第 18 回　関数空間セミナー報告集

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（会場：北海道大学）

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Seminar on Function Spaces, 2009

CONTENTS

The Lower Norm Of Multiplication Operators And Champagne Subdomains

Rikio Yoneda (Otaru University of Commerce)

Let D be the open unit disk in complex plane C. For $z, w \in D$, $0 < r < 1$, let $\varphi_w(z) = \dfrac{w-z}{1-\overline{w}z}$ and let $\rho(z,w) = \left|\dfrac{w-z}{1-\overline{w}z}\right|$ and $D(w,r) = \left\{z \in D, \rho(w,z) < r\right\}$. Then the Euclidean center and radius of $D(w,r)$ are

$$C = \frac{1-r^2}{1-r^2|w|^2}w, \quad R = \frac{1-|w|^2}{1-r^2|w|^2}r.$$

For $\alpha \geq 0$, the space L^∞_α is defined to be the Banach space of Lebesgue measurable functions f on the open unit disk D with

$$\| f \|_{L^\infty_\alpha} = \sup_{z \in D}\left(1-|z|^2\right)^\alpha |f(z)| < +\infty.$$

Let $H(D)$ be the space of all analytic functions on D. Then the Bloch-type space $\mathcal{B}^\alpha$ of D is defined by

$$\mathcal{B}^\alpha = H(D) \cap L^\infty_\alpha.$$

The space of bounded analytic functions on D will be denoted by H^∞.

Let $v : D \to R^+$, radial weight, continuous strictly positive function on D such that the weighted Banach spaces $H^\infty_v = \left\{f \in H(D) : \sup_{z\in D} v(z)|f(z)| < \infty\right\}$ contains a non-zero function, where $v(z) = v(|z|)$.

In the case of $v_1(z) = (1-|z|^2)^\alpha$, then it is clear that $H^\infty_{v_1} = \mathcal{B}^\alpha$ is the Bloch-type spaces.

For $\alpha > -1$ and $p \geq 1$, the space $L^p\left((1-|z|^2)^\alpha dA(z)\right)$ is defined to be the space of Lebesgue measurable functions f on D such that

$$\| f \|_{L^p((1-|z|^2)^\alpha dA(z))} = \left\{\int_D |f(z)|^p(\alpha+1)(1-|z|^2)^\alpha dA(z)\right\}^{\frac{1}{p}} < +\infty,$$

where $dA(z)$ denote the area measure on D. The weighted Bergman space is defined by

$$L_a^p\left((1-|z|^2)^\alpha dA(z)\right) = H(D) \cap L^p\left((1-|z|^2)^\alpha dA(z)\right).$$

For $\alpha > -1$ and $p \geq 1$, the weighted Dirichret space D_p^α is defined to be the space of analytic functions f on D such that

$$\int_D (1-|z|^2)^\alpha |f'(z)|^p (\alpha+1) dA(z) < +\infty.$$

In the case of $\alpha = 1$ and $p = 2$, then $D_2^1 = H^2$ is the Hardy space. In the case of $\alpha = 2$ and $p = 2$, then $D_2^2 = L_a^2$ is the Bergman space.

Let X, Y be Banach spaces and let T be a linear operator from X into Y. Then T is called to be bounded below from X to Y if there exists a positive constant $C > 0$ such that $\| Tf \|_Y \geq C \| f \|_X$ for all $f \in X$, where $\| * \|_X$, $\| * \|_Y$ be the norm of X, Y, respectively.
And the lower norm is defined by the following:

$$L(T) = \inf_{\|x\|_X = 1} \| Tx \|_Y .$$

For g analytic on D, the multiplication operators M_g is defined by the following:

$$M_g(f)(z) = g(z)f(z).$$

With respect to the lower norm, it is open problem to calculate it. In fact, although a large number of studies have been made on the closed range operator, little is known about the lower norm of the operator. The purpose of this paper is to evaluate the lower norm of the multiplication operator on some Banach spaces. And that we characterize "champagne subdomain" that $D \setminus D(w, r_w) = D \setminus \{z \in D, |\varphi_w(z)| < r_w\}$ is a sampling set for some Banach spaces.

Definition. *Let $\alpha > 0$. A set Γ of the open unit disk D is called a sampling set for $\mathcal{B}^\alpha$ if there exists a positive constant $C > 0$ such that*

$$\sup_{z \in D}(1-|z|^2)^\alpha |f(z)| \leq C \sup_{z \in \Gamma}(1-|z|^2)^\alpha |f(z)|,$$

for all $f \in \mathcal{B}^\alpha$.

In [20] we studied the multiplication operator M_h on the weighted Bergman spaces that have a closed range using sampling set for $\mathcal{B}^\alpha$.

Theorem. ([20]) *Let $\alpha > 0$. For any inner function φ, the following are equivalent.*
(1) *M_φ is bounded below on $L^2_a\left((1-|z|^2)^{2\alpha}dA(z)\right)$*
(2) *There exists a positive constant $\epsilon > 0$ such that $\left\{z \in D, |\varphi(z)| \geq \epsilon\right\}$ is a sampling set for $\mathcal{B}^\alpha$.*
(3) $\inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^\alpha|\varphi(z)|\right\} > 0.$
(4) *φ is a finite product of interpolating Blaschke products.*
(5) *For some $\epsilon > 0$ and $0 < r < 1$ the area of the subset of the disc $D_{z,r} = \left\{w : \rho(z,w) < r\right\}$ where $|\varphi(w)| > \epsilon$ is comparable to the area of the whole disc $D_{z,r}$, $z \in D$. Moreover*

$$L(H_h) \approx \inf_{w\in D}\left\{\sup_{z\in D}(1-|\varphi_w(z)|^2)^\alpha|h(z)|\right\}.$$

In [3], J.Bonet, P.Domanski and Lindstrom proved the following.

Theorem. ([3]) *Let $v(z) = e^{-\frac{1}{1-|z|}}$. Then the lower norm of $M_{\varphi_w} : H^\infty_v \to H^\infty_v$ tends to zero as $|w| \to 1^-$.*

In [5], K.Cichon and K.Seip proved the following.

Theorem. ([5]) *Let $v \in C^2(D)$ be a radial weight and set $-\Delta\log v(z) = f(z)$. Suppose $(1-r^2)^2 f(r) \nearrow +\infty$ as $r \to 1^-$. Then the lower norm of $M_{\varphi_w} : H^\infty_v \to H^\infty_v$ tends to zero as $|w| \to 1^-$.*

Proposition 1. *Let $\delta > 0$ and $v(z) = e^{-\frac{1}{(1-|z|)^\delta}}$. For $w \in D$, the following are equivalent:*

(i) The multiplication operator $M_{\varphi_w} : H_v^\infty \to H_v^\infty$ is bounded below.

(ii) There exists a positive $\epsilon > 0$ such that $\left\{z \in D, |\varphi_w(z)| \geq \epsilon\right\}$ is a sampling set for H_v^∞.

Theorem 2. *Let $\delta > 0$ and $v(z) = e^{-\frac{1}{(1-|z|)^\delta}}$, and $w \in D$. Let $L(M_{\varphi_w})$ be the lower norm of the operator $M_{\varphi_w} : H_v^\infty \to H_v^\infty$. Then there exists a positive constant $K > 0$ such that*

$$L(M_{\varphi_w}) \leq K(1 - |w|)^{\frac{\delta}{2}}.$$

Corollary 3. *Let $\delta > 0$ and $v(z) = e^{-\frac{1}{(1-|z|)^\delta}}$. If there exists a positive r_w such that $D \setminus D(w, r_w)$ is a sampling set for H_v^∞, then there exists a positive constant $K > 0$ such that $r_w \leq K(1 - |w|)^{\frac{\delta}{2}}$.*

As a general consequence for a general weight $v \in C^2(D)$, we have the following result.

Theorem 4. *Let $\eta \in D$. Let $v \in C^2(D)$ be a radial weight and set $-\Delta \log v(z) = f(z)$. Suppose $(1 - r^2)^2 f(r) \nearrow +\infty$ as $r \to 1^-$. Let $L(M_{\varphi_\eta})$ be the lower norm of the multiplication operator $M_{\varphi_\eta} : H_v^\infty \to H_v^\infty$. Then there exists a $\gamma_\eta \in (0, 1)$ such that for some constant $K > 0$,*

$$L(M_{\varphi_\eta}) \leq KC(\gamma_\eta)^{-\frac{1}{4}},$$

where $C(\gamma_\eta) = (1 - \gamma_\eta^2)^2 f(\gamma_\eta)$.

Corollary 5. *Let $\eta \in D$. Let $v \in C^2(D)$ be a radial weight and set $-\Delta \log v(z) = f(z)$. Suppose $(1 - r^2)^2 f(r) \nearrow +\infty$ as $r \to 1^-$. If there exists a* *positive* r_η *such that* $D \setminus D(\eta, r_\eta)$ *is a sampling set for* H_v^∞, *then there exists a* $\gamma_\eta \in (0,1)$ *such that for some constant* $K > 0$, $r_\eta \leq KC(\gamma_\eta)^{-\frac{1}{4}}$, *where* $C(\gamma_\eta) = (1-\gamma_\eta^2)^2 f(\gamma_\eta)$.

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Almost commuting three unitaries and $\mathbb{Z}^2$-actions of the Jiang-Su algebra, a joint work with Hiroki Matui

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In this note, we give a survey of the studies for almost commuting unitaries and we obtain a result of $\mathbb{Z}^2$ actions on the projectionless algebra, which is called the Jiang-Su algebra. This result is an application of the theorem for almost commuting three unitaries. Throughout the note, we use M_d, $d \in \mathbb{N}$ to denote the algebra of $d \times d$ matrices with complex entries, $U(M_d)$ the unitary group of M_d, and M_d^1 the set of norm one elements of M_d. We set $[a, b] = ab - ba$ for a, $b \in M_d$. We denote by tr_d the normalized trace of M_d. The branch of log is the negative part of the real axis.

1 Almost commuting three unitaries

First, for almost commuting two unitaries we recall the following example given by D. Voiculescu in [Vo1].

Example 1.1. We set unitaries $d_n \in U(M_n)$, $n \in \mathbb{N}$ by

$$d_n = \mathrm{diag}(1, \exp(2\pi\sqrt{-1}/n), \exp(4\pi\sqrt{-1}/n), ..., \exp(2(n-1)\pi\sqrt{-1}/n)),$$

and set $s_n \in U(M_n)$ as the shift unitary. Then it is well-known that $\lim_{n\to\infty} \|[d_n, s_n]\| = 0$, but there exist no two unitaries d_n' and $s_n' \in M_n$ such that $\lim \|d_n' - d_n\| = 0$, $\lim \|s_n' - s_n\| = 0$, and $[d_n', s_n'] = 0$ for any $n \in \mathbb{N}$, because of

$$\mathrm{tr}_n \circ \log(d_n s_n d_n^* s_n^*) = \exp(-2\pi\sqrt{-1}/n).$$

Regardless of this example, the following theorem was proved by S. Eilers, T. A. Loring, and G. K. Pedersen.

Theorem 1.2 (Corollary 6.15, [ELP]). *Let k_n, $n \in \mathbb{N}$ be an increasing sequence. Suppose that $u_{1,n}$, $u_{2,n} \in U(M_{k_n})$, $n \in \mathbb{N}$ satisfy that*

(i) $\lim\limits_{n\to\infty} \|[u_{1,n}, u_{2,n}]\| = 0,$

(ii) $\mathrm{tr}_{k_n} \circ \log(u_{1,n}u_{2,n}u_{1,n}^*u_{2,n}^*) = 0,\ n \in \mathbb{N}$,

Then there exist $u'_{1,n},\ u'_{2,n} \in U(M_{k_n}),\ n \in \mathbb{N}$ *such that*

$$\lim_{n\to\infty} \|u'_{i,n} - u_{i,n}\| = 0, \quad i = 1, 2,$$

$$[u'_{1,n}, u'_{2,n}] = 0, \text{ for any } n \in \mathbb{N}.$$

The original problem given by P. R. Halmos for almost commuting two self-adjoint elements was affirmatively solved by H. Lin, [Lin1].

On the other hand, for almost commuting three unitaries there occurred a seemingly different phenomenon. The existence of the following example was proved by D. Voiculescu in [Vo2], and concrete examples was given by K. Davidson in [Dav].

Example 1.3. There exist $k_n \in \mathbb{N}$, $n \in \mathbb{N}$ and three self-adjoint elements $a_{i,n} \in M_{k_n}^1$, $i = 1, 2, 3$, $n \in \mathbb{N}$, satisfying that: $\lim_{n\to\infty} \|[a_{i,n}, a_{j,n}]\| = 0$, i, $j = 1, 2, 3$, but there exist no three self-adjoint elements $a'_{i,n} \in M_{k_n}$, $i = 1, 2, 3$, such that $\lim \|a'_{i,n} - a_{i,n}\| = 0$, $i = 1, 2, 3$, and

$$[a'_{i,n}, a'_{j,n}] = 0, \quad i, j = 1, 2, 3, \text{ for any } n \in \mathbb{N}.$$

By these three self-adjoint elements $a_{i,n}$ we obtain almost commuting three unitaries $u_{i,n} = \exp(\pi\sqrt{-1}a_{i,n}/2)$, $i = 1, 2, 3$ satisfying that: $\lim_{n\to\infty} \|[u_{i,n}, u_{j,n}]\| = 0$, $i, j = 1, 2, 3$,

$$\mathrm{tr}_{k_n} \circ \log(u_{i,n}u_{j,n}u_{i,n}^*u_{j,n}^*) = 0, \quad i, j, = 1, 2, 3, \text{ for any } n \in \mathbb{N},$$

but there exist no three unitaries $u'_{i,n} \in U(M_{k_n})$, $i = 1, 2, 3$, such that $\lim \|u'_{i,n} - u_{i,n}\| = 0$, $i = 1, 2, 3$, and

$$[u'_{i,n}, u'_{j,n}] = 0, \ i, j = 1, 2, 3, \text{ for any } n \in \mathbb{N}.$$

Regardless this example, we have the following theorem which is a corollary of theorem proved by G. Gong and H. Lin, [GL]. If a sequence of unitaries $u_n \in U(M_{k_n})$, $n \in \mathbb{N}$ satisfies

$$\lim_{n\to\infty} \mathrm{tr}_{k_n}(f(u_n)) = \int_{\mathbb{T}} f(t)d\mu(t), \quad f \in C(\mathbb{T}),$$

where μ is the normalized Haar measure on $\mathbb{T}$ (see [BEK]), we say that $\mathrm{sp}(u_n)$ *is uniformly distributed on* $\mathbb{T}$.

Theorem 1.4 (A corollary of Theorem 1.5 in [GL])**.** *Let* k_n, $n \in \mathbb{N}$ *be an increasing sequence. Suppose that* $u_{i,n} \in U(M_{k_n})$, $i = 1, 2, 3$, $n \in \mathbb{N}$ *satisfy that:*

(i) $\lim_{n\to\infty} \|[u_{i,n}, u_{j,n}]\| = 0,\ i, j = 1, 2, 3,$

(ii) $\mathrm{tr}_{k_n} \circ \log(u_{i,n}u_{j,n}u_{i,n}^*u_{j,n}^*) = 0,\ i, j = 1, 2, 3,\ n \in \mathbb{N}$,

(iii) *for any* $(j_1, j_2, j_3) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$, $\mathrm{sp}(u_{1,n}^{j_1} u_{2,n}^{j_2} u_{3,n}^{j_3})$, $n \in \mathbb{N}$ *are uniformly distributed on* $\mathbb{T}$.

Then there exist $u'_{i,n} \in U(M_{k_n})$, $i = 1, 2, 3$, $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} \|u'_{i,n} - u_{i,n}\| = 0, \quad i = 1, 2, 3,$$

$$[u'_{i,n}, u'_{j,n}] = 0, \quad i, j = 1, 2, 3, \text{ for any } n \in \mathbb{N}.$$

2 The Jiang-Su algebra

In this section, we recall the construction of the Jiang-Su algebra. We define the *prime dimension drop algebra* by the following

$$\mathbf{I}(k, k+1) := \{f \in C([0,1], M_{k(k+1)}); f(0) \in M_k \otimes 1_{k+1},\ f(1) \in 1_k \otimes M_{k+1}\}.$$

It is easy to see that $I(k, k+1)$ is a projection less C^*-algebra which satisfies that

$$(K_0(\mathbf{I}(k,k+1)), K_0(\mathbf{I}(k,k+1))_+, [1_{\mathbf{I}(k,k+1)}]) = (\mathbb{Z}, \mathbb{Z}_+, 1), \quad K_1(\mathbf{I}(k,k+1)) = \{0\}.$$

The *Jiang-Su algebra* $\mathcal{Z}$ is the inductive limit C^*-algebra which is unital, simple, and with a unique trace τ. The following properties were proved by Jiang and Su.

$$\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}, \quad K_*(\mathcal{Z}) \cong K_*(\mathbb{C}),$$

and any $\alpha \in \mathrm{Aut}(\mathcal{Z})$ is approximated by inner automorphisms of unitaries $u_n \in \mathcal{Z}$ as

$$\alpha(a) = \lim u_n a u_n^*, \ a \in \mathcal{Z}.$$

Recently, Toms, Lin, Niu, and Winter showed the following classification theorem in [Tom], [LN], [Win].

Theorem 2.1 (Toms, Lin-Niu, Winter). *Let A and B be unital simple ASH-algebras with a unique tracial state. Suppose that*

$$A \otimes \mathcal{Z} \cong A, \ B \otimes \mathcal{Z} \cong B, \quad K_*(A) \cong K_*(B).$$

Then we have $A \cong B$.

3 $\mathbb{Z}^2$-actions

Let (π_τ, H_τ) be the GNS-representation associated with the unique trace τ of $\mathcal{Z}$. For $u \in U(\mathcal{Z})$, set $\mathrm{Ad}\,u(a) := uau^*$, $a \in \mathcal{Z}$, and set

$$\mathrm{WInn}(\mathcal{Z}) := \{\alpha \in \mathrm{Aut}(\mathcal{Z});\ \pi_\tau \circ \alpha = \mathrm{Ad}\, W \circ \pi_\tau, \quad W \in \overline{\pi_\tau(\mathcal{Z})}^w\}.$$

For $\mathbb{Z}$-actions of the Jiang-Su algebra $\mathcal{Z}$, we obtained the following theorem in [Sat]. In the proof of this theorem, we essentially use the result of almost commuting two unitaries, Theorem 1.2.

Theorem 3.1. *Suppose that α, $\beta \in \mathrm{Aut}(\mathcal{Z})$ satisfy $\alpha^k, \beta^k \notin \mathrm{WInn}(\mathcal{Z})$, for any $k \in \mathbb{N}$. Then there exist $\delta \in \mathrm{Aut}(\mathcal{Z})$ and $u \in U(\mathcal{Z})$ such that*

$$\alpha = \mathrm{Ad}\, u \circ \delta \circ \beta \circ \delta^{-1}.$$

Similarly using the result of almost commuting three unitaries, Theorem 1.4, for $\mathbb{Z}^2$-actions on the Jiang-Su algebra $\mathcal{Z}$ we can obtain the following main theorem in [MS].

Theorem 3.2. *Let α, β be $\mathbb{Z}^2$ actions on $\mathcal{Z}$. Suppose that $\alpha_g, \beta_g \notin \mathrm{WInn}(\mathcal{Z})$, for any $g \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Then there exist $\delta \in \mathrm{Aut}(\mathcal{Z})$ and $u_g \in U(\mathcal{Z})$, $g \in \mathbb{Z}^2$ such that*

$$\alpha_g = \mathrm{Ad}\, u_g \circ \delta \circ \beta_g \circ \delta^{-1}, \quad g \in \mathbb{Z}^2.$$

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C^*-algebras generated by composition operators with finite Blaschke product symbols (有限 Blaschke 積をシンボルとする合成作用素から作られる C^* 環)

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概要

This is a joint work with Y. Watatani.

Let R be a finite Blaschke product of degree at least two with $R(0) = 0$. Then there exists a relation between the associated composition operator C_R on the Hardy space and the C^*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system $(R^{\circ n})_n$ on the Julia set J_R. We study the C^*-algebra $\mathcal{TC}_R$ generated by both the composition operator C_R and the Toeplitz operator T_z to show that the quotient algebra by the ideal of the compact operators is isomorphic to the C^*-algebra $\mathcal{O}_R(J_R)$, which is simple and purely infinite.

1 はじめに

講演の内容は，九州大学の綿谷氏との共同研究で，[6] に基づくものである．

R を次数が 2 以上で $R(0) = 0$ を満たす有限 Blaschke 積とする．このとき，合成作用素 C_R と Toeplitz 作用素 T_z から生成される C^* 環 $\mathcal{TC}_R$ がどのような構造を持つかを調べた．$\mathcal{TC}_R$ をコンパクト作用素からなるイデアルで剰余した環 $\mathcal{OC}_R$ が，[8] で梶原氏と綿谷氏によって導入された複素力学系に付随する C^* 環 $\mathcal{O}_R(J_R)$ に同型であることがわかった．これにより，$\mathcal{OC}_R$ が単純で純無限であることを示すことができた．さらに同型を用いることで，$\mathcal{OC}_R$ が $\mathcal{OC}_{z^n}$ に同型であることもわかった．ただし n は R の次数とする．これは $\mathcal{OC}_R$ が R の次数のみに依存し，$\mathcal{OC}_R$ では R の解析的な情報が消えていることを意味している．

同型を示す過程では，主に 2 つの作用素の関係式を示すことが重要になり，そのためには合成作用素の共役作用素を計算することになる．合成作用素の共役作用素の研究は [1], [2], [3], [7], [9], [10] など多くなされているが，ここでは $\mathcal{L}_R$ という作用素を考え，これが Hardy 空間上では共役作用素になっていることを用いて計算を行う．

2 Toeplitz-composition C^*-algebras

$\mathbb{C}$ 内の単位開円板を $\mathbb{D}$，その境界を $\mathbb{T}$ とする．Hardy 空間 $H^2(\mathbb{D})$ を，$f \in H^2(\mathbb{D})$ に対して，動径極限値

$$\lim_{r \to 1-} f(re^{i\theta}) = f(e^{i\theta}) \quad \text{a.e.}\, \theta$$

を取ることにより $L^2(\mathbb{T})$ の部分空間 $H^2(\mathbb{T})$ と同一視し，$H^2 := H^2(\mathbb{D}) = H^2(\mathbb{T})$ と書く．

$\varphi : \mathbb{D} \to \mathbb{D}$ を解析的とする．このとき，合成作用素 $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ は

$$(C_\varphi f)(z) := f(\varphi(z)), \quad z \in \mathbb{D}$$

で定義される．$R(0) = 0$ を満たす有限 Blaschke 積 R の場合には合成作用素 C_R は等距離作用素になる (Ryff [12], Nordgren [11])．

今回考える合成作用素 C_φ から作られる C^* 環を定義する．Toeplitz 作用素 T_z も加えて以下のように定める．

定義. $\varphi : \mathbb{D} \to \mathbb{D}$ を解析的とする．

(1) T_z と C_φ で生成される C^* 環を $\mathcal{TC}_\varphi$ と書く．φ に対する Toeplitz-composition C^*-algebra という．

(2) $\mathcal{OC}_\varphi := \mathcal{TC}_\varphi / \mathcal{K}(H^2)$ と定める．ただし，$\mathcal{K}(H^2)$ は H^2 上のコンパクト作用素全体のなす C^* 環である．

このとき，$\mathcal{TC}_\varphi$ は $C(\mathbb{T})$ の元 a をシンボルとする Toeplitz 作用素 T_a を含むことに注意する．

3 複素力学系に付随する C^* 環

次数 2 以上で $R(0) = 0$ を満たす有限 Blaschke 積 R のとき，$\mathcal{OC}_R$ を解析するために，梶原氏と綿谷氏によって導入された複素力学系に付随する C^* 環について説明する [8]．この C^* 環は有理関数 R の反復合成系 $(R^{\circ n})_n$ を C^* 環の立場から調べるために導入されたものである．同じように複素力学系から C^* 環を構成する方法として Deaconu-Muhly [4] があるが，ここでの方法と少し異なっている．

R が有理関数，J_R を R の Julia 集合とする．$\operatorname{graph} R|_{J_R} := \{(z, w) \in J_R \times J_R \mid w = R(z)\}$ とおく．

定義. $A = C(J_R), X_R = C(\operatorname{graph} R|_{J_R})$ とおく．X_R に Hilbert 双加群の構造を以下で定める．$a, b \in A$, $\xi, \eta \in X_R$ に対して，

$$(a \cdot \xi \cdot b)(z, w) := a(z)\xi(z, w)b(w),$$
$$\langle \xi, \eta \rangle_A(w) := \sum_{z \in R^{-1}(w)} e(z)\overline{\xi(z, w)}\eta(z, w).$$

ただし，$e(z)$ は z における分岐指数である．

今回考えている場合では次のようになっている．

注意. R が次数 2 以上の有限 Blaschke 積で，$R(0)=0$ のとき

- $J_R = \mathbb{T}$,
- $z \in J_R$ に対して，$e(z)=1$.

