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and E. Yokoyama

Sapporo, 2010

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## PREFACE

It is well-known that a precise analysis of evolving patterns and structures is not only mathematically interesting but also important for understanding complicated phenomena in science and technology. In this minisemester, we focus on surface evolution problems arising in materials science and image processing. Our goals are not only to solve mathematical problems but also to identify new mathematical questions and directions which will have impact on other disciplines. Therefore the minisemester will involve researchers with diverse backgrounds.

The first intensive activity period (July 21-23) focuses on the viscosity methods. While they are relatively new in the field of nonlinear PDE, they have played an important role in the rigorous study of applications like materials science, image processing, optimizations, games, and finance. Viscosity methods are rather flexible. Not only they work with fully nonlinear equations but also they have very strong stability properties under seemingly singular asymptotic processes.

The second intensive activity period (July 26-30) focuses on the mathematics of crystal growth. For applications in materials science, we need improved models and understanding of the many mechanisms by which crystals grow or evolve - for example the mechanisms by which steps and dislocations nucleate and move. These problems are keys to the development of both theory and technology. New experimental methods permit us to go beyond the fundamental Burton-Cabrera-Frank theory, calling for improvement and extension of the theory.

The third intensive activity period (August 2-6) focuses on the singular diffusion. Singular diffusion arises naturally in both the evolution of crystals and the total variation flow. The former topic comes from materials science while the latter stems from image processing. Our understanding of these singular second-order diffusion problems has improved a lot in recent years, through analysis based for example on variational methods, viscosity solutions, and stochastic techniques.

All three activity periods are interdisciplinary in nature. We made a conscious effort to gather not only mathematicians but also experimental and theoretical physicists and material scientists working in this area. We hope to identify promising directions for future development through the interaction of a diverse group of leading specialists. This is the sincere wishes of the organizers and we hope the meetings are fruitful ones for all participants.

Y. Giga, H. Ishii, T. Funaki, Y. Tonegawa  
R. V. Kohn, P. Rybka

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Part 1

## 研究集会

「Viscosity methods and nonlinear PDE」



# 研究集会「Viscosity methods and nonlinear PDE」

組織委員: 儀我美一, 利根川吉廣, P. E. Souganidis, 石井仁司

Organizers: Y. Giga, Y. Tonegawa, P. E. Souganidis, H. Ishii

Period (期間): July 21 - 23, 2010

Venue (場所): Room 3-205, Faculty of Science Building #3, Hokkaido University  
北海道大学理学部3号館3-205室

## July 21, 2010 (Wednesday)

- 9:30-10:30 Panagiotis E. Souganidis (Univ. of Chicago)  
(Lecture I) An introduction to the theory of homogenization for first and second order elliptic pde in random environments
- 11:00-12:00 小池茂昭 (埼玉大学) Shigeaki Koike (Saitama Univ.)  
On the weak Harnack inequality for fully nonlinear PDEs with unbounded ingredients
- 12:05-13:05 Ki-Ahm Lee (Seoul National Univ.)  
Homogenizations of partial differential equations with oscillating boudary data
- 14:30-15:30 Claude L. Bris (Ecole Nationale des Ponts et Chaussees)  
(Lecture I) Stochastic Homogenization: An introduction to some recent variants and to numerical approaches
- 16:00-17:00 Andrzej Swiech (Georgia Institute of Technology)  
A Hamilton-Jacobi-Bellman equation in the space of probability measures
- 17:05-18:05 三上敏夫 (広島大学) Toshio Mikami (Hiroshima Univ.)  
The Knothe-Rosenblatt processes by the zero-noise limit of a class of stochastic controls

## July 22, 2010 (Thursday)

- 09:30-10:30 Panagiotis E. Souganidis (Univ. of Chicago)  
(Lecture II) An introduction to the theory of homogenization for first and second order elliptic pde in random environments
- 11:00-12:00 Inwon C. Kim (UCLA)  
Degenerate diffusion equation with a drift potential
- 12:05-13:05 Piotr Rybka (Warsaw Univ.)  
A Hamilton-Jacobi with discontinuous Hamiltonian arising from weighted mean curvature flow

- 14:30-15:30 Claude L. Bris (Ecole Nationale des Ponts et Chaussees)  
(Lecture II) Stochastic Homogenization: An introduction to some recent variants and to numerical approaches
- 16:00-17:00 Qing Liu (Univ. of Tokyo)  
A game-theoretic proof for fattening of motion by curvature and applications
- 17:05-18:05 Jean-M. Roquejoffre (Univ. Paul Sabatier)  
Travelling waves for the 3D forced mean curvature motion
- 18:30- Party at Enreiso (懇親会、エンレイソウ)

July 23, 2010 (Friday)

- 09:30-10:30 Panagiotis E. Souganidis (Univ. of Chicago)  
(Lecture III) An introduction to the theory of homogenization for first and second order elliptic pde in random environments
- 11:00-12:00 市原直幸 (広島大学) Naoyuki Ichihara (Hiroshima Univ.)  
Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients
- 12:05-13:05 Scott N. Armstrong (Univ. of Chicago)  
Liouville theorems for supersolutions of elliptic equations in unbounded domains
- 14:30-15:30 Claude L. Bris (Ecole Nationale des Ponts et Chaussees)  
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Existence of viscosity solutions for a nonlocal equation modelling polymer crystal growth
- 17:05-18:05 儀我美一 (東京大学) Yoshikazu Giga (Univ. of Tokyo)  
A few problems for Hamilton-Jacobi equations arising from step motions of crystal growth

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AN INTRODUCTION TO THE THEORY OF HOMOGENIZATION  
FOR FIRST AND SECOND ORDER ELLIPTIC PDE IN RANDOM  
ENVIRONMENTS

Panagiotis E. Souganidis  
Department of Mathematics  
University of Chicago

I will discuss the recently developed theory of homogenization for first and second order elliptic pde in random environments. I will begin reviewing the classical theory, then I will discuss the main difficulties one faces when going from periodic to random environment and I will present the basic results for Hamilton-Jacobi and fully nonlinear second order equations. I will conclude with some results about rates of convergence.

# On the weak Harnack inequality for fully nonlinear PDEs with unbounded ingredients

Shigeaki Koike (Saitama University)

In this talk, we discuss the weak Harnack inequality for  $L^p$ -viscosity supersolutions of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m \geq -f(x) \quad \text{in } \Omega,$$

where  $\mu \in L^q_+(\Omega)$  ( $q > n$ ), and  $f \in L^p_+(\Omega)$  ( $q \geq p > p_0$  for some  $p_0 \in [n/2, n)$ ) are given functions,  $\Omega \subset \mathbf{R}^n$  a domain, and  $m \geq 1$  a constant. Fixing  $0 < \lambda \leq \Lambda$ , we use the following Pucci operators:

$$\mathcal{P}^+(X) := \max\{-\text{trace}(AX) \mid \lambda I \leq A \leq \Lambda I\} \quad (X \in S^n)$$

The interior/boundary weak Harnack inequality is a key tool to establish Hölder continuity of  $L^p$ -viscosity solutions, strong maximum principle, maximum principle in unbounded domains, and also the local maximum principle.

Motivated by a pioneering work [1] by Caffarelli, the notion of  $L^p$ -viscosity solutions was introduced by Caffarelli-Crandall-Kocan-Świąch [3] to study fully nonlinear PDEs. Our aim is to establish the weak Harnack inequality even when  $f$  belongs to a wider space than  $L^n$ ,  $\mu$  is unbounded, and the superlinear growth in  $Du$  is considered.

After [3], Fok [4] first studied  $L^p$ -viscosity solutions of fully nonlinear PDEs with unbounded ingredients. In [6], we extend some results in [4], e.g. the ABP maximum principle and the strong solvability, by which we mean the existence of  $L^p$ -strong solutions. Under some restriction, we also obtain the ABP maximum principle in case when  $m > 1$  (see [5], [6]).

In order to prove the weak Harnack inequality, we follow Caffarelli's argument (cf. [2]). However, to this end, we need some modifications because we deal with unbounded coefficients. For instance, we cannot use "explicit" fundamental solutions associated with Pucci operators.

We note that Sirakov [10] obtained the Hölder continuity of  $L^p$ -viscosity solutions without the weak Harnack inequality when  $m = 1$ ,  $q > n$  and  $p \geq n$ .

Moreover, to establish the weak Harnack inequality in the superlinear case (i.e.  $m > 1$ ), we obtain the strong solvability of Pucci extremal PDEs with superlinear terms in [8]. We will mention the local maximum principle in [9].

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# HOMOGENIZATIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH OSCILLATING BOUNDARY DATA

Ki-ahm Lee  
Department of Mathematical Sciences  
Seoul National University

In this talk we are going to discuss Linear or Nonlinear Partial Differential Equations with Oscillating Dirichlet or Neumann Boundary data. First, we are going to overview the possible issues through examples: lower dimensional character of the boundary and the possibility of multiple limits. And then we will discuss the homogenization processes of the limits. Finally, we will find out the averaging of the boundary data with respect to the Nonlinear PDE.

# Stochastic Homogenization: An introduction to some recent variants and to numerical approaches

Claude Le Bris<sup>a</sup>

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The series of lectures will overview some recent contributions on several theoretical aspects and numerical approaches in stochastic homogenization. After an introduction to the elementary aspects of stochastic homogenization, a variant of the classical theory will be presented. It has been introduced in [6], and further studied in [11, 16]. The relation between stochastic homogenization problems and other multiscale problems in materials science [7] will be emphasized. Several numerical approaches will be presented: some for genuinely stochastic problems (where variance issues are a practical concern and need to be addressed for efficiency purposes, [9, 10, 13]), and some for approximations of stochastic problems when the random character is only a perturbation of a deterministic model [1, 2, 3, 12, 14]. Most of these contributions are summarized in [15, 4]. Further details will be available in [5, 11, 17].

The series of talks will be centered around a simple, *linear elliptic* situation, since the focus is

- (a) deliberately elementary,
- (b) more on the stochastic setting, its relation with situations relevant for applications and its requirements in terms of numerical approaches, than
- (c) on the complexity of the equation itself.

However, given the topic of the conference, one purpose of the series of talks will be to draw connections with more elaborate cases involving *nonlinear equations*. Several suggestions for application to the nonlinear setting of the ideas and techniques currently developed in the simple situation will be given. The idea is to hopefully spark interest in, and foster strong interaction with the audience, expert in *nonlinear equations*.

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## **A Hamilton-Jacobi-Bellman equation in the space of probability measures**

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We will present results on the Hamilton-Jacobi-Bellman equation related to an optimal control problem for a mixed flow of Hamiltonian and gradient type in the space of probability measures. The optimal control problem is motivated by a stochastic interacting particle model giving the 2-D Navier-Stokes equations in the vorticity formulation as mean-field equation. It can be interpreted as an optimal control problem for an abstract gradient-Hamiltonian flow in the Wasserstein space. We will introduce an appropriate definition of viscosity solution for the associated Hamilton-Jacobi-Bellman equation in the space of probability measures and we will present the basic tools needed to deal with such equations. We will discuss how to show a comparison theorem and how to prove that the value function is a viscosity solution. This is a joint work with Jin Feng.

# The Knothe-Rosenblatt processes by the zero-noise limit of a class of stochastic controls

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The Knothe-Rosenblatt rearrangement plays a crucial role in many fields, e.g., the Brunn-Minkowski inequality and statistics (see [3], [9] and the references therein) and can be constructed by a system of the Monge-Kantorovich problems for conditional probabilities (see [5]). In this talk we introduce the stochastic analogue of the Knothe-Rosenblatt rearrangement which we call the **Knothe-Rosenblatt process**, via the stochastic control (see [2] for stochastic control). We give a characterization of the Knothe-Rosenblatt process via the convergence result similar to below. Let  $d \geq 2$  and  $P_0, P_1 \in \mathcal{M}_1(\mathbf{R}^d) :=$  the set of all Borel probability measures on  $\mathbf{R}^d$  with a weak topology. Suppose that  $P_0(dx)/dx$  exists and that  $\int_{\mathbf{R}^d} |x|^2 (P_0(dx) + P_1(dx))$  is finite. Then for any  $\varepsilon > 0$ , the minimizer of the following is unique (see e.g. [8], [10], [11]):

$$\inf \left\{ \int_{\mathbf{R}^d \times \mathbf{R}^d} \sum_{k=1}^d \varepsilon^{2(k-1)} |y_k - x_k|^2 \mu(d\mathbf{x}_d d\mathbf{y}_d) \right. \quad (0.1)$$

$$\left. \mu(d\mathbf{x}_d \times \mathbf{R}^d) = P_0(d\mathbf{x}_d), \mu(\mathbf{R}^d \times d\mathbf{y}_d) = P_1(d\mathbf{y}_d) \right\},$$

where  $\mathbf{x}_d := (x_i)_{1 \leq i \leq d} \in \mathbf{R}^d$ . Suppose, in addition, that  $P_1$  does not have a pure point part. Then the unique minimizer of (0.1) weakly converges to  $P_0(d\mathbf{x}_d) \delta_{T_{KR}(\mathbf{x}_d)}(d\mathbf{y}_d)$  as  $\varepsilon \rightarrow 0$ , where  $\delta_x(dy)$  denotes the delta measure on  $\{x\}$  and  $T_{KR}$  denotes the Knothe-Rosenblatt rearrangement (see [1]).

Let  $\mathcal{A}_d$  denote the set of all  $\mathbf{R}^d$ -valued, continuous semimartingales  $\{X(t)\}_{0 \leq t \leq 1}$  on a (possibly different) complete filtered probability space such that there exists a Borel measurable  $\beta_X : [0, 1] \times C([0, 1]) \mapsto \mathbf{R}^d$  for which

- (i)  $\omega \mapsto \beta_X(t, \omega)$  is  $\mathcal{B}(C([0, t]))_+$ -measurable for all  $t \in [0, 1]$ ,
- (ii)  $X(t) = X(0) + \int_0^t \beta_X(s, X) ds + W_X(t)$  ( $0 \leq t \leq 1$ ).

Here  $\mathcal{B}(C([0, t]))_+ := \cap_{s > t} \mathcal{B}(C([0, s]))$ ,  $\mathcal{B}(C([0, t]))$  and  $W_X$  denote the Borel  $\sigma$ -field of  $C([0, t])$  and an  $(\mathcal{F}_t^X)$ -Brownian motion, respectively, and  $\mathcal{F}_t^X := \sigma[X(s) : 0 \leq s \leq t]$  (see e.g. [4]).

