alpha$,*
(2) *$\Gamma = \{z_n\}$ is an interpolation sequence for $L^p_a\left((1-|z|^2)^{2\alpha} dA(z)\right)$,*
(3) *$\Gamma = \{z_n\}$ is an interpolation sequence for $L^p_a(D)$,*
(4) $\inf_j (1-|z_j|^2)|H'(z_j)| > 0$.

Theorem 14.([24]) *Let $\alpha > 0$ and $0 < p < \infty$. Let $H(z)$ be an infinite product of the form* $H(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j}\left(\frac{z_j - z}{1-\overline{z_j}z}\right)\left(2 - \frac{|z_j|}{z_j}\left(\frac{z_j - z}{1-\overline{z_j}z}\right)\right)$ *that belongs to the Bloch space $\mathcal{B}$ and let $\{z_n\}$ be its zeros that there exists a $r < 1$ such that $D = \cup_k D(z_k, r)$. Then the following are equivalent:*
(1) *$\Gamma = \{z_n\}$ is an interpolation sequence for $\mathcal{B}^\alpha$,*
(2) *$\Gamma = \{z_n\}$ is an interpolation sequence for $L^2_a\left((1-|z|^2)^{2\alpha} dA(z)\right)$,*
(3) $\inf_{w\in D.}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^\alpha (1-|z|^2)|H'(z)|\right\} > 0$ *and* $\inf_{j\neq k} \rho(z_j, z_k) > 0$,
(4) *$\Gamma = \{z_n\}$ is an interpolation sequence for $L^p_a(D)$,*
(5) $\inf_j (1-|z_j|^2)|H'(z_j)| > 0$.

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Commutators on mixed invariant subspaces
(混合不変部分空間上の可換子について)

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This is a joint work with Kei Ji Izuchi and Kou Hei Izuchi.

Abstract

We study the structure of nontrivial closed subspaces N of Hardy space H^2 with $T_zN \subset N$ and $T_w^*N \subset N$, which called mixed invariant subspaces under T_z and T_w^*. If $[T_z, T_w^*] = 0$, we may describe N completely.

1 INTRODUCTION

Let $\mathbb{D}^2$ be the bidisk and Γ be the distinguished boundary of $\mathbb{D}^2$. We use z, w as variables over Γ. Let $L^2 = L^2(\Gamma^2)$ and $H^2 = H^2(\Gamma^2)$ be the usual Lebesgue and Hardy spaces over Γ. We denote by $H^2(z)$ and $H^2(w)$ the z and w variable Hardy spaces, respectively. For a closed subspace E of H^2, we denote by P_E the orthogonal projection from L^2 onto E.

For $\phi \in L^\infty(\Gamma)$, we define the Toeplitz operator T_ϕ on H^2 by $T_\phi f = P_{H^2}(\phi f)$. A closed subspace I of H^2 is called invariant if $T_zI \subset I$ and $T_wI \subset I$. For an invariant subspace I of H^2 with $I \neq 0$ and $I \neq H^2$, we have $T_z^*(H^2 \ominus I) \subset H^2 \ominus I$ and $T_w^*(H^2 \ominus I) \subset H^2 \ominus I$, that is, $H^2 \ominus I$ is a backward shift invariant subspace of H^2. We define the operators R_z and R_w on I by $R_z = T_z|_I$ and $R_w = T_w|_I$, and the compression operators S_z and S_w on $H^2 \ominus I$ by $S_z = P_{H^2\ominus I}T_z|_{H^2\ominus I}$ and $S_w = P_{H^2\ominus I}T_w|_{H^2\ominus I}$.

In the one variable case, the Beurling theorem says that if I is an invariant subspace of H^2, then $I = \phi H^2$ for an inner function ϕ. In two variables case, the structure of invariatn subspaces (backward shift invariant subspaces) is complicated.

We denote by $[T_1, T_2]$ the commutator of operator T_1 and T_2, that is, $[T_1, T_2] = T_1T_2 - T_2T_1$. It is known that $[T_z, T_w] = 0$ on H^2, so we have $[R_z, R_w] = 0$ on I and $[S_z, S_w] = 0$ on $H^2 \ominus I$. A motivation of this study is back to Mandrekar's work [4]. He proved that $[R_z, R_w^*] = 0$ if and only if $I = \phi H^2$ for an inner function ϕ so Beurling type invariant subspaces are characterized by the operator equation $[R_z, R_w^*] = 0$. In the backward shift invariant case, Izuchi, Nakazi, and Seto [3] gave a complete description of I satisfying $[S_z, S_w^*] = 0$, and Izuchi and Izuchi [1] gave a complete description of I satisfying $rank[S_z, S_w^*] = 1$.

Here, we introduce a concept of "mixed invariant subspaces" and study similar subjects.

Definition 1.1 *A closed subspace N of H^2 with $N \neq \{0\}$ and $N \neq H^2$ is said to be mixed invariant under T_z and T_w^* if*

$$T_zN \subset N \text{ and } T_w^*N \subset N.$$

Let N be a mixed invariant subspaces under T_z and T_w^*, and let M be the orthgonal complement of N in H^2. Then it is easy to check that M is a mixed invariant subspace under T_z^* and T_w. We define the operators V_z and V_w on N by $V_z = T_z|_N$, $V_w = P_N T_w|_N$, and U_z and U_w on M by $U_z = P_M T_z|_M$, $U_w = T_w|_M$.

2 RESULTS

We can describe N satisfying $[V_z, V_w] = 0$.

Theorem 2.1 *Let N be a mixed invariant subspace of H^2 under T_z and T_w^*. Then $[V_z, V_w] = 0$ if and only if $T_w^*(N \ominus zN) \subset N \ominus zN$. In this case, either $N = q_1(z)H^2$, where $q_1(z)$ is a one variable nonconstant inner function, or $N = q_1(z)(H^2 \ominus q_2(w)H^2)$, where $q_1(z)$ and $q_2(w)$ are inner function and $q_2(w)$ is nonconstant.*

We gave the form of N if $[V_z, V_w] = 0$. Next, we consider the case that commutator is not 0. For example, commutator is finite rank. But, we still don't know this case. We changed a viewpoint. By Wold decomposition theorem,

$$N = \sum_{n=0}^{\infty} \otimes (N \ominus zN)z^n = (N \ominus zN)H^2(z).$$

So we consider about the case that $\dim(N \ominus zN) = 1$. Under the assumption, we can describe the form of N. In this case, we have $rank[V_z, V_w] \leq 1$.

Theorem 2.2 *Let N be a mixed invariant subspaces of H^2 under T_z and T_w^*. Then* $\dim(N \ominus zN) = 1$ *if and only if there exist two one variable functions $a(z)$, $b(z)$ in $H^\infty(z)$ satisfying the following conditions;*

(1) $|a(z)| < 1$ *a.e. on* Γ,

(2) $|a(z)|^2 + |b(z)|^2 = 1$ *a.e. on* Γ,

(3) $N = GH^2(z)$, *where* $G = \dfrac{b(z)}{1 - wa(z)}$.

In this case, $[V_z, V_w] = 0$ if and only if $a(z)$ is constant, and $rank[V_z, V_w] = 1$ if and only if $a(z)$ is nonconstant.

3 RANK INEQUALITY

In [1],Izuchi and Izuchi had a following conjecture.

Conjecture 3.1 *Let I be a invariant subspace of H^2 with $I \neq \{0\}$ and $I \neq H^2$, then*

$$rank[S_z, S_w^*] - 1 \leq rank[R_z, R_w^*] \leq rank[S_z, S_w^*] + 1.$$

In [2], Izuchi and Izuchi proved that the conjecture holds under an additional condition. We considered the conjecture of mixed invariant case, and proved under an additional condition.

Theorem 3.2 *Let N be a mixed invariant subspace of H^2 under T_z and T_w^*. If* $\dim(N \ominus zN) < \infty$*, then*

$$rank[U_z, U_w] - 1 \leq rank[V_z, V_w] \leq rank[U_z, U_w].$$

Example 3.3

(1) Let $N = zH^2(z) \oplus wz^2 H^2(z)$. In this case,

$$rank[V_z, V_w] = 1 < 2 = rank[U_z, U_w].$$

(2) Let $N = H^2(z) \oplus wzH^2(z) \oplus w^2 z^2 H^2(z)$. In this case,

$$rank[V_z, V_w] = 2 = rank[U_z, U_w].$$

(3) Let $N = H^2(z) \oplus wzH^2(z) \oplus w^2 z^2 H^2$. In this case,

$$rank[V_z, V_w] = 2 > 1 = rank[U_z, U_w].$$

Note that $\dim(N \ominus zN) = \infty$.

We have the following conjecture for every mixed invariant subspace N of H^2 under T_z and T_w^*.

Conjecture 3.4

$$rank[U_z, U_w] - 1 \leq rank[V_z, V_w] \leq rank[U_z, U_w] + 1.$$

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Asymptotic properties of Bernoulli numbers and the numbers of real roots of Bernoulli polynomials

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Abstract

Let B_p and $B_p(x)$ $(p=1,2,\cdots)$ be the p-th Bernoulli numbers and the p-th Bernoulli polynomials respectively, and let $N(p)$ be the number of real roots of $B_p(x)=0$.

It is proved that we have asymtotic properties such as

1) $\lim\limits_{p\to\infty} \dfrac{B_p^{\frac{1}{2p}}}{p} = \dfrac{1}{\pi e}$ (Theorem 1),

2) $\lim\limits_{p\to\infty} \dfrac{N(p)}{p} = \dfrac{2}{\pi e}$ (Theorem 8)

The property 1) is based upon Euler's formula;

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{(2\pi)^{2p}}{(2p)!} \cdot \frac{B_p}{2}$$

and 1) plays a fundamental role in proving 2). Namely, 1) implies 2) in the process of Theorems 2, 3, 5, 6 and 7.

1 準備と基本事項

Abstractで述べた性質1)から性質2)を導く過程はいくつかの段階にわかれる．各段階の証明はすべて省略するが，重要と思われる段階を定理の形式（Theorems 1〜7）で述べることにする．Bernoulli数及びBernoulli多項式の定義とその基本的性質については[2]を参照と云うことですますことにする．

Bernoulli数 B_p $(p=1,2,\cdots)$ は正の有理数で，例えば最初の5項は $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}$, $B_5=\frac{5}{66}$ であるが，かなりの速さで発散する．Bernoulli多項式 $B_p(x)$ $(p=1,2,\cdots)$ は p 次の多項式で，その係数はBernoulli数 B_p に関連していて，次のようになる．

$$(0)\quad B_p(x)=x^p-\frac{p}{2}x^{p-1}+\sum_{k=1}^{[\frac{p}{2}]}\binom{p}{2k}(-1)^{k-1}B_k\,x^{p-2k}$$

Bernoulli多項式の基本的性質は次の三つである．

$$(1)\quad B_p(x+1)=B_p(x)+p\,x^{p-1},\quad p=1,2,\cdots,\ -\infty<x<+\infty$$

$$(2)\quad B_p(1-x)=(-1)^pB_p(x),\qquad〃$$

$$(3)\quad \frac{d}{dx}B_p(x)=p\,B_{p-1}(x).\qquad〃$$

特に(1)は，以下の各段階でしばしば利用されるが，例えば次の形で利用される．

$$(1)\quad B_p(x+n)=B_p(x)+p\left[x^{p-1}+(1+x)^{p-1}+\cdots+(n-1+x)^{p-1}\right],\quad n=1,2,\cdots$$