R が有理関数のとき，複素力学系に付随する C^* 環 $\mathcal{O}_R(J_R)$ を X_R を用いて次のように定義する.

定義. X_R から作られる Cuntz-Pimsner 環を $\mathcal{O}_R(J_R)$ と書く.

主定理では次の定理を用いる.

定理 1 (Kajiwara-Watatani [8]). R を次数 2 以上の有理関数とする．このとき，$\mathcal{O}_R(J_R)$ は単純である.

4 主定理

以上の準備のもと，今回得られた定理を述べる.

定理 2. R を次数 2 以上の有限 Blaschke 積で $R(0)=0$ とする．つまり $n \geq 2, |\lambda|=1, |z_k|<1$ に対して

$$R(z) = \lambda z \prod_{k=1}^{n-1} \frac{z-z_k}{1-\overline{z_k}z}$$

とする．このとき $\mathcal{OC}_R$ は単純，純無限で，$\mathcal{OC}_R \cong \mathcal{O}_R(J_R)$ となる.

4.1 $R(z)=z^n$ の場合の証明

実際の証明では，$R(z)=z^n$ の場合を特別に分ける必要はないが，この場合が最も簡単な場合であるのでここで説明する．ただし $n \geq 2$ である．このとき複素力学系に付随する C^* 環 $\mathcal{O}_R(J_R)$ は，以下の関係式を満たす $U, S_1, \ldots, S_n$ で生成される普遍的な C^* 環になることがわかる.

$$\begin{aligned} &U^*U = UU^* = I, \quad S_i^*S_i = I, \qquad i=1,\ldots,n, \\ &US_i = S_{i+1}, \qquad i=1,\ldots,n-1, \\ &US_n = S_1U, \quad \sum_{i=1}^{n} S_iS_i^* = I. \end{aligned}$$

$R(z)=z^n$ の場合の定理 *2* の証明. $\pi : \mathcal{TC}_R \to \mathcal{OC}_R$ を自然な商写像として,

$$U := \pi(T_z), \quad S_i := \pi(T_z)^{i-1}\pi(C_R)$$

とおく．このとき $U, S_1, \ldots, S_n$ が上に示した関係式を満たし，U のスペクトルが $\mathbb{T}$ であることが示せる．ただし $US_n = S_1U$ を示すときには下記の補題 3 を用いる．$\mathcal{O}_R(J_R)$ は単純で普遍的なので，同型であることが従う. □

次の式は簡単な計算で示すことができる.

補題 3. $\varphi : \mathbb{D} \to \mathbb{D}$ を解析的とする．このとき

$$C_\varphi T_z = T_\varphi C_\varphi$$

が成り立つ．

証明．$f \in H^2$ に対して，

$$(C_\varphi T_z f)(w) = (T_z f)(\varphi(w)) = \varphi(w) f(\varphi(w)) = \varphi(w)(C_\varphi f)(w) = (T_\varphi C_\varphi f)(w)$$

から従う． □

4.2 一般の場合の証明

R を次数 2 以上の有限 Blaschke 積で $R(0) = 0$ とする．この場合には，大きく分けて命題 5 と補題 6 の 2 つのことを示せばよい．あとは $\mathcal{O}_R(J_R)$ が単純で普遍的であることを使えば同型が言える．

まず始めに，C_R^* を計算する際に重要な役割を果たす次の作用素を定義する．

定義. $\mathbb{T}$ 上の関数 f に対して，

$$\mathcal{L}_R(f)(w) := \sum_{z \in R^{-1}(w)} \frac{1}{|R'(z)|} f(z) = \sum_{z \in R^{-1}(w)} \frac{R(z)}{zR'(z)} f(z), \quad w \in \mathbb{T}$$

と定める．

このとき，$\mathcal{L}_R$ は次の性質を満たす．

補題 4. (1) $\alpha : C(\mathbb{T}) \to C(\mathbb{T})$ を

$$(\alpha(a))(z) := a(R(z)), \quad a \in C(\mathbb{T}), z \in \mathbb{T}$$

と定める．このとき $\mathcal{L}_R$ は，$C(\mathbb{T})$ 上では $(C(\mathbb{T}), \alpha)$ に関する Exel の transfer [5] になる．つまり，$\mathcal{L}_R : C(\mathbb{T}) \to C(\mathbb{T})$ は正写像 (したがって特に連続) で，

$$\mathcal{L}_R(\alpha(a)b) = a\mathcal{L}_R(b), \quad a, b \in C(\mathbb{T})$$

が成り立つ．さらに $\mathcal{L}_R(1) = 1$ を満たす．

(2) $\mathcal{L}_R$ は $H^2(\mathbb{T})$ では，有界線形作用素になり，$C_R^* = \mathcal{L}_R$ が成り立つ．

これらの性質に加えて次の良く知られた事実を使えば，命題 5 を示すことができる．

事実. $n \geq 2, |\lambda| = 1$ で，$z_0 = 0, |z_k| < 1$ $(k = 1, \ldots, n-1)$ のとき，

$$R(z) = \lambda \prod_{k=0}^{n-1} \frac{z - z_k}{1 - \overline{z_k} z}$$

を考える．

$$Q_l(z) := \frac{\sqrt{1 - |z_l|^2}}{1 - \overline{z_l} z}, \quad R_l(z) := \begin{cases} 1, & l = 0, \\ \displaystyle\prod_{k=0}^{l-1} \frac{z - z_k}{1 - \overline{z_k} z}, & l \geq 1 \end{cases}$$

とし，$e_{kn+l} := Q_l R_l R^k \quad (k \geq 0\ ,\ 0 \leq l \leq n-1)$ とおく．このとき，$\{e_k\}_{k=0}^{\infty}$ は H^2 の完全正規直交系になる．

命題 5. $a \in C(\mathbb{T})$ に対して，
$$C_R^* T_a C_R = T_{\mathcal{L}_R(a)}$$
が成り立つ．

もう一つの重要な部分は次の式が成り立つことである．

補題 6. 上の事実で定めた記号を用いる．
$$\sum_{k=1}^{n} T_{Q_{k-1}R_{k-1}} C_R C_R^* (T_{Q_{k-1}R_{k-1}})^* = I.$$
が成り立つ．

証明の概略．簡単のため，$|z_1| < 1$ で
$$R(z) = z\frac{z - z_1}{1 - \overline{z_1}z}$$
のときをみる．H^2 の完全正規直交系 $\{e_k\}_{k=0}^{\infty}$ として
$$e_{2k} = R^k, \quad e_{2k+1} = Q_1 R_1 R^k = \frac{\sqrt{1 - |z_1|}}{1 - \overline{z_1}z} z R^k$$
が取れる．$k = 0, 1, \ldots$ に対して，$\xi_k(z) := z^k$ とおくと，
$$C_R \xi_k = R^k = e_{2k}, \quad T_{Q_1 R_1} C_R \xi_k = Q_1 R_1 R^k = e_{2k+1}.$$
よって，C_R と $T_{Q_1 R_1} C_R$ は等距離で，値域が直交し，
$$C_R C_R^* + (T_{Q_1 R_1} C_R)(T_{Q_1 R_1} C_R)^* = I$$
が成り立つ． □

4.3 主定理の系

系 7. R を次数 n が 2 以上の有限 Blaschke 積で $R(0) = 0$ とすると，$\mathcal{OC}_R \cong \mathcal{OC}_{z^n}$ となる．さらに $K_0(\mathcal{OC}_R) \cong \mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z}$, $K_1(\mathcal{OC}_R) \cong \mathbb{Z}$ が成り立つ．

証明の概略．主定理に加えて，次のことを使えば証明することができる．

- R_1 と R_2 を有理関数とする．$R_1|_{J_{R_1}}$ と $R_2|_{J_{R_2}}$ が位相共役ならば，$\mathcal{O}_{R_1}(J_{R_1}) \cong \mathcal{O}_{R_2}(J_{R_2})$ となる，
- $J_R = \mathbb{T}$, $J_{z^n} = \mathbb{T}$,
- (Shub [13]) $R|_{\mathbb{T}}$ と $z^n|_{\mathbb{T}}$ は位相共役である．

□

注意. 合成作用素の定義では，シンボルの $\mathbb{D}$ での情報が使われており，$\mathcal{OC}_\varphi$ は φ の $\mathbb{D}$ での情報を使って定義したと考えられる．．しかし主定理や系からわかるように，今考えている R の場合では，$\mathcal{OC}_R$ は $\mathbb{T}$ での情報のみによっていることがわかる．

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Morrey spaces for non-doubling measures

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Abstract

The aim of this report is to discuss some function spaces with the underlying measure μ non-doubling.

1 Metric measure space setting

Here we place ourselves in the setting of a metric measure space (X, d, μ). By doubling we mean that there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$, where $B(x, r)$ stands for the open ball centered at x of radius $r > 0$. So that, we have to work on the metric measure space (X, d, μ) such that μ may fail the doubling condition.

To consider how the doubling condition **was** important, we take up the following $5r$-coverling lemma.

Lemma 1.1. *Suppose that we are given a finite number of balls $B_1, B_2, \cdots, B_N$. Then after relabelling, we can find $M \leq N$ so that:*

1. *$B_1, B_2, \cdots, B_M$ are mutually disjoint;*
2. *any B_j is engulfed by $3B_{k(j)}$ with $k(j) \in \{1, 2, \cdots, M\}$. where $3B$ is the triple of B.*

Based on this covering lemma, we can prove the following theorem on the maximal operator.

$$Mf(x) = \sup_{B \in \mathcal{B}(x)} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),$$

where $\mathcal{B}(x)$ denotes the set of all open balls containing x. Suppose that (X, d, μ) is a metric measure space and that μ is a Radon measure. Assume in addition that μ is doubling: $\mu(B(x, 2r)) \leq D\, \mu(B(x, r))$ for all $x \in \mathbb{R}^n$ and $r > 0$. Then we have

$$\mu\{Mf > \lambda\} \leq \frac{D^2}{\lambda} \|f\|_1.$$

To illustrate how important the doubling condition was, let us prove this theorem.

Proof. We need to prove that $\mu\{Mf > \lambda\} \leq \dfrac{D^2}{\lambda}\|f\|_1$. By the Radon property, the matter is reduced to proving that $\mu(K) \leq \dfrac{D^2}{\lambda}\|f\|_1$ for all compact sets K included in $\{Mf > \lambda\}$. Let $x \in K \subset \{Mf > \lambda\}$. Then from the definition of the maximal operator we can find $y_x \in X$, $r_x > 0$ such that $x \in B(y_x, r_x)$, and that $\dfrac{1}{\mu(B(y_x, r_x))}\displaystyle\int_{B(y_x,r_x)} |f(y)|\, d\mu(y) > \lambda$. By the compactness of K, we can find a finite number of points $x_1, x_2, \cdots, x_N$ so that $K \subset \displaystyle\bigcup_{j=1}^{N} B(y_{x_j}, r_{x_j})$. To simplify the notatiion let us write $B_j = B(y_{x_j}, r_{x_j})$. By virtue of the lemma, if we change the label of B_j's, then we have $\displaystyle\sum_{j=1}^{L} \chi_{B_j} \leq 1$, $\chi_K \leq \displaystyle\sum_{j=1}^{L} \chi_{3B_j}$ for some $L \leq N$. Hence it follows that

$$\mu(K) \leq \sum_{j=1}^{L} \mu(3B_j). \tag{1}$$

By using the doubling condition twice, we have

$$\mu(3B_j) \leq \mu(4B_j) \leq D\mu(2B_j) \leq D^2\mu(B_j). \tag{2}$$

Also observe that

$$\frac{1}{\mu(B_j)}\int_{B_j} |f(y)|\, d\mu(y) > \lambda, \tag{3}$$

which is a consequence of $\dfrac{1}{\mu(B(y_x, r_x))}\displaystyle\int_{B(y_x,r_x)} |f(y)|\, d\mu(y) > \lambda$. (3) implies that

$$\mu(B_j) \leq \frac{1}{\lambda}\int_{B_j} |f(y)|\, d\mu(y) = \frac{1}{\lambda}\int_X \chi_{B_j}(y)|f(y)|\, d\mu(y). \tag{4}$$

Putting together (1), (2) and (4), we obtain $\mu(K) \leq \dfrac{D^2}{\lambda}\displaystyle\int_X \left(\sum_{j=1}^{N} \chi_{B_j}(y)\right) |f(y)|\, d\mu(y)$.

Now that we have $\displaystyle\sum_{j=1}^{L} \chi_{B_j} \leq 1$, it follows that $\mu(K) \leq \dfrac{D^2}{\lambda}\displaystyle\int_X |f(y)|\, d\mu(y)$. This is what we desired. □

In our theorem, it is absolutely necessary that μ is doubling. Otherwise, we can construct a counterexample. Indeed, let $(X, d) = (\mathbb{R}^2, d_{Euclid})$, $\mu(x_1, x_2) = \exp(|x_1| + |x_2|)\, dx_1\, dx_2$. Then (X, d, μ) is a counterexample showing that M does not satisfy the inequality of the theorem: More precisely, no matter how big the constant $C > 1$ is, the inequality

$$\mu\{Mf > \lambda\} \leq \frac{D^2}{\lambda}\|f\|_1$$

fails.

However, Volberg, Nazarov and Treil noticed that the doubling condition is superflous anyway [1, 2]. It just suffices to overcome the following disadvantage: We cannot use $\mu(3B_j) \le \mu(4B_j) \le D\mu(2B_j) \le D^2\mu(B_j)$. To bypass this inequality, we just need $\mu(3B_j) \le \frac{1}{\lambda}\int_{B_j} |f(y)|\,d\mu(y)$ in (4). So the definition of the maximal operator needs to be modified. We used $Mf(x) = \sup\limits_{B\in\mathcal{B}(x)} \frac{1}{\mu(1B)}\int_B |f(y)|\,d\mu(y)$, but we should have used $Mf(x) = \sup\limits_{B\in\mathcal{B}(x)} \frac{1}{\mu(3B)}\int_B |f(y)|\,d\mu(y)$. This was a breakthrough to the theory of non-doubling measures. I observed that the number 3 is sharp, while Terasawa noticed that the number 3 should be improved somehow in some other settings [3, 4]. To formulate our results, let us set

$$M_k f(x) = \sup_{B\in\mathcal{B}(x)} \frac{1}{\mu(kB)}\int_B |f(y)|\,d\mu(y)$$

and

$$\tilde{M}_k f(x) = \sup_{B\in\mathcal{B}(x),\, c(B)=x} \frac{1}{\mu(kB)}\int_B |f(y)|\,d\mu(y),$$

where $c(B)$ denotes the center of B.

Let us summarize what we can say about the modification rate in the inequality $\mu\{N_k f > \lambda\} \le \frac{C}{\lambda}\|f\|_1$ with $N = M, \tilde{M}$.

Theorem 1.2 (Nazarov, Treil and Volberg, [1]). *The number $k = 3$ suffices for M_k. Here C can be taken $C = 1$.*

Theorem 1.3 (S., [3]). *The number $k = 3$ is sharp for M_k in a certain metric measure space. Also, $C = 1$ is sharp.*

Theorem 1.4 (S. and Teresawa, [3, 4]). *The number $k = 2$ suffices for $\tilde{M}_k$. Here C can be taken $C = 1$.*

Theorem 1.5 (S., [3]). *The number $k = 2$ is sharp for $\tilde{M}_k$ in a certain metric measure space. Also, $C = 1$ is sharp.*

2 Euclidean setting

Let us turn to the theory of $\mathbb{R}^d$. The situation is quite different due to the covering lemma given below.

Theorem 2.1. *For all $k > 1$ there exists an integer $N = N_k$, depending only on the dimension and k, that satisfies the following:*

Let $\{B(x_\lambda, r_\lambda)\}_{\lambda\in L}$ be a family of balls in Euclidean space endowed with a standard distance. Suppose that $\sup\limits_{\lambda\in L} r_\lambda < \infty$.

Then we can take disjoint subfamilies

$$\{B(x_\rho, r_\rho)\}_{\rho\in L_1}, \{B(x_\rho, r_\rho)\}_{\rho\in L_2}, \ldots, \{B(x_\rho, r_\rho)\}_{\rho\in L_N}$$

such that $\bigcup_{\lambda\in L} B(x_\lambda, r_\lambda) \subset \bigcup_{j=1,\ldots,N} \bigcup_{\rho\in L_j} B(x_\rho, kr_\rho)$.

With this theorem, we can obtain a series of information on M_k.

Theorem 2.2 (S., [3]). *M_k is bounded, if $k > 1$.*

Theorem 2.3 (S., [3]). *If $1 < k < \infty$, $1 < p < \infty$, $1 < q\infty$, then we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} (M_k f_j)^q \right)^{1/q} \right\|_p \leq C_{p,q,k} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_p .$$

Theorem 2.4 (S., [3]). *We have the dual inequality*

$$\int_{\{M_b f > \lambda\}} |g(x)| d\mu(x) \leq \frac{C_{a,b}}{\lambda} \int_{\mathbb{R}^d} M_a g(x) |f(x)| d\mu(x)$$

if $b > a > 1$, where $C_{a,b}$ is a positive constant depending on a, b and d.

Proposition 2.5 (S., [5]). *The case when $a = b$ is not available in general in the above theorem.*

3 Morrey spaces

Now we turn to Morrey spaces. Morrey spaces are named after C. Morrey [6]. C. Morrey had been considering the elliptic differential equation and used the lemma together with a norm called the Morrey norm later. In the Lebesgue setting the Morrey norm is given by

$$\|f \,:\, \mathcal{M}^p_q\| = \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, dy \right)^{\frac{1}{q}}, \; 1 \leq q \leq p < \infty,$$

where $\mathcal{Q}$ denotes the set of all cubes with all sides parallel to the coordinate axis. Our aim here is to present a suitable definition of the Morrey norm adapted to the general measure μ. A natural candidate is

$$\|f \,:\, \mathcal{M}^p_q(1, \mu)\| = \sup_{Q\in\mathcal{Q}} \mu(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}.$$

However, the maximal operator, which is fundamental for analysis, is not bounded on this space. We noticed that the following definition is suitable.

$$\|f \,:\, \mathcal{M}^p_q(k, \mu)\| = \sup_{Q\in\mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}, \; k > 1,$$

where $\mathcal{Q}(\mu)$ is the subset of all $Q \in \mathcal{Q}$ such that $\mu(Q) > 0$. Actually, we have established the following proposition.

Proposition 3.1. *Let $k > 1$. Then*

$$\|f \,:\, \mathcal{M}^p_q(2k-1,\mu)\| \le \|f \,:\, \mathcal{M}^p_q(k,\mu)\| \le 2^d \|f \,:\, \mathcal{M}^p_q(2k-1,\mu)\|.$$

In particular, the norms

$$\|f \,:\, \mathcal{M}^p_q(k_1,\mu)\|, \ \|f \,:\, \mathcal{M}^p_q(k_2,\mu)\|$$

are mutually equivalent whenever $k_1, k_2 > 1$.

Proof. In view of the definition

$$\|f \,:\, \mathcal{M}^p_q(k,\mu)\| = \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}, \ k > 1.$$

and the fact that $1 \le q \le p < \infty$, the proof of

$$\|f \,:\, \mathcal{M}^p_q(2k-1,\mu)\| \le \|f \,:\, \mathcal{M}^p_q(k,\mu)\|$$

is simple. The crux is to prove the converse inequality. To this end, from the definition,

$$\|f \,:\, \mathcal{M}^p_q(k,\mu)\| = \sup_{Q \in \mathcal{Q}} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}, \ k > 1$$

we have only to prove

$$\mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \le 2^d \|f \,:\, \mathcal{M}^p_q(2k-1,\mu)\|. \tag{5}$$

To this end we bisect Q into $Q_1, Q_2, \cdots, Q_{2^d}$, which are of the same size. Then a geometric observation shows that

$$\mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_i} |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \le \|f \,:\, \mathcal{M}^p_q(2k-1,\mu)\|. \tag{6}$$

for each $i = 1, 2, \cdots, 2^d$. Thus, adding (5) over $i = 1, 2, \cdots, 2^n$, we obtain (6). $\square$

Proposition 3.2. *The norms*

$$\|\cdot \,:\, \mathcal{M}^p_q(2,\mu)\|, \ \|\cdot \,:\, \mathcal{M}^p_q(1,\mu)\|$$

are not mutually equivalent.

Theorem 3.3. *M_k is $\mathcal{M}(2,\mu)$-bounded if $k > 1$.*

4 Related function spaces

1. (Campanato spaces)

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2. (The predual)

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3. (Besov-Morrey spaces)

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A characterization of commutative convolution measure algebras

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Abstract We consider a theorem which gives necessary and sufficient conditions for a partially ordered commutative Banach algebra to be a convolution measure algebra.

A measure space $M(X,\Sigma)$ is a partially ordered Banach space of bounded complex measures on a measurable space (X,Σ) with the usual addition, scalar product, norm and an order defined by the cone of positive measures $M(X,\Sigma)_+$.

Definition 1 An L-subspace of $M(X,\Sigma)$ is a closed subspace $\mathfrak{M}$ which satisfies,
" $\omega \in M(X,\Sigma),\ \mu \in \mathfrak{M},\ \omega << \mu\ \Rightarrow \omega \in \mathfrak{M}$."

Although the original definition of complex L-spaces were given axiom-ally, we give below a definition following that of Taylor [4]. For other definitions and representation theorems of complex L-spaces, we refer to [2] and [3].

