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PREFACE

This volume is intended as the proceedings of Sapporo Guest House Symposium on Mathematics FINAL, Nonlinear Partial Differential Equations, held on March 29 and 30 in 2008 at Sapporo Guest House.

The first Sapporo Guest House Symposium was held in 1999 by Y. Giga. We kept the size of each meeting relatively small but international. However, this is the final Sapporo Guest House Symposium on Mathematics, as Sapporo Guest House will be closed and finish its operation on March 31, 2008. We are grateful for all the support and cooperation from the participants of this series of symposia, and the staff at Sapporo Guest House. The complete list of symposia at the Sapporo Guest House is in our website:

http://coe.math.sci.hokudai.ac.jp/sympo/sghs.html.en

Hideo Kubo
Tohru Ozawa

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SAPPORO GUEST HOUSE SYMPOSIUM ON MATHEMATICS, FINAL

Nonlinear Partial Differential Equations

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場 所：札幌天神山国際ハウス・研修室

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最寄駅：地下鉄南北線・澄川駅　または　南平岸駅

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組織委員: 小澤 徹（北海道大学）　Tel: 011-706-2672

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Asymptotic solutions of Hamilton-Jacobi equations with convex Hamiltonians

Hitoshi ISHII*

Abstract

The following discussion is based on joint work with N. Ichihara, especially on [14]. We are concerned with the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\,\cdot\,, 0) = u_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{1}$$

and study the long time behavior of the solution of (1).

We assume throughout the following (A1)–(A5).

(A1) $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$,

(A2) $\inf\{H(x,p) \mid x \in B(0,r),\ |p| \geq R\} \longrightarrow +\infty$ as $R \to +\infty$ for every $r > 0$,

(A3) $H(x,p)$ is convex with respect to p for every $x \in \mathbb{R}^n$,

(A4) for each $\phi \in \mathcal{S}_H^-$, there exist $C > 0$ and $\psi \in \mathcal{S}_{H-C}^-$ such that

$$\lim_{|x| \to \infty} (\phi - \psi)(x) = \infty,$$

(A5) $u_0 \in C(\mathbb{R}^n)$.

Here and henceforth, we denote by $\mathcal{S}_H$ (resp., $\mathcal{S}_H^-$, $\mathcal{S}_H^+$) the set of all continuous viscosity solutions (resp., subsolutions, supersolutions) of $H(x,\, Du(x)) = 0$ in $\mathbb{R}^n$. Similarly, given a domain Ω, we denote by $\mathcal{S}_H(\Omega)$ (resp., $\mathcal{S}_H^-(\Omega)$, $\mathcal{S}_H^+(\Omega)$) the set of all continuous viscosity solutions (resp., subsolutions, supersolutions) of $H(x,\, Du(x)) = 0$ in Ω.

Note that hypotheses (A1)–(A5) are not enough to assure the unique solvability of (1) in the sense of viscosity solution. To start our discussion with this generality, we

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Keywords : Hamilton-Jacobi equations, asymptotic solutions, long-time behavior, representation formula.

define the (unique) solution of (1) as follows. For any $\psi \in C(\mathbb{R}^n)$ and $t \geq 0$, we define the function $T_t\psi$ on $\mathbb{R}^n$ by

$$T_t\psi(x) = \inf\Big\{ \int_{-t}^{0} L(\eta(s), \dot\eta(s)),\, ds + \psi(\eta(-t)) \;\Big|\; \eta \in \mathcal{C}([-t,\, 0];\, x) \Big\}, \tag{2}$$

and refer the function $u(x,t) := T_t u_0(x)$ as the *solution* of (1). Here L is the *Lagrangian* of H defined by $L(x,\xi) = \sup\{\xi\cdot p - H(x,p)\}$ for $(x,\xi) \in \mathbb{R}^{2n}$ and $\mathcal{C}([a,\, b]; x)$, with $a < b$, denotes the space of all absolutely continuous functions (called curves) $\eta : [a,\, b] \to \mathbb{R}^n$ (i.e. $\eta \in \mathrm{AC}([a,\, b])$) such that $\eta(b) = x$. Also, $\mathcal{C}((-\infty,\, a]; x)$ denotes the space of all functions $\eta \in C((-\infty,\, 0])$ such that $\eta \in \mathcal{C}([c,\, a]; x)$ for all $c < a$.

We remark here that $T_t\psi(x)$ is well-defined with the following interpretation and $T_t\psi(x) \in [-\infty,\, 0)$ for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$. To see this, we fix $\psi \in C(\mathbb{R}^n)$. Note that $L(x,\xi)$ is lower semicontinuous on $\mathbb{R}^{2n}$, due to the assumption that $H \in C(\mathbb{R}^{2n})$, and hence that the function $s \mapsto L(\eta(s),\, \dot\eta(s))$ is Lebesgue measurable on $(-\infty,\, 0)$ for any $\eta \in \mathrm{AC}((-\infty,\, 0])$. Noting that $H(x,\, 0) = -\inf_{\xi\in\mathbb{R}^n} L(x,\xi)$ for all $x \in \mathbb{R}^n$, we observe that, for each $t > 0$, $r > 0$ and $\eta \in \mathrm{AC}([-t,\, 0])$ satisfying $\eta([-t, 0]) \subset B(0, r)$, we have $L(\eta(s),\, \dot\eta(s)) \geq -\max_{x\in B(0,r)} H(x,\, 0)$. Therefore, for $\eta \in \mathrm{AC}([-t,\, 0])$, it is natural to set

$$\int_{-t}^{0} L(\eta(s),\, \dot\eta(s))\, ds = \infty$$

if $s \mapsto L(\eta(s),\, \dot\eta(s))$ is not integrable on $(-t,\, 0)$. With this interpretation, $T_t\psi(x)$ is well-defined for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$. Next we see that for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$,

$$T_t\psi(x) \leq \int_{-t}^{0} L(x,\, 0)\, ds + \psi(x) = L(x,0)t + \psi(x) < \infty.$$

Now, let us consider an example where $T_t u_0(x) = -\infty$ for some (x,t). Let $n = 1$, $H(p) = (1/2)|p|^2$ and $u_0(x) = -|x|^2$. Then the Lagrangian L of H is given by $L(\xi) = (1/2)|\xi|^2$. Consider the curve $\eta \in \mathcal{C}((-\infty,\, 0];\, 0)$ given by $\eta(s) = cs$, with $c > 0$, and observe that for any $t > 0$,

$$T_t u_0(0) \leq \int_{-t}^{0} L(\dot\eta(s))\, ds + u_0(\eta(-t)) = \frac{c^2 t}{2} - (ct)^2 = \frac{c^2 t}{2}(1 - 2t),$$

which implies that $T_t u_0(0) = -\infty$ if $t > 1/2$. We recall that if the function $u(x,t) := T_t u_0(x)$ is continuous on an open set $U \subset \mathbb{R}^n \times (0,\, \infty)$, then u is a viscosity solution of $u_t + H(x, Du) = 0$ in U.

Our objective is to investigate the long-time behavior of the solution of (1). More precisely, we are concerned with the convergence of the form

$$u(x,t) + at - \phi(x) \longrightarrow 0 \qquad \text{in } \; C(\mathbb{R}^n) \qquad \text{as } t \to \infty \tag{3}$$

for some $a \in \mathbb{R}$ and $\phi \in C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ is equipped with the topology of locally uniform convergence. Note that if u satisfies (1) in the viscosity sense, then the function $\phi(x) - at$, which we call an *asymptotic solution* of (1), enjoys the following stationary Hamilton-Jacobi equation in the viscosity sense:

$$H(x, D\phi) = a \qquad \text{in } \mathbb{R}^n. \tag{4}$$

Thus a natural question related to (3) is the *additive eigenvalue problem* for H which seeks for a pair $(a, \phi) \in \mathbb{R} \times C(\mathbb{R}^n)$ such that ϕ is a solution of (4), i.e., $\phi \in \mathcal{S}_{H-a}$. The additive eigenvalue problem appears in ergodic control, in which it is called the ergodic control problem, and in homogenization, in which it is called the cell problem.

A standard approach to (3) is first to solve the additive eigenvalue problem for H and then to try to prove the convergence (3) for each fixed solution of the additive eigenvalue problem for H. However, to simplify our presentation, we will deal only with the latter step in the above approach and investigate if (3) holds for a fixed $a \in \mathbb{R}$. We assume moreover that $a = 0$. Indeed, convergence (3) with general a is equivalent to (3) with $a = 0$ once H and u are replaced by $H - a$ and $u(x,t) + at$, respectively. Then, one of fundamental observations is the following.

Proposition 1. *Let*

$$u_0^-(x) := \sup\{\, \phi(x) \mid \phi \in \mathcal{S}_H^-, \quad \phi \le u_0 \quad \textit{in } \mathbb{R}^n\},$$
$$u_\infty(x) := \inf\{\, \psi(x) \mid \psi \in \mathcal{S}_H, \quad \psi \ge u_0^- \quad \textit{in } \mathbb{R}^n\}.$$

Under the additional assumption

(A6) $\quad -\infty < u_0^-(x) \le u_\infty(x) < \infty$ *for all* $x \in \mathbb{R}^n$,

we have

$$\liminf_{t\to\infty} u(x,t) = u_\infty(x) \quad \textit{for all } x \in \mathbb{R}^n,$$

The condition (A6) is equivalent to saying that

$$\{\phi \in \mathcal{S}_H^- \mid \phi \le u_0 \ \text{ in } \mathbb{R}^n\} \ne \emptyset \quad \text{and} \quad \{\phi \in \mathcal{S}_H \mid \phi \ge u_0^- \ \text{ in } \mathbb{R}^n\} \ne \emptyset.$$

Thus our purpose here is to show the convergence:

$$u(\cdot, t) \longrightarrow u_\infty \quad \text{in } C(\mathbb{R}^n) \ \text{ as } t \to \infty. \tag{5}$$

Asymptotic problems of this type has been studied intensively in the last decade. As one of the most typical cases, it was proved that if H satisfies (A1)–(A3) and $H(x,p)$ is $\mathbb{Z}^n$-periodic with respect to x and is strictly convex with respect to p, then for each $\mathbb{Z}^n$-periodic initial function $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$ there exists a solution $(a, \phi) \in \mathbb{R} \times C(\mathbb{R}^n)$

of the additive eigenvalue problem for H such that (3) is valid and the constant a is determined independently of u_0. We refer to the literatures [2, 4, 5, 8, 9, 18, 19, 20] and references therein for more details. Remark that [2, 19] deal with non-convex Hamiltonians whereas most of others are concerned only with convex ones.

It has also been of interest in recent years on the long-time behavior of viscosity solutions to (1) that are not necessarily spatially periodic. As far as non-periodic solutions are concerned, the above (A1)-(A6) are insufficient to obtain the convergence (3) even if we admit strict convexity for H in any sense (see [3, 12]). The papers [1, 11, 12, 15] deal with some situations in which the solution of (1) has indeed the desired convergence of the form (3) for a suitable (a, ϕ).

Motivated by these earlier results, given a point $z \in \mathbb{R}^n$, we will introduce three general criteria (C1), (C2) and (C3), each of which, together mostly with (A1)–(A6) above, guarantees the pointwise convergence of the solution u of (1):

$$u(z,t) \longrightarrow u_\infty(z) \quad \text{as } t \longrightarrow \infty. \tag{6}$$

We then apply these criteria to obtain general convergence results of the form (5) and apply them to several examples. Our results cover most of existing results, and, on the other hand, involve a few observations which seem to be new.

One of our new observations is concerned with strict convexity for H. As pointed out in several literatures (see e.g. [2]), it is necessary in some situations to require a sort of strict convexity for H, as a genuine nonlinearity for H, so that the solution of (1) converges to an asymptotic solution as $t \to \infty$. We use condition $(A7)_+$ or $(A7)_-$ which guarantees, respectively, strict convexity of $H(x,p)$ in p at the zero level-set of H "upward" or "downward" (see below). We point out here that some of our convergence results involving $(A7)_-$ are not covered by the previous results. Another important feature here is in the observations of a "switch-back" motion of nearly optimal curves for the variational formula (2) for $T_t u_0$ in some situations. In one-dimensional case we have already encountered such a switch-back motion (see [13]), and here we investigate with greater generality the cases where such switch-back motions appear.

We use the dynamical approach in our investigations here which is based on tools from weak KAM theory (see e.g. [8, 10, 6, 7]) such as extremal curves, Aubry sets and representations formulas for solutions. We state our general criteria for the convergence to asymptotic solutions in terms of extremal curves, which seems inevitable to attain further generality.

Now, we introduce our criteria for the pointwise convergence. We begin with existence of extremal curves.

Theorem 2. *Let $\phi \in \mathcal{S}_H$ and $z \in \mathbb{R}^n$. Then there exists a curve $\gamma \in \mathcal{C}((-\infty,\, 0];\, z)$*

such that for all $t > 0$,

$$\int_{-t}^{0} L(\gamma(s),\, \dot\gamma(s))\, ds = \phi(z) - \phi(\gamma(-t)). \tag{7}$$

A curve $\gamma \in \mathcal{C}((-\infty, 0]; z)$ is said to be *extremal* for ϕ at z if it satisfies (7) for all $t > 0$. We denote by $\mathcal{E}_z(\phi)$ the set of all extremal curves for ϕ at z. Also, we use the notation: $\mathcal{E}(\phi) := \bigcup_{x\in\mathbb{R}^n} \mathcal{E}_x(\phi)$ and $\mathcal{E} := \bigcup_{\phi\in\mathcal{S}_H} \mathcal{E}(\phi)$.

We fix any $\gamma \in \mathcal{E}_z$ and introduce the first criterion

(C1) $\lim_{t\to\infty} (u_0 - u_\infty)(\gamma(-t)) = 0.$

Note that $u_0^- \le u_0$ in $\mathbb{R}^n$ by definition and that

$$\liminf_{t\to\infty} (u_\infty - u_0^-)(\gamma(-t)) = 0.$$

Hence condition (C1) is equivalent to the condition

$$\lim_{t\to\infty} (u_0 - u_0^-)(\gamma(-t)) = 0. \tag{8}$$

Theorem 3. *Under condition (C1), the convergence (6) holds.*

Next, we introduce our second criterion.

(C2) For each $\varepsilon > 0$ there exists a $\tau > 0$ such that for any $t > 0$ and for some $\eta \in \mathrm{AC}([-t,\, 0])$,

$$\eta(-t) = \eta(0) = \gamma(-\tau) \quad \text{and} \quad \int_{-t}^{0} L(\eta(s),\, \dot\eta(s))\, ds < \varepsilon.$$

Theorem 4. *Under condition (C2), the convergence (6) holds.*

Our third criterion is the following.

(C3) For any $\varepsilon > 0$, there exists a $\tau > 0$ and for each $t \ge \tau$, a $\sigma(t) \in [0,\, \tau]$ such that

$$u_\infty(\gamma(-t)) + \varepsilon > u(\gamma(-t), \sigma(t)).$$

Note that the above inequality is equivalent to the condition that there is an $\eta \in \mathcal{C}([-\sigma(t),\, 0];\, \gamma(-t))$ such that

$$u_\infty(\gamma(-t)) + \varepsilon > \int_{-\sigma(t)}^{0} L(\eta(s),\, \dot\eta(s))\, ds + u_0(\eta(-\sigma(t))).$$

In our next theorem, condition (C3) is used together with one of the conditions $(\mathrm{A7})_\pm$ on H, which are certain strict convexity requirements on H. We set $Q := \{(x,p) \in \mathbb{R}^{2n} \mid H(x,p) = 0\}$ and

$$S := \{(x,\xi) \in \mathbb{R}^{2n} \mid (x,p) \in Q,\ \ \xi \in D_2^- H(x,p) \ \text{ for some } p \in \mathbb{R}^n\},$$

where $D_2^- H(x,p)$ stands for the subdifferential of H with respect to the p-variable.

(A7)$_+$ (resp. **(A7)$_-$**) There exists a modulus ω satisfying $\omega(r) > 0$ for $r > 0$ such that for all $(x,p) \in Q$, $\xi \in D_2^- H(x,p)$ and $q \in \mathbb{R}^n$,

$$H(x,p+q) \geq \xi \cdot q + \omega((\xi \cdot q)_+) \qquad (\text{resp.} \quad \geq \xi \cdot q + \omega((\xi \cdot q)_-)), \tag{9}$$

where $r_\pm := \max\{\pm r, 0\}$ for $r \in \mathbb{R}$.

We remark here that condition (A7)$_+$ has already been used in [2, 12] to replace the strict convexity of $H(x,\cdot)$ in order to get the convergence (6) and also that condition (A7)$_-$ has been discussed in [13] when $n = 1$.

Theorem 5. *Assume that (C3) and either of (A7)$_+$ or (A7)$_-$ are satisfied. Then (6) holds.*

We will apply the criteria above, to obtain more specific conditions on (H, u_0) for the convergence (5).

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完全非線形楕円型方程式の L^p-粘性解の弱ハルナック不等式

小池　茂昭（埼玉大学)

2008年3月29〜30日

札幌天神山国際ハウス

1　序

本講演は A. Święch との共同研究 [12] である。

次の方程式の L^p-粘性**優解**の弱ハルナック不等式に関する結果を述べる。

$$\mathcal{P}^+(D^2u)+\mu(x)|Du|=f(x) \quad \text{in } \Omega \tag{1}$$

ここで、$\mathcal{P}^+(D^2u)$ は後で定義するが、およそ $-\triangle u$ と覚えておいて差し支えない。また、次の仮定は大切なので覚えてほしい。

$$\boxed{f\in L^p(\Omega),\ \mu\in L^q(\Omega)}$$

(1) は特殊な方程式に見えるが、多くの完全非線形方程式を扱うために、充分に一般的な方程式である。

2階楕円型方程式の弱解の regularity の第一段階を最大値原理とするならば、第二段階は弱ハルナック不等式である。実際、弱ハルナック不等式により、(局所最大値原理を用いることなく) 弱解の局所ヘルダー連続評価が可能であるだけでなく、境界付近での弱ハルナック不等式を導くことによって、境界までこめたヘルダー連続評価が得られる。また、2階微分の評価も可能である (Caffarelli[3], [2] を参照)。

更に、Liouville の定理、強最大値原理、Phragmén-Lindelöf の原理、非有界領域での最大値原理など、弱解の定性的性質の多くが導かれる。ここで注意しておくことは、係数 μ や非斉次項 f が非有界の場合、通常の "バリアー" など、Gilbarg-Trudinger[8] 等でおなじみの技法が使えないことである。また、非発散型方程式なので "部分積分" に依存する技法も使えない。

(1) のように1階微分項に非有界係数のある方程式に関しては、関数が $W^{2,n}_{\mathrm{loc}}$ に属し、(1) を殆どいたるところで満たす (即ち、L^p-強優解) ならば弱ハルナック不等式が $f,\mu\in L^n(\Omega)$ の場合に成立することが知られている (例えば、[8] を参照)。しかしながら、L^p-粘性解に

関しては、最近の Aleksandrov-Bakelman-Pucci(ABP と略す) 最大値原理に関する我々の結果 [11](2007 年) が知られているだけである。

ただし、$\mu \in L^{2n}(\Omega)$ を仮定すると、弱ハルナック不等式が粘性解に対しても示せる (Trudinger[19] で使われた方法、[9] も参照)。簡単のため $\mathcal{P}^+$ の代りに $-\triangle$ の場合に、この事実を確かめてみる。非負の粘性優解 u に対し、$\mu|Du| \leq \frac{1}{4}\mu^2 + |Du|^2$ に注意すると、$v = 1 - e^{-u} \geq 0$ とおけば、1 階微分項が消えて

$$-\triangle v \geq e^{-u}(f - \frac{1}{4}|\mu|^2)$$

となる。もし u が有界ならば、v に対して、粘性解に対する弱ハルナック不等式 (例えば、[2] 参照) を適用すれば、$u \approx v$ だから、u に対しても弱ハルナック不等式が示される。

しかし、古典的な結果は $\mu \in L^n(\Omega)$ であり、大きなギャップがある。

$p_0 = p_0(n, \lambda, \Lambda) \in [n/2, n)$ で次を満たす定数が知られている。

任意の $f \in L^p(\Omega)$ $(p > p_0)$ に対し、次を満たす関数 (後で L^p-強解と呼ぶ) $u \in C(\Omega) \cap W^{2,p}_{\text{loc}}(\Omega)$ が存在する。

$$\begin{cases} (i) & \mathcal{P}^-(D^2u) = f \quad a.e. \text{ in } \Omega, \\ (ii) & u = 0 \quad \text{on } \partial\Omega, \\ (iii) & -C\|f^-\|_{L^p(\Omega)} \leq u \leq C\|f^+\|_{L^p(\Omega)} \quad \text{in } \Omega \quad (\exists C > 0), \\ (iv) & \forall \Omega' \Subset \Omega, \exists C'(\text{dist}(\Omega', \partial\Omega)) > 0 \text{ s.t. } \|u\|_{W^{2,p}(\Omega')} \leq C'\|f\|_{L^p(\Omega)}. \end{cases} \tag{2}$$

【注意】ここで大切なのは、$p \geq n$ の場合には、ABP 最大値原理 ((iii) に対応する) が得られることである (更にこの場合、積分は Ω 全体である必要はない)。これによって、Caffarelli の L^p 理論を使えば (iv) は比較的易しく導かれる。$n > p > p_0$ まで、(2) を最初に示したのは Escauriaza[6] であり、p_0 を Escauriaza 定数と呼ぶことがある (p_0 の依存性に関しては [5] も参照)。

主結果の弱ハルナック不等式を述べる。Q_t を原点中心、一辺 $t > 0$ の n 次元立方体とする。$Q_4 \subset \Omega$ としておこう。

主結果　(弱ハルナック不等式)

> $q > n$ かつ $q \geq p > p_0$ の場合に (1) の非負 L^p-粘性優解 u に対し、
> $$\|u\|_{L^r(Q_1)} \leq C\left(\inf_{Q_1} u + \|f\|_{L^p(Q_4)}\right)$$
> となる u に依存しない定数 $r, C > 0$ が存在することある。

【注意】・正確には右辺の Q_4 は $Q_{3+\varepsilon}$ $(\forall\varepsilon \in (0,1])$ でよいことは議論をよくみるとすぐわかる。更に、Cabré[3] の "被覆法" を用いれば、$Q_{1+\varepsilon}$ $(\varepsilon > 0)$ でよいことも分かる。これは、きちんと書かれていないので付録に述べておく。