この形から，区間 $[n,n+1]$ $(n=1,2,\cdots)$ 上での $B_p(x)$ の状態は区間 $[0,1]$ 上での $B_p(t)$ $(0\leq t\leq 1)$ と $p\left[t^{p-1}+(1+t)^{p-1}+\cdots+(n-1+t)^{p-1}\right]$ $(0\leq t\leq 1)$ との和となっていることがわかる．この事が応用される．

次の(4)と(5)は(0)と(2)から出る．

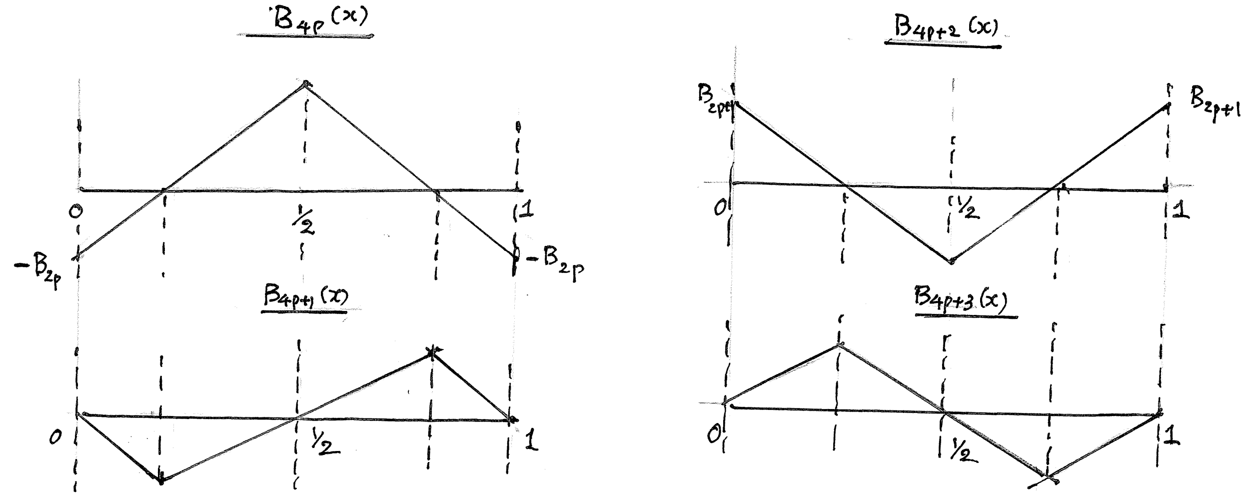
(4) $B_p(x)$ は $x=\frac{1}{2}$ にかんして，p が偶数ならば偶関数であり，p が奇数ならば奇関数である．

$$(5)\quad \text{a)}\ B_{2p}(0)=B_p(1)=(-1)^{p-1}B_p,\quad p=1,2,\cdots$$

$$\text{b)}\ B_{2p+1}(0)=B_{2p+1}\left(\tfrac{1}{2}\right)=B_{2p+1}(1)=0,\quad p=1,2,\cdots$$

Bernoulli 多項式 $B_p(x)$ の区間 $[0,1]$ での増減状況はある種の規則性を持っている。このことは p にかんする帰納法で容易にわかる。

(6) 区間 $[0,1]$ 上での増減状況の概形は下図のようである。



2. $B_p(x)$ の Fourier 級数展開と Euler の式

Euler の式については，二三の別証が知られているが，これが Bernoulli 多項式の Fourier 級数展開から自然に出ることが余り知られていないように思うので，こゝでそれを説明する。

Bernoulli 多項式 $B_p(x)$ を区間 $[0,1]$ 上に制限したものを周期1で全実数直線上に拡張して得られる周期関数を $\widetilde{B}_p(x)$ の記号で表わすことにする。$\widetilde{B}_p(x)$ は convolution（たたみ込み）にかんしてよく適応していて，次のことが成立することが容易にわかる。[1] を参照

(7) a) $\left(\widetilde{B}_p * \widetilde{B}_1\right)(x) = \dfrac{-1}{p+1}\widetilde{B}_{p+1}(x)$,

b) $\left(\underbrace{\widetilde{B}_1 * \cdots\cdots * \widetilde{B}_1}_{p\text{個}}\right)(x) = \dfrac{(-1)^{p-1}}{p!}\widetilde{B}_p(x)$

簡単な計算により，$\widetilde{B}_1(x)$ の Fourier 級数は $\dfrac{i}{2\pi}\sum_{\substack{-\infty<n<+\infty \\ n\neq 0}}\dfrac{1}{n}e^{i2n\pi x}$ であることがわかるから，(7) より $\widetilde{B}_p(x)$ の Fourier 級数が自動的に導かれる。

$$\widetilde{B}_p(x) = (-1)^{p-1}p!\left(\frac{i}{2\pi}\right)^p \sum_{\substack{-\infty<n<+\infty \\ n\neq 0}} \frac{1}{n^p} e^{i2n\pi x}$$

$\widetilde{B}_p(x)$ を区間 $[0,1]$ 上に戻せば，$B_p(x)$ の区間 $[0,1]$ 上の Fourier 級数展開が

得られる.

(8) (Complex form). $0 \leqq x \leqq 1$ と $p=1,2,\cdots$ に対して,

$$B_p(x) = (-1)^{p-1} p! \left(\frac{i}{2\pi}\right)^p \sum_{\substack{-\infty<n<+\infty \\ n \neq 0}} \frac{1}{n^p} e^{i2n\pi x}$$

収束は, $p \geqq 2$ のとき $[0,1]$ 上での一様収束, $p=1$ のとき区間 $[\varepsilon, 1-\varepsilon]$ ($\forall\, 0<\varepsilon<\frac{1}{2}$) 上での一様収束である.

(8)を変形すれば,

(9) (Real form). $0 \leqq x \leqq 1$, $p=1,2,\cdots$ に対して

a) $$B_{2p}(x) = \frac{(-1)^{p-1} 2 \cdot (2p)!}{(2\pi)^{2p}} \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \cos(2n\pi x)$$

b) $$B_{2p+1}(x) = \frac{(-1)^{p-1} \cdot 2 \cdot (2p+1)!}{(2\pi)^{2p+1}} \sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} \sin(2n\pi x)$$

この (9), a) において, $x=0$ とすれば Euler の式 (同値な式) が得られる. Complex form (8) に Parseval の等式を適用しても同様である.

(10) $$B_p = \frac{2 \cdot (2p)!}{(2\pi)^{2p}} \sum_{n=1}^{\infty} \frac{1}{n^{2p}}$$

また, (9), a) を $x=\frac{1}{2}$ 及び $x=\frac{1}{4}$ で評価すれば次が得られる

(11) a) $$B_{2p}\left(\tfrac{1}{2}\right) = (-1)^p \left(1 - \frac{1}{2^{2p-1}}\right) B_p,$$

b) $$B_{2p}\left(\tfrac{1}{4}\right) = \frac{1}{2^{2p}} B_{2p}\left(\tfrac{1}{2}\right)$$

3. B_p ($p=1,2,\cdots$) の漸近的性質と数列 M_p ($p=1,2,\cdots$) の導入

(10) より, Bernoulli 数列 B_p ($p=1,2,\cdots$) の漸近的性質が容易に示される.

Theorem 1 数列 $\frac{1}{p} B_p^{\frac{1}{2p}}$ ($p=1,2,\cdots$) は単調減少(狭義)であり, $\frac{1}{\pi e}$ に収束する.

Corollary $$\left(\frac{p}{\pi e}\right)^{2p} < B_p < \left(\frac{p}{\sqrt{6}}\right)^{2p} \quad \forall p = 2,3,\cdots$$

このことから, B_p ($p=1,2,\cdots$) がかなりの速さで発散する数列であることもわかる. 例えば $\lim_{p\to\infty} \frac{B_p}{p^{\delta p}} = +\infty$ ($\forall\, 0<\delta<2$).

M_pの定義　以下で（Theorem5の証明段階で），区間$[0,1]$上で$B_p(x)$の最小値が有効な働きをすることがわかる．よって，

$$M_p = \left| \min_{0 \leq x \leq 1} B_p(x) \right| > 0$$

と定義する．したがって，$-M_p < 0$が$B_p(x)$の区間$[0,1]$上での最小値である．

次が成立する．（(11)のa)，b)を利用する）

Theorem 2　$p=1,2,\cdots$に対して，

a)　$M_{4p} = B_{2p}$，　$M_{4p+2} = \left(1 - \frac{1}{2^{4p+1}}\right) B_{2p+1}$

b)　$\frac{2p+1}{16} \cdot B_p < M_{2p+1} < \frac{2p+1}{4} \cdot B_p$

Theorem 1の結果とTheorem 2より，次が成立する．

Theorem 3　$\displaystyle \lim_{p\to\infty} \frac{1}{p} M_p^{\frac{1}{p}} = \frac{1}{2\pi e}$

4.　数列 m_p $(p=1,2,\cdots)$ の導入

(6)の図で示したように，区間$[0,1]$は，pが偶数ならば$B_p(x)=0$の2個の実根をふくみ，pが奇数（ただし$p\neq1$）ならば$B_p(x)=0$の3個の実根（0と$\frac{1}{2}$と1）をふくんでいる．$x=\frac{1}{2}$にかんするBernoulli多項式の対称性を考慮すれば開半直線$(1,+\infty)$にふくまれる実根の個数が問題となる．次に，半直線$(1,+\infty)$を区間$(n,n+1]$，$n=1,2,\cdots$に分割して，各区間$(n,n+1]$にふくまれる実根の個数を問題とすることになる．ただ，次の事に注意する．あるm $(m=1,2\cdots)$に対して，もし$B_p(x)$が区間$(m,m+1]$上で正数値だけを取るならば$B_p(x)$は半直線$(m,+\infty)$上で正となる．また，あるmに対して区間$(m,m+1]$上で$B_p(x)$が負又は0の値を取るならば，$1 \leqq n \leqq m$なるすべてのnに対して，区間$(n,n+1]$上で$B_p(x)$は負又は0の値を取らねばならないことが示される．$B_p(x)$に対して，0又は自然数m_pを次のように定義する

m_pの定義　区間$(n,n+1]$ $(n=1,2,\cdots)$のうちで，その上で$B_p(x)$が負又は0の値を取るような区間の個数をm_pと定義する．

数列m_p $(p=1,2,\cdots)$と数列M_p $(p=1,2,\cdots)$との関連を求めることが出来る．それには，次の定理4で示す不等式が本質的である．$x>1$に対して，その整数部分を$[x]$とおくと，$t = x - [x]$，$0 \leqq t < 1$が分数部分となる．

次の不等式が成立する.

Theorem 4　$x>1$ と $p=2,3,\cdots$ に対して，次が成立する

$$B_p(x-[x])+(x-1)^p < B_p(x) < B_p(x-[x])+x^p$$

これを利用して，次が示される.

Theorem 5　$M_p^{\frac{1}{p}}-2 < m_p < M_p^{\frac{1}{p}}+1$, $p=1,2,\cdots$

… 定理3の結果を適用して,

Theorem 6　$\lim_{p\to\infty}\frac{1}{p}m_p=\frac{1}{2\pi e}$

5. $B_p(x)=0$ の実根の個数 $N(p)$ と m_p との関連

$N(p)$ が m_p を用いて表示出来ることを示すことが最終段階となる.

$m_p\geqq 1$ とする. m_p 個の区間 $(n,n+1]$, $1\leqq n\leqq m_p$ に対して，各区間上での $B_p(x)$ の増減状況をやゝ詳しく調べることが可能であって，各区間は2個の実根をふくむが，特別の場合 ($n=1$, $n=m_p$ のとき) 1個又は3個の実根をふくむことが示される. 次が成立する.