Definition 2 A partially ordered, complex Banach space $\mathfrak{M}$ is called a complex L-space if it is isomorphic and isometric, in an order-preserving fashion, to an L-subspace of some measure space $M(X,\Sigma)$.

The concept of convolution measure algebras was introduced in 1965 by J. L. Taylor [4] (see also [5]).

Definition 3 A convolution measure algebra is a partially ordered complex Banach algebra $(\mathfrak{M},\cdot,\|\ \|,\leq)$ such that
(i) $(\mathfrak{M},\|\ \|,\leq)$ is a complex L-space,
(ii) If we put $\mathfrak{M}_+ = \{\mu \in \mathfrak{M} : \mu \geq 0\}$, the following (a) and (b) hold.
(a) If $\mu,\nu \in \mathfrak{M}_+$, then $\mu\cdot\nu \geq 0$ and $\|\mu\cdot\nu\| = \|\mu\|\|\nu\|$,
(b) Let ω,μ,ν be in $\mathfrak{M}_+$ with $\omega \leq \mu\cdot\nu$. Then, for any $\varepsilon > 0$, there exist $\{\mu_j\}_{j=1}^n, \{\nu_k\}_{k=1}^m$

$\subseteq \mathfrak{M}_+$ and a set of numbers $\{a_{j,k}\}_{j,k=1}^{n,m} \subseteq [0,1]$ such that

$$\sum_{j=1}^{n} \mu_i \leq \mu, \ \sum_{k=1}^{m} \nu_k \leq \nu \text{ and } \left\|\sum_{j=1}^{n}\sum_{k=1}^{m} a_{j,k}\mu_j \cdot \nu_k - \omega\right\| < \varepsilon.$$

Group algebras and measure algebras on locally compact groups, measure algebras on locally compact topological semigroups as well as their L-subalgebras (i.e. subalgebras which are also L-subspaces) are examples of convolution measure algebras (cf. [5]).

In this note, we restrict ourselves to only commutative cases.

In [4], J. L. Taylor proved a remarkable representation theorem of commutative convolution measure algebras.

Definition 4 Suppose that S is a compact abelian topological semigroup. $M(S)$ denotes the usual measure algebra of bounded regular Borel measures on S with the convolution multiplication. A semicharacter on S is a complex-valued homomorphism on S. The set of semichacters on S will be denoted by $\hat{S}$.

Theorem 1 (Taylor's representation theorem [4, 5]) Suppose that $\mathfrak{M}$ is a commutative semisimple convolution measure algebra. Then there exist a compact abelian topological semigroup S and an order-preserving isometric isomorphism $\theta : \mu \to \mu_S : \mathfrak{M} \to M(S)$, and a bijection $\pi : \Delta \to \hat{S}$ such that

(a) The Gelfand trasform $\hat{\mu}$ of $\mu \in \mathfrak{M}$ is given by $\hat{\mu}(\varphi) = \int_S \pi(\varphi) d\mu_S \ \ (\mu \in \mathfrak{M}, \varphi \in \Delta)$,

(b) $\hat{S}$ separates points in S,

(c) θ has a weak*-dense range in $M(S)$.

S in Theorem 1 is called the structure semigroup of $\mathfrak{M}$, which is uniquely determined by $\mathfrak{M}$. Using Theorem 1, we are able to show that a commutative semisimple convolution measure algebra satisfies the following set of conditions (α).

Proposition 1 ([1]) *Let $(\mathfrak{M}, \leq, \Delta)$ be a commutative semisimple convolution measure algebra. Then $\mathfrak{M}$ satisfies the following set of conditions:*

(α) (i) *We can define a multiplication $\cdot$ and an involution $*$ in $\Delta_0 := \Delta \cup \{0\}$ such that $(\Delta_0, \cdot, *)$ becomes an abelian semigroup with involution, which has an identity element 1 with $1^* = 1$, and following statements are true*:

(a) *For $\mu \in \mathfrak{M}$, $\mu \in \mathfrak{M}_+ := \{\mu \in \mathfrak{M} : \mu \geq 0\}$ if and only if $\hat{\mu}$ is positive definite on Δ_0.*

(b) *For $\mu \in \mathfrak{M}$, there exist $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathfrak{M}_+$ such that*

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4 \ \textit{with} \ \|\mu_j\| \leq \|\mu\| \ (j = 1, 2, 3, 4).$$

(c) $\forall \psi \in \Delta$ *and* $\forall \mu \in \mathfrak{M}$, $\exists \mu^\psi \in \mathfrak{M}$ *such that* $\widehat{\mu^\psi}(\varphi) = \hat{\mu}(\varphi \cdot \psi) \ \ (\varphi \in \Delta)$.

(d) *For $\mu \in \mathfrak{M}_+, \psi \in \Delta$ and $c \in \{1, -1, i, -i\}$, we have $\mu + \frac{c}{2}\mu^\psi + \frac{\bar{c}}{2}\mu^{\psi^*} \in \mathfrak{M}_+$.*

(e) *For $\mu \in \mathfrak{M}_+$, we have $\|\mu\| = \hat{\mu}(1)$.*

(ii) (a) $\|p\|_{\mathfrak{M}^*} = \sup\{|\sum_j c_j \hat{\mu}(\varphi_j)| : \mu \in \mathfrak{M}_+, \|\mu\| \leq 1\} \ \ (p = \sum_j c_j \varphi_j \in \mathrm{span}(\Delta_0))$.

(b) $\|\mu\| = \sup\{| \sum_j c_j\hat{\mu}(\varphi_j) | : p = \sum_j c_j\varphi_j \in \mathrm{span}(\Delta), \|p\|_{\mathfrak{M}^*} \leq 1\} \quad (\mu \in \mathfrak{M})$.

Conversely, we can prove the following proposition.

Proposition 2([1]) *Let $(\mathfrak{M}, \Delta, \leq)$ be a partially ordered commutative semisimple Banach algebra. If $(\mathfrak{M}, \Delta, \leq)$ satisfies the set of conditions (α) in Proposition 1, then $(\mathfrak{M}, \Delta, \leq)$ is a convolution measure algebra.*

[outline of the proof] By the assumption (α), we can define multiplication $\cdot$ and an involution $*$ in $\Delta_0 := \Delta \cup \{0\}$ so that $(\Delta_0, \cdot, *)$ becomes an abelian semigroup with involution.

Let $D_{BE}(\Delta_0) := \{\sigma$: a complex function on Δ_0 s. t. $\sigma(0) = 0, \|\sigma\|_{BE} < \infty\}$, where $\|\sigma\|_{BE} := \sup\{| \sum_j c_j\sigma(\varphi_j) | : p = \sum_j c_j\varphi_j \in \mathrm{span}(\Delta_0), \|p\|_{\mathfrak{M}^*} \leq 1\}$. The set $D_{BE}(\Delta_0)$ is an algebra under the pointwise addition, scalar multiplication, and $\| \ \|_{BE}$-norm. If we introduce in $D_{BE}(\Delta_0)$ an order "$\sigma \geq 0$ if and only if σ is positive definite on Δ_0", then $(D_{BE}(\Delta_0), \| \ \|_{BE}, \leq)$ becomes a partially ordered commutative Banach algebra. Also, $D_{BE}(\Delta_0)$ is a locally convex space with respect to the simple convergence topology on Δ_0.

Write $D_{BE}(\Delta_0)^1_+ := \{\sigma \in D_{BE}(\Delta_0) : \sigma \geq 0, \sigma(1) = \|\sigma\|_{BE}\}$. Then $D_{BE}(\Delta_0)^1_+$ is a com pact convex set with respect to simple convergence topology on Δ_0. We write the set of extreme points of $D_{BE}(\Delta_0)^1_+$ by $\mathrm{ex}[D_{BE}(\Delta_0)^1_+]$. Then we can prove that $\mathrm{ex}[D_{BE}(\Delta_0)^1_+]$ coincides with $\widehat{\Delta_0}$, where $\widehat{\Delta_0} = \{\sigma \in D_{BE}(\Delta_0)^1_+ : \sigma(\varphi \cdot \psi) = \sigma(\varphi)\sigma(\psi), \sigma(\varphi^*) = \overline{\sigma(\varphi)}\}$. Thus $\mathrm{ex}[D_{BE}(\Delta_0)^1_+]$ becomes an abelian semigroup, which we write S. S is a compact abelian semigroup. For $\varphi \in \Delta$, we put $\pi(\varphi)(s) := s(\varphi) \quad (s \in S)$, then $\pi(\varphi) \in \hat{S}$, where $\hat{S}$ is the set of non-zero semichacters on S. Using the Krein-Milman theorem, we can define an order-preserving Banach algebra isomorphism $\theta_1 : D_{BE}(\Delta_0) \longrightarrow M(S)$. Since the Gelfand transform $\theta_2 : \mu \longrightarrow \hat{\mu} : \mathfrak{M} \longrightarrow D_{BE}(\Delta_0)$ is a Banach algebra isomorphism by (ii) (b) of (α), we get an order-preserving algebra isomorphism $\theta_1 \circ \theta_2 : \mu \longrightarrow \theta(\hat{\mu}) : \mathfrak{M} \longrightarrow M(S)$. Therefore we can get that $(\mathfrak{M}, \leq)$ is a convolution measure algebra. In this case, it turns out that S is the structure semigroup of $\mathfrak{M}$.

Combining Proposition 1 and 2, we get the following theorem which characterize a partially ordered commutative semisimple Banach algebra to be a convolution measure algebra.

Theorem 2 ([1]) *Let $(\mathfrak{M}, \Delta, \leq)$ be a partially ordered commutative semisimple Banach algebra. Then $(\mathfrak{M}, \Delta, \leq)$ is a convolution measure algebra if and only if it satisfies the set of conditions (α) in Proposition 1.*

The detailed proof of Theorem 2 will appear elsewhere in the near future.

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An inequality of measures of analytic type on certain compact groups

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Abstract

Let $\mathbb{T}$ and $\mathbb{Z}$ be the circle group and the integer group, respectively. Let $M(\mathbb{T})$ be the measure algebra on $\mathbb{T}$. For $\mu \in M(\mathbb{T})$ and $E \subset \mathbb{Z}$, put $\|\mu\|_E = \inf\{\|\nu\| : \hat{\nu} = \hat{\mu} \, \mathrm{on} \, E\}$, where $\hat{\mu}$ denotes the Fourier transform of μ. Pigno and Smith proved the following theorem:

Theorem ([8, Theorem V]). There is a constant $C > 0$ such that for any sequence $< D_n >_1^\infty$ of symmetric dyadic intervals in $\mathbb{Z}$ and any analytic measure μ on $\mathbb{T}$,

$$\sum_{k=1}^{\infty} \frac{\|\mu\|_{D_k}}{k} \leqq C\|\mu\|.$$

We state that an inequality similar to the one in the above theorem holds for certain (noncommutative) compact groups.

§ 1 序

$\mathbb{T}$ と $\mathbb{Z}$ はそれぞれ circle group, integer group とし、$\mathbb{Z}^+$ を非負な整数全体とする. $M(\mathbb{T})$ は通常の $\mathbb{T}$ 上の有界正則測度の Banach 空間とする. $\mu \in M(\mathbb{T})$ に対し、$\hat{\mu}$ を μ の Fourier 変換とする (i.e., $\hat{\mu}(n) = \int_{\mathbb{T}} e^{-in\theta} d\mu(\theta)$ $(n \in \mathbb{Z})$). $M_{\mathbb{Z}^+}(\mathbb{T}) = \{\mu \in M(\mathbb{T}) : \hat{\mu}(n) = 0 \ \ \forall n < 0\}$ とおく. また、$\mu \in M(\mathbb{T})$, $E \subset \mathbb{Z}$ に対し、

$$\|\mu\|_E = \sup\{|\sum_{-m}^{m} c_n \hat{\mu}(n)| : \|\sum_{-m}^{m} c_n e^{-in\theta}\|_\infty \leqq 1, \ \ n \in E\}$$

とおく.

$n_{k+1}/n_k \geqq 2$ $(\forall k \in \mathbb{N})$ なる $\mathbb{Z}^+$ の数列 $\{n_k\}$ が存在し、$D_k = \{m \in \mathbb{Z} : -n_{2k+1} < m \leqq -n_{2k}\} \bigcup \{m \in \mathbb{Z} : n_{2k} \leqq m < n_{2k+1}\}$ $(k = 1, 2, 3, \cdots)$ のとき、$< D_n >_1^\infty$ を $\mathbb{Z}$ における symmetric dyadic intervals の列という. Pigno と Smith は [8] で次の定理を得た.

定理 ([8, Theorem V]).　定数 $C > 0$ が存在し、任意の $\mathbb{Z}$ における symmetric dyadic intervals の列 $< D_n >_1^\infty$ と $\mu \in M_{\mathbb{Z}^+}(\mathbb{T})$ に対し、

$$\sum_{k=1}^{\infty} \frac{\|\mu\|_{D_k}}{k} \leqq C\|\mu\|.$$

ここでは、ある種の (非可換) コンパクト群においても、上の定理と類似の結果が成り立つことを述べる.

§ 2　準備

K をコンパクト群とし、Σ_K を K の dual object とする.　m_K は K の Haar measure とする. $Z(K)$ を K の中心とし、$G \subset Z(K)$ を K の閉部分群とする. また、$\hat{G}$ を G の dual group とする.
$\psi : \hat{G} \to \mathbb{R}$ を自明でない準同型写像とし、$\phi : \mathbb{R} \to G$ を ψ の dual homomorphism とする (i.e., $(\phi(t), \gamma) = exp(i\psi(\gamma)t)$ for $t \in \mathbb{R}$ and $\gamma \in \hat{G}$).

$\sigma \in \Sigma_K$ に対し、$U^{(\sigma)}$ を σ に属し、次元 d_σ の H_σ を表現空間として持つ、K の continuous irreducible unitary representation とする.
すると、Schur's lemma ([6, (27.10) Corollary]) より、$\exists \gamma : \Sigma_K \to \hat{G}$: map s.t.

$$U_x^{(\sigma)} = (x, \gamma(\sigma))I \quad (x \in G,\ \sigma \in \Sigma_K). \tag{2.1}$$

但し、$U^{(\sigma)} \in \sigma$, I は identity operator on H_σ.

$\mathfrak{T}(K)$ を K 上の trigonometric polynomials の空間とする. すなわち、$\mathfrak{T}(K)$ は関数　$x \to < U_x^{(\sigma)}\xi, \eta >$ $(\sigma \in \Sigma_K; \xi, \eta \in H_\sigma)$ の一次結合全体の空間.

$C(K)$: K 上の連続関数全体の空間.
$M(K)$: K 上の有界正則測度の Banach 空間 $(M(K) \cong C(K)^*)$.

$\mu \in M(K)$ に対し、$\hat{\mu}$ を μ の Fourier 変換とする.
i.e., $\sigma \in \Sigma_K, \xi, \eta \in H_\sigma$ に対し、

$$< \hat{\mu}(\sigma)\xi, \eta >= \int_K < \overline{U}_x^{(\sigma)}\xi, \eta > d\mu(x). \tag{2.2}$$

但し、$\overline{U}_x^{(\sigma)} = D_\sigma U_x^{(\sigma)} D_\sigma$ で、D_σ は H_σ 上の conjugation.

注意　$\{\xi_1^{(\sigma)}, \cdots, \xi_{d_\sigma}^{(\sigma)}\}$ を固定された H_σ の正規直交基底としたとき、D_σ は、$D_\sigma(\sum_{j=1}^{d_\sigma} \alpha_j \xi_j^{(\sigma)}) = \sum_{j=1}^{d_\sigma} \overline{\alpha_j} \xi_j^{(\sigma)}$ $(\alpha_j \in \mathbb{C})$ により定義されたもの.

$spec(\mu) = \{\sigma \in \Sigma_K : \hat{\mu}(\sigma) \neq 0\}$ とおく. $E \subset \Sigma_K$ に対し、

$$M_E(K) = \{\mu \in M(K) : spec(\mu) \subset E\}$$

とおく. また、$C_E(K)$, $\mathfrak{T}_E(K)$ も同様に定義する.

$\mu \in M(K)$, $E \subset \hat{G}$ に対し、

$$\|\mu\|_E := \sup\{|\int_K f(x)d\mu(x)| : f \in \mathfrak{T}_E(K), \|f\|_\infty \leqq 1\}$$

とおく.

§ 3　結果

○ $< D_n >_1^\infty$: $\mathbb{R}$ における symmetric dyadic intervals の列
$\underset{def.}{\Longleftrightarrow}$ $\exists\{\omega_k\}_1^\infty \subset (0, \infty)$: sequence s.t. $\omega_{k+1} \geqq 2\omega_k$ and
$D_k = (-\omega_{2k+1}, -\omega_{2k}] \cup [\omega_{2k}, \omega_{2k+1})$ $(\forall k \geqq 1)$.

○ $< D_n^{\psi\circ\gamma} >_1^\infty$: sequence of symmetric $\psi \circ \gamma$-dyadic intervals in Σ_K
$\underset{def.}{\Longleftrightarrow}$ $\exists < D_n >_1^\infty$: sequence of symmetric dyadic intervals in $\mathbb{R}$
s.t. $D_n^{\psi\circ\gamma} = \gamma^{-1}(\psi^{-1}(D_n))$ for $\forall\ n \geqq 1$.

○ $\Sigma_{K,+} := \{\sigma \in \Sigma_K : \psi(\gamma(\sigma)) \geqq 0\}$ とし、

$$M_{\Sigma_{K,+}}(K) = \{\mu \in M(K) : \hat{\mu}(\sigma) = 0 \text{ for}\forall\sigma \notin \Sigma_{K,+}\}$$

とおく.
次の定理は、[8, Theorem V] と同様に証明できる.

定理 3.1. 定数 $C>0$ が存在し、任意の Σ_K における symmetric $\psi\circ\gamma$-dyadic intervals の列 $< D_n^{\psi\circ\gamma} >_1^\infty$ と $\mu \in M_{\Sigma_{K,+}}(K)$ に対し、

$$\sum_{k=1}^{\infty} \frac{\|\mu\|_{D_k^{\psi\circ\gamma}}}{k} \leqq C\|\mu\|.$$

定理 3.1 より、次の系が得られる.

系 3.1. $\mu \in M_{\Sigma_{K,+}}(K) \Longrightarrow \lim_{t\to 0} \|\mu - \delta_{\phi(t)} * \mu\| = 0.$

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BOHR INEQUALITY FOR OPERATORS

MASATOSHI FUJII

1. Bohr inequalities for operators

Let H be a complex separable Hilbert space and $B(H)$ the algebra of all bounded operators on H. The absolute value of $A \in B(H)$ is denoted by $|A| = (A^*A)^{1/2}$.

The classical Bohr inequality [2] says that

$$|a+b|^2 \leq p|a|^2 + q|b|^2$$

for all scalars a, b and p, $q > 0$ with $1/p + 1/q = 1$. The equality holds if and only if $(p-1)a = b$.

For this, Hirzallah proposed an operator version of Bohr inequality:

Theorem 1.1. *If A and B are operators on a Hilbert space, and $q \geq p > 0$ satisfy $1/p + 1/q = 1$, then*

$$|A-B|^2 + |(p-1)A+B|^2 \leq p|A|^2 + q|B|^2.$$

Afterwards, several authors have presented generalizations of Bohr inequality.

Theorem 1.2. *If $A, B \in B(H)$, $\frac{1}{p} + \frac{1}{q} = 1$, and $1 < p \leq 2$, i.e., $q \geq p > 1$, then*

(i) $|A-B|^2 + |(p-1)A+B|^2 \leq p|A|^2 + q|B|^2$,

(ii) $|A-B|^2 + |A+(q-1)B|^2 \geq p|A|^2 + q|B|^2$.

On the other hand, if $p < 1$, then

(iii) $|A-B|^2 + |(p-1)A+B|^2 \geq p|A|^2 + q|B|^2$.

Theorem 1.3. *If $A, B \in B(H)$ and $|\alpha| \geq |\beta|$, then*

$$|A-B|^2 + \frac{1}{|\alpha|^2}||\beta|A + |\alpha|B|^2 \leq (1 + \frac{|\beta|}{|\alpha|})|A|^2 + (1 + \frac{|\alpha|}{|\beta|})|B|^2.$$

We note that it unifies the following inequalities:

(i) If $\alpha \geq |\beta| = -\beta$, then

$$|A-B|^2 + |\frac{|\beta|}{\alpha}A + B|^2 \leq (1 + \frac{|\beta|}{\alpha})|A|^2 + (1 + \frac{\alpha}{|\beta|})|B|^2.$$

(ii) If $0 < \alpha \leq -\beta$, then

$$|A-B|^2 + |\frac{\alpha}{|\beta|}A + B|^2 \leq (1 + \frac{\alpha}{|\beta|})|A|^2 + (1 + \frac{|\beta|}{\alpha})|B|^2.$$

Next we state Bohr inequalities for multi-operators.

Theorem 1.4. *Suppose that $A_i \in B(H)$, and $r_i \geq 1$ for $i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$. Then*

$$|\sum_{i=1}^{n} A_i|^2 \leq \sum_{i=1}^{n} r_i |A_i|^2.$$

In other words, it says that $K(z) = |z|^2$ satisfies the (operator) Jensen inequality:

$$K(\sum_{i=1}^{n} t_i A_i) \leq \sum_{i=1}^{n} t_i K(A_i)$$

holds for $t_1, \cdots, t_n > 0$ with $\sum_{i=1}^{n} t_i = 1$.

2. Matrix approach to Bohr inequalities

In this section, we present an approach to Bohr inequalities by the use of the matrix order.

For this, we introduce two notations: For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define an $n \times n$ matrix $\Lambda(x) = x^* x = (x_i x_j)$ and $D(x) = \mathrm{diag}(x_1, \cdots, x_n)$.

Theorem 2.1. *If $\Lambda(a) + \Lambda(b) \leq D(c)$ for $a, b, c \in \mathbb{R}^n$, then*

$$|\sum_{i=1}^{n} a_i A_i|^2 + |\sum_{i=1}^{n} b_i A_i|^2 \leq \sum_{i=1}^{n} c_i |A_i|^2$$

for arbitrary n-tuple (A_i) in $B(H)$. Incidentally, the statement is correct even if the order is replaced to the reverse.

Proof. We define a positive map Φ of $B(H)^n$ to $B(H)$ by

$$\Phi(X) = (A_1^* \cdots A_n^*) X \, {}^t(A_1 \cdots A_n).$$

Since $\Lambda(a) = {}^t(a_1, \cdots, a_n)(a_1, \cdots, a_n)$, we have

$$\Phi(\Lambda(a)) = (\sum_{i=1}^{n} a_i A_i)^* (\sum_{i=1}^{n} a_i A_i) = |\sum_{i=1}^{n} a_i A_i|^2,$$

so that

$$|\sum_{i=1}^{n} a_i A_i|^2 + |\sum_{i=1}^{n} b_i A_i|^2 = \Phi(\Lambda(a) + \Lambda(b)) \leq \Phi(D(c)) = \sum_{i=1}^{n} c_i |A_i|^2.$$

The additional part is easily shown by the same way. □

The meaning of Theorem 2.1 will be explained in the following theorem well.