**Definition 0.1** ([6]). *Let  $1 < k \leq d$ ,  $1 \leq d_1 < \dots < d_k = d$  and  $L_i : [0, 1] \times \mathbf{R}^{d_i} \times \mathbf{R}^{d_i - d_{i-1}} \mapsto [0, \infty)$  be Borel measurable ( $i = 1, \dots, k$ ). For  $P_0, P_1 \in \mathcal{M}_1(\mathbf{R}^d)$ ,*

$\{\mathbf{X}_d(t) = (X_i(t))_{1 \leq i \leq d}\}_{0 \leq t \leq 1} \in \mathcal{A}_d$  is called the **Knothe-Rosenblatt process** (for Brownian motion) if for all  $i = 1, \dots, k$ ,

$$\beta_{\mathbf{X}_d}(t, \mathbf{X}_d)_{d_i} = \beta_{\mathbf{X}_{d_i}}(t, \mathbf{X}_{d_i}), \quad (0.2)$$

and  $\mathbf{X}_{d_i}$  is the unique minimizer of

$$\begin{aligned} & V_i(P_{0,d_i}, P_{1,d_i} | \mathbf{X}_{d_{i-1}}) \\ := & \begin{cases} \inf\{E[\int_0^1 L_1(t, \mathbf{Y}_{d_1}(t); \beta_{\mathbf{Y}_{d_1}}(t, \mathbf{Y}_{d_1}))dt] | \mathbf{Y}_{d_1} \in \mathcal{A}_{d_1}, \\ \quad P\mathbf{Y}_{d_1}(0)^{-1} = P_{0,d_1}, P\mathbf{Y}_{d_1}(1)^{-1} = P_{1,d_1}\} =: V_1(P_{0,d_1}, P_{1,d_1}) & (i = 1), \\ \inf\{E[\int_0^1 L_i(t, \mathbf{Y}_{d_i}(t); \beta_{\mathbf{Y}_{d_i}}(t, \mathbf{Y}_{d_i})_{d_{i-1}, d_i})dt] | \mathbf{Y}_{d_i} \in \mathcal{A}_{d_i}, \\ \quad P\mathbf{Y}_{d_i}(0)^{-1} = P_{0,d_i}, P\mathbf{Y}_{d_i}(1)^{-1} = P_{1,d_i}, P\mathbf{Y}_{d_{i-1}}^{-1} = P\mathbf{X}_{d_{i-1}}^{-1}\} & (1 < i \leq k), \end{cases} \end{aligned} \quad (0.3)$$

where  $P_{t,d_i}(d\mathbf{x}_{d_i}) := P_t(d\mathbf{x}_{d_i} \times \mathbf{R}^{d-d_i})$  and  $\beta_{\mathbf{Y}_{d_i}}(t, \mathbf{Y}_{d_i}) =: (\beta_{\mathbf{Y}_{d_i}}(t, \mathbf{Y}_{d_i})_{d_{j-1}, d_j})_{j=1}^i \in \Pi_{j=1}^i \mathbf{R}^{d_j - d_{j-1}}$  for  $\mathbf{Y}_{d_i} \in \mathcal{A}_{d_i}$ . When it is not confusing, we do not mention the dependence of  $\{\mathbf{X}_d(t)\}_{0 \leq t \leq 1}$  on  $P_0, P_1$  and  $\{L_i\}_{1 \leq i \leq k}$ .

**Remark 0.1** (i) In the definition of  $\mathcal{A}_d$ , the diffusion matrix is an identity. One can consider the case where it is a variable matrix. In particular, the Knothe-Rosenblatt process generally depends on  $P_0, P_1, \{L_i\}_{1 \leq i \leq k}$  and the diffusion matrix. In this talk, for the sake of simplicity, we do not consider the case where the diffusion matrix is a variable matrix. (ii) If  $L_1 = |\mathbf{u}_{d_1}|^2$ , then  $\mathbf{X}_{d_1}$  is the h-path process for Brownian motion, provided  $V_1(P_{0,d_1}, P_{1,d_1})$  is finite. In this sense, **the Knothe-Rosenblatt process can be considered as a generalization of the h-path process.**

Let  $1 \leq k \leq d, 1 \leq d_1 < \dots < d_k = d$  and  $P_0, P_1 \in \mathcal{M}_1(\mathbf{R}^d)$ . For  $i = 2, \dots, k$  and  $\varepsilon > 0$ ,

$$\begin{aligned} V_i^\varepsilon(P_{0,d_i}, P_{1,d_i}) & := \inf \left\{ E \left[ \sum_{j=1}^i \varepsilon^{j-1} \int_0^1 L_j(t, \mathbf{Y}_{d_j}(t); \beta_{\mathbf{Y}_{d_i}}(t, \mathbf{Y}_{d_i})_{d_{j-1}, d_j}) dt \right] \right. \\ & \left. P\mathbf{Y}_{d_i}(0)^{-1} = P_{0,d_i}, P\mathbf{Y}_{d_i}(1)^{-1} = P_{1,d_i}, \mathbf{Y}_{d_i} \in \mathcal{A}_{d_i} \right\}. \end{aligned} \quad (0.4)$$

We prove that a minimizer of  $V_k^\varepsilon(P_0, P_1)$  converges to the Knothe-Rosenblatt process as  $\varepsilon \rightarrow 0$ . We first consider the case where  $k = 2$  under the following assumption.

(A.1)<sub>j</sub>. (i)  $L_j \in C([0, 1] \times \mathbf{R}^{d_j} \times \mathbf{R}^{d_j - d_{j-1}} : [0, \infty))$ . (ii) For  $(t, x) \in [0, 1] \times \mathbf{R}^{d_j}$ ,  $L_j(t, x; \cdot)$  is strictly convex.

(A.2)<sub>j</sub>. There exists  $\gamma > 1$  such that

$$\liminf_{u \in \mathbf{R}^{d_j - d_{j-1}}, |u| \rightarrow \infty} \frac{\inf\{L_j(t, x; u) : (t, x) \in [0, 1] \times \mathbf{R}^{d_j}\}}{|u|^\gamma} > 0. \quad (0.5)$$

(A.3)<sub>j</sub>.

$$\Delta L_j(\varepsilon_1, \varepsilon_2) := \sup \frac{L_j(t, x; u) - L_j(s, y; u)}{1 + L_j(s, y; u)} \rightarrow 0 \quad (0.6)$$

as  $\varepsilon_1, \varepsilon_2 \rightarrow 0$ , where the supremum is taken over all  $(t, x)$  and  $(s, y) \in [0, 1] \times \mathbf{R}^{d_j}$  for which  $|t - s| \leq \varepsilon_1$ ,  $|x - y| < \varepsilon_2$  and over all  $u \in \mathbf{R}^{d_j - d_{j-1}}$ .

**Theorem 0.1** *Suppose that  $k = 2$  and that (A.1)<sub>j</sub> - (A.3)<sub>j</sub> hold for  $j = 1, 2$ . Then for any  $P_0, P_1 \in \mathcal{M}_1(\mathbf{R}^d)$  for which the Knothe-Rosenblatt process  $\{\mathbf{X}_d(t)\}_{0 \leq t \leq 1}$  exists, a minimizer  $\{\mathbf{X}_d^\varepsilon(t)\}_{0 \leq t \leq 1}$  of  $V_2^\varepsilon(P_0, P_1)$  exists and weakly converges to  $\{\mathbf{X}_d(t)\}_{0 \leq t \leq 1}$  as  $\varepsilon \rightarrow 0$ . In addition,*

$$\lim_{\varepsilon \rightarrow 0} E \left[ \int_0^1 L_1(t, \mathbf{X}_{d_1}^\varepsilon(t); \beta_{\mathbf{X}_d^\varepsilon}(t, \mathbf{X}_d^\varepsilon)_{d_1}) dt \right] = V_1(P_{0, d_1}, P_{1, d_1}), \quad (0.7)$$

$$\lim_{\varepsilon \rightarrow 0} E \left[ \int_0^1 L_2(t, \mathbf{X}_d^\varepsilon(t); \beta_{\mathbf{X}_d^\varepsilon}(t, \mathbf{X}_d^\varepsilon)_{d_1, d}) dt \right] = V_2(P_0, P_1 | \mathbf{X}_{d_1}). \quad (0.8)$$

We consider the case where  $k > 2$ . We introduce new assumptions.

(A.1)<sub>j</sub>'. (i)  $L_j \in C([0, 1] \times \mathbf{R}^{d_j} \times \mathbf{R}^{d_j - d_{j-1}} : [0, \infty))$ . (ii) For  $(t, x) \in [0, 1] \times \mathbf{R}^{d_j}$ ,  $L_j(t, x; \cdot)$  is twice differentiable and there exists  $C_L > 0$  such that

$$\inf \{ \langle D_u^2 L_j(t, x; u) z, z \rangle \mid z \in \mathbf{R}^{d_j - d_{j-1}}, |z| = 1 \} \geq C_L,$$

for all  $(t, x, u) \in [0, 1] \times \mathbf{R}^{d_j} \times \mathbf{R}^{d_j - d_{j-1}}$ , where  $D_u^2 := (\partial^2 / \partial u_i \partial u_j)_{i, j=1}^{d_j - d_{j-1}}$ .

(A.4)<sub>j</sub>.  $L_j(t, x; o)$  and  $D_u L_j(t, x; u)$  are bounded on  $[0, 1] \times \mathbf{R}^{d_j} \times B_R$  for all  $R > 0$ , where  $B_R := \{u \in \mathbf{R}^{d_j - d_{j-1}} \mid |u| \leq R\}$ .

**Remark 0.2** ([7]). *Suppose that (A.1)<sub>j</sub> - (A.4)<sub>j</sub> hold for  $j = 1, \dots, k$ . Set*

$$H^\varepsilon(t, \mathbf{x}_{d_k}; \mathbf{z}_{d_k}) := \sup \left\{ \langle \mathbf{z}_{d_k}, \mathbf{u}_{d_k} \rangle - \sum_{j=1}^k \varepsilon^{j-1} L_j(t, \mathbf{x}_{d_j}; \mathbf{u}_{d_{j-1}, d_j}) \mid \mathbf{u}_{d_k} \in \mathbf{R}^{d_k} \right\}.$$

For the minimizer  $\{\mathbf{X}_d^\varepsilon(t)\}_{0 \leq t \leq 1}$  of  $V_k^\varepsilon(P_0, P_1)$ , there exists a sequence  $\{\varphi_n\}_{n \geq 1}$  of solutions to

$$\frac{\partial \varphi^\varepsilon(t, x)}{\partial t} + \frac{1}{2} \Delta \varphi^\varepsilon(t, x) + H^\varepsilon(t, x; D_x \varphi^\varepsilon(t, x)) = 0 \quad ((t, x) \in (0, 1) \times \mathbf{R}^d) \quad (0.9)$$

such that

$$b_{\mathbf{X}_d^\varepsilon}(t, \mathbf{X}_d^\varepsilon(t)) = \lim_{n \rightarrow \infty} D_z H^\varepsilon(t, \mathbf{X}_d^\varepsilon(t); D_x \varphi_n(t, \mathbf{X}_d^\varepsilon(t))) \quad a.s.. \quad (0.10)$$

Let  $\{\mathbf{X}_d(t)\}_{0 \leq t \leq 1}$  denote the Knothe-Rosenblatt process. Consider the following PDEs: for all  $i = 1, \dots, k$ ,

$$\begin{aligned} \frac{\partial v_i(t, \mathbf{x}_{d_i})}{\partial t} + \frac{1}{2} \Delta_{d_i} v_i(t, \mathbf{x}_{d_i}) + \langle \nabla_{d_{i-1}} v_i(t, \mathbf{x}_{d_i}), b_{\mathbf{x}_{d_{i-1}}}(t, \mathbf{x}_{d_{i-1}}) \rangle \\ + H_i(t, \mathbf{x}_{d_i}; \nabla_{d_{i-1}, d_i} v_i(t, \mathbf{x}_{d_i})) = 0 \end{aligned} \quad (0.11)$$

$((t, \mathbf{x}_{d_i}) \in (0, 1) \times \mathbf{R}^{d_i})$ , where  $b_{\mathbf{x}_{d_0}} := 0$  and

$$H_i(t, \mathbf{x}_{d_i}; z) := \sup \{ \langle z, u \rangle - L_i(t, \mathbf{x}_{d_i}; u) \mid u \in \mathbf{R}^{d_i - d_{i-1}} \} \quad (z \in \mathbf{R}^{d_i - d_{i-1}}).$$

If there exists a classical solution  $v_i(t, \mathbf{x}_{d_i})$  to (0.11) which attains the maximum in the duality theorem for  $V_i$  (see [5]), then the following holds:

$$b_{\mathbf{x}_{d_i}}(t, \mathbf{x}_{d_i})_{d_{i-1}, d_i} = D_z H_i(t, \mathbf{x}_{d_i}; \nabla_{d_{i-1}, d_i} v_i(t, \mathbf{x}_{d_i})), \quad (0.12)$$

where  $D_z = (\partial / \partial z_j)_{j=1}^{d_i - d_{i-1}}$ .

$P_{1, d_1}^\varepsilon := P_{1, d_1}$  and  $\mathbf{X}_{1, d_1}^\varepsilon := \mathbf{X}_{d_1}$  which is the unique minimizer of  $V_1(P_0, P_1)$ , provided it exists. For  $i = 2, \dots, k$ , let  $(\mathbf{Z}_{i, d_i}^\varepsilon)_{d_{i-1}} := \mathbf{X}_{i-1, d_{i-1}}^\varepsilon$  and

$$d(\mathbf{Z}_{i, d_i}^\varepsilon)_{d_{i-1}, d_i}(t) = b_{\mathbf{x}_{d_i}}(t, \mathbf{Z}_{i, d_i}^\varepsilon(t))_{d_{i-1}, d_i} dt + d(W_{\mathbf{Z}_{i, d_i}^\varepsilon})_{d_{i-1}, d_i}(t), \quad (0.13)$$

and  $P_{1, d_i}^\varepsilon := P Z_{i, d_i}^\varepsilon(1)^{-1}$ . Let  $\mathbf{X}_{i, d_i}^\varepsilon$  denote the unique minimizer of  $V_i^\varepsilon(P_0, P_{1, d_i}^\varepsilon)$ , provided it exists. Then we have

**Theorem 0.2** *Suppose that (A.1) $_j$ ', (A.3) $_j$  and (A.4) $_j$  hold for all  $j = 1, \dots, k$ . Then for any  $P_0, P_1 \in \mathcal{M}_1(\mathbf{R}^d)$  for which the Knothe-Rosenblatt process  $\{\mathbf{X}_d(t)\}_{0 \leq t \leq 1}$  exists and for which there exists a solution  $v_i(t, \mathbf{x}_{d_i}) \in C_b^{1,2}([0, 1] \times \mathbf{R}^{d_i})$  to (0.11) such that (0.12) holds for all  $i = 1, \dots, k$ , a minimizer  $\{\mathbf{X}_{k, d}^\varepsilon(t)\}_{0 \leq t \leq 1}$  of  $V_k^\varepsilon(P_0, P_{1, d}^\varepsilon)$  exists and converges to  $\{\mathbf{X}_d(t)\}_{0 \leq t \leq 1}$  in the sense of relative entropy as  $\varepsilon \rightarrow 0$ :*

$$H(P(\mathbf{X}_{k, d_k}^\varepsilon)^{-1} | P(\mathbf{X}_{d_k})^{-1}) := \frac{1}{2} E \left[ \int_0^1 |\beta_{\mathbf{x}_{k, d_k}^\varepsilon}(t, \mathbf{X}_{k, d_k}^\varepsilon) - b_{\mathbf{x}_{d_k}}(t, \mathbf{X}_{k, d_k}^\varepsilon(t))|^2 dt \right] \rightarrow 0 \quad (0.14)$$

$(\varepsilon \rightarrow 0)$ . For  $j = 1, \dots, k$ , we also have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} E \left[ \int_0^1 L_j(t, \mathbf{X}_{k, d_k}^\varepsilon(t)_{d_j}; \beta_{\mathbf{x}_{k, d_k}^\varepsilon}(t, \mathbf{X}_{k, d_k}^\varepsilon)_{d_{j-1}, d_j}) dt \right] \\ = E \left[ \int_0^1 L_j(t, \mathbf{X}_{d_j}(t); b_{\mathbf{x}_{d_j}}(t, \mathbf{X}_{d_j}(t))_{d_{j-1}, d_j}) dt \right]. \end{aligned} \quad (0.15)$$

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# Degenerate diffusion equation with a drift potential

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Consider a  $C^2$  function  $\Phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\Phi(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ . Also consider a nonnegative, continuous function  $\rho_0(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  which has compact support  $\Omega_0$ . We study the following equation:

$$(PMED) \quad \rho_t = \Delta(\rho^m) + \nabla \cdot (\rho \nabla \Phi)$$

with  $m > 1$  and with initial data  $\rho_0(x)$ . Note that the equation preserves the mass ( $\int \rho(\cdot, t) dt \equiv C$ ).