Theorem 7　$p\equiv i \pmod 4$, $i=0,1,2,3$ とする. このとき， $i=1,2$, 又は 3 ならば $N(p)=4m_p+i$ であり, $i=0$ ならば $N(p)=4m_p$ 又は $4m_p+4$ である.

我々の目的とした $N(p)$ の漸近的性質は定理6と7との結果である

Theorem 8　$\lim_{p\to\infty}\frac{1}{p}N(p)=\frac{2}{\pi e}$

以上

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Explicit and direct representations of the solutions of nonlinear simultaneous equations

Masato Yamada and Saburou Saitoh

Abstract

In this note we shall introduce practical, numerical and explicit representations of inverse mappings of n-dimensional mappings (of the solutions of n-nonlinear simultaneous equations). We derive those concrete formulas from very general ideas for the representation of the inverse functions.

Keywords: Inverse function, inverse mapping, integral transform, integral representation, Green-Stokes formula, singular integral, fundamental solution, reproducing kernel, Sobolev space, non-linear mapping, general non-linear equation.

Mathematics Subject Classification (2000): Primary 93B30, 93C10, 35A35

1 Introduction

The 2nd author of this paper considered that for any mapping ϕ from an arbitrary abstract set into an arbitrary set, he tried to consider the representation of the inversion ϕ^{-1} in terms of the direct mapping ϕ and he obtained some simple concrete formulas from some general ideas in ([1]). Here, from its general ideas, we shall introduce practical representation formulas of some general functions. Here, we shall give furthermore some general methods and ideas for the inversion formulas for some general non-linear mappings. We shall first state the principles for our methods for the representations of inverses of non-linear mappings based on ([1]):

We shall consider some representation of the inversion ϕ^{-1} in terms of some integral form - at this moment, we shall need a natural assumption for the mapping ϕ . Then, we shall transform the integral representation by the mapping ϕ to the original space that is the defined domain of the mapping ϕ. Then, we will be able to obtain the representation of the inverse ϕ^{-1} in terms of the direct mapping ϕ. In [1], we considered the representation of the inverse ϕ^{-1} in some reproducing kernel Hilbert spaces, and in [4], we considered the representations of the inverse ϕ^{-1} for a very concrete situation and we gave a very fundamental representation of the inverse for some general functions on 1 dimensional spaces.

Indeed, note that

$$K(y_1, y_2) = \frac{1}{2}e^{-|y_1-y_2|} \qquad y_1, \quad y_2 \in [A, B] \tag{1}$$

is the reproducing kernel for the Sobolev Hilbert space H_K whose members are real-valued and absolutely continuous functions on $[A, B]$ and whose inner product is given by

$$(f_1, f_2)_{H_K} = \int_A^B (f_1'(y)f_2'(y) + f_1(y)f_2(y))dy + f_1(A)f_2(A) + f_1(B)f_2(B) \tag{2}$$

([2]).

For a function $y = f(x)$ that is of C^1 class and a strictly increasing function and $f'(x)$ is not vanishing on $[a, b]$ $(f(a) = A, f(b) = B)$, of course, the inverse function $f^{-1}(y)$ is a single-valued function and it belongs to the space H_K and from the reproducing property, we obtain the representation, for any $y_0 \in [f(a), f(b)]$

$$\begin{aligned} f^{-1}(y_0) &= \left(f^{-1}(\cdot), K(\cdot, y_0)\right)_{H_K} \\ &= \int_{f(a)}^{f(b)} ((f^{-1})'(y)K_y(y, y_0) + f^{-1}(y)K(y, y_0))dy \\ &\quad + aK(f(a), y_0) + bK(f(b), y_0). \end{aligned} \tag{3}$$

From this representation, we obtained in ([4]) the very simple representation

$$f^{-1}(y_0) = \frac{a+b}{2} + \frac{1}{2}\int_a^b \operatorname{sign}(y_0 - f(x))dx. \tag{4}$$

Furthermore, by using the several reproducing kernel Hilbert spaces from [2] as in (3), we calculated similarly with the related assumptions, however, surprisingly enough, we obtain the same formula (4). For the formula (4), we note directly that we do not need any smoothness assumptions for the function $f(x)$, indeed, we need only the strictly increasing assumption. The assumption of integrability does not, even, need for the formula (4).

Now, we would like to obtain some multi-dimensional versions. At this moment, it seems that we can not find some simple representations as in (3) by some concrete known reproducing kernels for some general domains, and indeed, we know the reproducing kernels only for special domains and for special reproducing kernel Hilbert spaces.

In order to consider some general integral representations for some general functions, we shall recall the fundamental facts:

We can represent a function f in terms of the delta function δ in the form

$$f(q) = \int_D f(p)\delta(p - q)dp \tag{5}$$

in some domain, symbolically. Meanwhile, a fundamental solution $G(p-q)$ for some linear differential operator L is given by the equation, symbolically

$$LG(p - q) = \delta(p - q). \tag{6}$$

So, from (5) we obtain the representation

$$f(q) = \int_D f(p)LG(p - q)dp. \tag{7}$$

Then, we can obtain the representation symbolically, by using the Green-Stokes formula, for some adjoint operator L^* for L,

$$f(q)=\int_D L^*f(p)G(p-q)dp+\text{some}\quad\text{boundary}\quad\text{integrals}. \tag{8}$$

We shall firstly use this type representation. In this approach, we will meet the singular integral representation in the first term of (8), however, if $G(p-q)$ is integrable, then by a simple regularization for $G(p-q)$ we will be able to realize the representation in numerical treatments. In the separate paper [3] we discussed the natural regularization in the form, for example, for the singularity

$$\frac{1}{(|x-y|)^\alpha},$$

we consider the regularization

$$\frac{1}{(|x-y|+\delta)^\alpha}$$

for a small δ and we considered their error estimates.

We are interested in some very concrete results that may be realized by computers. So, we considered very concrete cases in the 2 dimensional spaces. It seems that the results will depend on dimensions, domains and functions spaces dealing with.

In [6], we considered the following typical problem:

Let $D\subset\mathbf{R}^2$ be a bounded domain with a finite number of piecewise C^1 class boundary components. Let f be a one-to-one C^1 class mapping from $\bar{D}$ into $\mathbf{R}^2$ and we assume that its Jacobian $J(x)$ is positive on D. We shall represent f as follows:

$$\begin{aligned} y_1&=f_1(x)=f_1(x_1,x_2)\\ y_2&=f_2(x)=f_2(x_1,x_2) \end{aligned} \tag{9}$$

and the inverse mapping f^{-1} of f as follows:

$$\begin{aligned} x_1&=(f^{-1})_1(y)=(f^{-1})_1(y_1,y_2)\\ x_2&=(f^{-1})_2(y)=(f^{-1})_2(y_1,y_2). \end{aligned} \tag{10}$$

Then, we represented

$$\begin{pmatrix}(f^{-1})_1(y^*)\\(f^{-1})_2(y^*)\end{pmatrix} \tag{11}$$

in terms of the direct mapping (9).

Of course, we are interested in some numerical and practical solutions of the non-linear simultaneous equations (9) and we obtained

Proposition 1 *([6]). For the mappings (9) and (10) with (11), we obtain the representation, for any $y^*=(y_1^*,y_2^*)\in f(D)$,*

$$\begin{aligned}\begin{pmatrix}(f^{-1})_1(y^*)\\(f^{-1})_2(y^*)\end{pmatrix}&=\frac{1}{2\pi}\oint_{\partial D}\begin{pmatrix}x_1\\x_2\end{pmatrix}d\,\mathrm{Arctan}\,\frac{f_2(x)-y_2^*}{f_1(x)-y_1^*}\\ &-\frac{1}{2\pi}\int\int_D\frac{1}{|f(x)-y^*|^2}\,\mathrm{adj}\,J(x)\begin{pmatrix}f_1(x)-y_1^*\\f_2(x)-y_2^*\end{pmatrix}dx_1dx_2. \end{aligned}\tag{12}$$

We can give the natural version for the 3 dimensional case by using the well-known Poisson integral formula and however, surprisingly enough we can give some unified and natural inversion formulas for the general dimensions that are better than the formula derived from the Poisson integral formula.

2 n-dimensional formulas

We shall give the very beautiful representation

Theorem 2 Let D be a bounded domain in $\mathbf{R}^n$ with a finite number ∂D of C^1 class boundary commponents. Let f be a C^1 class real-valued function on $\bar{D}$. For any $\hat{x} \in D$ and for any $n \in \mathbf{N}$ we have the representation

$$f(\hat{x}) = -c_n\big(df(x), dG_n(x-\hat{x})\big) + c_n \int_{\partial D} f(x) * dG_n(x-\hat{x}) \tag{13}$$

Here, for $n \leq 2, c_n = 1$ and for $n \geq 3, c_n = n-2$. $*$ is the Hodge star operator, G_n the fundamental solution of the Laplacian $\Delta_n = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, and $(\ ,\ )$ the inner product of the vector space $A^k(D)$ comprising of the k order differential forms over D with finite L^2 norms that is

$$(\omega, \eta) = \int_D \omega \wedge *\eta = \int_D \eta \wedge *\omega \quad (\omega, \eta \in A^k(D)).$$

Lemma 1 Let $U_\epsilon(0)$ be an ϵ neighbourhood with centre 0, then

$$\int_{\partial U_\epsilon(0)} *dG_n(x) = \frac{1}{c_n}.$$

Proof of Theorem 2.
Let $U_\epsilon(\hat{x})$ be a neighbourhood contained in D. Then, for $G_n(x-\hat{x}) \in C^\infty(D \setminus U_\epsilon(\hat{x}))$ and for $f \in C^1(D)$, $f(x) * dG_n(x-\hat{x}) \in A^1(D \setminus U_\epsilon(\hat{x}))$ that is a C^1 class differential. Hence, on $D \setminus U_\epsilon(\hat{x})$, for $f(x) * dG_n(x-\hat{x})$ we apply the Green-Stokes formula and we have

$$\begin{aligned}&\int_{D \setminus U_\epsilon(\hat{x})} d\{f(x) * dG_n(x-\hat{x})\} \\ &= \int_{\partial D} f(x) * dG_n(x-\hat{x}) - \int_{\partial U_\epsilon(\hat{x})} f(x) * dG_n(x-\hat{x}).\end{aligned}$$

Let δ be the Dirac distribution and ω_v be $dx_1 \wedge \cdots \wedge dx_n$. Then, by $d*dG_n(x-\hat{x}) = \Delta_n G_n(x-\hat{x})\omega_v = \delta(x-\hat{x})\omega_v$, we have

$$\begin{aligned} d\{f(x) * dG_n(x-\hat{x})\} &= df(x) \wedge *dG_n(x-\hat{x}) + (-1)^0 f(x) d * dG_n(x-\hat{x}) \\ &= df(x) \wedge *dG_n(x-\hat{x}) + \delta(x-\hat{x}) f(x)\omega_v. \end{aligned}$$

Hence,
$$\lim_{\epsilon\to 0}\int_{D\setminus U_\epsilon(\hat{x})} d\{f(x) * dG_n(x-\hat{x})\} = \big(df(x), dG_n(x-\hat{x})\big).$$
From the polar coordinate representation $\phi^*(f(x)*dG_n(x-\hat{x})) = \phi^* f(x)\phi^* * dG_n(x-\hat{x}) = f(\hat{x}+\epsilon\tilde{\phi}(\theta))\phi^* * dG_n(x-\hat{x})$,

$$\begin{aligned}\int_{\partial U_\epsilon(\hat{x})} f(x) * dG_n(x-\hat{x}) &= \int_{[0,\pi]\times\cdots\times[0,\pi]\times[0,2\pi]} f(\hat{x}+\epsilon\tilde{\phi}(\theta))\phi^* dG_n(x-\hat{x}) \\ &= \frac{f(\hat{x})}{c_n} \quad (\epsilon\to 0).\end{aligned}$$

We thus obtain the desired representation.