Theorem 2.2. *(i) If $0 < t \leq 1$, then*
$|A \mp B|^2 + |tA \pm B|^2 \leq (1+t)|A|^2 + (1 + \frac{1}{t})|B|^2;$
(ii) If $t \geq 1$ or $t < 0$, then
$|A \mp B|^2 + |tA \pm B|^2 \geq (1+t)|A|^2 + (1 + \frac{1}{t})|B|^2.$

Proof. We consider the order between corresponding matrices:

$$T = \begin{pmatrix} 1+t & 0 \\ 0 & 1+\frac{1}{t} \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix} = (1-t)\begin{pmatrix} t & 1 \\ 1 & \frac{1}{t} \end{pmatrix}.$$

Since $\det(T) = 0$, it is positive (resp. negative) if $0 < t < 1$ (resp. $t > 1$ or $t < 0$). □

Remark 2.3. *Note that Theorem 2.2 implies all of theorems cited in Section 1 easily. For instance, for (i) and (iii) of Theorem 1.2, we take $t = p - 1$. For (ii), we take $t = q - 1$ and permute A and B. Also Theorem 1.3 follows from taking $t = \frac{|\beta|}{|\alpha|}$.*

As another application of Theorem 2.1, we give a proof of Theorem 1.4.

Proof of Theorem 1.4. We check the order between the corresponding matrices $D = \operatorname{diag}(r_1, \cdots, r_n)$ and $C = (c_{ij})$ where $c_{ij} = 1$. Since

$$\det(D - C) = (r_1 \cdots r_n)(1 - r) \text{ (for arbitrary} n),$$

it follows that $C \leq D$. Hence we have the conclusion by Theorem 2.1.

Concluding this section, we remark the monotonicity of the operator function $F(a) = |\sum_{i=1}^{n} a_i A_i|^2$. It is proved by $F(a) = \Phi(a^* a)$, where Φ is as in the proof of Theorem 2.1.

Corollary 2.4. *For a fixed n-tuple (A_i) in $B(H)$, the operator function $F(a) = |\sum_{i=1}^{n} a_i A_i|^2$ for $a = (a_1, \cdots, a_n)$ is order preserving, that is, if $\Lambda(a) \leq \Lambda(b)$, then $F(a) \leq F(b)$.*

Corollary 2.5. *If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ satisfy $|a_i| \leq |b_i|$ for $i = 1, 2, 3$ and $a_i b_j = a_j b_i$ for $i \neq j$, then $F(a) \leq F(b)$.*

Proof. It suffices to check that $\Lambda(a) \leq \Lambda(b)$ by the preceding corollary. It follows from the fact that if $i \neq l$ and $j \neq k$, then

$$\begin{vmatrix} a_i a_j - b_i b_j & a_i a_k - b_i b_k \\ a_l a_j - b_l b_j & a_l a_k - b_l b_k \end{vmatrix} = a_k b_j (a_i b_l - b_i a_l) + a_j b_k (b_i a_l - a_i b_l) = 0.$$

□

3. Generalized parallelogram law for operators

Next we give another approach to Bohr inequality In our frame, the following generalization of the parallelogram law easily implies Theorem 2.2 which covers many previous results as discussed in the preceding section.

Theorem 3.1. *If A and B are operators on a Hilbert space and $t \neq 0$, then*

$$|A + B|^2 + \frac{1}{t}|tA - B|^2 = (1 + t)|A|^2 + (1 + \frac{1}{t})|B|^2.$$

Remark 3.2. *We immediately obtain Theorem 2.2 by noting the condition of t in Theorem 3.1. This means that Theorem 3.1 also implies Theorems 1.1 and 1.2.*

Next we extend Theorem 3.1 for multi-operators.

Theorem 3.3. *Suppose that $A_i \in B(H)$ and $r_i \geq 1$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$ for $i = 1, 2, ..., n$. Then*

$$\sum_{i=1}^{n} r_i |A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1 \leq i < j \leq n} \left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} A_j \right|^2.$$

Remark 3.4. *Theorem 1.4 is an easy consequence of Theorem 3.3.*

Incidentally we note that the condition $r_i \geq 1$ in Theorem 3.3 is not necessary.

Corollary 3.5. *Let $A_i \in \mathbb{B}(\mathbf{H})$ and $r_i \neq 0$ for $i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$. Then*

$$\sum_{i=1}^{n} r_i |A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1 \leq i \leq j \leq n} \frac{r_j}{r_i} \left| \frac{r_i}{r_j} A_i - A_j \right|^2.$$

4. Dunkl-Williams inequality

Dunkl and Williams showed that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}$$

for every nonzero element x, y in a normed linear space.

Pečarić and Rajić gave the following refinement: For every nonzero element x, y in a normed linear space X

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\sqrt{2\|x - y\|^2 + 2(\|x\| - \|y\|)^2}}{\max\{\|x\|, \|y\|\}}.$$

Furthermore they generalized it to an operator inequality as follow:

Theorem 4.1. *Let $A, B \in B(H)$ be operators where $|A|$ and $|B|$ are invertible, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

$$|A|A|^{-1} - B|B|^{-1}|^2 \;\leq\; |A|^{-1}(p|A - B|^2 + q(|A| - |B|)^2)|A|^{-1}.$$

The equality holds if and only if

$$p(A - B)|A|^{-1} = qB(|A|^{-1} - |B|^{-1}).$$

Very recently, Saito-Tominaga improved Theorem 4.1 without the assumption of the invertibility of the absolute value of operators.

Theorem 4.2. *Let $A, B \in B(H)$ be operators with polar decompositions $A = U|A|$ and $B = V|B|$, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|(U - V)|A||^2 \;\leq\; p|A - B|^2 + q(|A| - |B|)^2.$$

The equality holds if and only if

$$p(A - B) = qV(|B| - |A|) \quad \text{and} \quad V^*V \geq U^*U.$$

In this section, we consider Dunkl-Williams inequality for operators as an application of generalized parallelogram law of operators in Theorem 3.1:

$$|A-B|^2+\frac{1}{t}|tA+B|^2=(1+t)|A|^2+(1+\frac{1}{t})|B|^2.$$

for any nonzero $t \in \mathbb{R}$.

The following lemma follows from it easily.

Lemma 4.3. *Let $A, B \in B(H)$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$, Then for each $t>0$*

$$|A-B|^2 \leq (1+t)|A|^2+(1+\frac{1}{t})|B|^2.$$

The equality holds for t if and only if $tA+B=0$.

Lemma 4.4. *Let $A, B \in B(H)$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$ and $t>0$. If $t(A-B)+V(|A|-|B|)=0$ is satisfied, then*

$$t|A-B|^2 \leq |A|^2-|B|^2,$$

and so $|A| \geq |B|$ and $U^*U \geq V^*V$.

In addition, if $U^*U=V^*V$, then $t|A-B|^2=|A|^2-|B|^2$.

The following theorem is proved by the lemmas cited above, and it changes to Theorem 4.2 by taking $t=p-1$ and replacing A (resp. B) by $A-B$ (resp. $V(|A|-|B|)$).

Theorem 4.5. *Let $A, B \in B(H)$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$, and $t>0$. Then*

$$|(U-V)|A||^2 \ \leq \ (t+1)|A-B|^2+(1+\frac{1}{t})(|A|-|B|)^2.$$

The equality holds if and only if

$$t(A-B)=V(|B|-|A|) \quad \text{and} \quad V^*V=U^*U.$$

Lemma 4.6. *Let $A, B \in B(H)$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$, and $t>0$. Suppose that $V^*V=U^*U$. Then*

$$t(A-B)=V(|B|-|A|)$$

if and only if

$$|A|=|B|+t|A-B| \text{ and } A-B=-V|A-B|.$$

Lemma 4.7. *Let $A, B \in B(H)$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$, and $t>0$. Suppose that $V^*V=U^*U$. If $t(A-B)=V(|B|-|A|)$, then*

$$|B||A-B|+|A-B||B|=(1-t)|A-B|^2.$$

Theorem 4.8. *Let $A, B \in B(H)$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$, and $C=A-B=W|C|$ the polar decomposition of C. Assume that the equality*

$$|(U-V)|A||^2=(t+1)|A-B|^2+(1+\frac{1}{t})(|A|-|B|)^2.$$

holds for some $t>0$.

(1) If $t \geq 1$, then $A=B$.

(2) If $0 \leq t \leq 1$, then

$$A = B(I - \frac{2}{1-t}W^*W) \text{ and } |A| = |B|(I + \frac{2t}{1-t}W^*W),$$

and the converse is true.

We next state another application of Lemma 4.3:

Theorem 4.9. *Let $A, B \in B(H)$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$ and let $t > 0$ and $0 < \alpha \leq 1$ be arbitrary. Then*

$$|(U|A|^\alpha - V|B|^\alpha)|A|^{1-\alpha}|^2 \leq (1+t)|A-B|^2 + (1+\frac{1}{t})||B|^\alpha|A|^{1-\alpha} - |B||^2.$$

The equality holds if and only if $t(A-B) + V(|B|^\alpha|A|^{1-\alpha} - |B|) = 0$.

Thus we have an estimation of the operator p-angular distance.

Theorem 4.10. *Let $A, B \in B(H)$ such that $|A|$ and $|B|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $p \in \mathbb{R}$. Then*

$$|A|A|^{p-1} - B|B|^{p-1}|^2 \leq |A|^{p-1}(r|A-B|^2 + s||B|^p|A|^{1-p} - |B||^2|A|^{p-1}.$$

Moreover the equality holds if and only if

$$(r-1)(A-B)|A|^{p-1} = B(|A|^{p-1} - |B|^{p-1}).$$

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On further extension of grand Furuta inequality
- Mean theoretic approach -

Masatoshi Ito and Eizaburo Kamei

1. Introduction

Throughout this paper, A and B are positive operators on a complex Hilbert space, and we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. strictly positive) operator.

As an extension of Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$*," Furuta established the following inequality in [7]([2,8,11.17]).

Furuta inequality: If $A \geq B \geq 0$, then for each $r \geq 0$,

$$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

For $A > 0$ and $B \geq 0$, α-power mean $\sharp_{\alpha}$ for $\alpha \in [0,1]$ is defined by

$$A \sharp_{\alpha} B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\alpha} A^{\frac{1}{2}}.$$

Furuta inequality can be arranged in terms of α-power mean as follows [17]:
If $A \geq B \geq 0$ with $A > 0$, then

$$A \geq B \geq A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$

2. Extension of Furuta inequality

Furuta had shown an extension of Furuta inequality [9], which is called grand Furuta inequality [4,6,16]. We remark that grand Furuta inequality is also an extension of Ando-Hiai inequality [1] which is equivalent to the main result of log majorization.

Grand Furuta inequality: If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

By putting $\beta = (p-t)s + t$ and $\gamma = r - t$, we can arrange the grand Furuta inequality in terms of α-power mean as follows: If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$ with $p \neq t$,

$$A \geq B \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{for } \beta \geq p \text{ and } \gamma \geq 0,$$

where $A \natural_s B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}}$ for a real number s. (If $s \in [0,1]$, then $\natural_s = \sharp_s$.)

3. Further extension of Furuta inequality

Very recently, Furuta has dug for a further extension of grand Furuta inequality [13,14], which is the following Theorems. We call this "FGF inequalities" here.

Theorem F-1. Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \dots, p_{2n} \geq 1$ for natural number n. Then
$A^{1-t+r} \geq$
$[A^{\frac{r}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}} \cdots (A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4} \cdots A^{\frac{t}{2}}\}^{p_{2n-1}}A^{\frac{-t}{2}})^{p_{2n}}A^{\frac{r}{2}}]^{\frac{1-t+r}{q[2n]-t+r}}$ holds for $r \geq t$, where

$$q[2n] = (\{\cdots(\{(p_1 - t)p_2 + t\}p_3 - t)p_4 + \cdots + t\}p_{2n-1} - t)p_{2n} + t.$$

Theorem F-2. Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \dots, p_{2n-1}, p_{2n} \geq 1$ for natural number n. Then
$A \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{\frac{1}{q[2]}} \geq \dots$
$\cdots \geq [A^{\frac{t}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}} \cdots (A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}})^{p_4} \cdots A^{\frac{t}{2}}\}^{p_{2n-1}}A^{\frac{-t}{2}})^{p_{2n}}A^{\frac{t}{2}}]^{\frac{1}{q[2n]}}$,
where $q[2n] = (\{\cdots(\{(p_1 - t)p_2 + t\}p_3 - t)p_4 + \cdots + t\}p_{2n-1} - t)p_{2n} + t$.

We scrutinize FGF inequalities by mean theoretic methods. We attempt to trace a proof of grand Furuta inequality from the viewpoint of a sequence $\{B_i\}$ such that $B_i = (A^t \natural_{\frac{\beta_i - t}{p-t}} B^p)^{\frac{1}{\beta_i}}$.

Lemma. Let $A \geq B \geq 0$ with $A > 0$. Then

$$A \geq B \geq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$$

holds for $t \in [0,1]$, $\beta \geq p \geq 1$ and $p \neq t$.

For convenience, we prove this lemma given in [5].

Proof. We may also assume that B is invertible. Let $[p, \beta]$ divide into $p = \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n = \beta$ such that $1 \leq \frac{\beta_i - t}{\beta_{i-1} - t} \leq 2$. Let $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{p-t}} B^p)^{\frac{1}{\beta_i}}$ for $i = 1, 2, \dots, n$. Then we show

$$B_{i-1}^{\beta_i} \geq B_i^{\beta_i} \text{ for } i = 1, 2, \dots, n \quad \text{and} \quad A \geq B \geq B_1 \geq \cdots \geq B_n. \tag{3.1}$$

First, we can show $B^{\beta_1} \geq B_1^{\beta_1}$ as follows:

$$\begin{aligned} B_1^{\beta_1} &= A^t \natural_{\frac{\beta_1 - t}{p-t}} B^p = B^p \natural_{\frac{p-\beta_1}{p-t}} A^t = B^p(B^{-p} \sharp_{\frac{\beta_1 - p}{p-t}} A^{-t})B^p \\ &\leq B^p(B^{-p} \sharp_{\frac{\beta_1 - p}{p-t}} B^{-t})B^p = B^{\beta_1} \end{aligned}$$

since $1 \leq \frac{\beta_1 - t}{p-t} \leq 2$ and $t \in [0,1]$.

For some natural number k such that $k \leq n-1$, assume that $B_{i-1}^{\beta_i} \geq B_i^{\beta_i}$ for $i = 1, 2, \dots, k$. We note that $A \geq B \geq B_1 \geq \cdots \geq B_k$ is easily obtained by Löwner-Heinz theorem. Then we obtain

$$
\begin{aligned}
B_{k+1}^{\beta_{k+1}} &= A^t \natural_{\frac{\beta_{k+1}-t}{p-t}} B^p = A^t \natural_{\frac{\beta_{k+1}-t}{\beta_k-t}} (A^t \natural_{\frac{\beta_k-t}{p-t}} B^p) \\
&= A^t \natural_{\frac{\beta_{k+1}-t}{\beta_k-t}} B_k^{\beta_k} = B_k^{\beta_k} \natural_{\frac{\beta_k-\beta_{k+1}}{\beta_k-t}} A^t = B_k^{\beta_k}(B_k^{-\beta_k} \sharp_{\frac{\beta_{k+1}-\beta_k}{\beta_k-t}} A^{-t})B_k^{\beta_k} \\
&\leq B_k^{\beta_k}(B_k^{-\beta_k} \sharp_{\frac{\beta_{k+1}-\beta_k}{\beta_k-t}} B_k^{-t})B_k^{\beta_k} = B_k^{\beta_{k+1}}
\end{aligned}
$$

since $1 \leq \frac{\beta_{k+1}-t}{\beta_k-t} \leq 2$ and $t \in [0,1]$, so that the proof is complete. □

Theorem 1. Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. By applying Lemma ?? to that $A \geq B \geq 0$ with $A > 0$, we have

$$A \geq B \geq (A^t \natural_{\frac{\beta_1-t}{\alpha_1-t}} B^{\alpha_1})^{\frac{1}{\beta_1}} = B_1$$

for $t \in [0,1]$, $\beta_1 \geq \alpha_1 \geq 1$ and $\alpha_1 \neq t$, and also by applying Lemma ?? repeatedly to that $A \geq B_{i-1} \geq 0$ with $A > 0$ for $i = 1, 2, \ldots, n$, we have

$$B_{i-1} \geq (A^{t_i} \natural_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}} = B_i$$

for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$, so that

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n.$$

Hence the proof is complete. □

Theorem 2. Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$, $\gamma \geq 0$ and $\alpha_1 \neq t$,

$$
\begin{aligned}
A \geq B &\geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_1+\gamma}} B^{\alpha_1} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B_1^{\beta_1} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_2+\gamma}} B_1^{\alpha_2} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_2^{\beta_2} \\
&\geq \cdots \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{n-1}+\gamma}} B_{n-1}^{\beta_{n-1}} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_n+\gamma}} B_{n-1}^{\alpha_n} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_n^{\beta_n},
\end{aligned}
$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. Let $\beta_0 = 1$. By Proposition ??, $A \geq B_{i-1}$ holds for $i = 1, 2, \ldots, n$, so that we have

$$
\begin{aligned}
& A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{i-1}+\gamma}} B_{i-1}^{\beta_{i-1}} \\
& \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_i+\gamma}} B_{i-1}^{\alpha_i} && \text{by Theorem ??} \\
& \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} (A^t \natural_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i}) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_i^{\beta_i} && \text{by Theorem ??}
\end{aligned}
$$

since $\beta_i \geq \alpha_i \geq \beta_{i-1} \geq 1$. Hence the proof is complete. □

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Krein space numerical ranges and Morse theory

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Abstract We provide a Krein space analogue of Westwick's convexity theorem on the C-numerical range of a matrix T for C Hermitian. The convexity of the range is preoved by using More theory. The result is based on a joint work with N. Bebiano and J. da Providência. .

1. Toeplitz=Hausdorff の定理とその系譜

複素 Hilbert 空間 H からそれ自身への有界線形作用素 T の数域 $W(T)$ は、

$$W(T) = \{\langle T\xi, \xi\rangle : \xi \in H, ||\xi|| = 1\}, \tag{1.1}$$

で定義される. 1918 年に Toeplitz [6] は $W_C^J(T) = W_C(T) = W(T)$ の境界 $\partial W(T)$ が凸曲線となることを証明した. 翌 1919 年に, Hausdorff [4] は、$W(T)$ 自体が凸であると証明した.

数域の凸性の証明は、H が有限次元の場合に帰着されるので、$H = \mathbf{C}^n$ と仮定する. 各 $0 \leq \theta \leq 2\pi$ に対して

$$T_\theta = \cos\theta(T + T^*)/2 + \sin\theta(T - T^*)/(2i), \tag{1.2}$$

とするとき、各実数 a に対して複素数平面における集合

$$W(T) \cap \{z \in \mathbf{C} : \Re(z\exp(-i\theta)) = a\}, \tag{1.3}$$

を考える. 図 1 では、おむすび形の図形が、$W(T)$ を表し (曲線はその境界)、それと複素数 $\exp(i[\theta + \pi/2])$ を方向ベクトルとする直線との共通部分が表されている.「 この共通部分が、集合

$$\{\xi \in \mathbf{C}^n : \xi^*\xi = 1, \xi^* T_\theta \xi = a\}, \tag{1.4}$$

の連続写像による像であって、集合 (1.4) が連結であるから、集合 (1.3) が連結である．このことより $W(T)$ は、凸集合である」という論法を、Hausdorff は用いた．

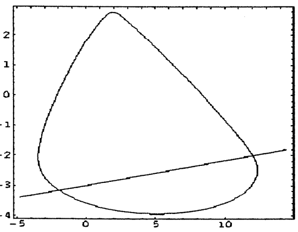

図 1:

C が正規行列である場合に対し、$\sigma(C)=\{c_1,c_2,\ldots,c_n\}$ とするとき、T の C-数域 $W_C(T)$ を

$$W_C(T)=\{\sum_{j=1}^{n}c_j\langle T\xi_j,\xi_j\rangle:\{\xi_1,\xi_2,\ldots,\xi_n\}\text{ は }\mathbf{C}^n\text{の正規直交系}\}, \quad (1.5)$$

と定義する．C が階数 1 の直交射影のときは、$W_C(T)$ は、(1.1) で定義された $W(T)$ と一致する．1963 年に、C. A. Berger [3] は学位論文で C が一般の直交射影のとき、すなわち、$\sigma(C)=\{1,1,\ldots,1,0,\ldots,0\}$ のとき、$W_C(T)$ が凸になることを示し、Hausdorff の結果を一般化した．さらに 1975 年に、カナダの Westwick [8] は、C がエルミット行列であって、T が同じサイズの正方行列であるとき、$W_C(T)$ が凸になることを証明した．C,T が、3×3 ユニタリ行列 $\mathrm{diag}(1,-1/2+i\sqrt{3}/2,-1/2-i\sqrt{3}/2)$ のとき、$W_C(T)$ はデルトイド

$$\{2\exp(i\theta)+\exp(-2i\theta):0\leq\theta\leq 2\pi\}$$

によって囲まれる閉領域となり、$W_C(T)$ は凸ではない．C がエルミット行列のとき、Westwick は、Hausdorff の証明にならってエルミット行列 T_θ を定義し、

$$W_C(T)\cap\{z\in\mathbf{C}:\Re(z\exp(-i\theta))=a\}$$
$$=\{\mathrm{Tr}(CUTU^{-1}):U^*U=I_n,\mathrm{Tr}(CUT_\theta U^{-1})=a\}$$

が連結集合の連続写像による像になるという論法を用いた．ここで、$\mathrm{Tr}(\cdot)$ は、行列のトレースである．集合 $\{U:U^*U=I_n,\mathrm{Tr}(CUT_\theta U^{-1})=a\}$ が

連結であることを証明するのに、Westwick は Morse 理論を使った. Morse 理論に頼らないで, majorization の方法で Westwick の結果を導くことを Y. T. Poon が [7] で行っている.

Poon の証明の核心部分は、次の命題である.

命題 1.1 $n \times n$-エルミット行列 A の固有値 $\{a_1, a_2, \ldots, a_n\}$ とし、凸集合

$$\mathrm{Conv}(\{UAU^{-1}) : UU^* = I\})$$

に属する任意のエルミット行列 B の固有値を $\{b_1, b_2, \ldots, b_n\}$ とすれば、

$$b_i = \sum_{j=1}^{n} |u_{ij}|^2 a_j$$

($1 \leq i \leq n$) となるような $n \times n$ ユニタリ行列 $U = (u_{ij})$ が存在する.