In pressure variable  $u = \frac{m}{m-1} \rho^{m-1}$ , the equation becomes

$$(PMED - P) \quad u_t = (m-1)u \Delta y + |\nabla u|^2 + \nabla u \cdot \nabla \Phi + (m-1)u \Delta \Phi.$$

Note that in this form the free boundary velocity  $V = \frac{u_t}{|\nabla u|}$  on  $\Gamma = \partial\{u > 0\}$  is given by

$$V = |\nabla u| + \frac{\nabla u}{|\nabla u|} \cdot \nabla \Phi.$$

Formally the solution of (PMED) is a gradient flow with the energy

$$(E) \quad E(\rho) = \int \rho^m(x) + \rho(x)\Phi(x) dx$$

with respect to the Wasserstein distance. Using this observation, it was shown in [CJMTU] (and earlier in [BH]) that  $\rho$  uniformly converges to one of the equilibrium solutions of (E),

$$\rho_\infty = (C - \Phi)_+$$

(or combination of different components of above function) in  $L^1$ -norm as time goes to infinity. Further, when  $\Phi$  is convex, one can show that the convergence is exponential.

On the other hand, maximum-principle type arguments hold for the equation. Therefore it is possible to introduce the notion of viscosity solutions for (PME - D), which is equivalent

to the notion of weak solutions defined in  $L^1$ -theory of the solutions. Using the maximum principle type arguments, we show that the free boundary of the solution uniformly converges to that of the equilibrium solution, with an exponential rate when  $\Phi$  is convex, as time goes to infinity.

Besides comparison principle, we have used estimates on the size of the solution with respect to the local  $L^1$ -norm, which can be summarized as below:

**Lemma -1.1.** *Let us fix  $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ . Then there exists  $k, k'$  and  $C$  depending on  $m, n, \Phi$  and  $\sup \rho_0$  such that, for  $0 < a < 1$ , the following holds:*

(a) If

$$a^{-n} \int_{B_a(x_0)} u(\cdot, t_0) dx \geq a^k.$$

Then  $u(\cdot, t_0 + a) \geq a^{k'}$  in  $B_a(x_0)$ .

(b) If

$$\int_{B_2(0)} u(\cdot, t) dx \leq a \text{ for } t_1 \leq t \leq t_2 := t_1 + \log(1/a)$$

Then  $u(\cdot, t_2) \leq Ca^{1/n+1}$  on  $B_1(0)$ .

These estimates are obtained via perturbation arguments and previously obtained regularity results on degenerate diffusion equations. It is possible to improve aforementioned estimates to obtain the Hölder regularity on the solutions of *(PMED)* for  $m \leq 2$ , however Hölder regularity for solutions of *(PMED)* with  $m > 2$  is still an open question.

As an application, we show a uniform convergence result on diffusion-aggregation equation

$$\rho_t = \Delta(\rho^m) + \nabla \cdot (\rho \nabla \Phi),$$

on a  $n$ -dimensional torus, with  $m = 2$ , where  $\Phi = u * K$  and  $K(x)$  is a  $C^2$ , radially symmetric potential.

This is joint work with L. Chayes, H. Lei and Y. Yao. ([KL], [CKY])

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## A Hamilton-Jacobi with discontinuous Hamiltonian arising from weighted mean curvature flow

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Our goal is to study the Hamilton-Jacobi equations

$$u_t + H(t, x, u, u_x) = 0 \quad \text{in } (0, T) \times \mathbb{R}, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}, \quad (1)$$

when  $H$  is discontinuous. There is a sizable amount of literature on this subject. These papers mostly deal with the case of  $H$  independent of  $u$ , so (1) is the heterogeneous eikonal equation, see [CH],[CR], [DE], [DZS], [T]. Much less is known when the Hamiltonian depends on  $u$ .

Our motivation comes from the singular weighted mean curvature (wmc) flow,

$$\beta V = \sigma + \kappa_\gamma \quad \text{on } \Gamma(t). \quad (2)$$

Here  $\kappa_\gamma$  is the weighted mean curvature and it has to be carefully interpreted, [GGR]. Formally, it is defined as

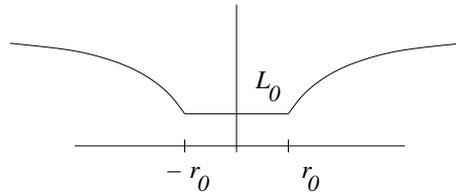
$$\kappa_\gamma = -\operatorname{div}_S (\nabla_X \gamma)(X)|_{X=\mathbf{n}(x)}, \quad (3)$$

so the wmc flow is a second order parabolic equation. When we consider a well-justified anisotropy function  $\gamma$  given by the following formula

$$\gamma(p_1, p_2) = |p_1| \gamma_\Lambda + |p_2| \gamma_R,$$

then it turns out that the (3) does not make any sense.

We want to consider a simple situation of a graph of a function of one variable having finite limits at infinity. This graph has a central facet,



When we properly interpret (3) for  $\gamma$  as above, then (1) becomes (2). Then two basic questions arise:

- (a) existence of solutions,
- (b) uniqueness of solutions.

The issue of existence is rather technical, in some interesting situations it relies essentially on the use of viscosity theory. The basic result was presented in [GGR]. In my talk I will briefly mention the main ideas, here I will omit the details, because I want to concentrate on uniqueness. More precisely we want to show a Comparison Principle for viscosity solutions to (1).

We will now present the properties of the Hamiltonian we study and we will state the main result. We consider the Hamiltonian  $H$  given by the following formula,

$$H(t, x, u, p) = \begin{cases} -\sigma(t, r^*(t), u)m(p), & \text{if } |x| < r_0(t), \\ -\sigma(t, x, u)m(p), & \text{if } |x| \geq r_0(t). \end{cases} \quad (4)$$

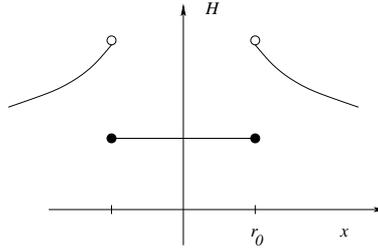
Function  $\sigma$  is even and increasing for  $x > 0$  and

$$0 < \frac{\partial \sigma}{\partial u}(t, x, u) \leq M. \quad (5)$$

We make the following assumption on the discontinuity line defined by the functions  $r_0$  and  $r^*$ :

(R1)  $r_0, r^*$  are bounded continuous functions on  $[0, T]$ , for all  $t \in [0, T]$   $r^*(t) > r_0(t)$  and  $\Gamma = \{(t, \pm r_0(t)) : t \in [0, T]\}$  is a Lipschitz curve.

This condition reflects our need to localize the discontinuity. It is in line with most of the literature. Thus, for each time instance, the graph of  $H$  looks like



The remaining properties of  $H$  that we need may be summarized as follows, please note that (H5) requires extra comments.

(H1) Hamiltonian  $H$  is lower semicontinuous in  $[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ ;

(H2)  $H$  is continuous away from  $\Gamma$  and it has a jump discontinuity at  $\Gamma$ .

(H3)  $H^*$  is continuous in  $G = \{(t, x) : |x| \geq r_0(t)\}$ , while  $H_*$  is continuous on the closure of  $([0, T] \times \mathbb{R}) \setminus G$ . Here  $H^*$  (resp.  $H_*$ ) is the standard *upper* (resp. *lower*) *semicontinuous envelope* of  $H$ .

(H4) For any  $\epsilon_1, \epsilon_2$  in  $\{-1, 1\}$  we have  $H(t, \epsilon_1 x, u, \epsilon_2 p) = H(t, x, u, p)$ .

(H5) Hamiltonian  $H$  is strictly increasing with respect to  $u$ , i.e. there is a positive  $h_0$ , such that the following inequality holds for all  $u_2, u_1, x, t$  and  $p$ ,

$$H(t, x, u_2, p) - H(t, x, u_1, p) \geq h_0(u_2 - u_1). \quad (6)$$

(H6) For all  $t, u$  and  $p$  function  $x \mapsto H(t, x, u, p)$  is decreasing for  $x > r_0(t)$ , moreover  $H(t, x, u, p) = H(t, r^*(t), u, p)$  for  $x \in [-r_0(t), r_0(t)]$ .

(H7)  $\lim_{\substack{|x| \rightarrow \infty \\ p \rightarrow 0}} H(t, x, u, p) = H^\infty \in C([0, T] \times \mathbb{R})$  locally uniformly with respect to  $(t, u) \in [0, T] \times \mathbb{R}$ , i.e.  $H^\infty$  does not depend upon  $p$ .

**Remarks.** Condition (H4) is just for convenience, but (H5) is essential. It is possible to convert  $H$  given by (4) into one satisfying (6), by means of the following change of variables  $v_{new} = e^{t\lambda} u_{old}$ , where  $\lambda = -2M$  and  $M$  is the constant appearing in (5).

Assumption (H7) stems from the fact that we do not impose explicit boundary condition, thus we have to control the behavior of the Hamiltonian and sub- super-solutions at space infinity. For this reason we introduce the following notion.

**Definition.** Let us suppose that  $H$  satisfies (H7). We shall say that a piecewise  $C^1$ -function  $w$  is a *supersolution at infinity* of (1) provided that  $w$  is a supersolution to (1), the following limits exist and they are uniform with respect to  $t \in [0, T]$ ,

$$w_t \rightarrow w_t^\infty, \quad w \rightarrow w^\infty, \quad w_x \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

and

$$w_t^\infty(t) + H^\infty(t, w^\infty(t)) \geq 0. \quad (7)$$

Here is our main result.

**Theorem.** Let us assume that a measurable function  $H$  satisfies (R1) and (H1–H7) and for  $u, v \in BUC([0, T] \times \mathbb{R})$  the following conditions are valid:

(a)  $v$  is a supersolution to (1),  $u$  is a subsolution to (1) and  $u(0, x) \leq v(0, x)$ .

(b)  $v$  is a piecewise  $C^1$ -function.

(c)  $v$  is a supersolution of (1) at infinity.

Then, for all  $t > 0$

$$u(t, x) \leq v(t, x).$$

The idea of the proof is to regularize  $H$ . Namely we define,

$$H^\delta(t, x, u, p) = \begin{cases} H(t, x, u, p) & |x| \geq r_0(t) + \delta, \\ (1 - \frac{\lambda}{\delta})H(t, r^*, u, p) + \frac{\lambda}{\delta}H(t, r_0 + \delta, u, p) & |x| = r_0(t) + \lambda, \lambda \in (0, \delta), \\ H(t, r^*(t), u, p) & |x| \leq r_0(t). \end{cases}$$

This implies that we have to shift the supersolution from the region where we change  $H$ . That is, we set

$$v^\delta(t, x) = \begin{cases} v(t, x - \delta) & \text{for } x > \delta, \\ v(t, 0) & \text{for } x \in [-\delta, \delta], \\ v(t, x + \delta) & \text{for } x < -\delta, \end{cases}$$

It is important to see that for any  $\epsilon > 0$ , one can choose  $\delta$  so that  $v^\delta(0, x) + \epsilon \geq u(0, x)$ .

Subsequently, we show that  $v^\delta + \epsilon$  is a supersolution while  $u$  is a subsolution to

$$d_t + H^\delta(t, x, d, d_x) = 0. \quad (8)$$

At this point we may apply the classical comparison to conclude that  $v^\delta(t, x) + \epsilon \geq u(t, x)$ . After passing to the limit with  $\epsilon$  our claim follows.

In order to complete the talk we will present the main points of construction of the viscosity solutions to (1), all details of this process can be found in [GGR].

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# A Game-theoretic Proof for Fattening of Motion by Curvature and Applications

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In this talk, we intend to present an application of the game method (e.g., [3], [2]) for the level-set mean curvature flow equation:

$$(1) \quad \begin{cases} \partial_t u - |\nabla u| \operatorname{div}(\frac{\nabla u}{|\nabla u|}) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, T) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases}$$

By comparing the optimal strategies of the game and its inverse one, we explain the fattening of level sets for examples of “figure eight” type, which are known to cause fat level sets instantly [1]. One of the advantages is that we can avoid general parabolic PDE theory, which is usually required in the rigorous proof of fattening.

Our interpretation can also be applied to the stationary Dirichlet boundary problem:

$$(2) \quad \begin{cases} -|\nabla U| \operatorname{div}(\frac{\nabla U}{|\nabla U|}) - 1 = 0 & \text{in } \Omega, \\ U(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

It is an open question whether a (weak) comparison principle of (2) holds for a general domain  $\Omega$  [3]. We show that the nonuniqueness of viscosity solutions of (2) is intimately connected, via games, with the fattening of positive mean curvature flow. We again take the example of a planar “figure eight” set to see the loss of comparison principle for (2) even in the weak sense.

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TRAVELLING WAVES FOR THE 3D FORCED MEAN  
CURVATURE MOTION

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We will explain how to construct travelling waves solutions for the forced mean curvature motion for graphs, that are asymptotic to those of the eikonal equation. The relevance of this construction for the unbalanced Allen-Cahn equation will also be discussed.

Joint work with R. Monneau and V. Roussier.

# Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients\*

Naoyuki Ichihara<sup>†</sup> (Hiroshima University)

The talk is concerned with the Cauchy problem for semi-linear parabolic equations of the form

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u + H(x, Du) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1)$$

where Hamiltonian  $H = H(x, p)$  and initial datum  $u_0$  satisfy the following assumptions:

**(A1)**  $H \in C_p^2(\mathbb{R}^{2N})$ , where  $C_p^k(\mathbb{R}^N)$  denotes the set of functions in  $C^k(\mathbb{R}^N)$  that are polynomially growing for any multi-index  $\alpha \in (\mathbb{Z}_+)^N$  with  $0 \leq |\alpha| \leq k$ .

**(A2)** There exists a  $g \in C_p(\mathbb{R}^N) := C_p^0(\mathbb{R}^N)$  such that

$$|D_x H(x, p)| \leq g(x)(1 + |p|^2), \quad (x, p) \in \mathbb{R}^{2N}.$$

**(A3)** There exist constants  $\kappa_1, \kappa_2 > 0$  such that

$$\kappa_1 |\eta|^2 \leq D_{pp}^2 H(x, p) \eta \cdot \eta \leq \kappa_2 |\eta|^2, \quad (x, p, \eta) \in \mathbb{R}^{3N}, \quad (2)$$

where  $D_{pp}^2 H(x, p)$  stands for the Hessian of  $H(x, p)$  with respect to  $p$ .