・$r \geq 1$ とは限らないので、左辺はノルムでないかもしれない。でも、後の議論では、そんなの関係ない。

・弱ハルナック不等式が登場すると、局所最大値原理が成り立つか気になるところだが、必要ないので右から左へ受け流す。

2　準備

以下の議論で $C>0$ は、与えられた条件 (n, λ, Λ 等) 以外には依存しない定数とする。

ここでは、$p>1$ は少なくとも次を満たすものとする。

$$p > \frac{n}{2}$$

これは、$W^{2,p}$ 関数が連続になるための条件だけでなく、$W^{2,p}$ 関数が殆ど至る所で2階のテイラー展開が可能である条件でもある。

$r>0$ と関数 $g: U \subset \mathbf{R}^n \to \mathbf{R}$ に対し、$\|g\|_{L^r(U)} = \left(\int_U |g|^r dx\right)^{\frac{1}{r}}$ とする。誤解がない場合は $\|g\|_r$ と書く。また、関数の定義域をはみ出して関数を考える時はゼロ拡張とする。

さて、定義を述べるために一般の完全非線形2階方程式

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega \tag{3}$$

を考える。ただし、$\Omega \subset \mathbf{R}^n$ は少なくとも開集合とする。

L^p-粘性解理論では、係数や非斉次項に可測関数を考えているので、$x \to F(x,r,p,X)$ や $x \to f(x)$ に連続性は仮定しない。

以下、関数 $F: \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$ は、次の意味で一様楕円型とする。任意の $x \in \Omega$, $r \in \mathbf{R}$, $p \in \mathbf{R}^n$, $X, Y \in S^n$ に対し

$$\mathcal{P}^-(X-Y) \le F(x,r,p,X) - F(x,r,p,Y) \le \mathcal{P}^+(X-Y)$$

が成り立つ。また、S^n は $n \times n$ 実対称行列で全体であり、S^n では通常の部分順序を考える。

また、ここでは一様楕円型定数

$$0 < \lambda \le \Lambda$$

を固定し、プッチの極大・極小作用素 $\mathcal{P}^\pm : S^n \to \mathbf{R}$ を次のように定義する。$X \in S^n$ に対し、

$$\mathcal{P}^+(X) := \max\{-\mathrm{trace}(AX) \mid \lambda I \le A \le \Lambda I\}$$
$$\mathcal{P}^-(X) := \min\{-\mathrm{trace}(AX) \mid \lambda I \le A \le \Lambda I\}$$

【注意】・定義からすぐに $\mathcal{P}^+(X) = -\mathcal{P}^-(-X)$ ($\forall X \in S^n$) が示せるので、$\mathcal{P}^-$ に関する性質から $\mathcal{P}^+$ に関する性質も簡単に分かる。

・$\mathcal{P}^-$ は凹、$\mathcal{P}^+$ は凸であることが分かる。また、$\mathcal{P}^\pm(tX) = t\mathcal{P}^\pm(X)$ ($t \ge 0$) も簡単に分かる。

・更に、定義から次の不等式がわかり、それらをうまく使えば、$\mathcal{P}^\pm$ は、$-\triangle$ と考えて大丈夫である。

$$\boxed{\mathcal{P}^-(X)+\mathcal{P}^-(Y) \le \mathcal{P}^-(X+Y) \le \mathcal{P}^+(X)+\mathcal{P}^-(Y) \le \mathcal{P}^+(X+Y) \le \mathcal{P}^+(X)+\mathcal{P}^+(Y)}$$

f が L^p 関数のときは、Carrafelli[3] による局所 L^p 評価が知られており、その研究を元に Caffarelli-Crandall-Kocan-Święch[4] によって、L^p 粘性解が導入された。実際、比較原理を中心とした通常の粘性解の技法は全く使えないことに注意しておく。実際、可測係数を念頭においており、その場合、一意性に関しては "反例の王様"Nadirashvili[13] や Safonov[17] による反例があるので、比較原理が成立すると矛盾してしまう。

本講演に関しては、L^p-粘性解の定義は "殆ど" 使わないが念のため述べる。

L^p-粘性解の定義

$u \in C(\Omega)$ が (3) の L^p-粘性劣解 (L^p-subsolution) とは、任意の $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ に対し、もし $u-\phi$ が $x \in \Omega$ で局所最大値を取るならば、次が成り立つこととする。 $$\lim_{\varepsilon\to 0} ess.\ \inf_{B_\varepsilon(x)} \{F(y,u(y),D\phi(y),D^2\phi(y)) - f(y)\} \le 0$$
$u \in C(\Omega)$ が (3) の L^p-粘性優解 (L^p-supersolution) とは、任意の $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ に対し、もし $u-\phi$ が $x \in \Omega$ で局所最小値を取るならば、次が成り立つこととする。 $$\lim_{\varepsilon\to 0} ess.\ \sup_{B_\varepsilon(x)} \{F(y,u(y),D\phi(y),D^2\phi(y)) - f(y)\} \ge 0$$
$u \in C(\Omega)$ が (3) の L^p-粘性解 (L^p-solution) とは、u が (3) の L^p-粘性劣解かつ L^p-粘性優解であることとする。

更に、L^p-強解の定義も述べておく。こちらは使うかもしれない。

L^p-強解の定義

$u \in C(\Omega)$ が (3) の L^p-強優解 (L^p-strong subsolution) とは、$u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ であり、次が成り立つこととする。 $$F(x,u(x),Du(x),D^2u(x)) \le f(x) \quad a.e. \text{ in } \Omega$$
$u \in C(\Omega)$ が (3) の L^p-強劣解 (L^p-strong subsolution) とは、$u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ であり、次が成り立つこととする。 $$F(x,u(x),Du(x),D^2u(x)) \ge f(x) \quad a.e. \text{ in } \Omega$$
$u \in C(\Omega)$ が (3) の L^p-強解 (L^p-strong solution) とは、u が (3) の L^p-強劣解かつ L^p-強優解であることとする。

【注意】・(3) の L^p-強解ならば L^p-粘性解になるのが、自然であるが自明ではない。しかし、充分一般的な F の仮定の下で成立する ([12] 参照)。よって、ここで得られる結果は当然 L^p-強解でも成り立つ。もちろん、L^p-強解の場合、$q=n$ で ABP 最大値原理が成り立つ

から、より強い結論が得られる。
・逆に、(3) の L^p-粘性解が $W^{2,p}_{\mathrm{loc}}(\Omega)$ に属すれば、L^p-強解になることが期待されるが、これも自明ではない ([12] 参照)。
・上記の二つの注意は、単純な方程式 (1) に関して言えば、$\mu \in L^\infty(\Omega)$ では、[4] で述べられているが、$\mu \in L^q(\Omega)$ の場合は知られてなかった。

また、Du に関して一次以上の増大度があっても上記の二つの注意は成立する。(証明は難しくないが、細かい評価が必要。例えば中川 [16] の結果を参照)

3　主結果の証明のあらすじ

以前の講演に比べて、だいぶ簡単になった。

古典的な強解に対する結果は $\mu, f \in L^n(\Omega)$ で成り立つが、主結果ではこの場合をカバーできていない。これは、ABP 型最大値原理 ((2) の (iii) のこと) が L^p-粘性解で知られていないことが原因である。L^p-粘性解に対する ABP 最大値原理は、主結果の p, q に対する条件で Święch との共同研究で得られている ([11] 参照)。

Caffarelli の補題を述べておく。

Caffarelli の補題　([2] 参照)

> Ω を滑らかな境界を持つ有界領域とし、$g \in C^\infty(\partial\Omega)$ とする。
> 次の Dirichlet 問題の $C^2(\overline{\Omega})$ 解が存在する。
> $$\begin{cases} \mathcal{P}^+(D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

【注意】・要するに、最も簡単な場合は完全非線形でも古典解があるということである。ただし、$\mathcal{P}^+$ が凸であることははずせない。実際、12 次元で古典解がない例 ([14])、24 次元で $W^{2,\infty}$ ですらない例 ([15]) によって最近得られた (読めない論文だが) からである。(愚痴になりますが、"この分野" には読めない良い論文を書く人が多い。Ca$\cdots$, Kr$\cdots$, Nad$\cdots$, $\cdots$、そういう観点からは Evans 氏は "ネ申" です。)
・Caffarelli の原著 [3](および、[2]) では、Ω が球の場合に、特殊な g に対して具体的に構成しており、幾つかの付随した性質も簡単に確かめられる。実際、新たに上述の補題を述べなくても [3] の補題でも充分である (但し、$B_{1/2} \subset Q_1 \subset \overline{B}_{\sqrt{n}/2}$ など、球と立方体の包含関係で煩わしい定数がつく)。

一般の Ω では、具体的には u が分からないが、強最大値原理を使えば、必要な性質が得られる ([12] 参照)。

証明の概略　$q \geq p > n$ の場合だけ考える。$n \geq p > p_0$ は [10], [11] で導入した "逐次比較関数法" を用いる (付録参照)。u の代わりに、充分大きな $t > 1$ をとり、$u/\{t\|f\|_p + \inf_{Q_1} u\}$ とすれば、$\inf_{Q_1} u \leq 1$ かつ、$\|f\|_p \ll 1$ と仮定してよい。すると、$\|u\|_{L^r(Q_1)} \leq C$ となる $r,\ C > 0$ を見つければよい。(実は、より弱い条件 $\inf_{Q_3} u \leq 1$ で充分。この 3 は、Calderon-Zygmmund の立方体分割の補題を使うので小さくできないが、最終的に Cabré の被覆法を使えば、あまり気にしなくて良くなる。)

分布関数の言葉に直せば、ある $\sigma > 0,\ C > 0$ に対し、

$$|\{x \in Q_1 \mid u(x) \geq s\}| \leq Cs^{-\sigma} \quad (s > 0)$$

となっていればよい。更に、離散的に任意の自然数 k に対し、

$$|\{x \in Q_1 \mid u(x) \geq M^k\}| \leq \theta^k$$

となる $M > 1, \theta \in (0,1)$ が存在すればよい。

$k = 1$ の時に、この評価を示せれば、立方体分割の補題と縮尺法で $k \geq 2$ でも示せるので、$k = 1$ に限って示してみる。

Q_4 は滑らかでないけど、適当に近似した領域で考えると Caffarelli の補題から次を満たす $\phi \in C^2(Q_4)$ が見つかる。

面倒だから、立方体は滑らかな境界を持つとしてしまう (領域を近似すればうまく行く)。Caffarelli の補題から、$\phi_0 \in C^2(Q_4 \setminus (Q_{1/2})^o)$ で、次の Dirichlet 問題の解がある。

$$\begin{cases} \mathcal{P}^-(D^2\phi_0) = 0 & \text{in } (Q_4)^o \setminus Q_{1/2}, \\ \phi_0 = 0 & \text{on } \partial Q_4, \\ \phi_0 = -1 & \text{on } \partial Q_{1/2}. \end{cases}$$

ϕ_0 は古典解だから、強最大値の原理から $-\sigma := \max_{Q_3} \phi_0 < 0$ となり、$\phi = 2\sigma^{-1}\phi_0$ とおくと、$\phi(x) \leq -2\ (x \in Q_3)$ を得る。ϕ を滑らかに $Q_{1/2}$ 内部に拡張したものを改めて ϕ と書けば、$\phi \in C^2(Q_4)$ は次を満たす。

$$\begin{cases} \mathcal{P}^-(D^2\phi) = \xi(x) & \text{in } (Q_4)^o, \\ \mathrm{supp}\xi \subset Q_1, & \\ \phi = 0 & \text{on } \partial Q_4, \\ \phi \leq -2 & \text{in } Q_3. \end{cases}$$

$w := u + \phi$ とおくと、$\mathcal{P}^\pm$ の不等式から w は次の L^p-優解になる。

$$\mathcal{P}^+(D^2w) + \mu(x)|Dw| \geq f(x) - \mu(x)|D\phi(x)| + \xi(x) \quad \text{in } \Omega_0$$

ただし、$\Omega_0 = \{x \in Q_4^o \mid w(x) < 0\}$ である。ABP 最大値原理より

$$\max_{Q_4}(-w) \leq C\exp(C\|\mu\|_n)\left(\|f\|_n + \|D\phi\|_\infty\|\mu\|_{L^n(\Omega_0)} + \|\xi\|_{L^n(\Omega_0)}\right)$$

$\mathrm{supp}\xi \subset Q_1$ だから、右辺第二項の積分は Ω_0 の変わりに $\{x \in Q_1 \mid w(x) < 0\}$ としてよい。

$\max_{Q_4}(-w) \geq \max_{Q_3}(-u - \phi) \geq \max_{Q_3}(-u + 2) \geq 1$ より、$\|\mu\|_n$ が小さければ ($\|f\|_n$ も小さいことを思い出そう)、次が分かる。

$$|\{x \in Q_1 \mid u(x) < -\min_{Q_1}\phi =: M\}| \geq \delta$$

となる $\delta \in (0,1)$ が存在する。よって、

$$|\{x \in Q_1 \mid u(x) \geq M\}| \leq \theta := 1 - \delta$$

となる。

後は、μ が小さいと言う仮定を Cabré の被覆法 (付録参照) で取り除けばよい。

4 応用

主結果から導かれる新しい結果を列挙する。

(A) 強最大値原理

『$f \equiv 0$ の時、(1) の L^p-粘性劣解 u が内部で最大値を取れば定数関数になる』ことを強最大値原理と呼ぶ。これが (1)(逆の不等式だが) で成り立つ。

(B) 境界付近での弱ハルナック不等式

ポテンシャル論の境界ハルナック不等式とは違うことに注意する。具体的な結果は例えば、[8] と同じなので略す。

(B) を使うことで、応用が広がる。

(C) L^p-粘性解の境界までこめたヘルダー連続評価

もちろん、(B) なしでも弱ハルナック不等式から内部ヘルダー連続評価は導かれる。最近、Sirakov[18] が弱ハルナック不等式を使わずに、同じ結果を得ている。ただし、$q \geq p > n$ の場合のみである ($q > n \geq p > p_0$ の場合は、我々の逐次比較関数法 [11] を使わないと最大値原理すら導かれない)。

(D) L^p-強解の存在

(C) を用いることで (よって、(B) を使っている)、$\mu \in L^q(\Omega)$ $(n < q < \infty)$ の時、次の L^p-強解の存在が示せる。

$$\begin{cases} \mathcal{P}^{\pm}(D^2u) + \mu(x)|Du| = f(x) & a.e. \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

ただし、$\mathrm{supp}\mu \Subset \Omega$ の場合は Fok[7] で知られている。なぜ、$\mathrm{supp}\mu \Subset \Omega$ という仮定が必要だったかというと、そうでないと近似解の一様収束がしめせない。詳しく言えば、μ_k を μ の近似、u_k をそれらを使った対応する近似方程式の解とすると、最大値原理から

$$\max_{\overline{\Omega}}(u_k - u_j) \leq C\|f_k - f_j\|_p + C\|\mu_k|Du_k| - \mu_j|Du_j|\|_p$$

右辺第 2 項が $k, j \to \infty$ でゼロに行くかどうか分からない。

内部 $W^{2,p}$ 評価は [8] や [4]、[11] の方法でできるので、$\mathrm{supp}\mu$ がコンパクトなら、$W^{2,p}_{\mathrm{loc}}$ 局所評価と合わせて右辺第 2 項がゼロに行くことを示せる。

(E) 非有界領域での ABP“型” 最大値原理

Cabré[3] が領域に適当な条件 (例えば、柱状領域や錘状領域) があれば、非有界領域でも ABP 最大値原理 (ただし、解の有界性を仮定して、ABP 型の評価を得ること) が成り立つことを示し、Vitolo 等によって、もっと色々な非有界領域を含むよう拡張されてきた。しかし、係数に関しては古典的な結果 [8] の範囲を出ていない。(B) を用いると、この点を改良できる。

更に、Phragmén-Lindelöf の原理や放物型への拡張等々は、中川氏 (院生) が鋭意努力中である。

5　付録　　（Cabréの被覆法）

まず、我々の主結果を正しいとする。Q_4 の代わりに Q_2 で成り立つことを示す。同じように $Q_{1+\varepsilon}$ $(\forall\varepsilon > 0)$ で成り立つことも簡単に分かる。

　Q^j $(j = 1, 2, \ldots, \exists N)$ を Q_1 の点を中心にした一辺が $1/4$ の立方体で、ある $\sigma \in (0,1)$ に対し、次を満たすとする (Q の上付きの番号は一辺の長さではなく、単なる番号である)。

$$Q_1 \subset \bigcup_{j=1}^{N} Q^j \subset Q_2, \quad |Q^j \cap Q^{j+1}| \geq \frac{\sigma}{4^n}$$

ただし、上の第 2 式では、$N+1=1$ とする。2 次元なら、簡単に Q^j を選べるが、高次元では、このように "一列" につなげるかどうか、微妙かもしれないが、第 2 の条件の代りに次の条件で充分なので気にしないことにする。

$$\left\{\begin{array}{l} \text{任意の } 1 \leq j < k \leq N \text{ に対し、次を満たす } \{j_\ell\}_{\ell=1}^{m} \text{ が存在する} \\ \qquad j_1 = j,\ j_m = k,\ |Q_{j_\ell} \cap Q_{j_{\ell+1}}| \geq \dfrac{\sigma}{4^n}\ (\ell = 1, \ldots, m-1) \end{array}\right.$$

　各 j に対し、弱ハルナック不等式から

$$\|u\|_{L^r(Q^j)} \leq C_1 \left(\inf_{Q^j} u + \|f\|_p \right)$$

が成り立つ。

　一方、Q^j の選び方から、

$$\inf_{Q^j} u \leq \inf_{Q^j \cap Q^{j+1}} u \leq \left(\frac{1}{|Q^j \cap Q^{j+1}|} \int_{Q^j \cap Q^{j+1}} u^r dx \right)^{1/r} \leq C_2 \|u\|_{L^r(Q^{j+1})}$$

が成り立っている。

　さて、$\inf_{Q_1} u = \inf_{Q^{j_0}} u$ となる j_0 を固定する。$j_0 = N$ としても一般性を失わない。$\alpha = C_1 C_2$ とおく ($\alpha \geq 1$ としてよい)。任意の $1 \leq j < N$ に対して、

$$\begin{array}{rcl} \|u\|_{L^r(Q^j)} & \leq & C_1 \left(\inf_{Q^j} u + \|f\|_p \right) \\ & \leq & C_1 \left(C_2 \|u\|_{L^r(Q^{j+1})} + \|f\|_p \right) \\ & \leq & \cdots \\ & \leq & \alpha^{N-j} \|u\|_{L^r(Q^N)} + C_1 (1 + \cdots + \alpha^{N-j-1}) \|f\|_p \\ & \leq & C_1 \left(\alpha^{N-j} \inf_{Q^N} u + \dfrac{1-\alpha^N}{1-\alpha} \|f\|_p \right) \end{array}$$

となり、$0 < r < 1$ の時は ($r \geq 1$ なら、通常の三角不等式を使う)、

$$\|u_1 + \cdots + u_N\|_r \leq 2^{(\frac{1}{r}-1)(N-1)} \sum_{j=1}^{N} \|u_j\|_r$$

が成り立つことに注意すれば (もっといい不等式を知ってたら教えてください)、

$$\begin{aligned}\|u\|_{L^r(Q_1)} &\leq 2^{(\frac{1}{r}-1)(N-1)}\sum_{j=1}^{N}\|u\|_{L^r(Q^j)}\\ &\leq 2^{(\frac{1}{r}-1)(N-1)}C_1\left(N\alpha^N\inf_{Q^N}u+\frac{1-\alpha^N}{1-\alpha}\|f\|_p\right)\end{aligned}$$

$\inf_{Q^N}u=\inf_{Q_1}u$ を思い出せば、証明が終わる。

6 付録 (逐次比較関数法)

[10], [11] の逐次比較関数法 (勝手に名付けてしまいました) による ABP"型" 最大値原理のおさらいをしておく。これは、非斉次項が n 以下のべき乗可積分 (つまり、$f\in L^p(\Omega)$, $p\in(p_0,n]$) の時に、f を n 以上のべき乗可積分関数と見なせるようにするための方法である。

私見では、このような技法は簡単ではあるが、今までなかったと思われる。というより、必要なかったから誰も作ろうとしなかった。

証明を簡単にするため、$q>n>p>p_0\geq n/2$ に加え、次を仮定する (一般には、以下の議論を有限回繰り返せばよい)。

$$q>\frac{pn}{2p-n} \tag{5}$$

この条件は、$p<n$ が充分 n に近いことを意味している。

さて、$\mathcal{P}^{\pm}$ の代わりに $-\triangle$ で計算してみよう。

$\Omega\subset B_1$ を "一様外錐条件" を満たすとする (境界を頂点とする、一様な開き具合の錐が Ω の外に取れること。滑らかなら OK)。$\mu\in L^q_+(\Omega)$ と $f\in L^p_+(\Omega)$ に対し、$u\in C(\overline{\Omega})$ を次の L^p-粘性劣解とする。

$$-\triangle u-\mu(x)|Du|\leq f(x)\quad\text{in }\Omega.$$

ABP"型" 最大値原理とは、次を満たす u に依存しない定数 $C_k>0$ $(k=1,2)$ と自然数 N が取れることである ([11] 参照)。

$$\max_{\overline{\Omega}}u\leq\max_{\partial\Omega}u+C_1\left\{\exp(C_2\|\mu\|_n)\|\mu\|_q^N+\sum_{k=0}^{N-1}\|\mu\|_q^k\right\}\|f\|_p.$$

【注意】$q>n$ かつ $q\geq p\geq n$ の場合は、右辺の "$\sum_{k=0}^{N-1}$" の項がない形で成立している ([4])。これが "型" なしの由緒正しい ABP 最大値原理である。

(5) の仮定の下でこれを示そう。(5) の場合は、$N=1$ となることが以下で分かる。

$R>1$ をとり、(2) により次の L^p-強解 $v\in C(\overline{B}_R)\cap W^{2,p}_{\mathrm{loc}}(B_R)$ を考える。

$$\begin{cases}-\triangle v=-f & \text{in }B_R,\\ \qquad v=0 & \text{on }\partial B_R.\end{cases}$$

(2) から、$0 \leq -v \leq C\|f\|_p$ in B_R かつ、$\|v\|_{W^{2,p}(\Omega)} \leq C\|f\|_p$ となっている。よって、$\|Dv\|_{L^{p^*}(\Omega)} \leq C\|f\|_p$ となる。ここで、$p^* = np/(n-p)$ である。