Theorem 3 In the situation of Theorem 2 and we assume furthermore that f is a sense preserving C^1 class function on $\bar{D}$ in $\mathbf{R}^n$ with a single-valued inverse. Then, for $\hat{y}\in f(D)$, we obtain the representation

$$f_i^{-1}(y_0) = -\int_D dx_i \wedge f^* * dG_n(y-y_0) + \int_{\partial D} x_i f^* * dG_n(y-y_0).$$
Here, f_i^{-1} denotes the i commponent of f^{-1}.

Proof. For the function f^{-1} on $f(\bar{D})$, we use the representation in Theorem 2 and we use the transform of the representation by f. Then, by using the formulas $f^* df_i^{-1}(y) = dx_i$, and $f^*(\omega,\eta) = \big(f^*\omega, f^*\eta\big)$, we obtain the desired representation.

In particular, for $n=1$, we obtain (4), directly.

For $n=2$, we obtain (13) and this formula may be represented as follows, from our general formula:

For any $\hat{y}\in f(D)$, we have

$$f_i^{-1}(\hat{y}) = \frac{1}{2\pi}\left(\int_{\partial D} x_i d\theta_i - \int_D dx_i\wedge d\theta_i\right) \qquad i=1,2.$$

Here, $\theta_1 = \operatorname{Arctan}\frac{f_1(x)-\hat{y}_1}{f_2(x)-\hat{y}_2}, \theta_2 = -\operatorname{Arctan}\frac{f_2(x)-\hat{y}_2}{f_1(x)-\hat{y}_1}$. In particular, furthermore, when D is a convex domain, we have the representation

$$\begin{aligned}f_i^{-1}(\hat{y}) = \frac{\hat{x}_i^{min}+\hat{x}_i^{max}}{2} &+ \frac{1}{2\pi}\left(\int_{\partial D}\theta_i dx_i\right. \\ &\left.- \int_D dx_i\wedge d\theta_i\right) \qquad i=1,2.\end{aligned}$$

Here, $\hat{x}_i^{min}$ and $\hat{x}_i^{max}$ are determined by $\hat{y}$ as the two points of ∂D ([6]).

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Some recent results on Gao's constants

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バナッハ空間 X の幾何学的性質に関連して，種々の幾何学的定数が知られている (cf.[5,13])．James 定数 $J(X)$ と Schäffer 定数 $S(X)$ は空間の non-squareness の度合いを表し，von Neumann-Jordan (NJ-) 定数 $C_{NJ}(X)$ はヒルベルト空間との類似度を表す．$J(X)<2$, $S(X)>1$, $C_{NJ}(X)<2$ は同値であり，これらは uniform non-squareness を特徴づける．また，$C_{NJ}(X)=1$ はヒルベルト空間を特徴づける．これらの定数は空間の２次元的性質を記述するが，n 次元的性質を記述するためには n 次の (NJ-) 定数 $C_{NJ}^{(n)}(X)$, James 定数 $J_n(X)$, Schäffer 定数 $S_n(X)$ 等が必要と思われる (cf.[7,8])．

本講演では，Gao [2] の導入した幾何学的定数 $E(X)$, $f(X)$ について最近の結果を紹介する．更に，n 次の幾何学的定数 $E^{(n)}(X)$, $f^{(n)}(X)$ を導入し，それらの定数と n 次の幾何学的性質との関係を考察する．

以下，$\dim X\geq 2$ とし，B_X は閉単位球，X^* は双対空間を表す．$J(X)$, $S(X)$, $C_{NJ}(X)$, Gao の定数 $E(X)$, $f(X)$ は次のように定義される：

$$J(X)=\sup\{\min(\|x+y\|,\|x-y\|):\ \|x\|=\|y\|=1\},$$

$$S(X)=\inf\{\max(\|x+y\|,\|x-y\|):\ \|x\|=\|y\|=1\},$$

$$C_{NJ}(X)=\sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)}:\ x,y\in B_X,\ (x,y)\neq(0,0)\right\},$$

$$E(X)=\sup\left\{\|x+y\|^2+\|x-y\|^2:\ \|x\|=\|y\|=1\right\},$$

$$f(X)=\inf\left\{\|x+y\|^2+\|x-y\|^2:\ \|x\|=\|y\|=1\right\}.$$

なお，$J(X)$, $E(X)$ の定義で，$x,y\in B_X$ としても同値であることが知られている．

次の結果は，容易に示される (cf.[13,15])．

Theorem 1. For any Banach space X, we have

(1)　$4\leq 2J(X)^2\leq\min\{E(X),E(X^*)\}\leq\max\{E(X),E(X^*)\}\leq 4C_{NJ}(X)\leq 8.$

(2)　$2\leq 4/C_{NJ}(X)\leq f(X)\leq 2S(X)^2=8/J(X)^2\leq 4.$

Corollary 1. Let $1\leq p\leq\infty, 1/p+1/p'=1, r=\min(p,p')$ and $1/r+1/r'=1$. If X is an L_p-space with dim $X\geq 2$, then $E(X)=2^{1+2/r}$ and $f(X)=2^{1+2/r'}$.

Corollary 2. X is uniformly non-square, i.e., $J(X) < 2$ if and only if $E(X) < 8$, and also if and only if $f(X) > 2$.

Recall that the modulus of smoothness of X is defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \ x, y \in S_X\right\},$$

where S_X is the unit sphere of X. Let us collect some known facts ([13]):

(1) $\rho_X(1) = \rho_{X^*}(1)$ and $J(X) - 1 \le \rho_X(1) \le J(X)/2$. Moreover, $J(X) - 1 = \rho_X(1)$ if X is an L_p-space, $1 \le p \le \infty$, and $\rho_X(1) = J(X)/2$ iff X is not uniformly non-square.

(2) $2(1+\rho_X(1))^2 \le E(X) \le 4(1+\rho_X(1)^2) \le 4 + J(X)^2$. We have equality in the last inequality iff X is not uniformly non-square. Moreover, there are uniformly non-square spaces X such that we have equality in each one of the first and second inequalities.

(3) $E(X^*) \le 4 + 4(\sqrt{E(X)/2} - 1)^2 \le 4 + E(X)/2$. Moreover, the second inequality is strict if and only if X is uniformly non-square. On the other hand, there exists a uniformly non-square space X such that equality in the first inequality holds.

(4) $C_{NJ}(X) \le 1 + \rho_X(1)(\sqrt{(1-\rho_X(1))^2 + 1} - (1 - \rho_X(1)))$. We have equality if X is not uniformly non-square. Moreover, there exists a uniformly non-square space X such that equality holds. Using this inequality, we easily have

$$C_{NJ}(X) \le \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4}(\sqrt{J(X)^2 - 4J(X) + 8} - 2).$$

The above estimate was proved independently by Maligranda [2003] and Nikolova-Persson-Zacharides [2003], see also [8]. We note that this inequality is strict if and only if X is uniformly non-square.

(5) If X is an L_p-space, $1 \le p \le \infty$, then $C_{NJ}(X) = E(X)/4$. There exists a Banach space X such that $C_{NJ}(X) > 1 + \rho_X(1)^2 \ge E(X)/4$. Hence, $C_{NJ}(X) \ne E(X)/4$ in general.

Finite representability and super-reflexive Banach spaces: For isomorphic Banach spaces X and Y, the *Banach-Mazur distance* between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\|\|T^{-1}\|$ taken over all bicontinuous linear operators T of X onto Y. A Banach space Y is said to be *finitely representable* (*f.r.*) in a Banach space X if for any finite dimensional subspace F of Y and for any $\epsilon > 0$ there exists a finite dimensional subspace E of X with dim E = dim F such that $d(E,F) < 1 + \epsilon$. It is well-known that X^{**} (bidual of X) is finitely representable in X, and ℓ_2 is finitely representable in any infinite-dimensional Banach space.

X is said to be super-reflexive if any Banach space Y *f.r.* in X is reflexive. It is known that uniformly convex spaces are super-reflexive, and conversely, any super-reflexive Banach space has an equivalent uniformly convex norm, see Enflo [1].

Denote by $\tilde{E}(X)$ ($\tilde{f}(X)$) the infimum (supremum) of all $E(X)$ ($f(X)$) for the equivalent norms of X. It is shown that if X is an L_p-space, $1 \leq p \leq \infty$, then $\tilde{E}(X) = E(X)$ and $\tilde{f}(X) = f(X)$. But in general, $\tilde{E}(X) \neq E(X)$ and $\tilde{f}(X) \neq f(X)$. It is known (cf.[6]) that X is super-reflexive if and only if $\tilde{C}_{NJ}(X) < 2$, where $\tilde{C}_{NJ}(X)$ is the infimum of all $C_{NJ}(X)$ for the equivalent norms of X. The following result can be proved similarly, see [15].

Theorem 2. The following are equivalent.
(1) X is super-reflexive.
(2) $\tilde{E}(X) < 8$.
(3) $\tilde{f}(X) > 2$.

Type and cotype of Banach spaces:
X is said to be of (Rademacher) type p, $1 \leq p \leq 2$, if there exists $M > 0$ such that

$$\left\{\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^2 dt\right\}^{1/2} \leq M\left\{\sum_{j=1}^n \|x_j\|^p\right\}^{1/p}$$

for all finite systems $\{x_j\}$ in X, where $r_j(t)$ are the Rademacher functions, i.e., $r_j(t) = \mathrm{sgn}(\sin 2^j \pi t)$. It is obvious that any Banach space is of type 1.

X is said to be of (Rademacher) cotype q, $2 \leq q < \infty$, if there exists $M > 0$ such that

$$M\left\{\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^2 dt\right\}^{1/2} \geq \left\{\sum_{j=1}^n \|x_j\|^q\right\}^{1/q}$$

for all finite systems $\{x_j\}$ in X.

It is well-known that if X is of type p, then its dual space X^* is of cotype q, and the converse is also true if ℓ_1 is not finitely representable in X, where $1/p + 1/q = 1$. It is also known that X is isomorphic to a Hilbert space if and only if X is of type 2 and cotype 2.

Theorem 3. If Y is finitely representable in X, then $E(Y) \leq E(X)$ and $f(X) \leq f(Y)$. In particular, if l_p is f.r. in X, then $E(X) \geq 2^{1+2/r}$ and $f(X) \leq 2^{1+2/r'}$, where

$1 \leq p < \infty, 1/p + 1/p' = 1, r = \min(p, p')$ and $1/r + 1/r' = 1$.

Corollary 3. Let $1 < t < 2$ and $1/t + 1/t' = 1$.
(1) If $E(X) \leq 2^{1+2/t}$, then X is of type p and cotype q for all $p < t$ and $q > t'$.
(2) If $f(X) \geq 2^{1+2/t'}$, then X is of type p and cotype q for all $p < t$ and $q > t'$.

Remark 1. In (1), if $E(X) = 4$, then X is isometric to a Hilbert space, and so X is of type 2 and cotype 2.