Krein 空間 $(\mathbf{C}^n, [\cdot, \cdot])$ は、不定計量

$$[(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)^t, (y_1, \ldots, y_r, y_{r+1}, \ldots, y_n)^t]$$

$$= x_1\overline{y_1} + \cdots + x_r\overline{y_r} - x_r\overline{y_{r+1}} - \cdots - x_n\overline{y_n}$$

を伴う複素線形空間である. この空間からそれ自身への線形作用素 T に対して、重みのついた数域を考える. $[T\xi, \eta] = [\xi, T^\sharp \xi]$ により、作用素 $T^\sharp$ を定義し、$T^\sharp = T$ のとき、T は、J-エルミット、$TT^\sharp = I$ のとき、T は J-ユニタリーであるという (ここでは、有限次元の線形空間を考えている). J-エルミット行列の固有値は必ずしも実数ではないが、C が、 J-ユニタリ行列で対角化される J エルミット行列であるとき、その固有値 $\{c_1, \ldots, c_r, c_{r+1}, \ldots, c_n\}$ はすべて実数となる. このような、C のスペクトルを

$$\sigma_+^J(C) = \{c \in \mathbf{R} : C\xi = c\xi \,\text{for some}\, \xi \in \mathbf{C}^n, [\xi, \xi] > 0\},$$

$$\sigma_-^J(C) = \{c \in \mathbf{R} : C\xi = c\xi \,\text{for some}\, \xi \in \mathbf{C}^n, [\xi, \xi] < 0\}$$

というように２つに分類する: $\sigma(C) = \sigma_+^J(C) \cup \sigma_-^J(C)$.さて、J−ユニタリで対角化されるJ−エルミット行列 C に対して

$$\max \sigma_+^J(C) < \min \sigma_-^J(C) \text{ または } \max \sigma_-^J(C) < \min \sigma_+^J(C)$$

であるとき、C のスペクトルは、非交差　non-interlacing であると言う.

Krein 空間　$(\mathbf{C}^n, [\cdot,\cdot])$ における作用素 C, T に対して、

$$W_C^J(T) = \{\mathrm{Tr}(CUTU^{-1}) : U \in M_n(\mathbf{C}), UU^\sharp = I\}, \tag{1.6}$$

により、Krein 空間の作用素としての C-数域 $W_C^J(T)$ を定義する. 特に、C が、J-ユニタリで対角化される J-エルミット行列であって

$$\sigma_+^J(C) = \{c_1, c_2, \ldots, c_r\},\ \sigma_-^J(C) = \{c_{r+1}, c_{r+2}, \ldots, c_n\}$$

であるとき、次のように表すこともできる.

$$W_C^J(T) = \{\sum_{j=1}^r c_j[T\xi_j, \xi_j] - \sum_{j=r+1}^n c_j[T\xi_j, \xi_j] : \xi_j \in \mathbf{C}^n,$$

$$[\xi_j, \xi_k] = 0\,(j \neq k), [\xi_j, \xi_j] = 1\,\mathrm{for}\, j \leq r, [\xi_j, \xi_j] = -1\,\mathrm{for}\, j \geq r+1\}, \tag{1.7}$$

定理 1.2 (cf. [1], [2]) $(r, n-r)$-型の Krein 空間 $\mathbf{C}^n$ における作用素 T, C に対して、或る $0 \leq \theta \leq 2\pi$ に対する

$$T_\theta = (\cos\theta)(T + T^\sharp)/2 + (\sin\theta)(T - T^\sharp)/(2i)$$

と C が、J-ユニタリで対角化される J-エルミット行列であって、2つの作用素のスペクトルがそれぞれ非交差ならば、Krein 空間の作用素としての C-数域 $W_C^J(T)$ は、閉凸集合であって、或る点を頂点とする凸錐に含まれる.

この定理は、Westwick が用いたと同じように、多様体上の Morse 関数と Morse 指数を用いて証明できる

領域 $W_C^J(T)$ の 例　$n=3$, $[(x_1, x_2, x_3)^T, (y_1, y_2, y_3)^T] = x_1\overline{y}_1 + x_2\overline{y}_2 - x_3\overline{y}_3$, $C = \mathrm{diag}(1, 0, -1)$,

$$T = \begin{pmatrix} 2 & i & 0 \\ i & 1 & i \\ 0 & -i & -2 \end{pmatrix}$$

のとき、$W_C^J(T)$ は、図 2 の黒塗りの部分のような閉領域（非有界）となる.

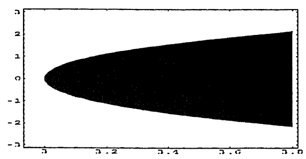

図 2:

2. Morse 理論の応用

次の命題が、定理 1.2 を証明する鍵となる.

命題 2.1 (Morse 理論の標準的な結果, 例えば [5]). M を連結な m-次元 の C^∞-級多様体とする. ただし $m \geq 2$ とする. f を M 上の 実数値 C^∞-関数とし、次の条件を f が満たすと仮定する.

(i) 関数 f は最小値を取る;

(ii) 集合 $\{x \in M : a \leq f(x) \leq b\}$ は各 $-\infty < a < b < +\infty$ に対してコンパクトである;

(iii) f の臨界点（停留点） $x_1, x_2, \ldots, x_n$ は有限個であって臨界値 $f(x_1), f(x_2), \ldots, f(x_n)$ は互いに相異なる;

(iv) 関数 f の各臨界点 x_j におけるヘッセ行列

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1}^{n}$$

は可逆な実対称行列である;

(v) 関数 f の各臨界点 x_j におけるヘッセ行列の負の固有値の重複度を込めた個数は 1 にも $m-1$ にもならない.

このとき水準集合 $\{x \in M : f(x) = a\}$ は各実数 a に対して連結である.

関数 f で条件 (i)-(iv) を満たすものが Morse 関数で, 各臨界点 x_j におけるヘッセ行列の負の固有値の重複度の総和を Morse 指数と言う.

Westwick の定理でも、B-N-P の主定理のいずれの場合でも C の固有値がすべて相異なり, T_θ の固有値もすべて相異なる場合に対して, 関数

$$f(U) = \mathrm{Tr}(CUT_\theta U^{-1})$$

が, 偶数次元多様体 $M = U(n)/D(n)$ または、$M = U(r, n-r)/D(r, n-r)$ 上の Morse 関数であって, 各臨界点における Morse 指数が偶数 であることより、集合の連結性を導き、これを用いて $W_C(T)$ または $W_C^J(T)$ の凸性が示される. ここで、$D(n)$ は、$g \in U(n)$ のうち対角行列であるような g からなる閉部分群、$D(r, n-r)$ は、$g \in U(r, n-r)$ のうちの対角行列であるような g からなる閉部分群である.

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Some results on new geometric constants of Banach spaces

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Let X be a real Banach space with dim $X \geq 2$. Geometric properties of X are determined by its unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ or its unit sphere $S_X = \{x \in X : \|x\| = 1\}$. X is called *uniformly non-square* (James, 1964) if there exists $\delta \in (0,1)$ such that for any $x, y \in S_X$ either $\|x+y\|/2 \leq 1-\delta$ or $\|x-y\|/2 \leq 1-\delta$. We shall consider the following constants:

$$J(X) = \sup\{\min(\|x+y\|,\ \|x-y\|) :\ x, y \in S_X\},$$

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} :\ x \in S_X,\ y \in B_X\right\},$$

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} :\ x, y \in S_X\right\},$$

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} :\ x, y \in S_X\right\}.$$

Recently many geometric constants for a Banach space X have been investigated, which enable us to make precise descriptions on various geometric properties of X. In particular the James constant $J(X)$ and the von Neumann-Jordan constant $C_{NJ}(X)$ are most widely treated. It is clear that X is uniformly non-square if and only if $J(X) < 2$. The first systematic work on the von Neumann-Jordan constant was done in Kato-Takahashi [8], and they showed that X is uniformly non-square if and only if $C_{NJ}(X) < 2$. The first result concerning the relation between $J(X)$ and $C_{NJ}(X)$ was given in 2001 by Kato-Maligranda-Takahashi [7]. Since then the relation between these two constants have been studied by several authors. The constant $C'_{NJ}(X)$ was introduced by Alonso et al. [1] (see also Takahashi [10]). It was shown in [1] that

$$C'_{NJ}(X) \leq J(X). \tag{1}$$

The constant $A_2(X)$ was introduced by Baronti et al. [2]. In 2009 Wang-Pang [12] showed that

$$A_2(X) \leq 1 + \sqrt{J(X) - 1}, \tag{2}$$

which yields that $C'_{NJ}(X) \leq 1 + (A_2(X) - 1)^2 \leq J(X)$ ([9]).

Here we shall introduce a new geometric constant $C(X)$ defined by

$$C(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} :\ x, y \in X,\ \|x\|^2 + \|y\|^2 = 2\right\}. \tag{3}$$

It is clear that for any Banach space X

$$J(X) \le A_2(X) \le C(X) \le \sqrt{2C_{NJ}(X)}, \tag{4}$$

where all the terms coincide if X is not uniformly non-square. We note that there are many uniformly non-square spaces X for which all the terms in (4) coincide. In fact, if X is an $L_p(\mu)$-space $(1 \le p \le \infty)$, then $J(X) = \sqrt{2C_{NJ}(X)} = 2^{1/r}$, where $r = \min\{p, p'\}$ and $1/p + 1/p' = 1$. For such a space X we have

$$J(X) = A_2(X) = C(X) = \sqrt{2C_{NJ}(X)}.$$

The constant $C(X)$ is closely related to the modulus of smoothness $\rho_X(\tau)$.

Proposition 1. *Let X be a Banach space. Then*

$$C(X) = \sup\left\{\frac{\sqrt{2}(1 + \rho_X(\tau))}{\sqrt{1+\tau^2}} :\ 0 \le \tau \le 1\right\}. \tag{5}$$

It is well-known that $A_2(X) = \sqrt{2}$ if X is a Hilbert space; but the converse is not true (see Example 1). By virtue of Proposition 1 we have

Theorem 1. *X is a Hilbert space if and only if $C(X) = \sqrt{2}$.*

As already mentioned, we always have $A_2(X) \le C(X)$. But in general these two constants are different. So we may ask how big the difference between these two constants can be. The next results give an answer to this question.

Theorem 2. *Let X be a Banach space. Then*

$$C(X) \le \sqrt{2}\sqrt{1 + (A_2(X) - 1)^2}. \tag{6}$$

Corollary 1. *Let X be a Banach space. Then*

$$C(X) - A_2(X) \le 2\sqrt{2 - \sqrt{2}} - \sqrt{2}. \tag{7}$$

Remark 1. The estimates given by Theorem 2 and Corollary 1 are sharp. The inequality in (6) attains equality if X is not uniformly non-square. It is shown that equality holds in (6) if and only if $\rho_X(\tau) = \rho_X(1)\tau$ for all $\tau \in [0,1]$. Hence the inequality in (6) also attains equality if X is one of the spaces of Examples 2 and 3. The inequality in (7) attains equality if X is the space of Example 1.

Now we shall improve (2).

Theorem 3. *Let X be a Banach space. Then*

$$C(X) \le 1 + \sqrt{J(X) - 1}, \tag{8}$$

where equality holds only when X is not uniformly non-square.

The von Neumann-Jordan constant can be reformulated in the following form, which is usefull to prove Theorem 4.

Proposition 2. *Let X be a Banach space. Then*

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : \ x, y \in X, \ \|x\|^2 + \|y\|^2 = 2\right\}. \tag{9}$$

Theorem 4. *Let X be a Banach space. Then*

$$C_{NJ}(X) \le 1 + (C(X) - 1)^2. \tag{10}$$

We note that there exists a two dimensional Banach space X such that $C_{NJ}(X) > 1 + (A_2(X) - 1)^2$ (see Example 2), while it is shown that

$$C'_{NJ}(X) \le 1 + (A_2(X) - 1)^2, \tag{11}$$

where equality holds if X is not uniformly non-square ([9], [10]). There are uniformly non-square spaces X such that equality holds in (11) (see Examples 1 and 2).

Proposition 3. *Let X be a Banach space. Then*

$$C(X) \le \sqrt{2C'_{NJ}(X^*)}, \tag{12}$$

where X^* is the dual space of X.

We note that the estimates given by Theorem 4 and Proposition 3 are sharp. It is clear that equality holds in (10) and (12) if X is not uniformly non-square. Moreover, there exists a uniformly non-square space X for which both inequalities (10) and (12) attain equality, while $C(X^*) > \sqrt{2C'_{NJ}(X^*)}$ (see Example 3).

By Theorem 4 and Proposition 3 we have the next result, which was proved by Alonso et al. [1].

Corollary 2. *Let X be a Banach space. Then*

$$C_{NJ}(X) \leq 1 + (\sqrt{2C'_{NJ}(X)} - 1)^2.$$

Theorems 3 and 4 yield the following simple inequality, which was recently proved by Takahashi-Kato [11].

Theorem 5. *Let X be a Banach space. Then*

$$C_{NJ}(X) \leq J(X), \tag{13}$$

where equality holds only when X is not uniformly non-square.

Finally, we shall present some examples.

Example 1. Let $X = \mathbb{R}^2$ with the norm defined by

$$\|x\| = \max\{|x_1| + (\sqrt{2}-1)|x_2|,\ |x_2| + (\sqrt{2}-1)|x_1|\} \quad \text{for } x = (x_1, x_2).$$

Then the unit sphere is a regular octagon and the dual space X^* is isometric to X. This is a non-Hilbertian space, but it is easy to see that $J(X) = A_2(X) = \sqrt{2}$ ([2]). Therefore, we have $C(X) \leq 2\sqrt{2-\sqrt{2}}$ by Theorem 2, and $C'_{NJ}(X) \leq 1 + (\sqrt{2}-1)^2$ by (11). It is not difficult to see that $C(X) = 2\sqrt{2-\sqrt{2}}$ and $C'_{NJ}(X) = 1 + (\sqrt{2}-1)^2$. This space is an example such that the inequalities (6), (7), (11) and (12) attain equality.

Example 2. Let $X = \mathbb{R}^2$ with the ℓ_∞-ℓ_1 norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1x_2 \geq 0, \\ \|x\|_1 & \text{if } x_1x_2 \leq 0 \end{cases}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. Then the unit sphere is a hexagon and the dual space X^* is isometric to X. It is well-known that $A_2(X) = 3/2$ ([2, Example 3.2], see also [7, Example 4]). Hence, we have $C(X) \leq \sqrt{10}/2$ by Theorem 2, and $C'_{NJ}(X) \leq 5/4$ by

(11). It is not difficult to see that $C(X) = \sqrt{10}/2$ and $C'_{NJ}(X) = 5/4$. It is also known that $C_{NJ}(X) = (3+\sqrt{5})/4$ ([10], [14]). This space is an example such that the inequalities (6), (11) and (12) attain equality.

Example 3. Let $X = \mathbb{R}^2$ with the ℓ_2-ℓ_1 norm defined by

$$\|x\| = \begin{cases} \|x\|_2 & \text{if } x_1x_2 \geq 0, \\ \|x\|_1 & \text{if } x_1x_2 \leq 0 \end{cases}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. The dual space X^* of X has the ℓ_2-ℓ_∞ norm defined by

$$\|x\|^* = \begin{cases} \|x\|_2 & \text{if } x_1x_2 \geq 0, \\ \|x\|_\infty & \text{if } x_1x_2 \leq 0. \end{cases}$$

It is easy to see that $A_2(X) = A_2(X^*) = 1+1/\sqrt{2}$ ([10]). Hence we have $C(X^*) \leq \sqrt{3}$ by Theorem 2, and $C'_{NJ}(X) \leq 3/2$ by (11). It is not difficult to see that $C(X^*) = \sqrt{3}$ and $C'_{NJ}(X) = 3/2$. In [1] it was shown that $C'_{NJ}(X^*) = (3+2\sqrt{2})/4$ (see also [10]). Hence we have $C(X) \leq \sqrt{2C'_{NJ}(X^*)} = 1 + 1/\sqrt{2}$ by (11). Since $C(X) \geq A_2(X) = 1 + 1/\sqrt{2}$, we have the equality $C(X) = 1 + 1/\sqrt{2}$. Finally, we show that $C_{NJ}(X) = 3/2$ ([10], [14]). In fact, $C_{NJ}(X) \leq 1 + (C(X)-1)^2 = 3/2$ by (10), and $C_{NJ}(X) \geq C'_{NJ}(X) = 3/2$. The space X is an example such that the inequalities (10), (11) and (12) attain equality. The dual space X^* is an example such that equality holds in (6). Moreover, we have $C(X^*) \neq C(X)$ and $C(X^*) = \sqrt{2C'_{NJ}(X)} > \sqrt{2C'_{NJ}(X^*)}$.

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Operator equations with continuous parameters – inverse problems

Saburou Saitoh

Abstract

In this note we shall introduce our new ideas for inversion formulas by using a finite number of output data, after reviewing the general linear inversion formulas using the theory of reproducing kernels. We derive very concrete formulas in the Laplace transform and the heat conduction problem.

Keywords: Linear inversion, reproducing kernel, inverse problem, finite number of output data, Laplace transform, heat conduction.

Mathematics Subject Classification (2000): Primary 30C40, 44A15, 35K05

1 Reproducing kernels and linear mappings

1.1 Identification of the image of a linear mapping

Let $\mathcal{F}(E)$ be the linear space comprising of all complex-valued functions on any fixed abstract set E. Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{H}}$. Let $\boldsymbol{h}$: $E \longrightarrow \mathcal{H}$ be a fixed $\mathcal{H}$ valued mapping on E. Then, we shall consider the linear mapping L from $\boldsymbol{f} \in \mathcal{H}$ into $\mathcal{F}(E)$ defined by the following

$$f(p) = (L\boldsymbol{f})(p) = (\boldsymbol{f}, \boldsymbol{h}(p))_{\mathcal{H}}. \tag{1}$$

The most fundamental problem in the linear mapping (1) will be the characterization of the image $f(p)$ of L and the next interest will be the interrelationship between the input $\boldsymbol{f}$ and the output $f(p)$. The key to consider these fundamental problems is to form the two variable complex-valued function

$$K(p,q) = (\boldsymbol{h}(q), \boldsymbol{h}(p))_{\mathcal{H}} \tag{2}$$

defined on $E \times E$. Then, following the theorem by Aronszajn-Moore, there exists a uniquely determined Hilbert space H_K satisfying and characterized by the properties:

(i) For any $q \in E$, $K(p,q)$ belongs to H_K as a function in p,

(ii) For any function $f \in H_K$ and for any point $q \in E$,

$$f(q) = (f(\cdot), K(\cdot,q))_{H_K}.$$

These two properties (i) and (ii) are called a reproducing property of $K(p,q)$ in the Hilbert space H_K and $K(p,q)$ is a reproducing kernel. Then we obtain the fundamental:

Proposition 1([7]). *The image space $\{f(p)\}$ by (1) of $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space H_K determined by $K(p,q)$ in (2). Then, we have the inequality*

$$\|f\|_{H_K} \leq \|\boldsymbol{f}\|_{\mathcal{H}}.$$

Furthermore, for any $f \in H_K$, there exists a uniquely determined function $\boldsymbol{f}^*$ in $\mathcal{H}$ satisfying:

$$f(p) = (\boldsymbol{f}^*, \boldsymbol{h}(p))_{\mathcal{H}} \quad \text{on} \quad E \quad \text{and} \quad \|f\|_{H_K} = \|\boldsymbol{f}^*\|_{\mathcal{H}}.$$

1.2 Example

For the simple integral transform

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} F(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi \tag{3}$$

for $F \in L_2(\mathbf{R})$, we have

Proposition 2([7]). *In the integral transform (3) of $L_2(\mathbf{R})$ functions F, the images $u(x,t)$ are extended analytically onto $\mathbf{C}$ in the form $u(z,t), (z = x+iy)$ and we obtain the isometric identities*

$$\begin{aligned} \int_{\mathbf{R}} |F(\xi)|^2 d\xi &= \frac{1}{\sqrt{2\pi t}} \iint_{\mathbf{C}} |u(z,t)|^2 \exp[\frac{-y^2}{2t}] dxdy \\ &= \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\mathbf{R}} |\partial_x^j u(x,t)|^2 dx \end{aligned}$$

for any fixed $t > 0$.

1.3 Inversion formulas

By Example:

Proposition 3([7]). *In the Weierstrass integral transform (3) and for any fixed $t > 0$ we have*

$$\begin{aligned} F(\xi) &= s - \lim_{N\to\infty} \frac{1}{\sqrt{2\pi t}} \iint_{E_N} u(z,t)\overline{k(z-\xi,t)} \exp[\frac{-y^2}{2t}] dxdy \\ &= s - \lim_{N\to\infty} \sum_{j=0}^{N} \frac{(2t)^j}{j!} \int_{\mathbf{R}} \partial_x^j u(x,t) \partial_x^j k(x-\xi,t) dx \\ &= s - \lim_{N\to\infty} \sum_{j=0}^{N} \frac{(-2t)^j}{j!} \int_{\mathbf{R}} u(x,t) \partial_x^{2j} k(x-\xi,t) dx, \end{aligned}$$

where $\{E_N\}$ is any exhaustion of $\mathbf{C}$ by compact sets and the limits mean the strong convergence in $L_2(\mathbf{R})$ and k is the integral kernel in (3).

1.4 Determination of the linear system

In our Proposition 1, conversely by using an isometric mapping $\widetilde{L}$ from a Hilbert space H_K admitting a reproducing kernel $K(p,q)$ on E onto a Hilbert space $\mathcal{H}$ and by using the reproducing kernel $K(p,q)$, we can determine the linear system function $\boldsymbol{h}(p)$ which is a Hilbert $\mathcal{H}$-valued function on E([7]). By our three Propositions, we can determine the relations among input, linear system and output. The author, indeed, thinks the theory of reproducing kernels is very fundamental and applicable in mathematics as in vector analysis and complex analysis, in the undergraduate level. See also [3].

2 Best approximations

Let L be any bounded linear operator from a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$. Then the following problem is a well-known classical and fundamental problem which is known as best approximate mean square norm problems: For any member b of $\mathcal{H}$

$$\inf_{f\in H_K} \|Lf - \mathrm{b}\|_{\mathcal{H}}. \tag{4}$$

This problem has, however, a complicated structure, when the Hilbert spaces are infinite dimensions and the problem leads to the generalized inverse in the sense of Moore-Penrose. By our theory, we can see its complicated structure, and the problem is well-established by the theory of reproducing kernels, that is the existence, uniqueness and representation of the solutions in the problem are well-formulated ([7]). However, the result is involved and so the Moore-Penrose generalized inverse is not good, when the data contain error or noises in some practical cases. So, we shall introduce the idea of the Tikhonov regularization.

3 Tikhonov regularization

Proposition 4([6]). *For any $b \in \mathcal{H}$ and for any $\alpha > 0$, in the Tikhonov functional*

$$\inf_{f\in H_K} \{\alpha\|f\|^2_{H_K} + \|b - Lf\|^2_{\mathcal{H}}\} \tag{5}$$

the extremal function $f_{b,\alpha}(p)$ exists uniquely and it is represented, by using the kernel $K_L(p,q;\alpha)$ as follows:

$$f_{b,\alpha}(p) = (b, LK_L(\cdot,p;\alpha))_{\mathcal{H}}. \tag{6}$$

Here, $K_L(p,q;\alpha)$ is the solution $\tilde{K}(p,q;\alpha)$ of the functional equation

$$\tilde{K}(p,q;\alpha) + \frac{1}{\alpha}(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\alpha}K(p,q), \tag{7}$$

that is corresponding to the Fredholm integral equation of the second kind for many concrete cases. Here,

$$\tilde{K}_q = \tilde{K}(\cdot,q;\alpha) \in H_K \quad \text{for} \quad q \in E, \qquad K_p = K(\cdot,p) \quad \text{for} \quad p \in E.$$

For the typical and famous problems of the inversion for the heat conduction and of the real inversions of the Laplace transform, we succeeded in their numerical experiments and for the last formulas, we are requesting the patents. See [4,5,6].