簡単な計算から、上の仮定のおかげで、$p' := pqn/(pn+qn-pq) > n$ とおくと、

$$\|\mu Dv\|_{p'} \leq C\|\mu\|_q\|Dv\|_{p^*} \leq C\|\mu\|_q\|f\|_p$$

となる。

(少なくとも形式的には)$w := u + v$ は次の L^p-粘性劣解になる。

$$-\triangle w - \mu(x)|Dw| \leq \mu(x)|Dv(x)| =: f_1(x) \quad \text{in } \Omega$$

$f \in L^n(\Omega)$ だから、通常の ABP 最大値原理より $\max_{\overline{\Omega}} w \leq \max_{\partial\Omega} w + C_1 \exp(C_2\|\mu\|_n)\|f_1\|_n$ が成り立つ。よって、$0 \geq v$ に注意して、次のように示せる。

$$\begin{aligned}
\max_{\overline{\Omega}} u &\leq \max_{\overline{\Omega}} w + \max_{\overline{\Omega}}(-v) \\
&\leq \max_{\partial\Omega} w + C\|f\|_p + C_1\exp(C_2\|\mu\|_n)\|f_1\|_n \\
&\leq \max_{\partial\Omega} u + C\{1 + \exp(C_2\|\mu|_n)\|\mu\|_q\}\|f\|_p.
\end{aligned}$$

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LARGE TIME BEHAVIOR OF SOLUTIONS OF THE CAUCHY-DIRICHLET PROBLEM FOR HAMILTON-JACOBI EQUATIONS WITH PERIODIC BOUNDARY DATA

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1. Introduction

We investigate the large time behavior of viscosity solutions of the Cauchy-Dirichlet problem with periodic boundary data for Hamilton-Jacobi (HJ) equations:

$$\text{(CD)} \qquad \begin{cases} u_t(x,t) + H(x, Du(x,t)) = 0 & \text{in } \Omega \times (0,\infty), & (1) \\ u(x,0) = f(x) & \text{in } \Omega, & (2) \\ u(x,t) = g_1(x,t) + g_2(x,t) + bt & \text{on } \partial\Omega \times (0,\infty), & (3) \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $H = H(x,p)$ is a real-valued function on $\overline{\Omega} \times \mathbb{R}^n$ which is assumed to satisfy

(A1) $H \in C(\overline{\Omega} \times \mathbb{R}^n)$,
(A2) the function $p \mapsto H(x,p)$ is strict convex for each $x \in \overline{\Omega}$,
(A3) the function H is coercive, i.e.

$$\lim_{r\to\infty} \inf\{H(x,p) \mid x \in \overline{\Omega}, p \in \mathbb{R}^n \setminus U(0,r)\} = \infty,$$

(B) for each $z \in \partial\Omega$, there are a constant $r > 0$, a C^1-diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and a function $b \in W^{1,\infty}(\mathbb{R}^{n-1})$ such that

$$\Phi(\Omega \cap U(z,r)) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > b(x')\} \cap \Phi(U(z,r)).$$

Here $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is the unknown function and $f : \overline{\Omega} \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \times [0,\infty) \to \mathbb{R}$ are given functions and $b \in \mathbb{R}$ is a given constant. We make throughout the following assumptions on f, g_1 and g_2:

(A4) $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and $g_1, g_2 \in C(\partial\Omega \times [0,\infty)) \cap W^{1,\infty}(\partial\Omega \times (0,\infty))$.
(A5) The function $G(x,t) := g_1(x,t) + g_2(x,t) + bt$ satisfies $f(x) \le G(x,0)$ for any $x \in \partial\Omega$.
(A6) There exists a function $g_\infty \in C(\partial\Omega)$ such that $g_1(x,t) \to g_\infty(x)$ uniformly for $x \in \partial\Omega$ as $t \to \infty$.
(A7) The function $g_2(x,\cdot)$ is a periodic function with period $T \ge 0$ for each $x \in \partial\Omega$, i.e. $g_2(x, t+T) = g_2(x,t)$ for all $x \in \partial\Omega$ $t \ge 0$. In the case where $T = 0$, we think that g_2 is a time-independent function and we write $\overline{g_2}(x)$ for $g_2(x,t)$.

We remark that our hypotheses (B) and (A4) are stronger than is really necessary, since we wish to focus on the large time behavior of viscosity solutions of (CD). See [13] for results for this asymptotic problem under weaker hypotheses.

Hereinafter, we mean by "solutions", "subsolutions", and "supersolutions" viscosity solutions, viscosity subsolutions and viscosity supersolutions, respectively.

2. Short history and Known results

The study of the large time behavior of solutions of the Cauchy problem for HJ equations goes back to Kružkov, Lions and Barles, where they studied the case when $\Omega = \mathbb{R}^n$ and $H = H(p)$ does not depend on the x variable. In the case when $H = H(x,p)$ depends both on x and p, the first general results were obtained by Namah and Roquejoffre [14] and Fathi [6], and their results are for HJ equations on compact manifolds Ω without boundary. Afterwards this subject has received much interest. (See [3, 15, 4], for example.)

The asymptotic problem of the same kind has been studied in the case where $\Omega = \mathbb{R}^n$ by Barles-Roquejoffre [2], Ishii [10] and Ichihara-Ishii [9]. The convergence rate has been investigated by Fujita [7] and Fujita-Uchiyama [8].

The large time asymptotic problem under the Dirichlet boundary condition has been treated in [1, 15, 12]. In particular, Roquejoffre ([15]) studied asymptotic behavior of solutions of the Cauchy-Dirichlet problem with periodic boundary data. In [15], the initial and the boundary data are assumed to satisfy a certain compatibility condition in order that solutions of Cauchy-Dirichlet problem satisfy the boundary condition in the pointwise sense.

The author has recently studied this type asymptotic problem for HJ equations under the state constraint boundary condition and the time-independent Dirichlet boundary condition. In the study for the Cauchy-Dirichlet problem, we do not assume a compatibility condition on the initial and the boundary data. In this case, we do not expect any more the existence of solutions of (CD) which satisfy the Dirichlet condition pointwise. We have to understand the boundary condition in a generalized sense. The definition of solutions of (CD) is written below. This gives a viewpoint which unifies the state constraint and Dirichlet boundary conditions.

In this paper, we do not assume a compatibility condition on the initial and the boundary value either, so we deal with the solution of (CD) which satisfies the boundary condition in a generalized sense.

In the last of this section, we write the result for the large time behavior of solutions of the Cauchy problem for HJ equations with state constraints

$$(\mathrm{C}) \qquad \begin{cases} u_t(x,t) + H(x, Du(x,t)) \le 0 & \text{in } \Omega \times (0,\infty), & (4)\\ u_t(x,t) + H(x, Du(x,t)) \ge 0 & \text{in } \overline{\Omega} \times (0,\infty), & (5)\\ u(x,0) = f(x) & \text{on } \overline{\Omega}. & (6) \end{cases}$$

Theorem 1 ([11, Theorem 2.1]). *Let $f \in C(\overline{\Omega})$ and u be the solution of* (C). *Then there exists a solution of $v \in C(\overline{\Omega})$ of $H(x, Du) \le c_H$ in Ω and $H(x, Du) \ge c_H$ in $\overline{\Omega}$ such that*

$$u(x,t) + c_H t - v(x) \to 0 \quad \textit{uniformly for } x \in \overline{\Omega} \textit{ as } t \to \infty,$$

where c_H is the constant defined below.

3. Main results

We define (viscosity) solutions of (CD) by the following.

Definition 1. *Let $u \in C(\overline{\Omega} \times [0,\infty))$. (i) We call u a subsolution (resp., supersolution) of* (CD) *if the following conditions* (1)-(3) *hold:* (1) *u is a (viscosity) subsolution (resp., (viscosity) supersolution) of* (1), (2) *$u(x,0) \le f(x)$ (resp., $u(x,0) \ge f(x)$) for all $x \in \Omega$, and* (3) *for any $\phi \in C^1(\overline{\Omega} \times (0,\infty))$ and any $(x_0,t_0) \in \partial\Omega \times (0,\infty)$ such that $u-\phi$ takes a local maximum (resp., minimum) at (x_0,t_0),*

$$\begin{aligned} \min\{(u-G)(x_0,t_0), \phi_t(x_0,t_0) + H(x_0, D\phi(x_0,t_0))\} &\le 0 \\ (\textit{resp., } \max\{(u-G)(x_0,t_0), \phi_t(x_0,t_0) + H(x_0, D\phi(x_0,t_0))\} &\ge 0). \end{aligned}$$

(ii) *We call u a solution of* (CD) *if it is a subsolution and a supersolution of* (CD).

We now state our main theorems.

Theorem 2. *Assume $T=0$ in* (A7). *Let $u \in C(\overline{\Omega} \times [0,\infty))$ be the solution of* (CD). (i) *If $c_H + b > 0$, then there exists a solution $v \in C(\overline{\Omega})$ of*

$$(\mathrm{SC})_{c_H} \qquad \begin{cases} H(x, Du(x)) \le c_H & \text{in } \Omega, \qquad (7) \\ H(x, Du(x)) \ge c_H & \text{on } \overline{\Omega} \qquad (8) \end{cases}$$

such that

$$u(x,t) - (v(x) - c_H t) \to 0 \quad \textit{uniformly for } x \in \overline{\Omega} \textit{ as } t \to \infty.$$

(ii) *If $c_H + b \le 0$, then there exists a solution $w \in C(\overline{\Omega})$ of*

$$(\mathrm{D})_{-b} \qquad \begin{cases} H(x, Du(x)) = -b & \text{in } \Omega, \qquad (9) \\ u(x) = g_\infty(x) + \overline{g_2}(x) & \text{on } \partial\Omega \qquad (10) \end{cases}$$

such that

$$u(x,t) - (w(x) + bt) \to 0 \quad \textit{uniformly for } x \in \overline{\Omega} \textit{ as } t \to \infty.$$

Theorem 3. *Assume $T > 0$ in* (A7). *Let $u \in C(\overline{\Omega} \times [0,\infty))$ be the solution of* (CD). (i) *If $c_H + b > 0$, then there exists a solution $v \in C(\overline{\Omega})$ of $(\mathrm{SC})_{c_H}$ such that*

$$u(x,t) - (v(x) - c_H t) \to 0 \quad \textit{uniformly for } x \in \overline{\Omega} \textit{ as } t \to \infty.$$

(ii) *If $c_H + b \le 0$, then there exists a solution $w \in C(\overline{\Omega} \times \mathbb{R})$ of*

$$(\mathrm{P}) \qquad \begin{cases} w_t + H(x, Dw(x)) = -b & \text{in } \Omega \times \mathbb{R}, \qquad (11) \\ w(x,t) = g_\infty(x) + g_2(x,t) & \text{on } \partial\Omega \times \mathbb{R}, \qquad (12) \\ w(x,t+T) = w(x,t) & \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R} \qquad (13) \end{cases}$$

such that

$$u(x,t) - (w(x,t) + bt) \to 0 \quad \textit{uniformly for } x \in \overline{\Omega} \textit{ as } t \to \infty. \qquad (14)$$

4. Proof of Main results

In this paper, we prove only Theorem 3. Hereinafter we assume the period T of g_2 is positive. Without loss of generality we can assume $T = 1$.

Theorem 4. *The function* $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ *defined by*

$$u(x,t) = \inf\{ \int_\tau^t L(\gamma(s), \dot\gamma(s))\, ds + \chi_{\{\tau=0\}} f(\gamma(0)) + \chi_{\{\tau\neq 0\}} G(\gamma(\tau), \tau) \mid \gamma \in \mathcal{C}(x,t), \tau \in [0,t], (\gamma(\tau), \tau) \in \partial Q_p\} \quad (15)$$

is a Lipschitz function of $\overline{\Omega} \times [0,\infty)$ *and gives a solution of* (CD), *where* $\chi_{\{\tau=0\}} = 1$ *if* $\tau = 0$, $\chi_{\{\tau=0\}} = 0$ *if* $\tau \neq 0$ *and* $\chi_{\{\tau\neq 0\}} = 0$ *if* $\tau = 0$, $\chi_{\{\tau\neq 0\}} = 1$ *if* $\tau \neq 0$, $\mathcal{C}(x,t)$ *denotes the set of all trajectories* $\gamma \in \mathrm{AC}([0,t], \overline{\Omega})$ *such that* $\gamma(t) = x$ *and* $\partial_p Q = \partial\Omega \times (0,\infty) \cup \Omega \times \{0\}$.

Theorem 5. *Let* $T > 0$ *and set* $Q_T := \Omega \times (0,T)$. *Let* $u, v \in C(\overline{Q_T})$ *be a subsolution and a supersolution of* (1) *and* (3), *respectively. Assume* $u \leq v$ *on* $\overline{\Omega} \times \{0\}$. *Then* $u \leq v$ *on* $\overline{Q_T}$.

Theorems 4 and 5 are standard in the theory of viscosity solution, so we do not give here the details of the proofs.

An immediate consequence of Theorems 4 and 5 is the following corollary.

Corollary 6. *The function* u *given by* (15) *is a unique solution of* (CD).

Let u be the solution of (CD). We define the constant c_H by

$$c_H := \inf\{a \in \mathbb{R} \mid \text{there exists a solution } v \in C(\Omega) \text{ of } H(x, Du) \leq a \text{ in } \Omega\}.$$

The constant c_H is important when we consider the existence of stationary HJ equations under some boundary conditions, as the following proposition shows.

Proposition 7 ([11, Theorems 3.3, 3.4], [12, Proposition 3.2]). *Let* $a \in \mathbb{R}$ *be a constant.*
(i) *There exists a solution in* $C(\overline{\Omega})$ *of*

$$(\mathrm{SC})_a \qquad \begin{cases} H(x, Du(x)) \leq a & \text{in } \Omega, \\ H(x, Du(x)) \geq a & \text{on } \overline{\Omega}, \end{cases}$$

if and only if $a = c_H$.
(ii) *For any* $g \in C(\partial\Omega)$, *there exists a solution in* $C(\overline{\Omega})$ *of*

$$\begin{cases} H(x, Du(x)) = a & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega, \end{cases}$$

if and only if $a \geq c_H$.

The following proposition gives a bound of the solution of (CD).

Proposition 8. (i) *If* $c_H + b > 0$, *then there exists a constant* $M_1 > 0$ *such that* $|u(x,t) + c_H t| \leq M_1$ *for all* $(x,t) \in \overline{\Omega} \times [0,\infty)$. (ii) *If* $c_H + b \leq 0$, *then there exists a constant* $M_2 > 0$ *such that* $|u(x,t) - bt| \leq M_2$ *for all* $(x,t) \in \overline{\Omega} \times [0,\infty)$.

Proof. We first consider the case where $c_H + b > 0$. By Proposition 7 (i), there exists a solution $\psi \in C(\overline{\Omega})$ of $(\mathrm{SC})_{c_H}$. We set $v^{\pm}(x,t) := \psi(x) - c_H t \pm C_1$, where $C_1 > 0$ is a constant chosen so that $v^-(x,0) \le f(x) \le v^+(x,0)$.

It is clear that v^+ is a supersolution of (CD). Choose $C_1 > 0$ again, if necessary, such that

$$C_1 \ge \sup_{(x,t)\in\overline{\Omega}\times[0,\infty)} \{|g_1(x,t) + g_2(x,t)| + |\psi(x)|\}.$$

Note that $-c_H t \le bt$. We have

$$v^-(x,t) = \psi(x) - c_H t - C_1 \le \psi(x) - C_1 + bt \le g_1(x,t) + g_2(x,t) + bt,$$

which implies that v^- is a subsolution of (CD). In view of Theorem 5, we obtain that

$$v^-(x,t) \le u(x,t) \le v^+(x,t) \quad \text{for all } (x,t) \in \overline{\Omega}\times[0,\infty).$$

Therefore we have $|u(x,t) + c_H t| \le M_1$ for some $M_1 > 0$ and all $(x,t) \in \overline{\Omega}\times[0,\infty)$.

Next we consider the case where $c_H + b \le 0$. In view of Proposition 7 (ii), we may choose a solution $\phi \in C(\overline{\Omega})$ of

$$\begin{cases} H(x, Du(x)) = -b & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

We set $w^{\pm}(x,t) = \phi(x) + bt \pm C_2$. It is clear that $w^{\pm}$ satisfies (1) in the viscosity sense. Choose $C_2 > 0$ such that

$$C_2 \ge \sup_{\overline{\Omega}\times[0,\infty)} \{|f(x)| + |g_1(x,t) + g_2(x,t)| + |\phi(x)|\}.$$

Then w^+ and w^- is a supersolution and a subsolution of (CD), respectively. In view of Theorem 5,

$$w^-(x,t) \le u(x,t) \le w^+(x,t) \quad \text{for all } (x,t) \in \overline{\Omega}\times[0,\infty).$$

Thus we have $|u(x,t) - bt| \le C_3$ for some $C_3 > 0$ and all $(x,t) \in \overline{\Omega}\times[0,\infty)$. □

In the case where $c_H + b > 0$, Problem (CD) may be reduced to (C). To be more precisely, we present

Proof of Theorem 3 (i). Let $v(x,t) = u(x,t) + c_H t$ and M_1 be the constant given in Proposition 8 (i). It follows that v satisfies

$$\begin{cases} v_t + H(x, Dv(x,t)) = c_H & \text{in } \Omega\times(0,\infty), \\ v(x,t) = f(x) & \text{in } \Omega\times\{0\}, \\ v(x,t) = G(x,t) + c_H t & \text{on } \partial\Omega\times(0,\infty) \end{cases}$$

in the viscosity sense. The dynamic programming principle yields for any $t, s \ge 0$,

$$\begin{aligned} v(x,t+s) = \inf\Big\{ &\int_\tau^t L_{c_H}(\gamma(\lambda), \dot\gamma(\lambda))\, d\lambda + \chi_{\{\tau=0\}} v(\gamma(0), s) \\ &+ \chi_{\{\tau\ne0\}}\{G(\gamma(\tau), \tau+s) + c_H(\tau+s)\} \mid \gamma \in \mathcal{C}(x,t),\ \tau\in[0,t],\ (\gamma(\tau),\tau)\in\partial_p Q\Big\}. \end{aligned} \tag{16}$$

Noting that $bt+c_Ht\to\infty$ as $t\to\infty$, we see that $G(x,t)+c_Ht\to\infty$ as $t\to\infty$. Note that v is bounded. Then there exists $\bar{t}>0$ such that

$$\int_\tau^t L_{c_H}(\gamma(\lambda),\dot\gamma(\lambda))\,d\lambda+G(\gamma(\tau),\tau+s)+c_H(\tau+s)-\{\int_0^t L_{c_H}(\gamma(\lambda),\dot\gamma(\lambda))\,d\lambda+v(\gamma(0),s)\}>0$$

for any $s\geq\bar{t}$ and $\tau\in[0,t]$. Thus we have

$$v(x,t+s)=\inf\{\int_0^t L_{c_H}(\gamma(\lambda),\dot\gamma(\lambda))\,d\lambda+v(\gamma(0),s)\mid\gamma\in\mathcal{C}(x,t)\}.$$

This implies that v satisfies

$$\begin{cases} v_t(x,t)+H(x,Dv(x,t))\leq c_H & \text{in } \Omega\times(\bar{t},\infty),\\ v_t(x,t)+H(x,Dv(x,t))\geq c_H & \text{on } \overline{\Omega}\times(\bar{t},\infty).\end{cases}$$

Therefore the large time behavior of the solution of (CD) in the case where $c_H+b>0$ follows from Theorem 1. □

Next we consider the case where $c_H+b\leq 0$. In view of Theorems 4 and 8 (ii), we may define the functions $w^\pm\in C(\overline{\Omega}\times\mathbb{R})$ by $w^+(x,t):=\limsup_{n\to\infty}\big(u(x,t+n)-b(t+n)\big)$ and $w^-(x,t):=\liminf_{n\to\infty}\big(u(x,t+n)-b(t+n)\big)$, respectively. It is clear that $w^\pm$ is time-periodic with period 1 on $\overline{\Omega}\times\mathbb{R}$. We see that w^+ and w^- are a subsolution and a supersolution of (P) with $T=1$, respectively. Moreover, since the functions $H(x,\cdot)$ are convex, we see that w^- is a solution of (P) with $T=1$.

The following lemma is standard in the viscosity theory.

Lemma 9. *Let $T>0$, $\lambda\in\mathbb{R}$, $g^i\in C(\partial_pQ)$ $(i=1,2)$ and $u,v\in C(\overline{\Omega}\times[0,T])$ be a subsolution and a supersolution of*

$$\begin{cases} u_t+\lambda u+H(x,Du)=0 & \text{in } \Omega\times(0,T),\\ u(x,t)=g^1(x,t) & \text{on } \partial\Omega\times(0,T),\end{cases}$$

$$\begin{cases} u_t+\lambda u+H(x,Du)=0 & \text{in } \Omega\times(0,T),\\ u(x,t)=g^2(x,t) & \text{on } \partial\Omega\times(0,T),\end{cases}$$

respectively. Then,

$$e^{\lambda t}\max_{\overline{\Omega}}(u-v)_+(x,t)\leq\max\{\max_{\overline{\Omega}}(u-v)_+(x,0),\max_{\partial\Omega\times[0,t]}e^{\lambda s}(g^1-g^2)_+(x,s)\}$$

for all $t\in[0,T]$. Here $a_+(x):=\max\{a(x),0\}$ for any function a.

Proposition 10. *Assume $c_H+b<0$. Then there exists a unique solution $\phi\in C(\overline{\Omega}\times\mathbb{R})$ of* (P).

Remark 1. We remark that in the case where $c_H+b=0$, the uniqueness of solutions of (P) does not hold in general.

Proof. The above observation shows the existence of a solution of (P). We prove here the uniqueness of a solution of (P). Suppose that there exist two solutions $\phi_1,\phi_2\in C(\overline{\Omega}\times\mathbb{R})$ of (P).