B-convexity and B_n-convexity: X is said to be B_n-*convex* (or *uniformly non-*ℓ_1^n) provided there exists ε $(0 < \varepsilon < 1)$ such that for all $x_1, ..., x_n \in B_X$ there exist ε_j $(\varepsilon_j = \pm 1)$ satisfying

$$\|\varepsilon_1 x_1 + ... + \varepsilon_n x_n\| \leq n(1 - \varepsilon).$$

X is said to be B-*convex* if X is B_n-convex for some $n \geq 2$. It is well-known that X is B-*convex* if and only if l_1 is not finitely representable in X; and if and only if X is of *type* p for some $p > 1$.

Theorem 4. Let $1 < p < 2$. Suppose that there exists ε $(0 < \varepsilon < 1)$ such that for all $x_1, ..., x_n \in B_X$ there exist ε_j $(\varepsilon_j = \pm 1)$ satisfying

$$\|\varepsilon_1 x_1 + ... + \varepsilon_n x_n\| \leq n^{1/p}(1 - \varepsilon).$$

Then X is of type r for some $r > p$.

n-th von Neumann-Jordan constant and Gao's constants: The n-th von Neumann-Jordan constant $C_{NJ}^{(n)}(X)$, $n \geq 2$, is defined by

$$C_{NJ}^{(n)}(X) := \sup\left\{ \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2 \Big/ 2^n \sum_{j=1}^n \|x_j\|^2;\ \sum_{j=1}^n \|x_j\| \neq 0 \right\}.$$

It was shown in [7] that X is B_n-convex, $n \geq 2$, if and only if $C_{NJ}^{(n)}(X) < n$; and for $1 < p \leq 2$, $C_{NJ}^{(n)}(l_p) = C_{NJ}^{(n)}(L_p) = n^{2/p-1}$ for all $n \geq 2$, where $\dim L_p = \infty$. Note that $C_{NJ}^{(2)}(l_p) = C_{NJ}^{(2)}(L_p) = 2^{2/p'-1}$ for $2 < p < \infty$.

The n-th Gao's constants $E^{(n)}(X)$ and $f^{(n)}(X)$ are defined by

$$E^{(n)}(X) := \sup\left\{ \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2 \Big/ 2^n \sum_{j=1}^n \|x_j\|^2;\ x_1, x_2, ..., x_n \in S_X \right\}.$$

$$f^{(n)}(X) := \inf\left\{ \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2 \Big/ 2^n \sum_{j=1}^n \|x_j\|^2;\ x_1, x_2, ..., x_n \in S_X \right\}.$$

It is shown that if X is an L_p-space with dim $X \geq n$, $1 \leq p \leq 2$, then $E^{(n)}(X) = C_{NJ}^{(n)}(X)$. In general, $E^{(n)}(X) \leq C_{NJ}^{(n)}(X)$ for any Banach space X. For $n = 2$, there exists a Banach space X such that $E^{(2)}(X) < C_{NJ}^{(2)}(X)$. We pose the following conjecture: For any $n \geq 3$ there exists a Banach space X such that $E^{(n)}(X) < C_{NJ}^{(n)}(X)$.

n-th James and Schäffer constants:

$$J_n(X) := \sup\left\{ \min_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|;\ x_1, x_2, ..., x_n \in B_X \right\}.$$

$$J_n^s(X) := \sup\left\{ \min_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|;\ x_1, x_2, ..., x_n \in S_X \right\}.$$

$$S_n(X) := \inf\left\{ \max_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|;\ x_1, x_2, ..., x_n \in S_X \right\}.$$

Theorem 5. If $n \geq 2$, then
(1) $J_n^s(X)^2/n \leq E^{(n)}(X) \leq C_{NJ}^{(n)}(X)$.
(2) $1/C_{NJ}^{(n)}(X^*) \leq f^{(n)}(X) \leq S_n(X)^2/n$.

Corollary 4. Let $1 \leq p \leq 2$ and $2 \leq q < \infty$. If $n \geq 2$, then
(1) If dim $L_p \geq n$, then $E^{(n)}(L_p) = n^{2/p-1}$.
(2) If dim $L_q \geq n$, then $f^{(n)}(L_q) = n^{2/q-1}$.

Remark 2. For $n = 2$, we have $E^{(2)}(L_p) = 2^{2/p'-1}$ if $2 \leq p \leq \infty$, and $f^{(2)}(L_q) = 2^{2/q'-1}$ if $1 \leq q \leq 2$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, see Corollary 1.

Theorem 6. Let $n \geq 2$. The following are equivalent:
(1) X is B_n-convex.
(2) $J_n(X) < n$.
(3) $J_n^s(X) < n$.
(4) $C_{NJ}^{(n)}(X) < n$.
(5) $E^{(n)}(X) < n$.

Theorem 7. Let $n \geq 2$. The following are equivalent:
(1) ℓ_∞ is not finitely representable in X, i.e., X is of cotype q for some $q < \infty$.
(2) $S_n(X) > 1$ for some n.
(3) $f^{(n)}(X) > 1/n$ for some n.

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Functions of the Nevanlinna Class on the Boundary

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Abstract. Let N and N_+ be a Nevanlinna class and a Smirnov class, respectively, on the open unit disc D in the complex plane. We describe a function in N whose real part is nonnegative on ∂D the boundary. We also show that for a function F in N there exists a function f in N_+ such that $\mathrm{Re} f = |F|$ on ∂D. Moreover we give a necessary and sufficient condition for that there exists such an outer function f.

D を $\mathbb{C}$ の単位開円板、∂D をその境界とする。$f \in N$ とは f が D 上で正則かつ

$$\sup_{0 \le r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta/2\pi < \infty$$

を満たすことである。N は Nevanlinna 族と呼ばれる。このとき、Fatou の定理は任意の N の元はほとんどいたる所 radial limit を持つことを示している。よって D 上の N と ∂D 上の radial limit を同一視できる。$f \in N_+$ とは $f \in N$ かつ

$$\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta/2\pi = \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta/2\pi$$

となるときをいう。N_+ は Smirnov 族と呼ばれる。$0 < p \le \infty$ のとき、$N_+ \cap L^p(d\theta/2\pi) = H^p$ と書き、H^p は Hardy 族と呼ばれる。

次の [1] は良く知られている。

[1] $f \in H^1$ は恒等的に零ではなくかつ $\mathrm{Re} f \ge 0$ on ∂D ならば、$|k| \le 1$ on ∂D となる $k \in H^\infty$ が存在して、$f = (1+k)/(1-k)$ と書ける。このとき、$1-k$ は outer 関数であり、$|f-1| \le |f+1|$ on D が成立する。

証明 $f \in H^1$ は

$$f(z) = \int_0^{2\pi} P(r, \theta - t) f(e^{it}) dt/2\pi \quad (z = re^{i\theta} \in D)$$

と書けるので、$\mathrm{Re} f \ge 0$ ならば、$\mathrm{Re} f > 0$ on D である。よって、$k = (f-1)/(f+1)$ とすると $k \in H^\infty$ かつ $|k| < 1$ on D であるから $f = (1+k)/(1-k)$ と書ける。このとき、f は outer 関数である。

定理1は[1]のNに属する関数への一般化である。ここでNに属する関数は必ずしもPoisson積分表現できるとは限らないので、[1]とは違う方法をとらなければならない。

定理1 $f \in N$は恒等的に零ではなくかつ$\mathrm{Re} f \geq 0$ on ∂Dならば、$|k| \leq 1$ on ∂Dとなる$k \in H^\infty$とinner関数qが存在して

$$f = \frac{q+k}{q-k} \text{ on } \partial D \text{ かつ } |q-k| > 0 \text{ on } D$$

とできる。このとき$|f-1||q| \leq |f+1|$ on Dかつ$\mathrm{Re} f = (1-|k|^2)/|q-k|^2$ on ∂Dである。

略証 $\phi = (f-1)/(f+1)$とおくと、$|\phi| \leq 1$ on ∂D。$f+1 = q_1 h_1/s_1$、$f-1 = q_2 h_2/s_2$とF. and R. Nevanlinnaの定理を用いて因数分解する。ここでq_iはinner関数、s_iは特異inner関数、h_iはouter関数かつ$s_i \wedge q_i = 1$ $(i=1,2)$である。このとき、$s_1 = s_2$とみることができ、$q = q_1$かつ$k = q_2 h_2/h_1$とすると、$f = (q+k)/(q-k), k \in H^\infty, |k| \leq 1$ on ∂Dかつ$|q-k| > 0$ on Dを示すことができる。

定理1でfがN_+に属するならば、$q-k$はouter関数である。しかし一般には$q-k$は特異なinner部分をもつ。qがn次finite Blaschke積ならば$q-k$のinner部分はℓ次finite Blaschke積かつ$\ell \leq n$となることをAdamyan-Arov-Kereinの定理[1, Chapter IV, Theorem 5.3]を用いて示すことができる。特に$q=1$ならば$q-k$はouterとなる。よって定理1のfはqがfinite BlaschkeならばN_+に属する。次の[2]は知られている。

[2] $F \in H^1$が恒等的に零でないならば$f \in \bigcap_{p<1} H^p$で、$\mathrm{Re} f = |F|$ on ∂Dとなるものが存在する。このとき$\mathrm{Re} f > 0$ on Dとできる。

証明 $F \in H^1$に対して

$$f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} |F(e^{it})| dt/2\pi \quad (z \in D)$$

とすると、$f \in \bigcap_{p<1} H^p$かつ$\mathrm{Re} f = |F|$ on ∂Dである。このとき$\mathrm{Re} f > 0$ on D。

定理2は[2]の$f \in N$への一般化である。Nに属する関数は必ずしも$L^1(d\theta/2\pi)$に属するとは限らないので、Herglotz積分を用いることができない。よって他の方法を用いなければならない。

定理 2 $f \in N$ が恒等的に零でないならば $f \in N_+$ で、$\mathrm{Re} f = |F|$ on ∂D となるものが存在する。

略証 Douglas-Rudin の定理 [1, Chapter V, Theorem 2.1] より

$$\left| \frac{(1+q)^2 F}{|(1+q)^2 F|} - q' \right| \leq 1$$

となる inner 関数 q, q' が存在する。Koosis の補題より

$$\frac{(1+q)^2 F}{|(1+q)^2 F|} = \frac{G}{|G|}$$

となる outer $G \in H^1$ が存在する。

$$f = \frac{qF}{G} \frac{1+Q}{1-Q}$$

かつ $(1+Q)/(1-Q)$ は $|G|$ の Herglotz 積分とする。このとき $f \in N_+$ かつ $\mathrm{Re} f = |F|$ on ∂D。

[2] における f は outer 関数であるが、定理 2 における f は outer とは限らない。f が outer である必要十分条件は $F_0/G \geq 0$ on ∂D となる outer である $G \in H^1$ が存在することを示すことができる。ここで F_0 は F の outer 部分である。

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Hilbert Functions and Infinite Sequences of Inner Functions

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Abstract

The Hardy space over the bidisk has Hilbert module structure over the bidisk algebra under pointwise multiplication, and its submodules are defined as closed subspaces invariant under the module action. In this article, we deal with submodules constructed of infinite sequences of inner functions, and their Hilbert functions are calculated.