4 Inversion from a finite number of data

4.1 A family of bounded linear operators

For an abstract set $\mathbf{\Lambda}$, we shall consider an operator-valued function L_λ on $\mathbf{\Lambda}$,

$$\mathbf{\Lambda} \longrightarrow L_\lambda \tag{8}$$

where L_λ are bounded linear operators from a Hilbert space $\mathcal{H}$ into various Hilbert spaces $\mathcal{H}_\lambda$,

$$L_\lambda : \mathcal{H} \longrightarrow \mathcal{H}_\lambda. \tag{9}$$

In particular, we are interested in the inversion formula

$$L_\lambda x \longrightarrow x, \quad x \in \mathcal{H}. \tag{10}$$

Here, we consider $\{L_\lambda x; \lambda \in \Lambda\}$ as information obtained from x and we wish to determine x from the information. However, the information $L_\lambda x$ belong to various Hilbert spaces $\mathcal{H}_\lambda$, and so, in order to unify the information in a sense, we shall take fixed elements $\mathbf{b}_{\lambda,\omega} \in \mathcal{H}_\lambda$ and consider the linear mapping from $\mathcal{H}$

$$\begin{aligned} X_{\mathbf{b}}(\lambda, \omega) : &= (L_\lambda x, \mathbf{b}_{\lambda,\omega})_{\mathcal{H}_\lambda} \\ &= (x, L_\lambda^* \mathbf{b}_{\lambda,\omega})_{\mathcal{H}}, \quad x \in \mathcal{H} \end{aligned} \tag{11}$$

into a linear space comprising functions on $\mathbf{\Lambda} \times \mathbf{\Omega}$. Then, we can apply the theory in Section 1.

4.2 Explicit real inversion formulas of the Laplace transform from finite data

We shall consider the Laplace transform

$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt} F(t)dt, \quad p > 0 \tag{12}$$

for functions F of some natural function space.

We shall introduce the simple reproducing kernel Hilbert space (RKHS) H_K comprised of absolutely continuous functions F on the positive real line $\mathbf{R}^+$ with finite norms

$$\left\{ \int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt \right\}^{1/2}$$

and satisfying $F(0) = 0$. This Hilbert space admits the reproducing kernel $K(t, t')$

$$K(t, t') = \int_0^{\min(t,t')} \xi e^{-\xi} d\xi. \tag{13}$$

Then we see that

$$\int_0^\infty |(\mathcal{L}F)(p)p|^2 dp \leq \frac{1}{2} \|F\|_{H_K}^2; \tag{14}$$

that is, the linear operator $(\mathcal{L}F)(p)p$ on H_K into $L_2(\mathbf{R}^+, dp) = L_2(\mathbf{R}^+)$ is bounded. Now we shall consider the bounded linear mapping $\mathbf{L}$, for any fixed different points $\{p_j\}_{j=1}^n$ of $\mathbf{R}^+$,

$$\mathbf{L} : H_K \ni F \longrightarrow ((LF)(p_1)p_1, (LF)(p_2)p_2, ..., (LF)(p_n)p_n) \in \mathbf{R}^n. \tag{15}$$

We shall take the standard orthonormal system $\{\mathbf{e}_j\}_{j=1}^n$ in the space $\mathbf{R}^n$. Then, we see $(LF)(p_j)p_j = (\mathbf{L}F, \mathbf{e}_j)_{\mathbf{R}^n} = (F, \mathbf{L}^*\mathbf{e}_j)_{H_K}$; that is the situation of (4). Therefore, the image space forms the Hilbert space H_A admitting the reproducing kernel $(\mathbf{L}^*\mathbf{e}_{j'}, \mathbf{L}^*\mathbf{e}_j)_{H_K} := a_{jj'}$. If the matrix $\| \ a_{jj'} \ \|$ is positive definite, then, the norm in H_A is given by $\|\mathbf{L}F\|_{H_A}^2 = (\mathbf{L}F)^* \tilde{A}(\mathbf{L}F)$, where $\tilde{A} = \overline{A^{-1}} = \|\tilde{a}_{jj'}\|$. Here note that this is clearly calculated as follows: $(\mathbf{L}^*\mathbf{e}_j)(t) = ((\mathbf{L}^*\mathbf{e}_j)(\cdot), K(\cdot, t))_{H_K}$ $= (\mathbf{e}_j, \mathbf{L}K(\cdot, t))_{\mathbf{R}^n}$

$$= e^{-tp_j} e^{-t} \left[\frac{-t}{(p_j + 1)} + \frac{-1}{(p_j + 1)^2} \right] + \frac{1}{(p_j + 1)^2}$$

from

$$K(t, t') = \begin{cases} -te^{-t} - e^{-t} + 1 & \text{for} \quad t \leq t' \\ -t'e^{-t'} - e^{-t'} + 1 & \text{for} \quad t \geq t' \end{cases}$$

and $(LK(\cdot, t'))(p)$

$$= e^{-t'p} e^{-t'} \left[\frac{-t'}{p(p + 1)} + \frac{-1}{p(p + 1)^2} \right] + \frac{1}{p(p + 1)^2}. \tag{16}$$

Hence,

$$a_{jj'} = \frac{1}{(p_j + p_{j'} + 1)^2}.$$

In particular, note that the matrix $A = ||a_{jj'}||$ is positive definite. We thus obtain the real inversion formula

Theorem 1([2]). *For any given n values $\mathbf{d} = \{d_j\}_{j=1}^n$, among the Laplace transforms taking the values $f(p_j) = d_j; j = 1, 2, ..., n$, and for their inverses, the function $F_{\mathbf{d}}^*(t)$ with the minimum norm $||F||_{H_K}$ is uniquely determined and it is represented as follows: $F_{\mathbf{d}}^*(t) = \sum_{j,j'=1}^n (d_j p_j) \tilde{a}_{jj'} (\mathbf{L}^*\mathbf{e}_{j'})(t)$, where $\tilde{A} = \overline{A^{-1}} = \|\tilde{a}_{jj'}\|$.*

5 Source inversion of heat conduction from a finite number of observation data

We shall consider the Gaussian convolution (the Weierstrass transform)

$$u_F(x, t) = (L_t F)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} F(\xi) \exp\left\{ -\frac{|\xi - x|^2}{4t} \right\} d\xi \tag{17}$$

for the Paley-Wiener reproducing kernel spaces H_{K_h}(see [6]). For different points $\{(x_j, t_j)\}_{j=1}^n$ as the points of the time and space, we shall consider the bounded linear operators from the RKHS H_{K_h} into $\mathbf{R}$: $H_{K_h} \ni F \longrightarrow u_F(x_j, t_j)$; $\quad j = 1, 2, ..., n$ and we shall write them

$$\mathbf{L} : H_{K_h} \ni F \longrightarrow (u_F(x_1, t_1), u_F(x_2, t_2), ..., u_F(x_n, t_n)) \in \mathbf{R}^n. \tag{18}$$

We shall take the standard orthonormal system $\{\mathbf{e}_j\}_{j=1}^n$ in the space $\mathbf{R}^n$. Then, we see $u_F(x_j,t_j) = (\mathbf{L}F, \mathbf{e}_j)_{\mathbf{R}^n} = (F, \mathbf{L}^*\mathbf{e}_j)_{H_{K_h}}$. Therefore, the image space of the linear mapping $\mathbf{L}$ forms the Hilbert space H_A admitting the reproducing kernel $(\mathbf{L}^*\mathbf{e}_{j'}, \mathbf{L}^*\mathbf{e}_j)_{H_{K_h}} := a_{jj'}$.

Here note that $a_{jj'}$ are calculated. In particular, note that the matrix $A = ||a_{jj'}||$ is positive definite. We can obtain the real inversion formula

Theorem 2([1]). *For any given n values $\mathbf{d} = \{d(x_j,t_j)\}_{j=1}^n$, among the Weirstrass transform taking the values*

$$u_F(x_j,t_j) = d(x_j,t_j) \quad j = 1,2,...,n, \tag{19}$$

and for their inverses, the function $F_{\mathbf{d}}^*(x)$ with the minimum norm $||F||_{H_{K_h}}$ is uniquely determined and it is represented as $F_{\mathbf{d}}^*(x) = \sum_{j,j'=1}^n d(x_j,t_j)\tilde{a}_{jj'}(\mathbf{L}^*\mathbf{e}_{j'})(x)$.

H. Fujiwara solved the related Fredholm integral equation based on Proposition 4 by using his very powerful computer systems and the concept of infinite precision algorithm, and he succeeded in numerical and real inversion formulas of the Laplace transform. He made a software for numerical and real inversions of the Laplace transform. He calculated there, for $\alpha = 10^{-400}$ and numerical calculations were done with 600 dicimal digits precision. The number of points $\{p_j\}$ is 6000. There, **the method of discretization** of the integral equation is a serious problem and so, we wonder how will be our discretizations as effective algorithms.

追記： セミナー中 著者の言葉として「数学とは関係である」と複数の方から述べられましたが、数学の本質について、次の著書の付録で、徹底的に論じましたので、ご参考にして下さい： **夜明け前 よっちゃんの想い**（著者 齋藤三郎・齋藤尚徳）文芸社 ISBN978-4-286-08451-0。

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The sets with binary operation which have a superstability for multiplicative complex functions

乗法的複素数値関数を超安定にさせる2項演算の決定

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Abstract. Let X be a set with binary operation $\circ$ and f be an unbounded complex function on X. The superstability for the multiplicativity of f is stated as follows: If there exists an $\varepsilon \geq 0$ such that $|f(x \circ y) - f(x)f(y)| \leq \varepsilon$ for all $x, y \in X$, then $f(x \circ y) = f(x)f(y)$ for all $x, y \in X$. We consider the condition on X and $\circ$ which guarantees this superstability. We give the best condition in some sense.

§1. いきさつ

1940年, S. M. Ulam は,「近似的な加法的写像の近くに真の加法的写像が存在するか」という主旨の問題を提起した ([5]). 翌年, D. H. Hyers は, これに対する一解答を与えた:

Hyersの定理 ([2]). X, Y を実 Banach 空間とする. このとき,

$$f : X \to Y, \quad \varepsilon \geq 0, \quad |f(x+y) - f(x) - f(y)| \leq \varepsilon \quad (x, y \in X)$$
$$\Longrightarrow \text{ 次の2式を満たす } g : X \to Y \text{ が存在する:}$$
$$g(x+y) = g(x) + g(y), \quad \|g(x) - f(x)\| \leq \varepsilon \quad (x, y \in X).$$

この定理の精神は, 多くの場面に応用でき, **Hyers-Ulam の安定性問題**と呼ばれる広い研究対象を生んだ. たとえば, X が2項演算 $\circ$ をもつ集合で, Y が複素数体 $\mathbb{C}$ の場合, 次のような安定性が考えられる:

$$f : X \to \mathbb{C}, \quad \varepsilon \geq 0, \quad |f(x \circ y) - f(x)f(y)| \leq \varepsilon \quad (x, y \in X)$$
$$\Longrightarrow \text{ 次の2式を満たす } g : X \to \mathbb{C} \text{ が存在する:}$$
$$g(x \circ y) = g(x)g(y), \quad |g(x) - f(x)| \leq \varepsilon \quad (x, y \in X).$$

この講演では, とくに g が f 自身として存在する場合を考える. つまり,

$$f : X \to \mathbb{C}, \quad \varepsilon \geq 0, \quad |f(x \circ y) - f(x)f(y)| \leq \varepsilon \quad (x, y \in X) \tag{1}$$
$$\Longrightarrow f(x \circ y) = f(x)f(y) \quad (x, y \in X)$$

という形の**超安定性**を考える.

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われわれは, 次の 2 つの定理に着目した:

定理 A (Baker [1, Theorem 1]). X を半群とする. このとき,

$$f : X \to \mathbb{C}\ (\text{非有界}),\ \varepsilon \geq 0,\ |f(xy) - f(x)f(y)| \leq \varepsilon \quad (x, y \in X) \\ \Longrightarrow f(xy) = f(x)f(y) \quad (x, y \in X). \tag{2}$$

定理 B (Najdecki [4, Theorem 1]). X を可換半群とし, ϕ を X から X への写像とする. このとき,

$$f : X \to \mathbb{C}\ (\text{非有界}),\ \varepsilon \geq 0,\ |f(x\phi(y)) - f(x)f(y)| \leq \varepsilon \quad (x, y \in X) \\ \Longrightarrow f(x\phi(y)) = f(x)f(y) \quad (x, y \in X). \tag{3}$$

[1] では, $f : X \to \mathbb{C}$ が有界な場合に, 次のことを導いている:

$$f : X \to \mathbb{C}\ (\text{有界}),\ \varepsilon \geq 0,\ |f(xy) - f(x)f(y)| \leq \varepsilon \quad (x, y \in X) \\ \Longrightarrow |f(x)| \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2} \quad (x \in X).$$

f が有界な場合, 超安定性 (1) はとうてい期待できない. また, (2), (3) においては,「非有界」という仮定と「$\leq \varepsilon$」という有界のような仮定が合わさっているところに, ポイントがある. そこで, ここでは, $f : X \to \mathbb{C}$ が非有界な場合を扱うことにする.

§2. 考察過程と得られた結果

定理 A の (2) は, 定理 B の (3) の特別な形 (ϕ が X の恒等写像の場合) だから, まずは定理 B に注目しよう. 定理 B の X において, 別の演算 $\circ$ を,

$$x \circ y = x\,\phi(y) \qquad (x, y \in X)$$

と定義すると,

$$(x \circ y) \circ z = x\,\phi(y)\,\phi(z) = x\,\phi(z)\,\phi(y) = (x \circ z) \circ y \qquad (x, y, z \in X)$$

が成り立つ. この演算公式 $(x \circ y) \circ z = (x \circ z) \circ y$ は右半劫とでも呼ぼうか? 同様に, 左半劫 $x \circ (y \circ z) = y \circ (x \circ z)$ も考えられる. このような観点から, 定理 B の拡張として, 次の定理が得られた:

> **定理 1** ([3]). 集合 X では, 2 項演算 $\circ$ が定義されていて, 次の条件を満たすとする:
>
> $$\text{任意の } x, y, z \in X \text{ に対して,} \\ x \circ (y \circ z) = y \circ (x \circ z) \ \text{ or } \ (x \circ y) \circ z = (x \circ z) \circ y. \tag{4}$$
>
> このとき,
>
> $$f : X \to \mathbb{C}\ (\text{非有界}),\ \varepsilon \geq 0,\ |f(x \circ y) - f(x)f(y)| \leq \varepsilon \quad (x, y \in X) \\ \Longrightarrow f(x \circ y) = f(x)f(y) \quad (x, y \in X). \tag{5}$$

定理 1 の仮定 (4) では, 3 元 x, y, z に関する演算公式が 2 つ出てくる. このような式は他にもたくさんある. たとえば, $(x \circ y) \circ z = x \circ (y \circ z)$ (結合律) や $(x \circ y) \circ z = z \circ (y \circ x)$

など.... 実際, 3元 x, y, z の演算は,

$$(x\circ y)\circ z,\quad (x\circ z)\circ y,\quad (y\circ x)\circ z,\quad (y\circ z)\circ x,\quad (z\circ x)\circ y,\quad (z\circ y)\circ x,$$
$$z\circ(x\circ y),\quad y\circ(x\circ z),\quad z\circ(y\circ x),\quad x\circ(y\circ z),\quad y\circ(x\circ z),\quad x\circ(z\circ y)$$

の12通りがあり, この中から異なるものを2つ選んでそれらを等号 = で結べば, ひとつの演算公式が作れるので, (x, y, z を区別した場合の) 演算公式は全部で ${}_{12}\mathrm{C}_2 = 66$ 通りある. この66個の演算公式を効率よく見るには, X の左正則表現 L と右正則表現 R:

$$L_w x = w\circ x,\qquad R_w x = x\circ w\qquad (x, w\in X)$$

を用いると便利である. 実際, 仮定 (4) は,

(4)′ 任意の $x, y, z \in X$ に対して, $L_xL_yz = L_yL_xz$ or $R_zR_yx = R_yR_zx$

と述べ換えられる. いま, L_w と R_w を同時に扱うにあたり, $A_w\ (A\in\{L, R\})$ と記すことにしよう. すると, (4)′ の2式は, $A_xB_yz = C_yD_xz,\ A_zB_yx = C_yD_zx\ \ (A, B, C, D\in\{L, R\})$ という形だといえる. そのような仮定を考えてみると, 次のことがいえた:

定理 2. 集合 X では, 2項演算 $\circ$ が定義されていて, 次の条件を満たすとする:

(6) 任意の $x, y, z\in X$ に対して, $A, B, C, D\in\{L, R\}$ が存在して,
$$A_xB_yz = C_yD_xz\ \text{ or }\ A_zB_yx = C_yD_zx.$$

このとき,

(7) $f : X\to\mathbb{C}$ (非有界), $\varepsilon\geq 0$, $|f(x\circ y) - f(x)f(y)|\leq\varepsilon\ \ (x, y\in X)$
$$\Longrightarrow\ f(x\circ y)\text{ or }f(y\circ x) = f(x)f(y)\quad (x, y\in X).$$

定理2では超安定性 (5) が得られていない. しかし, 仮定 (6) の第1式において $D = L$, 第2式において $D = R$ とすると, (5) を導くことができた:

定理 3. 集合 X では, 2項演算 $\circ$ が定義されていて, 次の条件を満たすとする:

(8) 任意の $x, y, z\in X$ に対して, $A, B, C\in\{L, R\}$ が存在して,
$$A_xB_yz = C_yL_xz\ \text{ or }\ A_zB_yx = C_yR_zx.$$

このとき,

(5) $f : X\to\mathbb{C}$ (非有界), $\varepsilon\geq 0$, $|f(x\circ y) - f(x)f(y)|\leq\varepsilon\ \ (x, y\in X)$
$$\Longrightarrow\ f(x\circ y) = f(x)f(y)\quad (x, y\in X).$$

§3. 定理の証明

定理の証明は初等的である.

定理2の証明 f は X 上の非有界な複素数値関数で, ある $\varepsilon\geq 0$ に対して,

(9) $$|f(x\circ y) - f(x)f(y)|\leq\varepsilon\qquad (x, y\in X)$$

が成り立つとする. また, 任意に $x, z\in X$ をとり固定する. f は非有界だから, $|f(y_n)|\nearrow\infty$ となる X の点列 $\{y_n\}$ が存在する. いま, 各 $A, B, C, D\in\{L, R\}$ に対して,

$$N_{ABCD} = \{n \in \mathbb{N} : A_x B_{y_n} z = C_{y_n} D_x z\}$$
$$N'_{ABCD} = \{n \in \mathbb{N} : A_z B_{y_n} x = C_{y_n} D_z x\}$$

とおこう. すると, 仮定 (6) から,

$$\mathbb{N} \ = \bigcup_{A,B,C,D\in\{L,R\}} \Big(N_{ABCD} \cup N'_{ABCD} \Big)$$

が成り立つ. この右辺は, 32 個 (有限個) の集合の和集合だから, その中の少なくともひとつは無限集合でなければならない. たとえば, それを N_{ABCD} としよう (N'_{ABCD} の場合も同様に証明できる). そして, $\{y_n : n \in N_{ABCD}\}$ の元を, 順序を変えずに, あらためて $y_1, y_2, \cdots, y_n, \cdots$ とかく. すると, $A_x B_{y_n} z = C_{y_n} D_x z$ だから,

$$\frac{f(C_{y_n} D_x z)}{f(y_n)} = \frac{f(A_x B_{y_n} z)}{f(y_n)}$$

である. ここで, $n \to \infty$ としよう. 左辺については, (9) より,

$$\left. \begin{array}{l} \big|f(L_{y_n} D_x z) - f(y_n) f(D_x z)\big| = \big|f(y_n \circ (D_x z)) - f(y_n) f(D_x z)\big| \\ \big|f(R_{y_n} D_x z) - f(y_n) f(D_x z)\big| = \big|f((D_x z) \circ y_n) - f(D_x z) f(y_n)\big| \end{array} \right\} \leq \varepsilon$$

だから,

$$\left| \frac{f(C_{y_n} D_x z)}{f(y_n)} - f(D_x z) \right| = \left| \frac{f(C_{y_n} D_x z) - f(y_n) f(D_x z)}{f(y_n)} \right| \leq \frac{\varepsilon}{|f(y_n)|} \ \longrightarrow \ 0$$

である. 右辺についても, 同様の考え方で,

$$\begin{aligned} &\left| \frac{f(A_x B_{y_n} z)}{f(y_n)} - f(x) f(z) \right| \\ &\quad = \left| \frac{f(A_x B_{y_n} z) - f(x) f(B_{y_n} z) + f(x) f(B_{y_n} z) - f(x) f(y_n) f(z)}{f(y_n)} \right| \\ &\quad \leq \left| \frac{f(A_x B_{y_n} z) - f(x) f(B_{y_n} z)}{f(y_n)} \right| + \left| f(x) \frac{f(B_{y_n} z) - f(y_n) f(z)}{f(y_n)} \right| \\ &\quad \leq \frac{\varepsilon}{|f(y_n)|} + |f(x)| \frac{\varepsilon}{|f(y_n)|} \ \longrightarrow \ 0 \end{aligned}$$

である. よって,

$$f(D_x z) = f(x) f(z) \tag{10}$$

となる (N'_{ABCD} の場合は, $f(D_z x) = f(x) f(z)$ となる). D は L or R だから,

$$f(x \circ z) \text{ or } f(z \circ x) = f(x) f(z)$$

である. □

定理 3 の証明 (6) と (8) の違いを意識して, 定理 2 の証明をたどっていくと, (10) 式のところで, $f(L_x z) = f(x) f(z)$ (N'_{ABCD} の場合は, $f(R_z x) = f(x) f(z)$) となるから,

$$f(x \circ z) = f(x) f(z)$$

を得る. □

定理 1 の証明 仮定 (4) は (4)$'$ と同値だから, 定理 1 は定理 3 に帰着できる. □

§4. 定理の仮定の吟味

定理 2, 3 の仮定 (6), (8) を, 左・右正則表現を用いずに, 演算 $\circ$ を用いて述べ直してみると, 次のようになる:

(6) $\iff$ 任意の $x, y, z \in X$ は, 下の演算公式のどれかを満たす.