Noting that $c_H+b<0$, there exist $\delta>0$, $\psi\in C(\overline{\Omega})$ such that

$$H(x,D\psi(x))\le -b-\delta \quad \text{in } \Omega,$$
$$\psi(x)\le g_\infty(x)+g_2(x,t) \quad \text{for all } (x,t)\in\partial\Omega\times\mathbb{R}.$$

Let $\lambda\in(0,1)$ and set $\phi_\lambda(x,t):=(1-\lambda)\phi_1(x,t)+\lambda\psi(x)$. In view of the convexity of H, ϕ_λ satisfies

$$\begin{cases} (\phi_\lambda)_t+H(x,D\phi_\lambda)\le -b-\lambda\delta & \text{in } \overline{\Omega}\times\mathbb{R}, \\ \phi_\lambda(x,t)=(1-\lambda)(g_\infty(x)+g_2(x,t))+\lambda\psi(x) & \text{on } \partial\Omega\times\mathbb{R} \end{cases}$$

in the viscosity sense. Note that $|\phi_\lambda(x,t)|\le C$ for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$ and some $C>0$ which is independent of λ. Then if $\lambda>0$ is small enough, we may assume

$$\lambda^2|\phi_\lambda(x,t)|\le\lambda\delta \quad \text{for all } (x,t)\in\overline{\Omega}\times\mathbb{R}.$$

This yields that

$$\begin{cases} (\phi_\lambda)_t+\lambda^2\phi_\lambda+H(x,D\phi_\lambda)\le -b & \text{in } \Omega\times\mathbb{R}, \\ \phi_\lambda(x,t)=(1-\lambda)(g_\infty(x)+g_2(x,t))+\lambda\psi(x) & \text{on } \partial\Omega\times\mathbb{R}. \end{cases}$$

Meanwhile, by adding some positive constant to ϕ_1 and ϕ_2, if necessary, we may assume that $\phi_1,\phi_2\ge 0$ on $\overline{\Omega}\times\mathbb{R}$, which implies that

$$\begin{cases} (\phi_2)_t+\lambda^2\phi_2+H(x,D\phi_2)\ge -b & \text{on } \overline{\Omega}\times\mathbb{R}, \\ \phi_2(x,t)=g_\infty(x)+g_2(x,t) & \text{on } \partial\Omega\times\mathbb{R} \end{cases}$$

in the viscosity sense.

Noting that

$$\big((1-\lambda)(g_\infty(x)+g_2(x,t))+\lambda\psi(x)-(g_\infty(x)+g_2(x,t))\big)_+=\lambda\big(\psi(x)-(g_\infty(x)+g_2(x,t))\big)_+=0,$$

we have by Lemma 9 that

$$e^{\lambda^2 t}\max_{\overline{\Omega}}(\phi_\lambda-\phi_2)_+(x,t)\le\max_{\overline{\Omega}}(\phi_\lambda-\phi_2)_+(x,0) \quad \text{for all } t\in\mathbb{R}. \tag{17}$$

Take $t=1$ in (17). By periodicity, we have $(\phi_\lambda-\phi_2)(x,1)=(\phi_\lambda-\phi_2)(x,0)$. Since $e^{\lambda^2}>1$, we deduce that $(\phi_\lambda-\phi_2)_+(x,0)=0$.

By the inequality (17), we get $\phi_\lambda(x,t)\le\phi_2(x,t)$ for all $\overline{\Omega}\times\mathbb{R}$. Sending $\lambda\to 0$ yields

$$\phi_1(x,t)\le\phi_2(x,t) \quad \text{for all } (x,t)\in\overline{\Omega}\times\mathbb{R}.$$

By symmetry, we obtain $\phi_1(x,t)\ge\phi_2(x,t)$ for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$. Therefore we get $\phi_1=\phi_2$ on $\overline{\Omega}\times\mathbb{R}$. □

We prove Theorem 3 (ii) in the case where $c_H+b<0$. The uniqueness of solutions of (P) makes the argument of asymptotic of solutions of (CD) easy. Set $v_n(x,t):=u(x,t+n)-b(t+n)$ for all $(x,t)\in\overline{\Omega}\times[0,1], n\in\mathbb{N}$. By Theorem 4 and Proposition 8 (ii), $\{v_n\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded on $\overline{\Omega}\times[0,1]$. By the Ascoli-Arzelá theorem, there exists a subsequence $\{v_{n_j}\}_{j\in\mathbb{N}}\subset\{v_n\}_{n\in\mathbb{N}}$ such that $\{v_{n_j}\}_{j\in\mathbb{N}}$ converges uniformly on $\overline{\Omega}\times[0,1]$. Set $\psi(x,t):=\lim_{j\to\infty}v_{n_j}(x,t)$ for $(x,t)\in\overline{\Omega}\times[0,1]$. It is clear that $\psi(x,0)=\psi(x,1)$. We write ψ for the periodic extension to $\overline{\Omega}\times\mathbb{R}$ of ψ again. It is now easy to see by the stability of viscosity property that ψ is a solution of (P). Moreover, noting the uniqueness of a solution of (P), we may conclude from this that $u(\cdot, \cdot + n)$ converges to ψ uniformly on $\overline{\Omega} \times [0, \infty)$ as $n \to \infty$. It is clear that this convergence is equivalent with (14).

Finally, we prove Theorem 3 in the case where $c_H + b = 0$. In order to do this, we provide some lemmas.

Lemma 11. *Let $x \in \overline{\Omega}$ and $\phi \in C(\overline{\Omega} \times \mathbb{R})$ satisfy*

$$\begin{cases} \phi_t + H(x, D\phi) = c_H & \text{in } \Omega \times \mathbb{R}, \quad (18) \\ \phi(x, t+1) = \phi(x, t) & \text{for all } (x, t) \in \overline{\Omega} \times \mathbb{R}. \quad (19) \end{cases}$$

Then there exists a curve $\gamma \in C((-\infty, 0], \overline{\Omega})$ such that $\gamma(0) = x$ and for any $[a, b] \subset (-\infty, 0]$,

$$\gamma \in \mathrm{AC}([a, b], \overline{\Omega}) \quad \textit{and} \quad \int_a^b L_{c_H}(\gamma(s), \dot{\gamma}(s))\, ds = \phi(\gamma(b), b) - \phi(\gamma(a), a), \tag{20}$$

where $L_{c_H}(x, \xi) := L(x, \xi) + c_H$ for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$.

Noting the periodic property of ϕ, Lemma 11 can be proved as in [11, Theorem 6.1]. So we skip presenting the details here. We call curves satisfying (20) *extremal curves* for ϕ and hereinafter we write $\mathcal{E}(\phi)$ to denote the set of all extremal curves for ϕ.

Lemma 12. *Set $v(x, t) := u(x, t) - bt$. There exist $\delta > 0$, $\omega \in C([0, \infty))$ which is $\omega(0) = 0$ and nondecreasing such that for any $\phi \in C(\overline{\Omega} \times \mathbb{R})$ which satisfies (18) and (19), $x \in \overline{\Omega}$, $\gamma \in \mathcal{E}(\phi)$ such that $\gamma(0) = x$, $0 \le s \le t$ and $s/(t - s) \le \delta$, then*

$$v(x, t) \le v(\gamma(-t), s) + \phi(x, 0) - \phi(\gamma(-t), -t) + \frac{ts}{t - s}\omega(\frac{s}{t - s}).$$

Proof. It is clear that v satisfies

$$\begin{cases} v_t + H(x, Dv(x, t)) = -b & \text{in } \Omega \times (0, \infty), \\ v(x, t) = f(x) & \text{in } \Omega \times \{0\}, \\ v(x, t) = g_1(x, t) + g_2(x, t) & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

in the viscosity sense.

Set $\epsilon := s/(t - s)$ and $t_\epsilon := (1 + \epsilon)^{-1} t$. Note $t = t_\epsilon + \epsilon t_\epsilon = t_\epsilon + s$. By the dynamic programming principle, we have

$$\begin{aligned} & v(x, t) = v(x, t_\epsilon + s) \\ = \;& \inf\{\int_\tau^{t_\epsilon} L(\eta(\lambda), \dot{\eta}(\lambda)) - b\, d\lambda + v(\eta(\tau), s + \tau) \mid \\ & \qquad\qquad \eta \in \mathcal{C}(x, t),\ 0 \le \tau \le t_\epsilon,\ (\eta(\tau), \tau) \in \partial_p Q\} \\ \le \;& \inf\{\int_0^{t_\epsilon} L_{c_H}(\eta(\lambda), \dot{\eta}(\lambda))\, d\lambda + v(\eta(0), s) \mid \eta \in \mathcal{C}(x, t_\epsilon)\}. \end{aligned}$$

Choosing $\eta(\lambda) := \gamma((1+\epsilon)\lambda - t)$ in the above formula, we get

$$\begin{aligned} v(x,t) &\leq \int_0^{t_\epsilon} L_{c_H}(\gamma((1+\epsilon)\lambda - t), (1+\epsilon)\dot{\gamma}((1+\epsilon)\lambda - t))\, d\lambda + v(\gamma(-t), s) \\ &= \int_{-t}^{0} L_{c_H}(\gamma(\lambda), (1+\epsilon)\dot{\gamma}(\lambda))(1+\epsilon)^{-1}\, d\lambda + v(\gamma(-t), s). \end{aligned}$$

In view of [10, Lemma 7.2], there exists $\delta > 0$ and $\omega \in C([0,\infty))$ which is $\omega(0) = 0$ and nondecreasing such that

$$\int_{-t}^{0} L_{c_H}(\gamma(\lambda), (1+\epsilon)\dot{\gamma}(\lambda))(1+\epsilon)^{-1}\, d\lambda \leq \int_{-t}^{0} L_{c_H}(\gamma(\lambda), \dot{\gamma}(\lambda)) + \epsilon\omega(\epsilon)\, d\lambda.$$

Thus we obtain

$$v(x,t) \leq v(\gamma(-t), s) + \phi(x,0) - \phi(\gamma(-t), -t) + \frac{ts}{t-s}\omega(\frac{s}{t-s}),$$

in view of Lemma 11. □

Proof of Theorem 3 (ii) *in the case where* $c_H + b = 0$. It is enough to show that $w^+(x,t) \leq w^-(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,1]$.

Set $v(x,t) := u(x,t) - bt$ and let $\delta > 0$ and ω be a given constant and a function in Lemma 12. Fix $(x,t) \in \overline{\Omega} \times [0,1]$. Choose a diverging sequence $\{n_j\}_{j\in\mathbb{N}}$ such that $v(x, t+n_j) \to w^+(x,t)$ as $j \to \infty$. Let $\gamma \in \mathcal{E}(w^-(\cdot, \cdot + t))$ such that $\gamma(0) = x$. Since $\overline{\Omega}$ is a compact set, by replacing $\{n_j\}_{j\in\mathbb{N}}$ by its subsequence if necessary, we may assume that $\gamma(-(t+n_j)) \to y$ as $j \to \infty$ for some $y \in \overline{\Omega}$. Fix any $\epsilon > 0$ and choose $N := N_{y,\epsilon} \in \mathbb{N}$ such that $w^-(y,0) + \epsilon > v(y,N)$. If j is large enough, we may assume that $t + n_j \geq N$, $N/(t+n_j - N) \leq \delta$ and by Lemma 12, we get

$$\begin{aligned} v(x, t+n_j) &\leq v(\gamma(-(t+n_j)), N) + w^-(x, 0+t) \\ &\quad - w^-(\gamma(-(t+n_j)), -(t+n_j)+t) + \frac{N(t+n_j)}{t+n_j-N}\omega(\frac{N}{t+n_j-N}). \end{aligned}$$

Sending $j \to \infty$ yields that

$$w^+(x,t) \leq v(y,N) + w^-(x,t) - w^-(y,0) \leq w^-(y,0) + \epsilon + w^-(x,t) - w^-(y,0).$$

Consequently, we get $w^+(x,t) \leq w^-(x,t)$ on $\overline{\Omega} \times [0,1]$.

□

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Drift-diffusion system in the critical spaces

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1　2次元臨界空間における drift-diffusion 方程式の可解性

2次元 drift-diffusion 方程式

$$
\begin{cases}
\partial_t n - \Delta n + \nabla \cdot (n\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\
\partial_t p - \Delta p - \nabla \cdot (p\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\
-\Delta\psi = \kappa(p - n), \quad x \in \mathbb{R}^2, \\
\qquad n(0,x) = n_0(x), \qquad p(0,x) = p_0(x),
\end{cases}
\tag{1.1}
$$

の時間局所解の存在は、熱核の半群表現と積分方程式を基礎にした縮小写像の方法で、解を構成することができることは、Kurokiba-Ogawa [25] によって論じた。この手法は2次元 Keller-Segel 系に対してもほぼ同様に考えることができる。

$$
\begin{cases}
\partial_t u - \Delta u + \nabla \cdot (u\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\
-\Delta\psi = \kappa u, \quad x \in \mathbb{R}^2, \\
\qquad u(0,x) = u_0(x),
\end{cases}
\tag{1.2}
$$

この問題においては選びうる関数空間として空間方向に $L^p(\mathbb{R}^n)$ (ただし $\frac{n}{2} < p < n$, $n = 2, 3$) という自由度がある。こうした空間で解く方法は実は Navier-Stokes 方程式の解の存在定理とほぼ平行な議論である。また $n = 2$ の場合には、別の方法 (エネルギー法) によって 一方の臨界である $p = 2$ でも解くことができる。これらのことは文献 [24] で取り扱っている。$p = 2$ のときには、特に Poisson 方程式の解の取り扱いから重み付きの L^2 空間で解く必要があることを注意する。

さて $n = 2$ の場合の他方の臨界である $p = 1$ は上記積分方程式の方法では単純には含まれない。他方、方程式 (1.1) あるいはそのより単純な形である (1.2) は2次元の渦度 Navier-Stokes 方程式

$$
\begin{cases}
\partial_t \omega - \Delta\omega + u\nabla\omega = 0, \qquad t > 0, x \in \mathbb{R}^2, \\
-\Delta u = \mathrm{rot}\ \omega = \partial_1\omega_2 - \partial_2\omega_1, \quad x \in \mathbb{R}^2, \\
\qquad \omega(0,x) = \mathrm{rot}\ u_0(x),
\end{cases}
\tag{1.3}
$$

と同一な Scaling 不変性を保つことがわかっている。2次元 渦度の方程式 (1.3) の 場合には、空間の基礎空間を $L^1(\mathbb{R}^2)$ と選んで時間局所解の存在が Giga-Miyakawa-Osada [16], Giga-Kanbe [17] らによって得られている。$L^1(\mathbb{R}^2)$ はこれら方程式を不変にする scaling

$$
\begin{cases}
n_\lambda(t,x) = \lambda^2 n(\lambda^2 t, \lambda x) \\
p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x) \\
\psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x)
\end{cases}
$$

において不変な空間、いわゆる臨界空間であって、こうした空間で方程式を考えることは Fujita-Kato の初期の結果に照らしてみれば重要なことであることがわかる。

まずはじめに 2 次元渦度の方程式が L^1 で可解となる方法を復習する。scaling 次元が同一の (1.1) や (1.2) に対しても同一の方法を用いることが可能であって、簡単のため、(1.2) に対して議論を行う。

2 次元 Keller-Segel 系 (1.2) を熱核を用いて積分方程式に直せば、

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla(u\nabla\psi)ds$$

ここで基本 norm として $\|u\| \equiv \sup_{t\in I} t^{1-1/p}\|u(t)\|_p$ ただし $I = [0, T)$ かつ $1 < p < 2$ と選ぶ。

$$\begin{aligned}
\|u(t)\|_p \le &\|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-(1/r-1/p)-1/2}\|u\nabla\psi\|_r ds \\
\le &\|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-(1/r-1/p)-1/2}\|u(s)\|_p\|\nabla\psi\|_s ds \\
\le &\|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-1/p}\|u(s)\|_p^2 ds \\
\le &\|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-1/p}s^{-2(1-1/p)}ds\big(\sup_{t\in I} t^{(1-1/p)}\|u(t)\|_p\big)^2
\end{aligned} \tag{1.4}$$

ここで $1/r = 1/p + 1/s$ かつ $1/s = 1/p - 1/2$ である。 従って 特に $1/r - 1/p + 1/2 = 1/p$。このとき

$$\frac{1}{r} = \frac{2}{p} - \frac{1}{2} \le 1$$

が必要なので、

$$\frac{2}{p} \le \frac{3}{2}$$

から $4/3 \le p$ また、 $p < 2$ を課しておく。今 $4/3 \le p < 2$ とすると以下の積分は収束して

$$\int_0^t |t-s|^{-1/p}s^{-2(1-1/p)}ds = t^{(1-1/p)}B.$$

ここで B は beta 関数から生じる定数である。従って

$$\sup_I t^{1-1/p}\|u(t)\|_p \le \sup_I t^{1-1/p}\|e^{t\Delta}u_0\|_p + B\big(\sup_{t\in I} t^{(1-1/p)}\|u(t)\|_p\big)^2$$

を得る。$t \to 0$ によって $\sup_I t^{1-1/p}\|e^{t\Delta}u_0\|_p$ はいくらでも小さくなりうるので (L^p が C_0^∞ を dense に含む事実を用い u_0 を C_0^∞ の関数で近似させることにより、正則性があれば $t \to 0$ に対して遅く近づくので) 上から bound を得ることができる。同様の方法で 差に対する評価することも得られる。次に解が $L^1(\mathbb{R}^2)$ に属することを示すためには、積分方程式を直接 L^1 で評価して L^1-L^1 有界

性を用いると

$$
\begin{aligned}
\|u(t)\|_1 \leq & \|e^{t\Delta}u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u\nabla\psi\|_1 ds \\
\leq & \|u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u(s)\|_{4/3}\|\nabla\psi\|_4 ds \\
\leq & \|u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u(s)\|_{4/3}\|u(s)\|_{4/3} ds \\
\leq & \|u_0\|_1 + C\int_0^t |t-s|^{-1/2}s^{-1/2}ds\big(\sup_{t\in I} t^{1/4}\|u(t)\|_{4/3}\big)^2
\end{aligned}
\tag{1.5}
$$

ここで

$$
\|\nabla(-\Delta)^{-1}u(s)\|_4 \leq C\|u(s)\|_{4/3}, \qquad \frac{1}{4}=\frac{3}{4}-\frac{1}{2}
$$

は 2 次元の Hardy-Littlewood-Sobolev の不等式である。最後の積分は収束して $B(1/2,1/2)$ となり特に $\sup_{t\in I} t^{1/4}\|u(t)\|_{4/3}$ の一様有界性を得ているので、解は すべての時間で L^1 に含まれることがわかる。

以上の方法を差に対しても適用することで Giga-Miyakawa-Osada [16] は初期値 $L^1(\mathbb{R}^2)$ の渦度に対して解を構成した。

Proposition 1.1 (Giga-Miyakawa-Osada) $\omega_0 \in L^1(\mathbb{R}^2)$ に対して 2 次元渦度の方程式 (1.3) の一意的な時間局所解 $\omega(t) \in C([0,T);L^1(\mathbb{R}^2)) \cap C((0,T);L^p(\mathbb{R}^2)$ が存在する。

まったく同一に近い議論により

Theorem 1.2 $u_0 \in L^1(\mathbb{R}^2)$ に対して 2 次元 *Keller-Segel* 方程式 (1.2) の一意的な時間局所解 $u(t) \in C([0,T);L^1(\mathbb{R}^2)) \cap C((0,T);L^p(\mathbb{R}^2)$ が存在する。

を得ることがわかる。従って Kurokiba-Ogawa [24] における可解性の議論は $n=2$ に関して言えば、$1 \leq p \leq 2$ に対して成立することがわかる。

さて 2 次元渦度方程式に対する可解性の場合には、$\omega(t)$ が最大値原理を満たすことから、その一様有界性から先見評価を得ることができて、解の時間大域存在定理を得ることができるが、Keller-Segel 系ないし Drift-diffusion 系に対しては、最大値原理が一般に成立しないために、先見評価を得るには entropy 評価を用いる必要がある。実際 Nagai-Senba-Yoshida [34], Biler [2], Nagai-Senba-Suzuki [33] らは、有界領域や全空間の場合の 2 次元の問題の時間大域存在性について entropy 汎関数：

$$
\int_{\Omega} u\log u dx - \frac{1}{2}\int_\Omega u\psi dx + \int_0^t\int_\Omega u\big|\nabla(\log u-\psi)\big|dxdt \leq \int_\Omega u_0\log u_0 dx - \frac{1}{2}\int_\Omega u_0\psi_0 dx
$$

を用いて証明を行っている。これらは Lyapunov 関数を構成し、解の時間大域挙動を決定するには不可欠であるが、初期時刻からこれらの各項の意味をもたせるには初期条件を $L^1(\mathbb{R}^2)$ から取るだけでは不十分である。

形式的には entropy 汎関数は次のように捉えることができる。

$$
\int_{\Omega} u\log u dx - \frac{1}{2}\int_\Omega u\psi dx = \int_{\Omega} u\log u dx - \frac{1}{2}\|\nabla\psi\|_2^2
$$

右辺第二項は形式的に Poisson 方程式を用いて書き直したものであるが、正値解を考えている限り、$\mathbb{R}^2$ においては、この第二項は意味をなさないことがわかる。実際 $|x| \to \infty$ において

$$\psi \simeq \log |x|^{-1}$$

と挙動することがわかるので $\nabla\psi \notin L^2(\mathbb{R}^2)$ である。この点を、drift-diffusion 方程式に対して考えるならば、解の差の平均が 0 となる特殊な状況の元で考えることにより、解決することができる。 より具体的には方程式の適切性を Hardy 空間 $\mathcal{H}^1$ で考えることにより、それら entropy 汎関数に意味を持たせることが可能になる。そのためには解の存在する空間を $L^1(\mathbb{R}^2)$ から Hardy 空間 $\mathcal{H}^1(\mathbb{R}^2)$ と取り直す必要がある。

そのために、まず $v = n + p$ and $w = n - p$ とおいて 方程式 (1.1) を以下のように書き直す:

$$\begin{cases} \partial_t v - \Delta v + \nabla \cdot (w \nabla \psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ \partial_t w - \Delta w + \nabla \cdot (v \nabla \psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ -\Delta \psi = -\kappa w, \quad x \in \mathbb{R}^2, \\ \qquad v(0,x) = n_0(x) + p_0(x), \qquad w(0,x) = n_0(x) - p_0(x), \end{cases} \tag{1.6}$$

を考える。w に対しては自然に平均 0 の仮定をおくことができる。v に対しては遠方の境界条件を犠牲にして $v - \bar{v}$ ただし

$$\bar{v} = \int_{\mathbb{R}^2} v(t,x)dx = \int_{\mathbb{R}^2} v_0(x)dx$$

とおきなおして

$$\begin{cases} \partial_t v - \Delta v + \nabla \cdot (w \nabla \psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ \partial_t w - \Delta w + \alpha \bar{v} w + \nabla \cdot (v \nabla \psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ -\Delta \psi = -\alpha w, \quad x \in \mathbb{R}^2, \\ \qquad v(t,x) \to 0, \quad w(t,x) \to 0, \quad |x| \to \infty, \\ \qquad v(0,x) = v_0 \equiv n_0(x) + p_0(x) - \overline{n_0 + p_0}, \\ \qquad w(0,x) = w_0 \equiv n_0(x) - p_0(x). \end{cases} \tag{1.7}$$