1 ヒルベルト関数とは

ヒルベルト関数とは，可換環論にて古くから研究されている重要な対象である．Douglas-Yan [1] により，多変数作用素論における不変量を探る目的で，ヒルベルト関数が関数解析学に導入された．ここでは，関数解析的ヒルベルト関数の定義を紹介する．

A を関数環，$\mathcal{H}$ を A が作用するヒルベルト加群とする．すなわち，$\mathcal{H}$ は

$$A \times \mathcal{H} \ni (f, h) \mapsto f \cdot h \in \mathcal{H}$$

という加群の作用が定義されているヒルベルト空間であり（特に，$1 \cdot h = h$ を仮定する），さらに，A と $\mathcal{H}$ のノルム位相に関し，この作用が各変数ごとに連続であるとする．次に，A の極大イデアル I に対し，I^k を $\{f_1 f_2 \cdots f_k : f_j \in I\}$ により生成されるイデアルとし，

$$[I^k \cdot \mathcal{H}] = \left[\left\{\sum_{j=1}^{n} f_j \cdot h_j : f_j \in I^k, h_j \in \mathcal{H}, n \in \mathbb{N}\right\}\right]_{\|\cdot\|_{\mathcal{H}}}$$

と定める．

定義 1 ヒルベルト関数 $H_{I,\mathcal{H}}$ を以下のように定義する：

$$H_{I,\mathcal{H}}(k) = \dim \mathcal{H}/[I^k \cdot \mathcal{H}] \quad (\dim \text{ は通常の線形空間としての次元とする}).$$

この定義は抽象的なものであるが，以下の例から，ヒルベルト関数は関数解析でも自然に現れる対象であることがわかる．

例 1.1 $\mathcal{H}$ をヒルベルト空間，T を $\|T\| \leq 1$ である $\mathcal{H}$ 上の線型作用素とする．また，$A(\mathbb{D})$ を disk algebra とする．このとき，von Neumann の不等式により，$f \in A(\mathbb{D})$ と $h \in \mathcal{H}$ に対して，$f \cdot h = f(T)h$ と定めれば，これはヒルベルト加群の作用となる．従って，$\mathcal{H}$ と T のペアを，$A(\mathbb{D})$ を係数環とするヒルベルト加群とみなすことができる．この設定でのヒルベルト関数は，I_λ を $\lambda \in \mathbb{D}$ に対応する A の極大イデアルとすると，$H_{I_\lambda,\mathcal{H}}(k) = \dim \mathcal{H}/[I_\lambda^k \cdot \mathcal{H}] = \dim \mathcal{H} \ominus [(T-\lambda)^k \mathcal{H}] = \dim \ker(T^* - \overline{\lambda})^k$ となる．

定義 2 係数環を A とするヒルベルト加群 $\mathcal{H}$ に対し，ある有限集合 $\{g_j\}_{j=1}^n \subset \mathcal{H}$ が存在して，$\left\{\sum_{j=1}^n f_j \cdot g_j : f_j \in A\right\}$ が $\mathcal{H}$ の中で稠密であるとき，$\mathcal{H}$ は有限生成であるいう．また，このような n の中で最小の数を $\mathcal{H}$ のランクといい，$\operatorname{rank} \mathcal{H}$ により表す．

次の定理は [1] で証明されているが，本質的な部分（以下の (ii) に相当する箇所）は可換環論におけるヒルベルトの有名な結果である．

定理 1 ([1]) $\mathcal{H}$ がヒルベルト加群として有限生成であり，さらに $\dim I/[I^2]_{\|\cdot\|_A} < \infty$ と仮定する．このとき，以下が成り立つ：

(i) $\dim \mathcal{H}/[I^k \cdot \mathcal{H}] < \infty \quad (k \geq 0)$,

(ii) $p_{I,\mathcal{H}} \in \mathbb{Q}[t]$ が存在し，十分大きな k に対して，$p_{I,\mathcal{H}}(k) = \dim \mathcal{H}/[I^k \cdot \mathcal{H}]$.

定理 1 の $p_{I,\mathcal{H}}$ はヒルベルト多項式とよばれる．可換環論において，ヒルベルト多項式の次数，最高次の係数は，加群に関する重要な不変量を与えることが知られている．そこで，可換環論の結果を参考にして，多変数作用素論におけるその対応物を探したい．これが，Douglas-Yan [1] に始まる，ヒルベルト関数を関数解析でも研究する動機である．

2 ヒルベルト関数の計算

Douglas-Yan [1] に続く研究として，Eschmeier [2], Fang [3] などがあるが，ここでは，彼らの研究とは異なる観点からヒルベルト関数を調べてみたい．以下では，$\mathbb{D}$ を $\mathbb{C}$ 内の開円板とし，$A(\mathbb{D}^2)$ を bidisk 環，$H^2(\mathbb{D}^n)$ を $\mathbb{D}^n$ $(n=1,2)$ 上の Hardy 空間とする．このとき，通常の掛け算で $H^2(\mathbb{D}^2)$ は $A(\mathbb{D}^2)$ を係数環とするヒルベルト加群となる．また，q_j/q_{j+1} が有界正則であるような inner function の列 $\{q_j\}_{j\geq 0}$ に対し，

$$\mathcal{H} = \sum_{j=0}^{\infty} \oplus q_j H^2(\mathbb{D}) \otimes w^j$$

と定めると $\mathcal{H}$ は $H^2(\mathbb{D}^2)$ の $A(\mathbb{D}^2)$-部分加群となる．ただし，上記のテンソルは $H^2(\mathbb{D}^2) = H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$（変数は z, w とする）の意味でとる．

定理 2 I_λ を $(\lambda,0)\in\mathbb{D}^2$ に対応する $A(\mathbb{D}^2)$ の極大イデアルとする．このとき，$\mathcal{H}=\sum_{j=0}^{\infty}\oplus q_jH^2(\mathbb{D})\otimes w^j$ に対し，

$$\begin{aligned}H_{I_\lambda,\mathcal{H}}(k)=&\frac{k(k+1)}{2}\\&+\left|\left\{j\geq 1:\deg_\lambda\frac{q_{j-k}}{q_j}\geq k,\deg_\lambda\frac{q_{j-k+1}}{q_j}\geq k-1,\dots,\deg_\lambda\frac{q_{j-1}}{q_j}\geq 1\right\}\right|\\&+\left|\left\{j\geq 2:\deg_\lambda\frac{q_{j-k}}{q_j}\geq k-1,\deg_\lambda\frac{q_{j-k+1}}{q_j}\geq k-2,\dots,\deg_\lambda\frac{q_{j-2}}{q_j}\geq 1\right\}\right|\\&\vdots\\&+\left|\left\{j\geq k:\deg_\lambda\frac{q_{j-k}}{q_j}\geq 1\right\}\right|\end{aligned}$$

が成り立つ．ここで，$\deg_\lambda f$ は正則関数 f の $z=\lambda$ における零点の重複度を表す．

この定理 2 は非常に限定的な結果であるが，注目すべき点が一つある．それは，定理 1 に含まれない場合も扱えるということである．以下の例 2.1 からそれがわかる．

例 2.1 (Rudin の例) $\alpha_n=1-n^{-3}$ $(n\in\mathbb{N})$ とし，α_n に零点をもつ Blaschke factor を b_n とする．定理 2 を $q_j=\prod_{n=j}^{\infty}b_n^{n-j}$ の場合に適用すると，以下のランク評価を得る．

$$H_{I_{\alpha_n},\mathcal{H}}(k)=\frac{k(k+1)}{2}+nk\leq\frac{k(k+1)}{2}\operatorname{rank}\mathcal{H}\quad(n\in\mathbb{N}).$$

この評価から，$n\to\infty$ とすれば，$\mathcal{H}$ は無限生成であることが導かれる（1 次の係数がランクと関係しているということか?）．これは，Rudin のよく知られた結果 ([4]) に対する別証明である．

例 2.2 (重複度 1 の場合) $\{\lambda_j\}_{j=0}^{\infty}$ を Blaschke sequence とし，さらに，$\lambda_i\neq\lambda_j$ $(i\neq j)$ を仮定する．λ_j に零点をもつ Blaschke factor を b_j とする．$q_j/q_{j+1}=b_j$ の場合，

$$H_{I_{\lambda_n,0},\mathcal{H}}(k)=\frac{k(k+1)}{2}+k\leq\frac{k(k+1)}{2}\operatorname{rank}\mathcal{H}\quad(n\in\mathbb{N}).$$

残念ながら，この例では，例 2.1 で用いたランク評価の方法は機能しない．

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行列の多項式型数域
Polynomial numerical hull of a matrix

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Abstract The polynomial numerical hull of a matrix is introduced and characterized. Some examples of polynomial numerical hulls are provided.

1. Numerical range of a matrix

Let $\mathbf{M_n}$ be the associative algebra of all $n \times n$ complex matrices. Many numerical objects are defined for a matrix $A \in \mathbf{M_n}$. Probably, the most important one is their *eigenvalues* or the *spectrum*. The spectrum $\sigma(A)$ of a matrix A is defined as

$$\sigma(A) = \{\lambda \in \mathbf{C} : A\xi = \lambda\xi \text{ for some } \xi \in \mathbf{C}^n, \xi \neq 0\}$$
$$= \{\lambda \in \mathbf{C} : \det(\lambda I_n - A) = 0\}. \tag{1.1}$$

The spectra of matrices have good properties such as $\sigma(p(A)) = p(\sigma(A))$ for a polynomial $p(\lambda)$ and $\sigma(AB) = \sigma(BA)$. We also consider the operator norm $||A||$ defined as

$$||A|| = \max\{\sqrt{(A\xi)^*(A\xi)} : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}$$
$$= \max\{\sqrt{\lambda} : \lambda \geq 0, \det(\lambda I_n - A^*A) = 0\}. \tag{1.2}$$

The numerical range $W(A)$ of a matrix $A \in \mathbf{M_n}$ is defined as

$$W(A) = \{\xi^*A\xi : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}. \tag{1.3}$$

In 1919 Hausdorff proved the convexity of $W(A)$ (cf.[6]). He pvoved that

$$\{\xi \in \mathbf{C}^n : \xi^*\xi = 1, \xi^*H\xi = a\}$$

was connected for every Hermitian matrix H and $a \in W(H)$. In 1975, Westwick generalized Hausdorff's result to c-numerical range $W_c(A)$ ($c \in \mathbf{R}^n$) by using Morse theory. He showed that the function

$$f((u_{ij})) = \sum_{i,j=1}^{n} c_i\lambda_j|u_{ij}|^2$$

was a Morse function on $U(n)/D(n)$ and all Morse indices for f are even numbers for real diagonal matrices C, H with distinct diagonal entries.

If $\lambda \in \mathbf{C}$ does not belong to $W(A)$, then the separation theorem for convex sets implies that there exist $0 \leq \theta < 2\pi$ and $b \in \mathbf{R}$ satisfying

$$\Re(\exp(-i\theta)\xi^* A\xi) \leq b < \Re(\exp(-i\theta)\lambda).$$

for every $\xi \in \mathbf{C}^n$, $\xi^*\xi = 1$. This inequality implies that the relation

$$\xi^*[I_n+\frac{1}{a}(\exp(-i\theta)A+\exp(i\theta)A^*)+\frac{1}{a^2}A^*A]\xi < 1+\frac{2}{a}\Re(\exp(-i\theta)\lambda)+\frac{1}{a^2}|\lambda|^2$$

($\xi \in \mathbf{C}^n$, $\xi^*\xi = 1$) holds for sufficiently large $a > 0$. Hence the inequality

$$||a\exp(-i\theta)I_n + A|| < |a\exp(-i\theta) + \lambda|$$

holds for sufficiently large $a > 0$.