**(A4)** There exists a function  $\phi_0 \in C_p^3(\mathbb{R}^N)$  such that  $\lim_{|x| \rightarrow \infty} F[\phi_0](x) = -\infty$ , where  $F[\cdot]$  is defined by

$$F[\psi](x) := -\frac{1}{2}\Delta\psi(x) + H(x, D\psi(x)), \quad x \in \mathbb{R}^N, \quad \psi \in C^2(\mathbb{R}^N).$$

**(A5)** There exists a function  $\phi_1 \in C_p^3(\mathbb{R}^N)$  such that

$$\lim_{|x| \rightarrow \infty} (\phi_0 - \phi_1)(x) = \infty, \quad \inf_{x \in \mathbb{R}^N} (F[\phi_0](x) - F[\phi_1](x)) > -\infty.$$

**(B1)**  $u_0 \in \Phi_0 := \{v \in C_p(\mathbb{R}^N) \mid \inf_{\mathbb{R}^N} (v - \phi_0) > -\infty\}$ .

As a typical example satisfying (A1)-(A5), we have in mind Hamiltonians of the form

$$H(x, p) = \frac{1}{2}a(x)p \cdot p + b(x) \cdot p - V(x), \quad (x, p) \in \mathbb{R}^{2N},$$

with appropriate  $a \in C_p^\infty(\mathbb{R}^N, \mathbb{R}^N \otimes \mathbb{R}^N)$ ,  $b \in C_p^\infty(\mathbb{R}^N, \mathbb{R}^N)$  and  $V \in C_p^\infty(\mathbb{R}^N)$ .

Under these assumptions, we study the large time behavior of solutions of (1), specifically, convergence of the form

$$u(T, \cdot) - (\phi(\cdot) - \lambda T) \longrightarrow 0 \quad \text{in } C(\mathbb{R}^N) \text{ as } T \rightarrow \infty, \quad (3)$$

where  $\lambda$  is a real constant which represents the growth rate of the solution and  $\phi$  is a function on  $\mathbb{R}^N$  regarded as a stationary state of the normalized solution  $u(T, \cdot) + \lambda T$  as  $T \rightarrow \infty$ . Pair  $(\lambda, \phi)$  in (3) turns out to be a solution of the time-independent equation, or ergodic problem

$$-\frac{1}{2}\Delta\phi + H(x, D\phi) = \lambda \quad \text{in } \mathbb{R}^N. \quad (4)$$

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\*A part of this talk is based on a joint work with S.-J. Sheu (Academia Sinica, Taiwan).

<sup>†</sup>Supported in part by Grant-in-Aid for Young Scientists (B), No. 21740076, MEXT.

Asymptotic behavior of type (3) has been studied in [1, 2] for similar types of second-order parabolic equations. Paper [1] stays in the periodic setting, namely, equations are considered in the torus  $\mathbb{T}^N$  instead of  $\mathbb{R}^N$ , and convergence (3) is proved for more general, possibly time-inhomogeneous, quasi-linear parabolic equations (see [1, Theorem 4.1]). Literature [2] studies equations whose Hamiltonian is given by

$$H(x, p) = \alpha x \cdot p + H(p) - f(x), \quad \alpha > 0,$$

and obtain (3) under the assumption that  $f$  and  $u_0$  are globally Lipschitz on  $\mathbb{R}^N$  (see [2, Theorem 6.5]).

The principal difference between their works and ours lies in the growth of solutions as  $|x| \rightarrow \infty$ . Under their settings, solutions become globally Lipschitz continuous with respect to  $x$ . This fact especially leads to a uniform gradient bound on  $(0, \infty) \times \mathbb{R}^N$  of solutions. In particular, growth property for  $H(x, p)$  as  $|p| \rightarrow \infty$  does not affect the large time behavior of solutions. Contrary to their cases, solution  $u$  to (1) is locally Lipschitz in general and quadratic nonlinearity in  $p$  of  $H$  plays a crucial role in the large time behavior of  $u$ .

We now state our main theorems.

**Theorem 1.** *Assume (A1)-(A4). Then, there exists a unique real constant  $\lambda^*$  such that (4) with  $\lambda = \lambda^*$  has a solution  $\phi \in C^2(\mathbb{R}^N)$  in the class  $\Phi_0$ . Moreover, if  $\phi, \psi \in C^2(\mathbb{R}^N)$  are two solutions of (4) with  $\lambda = \lambda^*$ , then  $\phi - \psi$  is constant in  $\mathbb{R}^N$ .*

**Theorem 2.** *Assume (A1)-(A5) and (B1). Then, there exists a unique solution  $u \in C^{1,2}((0, \infty) \times \mathbb{R}^N) \cap C([0, \infty) \times \mathbb{R}^N)$  of (1) such that  $\inf_{0 \leq t \leq T} \inf_{x \in \mathbb{R}^N} (u(t, x) - \phi_0(x)) > -\infty$  for all  $T > 0$ .*

**Theorem 3.** *Assume (A1)-(A5) and (B1). Let  $u$  be the solution of (1), and let  $\lambda^*$  be the constant in Theorem 1. Suppose also that*

$$\{u(T, \cdot) + \lambda^* T \mid T > 1\} \text{ is bounded below on any compact subset of } \mathbb{R}^N. \quad (5)$$

*Then, convergence (3) holds for some solution  $\phi \in \Phi_0$  of (4).*

We emphasize here that Theorem 3 is not obvious at all since solutions of (4) admit ambiguity of additive constants. In fact,  $\phi$  in (3) depends on the choice of  $u_0$ .

Unfortunately, we do not know if (5) is always true. The following (A6) is a sufficient condition for the validity of (5).

**(A6)** There exist a function  $\phi_2 \in C_p^3(\mathbb{R}^N)$  such that  $\lim_{|x| \rightarrow \infty} F[\phi_2](x) = -\infty$  and

$$(\phi - \phi_0)(x) \leq \alpha(\phi - \phi_2)(x) + C, \quad x \in \mathbb{R}^N,$$

for some  $C > 0$  and  $0 < \alpha < \kappa_1/\kappa_2$ , where  $\kappa_1, \kappa_2 > 0$  are the constants in (A3).

**Theorem 4.** *Assume (A1)-(A5), (B1), and either (A6) or  $\kappa_1 = \kappa_2$  in (A3). Then, (5) is valid.*

To prove these theorems, we employ both analytical and probabilistic arguments.

**Remark.** Initial value problem (1) has a stochastic control interpretation. For each  $T > 0$  and  $x \in \mathbb{R}^N$ , we consider the following minimizing problem:

$$\begin{aligned} \text{Minimize} \quad & J(T, x; \xi) := E^x \left[ \int_0^T L(\xi_t, X_t^\xi) dt + u_0(X_T^\xi) \right], \\ \text{subject to} \quad & X_t^\xi = X_0 - \int_0^t \xi_s ds + W_t, \quad 0 \leq t \leq T, \end{aligned}$$

where  $L(x, \xi) := \sup_{p \in \mathbb{R}^N} (p \cdot \xi - H(x, p))$ ,  $\xi = (\xi_t)_{0 \leq t \leq T}$  is a given admissible control and  $W = (W_t)_{0 \leq t \leq T}$  denotes an  $N$ -dimensional standard Brownian motion. Then, value function  $u(T, x) := \inf_{\xi} J(T, x; \xi)$  is characterized under suitable assumptions as the unique solution to Hamilton-Jacobi-Bellman equation (1). From this point of view, it is natural, eventually indispensable, to consider Hamiltonians and initial data polynomially growing in  $x$  since linear growth condition excludes the so-called LQG (Linear Quadratic Gaussian) control. In the LQG case,  $H(x, p)$  and  $u_0(x)$  are quadratically growing both in  $x$  and  $p$ .

**Remark.** Papers [5, 6] deal with similar types of asymptotic problems in the context of mathematical finance. In those papers, more specific Hamilton-Jacobi-Bellman equations with constant initial data are discussed under slightly different types of assumptions. Concerning the large time behavior of solutions, they prove the following:

$$\frac{u(T, \cdot)}{T} \longrightarrow -\lambda, \quad u(T, \cdot) - u(T, 0) \longrightarrow \phi \quad \text{in } C(\mathbb{R}^N) \quad \text{as } T \rightarrow \infty.$$

These convergences are automatically valid if our “unnormalized” (3) is true. Notice here that  $\phi$  in the second convergence does not rely on the choice of initial function, whereas  $\phi$  in (3) does depend on  $u_0$ .

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LIIOUVILLE THEOREMS FOR SUPERSOLUTIONS OF  
ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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In joint work with Boyan Sirakov, we introduce a new maximum principle-based method for proving the nonexistence of positive supersolutions of elliptic equations. The model equation is  $-\Delta u = f(u)$  in an exterior domain, where  $f(u)$  behaves like a power of  $u$ . Even for this semilinear equation we obtain new (and sharp) results, although our method extends easily to other types of elliptic equations, such as degenerate quasilinear equations, fully nonlinear equations, systems of Lane-Emden type, as well as to more general unbounded domains.

# EXISTENCE OF VISCOSITY SOLUTIONS FOR A NONLOCAL EQUATION MODELLING POLYMER CRYSTAL GROWTH

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Joint work with Pierre Cardaliaguet and Aurélien Monteillet

The talk is based on [8] and is concerned with the construction of viscosity solutions for the coupled system

$$\begin{cases} i) & u_t(x, t) = \bar{g}(v(x, t))|Du(x, t)| & \text{in } \mathbb{R}^N \times (0, +\infty), \\ ii) & v_t(x, t) - \Delta v(x, t) + \kappa \bar{g}(v(x, t))\mathcal{H}^{N-1}[\{u(\cdot, t) = 0\}] = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\ iii) & v(x, 0) = v_0(x), u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \quad (0.1)$$

where the unknowns are  $u, v : \mathbb{R}^N \times (0, +\infty) \rightarrow \mathbb{R}$ ,  $N \geq 1$ ;  $u_t, v_t, Du, Dv$  and  $\Delta v$  denote respectively the time derivatives, the gradient and the Laplacian. The term  $\mathcal{H}^{N-1}[\{u(\cdot, t) = 0\}] = 0$  is the  $N - 1$ -Hausdorff measure restricted to the set

$$\Gamma(t) := \{x \in \mathbb{R}^N ; u(x, t) = 0\}. \quad (0.2)$$

The data  $\kappa, \bar{g}, u_0$  and  $v_0$  satisfy Assumption **(A)** below.

Following [7, 10], the 3-dimensional version of this system modelizes the growth of the surface  $\Gamma(t)$  of a polymer crystal in a nonhomogeneous temperature field  $v(x, t)$ . In this model one describes the evolving surface  $\Gamma(t)$  of the crystal by (0.2), i.e., as the 0-level-set of the auxiliary function  $u$ . This is the level-set approach, see [11] and references therein. It has experimentally been observed that the normal velocity  $V_n$  of the crystal is a known, positive function of the temperature:

$$V_n = \bar{g}(v(x, t)),$$

where  $\bar{g}$  is a bell-shaped function depending on the specific polymer ([9]). Expressing the normal velocity  $V_n$  in terms of the function  $u$  gives the eikonal equation (0.1)-i), which holds at least on the set  $\{u(\cdot, t) = 0\}$ . As for the temperature field  $v$  it has to follow a heat equation with a (negative) heat source proportional to  $V_n \mathcal{H}^{N-1}[\Gamma(t)]$ . Whence (0.1)-ii).

Similar systems, coupling eikonal and diffusion equations, appear in many applications: shape optimization, image segmentation, etc. However the mathematical analysis of such couplings is delicate and few existence or

uniqueness results are available in the literature. Most of them are concerned with classical solutions on a short time interval. For instance short time existence and uniqueness of smooth solutions are obtained for system (0.1) in [10].

The point is that, in general, one cannot expect such a system to have classical solutions when the time becomes large: indeed the front  $\Gamma(t)$  usually develops singularities in finite time. For this reason a good description of this front is obtained by its representation as the 0-level-set of the solution of an eikonal equation, which has to be understood in the sense of viscosity solutions. However this approach (which is satisfactory from a numerical view point) raises severe mathematical difficulties. Such issues have been overcome in only a very few number of situations: for a dislocation dynamics model, introduced in [1] and analyzed in [2, 3, 4], or for a system arising in the study of the asymptotics of a Fitzhugh-Nagumo model [5, 12, 13]. In this later framework, the associated heat equation is of the form

$$v_t(x, t) - \Delta v(x, t) - \bar{g}(v(x, t))\mathbf{1}_{\{u(\cdot, t) \geq 0\}} = 0, \quad (0.3)$$

where  $\mathbf{1}_E$  is the indicator function of a set  $E$ . In [5, 12, 13] existence of generalized solutions for this Fitzhugh-Nagumo system is proved, while [6] contains some uniqueness results. However, system (0.1) turns out to be much more challenging than the coupling in the Fitzhugh-Nagumo system. Indeed the surface term  $\mathcal{H}^{N-1}[\{u(\cdot, t) = 0\}]$  in (0.1)-*ii* is more singular than the volume term  $\mathbf{1}_{\{u(\cdot, t) \geq 0\}}$  in (0.3). For this reason, up to now, only the long time existence in space dimension  $N = 2$  is known [14].

The aim is to obtain a similar existence result for the physical dimension  $N = 3$  (and in fact in any dimension). In order to state precisely our main result, let us introduce the definition of a solution to (0.1).

**Definition 0.1.** *A solution  $(u, v)$  of (0.1) on the time interval  $[0, T]$  is a map  $(u, v) : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}^2$  which is bounded, uniformly continuous, such that  $u$  satisfies the equation*

$$u_t(x, t) = \bar{g}(v(x, t))|Du(x, t)| \text{ in } \mathbb{R}^N \times (0, T), \quad u(x, 0) = u_0(x) \text{ in } \mathbb{R}^N$$

*in the viscosity sense, with*

$$\int_0^T \mathcal{H}^{N-1}(\{u(\cdot, t) = 0\}) < +\infty,$$

*and such that  $v(\cdot, 0) = v_0$  and  $v$  satisfies in the sense of distributions*

$$v_t(x, t) - \Delta v(x, t) + \kappa \bar{g}(v(x, t))\mathcal{H}^{N-1}[\{u(\cdot, t) = 0\}] = 0 \quad \text{in } \mathbb{R}^N \times (0, T).$$

In order to explain more precisely our result, we then give the set of assumptions on the datas, denoted by **(A)**.

- (A1)**  $\kappa$  is a fixed real number ( $\kappa$  is positive in the case of a negative heat source and negative otherwise),  $\bar{g} : \mathbb{R}^N \rightarrow \mathbb{R}$  is Lipschitz continuous,

bounded, and there exist  $A, B > 0$  such that

$$A \leq \bar{g}(z) \leq B \quad \text{for all } z \in \mathbb{R}.$$

**(A2)**  $v_0 : \mathbb{R}^N \rightarrow \mathbb{R}$  is Lipschitz continuous and bounded.

**(A3)**  $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$  is Lipschitz continuous and satisfies  $\{u_0 = 0\} = \partial\{u_0 > 0\}$ . Moreover, we assume that  $\{u_0 \geq 0\}$  is compact and has the interior ball property of radius  $r_0 > 0$ , that is,

$$\text{For all } x \in K_0, \text{ there exists } y \in K_0, \text{ with } x \in \overline{B}(y, r_0) \subset K_0, \quad (0.4)$$

where  $\overline{B}(y, r_0)$  is the closed ball of radius  $r_0$  centered at  $y$ .

Our result states that, under the above assumptions, system (0.1) has a solution:

**Theorem 0.2.** *Under Assumption (A), for any  $T > 0$ , there exists at least one solution to System (0.1). This solution is bounded on  $\mathbb{R}^N \times [0, T]$  and satisfies, for all  $x, y \in \mathbb{R}^N$ ,  $0 \leq s, t \leq T$ ,*

$$|v(x, t) - v(y, t)| \leq C|x - y|(1 + |\log|x - y||),$$

and

$$|v(x, t) - v(x, s)| \leq C|t - s|^{\frac{1}{2}}(1 + |\log|t - s||).$$

for some constant  $C$  which only depends on the data appearing in Assumption (A) and  $T$ .

Note that uniqueness of the solution is an open problem (even in dimension 2).