を考える。この系に対して 空間基礎空間を $\mathcal{H}^1(\mathbb{R}^2)$ と選ぶことにより、時間局所解の存在を示した。

Definition. $\lambda > 0$ と $\phi \in \mathcal{S}(\mathbb{R}^2)$ に対して $\phi_\lambda = \lambda^{-2}\phi(\lambda^{-1}x)$ とおく。 このとき $0 < p < \infty$ に対して

$$\mathcal{H}^p = \mathcal{H}^p(\mathbb{R}^2) = \left\{ f \in L^p_{loc}(\mathbb{R}^2); \quad \|f\|_{\mathcal{H}^p} \equiv \left\| \sup_{\lambda>0} |\phi_\lambda * f| \right\|_p < \infty \right\}.$$

とおいてこれを Hardy 空間 と呼ぶ。

特に $p = 1$ と選んだときの、Hardy 空間 $\mathcal{H}^1$ の双対空間は 有界平均振動 (*bounded mean oscillation*)

$$BMO = \{f \in L^1_{loc}(\mathbb{R}^2); \|f\|_{BMO} < \infty\},$$

のクラスであることが知られている。

Theorem 1.3 (Ogawa-Shimizu[36]) $\alpha = \pm 1$ とする。初期値 $(v_0, w_0) \in \mathcal{H}^1(\mathbb{R}^2) \times \mathcal{H}^1(\mathbb{R}^2)$, に対してある時刻 $T > 0$ があって *(1.7)* に対する時間一意解 (v, w) が存在して $v,\, w \in C([0,T); \mathcal{H}^1) \cap L^2(0,T; \dot{\mathcal{H}}^{1,1}) \cap C((0,T); \dot{\mathcal{H}}^{2,1}) \cap C^1((0,T); \mathcal{H}^1)$ を満たす。特に初期値から解への軌道写像 $(v_0, w_0) \to (v, w)$ は $\mathcal{H}^1(\mathbb{R}^2)^2 \to C([0,T); \mathcal{H}^1)^2$ への *Lipschitz* 連続となる。

Hardy 空間 $\mathcal{H}^1(\mathbb{R}^2)$ が方程式 (1.7) に対して臨界空間であることから、いわゆる Fujita-Kato の原理によって同様の関数空間において小さい初期値に対する時間大域解の存在も議論することができるが、この問題に関して、そうした議論は問題を本質を捉えた結果ではない。実際、drift-diffusion 型方程式であるならば、解の存在は大きな初期値に対して可能であるし、また Keller-Segel 系に対しても大域可解性のための初期値の大きさの臨海値を求めなければ問題の本質を捉えたことにはならない。これは自己相似解を捉える際にも、同様のことがいえるはずである。

$f \in \mathcal{H}^1(\mathbb{R}^2)$ であるとき次の評価が成り立つことが知られている。$f \in \mathcal{H}^1(\mathbb{R}^2)$ ならば そのフーリエ変換 $\hat{f}$ に対して

$$\int_{\mathbb{R}^2} \frac{|\hat{f}(\xi)|^2}{|\xi|^2} d\xi \le C\|f\|_{\mathcal{H}^1}^2. \tag{1.8}$$

このことから $-\Delta\psi = w$ に対して $w \in \mathcal{H}^1(\mathbb{R}^2)$ ならば、

$$\|\nabla\psi\|_2^2 = \int_{\mathbb{R}^2} |\xi\widehat{\psi}|^2 d\xi = \int_{\mathbb{R}^2} \frac{|\widehat{w}|^2}{|\xi|^2} d\xi \le C\|w\|_{\mathcal{H}^1}^2$$

が従い、entropy 汎関数が有限になることがわかる。

2 L^1 型エネルギー不等式

Theorem の証明には、最大正則性に対する端点評価に関連する、臨界型の消散型評価を確立することが重要な役割を果たす。ここではその詳細には至らないが、以下のように熱方程式の初期値問題の解に対するエネルギー評価としてとらえることができる.

$$\begin{cases} \partial_t u - \Delta u = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

の解に対してはエネルギー不等式

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \le \|u_0\|_2^2$$

が成り立つことが良く知られている。特に線形方程式の場合には解の正則性が直ちに得られることから、上記の不等式は等式で得られる。この不等式は解の基礎空間を L^2 に選んだ際に有効であるが、解の基礎空間を L^p とした際には、同等の不等式が放物型評価として得られる。その際 $p=1$ として選んだものに対応するのがここで述べる初期条件に対する最大正則性評価である。以下を得る。

Theorem 2.1 $e^{t\Delta}$ を熱方程式の解を与える半群、 $\phi \in \mathcal{H}^1$ とする。このとき

$$\left(\int_0^T \|\nabla e^{t\Delta}\phi\|_{\mathcal{H}^1}^2 dt\right)^{1/2} \le C\|\phi\|_{\mathcal{H}^1}. \tag{2.2}$$

が成り立つ。ここで C は $T>0$ に依存しない定数である。

Corollary 2.2 (一般化されたエネルギー不等式) $1<\theta<\infty$ とする。u を非斉次熱方程式

$$\begin{cases} \partial_t u - \Delta u = f, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = \phi(x). \end{cases} \tag{2.3}$$

の解とする。このとき

$$\|\nabla u\|_{L^2(I;\mathcal{H}^1)} \le C\left(\|\phi\|_{\mathcal{H}^1} + \|f\|_{L^2(I;\mathcal{B}^{-1}_{1,2})}\right).$$

ただし $I=\mathbb{R}_+$.

この評価は本質的には端点 Besov 空間における最大正則性理論と同等のものである (cf. Ogawa-Shimizu [37]). 証明の詳細には、実補間を用いた議論が必要である。

Theorem 2.1 の証明は実補間や関数解析の様々な性質を用いているため、直感的にその評価の是非については理解しづらい. そこで Theorem 2.1 の直感的理解の補助的事実として次の Proposition があげられる:

Proposition 2.3 $e^{t\Delta}$ を熱方程式の解を与える半群、$\phi \in L^1(\mathbb{R}^2)$ とする。このとき一般に不等式

$$\left(\int_0^T \|\nabla e^{t\Delta}\phi\|_1^2 dt\right)^{1/2} \le C\|\phi\|_1 \tag{2.4}$$

は成り立たない.

Proof of Proposition 2.3. Fourier 変換についての L^1-L^∞ 評価

$$\sup_\xi |\hat{f}(\xi)| \le \frac{1}{2\pi}\|f\|_1$$

から $f=\nabla e^{t\Delta}\phi$ と選んで

$$\begin{aligned}
\int_0^\infty \|\nabla e^{t\Delta}\phi\|_1^2 dt \ge& 4\pi^2 \int_0^\infty \left(\sup_\xi \left|\xi e^{-t|\xi|^2}\hat{\phi}(\xi)\right|\right)^2 dt \\
=& 4\pi^2 \int_0^\infty \frac{1}{t}\left(\sup_\xi \left|\sqrt{t}\xi e^{-t|\xi|^2}\hat{\phi}(\xi)\right|\right)^2 dt \\
& (\text{ここで}\ \xi\ \text{ついての sup をはずして適当なベクトル}\ \eta\ \text{を選べば}) \\
\ge& 4\pi^2 \int_0^\infty \frac{1}{t}\left(\left|\eta e^{-|\eta|^2}\right|\right)^2 |\hat{\phi}(\sqrt{t}^{-1}\eta)|^2 dt \\
\ge& C\int_0^\infty \frac{|\hat{\phi}(\sqrt{t}^{-1}\eta)|^2}{t} dt \\
& \frac{|\eta|}{\sqrt{t}} = r,\ \text{とおいて} \qquad dt = -2\frac{|\eta|^2}{r^3}dr\ \text{から} \\
=& C\int_0^\infty \frac{|\hat{\phi}(r\omega)|^2}{r} dr.
\end{aligned}$$

ここで $\omega=\eta/|\eta|$ なる単位ベクトルである. 両辺 $\omega \in \mathbb{S}^1$ について積分平均をとれば、

$$\int_0^\infty \|\nabla e^{t\Delta}\phi\|_1^2 dt \ge C(\eta)\int_{\mathbb{R}^2} \frac{|\hat{\phi}(\xi)|^2}{|\xi|^2} d\xi \tag{2.5}$$

を得る．一般に ϕ を適当に選べば右辺 $=\infty$ となる．(たとえば $\hat{\phi}(0)\neq 0$ とすればよい)．もし ϕ の積分平均が 0 であっても一般に $\phi\in L^1(\mathbb{R}^2)$ だけでは右辺は有限になるとは限らない． $\square$

以上の証明から、 Proposition 2.3 が評価式 (1.8) の証明も与えていることに注意する。実際 Proposition 2.3 は (1.8) を用いることなしに証明される。

Hardy 空間での可解性を得るにはもう一つ、非線形項に対する処理を行うために、こうした空間における双線形評価が必要である。以下の評価は発散が 0 となる場 (いわゆる divergence free 場) と 回転 が 0 となる場 (rotation free 場) 同士の積に対する著名な Coifman-Lions-Mayer-Semmes [9] による評価を一般化したものである。

Proposition 2.4 $\nabla w\in\mathcal{H}^1(\mathbb{R}^2)$ かつ $\nabla\psi\in\dot{H}^1\cap L^\infty$ とする。このとき以下の評価が成り立つ。

$$\|\nabla\cdot(w\nabla\psi)\|_{\mathcal{H}^1}\le C\big(\|w\|_2\|\Delta\psi\|_2+\|\nabla w\|_{\mathcal{H}^1}\|\nabla\psi\|_\infty\big).$$

類似の評価は Triebel-Lizorkin space においては知られているがそれらは $f,\,g\in\dot{F}^s_{p,\sigma}\cap L^r$ が $1<p\le\infty,\,1\le\sigma\le\infty$ となるときに $1/p=1/r+1/q$ and $s>0$ に対して

$$\|fg\|_{\dot{F}^s_{p,\sigma}}\le C\big(\|f\|_r\|g\|_{\dot{F}^s_{q,\sigma}}+\|f\|_{\dot{F}^s_{q,\sigma}}\|g\|_r\big)$$

となるというものでその証明にはベクトル値関数の極大関数に対する L^p における用いているために $p=1$ の場合を得ることができない。我々の評価はこの $p=1$ の場合に対応していることが

$$\|\nabla(fg)\|_{\dot{F}^0_{1,2}}\simeq\|\nabla(fg)\|_{\mathcal{H}^1}.$$

からわかる。

3　3次元臨界型 drift-diffusion 方程式の解の大域挙動

以下では $n=3$ の場合の解の時間大域的な挙動について論ずる．Carpio [7] は $n=2,3$ の場合に Navier-Stokes 方程式の初期値問題の解の漸近形が熱核で表現出来ることを示し、解の減衰の lower bound を示した．このことは drift-diffusion system (1.1) に対しても成り立つ．実際非常に類似の性格を持つ、放物型 Keller-Segel 系に対して、Nagai-Yamada [35] は解の漸近系の評価を与えた．ここでは $n=3$ の場合に初期値問題

$$\begin{cases}\partial_t n-\Delta n+\nabla\cdot(n\nabla\psi)=0, & t>0,x\in\mathbb{R}^3,\\ \partial_t p-\Delta p-\nabla\cdot(p\nabla\psi)=0, & t>0,x\in\mathbb{R}^3,\\ -\Delta\psi=\kappa(p-n), \quad x\in\mathbb{R}^3, & \\ \quad n(0,x)=n_0(x), \qquad p(0,x)=p_0(x), &\end{cases}\tag{3.1}$$

時間大域解を考えて、その漸近形を考える．考える問題を積分方程式

$$\begin{cases}n(t)=e^{t\Delta}n_0-\displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot(n(s)\nabla\psi(s))ds,\\ p(t)=e^{t\Delta}p_0+\displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot(p(s)\nabla\psi(s))ds,\\ \quad -\Delta\psi=\kappa(p-n).\end{cases}$$

と考えると、もし非線形項からの寄与である、積分方程式右辺第2項の時間減衰評価が、熱の減衰よりも早いことがあれば、解の時間無限遠での漸近挙動は右辺第1項から熱核の漸近形に近づくことがわかる. 実際非線形項からの寄与は非線形性 (線形近似の解が二つかかっていると見なす) から、右辺第一項からの減衰よりも早く減衰するために一次近似では非線形効果の寄与は現れない. すなわち初期条件に適当な遠方での減衰を仮定すれば、$1 \le q \le \infty$ に対して

$$\|n(t) - M_n G(t)\|_q = o(t^{-\gamma}), \quad \|p(t) - M_p G(t)\|_q = o(t^{-\gamma}), \qquad \gamma = \frac{n}{2}\left(1 - \frac{1}{q}\right)$$

を得ることができる. ここで $M_n = \int_{\mathbb{R}^3} n(t)dx$, $M_p = \int_{\mathbb{R}^3} p(t)dx$ は初期全電荷であって、$n_0 \ge 0$, $p_0 \ge 0$ の条件下ではこれらは保存するので、定数である. さてここで問題にするのは2次漸近形である. $n = 3$ の場合には次に述べる理由によって、この問題は臨界状況となる. 今上記の1漸近形を用いて、非線形項の減衰が t の一次分だけ早いことに着目すれば、2次展開は熱核の微分で与えられると予想される. 実際 $\|n(t)\|_q = O(t^{-\gamma})$ $(\gamma = \frac{n}{2}(1 - \frac{1}{q}))$ からベクトル

$$V = \int_0^\infty \int_{\mathbb{R}^n} n(s,x)\nabla(-\Delta)^{-1}(n(s,x) - p(s,x))dxds$$

が well-defined となれば これと $\nabla G(t)$ との内積が2次展開の候補となる。これは Convection diffusion 方程式の解に対して Escobedo-Zuazua [10] がまた $n = 2, 3$ のときに Navier-Stokes 方程式の解に対して Carpio [7], Fujigaki-Miyakawa [11] らが示していることに対応する。ところが今の場合、$n = 3$ のときは $\|n(s)\|_\infty = O(s^{-3/2})$ なので $\|\nabla(-\Delta)^{-1}(n(s) - p(s))\|_q = O(s^{-\gamma+\frac{1}{2}})$ で $q = 1$ とあわせると、形式的には

$$V = \int_0^\infty (1+s)^{-1}ds = \infty$$

となり、漸近形の係数が発散する. このことは Naveir-Stokes 方程式の解や、Nagai-Yamada によって扱われた放物型 Keller-Segel 方程式

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ \partial_t \psi - \Delta \psi + \psi = u, & t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x), & \psi(0,x) = \psi_0(x), \end{cases} \tag{3.2}$$

では $n \ge 2$ では現れなかった問題である。しかし実際にはもう一つの非線形項から現れる補正を加えることにより、ここでは問題 (3.1) の時間大域解について次の時間大域漸近挙動を示す。

Theorem 3.1 (Ogawa-Yamamoto[38]) $\kappa = -1$, $3/2 \le r < 3$ とする。初期値 $(n_0, p_0) \in \left(L^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\right)^2$ に対して *(3.1)* に対する大域的一意解 (n, p) が存在して $1 \le q \le \infty$ に対して $n, p \in C([0,\infty); L^q) \cap C^1(0,\infty; L^r) \cap C((0,\infty); W^{2,r})$ を満たす。特に解は次の漸近挙動を示す*:* $1 \le p \le \infty$ として

$$\|n(t) - M_n G(t) - m_n \cdot \nabla G(t) - V_n \cdot \nabla G(t) + M_n W J(t)\|_q = o(t^{-\gamma-\frac{1}{2}}) \quad t \to \infty$$
$$\|p(t) - M_p G(t) - m_p \cdot \nabla G(t) - V_p \cdot \nabla G(t) + M_p W J(t)\|_q = o(t^{-\gamma-\frac{1}{2}}) \quad t \to \infty$$

ただし $\gamma = \frac{3}{2}\left(1 - \frac{1}{q}\right)$. さらに $J(t)$ は熱核 $G(t) = \left(\frac{1}{4\pi t}\right)^{3/2} \exp(-\frac{|x|^2}{4t})$ によって

$$J(t) = \int_0^t \nabla G(t-s) * \left(G(s)\nabla(-\Delta)^{-1}G(s)\right)ds$$

により与えられる．また $W=\int_{\mathbb{R}^3}(n_0(x)-p_0(x))dx$, $m_n=\int_{\mathbb{R}^3}xn_0(x)dx$, $m_p=\int_{\mathbb{R}^3}xp_0(x)dx$

$$\begin{cases} V_n = \int_0^\infty n(s)\nabla(-\Delta)^{-1}(n(s)-p(s))ds \\ V_p = \int_0^\infty p(s)\nabla(-\Delta)^{-1}(n(s)-p(s))ds \end{cases}$$

とする。

最後に与えた V_p や V_n は形式的には発散する量ではあるが、この問題の場合には実は well-defined となる．これは問題の本質が $n=3$ の Navier-Stokes 方程式の解のような状況になっていることを示唆している。なお、電気的に中和されている設定、すなわち $W=0$ であるような場合には臨界状況とはならず、高次元の場合と同等の結果を得ることができる。

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ON A REGULARITY CRITERION BY SMALL BMO FOR WEAK SOLUTIONS TO THE HARMONIC MAP FLOWS †

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1. Introduction

Let Ω be a bounded domain in $m-$dimensional Euclidean space R^m, $m \geq 2$, with smooth boundary $\partial\Omega$, T be a positive number and S^{n-1} be the $(n-1)-$dimensional unit sphere in R^n, $n \geq 2$. In this talk we study a regularity of the harmonic map flow, which is a map $u(z) = (u^1(z), \ldots, u^n(z))$ defined for $z = (t, x)$ in time-space cylinder $(0, T) \times \Omega$ with values into R^n and satisfying the parabolic system of second-ordered partial differential equations with constraint on the target

$$\begin{cases} \partial_t u - \Delta u = u|\nabla u|^2 & \\ |u| = 1 & \text{in } (0,T) \times \Omega \end{cases}$$

The harmonic map flow is a natural approach to look for harmonic maps, firstly introduced by Eells and Sampson. In fact, a solution of the harmonic map flow is the trajectry of the negative L^2-gradient flow of the Dirichlet energy

$$E(u) = \int_\Omega \tfrac{1}{2} |\nabla u|^2 \, dx \tag{1.1}$$

in the class of maps u from Ω to $S^{n-1} \subset R^n$. So, if a solution of the harmonic map flow is time-globally exists, the limit map of the solution as $t \to \infty$ is the harmonic map, which is a solution of the Euler-Lagrange equation

$$\begin{cases} -\Delta u = u|\nabla u|^2 & \\ |u| = 1 & \text{in } (0,T) \times \Omega. \end{cases}$$

In general, for "large initial and (Dirichlet-)boundary data", we know only the local in time existence of the harmonic map flow and then, the singularity behavior of the time-local solution at a finite singular time is characterized by non-trivial stationary solutions ([7, 24, 25, 26]). Under such motivation we will study the regularity of solutions of the harmonic map flow. Hélein discovered the special structure of the harmonic map equation, called "gradient-curl" structure, and firstly showed the validity of the regulairty of weak solutions of the harmonic maps on two dimensional domain ([20, 21, 12, 1]). The structure was also made use of studying a regularity of solutions in some class of the harmonic map

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flow ([14, 8, 9, 30]). In this report we study some regularity criterion for weak solutons of the harmonic map flow, which generalizes the regularity results obtained previously.

To state our main result, let us set some preliminaries. Let $B_r(x_0)$ be a open ball of radius r with center x_0 in R^m and $Q_{\tau,r}(t_0,x_0) = (t_0-\tau,t_0)\times B_r(x_0)$ be a cylinder in $(0,\infty)\times R^m$. For brevity, let $Q_r(z_0) = (t_0-r^2,t_0)\times B_r(x_0)$ be a parabolic cylinder with vertex $z_0 = (t_0,x_0)$ in $(0,\infty)\times R^m$. We use the class of real-valued functions of bounded mean oscillation in a parabolic cylinder $Q = Q_R(z_0)\subset(0,\infty)\times R^m$, denoted by $BMO(Q)$, defined to be functiions $f\in L^1(Q)$ such that

$$|f|_{BMO(Q)} = \sup_{z\in Q}\sup_{0<r<2R}\frac{1}{|Q_r|}\int_{Q_r(z)\cap Q}|f(y)-(f)_{B_r(z)\cap Q}|dy \tag{1.2}$$

is finite, and referred to be one of the so-called parabolic Campanato spaces of which the members have some fundamental properties on regularity (see [3, 15, 11]). The class BMO is a Banach space with the norm $|f|_{L^1(Q)}+|f|_{BMO(Q)}$ and $|f|_{BMO(Q)}$ is the semi-norm in $BMO(Q)$. Also we need to use the space of functions $f\in L^1_{loc}(R^{m+1})$ such that $|f|_{BMO(R^{m+1})}$ is finite. The class $BMO(R^{m+1})$, modulo constants, (referred to the homogeneous BMO) is a Banach space with the norm (1.2) with Q and R replaced by R^{m+1} and ∞, respectively. For more properties of functions of bounded mean oscillation, we can refer to [4, 5, 13, 15].Our main theorem in this report is the regualrity criterion for a weak solution defined to be a map $u\in L^2(Q,R^n)$ such that $\nabla u\in L^2(Q,R^{mn})$, $\partial_t u\in L^2(Q,R^n)$ and the harmonic map flow equation is satisfied in the distribution sense.

Theorem 1 *There exists a positive constant ϵ_0 depending only on m and n such that the following holds for any weak solution u in $Q=(0,T)\times\Omega$ for a positive number T: If it holds for some $Q_{(R_0)^2,R_0}(t_0,x_0)\subset Q$ that*

$$|u|_{BMO(Q_{(R_0)^2,R_0}(t_0,x_0))} < \epsilon_0, \tag{1.3}$$

then, for any positive number $\alpha<1$, there exists a positive constant C depending only on α, R_0 and $|\nabla u|_{L^2(Q_{(R_0)^2,R_0}(t_0,x_0))}$ such that

$$|u|_{C^{\frac{\alpha}{2},\alpha}(Q_{\frac{(R_0)^2}{4},\frac{R_0}{2}}(t_0,x_0))} \le C, \tag{1.4}$$

where $C^{\frac{\alpha}{2},\alpha}(P)$ denotes the Hölder space on $P\subset R^{m+1}$ of Hölder exponent α with respect to the usual parabolic metric.