On the other hand, if the equation $\lambda = \xi^* A\xi$ holds for some unit vector ξ, then the inequality

$$|\lambda + \tilde{a}| \leq ||A + \tilde{a}I_n||$$

holds for any $\tilde{a} \in \mathbf{C}$. Thus we have one characterization of $W(A)$ as the following:

$$W(A) = V^1(A) = \{\lambda \in \mathbf{C} : |\lambda + a| \leq ||A + aI_n|| \text{ for all } a \in \mathbf{C}\}. \tag{1.4}$$

2. Polynomial numerical hulls

The object $W(A) = V^1(A)$ for a matrix A is generalized as

$$V^k(A) = \{\lambda \in \mathbf{C} : |a_k\lambda^k+a_{k-1}\lambda^{k-1}+\cdots+a_0| \leq ||a_kA^k+a_{k-1}A^{k-1}+\cdots+a_0I_n|| \\ \text{for all } a_k, a_{k-1}, \ldots, a_0 \in \mathbf{C}\}, \tag{2.1}$$

where k is a positive integer. It is also characterized as

$$V^k(A) = \{\lambda \in \mathbf{C} : a_k\lambda^k+a_{k-1}\lambda^{k-1}+\ldots+a_0 \in W(a_kA^k+a_{k-1}A^{k-1}+\ldots+a_0I) \\ \text{for all } a_k, a_{k-1}, \ldots, a_0 \in \mathbf{C}\}. \tag{2.2}$$

This numerical object $V^k(A)$ is said to be the *polynomial numerical hull* of order k of A. It satisfies $V^k(A^T) = V(A)$ for the transpose A^T of A. The set $V^k(A)$ is not necessarily connected. Each connected component of

$V^k(A)$ is simply connected. A crucial characterization is found by V. Faber et al. in 1996 ([4]) and also by A. Greenbaum in 2002 (cf. [5]):

$$V^k(A) = \{\lambda \in \mathbf{C} : (0,0,\ldots,0) \in \mathrm{Conv}(W(A-\lambda I_n, (A-\lambda I_n)^2, \ldots, (A-\lambda_n)^k))\}. \tag{2.3}$$

Thus it also satisfies

$$V^k(A) = \{\lambda \in \mathbf{C} : \min_{y_1^2+\cdots+y_{2k}^2=1} \max \sigma(y_1 \Re(A-\lambda I_n) + y_2 \Im(A-\lambda I_n) + \ldots$$

$$+ y_{2k-1}\Re((A-\lambda I_n)^k) + y_{2k}\Im((A-\lambda I_n)^k)) \geq 0\} \tag{2.4}$$

(cf.[7]). If $\lambda_0 \in \mathbf{C}$ satisfies

$$\max \sigma(y_1 \Re(A-\lambda I_n) + \ldots + y_{2k}\Im((A-\lambda I_n)^k)) \geq \epsilon > 0$$

for every point $(y_1, \ldots, y_n)$ of the unit sphere, then this inequality holds on a neighborhood of λ_0. Thus every boundary point λ_0 of $V^k(A)$ satisfies

$$\min_{y_1^2+\cdots+y_{2k}^2=1} \max \sigma(y_1 \Re(A-\lambda_0 I_n) + \ldots + y_{2k}\Im((A-\lambda_0 I_n)^k)) = 0.$$

We assume that $(y_1^{(0)}, \ldots, y_{2k}^{(0)})$ is a minimum point of the function $\max \sigma(y_1 \Re(A-\lambda_0 I_n) + \ldots)$ with $y_1^{(0)} \neq 0$. Set $Y_j^{(0)} = y_j^{(0)}/y_1^{(0)}$ for $2 \leq j \leq 2k$. Then the equations

$$\det(\Re(A-\lambda_0 I_n) + Y_2^{(0)}\Im(A-\lambda_0 I_n) + \ldots + Y_{2k}^{(0)}\Im((A-\lambda_0 I_n)^k)) = 0, \tag{2.5}$$

$$\frac{\partial}{\partial Y_j}\det(\Re(A-\lambda_0 I_n) + Y_2\Im(A-\lambda_0 I_n) + \ldots + Y_{2k}\Im((A-\lambda_0 I_n)^k)) = 0, \tag{2.6}$$

$(j = 2, \ldots, 2k)$ hold where the derivatives are taken at the point $(Y_2^{(0)}, \ldots, Y_n^{(0)})$. Using (2.5),(2.6), we eliminate $Y_2, \ldots, Y_{2k}$ and obtain an equation in λ. To perform such a process, we compute the descriminants of some polynomials $2k-1$ times. Any way, (2.4) show that $\partial V^k(A)$ lies on an algebraic curve (cf. [1]). Seidenberg's decision method provides a general principle. In the case $k = n$, let

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

be the characteristic polynopmial of A. Then we have

$$V^n(A) \subset \{\lambda \in \mathbf{C} : p(\lambda) = 0\},$$

and hence $V^n(A) \subset \sigma(A)$. If λ is an eigenvalue of A and $\xi \in \mathbf{C}^n$ is a unit eigenvector of A corresponding to λ, then the equation

$$\xi^* q(A)\xi = q(\lambda)$$

holds for an arbitrary polynomial $q(z)$. Hence the inequality $|q(\lambda)| \leq ||q(A)||$ holds for arbitrary polynomial. Thus the equation $V^n(A) = \sigma(A)$ holds. The numerical range $W(A) = V^1(A)$ of a matrix A is used to locate the eigenvalues of A. The polynomial numerical hulls $V^k(A)$ $(k = 2, 3, \dots, n)$ provide finer informations of the location of the eigenvalues of A. Some interesting results of $V^k(Q)$ has been found by C. Davis, A. Salemi and C. K. Li (cf. [2], [3]).

3. Numerical method to study $V^2(A)$

The equations (2.5), (2.6) are necessary conditions for $\lambda_0 \in \mathbf{C}$ to belong to $\partial V^k(A)$. We shall provide a performable numerical method to confirm that a point λ does not belongs to $V^2(A)$. We consider a polar coordinate system on the 3-dimensional unit sphere $\mathbf{S}^3$:

$$y_1 = \cos u, y_2 = \sin u \cos v, y_3 = \sin u \sin v \cos w, y_4 = \sin u \sin v \sin w$$

$(0 \leq u, v \leq \pi, 0 \leq w \leq 2\pi)$. For a random point $(y_1, y_2, y_3, y_4) \in \mathbf{S}^3$, we consider the characteristic polynomial $F = F(t, y_1, y_2, y_3, y_4)$ defined by

$$\det(tI_n - y_1\Re(A - \lambda I_n) - y_2\Im(A - \lambda I_n) - y_3\Re((A - \lambda I_n)^2) - y_4\Im((A - \lambda I_n)^2)).$$

In the case we use a software "Mathematica", a numerical maximal root of this polynomial in t is computed by the command

$$\text{Flatten[NSolve}[F == 0, t]][[n]][[2]].$$

We shall make a list of these largest roots for random points (y_1, y_2, y_3, y_4) on the unit sphere. We take the minimum of these values. If the minimum is negative, it show that λ does not belongs to $V^2(A)$. For instance, $20^3 = 8000$ points on $\mathbf{S}^3$ form a rather coarse net. A net composed of $100^3 = 10^6$ points is also not so fine. However these net provide a rough test for a point $\lambda \in \mathbf{C}$ not to belongs to $V^2(A)$. With the help of this method, we can determine the boundary of $V^2(A)$ for some 3×3 matrices, for instance

$$A = \begin{pmatrix} 0 & 2 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

The boundary of $V^2(A)$ for this matrix satisfies

$$\partial V^2(A) \subset \{z_1 + iz_2 : (z_1, z_2) \in \mathbf{R}^2, Q(z_1, z_2) = 0\},$$

where

$$Q(z_1, z_2) = 1358954496 + 2717908992z_1 - 1245708288z_1^2 - 1698693120z_1^3$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+87175416z_2^{14} + 56433384z_1z_2^{14} + 28872776z_1^2z_2^{14} + 10804327z_2^{16}.$$

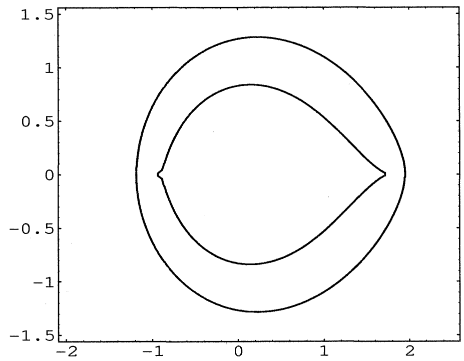

図 1: outer arc is the boundary of $V^2(A)$

We shall consider a simpler case. Let

$$S_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Set $P(x, y) = \Re((S_4 - xI_4)^2) + y\Re((S_4 - xI_4))$. Then,

$$16\det(P(x, y)) = G(x, y)G(-x, -y),$$

$$G(x, y) = 4x^2y^2 - 2xy^2 - y^2 - 8x^3y + 6x^2y + 4xy - 2y + 4x^4 - 4x^3 - 4x^2 + 4x - 1.$$

The resultnt of $G(x, y)$ and $G_y(x, y)$ with respect to y is given by

$$H(x) = -4x\,(5x^3 - 8x^2 + 6x - 2)(4x^2 - 2x - 1).$$

The resultant of $G(-x, -y)$ and $G_y(-x, -y)$ is given by $H(-x)$. Denote by a_4 the unique real root of $5x^3 - 8x^2 + 6x - 2$. An approximate value of a_4 is given by 0.702319. The roots of $4x^2 + 2x - 1$, that is, $(-1 \pm \sqrt{5})/4$ are approximated by 0.309017, -0.809017. The 3 values a_4, $|-1 \pm \sqrt{5}|/4$ are candidates of the radius of the disc $V^2(S_4)$. By a numerical method, we find that a_4 is the radius. The range $V^2(S_4)$ is given by $\{z \in \mathbf{C} : |z| \leq a_4\}$. By a similar method, we find that the 10×10 shift matrix

$$S_{10} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

satisfies

$$V^2(S_{10}) = \{z \in \mathbf{C} : |z| \leq a_{10}\},$$

where a_{10} is the largest real value of a polynomial

$$K(x) = 59969536x^{39} - 436142080x^{38} + \cdots + 230472x - 13500.$$

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ON FIXED POINTS OF RESOLVENTS OF MONOTONE OPERATORS IN BANACH SPACES
(バナッハ空間における単調作用素のレゾルベントの不動点について)

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ABSTRACT. We obtain a strong convergence theorem of Browder's type for firmly nonexpansive-type mappings in Banach spaces.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a (real) Hilbert space H and T a mapping from C into H. Then T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in C$. The mapping T is also said to be *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \tag{1.2}$$

for all $x, y \in C$. It is obvious that every firmly nonexpansive mapping is nonexpansive. The set of fixed points of T is denoted by $F(T)$. It is known that a mapping T from C into itself is firmly nonexpansive if and only if there exists a monotone operator $A\colon H \to 2^H$ such that $D(A) \subset C \subset R(I + A)$ and

$$Tx = (I + A)^{-1}x$$

for all $x \in C$, that is, T is the *resolvent* of A; see Bruck and Reich [7]. In this case, the following holds:

$$z \in F(T) \Longleftrightarrow 0 \in Az. \tag{1.3}$$

Browder [6] obtained the following strong convergence theorem for nonexpansive mappings in Hilbert spaces; see also Takahashi [20–22]:

Theorem 1.1 ([6]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, T a nonexpansive mapping from C into itself such that $F(T)$ is nonempty, and $x \in C$. Then the following hold:*

(1) *For each $t \in (0, 1)$, there exists a unique $u_t \in C$ such that*

$$u_t = tx + (1 - t)Tu_t;$$

(2) *the net $\{u_t\}$ converges strongly to $P_{F(T)}(x)$ as $t \downarrow 0$, where $P_{F(T)}$ denotes the metric projection from H onto $F(T)$.*

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In the part (1) of Browder's theorem, Banach's contraction principle ensures the unique existence of u_t. Browder's theorem was extended to accretive operators in Banach spaces by Reich [15] and Takahashi and Ueda [23]; see also Takahashi [20,21] for more details.