Let us now briefly describe the method of proof. The main difficulty in (0.1) is the singular surface term in the heat equation: to deal with this term, one has to obtain fine regularity estimates for the level-sets of  $u$ . Such estimates, which cannot be derived from the usual regularity results on the eikonal equation, have been investigated through several works. When the velocity  $x \mapsto \bar{g}(v(x, t))$  is positive of class  $\mathcal{C}^{1,1}$ , the front enjoys the interior ball property (0.4) [2, 4]; it has an interior cone property when the velocity is positive and Lipschitz continuous [6]. Unfortunately, for System (0.1), the interior cone property is not sufficient for guarantying the stability of the surface term  $\mathcal{H}^{N-1}[\{u(\cdot, t) = 0\}]$ . Moreover we were only able to prove that the map  $x \mapsto v(x, t)$  has a modulus of continuity of the form  $\omega(\rho) = \rho(1 + |\log(\rho)|)$  (even when the front is smooth this map is at most Lipschitz continuous [10]). Our main and new estimate on the eikonal equation is an interior paraboloid property for the level-sets of  $u$ . We call paraboloid a solid deformation of the set

$$\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}; x_N \geq c|x'|^{1+\gamma}\}, \quad c > 0, \gamma \in (0, 1).$$

This property is obtained under the (weak) assumption that the velocity  $x \mapsto \bar{g}(v(x, t))$  is of class  $\mathcal{C}^{0,\alpha}$ . For this, we use a representation formula for the solutions of (0.1)-i) in terms of optimal control as well as sharp regularity properties of optimal solutions for this control problem. As a

direct consequence of the interior paraboloid property one obtains that the front has an interior cone property. These interior paraboloid and cone properties are the two key ingredients which allow us to obtain *a priori* estimates on the heat flow: indeed, because of the cone property, the front  $\Gamma(t)$  can be covered by a finite (and controlled) number of Lipschitz graphs. The stability result on the surface term  $\mathcal{H}^{N-1}[\{u(\cdot, t) = 0\}]$  is a consequence of the interior paraboloid property.

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# A few problems for Hamilton-Jacobi equations arising from step motions of crystal growth

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The growth of crystal is often explained by step motions of crystal surfaces. This idea goes back to W. K. Burton, N. Cabrera and F. C. Frank [BCF]. Several Hamilton-Jacobi equations for heights of crystals are derived as a continuum limit as each step height goes to zero [EY].

A typical problem in the theory of crystal growth is stability of a facet (flat portion). This is qualitatively studied by [C] and more quantitatively by [KIO]. The issue is whether perfectly flat crystal surfaces grow keeping its flatness. It seems that there are two kinds of facets—facet due to interfacial energy and facet due to kinetics. The first one is found in an equilibrium shape and it is explained as singularity of surface energy. Its evolution and stability is also studied, see eg. [GR]. The second one is a facet due to kinetics and studied in crystal growth literature [C], [KIO]. It is formulated as follows. We consider an evolution of height  $h(=h^\varepsilon)$  of a crystal surface at  $x \in \mathbf{R}^2$  and at time  $t$  which is determined by

$$h_t - \sigma(x)m\left(\frac{|\nabla h|}{\varepsilon}\right)\sqrt{|\nabla h|^2 + 1} = 0$$

Here  $m(p) = p \tanh(1/p)$  and  $\varepsilon$  is a criterion of (microscopic) local slope which is very small. The function  $\sigma \geq 0$  is concentration of adatom at the crystal surface. We consider microscopic time approximation proposed in [YGR] by introducing microscopic time  $\tau$  so that

$$h^\varepsilon(x, \varepsilon\tau) = \varepsilon u(x, \tau) + o(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0$$

to get

$$u_t - \sigma(x)m(|\nabla u|) = 0 \tag{1}$$

Stability of facets in kinetic sense corresponds to the large time asymptotics of the equation (1) when initial data equals zero which corresponds to a flat surface. However, if the initial data equals zero,  $u \equiv 0$  is the solution so it is not interesting. This is because we did not consider step source at a crystal surface so it is natural to see that crystal surface does not move at all.

Instead, we consider the one-dimensional Dirichlet problem for (1) in  $(0, \infty)$  by assigning the speed at zero. A typical problem is to consider  $\sigma(x) = \sigma_0(1 - x^2)_+$ ,  $\sigma_0 > 0$  and  $u(0, t) = ct$  with  $0 < c < \sigma_0$ . Since the Hamiltonian is non coercive, conventional theory for the large time asymptotics for the Hamiltonian does not apply. As printed out in [YGR] when  $u = 0$  initially, one observes that

$$u(x, t) \sim ct \quad \text{in } (0, x_c) \quad \text{as } t \rightarrow \infty,$$

where  $x_c > 0$  is the point such that  $\sigma_0(1 - x_c^2)_+ = c$ . Outside  $(0, x_c)$ ,  $u(x, t)$  grows slowly with respect to  $O(t)$ . Physically, the region  $(0, x_c)$  is the stable region of the facet. The conventional theory for coercive Hamiltonians yields a uniform asymptotics for all domain.

This result is generalized by Q. Liu, H. Mitake and the author [GLM1] by extending a notion of viscosity solutions defined in a part of a domain for more general setting. The asymptotics of the Cauchy problem (1) with slightly different  $m$  satisfying  $m(0) > 0$  is studied by Q. Liu, H. Mitake and the author [GLM2] by introducing a singular Neumann problem. In this talk we shall explain some of these results.

Finally, we mention the issue of step source. In [SK] an explicit ‘solution’ of evolution with step source is given without defining the notion of solutions of the equation. A typical example is

$$h_t - |\nabla h| = \sum_{j=1}^m v_j I(x - a_j), \tag{2}$$

where  $v_j > 0$  and  $I(x) = 1$  for  $x = 0$ ,  $I(x) = 0$  otherwise and  $a_j \in \mathbf{R}^2$  is  $j$ -th step source place. A suitable notion of the solution was not known. In a work with progress the author with his student N. Hamamuki introduced a new notion for the solution which leads a unique global existence of Lipschitz

solutions for Lipschitz initial data. One important observation is that we interpret (2) as

$$h_t - |\nabla h| = \sum_{j=1} (v_j - |\nabla h|)_+ I(x - a_j).$$

In this talk we also plan to mention these topics.

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Part 2

Tutorial Lectures and Interdisciplinary  
Conference  
“Mathematical Aspects of Crystal Growth”

学際的国際会議とチュートリアルセミナー  
「結晶成長の数学的側面」



# Tutorial Lectures and Interdisciplinary Conference “ Mathematical Aspects of Crystal Growth ”

## 学際的国際会議とチュートリアルセミナー 「結晶成長の数学的側面」

組織委員: 石井仁司, 儀我美一, 坂上貴之, 横山悦郎, R. V. Kohn, P. Rybka

Organizers: H. Ishii, Y. Giga, T. Sakajo, E. Yokoyama, R. V. Kohn, R. Rybka

Period (期間): July 26 - 30, 2010

Venue (場所): Centennial Hall, Hokkaido University

(Room 309, Faculty of Science Building 3 only on July 30)

北海道大学百年記念会館 (7月30日のみ理学部3号館3-309室)

July 26, 2010 (Monday)

09:00-09:50 Registration

09:50-10:00 Opening

10:00-10:50 Dionisios Margetis (Univ. of Maryland)

(Lecture I) A tale of two scales: Fundamentals of crystal surface morphological evolution

11:10-12:00 Russel E. Caflisch (UCLA)

(Lecture I) Modeling and simulation for the growth of thin films

14:00-14:30 大塚岳 (群馬大学) Takeshi Ohtsuka (Gunma Univ.)

A level set method for spiral crystal growth and growth rate of crystal surface

14:40-15:10 佐崎元 (北海道大学) Gen Sazaki (Hokkaido Univ.)

Direct observation of elementary growth processes of ice crystals by advanced optical microscopy

15:30-16:00 須藤孝一 (大阪大学) Koichi Sudoh (Osaka Univ.)

Evolution of microstructures on silicon substrates by surface diffusion

16:20- Poster and Party

July 27, 2010 (Tuesday)

- 10:00-10:50 Dionisios Margetis (Univ. of Maryland)  
(Lecture II) From discrete schemes to macroscopic evolution laws: I.  
Coarse graining and homogenization in epitaxial relaxation
- 11:10-12:00 Russel E. Caflisch (UCLA)  
(Lecture II) Strain and structure of thin films
- 13:40-14:10 Follow-up time for tutorials
- 14:20-14:50 八木厚志 (大阪大学) Atsushi Yagi (Osaka Univ.)  
On the longtime behavior of solutions to a model for epitaxial growth
- 15:00-15:50 Peter G. Vekilov (Univ. of Houston)  
Kink generation by the association of 2D clusters
- 16:00- Visit to Furukawa's Lab

July 28, 2010 (Wednesday)

- 10:00-10:50 James W. Evans (Iowa State Univ.)  
Atomistic and coarse-grained modeling of epitaxial thin film growth
- 11:10-12:00 Regis Monneau (ENPC)  
(Lecture I) Introduction to dislocation dynamics

July 29, 2010 (Thursday)

- 10:00-10:50 Regis Monneau (ENPC)  
(Lecture II) Mean curvature motion (MCM) as a singular limit of  
dislocation dynamics
- 11:10-12:00 Russel E. Caflisch (UCLA)  
(Lecture III) Pattern formation on thin films
- 14:00-14:30 Follow-up time for tutorials
- 14:40-15:10 石渡哲哉 (芝浦工業大学) Tetsuya Ishiwata (Shibaura Institute of Technology)  
Behavior of solutions to an area-preserving crystalline motion
- 15:20-15:50 三竹大寿 (広島大学) Hiroyoshi Mitake (Hiroshima Univ.)  
Short time uniqueness results for solutions of nonlocal and  
non-monotone geometric equations
- 16:00-16:30 Harald Garcke (Univ. of Regensburg)  
Efficient computation of crystal growth using sharp interface methods

July 30, 2010 (Friday)

- 10:00-10:50 Regis Monneau (ENPC)  
(Lecture III) Homogenization of dislocation dynamics and of particle systems
- 11:10-12:00 Dionisios Margetis (Univ. of Maryland)  
(Lecture III) From discrete schemes to macroscopic evolution laws: II.  
Crystal facets and boundary conditions
- 14:00-14:30 Follow-up time for tutorials
- 14:40-15:10 高坂良史 (室蘭工業大学) Yoshihito Kohsaka (Muroran Institute of Technology)  
Stability analysis of steady states for surface diffusion equation  
in a bounded domain
- 15:20-16:00 Discussion: 'What is the next step?'

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**Statement by Robert Kohn, on the occasion of the meeting  
"Mathematical Aspects of Crystal Growth"  
at Hokkaido University, July 26-30, 2010.**

It has been a great pleasure to help organize this meeting, part of a month-long minisemester at Hokkaido University addressing the mathematics of evolving interfaces. I regret that I cannot participate in person.

Crystal growth is a widely-studied area of physics, with many applications and much intrinsic beauty. Through advances in microscopy and quantum chemistry, we know quite a bit about the atomic-scale mechanisms by which crystals grow, evaporate, or evolve. Through stochastic and continuum modeling, we have many approaches to the evolution of crystals on mesoscopic and macroscopic time and length scales. But we understand relatively little about the macroscopic consequences of our atomic-scale models, or the physical validity of our continuum models. Overall, our understanding is shockingly incomplete with regard to the relationships between different approaches to crystal growth.

This area offers many mathematical challenges. Improved algorithms can make simulation more efficient, permitting microscopic models to be used on longer time and length scales. Systematic coarse-graining can justify existing continuum models or produce better ones. Rigorous analysis can tell us whether a model is well-posed, and reveal why its solutions behave as they do.

Mathematics has much to offer, as the talks at this workshop will demonstrate. It also has much to gain, since the work addressing these challenges is pushing the research frontier.

This is fertile territory. I know you'll find the workshop thought-provoking. In addition, I hope that interaction with other participants will lead to fresh ideas and further progress.

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**Tutorial Lectures on "Mathematical Aspects of Crystal Growth"**  
**Dionisios Margetis**  
**Department of Mathematics,**  
**and Institute for Physical Science & Technology, and**  
**Center for Scientific Computation & Mathematical Modeling,**  
**University of Maryland, College Park**

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**Lecture 1: A tale of two scales: Fundamentals of crystal surface morphological evolution**

Abstract:

The goal with this lecture is to introduce basic physical and mathematical concepts permeating epitaxial relaxation and growth. This area of research encompasses mathematically rich phenomena and at the same time is strongly driven by laboratory experiments. In materials science, the design of novel devices requires understanding how structures on crystal surfaces evolve and fluctuate across several scales, from the atomistic to the continuum. In the last few decades, considerable theoretical efforts have focused on describing the motion of crystal surfaces. In this talk, I will review a few related models and their underlying principles. First, I will introduce the main mechanisms of crystal surface motion from a physics perspective, exemplifying the role of surface diffusion. Second, I will review past theories that aim to describe crystal surface morphological evolution above and below the roughening transition temperature. Emphasis will be placed on two scales, macroscale and nanoscale, and corresponding models: (i) a thermodynamics approach, which stems from the pioneering works of Mullins, Herring and others, on the basis of a continuum surface energy; and (ii) step flow models for temperatures below the roughening transition, according to the celebrated theory of Burton, Cabrera and Frank (BCF). The latter approach invokes the motion of line defects of atomic size. A third approach involves kinetic Monte Carlo simulations, which aim to capture aspects of the atomistic scale. I will discuss merits and limitations of these points of view; and mention germane issues of modeling and analysis, thus setting the stage for the following two lectures.

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## **Lecture 2: From discrete schemes to macroscopic evolution laws: I. Coarse graining and homogenization in epitaxial relaxation**

Abstract:

In this lecture, I will focus on derivations and implications of deterministic macroscopic laws for the relaxation of crystal surface morphologies at temperatures below the roughening transition. At the nanoscale, the surface motion is described by discrete equations for the positions of steps. At the macroscale, it is plausible to use Partial Differential Equations (PDEs) for the surface height or slope. Such PDEs are usually of fourth order (under surface diffusion) and fully nonlinear. The focus of this talk will be the linkage between descriptions at the nanoscale and the macroscale, especially in 2+1 dimensions where the curvature of steps and various anisotropies play an important role. My exposition will address: (i) Basic coarse-graining techniques for the formal derivation of PDEs from discrete schemes for steps; (ii) the case of surface reconstructions, where a particular homogenization procedure is applicable; (iii) the connection of the derived PDEs to thermodynamics principles, especially to continuum singular interfacial energies; and (iv) predictions of PDEs in 2+1 dimensions on the basis of numerical simulations, and their possible implications to experiments. For a large part of this lecture, I will restrict attention to monotone step trains, in the absence of macroscopically flat surface regions (facets). Issues in the modeling of material deposition (growth) and stochastic effects will be outlined.

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## **Lecture 3: From discrete schemes to macroscopic evolution laws: II. Crystal facets and boundary conditions**

Abstract:

In this talk, I will address subtle physical and mathematical issues in the global interpretation of PDEs that govern crystal surface motion below the roughening transition. In such temperature regimes, crystal surfaces often develop macroscopically flat surface regions, called facets. The existence of facets greatly complicates the interpretation of macroscopic evolution laws, since facets usually host microscopic

phenomena not captured by continuum theories. First, I will introduce the concept of a static facet from a thermodynamics perspective with recourse to the equilibrium shapes of crystals. Second, I will discuss the evolution of facets from a macroscopic viewpoint, starting with the pioneering work of Spohn who treated facets as free boundaries. This approach is intimately connected to the subgradient formulation for macroscopic evolution PDEs. Third, I will elaborate on the nature of facets from a kinetic, microscopic viewpoint. The connection of discrete schemes for steps to macroscopic evolution laws in the presence of facets will be illustrated in this context. I will show that, in principle, the discrete schemes are not consistent with the traditional thermodynamics interpretation of the macroscopic laws. This observation alludes to interesting mathematical questions.