Using the Hölder regularity as above, we also obtain a Hölder estimate for spatial gradient and then, the Schauder estimate for the parabolic operator can be applied to have the boundedness in the class $C^{1,2}_{\rm loc}(Q)$ of classical solutions and the smoothness of solutions is derived from the usual bootstrap argument.

2. Outline of Proof.

Now we explain the outline on the proof of our main theorem. Our proof follows the so-called direct approach, which is used in [18] to make some regularity estimates for solutions of "small image" on the target of the the harmonic map flow. For related topics we refer to [15]. The Hölder estimate is based on the growth estimation of the local energy on the scale parameter, which is the parabolic version of Morrey's Dirichlet growth condition well-known in the elliptic case.

Lemma 2 *Let u be a weak solution of the harmonic map flow in $Q=(0,T)\times\Omega$. Suppose that (1.3) holds for some $Q_{R_0}=Q_{(R_0)^2,R_0}\subset Q$ with some positive number R_0 and vertex in Q. Then there exist positive constants C and $q>2$ depending only on m and n such that*

$$\int_{Q_r}|\nabla u|^2dz\le C\left\{(\tfrac{r}{R})^{m+2}+\right. \tag{1.5}$$

$$\left.+C\,|u|_{L^\infty(Q_R)}^{\frac{2}{q}}\left(\frac{1}{|Q_R|}\int_{Q_R}|u-(u)_{Q_R}|^2dz\right)^{1-\frac{2}{q}}\right\}\int_{Q_R}|\nabla u|^2dz$$

hold for all positive numbers $r<R\le R_0$.

Substituting the smallness (1.3) in BMO into (1.5), we can make iteration of the resulting inequality along the appropriate sequences of scale parameter R to have the growth estimate of the local scaled energy:

$$\int_{Q_r}|\nabla u|^2dz\le C\,r^{m+2-\delta} \tag{1.6}$$

holds for all positive numbers δ and $r\le R_0$, where the positive constant C depends only on $(R_0)^{-1}$, $|\nabla u|_{L^2(Q_{R_0})}$, m and n. Combining (1.6) with the Poincaré type inequality (1.8) in Lemma 4 available for a weak solution of the harmonic map flow, we can apply the isomorphism theorem for Campanato spaces(see [11, Thorem 3.1]) in the parabolic regime to arrive at the assertion (1.4).

The key ingredient for (1.5) is the so-called Caccioppoli inequality and the Gehring type reverse Hölder inequality, obtained from Caccioppoli inequality. These are exactly the "nonlinear" estimations adapted to the "nonlinear" equation of the harmoinic map flow.

Lemma 3 *Let u be a weak solution of the harmonic map flow. Suppose that (1.3) holds for some $Q_{R_0}=Q_{(R_0)^2,R_0}\subset Q$ with a positive number R_0 and some vertex in Q. For any positive number $\theta<1$, there exists a posiitive constant C depending only on θ, m and n such that*

$$\int_{Q_{\tau_1,\rho_1}}|\nabla u|^2dz\ \le\ C\left((\tau_2-\tau_1)^{-1}+(\rho_2-\rho_1)^{-2}\right)\int_{Q_{\tau_2,\rho_2}}|u-(u(t))_{B_{\rho_2}}|^2dz+ \tag{1.7}$$

$$+\theta\int_{Q_{\tau_2,\rho_2}}|\nabla u|^2dz$$

holds for any $Q_{\tau_2,\rho_2} = Q_{\tau_2,\rho_2}(t_0, x_0) \subset Q_{R_0}$ *any positive number* $\tau_1 < \tau_2$ *and* $\rho_1 < \rho_2$.

Here we emphasize that the hypothesis (1.3) that the solution is small in BMO is only required to obtain the Caccioppoli inequality. The Poincaré-type inequality in the parabolic regime is also available for weak solutions of the harmonic map flow without small BMO.

Lemma 4 *Let* u *be a weak solution of the harmonic map flow. Then there exists a positive constant* C *depending only on* m *and* n *such that*

$$\sup_{t_0-\rho^2<t<t_0} \int_{B_\rho(x_0)} \left|u - (u(t))_{B_\rho}\right|^2 dx \le C \int_{Q_{(2\rho)^2,2\rho}(t_0,x_0)} |\nabla u|^2 dz \tag{1.8}$$

holds for any $Q_{(2\rho)^2,2\rho}(t_0, x_0) \subset Q$.

Let 2^* be $2m/(m+2)$ if $m \ge 3$ and any number $\gamma \ge 1$ if $m = 2$, which is referred to be the Sobolev conjugate exponent of 2. Applying the Sobolev inequality available for functions in $W^{1,2^*}(B_\rho)$ and the Poincaré type inequality (4), the first integral in the right hand side of (1.7) is bounded by

$$\begin{aligned}
&\sup_{t_0-\tau_2<t<t_0} \left(\int_{B_{\rho_2}} \left|u - (u(t))_{B_{\rho_2}}\right|^2 dx \right)^{1-\frac{2^*}{2}} \int_{t_0-\tau_2}^{t_0} \left(\int_{B_{\rho_2}(x_0)} |u - (u(t))_{B_{\rho_2}}|^2 dx \right)^{\frac{2^*}{2}} dt \\
&\le C \left(\int_{Q_{(2\rho_2)^2,\rho_2}} |\nabla u|^2 dz \right)^{1-\frac{2^*}{2}} \int_{Q_{\tau_2,\rho_2}} |\nabla u|^{2^*} dz.
\end{aligned}$$

For any $Q_{(2r)^2,2r} = Q_{(4r)^2,4r}(t_0, x_0) \subset Q$, let $\rho_1 = r < 2r = \rho_2$ and $\tau_1 = r^2 < (2r)^2 = \tau_2$ in the Caccioppoli inequality (1.7) and combine the resulting inequality with the one above. Then we see that the inequality of reverse Hölder type hold: For any positive number δ, there exists a positive constant $C = C(\delta^{-1})$ depending only on δ^{-1}, m and n such that

$$\int_{Q_{r^2,r}} |\nabla u|^2 dz \le C(\delta^{-1}) \int_{Q_{(2r)^2,2r}} |\nabla u|^{2^*} dz + \delta \int_{Q_{(2r)^2,2r}} |\nabla u|^2 dz \tag{1.9}$$

holds for any $Q_{(2r)^2,2r} = Q_{(4r)^2,4r}(t_0, x_0) \subset Q$. Therefore we can invoke the Gehring-type lemma in the parabolic regime (see [18, 15] and [22] for a flexible proof) to have the desicive higher integrablity of the spatial gradient of weak solutions of the harmonic map flow.

Lemma 5 *Let* u *be a weak solution of the harmonic map flow. Suppose that (1.3) holds for some* $Q_{R_0} = Q_{(R_0)^2,R_0} \subset Q$. *Then there exist positive constant* $q > 2$ *and* C *depending only on* m *and* n *such that*

$$\int_{Q_{r^2,r}(t_0,x_0)} |\nabla u|^q dz \le C \left(\int_{Q_{(4r)^2,4r}(t_0,x_0)} |\nabla u|^2 dz \right)^{\frac{q}{2}} \tag{1.10}$$

holds for any $Q_{(4r)^2,4r}(t_0, x_0) \subset Q_{(R_0)^2,R_0}$.

To show the validity of the Caccioppoli inequality (1.7), we make use of the Hardy space on the parabolic space. Let $\mathrm{dist}(z, z') = \max\{|t - t'|^{1/2}, |x - x'|\}$ for $z = (t, x)$ and $z' = (t', x')$ in $R \times R^m$, which is exactly the parabolic distance associated with the continuous group of affine transformations on $R \times R^m$ with parameter $r > 0$

$$A_r = \begin{pmatrix} r^2 & 0 \\ 0 & rI_m \end{pmatrix},$$

where I_m denotes the $m \times m-$identity matrix. Then we clearly have that $\det A_r = r^{m+2}$, the transposed A_r^* of A_r is A_r and the infinitesimal generator of A_r is

$$P = \begin{pmatrix} 2 & 0 \\ 0 & I_m \end{pmatrix}.$$

In the "parabolic" regime, we can follow the argument due to Calderón and Torchinsky in [4, 5]. In particular, we have that the L^p-boundedness, the Hardy-Littewood maximal theorem, holds for the non-tangential maximal function with openning $a \geq 0$ on R^{m+1} and for any $p > 1$ (see [4, Theorem 1.7, p.8; Corollary 1.8, p.10]) and that, for each positive p, the L^p-norm of the non-tangential maximal function is equivalent on different kernels in class of rapidly decreasing smooth functions and also, on different openning $a \geq 0$ (see [4, Theorem 4.7, pp.29–30;Theorem 2.4, pp.12–13;Theorem 4.6, pp.27–29; Theorem 4.3 and the proof, pp.23–24]). We also notice that the "ball" with radius r and center of the origin 0 in the "parabolic" regime is $P_r = P_{r^2,r}(0) = \{z = (t, x) \in R^{m+1} : \mathrm{dist}(0, z) < r\}$ and that $Q_r = Q_{r^2,r}(0)$ with vertex 0, referred to be the parabolic cylinder. Now we recall the Hardy space in the parabolic regime (see [5, Definition 1.1, p.102; Theorem 1.2, p.102]), which is the class of functions $f \in L^1(R^{m+1})$ such that

$$|f|_{\mathcal{H}^1(R^{m+1})} = \int_{R^{m+1}} \sup_{r>0} |\phi_r * f(z)| \, dz \tag{1.11}$$

is finite, where $\phi_r(z) = r^{-(m+2)}\phi(z) = (\det A_r)^{-1}\phi(A_r^{-1}z)$ and ϕ is a smooth function on R^{m+1} with compact support in $Q_1 = (-1, 1) \times B_1(0)$. Again we know that the definition does not depend on choice of a function ϕ ([4, Theorem 4.7, pp.29–30]). We also use the "local" Hardy space on $Q = (0, T) \times B \subset R^{m+1}$, which is functions $f \in L^1(Q)$ such that, for some positive number δ,

$$|f|_{\mathcal{H}^1(Q,\delta)} = \int_Q \sup_{0<r<\delta} |\phi_r * f(z)| \, dz \tag{1.12}$$

is finite. We also have the duality between the Hardy space and the BMO space in the parabolic regime. In particular, we have the decisive inequality referred to be a generalization of the Hölder inequality (see [5, Theorem 2.5, p. 125; Remark, pp. 134-135]).

Lemma 6 *It holds that* $(\mathcal{H}^1(R^{m+1}))^* = BMO(R^{m+1})$. *Moreover, there exists a positive constant* C *depending only on* m *such that*

$$\left|\int_{R^{m+1}} f g\, dz\right| \le C\, |g|_{BMO(R^{m+1})} |f|_{\mathcal{H}^1(R^{m+1})} \tag{1.13}$$

holds for any $f \in \mathcal{H}^1(R^{m+1})$ *and* $g \in BMO(R^{m+1})$.

To invoke the Fefferman-Stein inequality (1.13) in a "local" time-space region, we need to use the local version of the parabolic Hardy space. The following estimate in the "local" Hardy space is derived from a simply modification of the observation in [29, Proposition 1.92 and the proof, pp.299–301].

Lemma 7 *Let* $Q = (0,T)\times B \subset R^{m+1}$. *Then* $f \in \mathcal{H}(Q,\delta)$ *if and only if, for any smooth function* η *defined on* R^{m+1} *with support in* Q *and non-vanishing integral, it holds that* $\eta(f-\overline{f}) \in \mathcal{H}^1((0,\infty)\times R^m)$, *where* $\overline{f}$ *is a weighted integral mean* $\int \eta f dz / \int \eta dz$. *In this case, we have the estimate*

$$\left|\eta(f-\overline{f})\right|_{\mathcal{H}^1((0,\infty)\times R^m)} \le C(1+\delta^{-\mu})\, |f|_{\mathcal{H}^1(Q,\delta)}, \tag{1.14}$$

where positive numbers C *and* μ *depend only on* m.

Proof of Lemma 3. Let $\sigma = \sigma(t)$ be a real-valued function on R which is smooth on $[t_0,\infty)$, equal to 1 on $[t_0, t_0-\tau_1]$ and has a compact support in $[t_0, t_0-\tau_2)$ and $\eta = \eta(x)$ be a real-valued smooth function defined on R^m such that $0 \le \eta \le 1$ on R^m, $\eta = 1$ on $B_{\rho_1}(x_0)$ and the support of η is compactly contained in $B_{\rho_2}(x_0)$. We use a test function $\sigma\,\eta^2\,(u-(u(t))_{B_{\rho_2}(x_0)})$ for the harmonic map flow equation, which is shown to be admissible by the boundedness $|u| = 1$ almost everywhere in Ω and the usual approximation with, for instance, the Steklov averagings on time-variable.

$$\begin{aligned}
&\int_{Q_{\tau_2,\rho_2}} -u\cdot\partial_t\left(\sigma\,(u-(u(t))_{B_{\rho_2}(x_0)})\right)\eta^2 + \sigma\,\nabla u\cdot\nabla\left(\eta^2(u-(u(t))_{B_{\rho_2}(x_0)})\right) dz \\
&= \int_{Q_{\tau_2,\rho_2}} \sigma\,\eta^2\,(u-(u(t))_{B_{\rho_2}(x_0)})\cdot u\,|\nabla u|^2\, dz.
\end{aligned}\tag{1.15}$$

Each term in the left hand side of (1.15) can be estimated in the fundamental way. The first term is calculated to be

$$\begin{aligned}
&\textstyle\int_{Q_{\tau_2,\rho_2}} -\left(u-(u(t))_{B_{\rho_2}(x_0)}\right)\cdot\partial_t\left(\sigma\,(u-(u(t))_{B_{\rho_2}(x_0)})\right)\eta^2 dz \\
&= -\tfrac{1}{2}\textstyle\int_{\{t=t_0\}\times B_{\tau_2}(x_0)} \left|u(t)-(u(t))_{B_{\rho_2}(x_0)}\right|^2 \eta^2 dx - \tfrac{1}{2}\int_{Q_{\tau_2,\rho_2}} \left|u-(u(t))_{B_{\rho_2}(x_0)}\right|^2 \partial_t\sigma\,\eta^2 dz,
\end{aligned}\tag{1.16}$$

where we use the fact that $(u(t))_{B_{\rho_2}}(x_0)$ is absolutely continuous on $t \in [0,T]$ and thus, $\partial_t (u(t))_{B_{\rho_2}}(x_0)$ integrable on $t\in[0,T]$ and so,

$$\textstyle\int_{Q_{\tau_2,\rho_2}} (u(t))_{B_{\rho_2}(x_0)}\cdot\partial_t\left(\sigma\,(u-(u(t))_{B_{\rho_2}(x_0)})\right)\eta^2 dz = 0. \tag{1.17}$$

Here and in what follows we have only to let the support of the function σ to be $[t', t_0 - \tau_2)$ for any t', $t_0 - \tau_1 \le t' \le t_0$ to make estimation of the supremum on time-varable of the integral on space of $\left|u - (u(t))_{B_{\rho_2}(x_0)}\right|^2 \eta^2$. Hölder's and Cauchy's inequalities make the second term to be bounded below by

$$\frac{1}{2}\int_{Q_{\tau_2,\rho_2}} |\nabla u|^2 \eta^2 \sigma dz - 2\int_{Q_{\tau_2,\rho_2}} \left|u - (u(t))_{B_{\rho_2}(x_0)}\right|^2 \sigma |\nabla \eta|^2 dz. \tag{1.18}$$

The main task is to estimate the right hand side of (1.15). For this purpose, we use the special structure of the nonlinear term of the harmonic map flow equation, which Helein originally discovered in elliptic case: For each $i = 1, \dots, n$, the equality

$$\begin{aligned} u^i |\nabla u|^2 &= \nabla u \cdot \left(u^i \nabla u - u \nabla u^i\right) \\ &= \nabla\left(u - (u(t))_{B_{\rho_2}(x_0)}\right) \cdot \left(u^i \nabla u - u \nabla u^i\right), \end{aligned} \tag{1.19}$$

holds almost everywhere in $B_{\rho_2}(x_0)$ for almost all t, $t_0 - \tau_2 < t < t_0$, where we use that $|u| = 1$ almost everywhere. For brevity let $\overline{u}(t) = (u(t))_{B_{\rho_2}(x_0)}$ in the following. For each $i = 1, \dots, n$, we use (1.19) to estimate the last term in (1.15) by

$$\begin{aligned} &\left|\int_{Q_{\tau_2,\rho_2}} \sigma \eta^2 (u^i - \overline{u}^i(t)) u^i |\nabla u|^2 dz\right| \\ &\le \left|\int_{Q_{\tau_2,\rho_2}} \sigma \eta (u^i - \overline{u}^i(t)) \left(u^i \nabla u - u \nabla u^i\right) \cdot \nabla\left(\eta (u - (u(t))_{B_{\rho_2}(x_0)})\right) dz\right| \\ &\quad + \left|\int_{Q_{\tau_2,\rho_2}} \sigma \eta (u^i - \overline{u}^i(t)) \left(u^i \nabla u - u \nabla u^i\right) \cdot (\nabla \eta\, (u - \overline{u}(t)))\, dz\right| \\ &\le \left|\int_{Q_{\tau_2,\rho_2}} (u^i - \overline{u}^i(t)) \sigma \eta \left\{\left(u^i \nabla u - u \nabla u^i\right) \cdot \nabla\left(\eta (u - (u(t))_{B_{\rho_2}(x_0)})\right) - \overline{H}\right\} dz\right| \\ &\quad + \left|\overline{H}\right| \left|\int_{Q_{\tau_2,\rho_2}} (u^i - \overline{u}^i(t)) \sigma \eta\, dz\right| \\ &\quad + \left|\int_{Q_{\tau_2,\rho_2}} \sigma \eta (u^i - \overline{u}^i(t)) \left(u^i \nabla u - u \nabla u^i\right) \cdot (\nabla \eta\, (u - \overline{u}(t)))\, dz\right|, \end{aligned} \tag{1.20}$$

where we put $\overline{H} = \int_{Q_{\tau_2,\rho_2}} \sigma \eta \,(u^i \nabla u - u \nabla u^i) \cdot \nabla(\eta(u - (u(t))_{B_{\rho_2}(x_0)}))\, dz / \int_{R^{m+1}} \sigma \eta\, dz$. Noting that $|u| = 1$ alomost everywhere and using Hölder's and Cauchy's inequalities, the second and third terms in the right hand side of (1.20) are bounded by

$$\begin{aligned} &C \int_{Q_{\tau_2,\rho_2}} \sigma \eta |\nabla \eta| |(u - \overline{u}(t))| |\nabla u|\, dz \\ &\le \delta \int_{Q_{\tau_2,\rho_2}} \sigma \eta^2 |\nabla u|^2 dz + C(\delta^{-1}) \int_{Q_{\tau_2,\rho_2}} \sigma |\nabla \eta|^2 |(u - \overline{u}(t))|^2 dz \end{aligned} \tag{1.21}$$

We now make estimation of the first term in the right hand side of (1.20) and claim it to be valid that

$$\begin{aligned}&\left|\int_{Q_{\tau 2,\rho 2}} (u^i-\overline{u}^i(t))\sigma\eta\left\{\left(u^i\nabla u-u\nabla u^i\right)\cdot\nabla\left(\eta(u-(u(t))_{B_{\rho 2}(x_0)})\right)-\overline{H}\right\}\,dz\right|\\ &\quad\le C\,|u^i|_{BMO(Q_{\tau_2,\rho_2})}|\sigma\nabla\left(\eta(u-\overline{u}(t))\right)|_{L^2(Q_{\tau_2,\rho_2})}|\nabla u|_{L^2(Q_{\tau_2,\rho_2})}.\end{aligned}\tag{1.22}$$

We now show the validity of (1.22) for each $i=1,\dots,n$. Firstly we apply the Fefferman-Stein inequality (1.13) to make the left hand side of (1.22) bounded by

$$\widetilde{C}\,\left|\sigma\eta(u^i-\overline{u}^i(t))\right|_{BMO(Q_{\tau_2,\rho_2})}\left|\left(u^i\nabla u-u\nabla u^i\right)\cdot\nabla\left(\eta(u-\overline{u}(t))\right)\right|_{\mathcal{H}^1(Q_{\tau_2,\rho_2},R_0)},\tag{1.23}$$

where $\widetilde{C}=C\,(R_0)^{-\mu}$ for positive numbers C and μ depending only on m, and let $\delta=R_0$ in Lemma 7 and we crucially use (1.14). To make control of the right hand side of (1.23), we divide the estimations needed into some lemmata.

The following estimation is obtained similarly as the one in $\mathcal{H}^1-$space due to Coifman, Lions, Meyer and Semmes ([10]), which is now the fundamental estimate for the nonlinear quantity of the so-called "gradient-curl" structure (also refer to [20, 12]).