Let C be a nonempty closed convex subset of a smooth Banach space E, J the normalized duality mapping from E into E^*, and T a mapping from C into E. Then T is said to be of *firmly nonexpansive type* [10] if

$$\langle Tx - Ty, JTx - JTy\rangle \leq \langle Tx - Ty, Jx - Jy\rangle \tag{1.4}$$

for all $x, y \in C$. If E is a Hilbert space, then J is the identity operator on E and hence (1.4) reduces to (1.2). The class of firmly nonexpansive-type mappings is contained in the classes of *D-firm operators* introduced by Bauschke, Borwein, and Combettes [4] and *F-firmly nonexpansive operators* introduced by Bauschke, Wang, and Yao [5], where D stands for a Bregman distance and F stands for a single-valued strictly monotone operator from E into E^* with some assumptions.

We know that T is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x) \tag{1.5}$$

for all $x, y \in C$; see [4,10], where the function ϕ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{1.6}$$

for all $x, y \in E$; see Alber [2] and Kamimura and Takahashi [8]. If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$ and hence (1.5) is reduced to

$$2\|Tx - Ty\|^2 + \|Tx - x\|^2 + \|Ty - y\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.7}$$

for all $x, y \in C$. In the particular case where T has a fixed point, then (1.5) implies that

$$\phi(y, Tx) + \phi(Tx, x) \leq \phi(y, x) \tag{1.8}$$

for all $y \in F(T)$ and $x \in C$.

In this paper, using some results on firmly nonexpansive-type mappings (Theorem 2.2 and Lemma 2.4), we obtain a strong convergence theorem of Browder's type for firmly nonexpansive-type mappings in Banach spaces.

2. Preliminaries

Throughout the paper, every linear space is real. The set of real numbers is denoted by $\mathbb{R}$. Let E be a Banach space with norm $\|\cdot\|$ and E^* the conjugate space of E. Then the value of $x^* \in E^*$ at $x \in E$ is denoted by $\langle x, x^*\rangle$. For a sequence $\{x_n\}$ of E and $x \in E$, the strong convergence of $\{x_n\}$ to x and the weak convergence of $\{x_n\}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. A Banach space E is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$, where $S(E)$ denotes the unit sphere of E. The norm of E is also said to be *uniformly Gâteaux differentiable* if for each $y \in S(E)$, the limit (2.1) converges uniformly in $x \in S(E)$. A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. A Banach space E is said to have the *Kadec–Klee* property if $x_n \to x$ whenever $\{x_n\}$ is a sequence of E such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. The duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.2}$$

for all $x \in E$. If C is a nonempty closed convex subset of a strictly convex and reflexive Banach space E, then for each $x \in E$, there exists a unique $z \in C$ (denoted by $P_C x$) such that $\|z - x\| = \min_{y \in C} \|y - x\|$. The mapping P_C is called the *metric projection* from E onto C.

Let E be a Banach space and $A\colon E \to 2^{E^*}$. The graph of A is defined by

$$G(A) = \{(x, x^*) : x^* \in Ax\}. \tag{2.3}$$

The domain and the range of A are defined by $D(A) = \{x \in E : Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in E} Ax$, respectively. The mapping A is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in G(A)$. A monotone operator $A\colon E \to 2^{E^*}$ is said to be *maximal monotone* if there is no other monotone operator $B\colon E \to 2^{E^*}$ such that $G(A) \subset G(B)$. If $A\colon E \to 2^{E^*}$ is a monotone operator such that $D(A) \subset C \subset J^{-1}R(J + A)$ for some nonempty closed convex subset C of E, then we can define a single-valued mapping T from C into itself by

$$Tx = (J + A)^{-1} Jx \tag{2.4}$$

for all $x \in C$. The mapping T is called the *resolvent* of A. In the particular case where A is maximal monotone, then it holds that $R(J + A) = E$ and hence the mapping T is defined on the whole space E; see Takahashi [21].

The following lemma shows that the class of firmly nonexpansive-type mappings is coincident with that of resolvents of monotone operators:

Lemma 2.1 ([11]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and T a mapping from C into itself. Then the following are equivalent:*

(1) *T is of firmly nonexpansive type;*
(2) *there exists a monotone operator $A\colon E \to 2^{E^*}$ such that $D(A) \subset C \subset J^{-1}R(J + A)$ and*
$$Tx = (J + A)^{-1} Jx$$
for all $x \in C$.

In Lemma 2.1, it also holds that

$$z \in F(T) \Longleftrightarrow 0 \in Az. \tag{2.5}$$

Thus, the problem of finding zero points of monotone operators in Banach spaces is equivalent to that of finding fixed points of firmly nonexpansive-type mappings.

Motivated by the proof of Takahashi's fixed point theorem [19] for amenable semigroups of nonexpansive mappings in Hilbert spaces, we recently obtained the following fixed point theorem:

Theorem 2.2 ([10]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and T a firmly nonexpansive-type mapping from C into itself. Then the following are equivalent:*

(1) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*
(2) *T has a fixed point.*

In particular, if C is bounded, then T has a fixed point.

Let C be a nonempty closed convex subset of a smooth Banach space E and T a mapping from C into E. Following Reich [16], we say that a point $u \in C$ is an *asymptotic fixed point* of T if there exists a sequence $\{x_n\}$ of C such that $x_n \rightharpoonup u$ and $\|x_n - Tx_n\| \to 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. Following Matsushita and Takahashi [13, 14], we say that the mapping T is *relatively nonexpansive* if the following conditions are satisfied:

(R1) $F(T)$ is nonempty;
(R2) $\phi(y, Tx) \leq \phi(y, x)$ for all $y \in F(T)$ and $x \in C$;
(R3) $\widehat{F}(T) = F(T)$.

The mapping T is also said to be *strongly relatively nonexpansive* [1, 3, 9, 16] if T is relatively nonexpansive and $\phi(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a bounded sequence of C such that $\phi(u, x_n) - \phi(u, Tx_n) \to 0$ for some $u \in F(T)$.

We also know the following lemma:

Lemma 2.3 ([14]). *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, and T a mapping from C into itself such that $F(T)$ is nonempty and*

$$\phi(y, Tx) \leq \phi(y, x)$$

for all $y \in F(T)$ and $x \in C$. Then $F(T)$ is closed and convex.

It immediately follows from (1.8) and $\phi(x, y) \geq (\|x\| - \|y\|)^2$ for all $x, y \in E$ that if T is a firmly nonexpansive-type mapping which has a fixed point, then T satisfies (R2). The following lemma ensures that such a mapping also satisfies (R3) under some additional assumptions on the space E:

Lemma 2.4 ([10]). *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, and T a firmly nonexpansive-type mapping from C into itself. Then the following hold:*

(1) *$\widehat{F}(T) = F(T)$;*
(2) *if $F(T)$ is nonempty, then T is strongly relatively nonexpansive.*

3. Results

Using Theorem 2.2, Lemmas 2.3 and 2.4, we obtain the following strong convergence theorem of Browder's type for firmly nonexpansive-type mappings in Banach spaces; see [12] for the proof:

Theorem 3.1 ([12]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty bounded closed convex subset of E with $0 \in C$, and T a firmly nonexpansive-type mapping from C into itself. Then the following hold:*

(1) *For each $t \in (0, 1)$, there exists a unique $u_t \in C$ such that*

$$u_t = (1 - t)Tu_t;$$

(2) *if E has the Kadec–Klee property and the norm of E is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{F(T)}(0)$ as $t \downarrow 0$, where $P_{F(T)}$ denotes the metric projection from E onto $F(T)$.*

Applying Lemma 2.1 and Theorem 3.1, we can study the problem of finding zero points of monotone operators in Banach spaces.

Theorem 3.2 ([12]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty bounded closed convex subset of E with $0 \in C$. Let r be a positive real number, $A\colon E \to 2^{E^*}$ a monotone operator such that $D(A) \subset C \subset J^{-1}R(J + rA)$, and $Q_r x = (J + rA)^{-1}Jx$ for all $x \in C$. Then the following hold:*

1. *For each $t \in (0,1)$, there exists a unique $u_t \in C$ such that*
$$u_t = (1-t)Q_r u_t;$$
2. *if E has the Kadec–Klee property and the norm of E is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{A^{-1}0}(0)$ as $t \downarrow 0$, where $P_{A^{-1}0}$ denotes the metric projection from E onto $A^{-1}0$.*

Corollary 3.3. *Let E be a smooth, strictly convex, and reflexive Banach space and $A\colon E \to 2^{E^*}$ a maximal monotone operator such that $D(A)$ is bounded and $0 \in \overline{D(A)}$, where $\overline{D(A)}$ denotes the norm closure of $D(A)$. Let r be a positive real number and $Q_r x = (J + rA)^{-1}Jx$ for all $x \in \overline{D(A)}$. Then the following hold:*

1. *For each $t \in (0,1)$, there exists a unique $u_t \in \overline{D(A)}$ such that*
$$u_t = (1-t)Q_r u_t;$$
2. *if E has the Kadec–Klee property and the norm of E is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{A^{-1}0}(0)$ as $t \downarrow 0$, where $P_{A^{-1}0}$ denotes the metric projection from E onto $A^{-1}0$.*

Let E be a Banach space and f a function from E into $(-\infty, \infty]$. Then f is said to be *proper* if the *effective domain* $D(f) = \{x \in E : f(x) \in \mathbb{R}\}$ of f is nonempty. It is said to be *convex* if

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) \tag{3.1}$$

whenever $x, y \in E$ and $\alpha \in (0,1)$. It is also said to be *lower semicontinuous* if $\{x \in E : f(x) \le r\}$ is closed in E for all $r \in \mathbb{R}$. Let $x \in E$ be given. Then a point $x^* \in E^*$ is said to be a *subgradient* of f at x if

$$f(x) + \langle y - x, x^* \rangle \le f(y) \tag{3.2}$$

for all $y \in E$. We denote by $\arg\min_{y \in E} f(y)$ the set $\{x \in E : f(x) = \inf_{y\in E} f(y)\}$. The set of subgradients of f at x is said to be the *subdifferential* of f at x and denoted by $\partial f(x)$. The mapping $\partial f\colon E \to 2^{E^*}$ is called the *subdifferential mapping* of f. Rockafellar's theorem [17, 18] ensures that ∂f is maximal monotone; see also Takahashi [21]. In this case, it holds that $(\partial f)^{-1}(0) = \arg\min_{y\in E} f(y)$.

Using Corollary 3.3, we can also show the following corollary:

Corollary 3.4 ([12]). *Let E be a smooth, strictly convex, and reflexive Banach space, r a positive real number, and f a proper lower semicontinuous convex function from E into $(-\infty, \infty]$ such that $D(f)$ is bounded and $0 \in \overline{D(f)}$. Then the following hold:*

1. *For each $t \in (0,1)$, there exists a unique $u_t \in \overline{D(f)}$ such that*
$$u_t = (1-t) \cdot \arg\min_{y \in E} \Big\{ f(y) + \frac{1}{2r}\phi(y, u_t) \Big\};$$
2. *if E has the Kadec–Klee property and the norm of E is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P(0)$ as $t \downarrow 0$, where P denotes the metric projection from E onto $\arg\min_{y\in E} f(y)$.*

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