# Growth, Structure and Pattern Formation for Thin Films

Russel E. Caflisch

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**Abstract** An epitaxial thin film consists of layers of atoms whose lattice properties are determined by those of the underlying substrate. This paper reviews mathematical modeling, analysis and simulation of growth, structure and pattern formation for epitaxial systems, using an island dynamics/level set method for growth and a lattice statics model for strain. Epitaxial growth involves physics on both atomistic and continuum length scales. For example, diffusion of adatoms can be coarse-grained, but nucleation of new islands and breakup for existing islands are best described atomistically. In heteroepitaxial growth, mismatch between the lattice spacing of the substrate and the film will introduce a strain into the film, which can significantly influence the material structure, for example leading to formation of quantum dots. Technological applications of epitaxial structures, such as quantum dot arrays, require a degree of geometric uniformity that has been difficult to achieve. Modeling and simulation may contribute insights that will help to overcome this problem. We present simulations that combine growth and strain showing the structure of nanocrystals and the formation of patterns in epitaxial systems.

**Keywords** Epitaxial growth · Level set method · Island dynamics · Lattice statics · Strain energy · Nanocrystals · Quantum dots

## 1 Simulation of Epitaxial Growth

Epitaxy is the growth of a thin film on a substrate in which the crystal properties of the film are inherited from those of the substrate. Since an epitaxial film can (at least in principle) grow as a single crystal without grain boundaries or other defects, this method produces crystals of the highest quality. In spite of its ideal properties, epitaxial growth is still challenging to mathematically model and numerically simulate because of the wide range of

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length and time scales that it encompasses, from the atomistic scale of Ångströms and picoseconds to the continuum scale of microns and seconds. This paper reviews our work on simulation of epitaxial growth and of strain in epitaxial system, with applications to the structure of nanocrystals and the formation of patterns on epitaxial surfaces. For simulation of growth we use an island dynamics model with a level set simulation method. Atomistic strain is computed from a linearized lattice statics model.

## 1.1 Epitaxial Growth

The geometry of an epitaxial surface consists of step edges and island boundaries, across which the height of the surface increases by one crystal layer, and adatoms which are weakly bound to the surface. Epitaxial growth involves deposition, diffusion and attachment of adatoms on the surface. Deposition is from an external source, such as a molecular beam. The principal dimensionless parameter (for growth at low temperature) is the ratio  $D/(a^4 F)$ , in which  $a$  is the lattice constant and  $D$  and  $F$  are the adatom diffusion coefficient and deposition flux. It is conventional to refer to this parameter as  $D/F$ , with the understanding that the lattice constant serves as the unit of length. Typical values for  $D/F$  are in the range of  $10^4$  to  $10^8$ .

The models that are typically used to describe epitaxial growth include the following: *Molecular dynamics* (MD) consists of Newton's equations for the motion of atoms on an energy landscape. A typical *Kinetic Monte Carlo* (KMC) method simulates the dynamics of the epitaxial surface through the hopping of adatoms along the surface. The hopping rate comes from an Arrhenius rate of the form  $e^{-E/kT}$  in which  $E$  is the energy barrier for going from the initial to the final position of the hopping atom. *Island dynamics models*, one of the subjects of this article, describe the surface through continuum scaling in the lateral directions but atomistic discreteness in the growth direction. *Continuum equations* approximate the surface using a smooth height function  $h = h(x, y, t)$ , obtained by coarse graining in all directions. *Rate equations* describe the surface through a set of bulk variables without spatial dependence.

The island dynamics model described here is solved using a level set simulation method. Within the level set approach [28, 29, 40], the union of all boundaries of islands of height  $k + 1$ , can be represented by the level set  $\varphi = k$ , for each  $k$ . For example, the boundaries of islands in the submonolayer regime then correspond to the set of curves  $\varphi = 0$ . Growth of these islands is described by a smooth evolution of the function  $\varphi$ .

Validation of the island dynamics/level set method will be detailed in this article by comparison to results from an atomistic KMC model. The KMC model employed is a simple cubic pair-bond solid-on-solid (SOS) model [46]. In this model, atoms are randomly deposited at a deposition rate  $F$ . Any surface atom is allowed to move to its nearest neighbor site at a rate  $r$  that is determined by  $r = r_0 \exp\{-(E_S + nE_N)/k_B T\}$ , where  $r_0$  is a prefactor which is chosen to be  $10^{13} \text{ s}^{-1}$ ,  $k_B$  is the Boltzmann constant, and  $T$  is the surface temperature.  $E_S$  and  $E_N$  represent the surface and nearest neighbor bond energies, and  $n$  is the number of nearest neighbors.

Level set methods have been used for a number of thin film growth problems that are related to the applications described below. In [13] a level set method was used to simulate coarsening, and in [43] a level set method was used to describe spiral growth in epitaxy. A general level set approach to material processing problems, including etching, deposition and lithography, was developed in [1, 2] and [3]. A similar method was used in [30] for deposition in trenches and vias.

### 1.2 Island Dynamics

Burton, Cabrera and Frank [7] developed the first detailed theoretical description for epitaxial growth. In this “BCF” model, the adatom density solves a diffusion equation with an equilibrium boundary condition ( $\rho = \rho_{eq}$ ), and step edges (or island boundaries) move at a velocity determined from the diffusive flux to the boundary. Modifications of this theory were made, for example in [6, 11, 16, 20, 23], to include line tension, edge diffusion and nonequilibrium effects. These are “island dynamics” models, since they describe an epitaxial surface by the location and evolution of the island boundaries and step edges. They employ a mixture of coarse graining and atomistic discreteness, since island boundaries are represented as smooth curves that signify an atomistic change in crystal height.

Adatom diffusion on the epitaxial surface is described by a diffusion equation of the form

$$\partial_t \rho - D \nabla^2 \rho = F - 2(d/dt)N_{nuc} \tag{1}$$

in which the last term represents loss of adatoms due to nucleation and desorption from the epitaxial surface has been neglected. Attachment of adatoms to the step edges and the resulting motion of the step edges are described by boundary conditions at an island boundary (or step edge)  $\Gamma$  for the diffusion equation and a formula for the step-edge velocity  $v$ .

For the boundary conditions and velocity, several different models are used. The simplest of these is

$$\begin{aligned} \rho &= \rho_*, \\ v &= D[\partial \rho / \partial n] \end{aligned} \tag{2}$$

in which the brackets indicate the difference between the value on the upper side of the boundary and the lower side. Two choices for  $\rho_*$  are  $\rho_* = 0$ , which corresponds to irreversible aggregation in which all adatoms that hit the boundary stick to it irreversibly, and  $\rho_* = \rho_{eq}$  for reversible aggregation. For the latter case,  $\rho_{eq}$  is the adatom density for which there is local equilibrium between the step and the terrace [7].

Line tension and edge diffusion can be included in the boundary conditions and interface velocity as in

$$\begin{aligned} \partial \rho / \partial n_{\pm} &= D_T(\rho_{\pm} - \rho_*) - \mu \kappa, \\ v &= D_T \mathbf{n} \cdot [\nabla \rho] + \beta \rho_{*ss} + (\mu / D_E) \kappa_{ss}, \end{aligned} \tag{3}$$

in which  $\rho_{\pm}$  and  $\partial \rho / \partial n_{\pm}$  are the limiting values of the adatom density and its normal derivative at a step from the upper (+) and lower (−) terraces,  $\mathbf{n}$  is the normal direction at a step (pointing into the lower terrace),  $\kappa$  is curvature,  $s$  is the variable along the boundary, and  $D_E$  is the coefficient for diffusion along and detachment from the boundary. The term  $\beta \rho_{*ss}$  was derived in [20] due to edge diffusion.

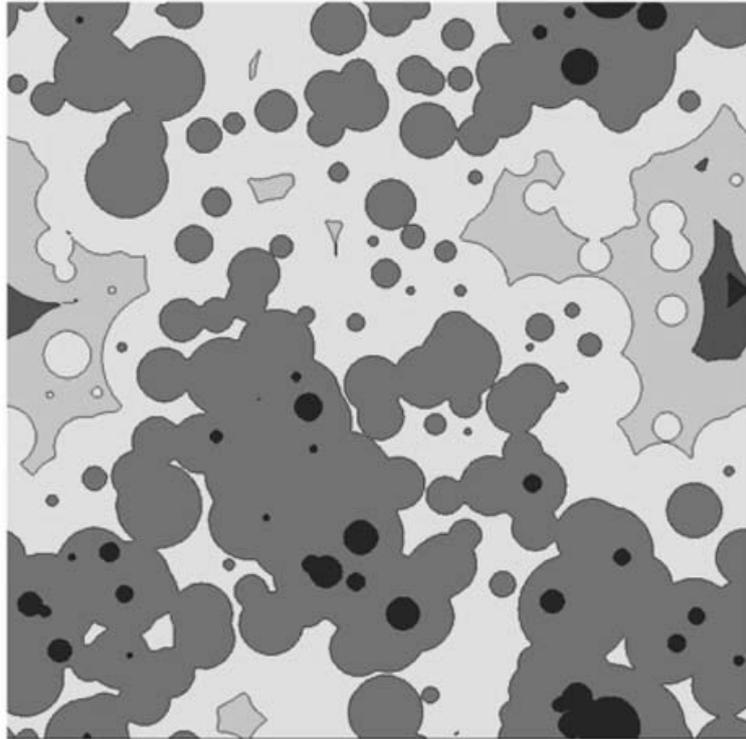
A snapshot of the results from a typical level-set simulation is shown in Fig. 1. This figure shows the epitaxial surface, consisting of islands of various heights, after deposition of 40 layers. Numerical details on implementation of the level set method for thin film growth are provided in [12].

### 1.3 Nucleation and Submonolayer Growth

For the case of irreversible aggregation, a dimer (consisting of two atoms) is the smallest stable island, and the nucleation rate is

$$\frac{dN_{nuc}}{dt} = D \sigma_1 \langle \rho^2 \rangle, \tag{4}$$

**Fig. 1** Snapshot of a typical level-set simulation after deposition of 40 layers



where  $\langle \cdot \rangle$  denotes the spatial average of  $\rho(\mathbf{x}, t)^2$  and

$$\sigma_1 = \frac{4\pi}{\ln[(1/\alpha)\langle \rho \rangle D/F]} \tag{5}$$

is the adatom capture number as derived in [5, 45]. Without the factor  $\sigma_1$ , (4) describes the rate of collisions for a system of uniformly distributed particles that do not stick together; the factor  $\sigma_1$  provides the leading order correction for particles that stick together. The parameter  $\alpha$  reflects the island shape, and  $\alpha \simeq 1$  for compact islands. Expression (4) for the nucleation rate implies that the time of a nucleation event is chosen deterministically. Whenever  $N_{\text{nuc}}L^2$  passes the next integer value ( $L$  is the system size), a new island is nucleated. Numerically, this is realized by raising the level-set function to the next level at a number of grid points chosen to represent a dimer.

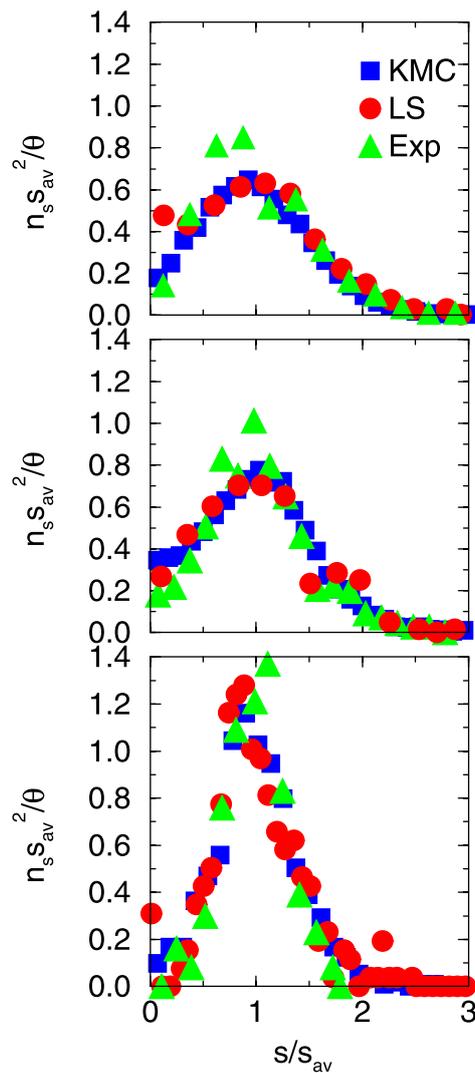
The choice of the location of the new island is determined by probabilistic choice with spatial density proportional to the nucleation rate  $\rho^2$ . This probabilistic choice constitutes an atomistic fluctuation that must be retained in the level set model for faithful simulation of the epitaxial morphology. For growth with compact islands, computational tests have shown additional atomistic fluctuations can be omitted [36].

Additions to the basic level set method, such as terms that represent finite lattice constant effects and edge diffusion (not the term  $\kappa_{ss}$  but a surrogate term that has a similar effect), are easily included [37]. The level set method with these corrections is in excellent agreement with the results of KMC simulations. For example, Fig. 2 shows the island size distribution (ISD)

$$n_s = \frac{\Theta}{s_{av}^2} g(s/s_{av}), \tag{6}$$

where  $n_s$  is the scaled density of islands of size  $s$ ,  $s_{av}$  is the average island size, and  $g(x)$  is a scaling function. The top panel of Fig. 2 is for irreversible attachment; the other two panels

**Fig. 2** The island size distribution, as given by KMC (squares) and LS (circles) methods, in comparison with STM experiments (triangles) on Fe/Fe(001) [44]. The reversibility increases from top to bottom



include reversibility that will be discussed below. All three panels show excellent agreement between the results from level set simulations, KMC and experiment.

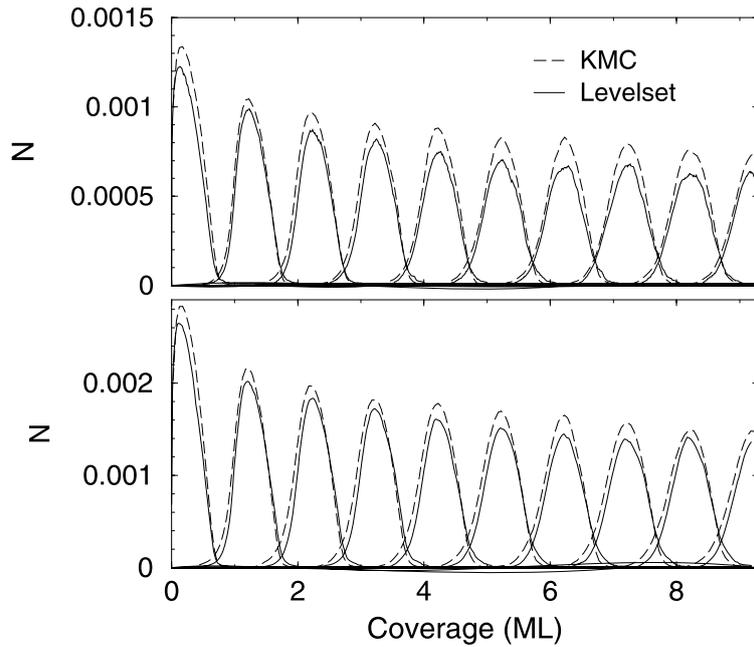
### 1.4 Multilayer Growth

In ideal layer-by-layer growth, a layer is completed before nucleation of a new layer starts. In this case, growth on subsequent layers would essentially be identical to growth on previous layers. In reality, however, nucleation on higher layers starts before the previous layer has been completed and the surface starts to roughen. This roughening transition depends on the growth conditions (i.e., temperature and deposition flux) and the material system (i.e., the value of the microscopic parameters). At the same time, the average lateral feature size increases in higher layers, which we will refer to as coarsening of the surface.