Lemma 8 *It holds for any $i=1,\dots,n$ that*

$$\begin{aligned}&\left|\left(u^i\nabla u-u\nabla u^i\right)\cdot\nabla\left(\eta(u-\overline{u}(t))\right)\right|_{\mathcal{H}^1(Q_{\tau_2,\rho_2},R_0)}\\ &\le C\,|\nabla\left(\eta(u-\overline{u}(t))\right)|_{L^2(Q_{\tau_2,\rho_2})}|\nabla u|_{L^2(Q_{\tau_2,\rho_2})}\\ &\qquad +C\,|Q_{\tau_2,\rho_2}|\,|\partial_t u|_{L^2(Q_{\tau_2,\rho_2})}|\nabla\left(\eta(u-\overline{u}(t))\right)|_{L^2(Q_{\tau_2,\rho_2})}.\end{aligned}\tag{1.24}$$

In the proof of (1.24), we proceed our estimation similarly as in [10, Lemma II.1 and the proof, pp.251-252]. The main ingredients for the estimation are the formula, which is the parabolic version of Hélein's elegant trick,

$$\nabla\cdot\left(u^i\nabla u-u\nabla u^i\right)=u^i\partial_t u-u\partial_t u^i,\tag{1.25}$$

which holds in the distribution sense, and the local energy estimate

$$\int_{Q_{\tau_1,\rho_1}(z_0)}|\partial_t u|^2\,dz\le C\,\left((\tau_2-\tau_1)^{-1}+(\rho_2-\rho_1)^{-2}\right)\int_{Q_{\tau_2,\rho_2}(z_0)}|\nabla u|^2\,dz,\tag{1.26}$$

which holds for any $Q_{\tau_2,\rho_2}(z_0)\subset Q$ and all positive numbers $\tau_1<\tau_2$ and $\rho_1<\rho_2$ (refer to [14, 8]). In addition, we make use of the Hardy-Littlewood maximal inequality for the weighted maximal function ([28, Theorem A, p.3; Theorem B, p.7]). Let

$$M_\delta(f)(z_0)=\sup_{r>0}\left(\frac{r^\delta}{|Q_r|}\int_{Q_r(z_0)}|f(z)|\,dz\right).$$

Then, if $f \in L^\beta(R^{m+1})$, it holds that

$$|M_\delta(f)|_{L^\alpha(R^{m+1})} \leq C\ |f|_{L^\beta(R^{m+1})}\,, \tag{1.27}$$

where the positive numbers α and β are determined by the relation

$$1 < \beta \leq \alpha, \qquad \tfrac{1}{\beta} = \tfrac{1}{\alpha} + \tfrac{\delta}{m+2}.$$

Now let $p > 2$, $p < q < 2(m+2)/m$ and $p^* = \frac{p(m+2)}{m+2+p}$. Then we choose the function f and the exponents in (1.27) to be

$$\begin{aligned} &f = |\partial_t u|^{p^*}, \\ &\delta = p^*, \qquad \alpha = \tfrac{q}{p^*}, \qquad \beta = \tfrac{q(m+2+p)}{p(m+2+q)}\ . \end{aligned}$$

We have the local estimation for the semi-norm of BMO.

Lemma 9 *Let* $u \in \mathrm{BMO}(Q_{\tau_2,\rho_2})$ *defined as in (1.2). Then it holds that*

$$|\chi_{Q_{\tau_2,\rho_2}}(u^i - \overline{u}^i(t))|_{BMO(R^{m+1})} \leq C\,|u^i(t)|_{BMO(Q_{\tau_2,\rho_2})}, \tag{1.28}$$

where $\chi_{Q_{\tau_2,\rho_2}}$ *is the characteristic function of* Q_{τ_2,ρ_2} *and the positive constant* C *depends only on* m *and* n.

Finally, we combine the estimations (1.24), (1.26) and (1.28) with (1.23) to arrive at (1.22).

Remark. We use the identity (1.25) to see that, for each $i = 1, \ldots, n$,

$$\begin{aligned} \partial_t u^i - \Delta u^i \quad &= \quad \nabla \cdot \Big((u - \overline{u}(t)) \cdot (u^i \nabla u - u \nabla u^i)\Big) \\ &\qquad -(u - \overline{u}(t)) \cdot (u^i \partial_t u - u \partial_t u^i) \end{aligned} \tag{1.29}$$

holds in the distribution sense. We can apply the Calderón-Zygmund estimate ([23, 19]) available for the equation (1.29) to show the validity of our Theorem 1 (refer to [6]). Such regularity estimate will be studied in the forthcoming paper.

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Propagation of singularities for semilinear wave equations satisfying null condition *

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1 Introduction

In this talk, we consider the propagation of singularities of solutions to the following system of semilinear wave equations with the null condition in one space dimension,

$$\begin{cases} \Box u = h_1(u,v)Q_0(u,u) + h_2(u,v)Q_0(u,v) + h_3(u,v)Q_0(v,v) + h_4(u,v)Q_1(u,v), \\ \Box v = h_5(u,v)Q_0(u,u) + h_6(u,v)Q_0(u,v) + h_7(u,v)Q_0(v,v) + h_8(u,v)Q_1(u,v), \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \\ v(0,x) = v_0(x),\ \partial_t v(0,x) = v_1(x), \end{cases} \tag{1.1}$$

where $u = u(t,x) : \mathbb{R}^2 \to \mathbb{R}$, $u_0(x), u_1(x), v_0(x), v_1(x)$ are real valued functions, $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$, $Q_0(f,g) = (\partial_t f)(\partial_t g) - (\partial_x f)(\partial_x g)$, $Q_1(f,g) = (\partial_t f)(\partial_x g) - (\partial_x f)(\partial_t g)$ and $h_j(u,v)$ are polynomials of u and v with real coefficients for $j = 1, 2, \ldots, 8$. Let $3/2 < s < 2$ and we assume that intial data u_0 and v_0 are in $H^s(\mathbb{R})$ and u_1 and v_1 are in $H^{s-1}(\mathbb{R})$. We study H^r-regularities for the solutions u and v with $s < r < 3r - 2$.

In order to state the main results prescisely, we define the following microlocal regularity with respect to Sobolev space H^s.

Definition 1 *Suppose that $u \in D'(\mathbb{R}^2)$ and $U \subset \mathbb{R}^2$ is open. We say that u is in $H^s(U)$ if there exists a $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi(x_0) = 1$ such that*

$$\langle \tau, \xi \rangle^s |\widehat{\psi u}(\tau, \xi)| \in L^2(\mathbb{R}^2), \tag{1.2}$$

where $\langle \tau, \xi \rangle = (1 + |\tau|^2 + |\xi|^2)^{1/2}$.

Let $(t_0, x_0, \tau_0, \xi_0) \in \mathbb{R}^2_{t,x} \times (\mathbb{R}^n_{\tau,\xi} \backslash \{0\})$. We say that u is in microlocally H^r at $(t_0, x_0, \tau_0, \xi_0)$ if there exists a $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi(x_0) = 1$ and a conic neighborhood K of (τ_0, ξ_0) in $\mathbb{R}^2 \backslash \{0\}$ such that

$$\langle \tau, \xi \rangle^r \chi_K(\tau, \xi) |\widehat{\psi u}(\tau, \xi)| \in L^2(\mathbb{R}^2), \tag{1.3}$$

where χ_k is the characteristic function of K. We write $u \in H^r_{ml}(t_0, x_0, \tau_0, \xi_0)$ if u is in microlocally H^r at $(t_0, x_0, \tau_0, \xi_0)$. If Γ is a closed conic set in $\mathbb{R}^2_{t,x} \times (\mathbb{R}^n_{\tau,\xi} \backslash \{0\})$, we say that $u \in H^r_{ml}(\Gamma)$ if $u \in H^r_{ml}(t, x, \tau, \xi)$ for all $(t, x, \tau, \xi) \in \Gamma$.

*This is a joint work with Shingo Itoh (Department of mathematics, Tokyo University of Science).

It is known that the microlocal regularity with respect to Sobolev sapce propagate along some integral curve which is called a null bicharacteristic for the linear partial differential equations of principal type ([3]). Let $p(t,x,\tau,\xi)$ is a characteristic polynomial of differential operator P. Recall that for points with $p(t_0,x_0,\tau_0,\xi_0)=0$, the null bicharacteristic through (t_0,x_0,τ_0,ξ_0) is the curve defined by the Hamiltonian system

$$\frac{dt}{ds}=\frac{\partial p}{\partial\tau}(t,x,\tau,\xi),\ \frac{dx}{ds}=\frac{\partial p}{\partial\xi}(t,x,\tau,\xi),\ \frac{d\tau}{ds}=-\frac{\partial p}{\partial t}(t,x,\tau,\xi),\ \frac{d\xi}{ds}=-\frac{\partial p}{\partial x}(t,x,\tau,\xi)$$

$$t(0)=t_0,\ x(0)=x_0,\ \tau(0)=\tau_0,\ \xi(0)=\xi_0.$$

Our main result is the following.

Theorem 1.1 *Let $3/2<s<2$ and $u,v\in H^s$ be a time local solution of the initial value problem (1.1). Suppose that $u,v\in H^r_{ml}(t_0,x_0,\tau_0,\xi_0)$ with $\tau_0^2-\xi_0^2=0$ for $s<r<3s-2$. Let Γ be a null bicharacteristic starting from $(0,x_0,\tau_0,\xi_0)$. Then the solutions u,v are in $H^r_{ml}(\Gamma)$ for $|t|<<1$.*

We introduce the known results briefly. Hörmander [3] has shown that the singularities of solutions to linear partial differential equations of principal type propagate along the null bicharcteristics.

In 1979, J. Rauch [7] has shown that the H^r-regularity propagates along the null bicharacteristics for n-space dimensions semilinear wave equation $\Box u=f(u)$ with $u\in H^s(s>\frac{n+1}{2})$ and $s<r<2s-\frac{n+1}{2}$, where $f(u)$ is a polynomial of u.

In 1982, M. Beals and M. Reed [2] have shown that the H^r-regularity propagates along the null bicharacteristics for n-space dimensions semilinear wave equation $\Box u=f(u,Du)$ with $u\in H^s(s>\frac{n+1}{2}+1)$ and $s<r<2s-n-1$, where $f(u,Du)$ is a C^∞ function of u and Du([2]). (They have treated not only semilinear wave equations but also general semilinear strictly hyperbolic equations.)

In 1993, S. Klainerman and M. Machedon have shown the local existence of solutions to the semilinear wave equations with nonlinearity satisfying null conditions.

The co-worker S. Itoh has shown that H^r-regularity of solutions to the wave equation of the form $\Box u=f(u,Du)$ propagates along the null bicharcteristics with $u\in H^s(s>n/2)$, $s<r<3s-n-1$ if $f(u,Du)$ satisfying the null condition ([4], [5]).

2 Key lemma

Let $\varphi(x)$ be a C_0^∞ function satisfying $\varphi\equiv 1$ near x_0 and $\chi(\tau,\xi)$ be a homogeneous function of order 0 satisfying $\chi\equiv 1$ on conic neighborhood of (τ_0,ξ_0) with $\tau_0^2-\xi_0^2=0$. We consider the operator P with its symbol $p(t,x,\tau,\xi)=\varphi(x+(\xi/\tau)t)\chi(\tau,\xi)$, which is defined by

$$Pf=\int_{\mathbb{R}^2}p(t,x,\tau,\xi)\hat{f}(\tau,\xi)e^{i(t\tau+x\xi)}d\tau d\xi. \tag{2.1}$$

Simple calculation yields that the symbol of the commutator $[\Box,P]=\Box P-P\Box$ is $(\xi^2/\tau^2-1)\varphi''(x+(\xi/\tau)t)\chi$, which is the operator of order 0.

In order to prove the theorem 1.1, we multiply the operator P to the both sides of the equations (1.1) and use the energy estimate for Pu. To do so, we prepare the following commutator estimate, which is essentially due to M. Beals and M. Reed [2].

Lemma 2.1 *Let P be the operator defined in (2.1), $v\in H^s$, $\Box v\in H^{s-1}$ and $3/2<s<2$. Then, for any $0\le\epsilon<s-1$, we have*

$$[P,v_t\partial_t-v_x\partial_x](\partial_t\pm\partial_x)v\in H^{s-2+\epsilon}.$$

Proof. Assume that p depend on τ, ξ only (the general case requires some obvious modifications). Let $v_\pm = (\partial_t \pm \partial_x)v$, then we can write $v_t\partial_t - v_x\partial_x = \frac{1}{2}v_+(\partial_t - \partial_x) + \frac{1}{2}v_-(\partial_t + \partial_x)$. Hence it suffices to consider the following four cases $[P, v_+(\partial_t - \partial_x)]v_+$, $[P, v_+(\partial_t - \partial_x)]v_-$, $[P, v_-(\partial_t + \partial_x)]v_+$ and $[P, v_-(\partial_t + \partial_x)]v_-$. We only consider the case $[P, v_+(\partial_t - \partial_x)]v_+$ here. For abbreviation, we put $\eta = (\tau, \xi)$, $\eta' = (\tau', \xi')$. Then

$$\begin{aligned}
\mathcal{F}_{t,x}\left[[P, v_+(\partial_t - \partial_x)]v_+\right] &= ip(\eta)\int \widehat{v_+}(\eta - \eta')(\tau' - \xi')\widehat{v_+}(\eta')d\eta' \\
&\quad - i\int \widehat{v_+}(\eta - \eta')(\tau' - \xi')p(\eta')\widehat{v_+}(\eta')d\eta' \\
&= i\int \widehat{v_+}(\eta - \eta')(p(\eta) - p(\eta'))(\tau' - \xi'))\widehat{v_+}(\eta').
\end{aligned}$$

Putting $f(\eta) = \langle \eta\rangle^{s-1}(1 + |\tau - \xi|)\widehat{v_+}(\eta)$, we have

$$\langle \eta\rangle^{s-2+\epsilon}\mathcal{F}_{t,x}[P, v_+(\partial_t - \partial_x)]v_+ = \int K(\eta, \eta')f(\eta')f(\eta - \eta')d\eta',$$

where

$$|K(\eta, \eta')| \leq \frac{C\langle \eta\rangle^{s-2+\epsilon}(p(\eta) - p(\eta'))}{\langle \eta'\rangle^{s-1}\langle \eta - \eta'\rangle^{s-1}(1 + |(\tau - \tau') - (\xi - \xi')|)}.$$

By the same method as in the paper of M. Beals and M. Reed [2], we have

$$\langle \eta\rangle^{s-2+\epsilon}\mathcal{F}_{t,x}[P, v_+(\partial_t - \partial_x)]v_+ \in H^{s-2+\epsilon}.$$

The proofs are similar in the case of $[P, v_+(\partial_t - \partial_x)]v_+$, $[P, v_+(\partial_t - \partial_x)]v_-$ and $[P, v_-(\partial_t + \partial_x)]v_-$.

3 Outline of the proof of Theorem 1.1

Differentiating of the both side of (1.1), it follows with $(\partial_t \pm \partial_x)u = u_\pm$ that

$$\begin{aligned}
\Box u_\pm &= (\partial_t \pm \partial_x)\Box u \\
&= (h_{1,1}u_\pm + h_{1,2}v_\pm)Q_0(u, u) + 2h_1Q_0(u, u_\pm) + (h_{2,1}u_\pm + h_{2,2}v_\pm)Q_0(u, v) \\
&\quad + h_2(Q_0(u, v_\pm) + Q_0(u_\pm, v)) + (h_{3,1}u_\pm + h_{3,2}v_\pm)Q_0(v, v) \\
&\quad + 2h_3Q_0(v, v_\pm)(h_{4,1}u_\pm + h_{4,2}v_\pm)Q_1(u, v) + h_4(Q_1(u, v_\pm) + Q_1(u_\pm, v)) \qquad (3.1)
\end{aligned}$$

where the $h_{i,j}$ denote the partial derivatives of h_i. Let $p_0 \in S^0_{1,0}$ as before. Then from (3.1),

$$\begin{aligned}
\Box Pu_\pm =& [\Box, P]u_\pm + P\big(h_{1,1}u_\pm + h_{1,2}v_\pm)Q_0(u, u) + 2h_1Q_0(u, u_\pm) + (h_{2,1}u_\pm + h_{2,2}v_\pm)Q_0(u, v) \\
&+ h_2(Q_0(u, v_\pm) + Q_0(u_\pm, v)) + (h_{3,1}u_\pm + h_{3,2}v_\pm)Q_0(v, v) \\
&+ 2h_3Q_0(v, v_\pm)(h_{4,1}u_\pm + h_{4,2}v_\pm)Q_1(u, v) + h_4(Q_1(u, v_\pm) + Q_1(u_\pm, v))\big). \qquad (3.2)
\end{aligned}$$

Let $\alpha(t) \geq 0$ be in C_0^∞, $\alpha(0) = 1$. Let $\alpha_t(\tilde{t}) = \alpha(t - \tilde{t})$ and define

$$\begin{aligned}
E(t) \equiv & \left\|a(\tilde{t})\Lambda^{r-1}_{\tilde{t},x}\partial_{\tilde{t}}(Pu(t - \tilde{t}))\right\|^2_{L^2_{\tilde{t}}} + \left\|a(\tilde{t})\Lambda^{r-1}_{\tilde{t},x}\partial_x(Pu(t - \tilde{t}))\right\|^2_{L^2_{\tilde{t}}} + \left\|a(\tilde{t})\Lambda^{r-1}_{\tilde{t},x}(Pu(t - \tilde{t}))\right\|^2_{L^2_{\tilde{t}}} \\
&+ \left\|a(\tilde{t})\Lambda^{r-1}_{\tilde{t},x}\partial_{\tilde{t}}(Pv(t - \tilde{t}))\right\|^2_{L^2_{\tilde{t}}} + \left\|a(\tilde{t})\Lambda^{r-1}_{\tilde{t},x}\partial_x(Pv(t - \tilde{t}))\right\|^2_{L^2_{\tilde{t}}} + \left\|a(\tilde{t})\Lambda^{r-1}_{\tilde{t},x}(Pv(t - \tilde{t}))\right\|^2_{L^2_{\tilde{t}}}.
\end{aligned}$$

Let $\varphi \in C_0^\infty$, $\varphi_\delta = \dfrac{1}{\delta}\varphi\left(\dfrac{t}{\delta}\right)$ and $v_\delta = v * \varphi_\delta$. By the usual energy estimate and the lemma 2.1, we have

$$\begin{aligned}
\frac{1}{2}\frac{dE_\delta}{dt} =& -\langle a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\Box P u_{\pm,\delta}(t-\tilde{t}),\ a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_{\tilde{t}}(Pu_\delta(t-\tilde{t}))\rangle \\
&+ \langle a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_t(Pu_\delta(t-\tilde{t})),\ a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}Pu_\delta(t-\tilde{t})\rangle \\
&- \langle a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\Box P v_{\pm,\delta}(t-\tilde{t}),\ a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_{\tilde{t}}(Pv_\delta(t-\tilde{t}))\rangle \\
&+ \langle a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_t(Pv_\delta(t-\tilde{t})),\ a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}Pv_\delta(t-\tilde{t})\rangle \\
\leq& \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\Box P u_{\pm,\delta}(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_{\tilde{t}}(Pu_\delta(t-\tilde{t}))\right\|_{L^2_{\tilde{t},x}} \\
&+ \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_t(Pu_\delta(t-\tilde{t}))\right\|_{L^2_{\tilde{t},x}} \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}Pu_\delta(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} \\
&+ \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\Box P v_{\pm,\delta}(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_{\tilde{t}}(Pv_\delta(t-\tilde{t}))\right\|_{L^2_{\tilde{t},x}} \\
&+ \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}\partial_t(Pv_\delta(t-\tilde{t}))\right\|_{L^2_{\tilde{t},x}} \left\|a(\tilde{t})\Lambda_{\tilde{t},x}^{r-1}Pv_\delta(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} \\
\leq& C_1(E(t)+C_2).
\end{aligned}$$

Hence Gronwall's inequality implies

$$E(t) \leq C_t E(0) < \infty.$$

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A coupled system of nonlinear Schrödinger equations with delta functions as initial data

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1 Introduction

We consider the following system of nonlinear Schrödinger equations

$$(1) \qquad \begin{cases} i\partial_t u_1 + \partial_x^2 u_1 = \lambda_1 |u_2|^{p_1-1} u_1 & \text{in } [0,\infty) \times \mathbf{R}, \\ i\partial_t u_2 + \partial_x^2 u_2 = \lambda_2 |u_1|^{p_2-1} u_2 & \text{in } [0,\infty) \times \mathbf{R} \end{cases}$$

with initial data

$$(2) \qquad u_1(0,x) = \mu\delta_a(x), \quad u_2(0,x) = \nu\delta_b(x) \quad \text{for } x \in \mathbf{R},$$

where $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, and the unknown functions $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ are complex-valued. For $a \in \mathbf{R}$, let $\delta_a(x)$ denote the Dirac mass at a. The aim of this talk is to study how the interaction between the nonlinearities effects on solutions, under the following condition:

$$(3) \qquad 2 \le p_j < 3, \quad a, b \in \mathbf{R}, \quad \lambda_j, \mu, \nu \in \mathbf{C} \quad (j = 1, 2).$$

This work is inspired by the pioneering work of Kita [2, 3], where the evolution of the superposition of delta functions under the nonlinear Schrödinger flow was studied. More precisely, Kita [2, 3] considered, among other things, the single Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p-1} u, & \text{in } [0,\infty) \times \mathbf{R}^n, \\ u(0,x) = \sum\limits_{j\in\mathbf{Z}} \mu_j \delta_{ja} & \text{for } x \in \mathbf{R}^n, \end{cases}$$

where $n \ge 1$, $\Delta = \sum_{k=1}^n \partial/\partial x_k^2$, $1 < p < 1 + 2/n$, $\lambda \in \mathbf{C}$ and $\{\mu_j\}_{j\in\mathbf{Z}} \subset \mathbf{C}$ satisfying $\sum_{j\in\mathbf{Z}} (1 + |j|^2)|\mu_j|^2 < \infty$, and showed the following results:

(i) there exists a local solution of (1) which has a special form;

(ii) if $\mathrm{Im}\lambda > 0$, then the local solution blows up in finite time;

(iii) if $\mathrm{Im}\lambda \leq 0$, then the solution exists globally in time.

In particular, we see that if $\mathrm{Im}\lambda > 0$, then the nonlinearity makes the solution unstable, and that if $\mathrm{Im}\lambda \leq 0$, then the nonlinearity has the dissipative nature. Therefore, the following question naturally arises: When either $\mathrm{Im}\lambda_1 > 0$ and $\mathrm{Im}\lambda_2 \leq 0$ or $\mathrm{Im}\lambda_1 \leq 0$ and $\mathrm{Im}\lambda_2 > 0$, which does effect coming from the nonlinear term dominate the other? This kind of consideration has been done by the author [1], where the following damped nonlinear Schrödinger equation is handled: $i\partial_t u + \Delta u + idu = \lambda |u|^{p-1}u$ ($\Delta = \sum_{k=1}^{n} \partial^2/\partial x_k^2$, $n \geq 1$, $d \in \mathbf{R}$, $\lambda \in \mathbf{C}$, $1 < p < 1 + 2/n$). Especially, when we take $\mu\delta_a(x)$ ($\mu \in \mathbf{C}$, $a \in \mathbf{R}^n$) as the initial data, even if $\mathrm{Im}\lambda > 0$, there exists a positive number d_0 such that

(i) for all $d < d_0$, the solution blows up in finite time;

(ii) for all $d \geq d_0$, the solution exists globally in time.

This result means that the linear term idu, which provides dissipative nature, violates the nonlinear effect if $d \geq d_0$. This is a motivation to study the above question.