(8) $\iff$ 任意の $x, y, z \in X$ は, 下の下線つきの演算公式のどれかを満たす.

$$
\begin{array}{llll}
\underline{x \circ (y \circ z) = y \circ (x \circ z)}, & x \circ (y \circ z) = y \circ (z \circ x), & \underline{x \circ (y \circ z) = (x \circ z) \circ y}, & x \circ (y \circ z) = (z \circ x) \circ y, \\
\underline{x \circ (z \circ y) = y \circ (x \circ z)}, & x \circ (z \circ y) = y \circ (z \circ x), & \underline{x \circ (z \circ y) = (x \circ z) \circ y}, & x \circ (z \circ y) = (z \circ x) \circ y, \\
\underline{(y \circ z) \circ x = y \circ (x \circ z)}, & (y \circ z) \circ x = y \circ (z \circ x), & \underline{(y \circ z) \circ x = (x \circ z) \circ y}, & (y \circ z) \circ x = (z \circ x) \circ y, \\
\underline{(z \circ y) \circ x = y \circ (x \circ z)}, & (z \circ y) \circ x = y \circ (z \circ x), & \underline{(z \circ y) \circ x = (x \circ z) \circ y}, & (z \circ y) \circ x = (z \circ x) \circ y, \\
z \circ (y \circ x) = y \circ (z \circ x), & \underline{z \circ (y \circ x) = y \circ (x \circ z)}, & z \circ (y \circ x) = (z \circ x) \circ y, & \underline{z \circ (y \circ x) = (x \circ z) \circ y}, \\
z \circ (x \circ y) = y \circ (z \circ x), & \underline{z \circ (x \circ y) = y \circ (x \circ z)}, & z \circ (x \circ y) = (z \circ x) \circ y, & \underline{z \circ (x \circ y) = (x \circ z) \circ y}, \\
(y \circ x) \circ z = y \circ (z \circ x), & \underline{(y \circ x) \circ z = y \circ (x \circ z)}, & (y \circ x) \circ z = (z \circ x) \circ y, & \underline{(y \circ x) \circ z = (x \circ z) \circ y}, \\
(x \circ y) \circ z = y \circ (z \circ x), & \underline{(x \circ y) \circ z = y \circ (x \circ z)}, & (x \circ y) \circ z = (z \circ x) \circ y, & \underline{(x \circ y) \circ z = (x \circ z) \circ y}
\end{array}
$$

定理 1 (4) の 2 つの演算公式は, 上の第 1 列最上段・第 4 列最下段にある. だから, 定理 3 は定理 1 の (それゆえ, 定理 B の) 一般化である. また, 第 2 列第 7 段は結合律である. 半群とは結合律を満たす 2 項演算ができる集合だから, 定理 3 は定理 A の一般化でもある.

さて, 2 項演算が定義された集合 X において,

任意の $x, y, z \in X$ が, 66 個の演算公式のどれを満たせば,
超安定性 (7) または (5) が成り立つのだろうか?

実は, 次の 2 つのことがいえる:

- 任意の $x, y, z \in X$ が, ある演算公式群のどれかを満たすときに, 超安定性 (7) がつねに成立すれば, その公式群を, 上の公式群に含まれるように選び直すことができる.
- 任意の $x, y, z \in X$ がどれかを満たしているような演算公式群を, どのように選んでも, 上以外の公式が必要な場合は, いつでも, 超安定性 (7) が成り立たなくなることがある (そのような反例が作れる).

こういう意味で, 定理 2 において, 仮定 (6) は最良といえる. また, 同様な意味で, 定理 3 の仮定 (8) は最良といえる. こうして, 「非有界な複素数値関数の乗法性に関する超安定性」を導く 2 項演算の集合が, 決定できた.

注: スペースの都合で, ここでは, あらましだけを述べた. 詳細は別の機会にゆずる.

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SAITOH'S INEQUALITY AND OPIAL'S INEQUALITY

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ABSTRACT. We prove an elementary integral inequality which extends a norm inequality of Saitoh concerning absolutely continuous functions on an interval of the real axis. Our inequality immediately yields well-known Opial-type inequalities.

1. INTRODUCTION

This is an expository article on the paper:
A. Yamada, *Saitoh's inequality and Opial's inequality*, to appear in Math. Inequal. Appl.

Let $\mathrm{AC}[a,b]$ denotes the space of real-valued absolutely continuous functions on $[a,b]$. The following inequality is well-known [4]:

FACT 1 (Opial's inequality). *For each $f \in \mathrm{AC}[0,a]$ with $f(0)=0$, we have*

$$\int_0^a |f(x)f'(x)|\,dx \le \frac{a}{2}\int_0^a |f'(x)|^2\,dx. \tag{1}$$

Equality holds if and only if there exist a constant C and $y \in [0,a]$ such that $f(x) = C\min\{x,y\}$.

On the other hand, as an example of the general norm inequalities for reproducing kernel Hilbert spaces [6], S. Saitoh proved the following inequality [5]:

FACT 2 (Saitoh's inequality). *For each $f \in \mathrm{AC}[0,1]$ with $f(0)=0$ and $\int_0^1 f'(x)^2dx < 1$, we have*

$$\int_0^1 \Big(\frac{f(x)}{1-f(x)}\Big)'^2 (1-x)^2dx \le \frac{\int_0^1 f'^2(x)\,dx}{1-\int_0^1 f'^2(x)\,dx}, \tag{2}$$

Equality holds if $f(x) = \min\{x,y\}$ for some $y \in [0,1)$.

He raised a question whether one can prove elementarily the above inequality without using the theory of kernel functions. Our answer is affirmative [7] and is given by Theorem 1 below which can be proved by using Hölder's inequality, so that the proof is considered to be elementary.

Theorem 1. *Let $G(x)$ be a function of class C^1 on an interval $(-R,R)$ $(0<R\le +\infty)$ satisfying the conditions*
(i) $G(0)=0$,
(ii) $|G'(x)| \le G'(|x|)$, *for all $x \in (-R,R)$, and*
(iii) *if $x^2 \le yz$, $(0<x,y,z<R)$, then $0 < G'^2(x) \le G'(y)G'(z)$.*
Assume that functions F, $f \in \mathrm{AC}[a,b]$ with $F(a)=f(a)=0$ satisfy

(iv) $F'(x) > 0$ *a.e. on* $[a, b]$, *and*

(v) $F(b) \leq R$, *and* $\int_a^b |f'(t)|^p / F'(t)^{p-1}\, dt < R$ *for some* $p > 1$.

Then,

$$\int_a^b \frac{|(G \circ f)'(x)|^p}{(G \circ F)'(x)^{p-1}}\, dx \leq G\Big(\int_a^b \frac{|f'(x)|^p}{F'(x)^{p-1}}\, dx\Big). \tag{3}$$

If $f(x) = C \cdot F(\min\{x, y\})$ $(a < y \leq b,\ C = 0, 1)$, *then equality holds in* (3).

Remark. (1) By setting $G(x) = x/(1-x)$, $F(x) = x$, $p = 2$, $a = 0$ and $b = R = 1$ in Theorem 1, we easily obtain Saitoh's inequality (2).

(2) The condition (iii) in Theorem is equivalent to the one that G' is positive, monotone increasing, and geometrically convex (i.e. $\log \circ G' \circ \exp$ is convex).

2. Equality conditions and Application

For most cases equality in our inequality (3) is attained only for functions stated in Theorem 1, i.e. the function 0 or $F(\min\{x, y\})$. We are able to show this by adding further assumptions on the function $G(x)$.

Theorem 2. *Under the same hypothesis as in Theorem 1, assume, moreover, that*

(vi) G' *is strictly monotone increasing on* $(0, R)$.

Then, if equality holds in inequality (3), *then there exist constants* C *and* y $(a < y \leq b)$ *such that*

$$f(x) = C \cdot F(\min\{x, y\}). \tag{4}$$

If, in addition, we assume that

(vii) $|G'(-x)| \neq G'(x)$ *for some* $x \in (0, F(y))$, *and*

(viii) *there exist no constants* $\alpha > 0$ *and* $\beta \geq 0$ *such that* $G'(x) = \alpha x^\beta$ *on* $(0, F(y))$,

then equality holds in inequality (3) *if and only if* f *is of the form* (4) *with* C *either 0 or 1.*

As an application of Theorem 1, we show that Opial's inequality is an immediate consequece of our inequality. In fact, if $G(x) = x^2$, $F(x) = x$, $p = 2$ and $R = +\infty$, then Theorem 1 implies that, for $f \in \mathrm{AC}[0, a]$ with $f(0) = 0$, we have

$$\int_0^a \frac{|ff'|^2}{t}\, dt \leq \frac{1}{2}\Big(\int_0^a |f'|^2\, dt\Big)^2.$$

On the other hand, by Cauchy-Schwarz inequality we have

$$\Big(\int_0^a |ff'|\, dt\Big)^2 \leq \int_0^a t\, dt \int_0^a \frac{|ff'|^2}{t}\, dt.$$

Combining these inequalities we obtain Opial's inequality (1). This observation, thus, leads us to obtain the following extension of Opial's inequality from Therem 1.

Theorem 3. *Let* $s(x)$, $t(x)$ *be nonnegative, measurable functions on* $[a, b]$ *such that* $\int_a^b t(x)^{-1/(p-1)}\, dx < +\infty$ *for some* $p > 1$. *Set* $F(x) = \int_a^x t(\xi)^{-1/(p-1)}\, d\xi$ *and assume that the functions* $G(x)$, $F(x)$ *and* $f(x)$ *satisfy the same conditions as stated in Theorem 1 with* $R = +\infty$. *Then, if* $K < +\infty$, *we have*

$$\Big\{\int_a^b |(G \circ f)'(x)|^q s(x)\, dx\Big\}^{1/q} \leq K \cdot G\Big(\int_a^b |f'(x)|^p t(x)\, dx\Big)^{1/p}, \tag{5}$$

where $1/p + 1/r = 1/q$, $r > 0$ *and*

$$K = \left\{ \int_a^b (G \circ F)'(x)^{r(1-1/p)} s(x)^{r/q} \, dx \right\}^{1/r}.$$

If, in addition, we assume the conditions (vi), (vii) *and* (viii) *in Theorem 2, then equality holds in the inequality* (1) *if and only if either* $f = 0$ *or there exist constants* C (≥ 0) *and* y $(a < y \leq b)$ *such that* $f(x) = F(\min\{x, y\})$ *and* $s(x) = C \cdot (G \circ F)'(x)^{1-q}$.

Remark 1. The existence of the multiplicative constant K is a merit of our inequality (1). Cf. [2], [3]

Remark 2. Let $G(x) = |x|^{p/q}$, $p > q$, $k > 1$ and $k > q > 0$. If $\int_a^b t(x)^{-1/(k-1)} dx < +\infty$, then from Theorem 3 we obtain Opial-type inequality

$$\int_a^b |f(x)|^{p-q} |f'(x)|^q s(x) \, dx \leq K \cdot \left\{ \int_a^b |f'(x)|^k t(x) \, dx \right\}^{p/k}, \tag{6}$$

where we assume that the constant

$$K = \left(\frac{q}{p}\right)^{q/k} \left\{ \int_a^b s^{k/(k-q)} t^{-q/(k-q)} \left(\int_a^x t^{-1/(k-1)} \, d\xi \right)^{(p-q)(k-1)/(k-q)} dx \right\}^{(k-q)/k}$$

is finite. Note that this is the same inequality as (2.6) in [1] except for notation. Equality condition can be given easily as above.

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Representing sequences on parabolic Bergman spaces

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H を $n+1$ 次元実 Euclid 空間 $\mathbb{R}^{n+1}(n \geq 1)$ の上半空間とする. すなわち,

$$H = \mathbb{R}^n \times \mathbb{R}_+ = \{(x,t) \in \mathbb{R}^{n+1}; x \in \mathbb{R}^n, t > 0\}.$$

$0 < \alpha \leq 1$ に対し, $L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha$ とする. ここで, $\partial_t = \partial/\partial t$, Δ_x は x に関する Laplacian である. また, H 上の連続関数 u が $L^{(\alpha)}$ 調和であるとは, 超関数の意味で $L^{(\alpha)}u = 0$ となるときをいう. $W^{(\alpha)}$ は $L^{(\alpha)}$ の基本解を表す.

$1 \leq p < \infty$, $\lambda > -1$ に対して, 放物型 Bergman 空間 $\boldsymbol{b}^p_\alpha(\lambda)$ を次のように定義する.

$$\boldsymbol{b}^p_\alpha(\lambda) := \{u \in C(H); L^{(\alpha)}\text{調和}, \|u\|_{L^p(\lambda)} := \left(\int_H |u(x,t)|^p t^\lambda dV(x,t)\right)^{1/p} < \infty\}.$$

ここで dV は Lebesgue 体積測度を表す. 注意として, $\boldsymbol{b}^p_\alpha(\lambda)$ はノルム $\|\cdot\|_{L^p(\lambda)}$ に関して Banach 空間となる. 特に $p=2$ のとき, $\boldsymbol{b}^2_\alpha(\lambda)$ はヒルベルト空間となる. $\alpha = 1/2$ のとき, $\boldsymbol{b}^p_{1/2}(\lambda)$ は調和 Bergman 空間となる.

κ を実数とする. t に関する fractional 微分作用素 (Riemann-Liouville operator) を $\mathcal{D}^\kappa_t = (-\partial_t)^\kappa$ とする. この $\mathcal{D}^\kappa_t$ を κ 次の fractional 微分と呼ぶことにする.

本研究の目的は, 放物型 Bergman 空間上の representing sequence について研究することである. 放物型 Bergman 空間上の representing sequence とは, 放物型 Bergman 関数の離散的な表現公式を与える H 内の点列である (定義 2). ところで, [2] では放物型 Bergman 空間上の再生公式を基本解 $W^{(\alpha)}$ の fractional 微分によって与えている. この再生公式は, 積分による連続的な表現公式である. 我々はこの再生公式との比較として, 放物型 Bergman 関数の離散的な表現公式に興味を持ち, 研究を進めている. 本講演では, 放物型 Bergman 関数の離散的な表現公式を満たす representing sequence に関する結果を述べた.

はじめに, 放物型 Bergman 空間上の再生公式を紹介する.

定理 A. $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$ とする. また, $\kappa > \frac{\lambda+1}{p}$ とする. このとき,

$$u(x,t) = C_\kappa \int_H u(y,s) \mathcal{D}^\kappa_t W^{(\alpha)}(x-y, t+s) s^{\kappa-1} dV(y,s) \tag{1.1}$$

が任意の $u \in \boldsymbol{b}^p_\alpha(\lambda)$ について成り立つ. ここで, $C_\kappa = 2^\kappa/\Gamma(\kappa)$, $\Gamma(\cdot)$ はガンマ関数を表す. 特に, $p=1$ かつ $\kappa = \lambda + 1$ の場合でも, 再生公式 (1.1) は成り立つ.

定理 A をもとに, 次の作用素を定義する.

定義 1. $0 < \alpha \leq 1$, $1 < p < \infty$, $\lambda > -1$, κ を実数とする. また $\mathbb{X} = \{(x_j, t_j)\}$ を H 内の点列, $\{\eta_j\}$ を実数列とする. このとき, 作用素 $U^\kappa_{p,\mathbb{X}}$ を次のように定義する.

$$U^\kappa_{p,\mathbb{X}}(\{\eta_j\})(x,t) = \sum_j \eta_j t_j^{\frac{n}{2\alpha}+\kappa-\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}} \mathcal{D}^\kappa_t W^{(\alpha)}(x-x_j, t+t_j), \quad (x,t) \in H.$$

放物型 Bergman 空間上の representing sequence を次のように定義する.

定義 2. $0 < \alpha \leq 1, 1 < p < \infty, \lambda > -1,\ \kappa$ を実数とする. また, $\mathbb{X} = \{(x_j, t_j)\}$ を H 内の点列とする. $\mathbb{X}$ が次の 2 条件を満たすとき, $\boldsymbol{b}_\alpha^p(\lambda)$–representing sequence であると呼ぶ：

(1) $U_{p,\mathbb{X}}^\kappa$: $\ell^p \to \boldsymbol{b}_\alpha^p(\lambda)$, 有界.

(2) 任意の $u \in \boldsymbol{b}_\alpha^p(\lambda)$ に対して, H 上で $u = U_{p,\mathbb{X}}^\kappa(\{\eta_j\})$ を満たす $\{\eta_j\} \in \ell^p$ が存在する.

$\boldsymbol{b}_\alpha^p(\lambda)$–representing sequence $\{(x_j, t_j)\}$ を調べるために, 次の α–parabolic cylinder と呼ばれる集合を導入する. $0 < \delta < 1$, $(x, t) \in H$ とする.

$$S_\delta^{(\alpha)}(x,t) := \left\{ (y,s) \in H, |x-y| < \left(\frac{2\delta}{1-\delta^2}t\right)^{\frac{1}{2\alpha}}, \frac{1-\delta}{1+\delta}t \leq s < \frac{1+\delta}{1-\delta}t \right\}.$$

点列 $\{(x_j, t_j)\} \subset H$ が δ–separated であるとは, $S_\delta^{(\alpha)}(x_j, t_j) \cap S_\delta^{(\alpha)}(x_i, t_i) = \phi\ (j \neq i)$ となるときをいう。また, 点列 $\{(x_j, t_j)\} \subset H$ が次の 2 条件を満たすとき, δ–lattice であると呼ばれている：

(1) $\bigcup_j S_\delta^{(\alpha)}(x_j, t_j) = H$.

(2) $0 < \varepsilon < \delta$ となる ε が存在して, $\{(x_j, t_j)\}$ は ε–separated である.

次が本講演における主定理である.

定理 1. $0 < \alpha \leq 1, 1 < p < \infty, \lambda > -1,\ \kappa > \frac{\lambda+1}{p}$ とする. このとき, 次を満たすような $0 < \delta_0 < 1$ が存在する：$0 < \delta \leq \delta_0$ に対して, $\mathbb{X} = \{(x_j, t_j)\}$ が δ–lattice ならば, 次が成り立つ：

(1) $U_{p,\mathbb{X}}^\kappa$ は ℓ^p から $\boldsymbol{b}_\alpha^p(\lambda)$ への有界線形作用素となる. 特に, δ に依存する定数 $C > 0$ が存在して, $\|U_{p,\mathbb{X}}^\kappa(\{\eta_j\})\|_{L^p(\lambda)} \leq C\|\{\eta_j\}\|_{\ell^p}$ が任意の $\{\eta_j\} \in \ell^p$ で成り立つ.

(2) 任意の $u \in \boldsymbol{b}_\alpha^p(\lambda)$ に対して, H 上で $u = U_{p,\mathbb{X}}^\kappa\{\eta_j\}$ を満たす $\{\eta_j\} \in \ell^p$ が存在する. また, δ_0 に依存する定数 $C > 0$ が存在して, $\|\{\eta_j\}\|_{\ell^p} \leq C\|u\|_{L^p(\lambda)}$ を満たす.

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Carleson measures on parabolic Hardy spaces
(joint work with Noriaki Suzuki)

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概要

上半空間において放物型作用素を導入した Hardy 空間を放物型 Hardy 空間とよぶ．ここではこの関数空間に対応した Carleson 測度を導入し，Carleson 不等式が成立するための必要十分条件がこの測度によって表現できることを紹介する．

1　放物型 Hardy 空間

$(n+1)$ 次元 Euclidean 空間を $\mathbb{R}^{n+1}$，その上半空間を $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n, t > 0\}$ $(n \geq 1)$, 上半空間の境界 $\mathbb{R}^n$ における中心が x で半径が r の球を $B(x,r) := \{y \in \mathbb{R}^n \mid |x-y| < r\}$ で表す．また，α について特に指定のないときには $0 < \alpha \leq 1$ であるものとする．なお，本稿では C は定数を表すが，たとえ同一の行にあっても必ずしも同じとは限らない．

$\mathbb{R}^{n+1}_+$ 上において，放物型作用素 $L^{(\alpha)}$ は，

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha$$

で定義される．ここで Δ_x は x に関する Laplace 作用素である（cf. [NSS]）．$1 < p \leq \infty$ として，上半空間における放物型作用素を導入した Hardy 空間 $h^p_\alpha(\mathbb{R}^{n+1}_+)$ を

$$h^p_\alpha(\mathbb{R}^{n+1}_+) = \{u\colon \mathbb{R}^{n+1}_+ \text{上連続関数} \mid L^{(\alpha)}u = 0, \|u\|_{h^p_\alpha} < \infty\}$$

と定義する．ここでノルムは古典的な Hardy 空間と同様に

$$\|u\|_{h^p_\alpha} := \begin{cases} \sup\limits_{t>0} \left(\displaystyle\int_{\mathbb{R}^n} |u(x,t)|^p dx \right)^{\frac{1}{p}} & (1 < p < \infty), \\ \sup\limits_{(x,t)\in\mathbb{R}^{n+1}_+} |u(x,t)| & (p = \infty), \end{cases}$$

と定める．以降，$h^p_\alpha(\mathbb{R}^{n+1}_+)$ を h^p_α と略記し，単に放物型 Hardy 空間と呼ぶことにする．放物型 Hardy 空間は Banach 空間になる．

$1 \leq p < \infty$ として，上半空間において Bergman 空間に放物型作用素を導入した空間 b^p_α は

$$b^p_\alpha := \{u\colon \mathbb{R}^{n+1}_+ \text{上連続関数} \mid L^{(\alpha)}u = 0, \|u\|_{L^p(\mathbb{R}^{n+1}_+, dV)} < \infty\}$$

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で定義され，放物型 Bergman 空間と呼ばれる．ここで dV は体積要素による測度である．b^p_α は Banach 空間であり，$b^p_{1/2}, b^p_1$ はそれぞれ L^p である調和関数全体，熱方程式の解全体に一致する．

放物型作用素 $L^{(\alpha)}$ の基本解 $W^{(\alpha)}$ は，

$$W^{(\alpha)}(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t|\xi|^{2\alpha}} e^{ix\cdot\xi}\, d\xi$$

で定義される．ここで $x\cdot\xi$ は x と ξ との内積であり，$|\xi| = (\xi\cdot\xi)^{1/2}$ である．$W^{(\alpha)}(x,t)$ は $\alpha = 1/2, 1$ のときそれぞれ Poisson 核，熱核に一致する：

$$W^{(1/2)}(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(|x|^2+t^2)^{\frac{n+1}{2}}},\ W^{(1)}(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

$\mathbb{R}^{n+1}_+$ における任意のコンパクト集合 K に対して，$\inf_{(x,t)\in K} W^{(\alpha)}(x,t) > 0$ である．古典的な積分核と同様に基本解は正規性を持つ：任意の $t>0$ に対して，

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x,t)dx = 1. \tag{1}$$

また，基本解は半群の性質も持つ：任意の $0<s<t$ に対して，

$$W^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t-s) W^{(\alpha)}(y,s)dy. \tag{2}$$

非負整数 k と multi-index $\beta = (\beta_1, \ldots, \beta_n)$ (各 β_i は非負整数) に対して $|\beta| := \beta_1 + \cdots + \beta_n$ とするとき，関数 f の多重微分を

$$\partial_x^\beta \partial_t^k f(x,t) := \frac{\partial^{|\beta|+k}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n} \partial t^k} f(x,t)$$

で表すことにする．簡単な変数変換により，

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t) = t^{-\frac{n+|\beta|}{2\alpha}+k} (\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-\frac{1}{2\alpha}}x, 1) \tag{3}$$

であることがわかる．次の命題は基本的な評価である．

補題 1 ([NSS], Lemma 3.1)．ある定数 $C>0$ が存在して，$(x,t) \in \mathbb{R}^{n+1}_+$ について，

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| \le C \frac{t^{1-k}}{(t+|x|^{2\alpha})^{\frac{n+|\beta|}{2\alpha}+1}}$$

である．

補題 1 により，例えば

$$W^{(\alpha)}(x,t) \le C \frac{t}{(t+|x|^{2\alpha})^{\frac{n}{2\alpha}+1}} \tag{4}$$