These features of multilayer growth and the effectiveness of the level set method in reproducing them is illustrated in Fig. 3 which shows the island number density  $N$  as a function of time for two different values of  $D/F$  from both a level set simulation and from KMC. The results show near perfect agreement. The KMC results were obtained with a value for the edge diffusion that is 1/100 of the terrace diffusion constants. The island density decreases as the film height increases which implies that the film coarsens.

The simulation results presented above have been for the case of irreversible aggregation. If aggregation is reversible, the KMC method must simulate a large number of detachment

**Fig. 3** Island densities  $N$  on each layer for  $D/F = 10^6$  (lower panel) and  $D/F = 10^7$  (upper panel) obtained with the level-set method and KMC simulations. For each data set there are 10 curves in the plot, corresponding to the 10 layers



and reattachment events that can slow down the simulations significantly. On the other hand, in a level set simulation these events can be directly replaced by their time average and therefore the simulation only needs to include detachment events that do not lead to a subsequent reattachment, making the level set method much faster than KMC, as shown in [33]. Reversibility can be included in the level set method using the boundary conditions (2) with  $\rho_* = \rho_{eq}$  in which  $\rho_{eq}$  depends on the local environment of the island, in particular the edge atom density [8]. For islands consisting of only a few atoms, however, the stochastic nature of detachment becomes relevant and is included through random detachment and breakup for small islands, as detailed in [33].

## 2 Strain in Thin Films

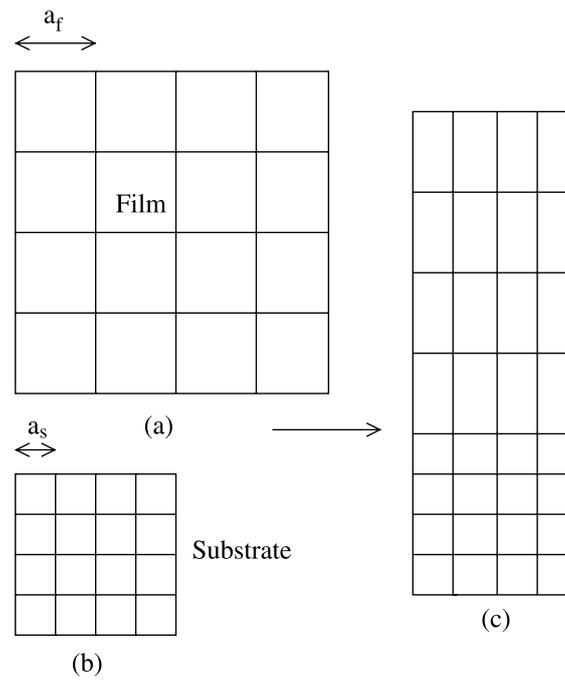
### 2.1 Numerical Simulations for Thin Films

In heteroepitaxial growth, a thin film of one material (e.g., Ge) is grown on top of a substrate of a second material (e.g., Si), with perfect, single crystalline structure in both materials and with the lattice structure of the film determined by the substrate. If the lattice constants  $a_f$  and  $a_s$  for the film and substrate are different (e.g.,  $a_{Ge} = 1.04 \times a_{Si}$ ) then strain is generated in the film. Figure 4 illustrates the horizontal compression, as well as the resulting vertical expansion, in the lattice of the film atoms, when they are placed on a substrate of smaller lattice constant. This strain has important effects on the material structure, as well as on its electronic properties.

For this system, it is most convenient to define the atomic displacement relative to a single reference lattice, for example the equilibrium lattice of the substrate, so that the displacement  $\mathbf{u}$  in the film is defined relative to a nonequilibrium reference lattice. The bond displacement  $\mathbf{d}^{k\pm}$  is then

$$\mathbf{d}^{k\pm}(\mathbf{i}) = (d_1^{k\pm}, d_2^{k\pm}, d_3^{k\pm}) = D_k^\pm \mathbf{u}(\mathbf{i}) - \epsilon \mathbf{e}_k \chi \tag{7}$$

**Fig. 4** (a) Equilibrium lattice for film with lattice size  $a_f$ . (b) Equilibrium lattice for substrate with lattice size  $a_s$ . (c) The reference (non-equilibrium) lattice in which strain is introduced due to lattice mismatch



and the discrete strain components at a point  $\mathbf{i}$  are defined as

$$S_{k\ell}^{pq} = \frac{D_k^p u_\ell + D_\ell^q u_k - \epsilon \delta_{k\ell} \chi}{2}. \tag{8}$$

In these equations  $D_k^+$  and  $D_k^-$  denote forward and backward difference operators,  $\epsilon = \frac{a_f - a_s}{a_s}$  is the relative lattice displacement and  $\chi$  is 0 in the substrate and 1 in the film. The resulting discrete strain equations have a force of size  $\epsilon$  along the film/substrate interface. The atomistic strain energy at a point  $\mathbf{i}$  has the form

$$E(\mathbf{i}) = \alpha \sum_{p=\pm, k=1,2} (S_{kk}^p)^2 + \sum_{p=\pm, q=\pm} (2\beta (S_{12}^{pq})^2 + \gamma S_{11}^p S_{22}^q), \tag{9}$$

where the elastic constant  $\alpha$ ,  $\beta$  and  $\gamma$  are chosen so that the model is consistent with continuum elasticity, namely,

$$\alpha = C_{11}/4, \quad \beta = C_{44}/4, \quad \gamma = C_{12}/4 \tag{10}$$

in which  $C_{ij}$  are the Voigt constants.

The total energy can be obtained by summing up all the energy densities to get

$$\mathcal{E}^{total} = \sum_{\mathbf{i}} E(\mathbf{i}). \tag{11}$$

Minimization of this energy leads to a force balance equation

$$\frac{\partial \mathcal{E}^{total}}{\partial u_k} = 0, \quad \text{for } k = 1, 2. \tag{12}$$

Away from boundaries and interfaces, these equations are equivalent to a finite difference approximation to the equations of continuum elasticity. At a step on a boundary or interface,

there are deviations from the continuum boundary conditions which can be interpreted as force distributions on the boundary. Details of the energy density and the discrete force balance equations can be found in [14, 39].

Computational solution of the strain equations can be computationally challenging. We apply two methods to reduce the computational complexity. First we use an artificial boundary condition along a plane in the substrate that is everywhere below the film. This greatly reduces the extent of the computational domain, with no loss of accuracy [19]. Second, we apply an algebraic multigrid method to solve the strain equations, that greatly accelerates the computations [9]. Similar methods have been developed and implemented by Smereka and Russo [38]. Simulations of island dynamics models including elastic effects have been performed by Hauser, Jabbour and Voigt [18], as well as by Niu et al. [27].

### 3 Modeling and Simulation for the Structure of Nanocrystals

Layered nanocrystals consist of a core of one material surrounded by a shell of a second material. Synthesis of layered nanocrystals with precise control over their size and shape has been achieved by a number of research groups [10, 22, 25, 32] and provides an effective method for designing material systems with desired optoelectronic properties [22].

Because of the small size of these systems, their atomic structure is epitaxial in many cases. Lattice mismatch between the materials in the core and shell leads to elastic strain in a layered nanocrystal. This strain has both structural and optoelectronic consequences. If the strain is large enough, then it is relieved by irregular growth of the shell [22]; i.e., the epitaxial structure is lost. As a result, the shell may break off from the core [22].

The present study from [4] employs a simple model for the structure and strain of a layered nanocrystal. Simulation of this model for a range of geometric and elastic parameters shows that there is a critical shell size at which strain has maximal influence.

#### 3.1 Core/Shell Model

Denote the lattice constants in the core and shell as  $l_c$  and  $l_s$ , respectively. For bonds connecting a core atom and a shell atom, the rest length is taken to be the average  $(l_c + l_s)/2$ . Similarly the elastic coefficients for the bonds connecting a core atom and a shell atom are taken to be the averages of the elastic coefficients for the pure materials.

The significant geometric parameters are the core radius  $r_c$ , the shell thickness  $r_s$  and the lattice mismatch

$$\epsilon = \frac{l_c - l_s}{l_c}. \quad (13)$$

The core consists of atoms whose lattice position  $x$  (before displacement) satisfies  $|x| \leq r_c$ , and the shell consists of atoms with  $r_c < |x| \leq r_c + r_s$ , as shown in Fig. 5.

#### 3.2 Critical Thickness: Simulation Results

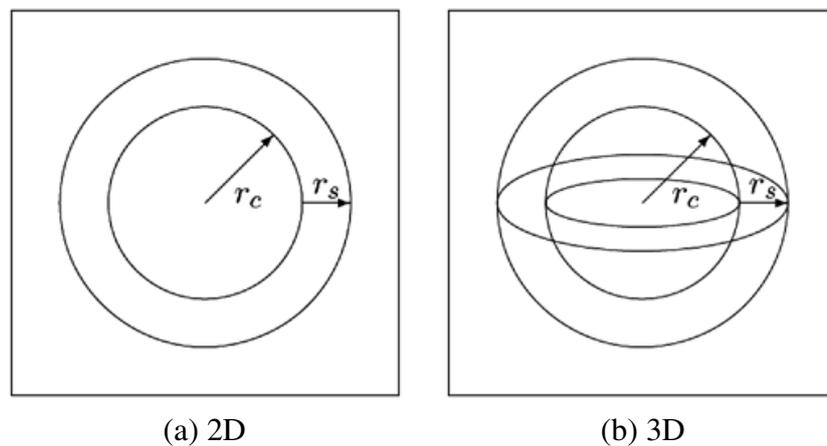
Computational results are presented here from minimization of the total elastic energy (after removing degenerate modes corresponding to translation and rotation), corresponding to balance of all of the forces in the system, for 2D (circular, or equivalently rods of infinite length) and 3D (spherical) nanocrystals. For the harmonic potentials used here, this amounts to solving a linear system of equations, in which the forcing terms come from the lattice

mismatch  $\epsilon$ . The simulation results include values of the displacements, the forces and the energy density. Graphical results will be presented for the last of these. As a figure of merit for the atomistic strain field in a nanocrystal, we shall use the maximum value  $E_m$  of the discrete energy density. Since the energy at each atom consists of elastic energy and bond energy, the maximum elastic energy may be a good indicator of strain-driven instability.

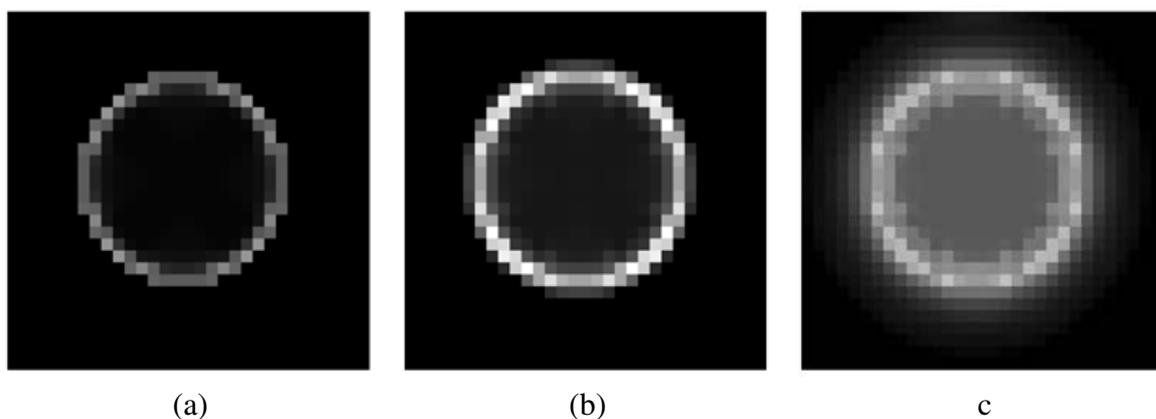
### 3.2.1 Elastic Energy Density

Figure 6 show the elastic energy density of 3D layered nanocrystals, of fixed core size  $r_c$  for various values of shell thickness  $r_s$ . In these simulations, the shell has thickness values  $r_s = 1, 2$  and  $7$  monolayers, on a core of radius  $r_c = 8$  monolayers. For all of these simulations, the elastic constants are  $\alpha = 5, \beta = 1$  and  $\gamma = 3$  and lattice mismatch is  $\epsilon = 0.04$ .

In this figure, the gray scale ranges from black for  $E = 0$  to white for  $E = E_m$  in which  $E_m$  is the largest value of  $E$  among the three subfigures; i.e., the scales are same for the different subfigures. The black region outside of each nanocrystal is a vacuum where there is no energy. Figure 6 shows that the energy is concentrated in the region of the shell, along the interface with the core. As the shell thickness increases, the strain energy becomes more concentrated near the shell/core interface, even though the maximum energy density decreases for larger shell thickness. In addition the largest values of the energy density are close to the diagonal.



**Fig. 5** Basic geometry of core/shell nanocrystal model



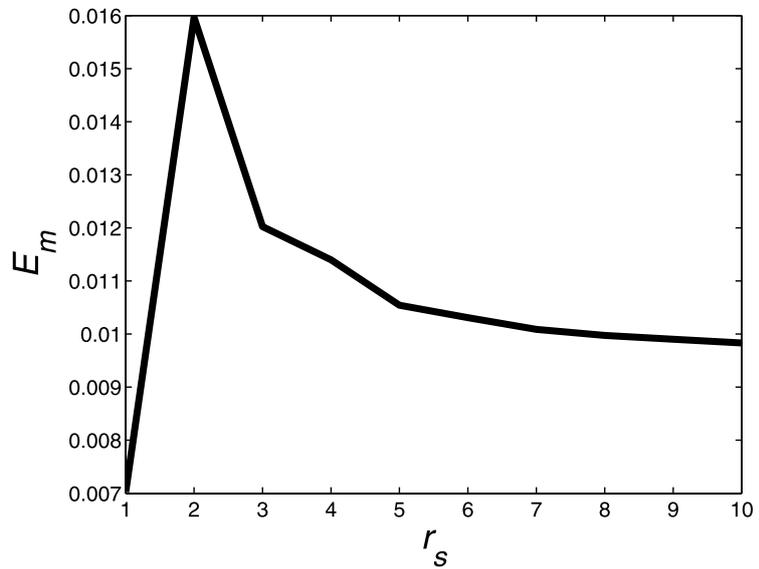
**Fig. 6** Elastic energy density on an equatorial cross section for 3D layered nanocrystals with core size  $r_c = 8$  monolayers and with shell thickness  $r_s$  of size (a) 1 monolayer, (b) 2 monolayers and (c) 7 monolayers

### 3.2.2 Critical Thickness

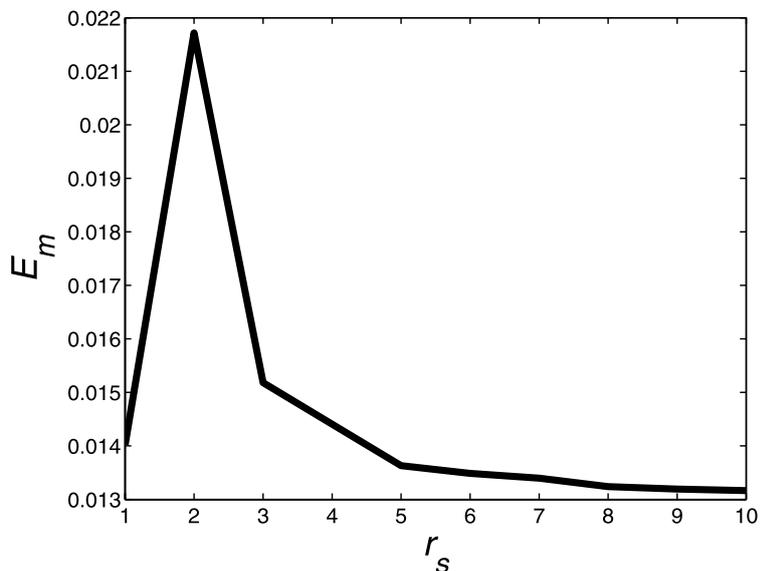
Figure 7 shows the maximum energy density for a layered nanocrystal, as a function of shell thickness  $r_s$ , for fixed values of the other parameters, core size  $r_c$  and elastic constants  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\epsilon$ . Figure 7 shows that the maximum energy density increases with increasing shell thickness  $r_s$  up to a critical shell thickness  $r_s^*$ . For  $r_s > r_s^*$ , the maximum energy density is decreasing as a function of  $r_s$ . The general similarity between the critical shell thickness in 2D and 3D is indicative of the robustness of this result. The physical core radius of CdSe/CdS core/shell nanocrystal is ranging from 11.5Å to 19.5Å which is equivalent to core radius of 3 monolayers to 6 monolayers, since one full monolayer is approximately 3.5Å [32].