First of all, we state a result on the local solution. For $a \in \mathbf{R}$, let $U(t)\delta_a$ denote the solution of the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0 & \text{in } [0,\infty) \times \mathbf{R}, \\ u(0,x) = \delta_a(x) & \text{for } x \in \mathbf{R}, \end{cases}$$

that is, $U(t)\delta_a = \exp(it\partial_x^2)\delta_a = (4\pi i t)^{-1/2} \exp(i|x-a|^2/4t)$.

Theorem 1 (local result). *Assume* (3). *Then there exist a time $T > 0$ and a unique solution of the Cauchy problem* (1)-(2) *described as*

$$\begin{cases} u_1(t,x) = A(t)U(t)\delta_a \in L^\infty_{\mathrm{loc}}((0,T];L^\infty(\mathbf{R})), \\ u_2(t,x) = B(t)U(t)\delta_b \in L^\infty_{\mathrm{loc}}((0,T];L^\infty(\mathbf{R})), \end{cases} \tag{4}$$

such that

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \in (C([0,T]) \cap C^1((0,T]))^2 \tag{5}$$

with $A(0) = \mu$ *and* $B(0) = \nu$.

Our next interest is to study global behavior of the solution to (1). We state blowup and global results.

Theorem 2. *Assume* (3). *Then the following hold:*

(i) (blowup result) *If* $\mathrm{Im}\lambda_1 > 0$, $\mathrm{Im}\lambda_2 > 0$ *and* $\mu\nu \neq 0$, *then the solution* (4) *blows up in finite time, that is, there exists a finite time* $T^* > 0$ *such that* $\lim_{t\to T^*-0} |A(t)| = \infty$ *or* $\lim_{t\to T^*-0} |B(t)| = \infty$.

(ii) (global result) *If either* $\mathrm{Im}\lambda_1 \leq 0$ *or* $\mathrm{Im}\lambda_2 \leq 0$, *then there exists a unique global solution of* (1)-(2) *described as* (4), *such that*

$$(A(t), B(t)) \in (C([0,\infty)) \cap C^1(0,\infty))^2.$$

Remark 1. When $\mu \neq 0$ and $\nu = 0$ (resp. $\mu = 0$ and $\nu \neq 0$) with (3), $\mathrm{Im}\lambda_1 > 0$ and $\mathrm{Im}\lambda_2 > 0$, the solutions $A(t)$ and $B(t)$ exist globally in time. More precisely, we have $|A(t)| \equiv |\mu|$ and $B(t) \equiv 0$ (resp. $A(t) \equiv 0$ and $|B(t)| \equiv |\nu|$) for all $t \geq 0$. We shall prove this assertion after the proof of Theorem 2 (i).

In the next section we shall prove the above theorems. The idea of proof of Theorem 1 is based on the reduction of the original problem (1)-(2) to (6)-(7) below. We show the part (i) of Theorem 2 by constructing two monotone increasing sequences which push the solutions $A(t)$ and $B(t)$ to blow up, respectively, and the part (ii) of the theorem by establishing suitable *a priori* estimates.

2 Proofs of Theorems 1 and 2

Proof of Theorem 1. Substitution of (4) into (1) yields the ODE system of $A(t)$ and $B(t)$:

$$i\frac{dA}{dt}(t) = \lambda_1 \varphi_{p_1}(t)|B(t)|^{p_1-1}A(t), \quad A(0) = \mu, \tag{6}$$

$$i\frac{dB}{dt}(t) = \lambda_2 \varphi_{p_2}(t)|A(t)|^{p_2-1}B(t), \quad B(0) = \nu, \tag{7}$$

where $\varphi_p(t) = (4\pi t)^{-(p-1)/2}$. It suffices to prove the existence and uniqueness of (6)-(7) for proving Theorem 1. To solve (6)-(7), we deal with the following integral equation:

$$\begin{aligned} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} &= \begin{pmatrix} \mu \\ \nu \end{pmatrix} - i\int_0^t \begin{pmatrix} \lambda_1 \varphi_{p_1}(\tau)|B(\tau)|^{p_1-1}A(\tau) \\ \lambda_2 \varphi_{p_2}(\tau)|A(\tau)|^{p_2-1}B(\tau) \end{pmatrix} d\tau \\ &=: \begin{pmatrix} \Psi_1\binom{A(t)}{B(t)} \\ \Psi_2\binom{A(t)}{B(t)} \end{pmatrix} =: \Psi\begin{pmatrix} A(t) \\ B(t) \end{pmatrix}, \quad t \in [0,T]. \end{aligned} \tag{8}$$

We shall solve (8) by the contraction mapping principle for Ψ.

Let $\rho = \max\{|\mu|, |\nu|\}$ and let $\overline{U_{2\rho}}$ be the following complete metric space:

$$\overline{U_{2\rho}} = \left\{ \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \in (L^\infty([0,T]))^2; \right.$$
$$\left. \left\| \begin{pmatrix} A \\ B \end{pmatrix} \right\| := \max\left\{ \sup_{0\le t\le T} |A(t)|, \sup_{0\le t\le T} |B(t)| \right\} \le 2\rho \right\}.$$

We then see that

$$(9) \quad \begin{aligned}
\left| \Psi_1 \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \right| &\le |\mu| + |\lambda_1| \int_0^t \varphi_{p_1}(\tau) |B(\tau)|^{p_1-1} |A(\tau)|\, d\tau \\
&\le |\mu| + |\lambda_1| \int_0^T \varphi_{p_1}(\tau) |B(\tau)|^{p_1-1} |A(\tau)|\, d\tau \\
&\le \rho + |\lambda_1| (2\rho)^{p_1} \int_0^T \varphi_{p_1}(\tau)\, d\tau \\
&= \rho + \frac{2|\lambda_1|(2\rho)^{p_1}}{3-p_1} \varphi_{p_1}(T) T, \\
\left| \Psi_2 \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \right| &\le |\nu| + |\lambda_2| \int_0^t \varphi_{p_2}(\tau) |A(\tau)|^{p_2-1} |B(\tau)|\, d\tau \\
&\le |\nu| + |\lambda_2| \int_0^T \varphi_{p_2}(\tau) |A(\tau)|^{p_2-1} |B(\tau)|\, d\tau \\
&\le \rho + |\lambda_2| (2\rho)^{p_2} \int_0^T \varphi_{p_2}(\tau)\, d\tau \\
&= \rho + \frac{2|\lambda_2|(2\rho)^{p_2}}{3-p_2} \varphi_{p_2}(T) T,
\end{aligned}$$

for all $0 \le t \le T$. Therefore we have $\Psi(\overline{U_{2\rho}}) \subset \overline{U_{2\rho}}$ if $T > 0$ is sufficiently small. Also for sufficiently small $T > 0$, we find that Ψ is a contraction mapping on $\overline{U_{2\rho}}$. Indeed, if $p_1 = 2$, then for any $0 \le t \le T$, we have

$$(10) \quad \left| \Psi_1 \begin{pmatrix} A_1(t) \\ B_1(t) \end{pmatrix} - \Psi_1 \begin{pmatrix} A_2(t) \\ B_2(t) \end{pmatrix} \right|$$
$$\le |\lambda_1| \int_0^t \varphi_2(\tau) \big| |B_1(\tau)| A_1(\tau) - |B_2(\tau)| A_2(\tau) \big|\, d\tau$$
$$\le |\lambda_1| \Big(\big(\sup_{0\le s\le T} |B_1(s)| \big) \sup_{0\le s\le T} |A_1(s) - A_2(s)|$$
$$+ \big(\sup_{0\le s\le T} |A_2(s)| \big) \sup_{0\le s\le T} |B_1(s) - B_2(s)| \Big) \int_0^T \varphi_2(\tau)\, d\tau$$
$$\le 2(2\cdot 2\rho)|\lambda_1| \varphi_2(T) T \left\| \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} - \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \right\|.$$

If $2 < p_1 < 3$, then for any $0 \le t \le T$, we have

(11)

$$\begin{aligned}
&\left|\Psi_1\begin{pmatrix}A_1(t)\\B_1(t)\end{pmatrix} - \Psi_1\begin{pmatrix}A_2(t)\\B_2(t)\end{pmatrix}\right| \\
&\le |\lambda_1| \int_0^t \varphi_{p_1}(\tau)\big||B_1(\tau)|^{p_1-1}A_1(\tau) - |B_2(\tau)|^{p_1-1}A_2(\tau)\big|\, d\tau \\
&\le |\lambda_1|\Big(\big(\sup_{0\le s\le T}|B_1(s)|\big)^{p_1-1}\sup_{0\le s\le T}|A_1(s)-A_2(s)| \\
&\qquad + \big(\sup_{0\le s\le T}|A_2(s)|\big)\sup_{0\le s\le T}\big(|B_1(s)|^{p_1-1} - |B_2(s)|^{p_1-1}\big)\Big)\int_0^T \varphi_{p_1}(\tau)\, d\tau \\
&\le \frac{2|\lambda_1|}{3-p_1}\varphi_{p_1}(T)T\Big((2\rho)^{p_1-1}\sup_{0\le s\le T}|A_1(s)-A_2(s)| \\
&\qquad + (2\rho)\sup_{0\le s\le T}\big(\int_0^1 \frac{d}{d\theta}|\theta B_1(s) + (1-\theta)B_2(s)|^{p_1-1}\, d\theta\big)\Big) \\
&\le \frac{2|\lambda_1|}{3-p_1}\varphi_{p_1}(T)T\Big((2\rho)^{p_1-1}\sup_{0\le s\le T}|A_1(s)-A_2(s)| \\
&\qquad + (2\rho)(p_1-1)\sup_{0\le s\le T}\big((|B_1(s)|+|B_2(s)|)^{p_1-2}|B_1(s)-B_2(s)|\big)\Big) \\
&\le \frac{2|\lambda_1|}{3-p_1}\varphi_{p_1}(T)T\Big((2\rho)^{p_1-1}\sup_{0\le s\le T}|A_1(s)-A_2(s)| \\
&\qquad + 2(4\rho)^{p_1-1}\sup_{0\le s\le T}|B_1(s)-B_2(s)|\Big) \\
&\le \frac{6|\lambda_1|(4\rho)^{p_1-1}}{3-p_1}\varphi_{p_1}(T)T\left\|\begin{pmatrix}A_1\\B_1\end{pmatrix} - \begin{pmatrix}A_2\\B_2\end{pmatrix}\right\|.
\end{aligned}$$

Similarly, if $p_2 = 2$, then for all $0 \le t \le T$, we find

$$(12)\quad \left|\Psi_2\begin{pmatrix}A_1(t)\\B_1(t)\end{pmatrix} - \Psi_2\begin{pmatrix}A_2(t)\\B_2(t)\end{pmatrix}\right| \le 2(2\cdot 2\rho)|\lambda_2|\varphi_2(T)T\left\|\begin{pmatrix}A_1\\B_1\end{pmatrix} - \begin{pmatrix}A_2\\B_2\end{pmatrix}\right\|,$$

and if $2 < p_2 < 3$, then for all $0 \le t \le T$, we have

$$(13)\quad \left|\Psi_2\begin{pmatrix}A_1(t)\\B_1(t)\end{pmatrix} - \Psi_2\begin{pmatrix}A_2(t)\\B_2(t)\end{pmatrix}\right| \le \frac{6|\lambda_2|(4\rho)^{p_2-1}}{3-p_2}\varphi_{p_2}(T)T\left\|\begin{pmatrix}A_1\\B_1\end{pmatrix} - \begin{pmatrix}A_2\\B_2\end{pmatrix}\right\|.$$

These inequalities (9) through (13) imply that for $T > 0$ sufficiently small, the mapping Ψ is a contraction on $\overline{U_{2\rho}}$. Thus a solution of (8) exists in $(L^\infty([0,T]))^2$. Moreover, we have (5), i.e., $\binom{A(t)}{B(t)} \in (C([0,T])\cap C^1((0,T]))^2$, and the uniqueness of $\binom{A(t)}{B(t)}$ in $(C([0,T]))^2$ in a similar way as [3]. This completes the proof. $\square$

In order to show Theorem 2, multiplying the equation in (6) by $\overline{A(t)}$ and the one in (7) by $\overline{B(t)}$, and taking the imaginary part, we obtain

$$\frac{d}{dt}|A(t)|^2 = 2(\mathrm{Im}\lambda_1)\varphi_{p_1}(t)|B(t)|^{p_1-1}|A(t)|^2, \tag{14}$$

$$\frac{d}{dt}|B(t)|^2 = 2(\mathrm{Im}\lambda_2)\varphi_{p_2}(t)|A(t)|^{p_2-1}|B(t)|^2. \tag{15}$$

Now we shall prove Theorem 2.

Proof of Theorem 2 (i). Assume that the conclusion is false, that is, $A(t)$ and $B(t)$ exist globally in time. We shall show that this leads to a contradiction. Define the following sequences of real positive numbers $\{a_k\}_{k\in\mathrm{N}}$ and $\{b_k\}_{k\in\mathrm{N}}$:

$$\begin{cases} a_1 = |\mu|^2, \quad b_1 = |\nu|^2, \\ a_{k+1} = a_k\big(1 + 2(\mathrm{Im}\lambda_1)b_k^{(p_1-1)/2}\varphi_{p_1}(1)\big) \quad \text{for } k \geq 1, \\ b_{k+1} = b_k\big(1 + 2(\mathrm{Im}\lambda_2)a_k^{(p_2-1)/2}\varphi_{p_2}(1)\big) \quad \text{for } k \geq 1. \end{cases}$$

We remark that both $\{a_k\}_{k\in\mathrm{N}}$ and $\{b_k\}_{k\in\mathrm{N}}$ are monotone increasing sequences. Indeed, by $a_k > 0$ and $b_k > 0$ for all $k \geq 1$, we have

$$a_{k+1} - a_k = 2(\mathrm{Im}\lambda_1)a_k b_k^{(p_1-1)/2}\varphi_{p_1}(1) > 0, \tag{16}$$

$$b_{k+1} - b_k = 2(\mathrm{Im}\lambda_2)b_k a_k^{(p_2-1)/2}\varphi_{p_2}(1) > 0. \tag{17}$$

For all $t \geq 0$, we see from (14) and (15) that

$$\frac{d}{dt}|A(t)|^2 \geq 0, \quad \frac{d}{dt}|B(t)|^2 \geq 0.$$

Hence we obtain

$$|A(t)|^2 \geq |\mu|^2 = a_1, \quad |B(t)|^2 \geq |\nu|^2 = b_1, \quad t \geq 0.$$

Therefore, for all $t \geq 0$, we have by (14) and (15) again

$$\frac{d}{dt}|A(t)|^2 \geq 2(\mathrm{Im}\lambda_1)a_1 b_1^{(p_1-1)/2}\varphi_{p_1}(t),$$
$$\frac{d}{dt}|B(t)|^2 \geq 2(\mathrm{Im}\lambda_2)b_1 a_1^{(p_2-1)/2}\varphi_{p_2}(t).$$

Thus we deduce that

$$
\begin{aligned}
|A(t)|^2 &\geq |\mu|^2 + 2(\mathrm{Im}\lambda_1)a_1 b_1^{(p_1-1)/2}\int_0^t \varphi_{p_1}(\tau)\,d\tau \\
&= a_1 + 2(\mathrm{Im}\lambda_1)a_1 b_1^{(p_1-1)/2}\frac{2}{3-p_1}\varphi_{p_1}(t)t \\
&\geq a_1\bigl(1 + 2(\mathrm{Im}\lambda_1)b_1^{(p_1-1)/2}\varphi_{p_1}(t)t\bigr) \\
&\geq a_1\bigl(1 + 2(\mathrm{Im}\lambda_1)b_1^{(p_1-1)/2}\varphi_{p_1}(1)\bigr) = a_2, \\
|B(t)|^2 &\geq |\nu|^2 + 2(\mathrm{Im}\lambda_2)b_1 a_1^{(p_2-1)/2}\int_0^t \varphi_{p_2}(\tau)\,d\tau \\
&= b_1 + 2(\mathrm{Im}\lambda_2)b_1 a_1^{(p_2-1)/2}\frac{2}{3-p_2}\varphi_{p_2}(t)t \\
&\geq b_1\bigl(1 + 2(\mathrm{Im}\lambda_2)b_1^{(p_2-1)/2}\varphi_{p_2}(t)t\bigr) \\
&\geq b_1\bigl(1 + 2(\mathrm{Im}\lambda_2)a_1^{(p_2-1)/2}\varphi_{p_2}(1)\bigr) = b_2,
\end{aligned}
$$

for all $t \geq 1$. Repeating the above argument, we obtain

$$
|A(t)|^2 \geq a_k, \quad |B(t)|^2 \geq b_k, \quad t \geq 1, \quad k \geq 2. \tag{18}
$$

Since both $\{a_k\}_{k\in\mathrm{N}}$ and $\{b_k\}_{k\in\mathrm{N}}$ are monotone increasing sequences, we see by (16) and (17)

$$
\begin{aligned}
a_{k+1} - a_k &= 2(\mathrm{Im}\lambda_1)a_k b_k^{(p_1-1)/2}\varphi_{p_1}(1) \\
&\geq 2(\mathrm{Im}\lambda_1)a_1 b_1^{(p_1-1)/2}\varphi_{p_1}(1) > 0, \\
b_{k+1} - b_k &= 2(\mathrm{Im}\lambda_2)b_k a_k^{(p_2-1)/2}\varphi_{p_2}(1) \\
&\geq 2(\mathrm{Im}\lambda_2)b_1 a_1^{(p_2-1)/2}\varphi_{p_2}(1) > 0,
\end{aligned}
$$

and hence we have

$$
\begin{aligned}
a_k - a_1 &= \sum_{j=1}^{k-1}(a_{j+1} - a_j) \geq 2(k-1)(\mathrm{Im}\lambda_1)a_1 b_1^{(p_1-1)/2}\varphi_{p_1}(1), \\
b_k - b_1 &= \sum_{j=1}^{k-1}(b_{j+1} - b_j) \geq 2(k-1)(\mathrm{Im}\lambda_2)b_1 a_1^{(p_2-1)/2}\varphi_{p_2}(1).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
a_k &\geq a_1 + 2(k-1)(\mathrm{Im}\lambda_1)a_1 b_1^{(p_1-1)/2}\varphi_{p_1}(1), \\
b_k &\geq b_1 + 2(k-1)(\mathrm{Im}\lambda_2)b_1 a_1^{(p_2-1)/2}\varphi_{p_2}(1),
\end{aligned}
$$

which imply that $a_k \to \infty$ and $b_k \to \infty$ as $k \to \infty$. This and (18) lead to a contradiction. This completes the proof of the part (i). $\square$

Proof of Remark 1. Suppose that $\mu \neq 0$ and $\nu = 0$ with (3), $\mathrm{Im}\lambda_1 > 0$ and $\mathrm{Im}\lambda_2 > 0$. We derive *a priori* estimates on $|A(t)|$ and $|B(t)|$ (we refer to [4], Section 1-2, pp. 3–10, for instance). Let $T > 0$. Suppose that there exists $0 < S \leq T$ such that the unique solution of (6)-(7) satisfying (5) (i.e., $\binom{A(t)}{B(t)} \in (C([0,S]) \cap C^1((0,S]))^2$) exists. We then see from (15) that

$$|B(t)|^2 \leq |\nu|^2 \exp\Big(2(\mathrm{Im}\lambda_2)\big(\sup_{0\leq s\leq t}|A(s)|^{p_2-1}\big)\int_0^t \varphi_{p_2}(\tau)\,d\tau\Big)$$
$$= 0, \quad 0 \leq t \leq S,$$

by the assumption $\nu = 0$. Therefore we obtain by (14)

$$\frac{d}{dt}|A(t)|^2 = 0, \quad 0 \leq t \leq S.$$

Consequently we have

$$|A(t)| \equiv |\mu|, \quad B(t) \equiv 0, \quad 0 \leq t \leq S. \tag{19}$$

Combining (19) and Theorem 1, we find that the solution exists globally in time, that is, $\binom{A(t)}{B(t)} \in (C([0,\infty)) \cap C^1(0,\infty))^2$. Moreover by (19), for $t \geq 0$, we have $|A(t)| \equiv |\mu|$ and $B(t) \equiv 0$.

When $\mu = 0$ and $\nu \neq 0$, we obtain $A(t) \equiv 0$ and $|B(t)| \equiv |\nu|$ by analogous argument mentioned in the above. □

Proof of Theorem 2 (ii). We derive *a priori* estimates on $|A(t)|$ and $|B(t)|$ (we refer to [4], Section 1-5, pp. 19–23, for instance). Let $T > 0$. Suppose that there exists $0 < S \leq T$ such that the unique solution of (6)-(7) satisfying (5) (i.e., $\binom{A(t)}{B(t)} \in (C([0,S]) \cap C^1((0,S]))^2$) exists.

Firstly, if $\mathrm{Im}\lambda_1 \leq 0$ and $\mathrm{Im}\lambda_2 \leq 0$, then we have from (14) and (15)

$$|A(t)| \leq |\mu|, \quad |B(t)| \leq |\nu|, \quad 0 \leq t \leq S. \tag{20}$$

Combining (20) and Theorem 1, we obtain that the solution exists globally in time, that is, $\binom{A(t)}{B(t)} \in (C([0,\infty)) \cap C^1(0,\infty))^2$.

We next consider the cases "$\mathrm{Im}\lambda_1 > 0$ and $\mathrm{Im}\lambda_2 \leq 0$" and "$\mathrm{Im}\lambda_1 \leq 0$ and $\mathrm{Im}\lambda_2 > 0$." One can assume without loss of generality that $\mathrm{Im}\lambda_1 > 0$ and $\mathrm{Im}\lambda_2 \leq 0$. Then we see from (15) that

$$|B(t)| \leq |\nu|, \quad 0 \leq t \leq S, \tag{21}$$

and hence for all $0 \leq t \leq S$, we obtain from (14)

$$\begin{aligned} |A(t)| &\leq |\mu| \exp\Big(|\nu|^{p_1-1}(\mathrm{Im}\lambda_1)\int_0^t \varphi_{p_1}(\tau)\,d\tau\Big) \\ &\leq C\exp(C'\varphi_{p_1}(t)t) \\ &\leq C\exp(C'\varphi_{p_1}(T)T). \end{aligned} \tag{22}$$

Combining (22), (21) and Theorem 1, we see that the solution exists globally in time, that is, $\binom{A(t)}{B(t)} \in (C([0,\infty)) \cap C^1(0,\infty))^2$. This completes the proof of the theorem. $\square$

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