および

$$\left|\frac{\partial}{\partial x} W^{(\alpha)}(x,t)\right| \le C \frac{t}{(t+|x|^{2\alpha})^{\frac{n+1}{2\alpha}+1}} \tag{5}$$

であることがわかる．

$f \in L^p(\mathbb{R}^n)$ について

$$P^{(\alpha)}[f](x,t) := \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)f(y)dy$$

とおく．$W^{(1/2)}$ は Poisson 核であったので，$P^{(1/2)}[f]$ は f の Poisson 積分である．放物型 Hardy 空間に属する関数と境界上の関数について以下のことがわかる．

命題 2. $1 < p \leq \infty$ として，h^p_α に属する関数 u に対して，$u = P^{(\alpha)}[f]$ を満たす関数 $f \in L^p(\mathbb{R}^n)$ がただ一つ存在する．逆に，関数 $f \in L^p(\mathbb{R}^n)$ に対して $P^{(\alpha)}[f] \in h^p_\alpha$ である．

2 測度の導入

放物型 Hardy 空間に対応した Carleson 測度として T_τ-Carleson 測度を提唱する．

定義 3. $0 < \alpha \leq 1$, μ を $\mathbb{R}^{n+1}_+$ 上正 Borel 測度，$\tau > 0$ とする．このとき，ある定数 $C > 0$ が存在して，

$$\mu(T^{(\alpha)}(x,t)) \leq Ct^{(\frac{n}{2\alpha}+1)\tau} \tag{6}$$

が満たされているとき，μ を（$L^{(\alpha)}$ に関する）T_τ-Carleson 測度と呼ぶ．ここで，

$$T^{(\alpha)}(x,t) := \{(y,s) \in \mathbb{R}^{n+1}_+ \mid |y-x|^{2\alpha} + s \leq t\}$$

である．また，

$$\kappa^{(\alpha)}_\tau[\mu] = \sup_{(x,t)\in\mathbb{R}^{n+1}_+} \frac{\mu(T^{(\alpha)}(x,t))}{t^{(\frac{n}{2\alpha}+1)\tau}}$$

とおき，T_τ-Carleson 定数と呼ぶ．

定義 3 において，$\alpha = 1/2$ とすると従来の Carleson measure およびその定数になる．次の定理は上半空間における測度の特徴づけを Carleson 不等式が成立するための必要十分条件として行えることを示すものである．

定理 4. $1 < p \leq q < \infty$, μ を $\mathbb{R}^{n+1}_+$ 上の正 Borel 測度とする．このとき，放物型 Hardy 空間において Carleson 不等式が成立する，すなわち，ある定数 $C > 0$ が存在して $u \in h^p_\alpha$ に対して，次の不等式

$$\|u\|_{L^q(\mathbb{R}^{n+1}_+, d\mu)} \leq C\|u\|_{h^p_\alpha}$$

が常に成立するための必要十分条件は，$\tau = \frac{n}{2\alpha} \cdot \frac{q}{p}/(\frac{n}{2\alpha}+1)$ として μ が T_τ-Carleson 測度であることである．

注意 5. 定理 4 において測度に T_τ-Carleson 測度を仮定するとき，Carleson 不等式の定数部分は $C = C_{\alpha,p,q,n}(\kappa^{(\alpha)}_\tau[\mu])^{1/q}$ と表されるが，$2 < p = q < \infty$ のときに限り $C_{\alpha,p,q,n}$ は $p(=q)$ と n によらずに取れる．すなわち，$C = C_\alpha(\kappa^{(\alpha)}_\tau[\mu])^{1/p}$ と表される．証明するには Lorentz 空間に関する補間定理を使うが，その方法はここでは省略する．

放物型 Bergman 空間においても Carleson 不等式についての結果が得られている[*1]．こちらは必要十分条件として τ-Carleson 測度が提唱されている．

[*1] $p > q$ のときの放物型 Bergman 空間における Carleson 不等式成立のための必要十分条件は既に知られている．ただし，$p \leq q$ のときとはその条件は同じではない．

定義 6 ([NSY]). μ を $\mathbb{R}^{n+1}_+$ 上正 Borel 測度，$\tau > 0$ とする．このとき，ある定数 $C > 0$ が存在して，

$$\mu(Q^{(\alpha)}(x,t)) \le Ct^{(\frac{n}{2\alpha}+1)\tau} \tag{7}$$

が満たされているとき，μ を（$L^{(\alpha)}$ に関する）τ-Carleson 測度と呼ぶ．ここで，$Q^{(\alpha)}(x,t)$ は α-Carleson box と呼ばれ，

$$Q^{(\alpha)}(x,t) := \{(y_1, y_2, \ldots, y_n, s) \in \mathbb{R}^{n+1}_+ \mid t \le s \le 2t, |y_i - x_i| \le 2^{-1}t^{\frac{1}{2\alpha}}, i = 1, 2, \ldots, n\}$$

と定義する．

定理 7 ([NSY]). $1 < p \le q < \infty$，μ を $\mathbb{R}^{n+1}_+$ 上正 Borel 測度とする．このとき，放物型 Bergman 空間において Carleson 不等式が成立する，すなわち，ある定数 $C > 0$ が存在して $u \in b^p_\alpha$ に対して，次の不等式

$$\|u\|_{L^q(\mathbb{R}^{n+1}_+, d\mu)} \le C\|u\|_{b^p_\alpha}$$

が常に成立するための必要十分条件は，μ が $\frac{q}{p}$-Carleson 測度であることである．

T_τ-Carleson measure と τ-Carleson measure との関係は以下のようになる．

命題 8. $\frac{n}{2\alpha}/(\frac{n}{2\alpha}+1) < \tau$ のとき，μ が T_τ-Carleson measure であることと μ が τ-Carleson measure であることとは同値である．

なお，$\frac{n}{2\alpha}/(\frac{n}{2\alpha}+1) \ge \tau$ のときには，μ が T_τ-Carleson measure であることと μ が τ-Carleson measure であることとは異なる．特に，T_τ-Carleson measure は τ-Carleson measure であるが，逆の真偽は不明である．

3 定理 4 の証明

ここでは放物型 Hardy 空間における Carleson 不等式が成立するための必要十分条件に関する定理 4 の証明を行う．

開集合 $E(\subset \mathbb{R}^n)$ に対して，

$$\widehat{E} := \{(x,t) \in \mathbb{R}^{n+1}_+ \mid B(x, t^{\frac{1}{2\alpha}}) \subseteq E\}$$

とおく．証明の見通しを良くするため，定理 4 の T_τ-Carleson 測度と同値である次の測度条件を考える：

$$\mu(\widehat{B(x, t^{\frac{1}{2\alpha}})}) \le Ct^{\frac{n}{2\alpha}\frac{q}{p}}. \tag{8}$$

まず，Carleson 不等式が成立するときに測度の持つ性質を調べる．$(x,t) \in \mathbb{R}^{n+1}_+$ を固定し，$u(y,s) = W^{(\alpha)}(x-y, t+s)$ とする．

Carleson 不等式の左辺について評価する．

$$\|W^{(\alpha)}(x-\cdot, t+\cdot)\|^q_{L^q(\mathbb{R}^{n+1}_+, d\mu)} \ge \int_{\widehat{B(x,t^{\frac{1}{2\alpha}})}} (t+s)^{-\frac{n}{2\alpha}q} \left| W^{(\alpha)}\left(\frac{x-y}{(t+s)^{\frac{1}{2\alpha}}}, 1\right)\right|^q d\mu(y,s).$$

である．2 つ目の等式は (3) による．$\widehat{B(x, t^{\frac{1}{2\alpha}})}$ の決め方により $0 \le \dfrac{|x-y|}{(t+s)^{\frac{1}{2\alpha}}} < 1$ である．よって，基本解の下からの評価を用いることが出来，

$$\|W^{(\alpha)}(x-\cdot, t+\cdot)\|^q_{L^q(\mathbb{R}^{n+1}_+, d\mu)} \ge Ct^{-\frac{n}{2\alpha}q}\mu(\widehat{B(x, t^{\frac{1}{2\alpha}})})$$

であるから，

$$Ct^{-\frac{n}{2\alpha}}\left(\mu(\widehat{B(x,t^{\frac{1}{2\alpha}})})\right)^{1/q} \leq \|W^{(\alpha)}(x-\cdot,t+\cdot)\|_{L^q(\mathbb{R}^{n+1}_+,d\mu)} \tag{9}$$

である．次に右辺の評価をする．補題 1 と変数変換により，

$$\|W^{(\alpha)}(x-\cdot,t+\cdot)\|^p_{h^p_\alpha} \leq C\sup_{s>0}(t+s)^{\frac{n}{2\alpha}(1-p)}\int_0^\infty \frac{\eta^{\frac{n}{2\alpha}-1}}{(1+\eta)^{(\frac{n}{2\alpha}+1)p}}d\eta$$

であり，$p>1$ より $\displaystyle\int_0^\infty \frac{\eta^{\frac{n}{2\alpha}-1}}{(1+\eta)^{(\frac{n}{2\alpha}+1)p}}d\eta<\infty$ および $\dfrac{n}{2\alpha}(1-p)\leq 0$ であるから，

$$\|W^{(\alpha)}(x-\cdot,t+\cdot)\|_{h^p_\alpha} < Ct^{\frac{n}{2\alpha}(\frac{1}{p}-1)} \tag{10}$$

が得られる．

Carleson 不等式が成立するという仮定と (9)(10) の評価より結論が得られる．

次に，T_τ-Carleson 測度の条件を課したとき，$1<p\leq q$ のもとでは Carleson 不等式が成立することを示す．

$u\in h^p_\alpha$ について，

$$\Gamma(x)=\{(y,s)\in\mathbb{R}^{n+1}_+ \mid |y-x|<s^{\frac{1}{2\alpha}}\}$$
$$u^*(x)=\sup_{(y,s)\in\Gamma(x)}|u(y,s)|$$

とおく．$\lambda>0$ として $E_\lambda:=\{x\in\mathbb{R}^n \mid u^*(x)>\lambda\}$ とすると，u^* が下半連続であることから E_λ は開集合であり，

$$E_\lambda=\bigcup_{Q\in\mathfrak{F}}Q$$

と Whitney 分解できる（cf. [S]）．ここで，$\mathfrak{F}$ は $\mathbb{R}^n$ 内の cube 全体の集合であり，各 Q は共通部分を持たず，$\mathrm{diam}(Q)$ を Q の diameter として，

$$C_1\mathrm{diam}(Q)\leq \mathrm{dist}(Q,\partial E_\lambda)\leq C_2\mathrm{diam}(Q) \tag{11}$$

と定数 C_1,C_2 がとれる．$G_\lambda:=\{(x,t)\in\mathbb{R}^{n+1}_+ \mid |u(x,t)|>\lambda\}$ とすると $G_\lambda\subset\widehat{E_\lambda}$ が成立する．また，C_3Q は Q と中心を共通とし diameter が C_3 倍の cube を表すものとしたとき，$\widehat{E_\lambda}\subset\bigcup_{Q\in\mathfrak{F}}(\widehat{C_3Q})$ が成立する．よって，

$$\mu(G_\lambda)\leq\mu(\widehat{E_\lambda})\leq\sum_{Q\in\mathfrak{F}}\mu(\widehat{C_3Q})\leq C_0\sum_{Q\in\mathfrak{F}}|C_3Q|^{q/p}\leq C\sum_{Q\in\mathfrak{F}}|Q|^{q/p}\leq C|E_\lambda|^{q/p}$$

である．この評価により，

$$\begin{aligned}
\|u\|^q_{L^q(\mathbb{R}^{n+1}_+,d\mu)} &= \int_{\mathbb{R}^{n+1}_+}|u(x,t)|^q\,d\mu(x,t)\\
&= \int_0^\infty \mu(\{|u(x,t)|^q>\lambda\})\,d\lambda\\
&= q\int_0^\infty \mu(G_\lambda)\lambda^{q-1}d\lambda\\
&\leq Cq\int_0^\infty |E_\lambda|^{q/p}\lambda^{q-1}d\lambda
\end{aligned}$$

である．次の 2 つの評価式

$$\int_0^\infty |E_\lambda|^{q/p}\lambda^{q-1}d\lambda \le \sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}}|E_{2^k}|^{q/p}2^{(k+1)(q-1)}d\lambda \le \sum_{k=-\infty}^{\infty}2^{(k+1)q}|E_{2^k}|^{q/p}$$

$$\int_{\mathbb{R}^n}|u^*(x)|^p dx = p\int_0^\infty |E_\lambda|\lambda^{p-1}d\lambda \ge p\sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}}|E_{2^{k+1}}|2^{k(p-1)}d\lambda = p\sum_{k=-\infty}^{\infty}2^{kp}|E_{2^{k+1}}|$$

より，$p \le q$ に注意すると

$$\left(\int_0^\infty |E_\lambda|^{q/p}\lambda^{q-1}d\lambda\right)^{p/q} \le \sum_{k=-\infty}^{\infty}2^{(k+1)p}|E_{2^k}| \le \frac{2^{2p}}{p}\int_{\mathbb{R}^n}|u^*(x)|^p dx$$

が得られる．ゆえに，

$$\|u\|_{L^q(\mathbb{R}^{n+1}_+,d\mu)} \le Cq^{1/q}\left(\int_0^\infty |E_\lambda|^{q/p}\lambda^{q-1}d\lambda\right)^{1/q} \le 4Cp^{-1/p}q^{1/q}\|u^*\|_{L^p(\mathbb{R}^n)} \tag{12}$$

である．$f \in L^p(\mathbb{R}^n)\ (1 < p \le \infty)$ に対して，

$$Mf(x) := \sup_{r>0}\frac{1}{r^n}\int_{B(x,r)}|f(y)|\,dy$$

は f の極大関数と呼ばれ，$Mf \in L^p(\mathbb{R}^n)$ である．$u = P^{(\alpha)}[f]$ を満たすように関数 $f \in L^p(\mathbb{R}^n)$ をとるとき，ある定数 $C > 0$ が存在して $u^* \le C\,Mf$ とできる．これと (12) とにより，ある定数 $C > 0$ が存在して $\|u\|_{L^q(\mathbb{R}^{n+1}_+,d\mu)} \le C\|Mf\|_{L^p(\mathbb{R}^n)}$ とできる．極大関数のノルムの評価として，$C_{p,n} = 2\left(\dfrac{5^n p}{p-1}\right)^{1/p}$ とすると，$\|Mf\|_{L^p(\mathbb{R}^n)} \le C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}$ が成立することが知られている（cf. [SW]）．したがって，

$$\|u\|_{L^q(\mathbb{R}^{n+1}_+,d\mu)} \le C\|f\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{h^p_\alpha}$$

であり定理は証明される．

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Szegö's Theorem and Uniform Algebras
(Application for Bergman Space)

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Let A be a uniform algebra on X, τ a nonzero complex homomorphism on A and m a representing measure on X. For a nonnegative function W in $L^1(X, dm)$, put

$$S(W) = \inf_{g \in A_\tau} \int_X |1-g|^2 W dm$$

where $A_\tau = \{f \in A : \tau(f) = 0\}$. $S(W)$ is called a Szegö's infimum. In this lecture, we are interested in when A is a disc algebra on $X = \{x \in \mathbb{C} : |z| \leq 1\}$. When $m = d\theta/2\pi$, Szegö [4] proved that $S(W) = \exp \int_0^{2\pi} \log W(e^{i\theta}) d\theta/2\pi$. In this lecture, we consider $S(W)$ when $m = rdrd\theta/2\pi$.

§1 一般の uniform algebra

X を compact Hausdorff space、A を X 上の uniform algebra とする。また τ を A 上の零でない complex homomorphism、m を X 上の τ の表現測度とする。すなわち

$$\tau(f) = \int_X f dm \quad (f \in$$

が成立している。

$1 \leq p \leq \infty$ のとき、$L^p(X, dm)$ を X 上の m に関する Lebesgue space とする。$W \in L^1(X, dm)$ は $W \geq 0$ a.e.m on X とするとき、

$$S(W) = \inf_{g \in A_\tau} \int_X |1-g|^2 W dm$$

とする。ここで A_τ は τ の kernel である。$S(W)$ は Szegö の infimum と呼ばれる。

$p = 1, 2$ のとき $H^p(X, dm)$ を A の $L^p(X, dm)$ における閉包とし、$W \in L^1(X, dm)$ かつ $W \geq 0$ に対して $H^p(X, Wdm)$ を A の $L^p(X, Wdm)$ における閉包とする。

$h \in H^2(X, dm)$ のとき h が generator とは hA が $H^2(X, dm)$ で稠密なときをいう。

m が複素解析における Jensen の不等式とは任意の $f \in H^p(X, dm)$ に対して

$$\int_X \log |f| dm \geq \log \left| \int_X f dm \right|$$

が成立することをいう。このとき m は Jensen 測度という。実解析における Jensen の不等式により、一般に

$$\int_X W dm \geq \exp \int_X \log W dm$$

が成立するが、m が Jensen 測度ならば

$$\int_X W dm \geq S(W) \geq \exp \int_X \log W dm$$

が成立する。

$W = W_1 W_2, W_j \geq 0$ かつ $W_j \in L^1(X, dm)$ $(j = 1, 2, \cdots)$ とし、$W_1 dm / \int_X W_1 dm$ は τ の表現測度でありかつ Jensen 測度ならば

$$\int_X W_1 W_2 dm \geq S(W_1 W_2) \geq \int_X W_1 dm \exp \int_X \log W_2 dm \geq \exp \int_X \log W_1 W_2 dm$$

が成立する。

$\boxed{\text{定理1}}$

$S(W) = \int_X W dm$ が成立する必要かつ十分条件は $W dm / \int_X W dm$ は τ の表現測度となることである。

$\boxed{\text{定理2}}$

m を Jensen 測度とする。

(1) $W = |h|^2$ かつ $h \in H^2(X, dm)$ が generator ならば

$$S(W) = \exp \int_X \log W dm > 0 \text{ 。}$$

(2) $S(W) = \exp \int_X \log W dm > 0$ ならば $k \in H^2(X, dm)$ と正の整数 c が存在して $|k|^2 W = c$ が成立する。

$\boxed{\text{定理3}}$

$W_1 \in L^\infty(X, dm)$ とする。

$W_1 dm / \int_X W_1 dm$ は τ の表現測度、$W_2 = |h|^2$ かつ $h \in H^2(X, dm)$ は generator とすると、

$$S(W) = \int_X W_1 dm \exp \int_X \log W_2 dm > 0$$

が成立する。

問題 I： 定理 2 の (2) でもし $S(W^{-1}) = \exp \int_X \log W^{-1} dm > 0$ も成立するならば、$h = k^{-1}$ は $H^2(X, dm)$ の generator となる。これはその仮定なしには成立するだろうか。

問題 II： 定理 3 は $W_1 \in L^1(X, dm)$ として成立するだろうか。

問題 III： 定理 3 と関係して、一般の W が定理 3 における様に分解できるだろうか。

§2 disc algebra

$D = \{z \in \mathbb{C} : |z| < 1\}$、$\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ かつ $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ とする。A を $\bar{D}$ 上で連続かつ D で正則な関数の全体とするとき、A は $X = \bar{D}$ 上の uniform algebra となる。A は disc algebra と呼ばれる。$\tau(f) = f(0)$ $(f \in A)$ とすると、τ に対する $X = \bar{D}$ 上の表現測度は沢山ある。例えば、$dm = d\sigma(r)d\theta/2\pi$ かつ σ は $[0,1]$ 区間上の確率測度のとき m は τ に対する表現測度である。

Szegö は 1920 年に $\sigma(r) = \delta_r = 1$ のとき、すなわち $dm = d\theta/2\pi$ のときに、Szegö の infimum を決定した。すなわち、$W \in L^1(\bar{D}, d\theta/2\pi)$ が非負ならば

$$S(W) = \exp \int_0^{2\pi} \log W(e^{i\theta}) d\theta/2\pi$$

となることを証明した。この結果は Szegö の定理として有名である。この定理は $S(W) > 0$ である必要十分条件は $\log W \in L^1(\bar{D}, d\theta/2\pi)$ であることを示している。よって問題 I∼III を全て肯定的に解いている。

我々は $d\sigma(r) = 2rdr$ のとき、すなわち $dm = rdrd\theta/\pi$ のとき Szegö の infimum を考察する。特に問題 I∼III を考察する。

定理 1 は適用できる。よって $Wdm/\int_X Wdm$ が τ の表現測度となる W を決定することは興味ある。$W = |Q|^2$ かつ $Q \in H^2(\bar{D}, dm)$ のとき、$Wdm/\int_X Wdm$ が表現測度である必要十分条件は Q が Bergman 空間における inner 関数となることである。

定理 2 の (2) に関係する問題 I は肯定的に解くことができる。

定理 3 と関係する問題 II は $W_1(re^{i\theta}) = W_1(r)$ のときには肯定的に解けるが一般的にはまだ解けない。

定理 3 と関係する問題 III については、Alman-Richter-Sundberg [1] によって、$W = |F|^2$ かつ $F \in H^2(\bar{D}, rdrd\theta/\pi)$ のときに解かれた。これは Bergman 空間における inner-outer 分解に他ならない。

§3 注意

注意 1 Szegö の infimum は disc algebra (Szegö), Dirichlet algebra (Wermer), hypo-Dirichlet algebra (Gamelin), bidisc algebra (Nakazi) 等に研究された。この §2 では Szegö の考えた disc algebra の場合を再び考察している。

注意 2 μ が $\bar{D}$ 上の確率測度とするとき、$\{z^n\}_{n=-\infty}^{\infty}$ は $L^2(\mu)$ で確率過程となるが、supp$\mu \subseteq \partial D$ のときを除いて定常確率過程ではない。supp$\mu \subseteq \partial D$ のとき Wiener は Szegö の定理を一般的な定常確率過程の予測理論に用いた。§2 では必ずしも定常ではない確率過程を扱っていることになる。

注意 3 $dm = rdrd\theta/\pi$ のとき、次の不等式は二つの Jensen の不等式より示すことができる。

$$\begin{aligned}\int_0^1 2rdr \int_0^{2\pi} W(re^{i\theta})d\theta/2\pi \geq S(W)& \\ \geq \int_0^1 2rdr \left(\exp \int_0^{2\pi} \log W(re^{i\theta})d\theta/2\pi\right)& \\ \geq \exp \int_0^1 2rdr \int_0^{2\pi} \log W(re^{i\theta})d\theta/2\pi \text{。}&\end{aligned}$$

$S(W) = \int_0^1 2rdr \left(\exp \int_0^{2\pi} \log W(re^{i\theta})d\theta/2\pi\right) > 0$ となる W を描くことは、定理 3 の特別の場合である。実際、必要条件としては $W(re^{i\theta}) = \phi(r)|h(re^{i\theta})|^2$、$\phi \in L^1([0,1], dr)$、$\exp \int_0^{2\pi} \log|h(re^{i\theta})|d\theta/2\pi = |h(0)|$ かつ h は D 上正則となることを示すことができるが、逆は示せなかった。h が $H^2(\bar{D}, \phi dm)$ の generator ならば逆を示すことができる。

注意 4 この講演の内容は参考文献の [4] にあり、印刷中である。

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