Our simulations show weak sensitivity of critical shell thickness  $r_s^*$  on the core radius  $r_c$ . The critical thickness  $r_s^*$  is uniformly 2 monolayers as long as the core size is big enough. In simulation, for smaller core size than 3 monolayers for 2D layered nanocrystals and 5

**Fig. 7** Maximum energy density  $E_m$  vs. shell thickness  $r_s$  for (a) 2D and (b) 3D nanocrystal of core radius  $r_c = 8$  monolayers



(a)



(b)

monolayers for 3D layered nanocrystals, the maximum elastic energy density  $E_m$  occurs at 1 monolayer of shell thickness  $r_s$ . We also find that the critical shell thickness  $r_s^*$  is roughly independent of the lattice misfit  $\epsilon$  and the elastic parameters  $\alpha$ ,  $\beta$  and  $\gamma$ .

## 4 Patterns in Epitaxial Systems

### 4.1 Self-Assembly

Highly ordered and uniformly sized nano patterns play an increasingly important role for many technological applications. A critical factor for the performance of all such devices is that the patterns are all within a certain size range (which depends on the material), and that the dots are essentially all equal in size. It is therefore the focus of a large number of studies to understand the formation and growth of nano patterns (for recent reviews, see [41, 42]), and to control their formation and size distribution.

There are various approaches to obtaining arrays of equally sized and spaced nano patterns. In the top-down approach, islands nucleate in previously fabricated nucleation sites. But structures as small as a few nm are difficult to obtain with standard lithographic techniques. In the bottom-up approach, kinetic and/or thermodynamic factors spontaneously lead to the formation of quantum dots [15, 17, 24]. Guided or directed self-assembly is somewhere in-between. In this approach, the goal is to control pattern formation by manipulating the epitaxial growth process. For example, introduction of subsurface dislocation arrays introduces a long-range strain field, which alters the potential energy surface (PES), changing both the adsorption energy  $E_{ad}$  and the transition energy  $E_{trans}$  of the PES [31, 35]. The results presented here from [26] simulate the process of directed self-assembly, starting from a spatially varying PES.

### 4.2 Epitaxial Growth with Spatially Varying Potential Energy Surface

In this section, we discuss simulations that demonstrate that a properly modified PES for adatom diffusion can lead to self organization of nano patterns. Adatom diffusion is described by a rate for surface diffusion, which is  $D = D_0 \exp(-\Delta E/k_B T)$ , where  $D_0$  is a prefactor (chosen to be  $10^{13} \text{ s}^{-1}$ ),  $k_B$  is the Boltzmann constant,  $T$  is the temperature, and  $\Delta E$  is the energy barrier for surface diffusion, given by  $\Delta E = E_{trans} - E_{ad}$ . We study growth systematically as a function of  $E_{ad}$  and  $E_{trans}$ , which are treated as independent parameters.

For a surface with a spatially varying, anisotropic PES, the diffusion equation (1) becomes

$$\frac{\partial \rho}{\partial t} = F + \nabla \cdot (\mathbf{D} \nabla \rho) - 2 \frac{dN}{dt} + \nabla \cdot \left( \frac{\rho}{k_B T} \mathbf{D} (\nabla E_{ad}) \right). \tag{14}$$

In (14),  $\mathbf{D}$  is a diffusion tensor where the diagonal entries are labeled  $D_i(\mathbf{x})$  and  $D_j(\mathbf{x})$ , and correspond to diffusion along the two directions  $i$  and  $j$ . For simplicity no other direction for diffusion is included (but could easily be incorporated).  $F$  is the deposition flux,  $dN/dt$  is the nucleation rate, and the last term is the thermodynamic drift, where  $k_B$  is the Boltzmann constant, and  $T$  is the temperature. On island boundaries there is rapid attachment and detachment of adatoms, so that the correct boundary condition is  $\rho(\mathbf{x}) = \rho_{eq}(D_{det}(\mathbf{x}), \mathbf{x})$ , where  $D_{det}(\mathbf{x})$  is a (spatially varying) detachment rate [8]. The nucleation rate is given by [37]

$$dN/dt = \sigma_1 \langle [(D_i(\mathbf{x}) + D_j(\mathbf{x}))/2] \rho^2(\mathbf{x}) \rangle, \tag{15}$$

where  $\sigma_1$  is the capture number as in (5), and the average  $\langle \cdot \rangle$  is taken over all lattice sites.

### 4.3 Simulation Results

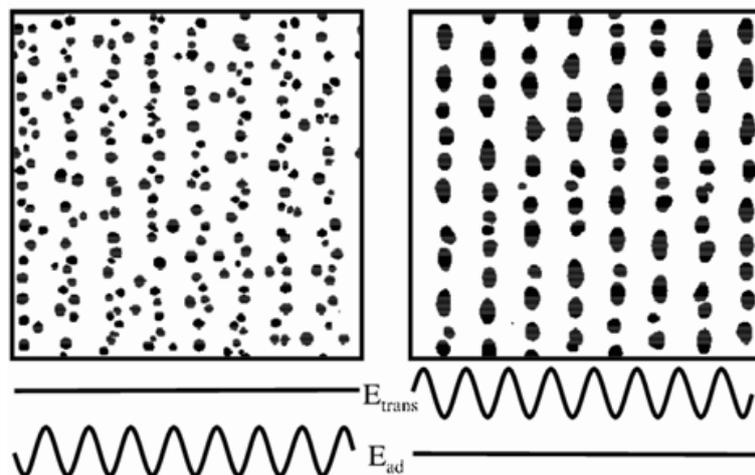
We assume a simple sinusoidal variation of  $E_{ad}$  and  $E_{trans}$ . More precisely, for the results shown in Fig. 8, we assume that the diffusion constant varies between  $D = 10^5 \text{ s}^{-1}$  and  $D = 10^7 \text{ s}^{-1}$  along the  $i$ -direction, and that in fact  $\text{LOG}_{10} D$  varies sinusoidally. Diffusion is isotropic but spatially varying, and we use the notation  $D = D_i(\mathbf{x}) = D_j(\mathbf{x})$ . A schematic of the variations of the PES is shown in the bottom panels of Fig. 8. The periodicity of the variations of the PES in the  $i$ -direction was chosen to be 50 atomic spacings. We also use a simplified spatial variation of  $D_{det}$ , and vary it between  $422 \text{ s}^{-1}$  and  $750 \text{ s}^{-1}$ . Smaller or larger numbers for  $D_{det}$ , or even a constant  $D_{det}$  in the same range, lead to very similar results.

The results shown in Fig. 8 correspond to the thermodynamic limit (left panel), where only  $E_{ad}$  is varied, and the kinetic limit (right panel), where only  $E_{trans}$  is varied. For the particular choices presented here, the spatial variation of the diffusion constant  $D$  is identical in both cases. The PES is varied only along the  $i$ -direction, and is constant along the  $j$ -direction. It is immediately evident from the morphologies that islands almost exclusively form along stripes in either limit. But in the kinetic limit, the islands are rather large, while they are much smaller in the thermodynamic limit. Closer inspection shows that the positions of most islands are inverted. In fact the islands nucleate in the region of fast diffusion (low potential energy barrier) in the kinetic limit, but nucleate in the region of slow diffusion in the thermodynamic limit, and that correspondingly all the mass is in these regions.

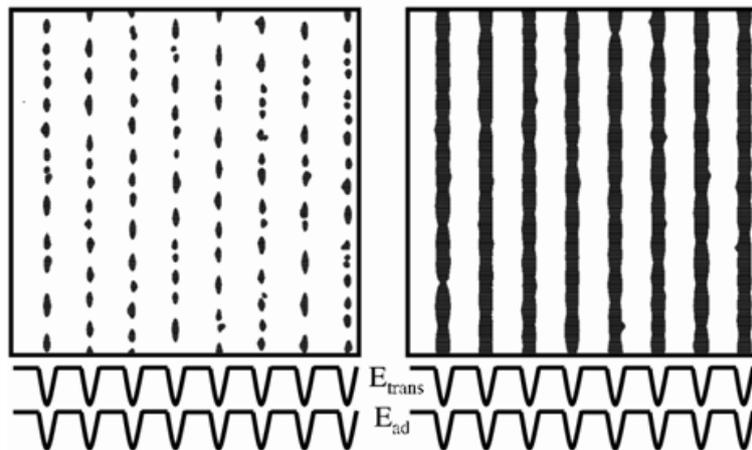
The explanation for this is the following: In the nucleation rate  $dN/dt$  in (14), the parameter  $\sigma_1$  is essentially constant, so that  $dN/dt$  increases either when  $D$  increases, or when  $\rho(\mathbf{x})$  increases. In the kinetic limit (without a thermodynamic drift),  $\rho(\mathbf{x})$  is spatially constant (at least before islands start acting as sinks on the surface, which is the case in the nucleation phase), so that the nucleation rate is dominant in regions where  $D$  is large. However, once a thermodynamic drift is present, the adatom concentration is not constant, and is in fact largest in regions where  $E_{ad}$  has its minimum. If the drift term is large enough,  $dN/dt$  is dominated by a large  $\rho$ , which is in regions where  $D$  is small (large barrier).

We can now also understand why the islands are much larger in the kinetic limit: Here, nucleation is determined by a large diffusion constant. But the diffusion constant  $D$  also determines a characteristic length  $l_{char} \sim D^\chi$ , which characterizes the size of and spacing between islands. The positive exponent  $\chi$  depends on the degree of reversibility (i.e.,  $D_{det}$  and  $F$ ) [34]. This means that in regions of large  $D$ , islands are on average larger and fewer.

**Fig. 8** Morphologies as obtained in the kinetic limit (*right*) and the thermodynamic limit (*left*) (top panels). A schematic of the envelope of the underlying variations of the PES is shown at the bottom for each case. Note that each period of the sinusoidal variation corresponds to 50 lattice constants



**Fig. 9** Morphologies at coverages  $\Theta = 0.1$  ML (*left*) and  $\Theta = 0.3$  ML (*right*) obtained with a PES (below) that has a much narrower variation



On the other hand, in the thermodynamic limit, islands nucleate in the region of small  $D$ , where  $l_{\text{char}}$  is smaller, and hence there are more and smaller islands.

The morphologies shown so far were all obtained at a sub-monolayer pre-coalescence coverage of  $\Theta = 0.2$  ML and with a PES that varies sinusoidally. In Fig. 9 we show the morphology at different coverages obtained with PES that varies more sharply in certain regions, and is essentially constant in others. At  $\Theta = 0.1$  ML, the islands are aligned even better than in the previously discussed cases. Moreover, at  $\Theta = 0.3$  ML, all the islands that are aligned along the  $j$ -direction have coalesced in this direction, while they do not touch at all along the  $i$ -direction. In fact, we get a very regular array of one-dimensional, monolayer-high nano-wires on the surface. Our simulations suggest a new mechanism by which quantum wires can be obtained, with a width that can be much smaller.

## 5 Conclusions

The simulations described above have established the validity of the level set method for simulation of epitaxial growth. Moreover, the level set method makes possible simulations that would be difficult for atomistic methods such as KMC; e.g., systems with large rates of attachment/detachment due to strain [33]. This method can now be used with confidence in many applications that include epitaxy along with additional phenomena and physics.

Atomistic strain due to lattice mismatch in heteroepitaxy is an important feature of thin films. The mathematical model and computational method described here make possible effective simulation of epitaxial systems with strain even in three dimensions; e.g., [27]. This may allow application to many epitaxial phenomena of scientific and technological interest. Two examples presented here are the structure of nanocrystals and pattern formation on an epitaxial surface.

We have examined the elastic energy density of a nanocrystal and the corresponding critical shell thickness. The simulation results presented above are for a highly idealized model of a layered nanocrystal. The robustness of these results with respect to variation of dimension, geometry and material parameters suggests that these results are qualitative and generally applicable.

The results on pattern formation suggest an approach to guiding self-assembly of nano patterns. Application of this approach, even in simulation, will require several additional ingredients, including microscopic models of elasticity and of the strain dependence of the PES and other properties. Also, strain induced changes of the PES due to the developing surface morphologies should be included in a more comprehensive model [21].

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# A level set method for spiral crystal growth and growth rate of crystal surface

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## 1. INTRODUCTION

Burton, Cabrera and Frank proposed the theory of step motion on a crystal surface and evolution of the crystal surface with aid of screw dislocations in [BCF]. According to their theory steps evolves with the step velocity  $V$  of the form

$$(1) \quad V = v_\infty(1 - \rho_c \kappa),$$

where  $\kappa$  is the curvature of the step,  $v_\infty$  is the velocity of straight line step, and  $\rho_c$  is the critical radius for generation of two dimensional kernel. They also pointed out that, if screw dislocations appear on the crystal surface, then the steps provided by screw dislocations describe spiral patterns whose centers are the screw dislocations. Consequently the steps form pyramids, evolve like as a rotating spirals, and then the surface evolves.

In this talk we consider the situation such that a lot of screw dislocations are on the crystal surface. Some mathematical models for such a situation are proposed from two points of view; one is the phase field model with Allen–Cahn type equation, and the other is the level set formulation. Karma and Plapp [KP], or Kobayashi [Ko] proposed phase field models by Allen–Cahn type equation with multiple-well potentials and the sheet structure function which expresses the initial surface. On the other hand, Smereka [S] or the author [O] proposed level set formulations. Generally, the level set formulation (and also the Allen–Cahn equation) is the methods to describe a motion of interfaces which divide the domain into two regions. However, spiral curves do not divide the domain into two regions. To overcome this difficulty, Smereka introduce two auxiliary functions to describe spiral curves by level sets. However, his model is then a system of partial differential equation, and does not include some complicated situations, for examples, there exist two ore more screw dislocations with multiple steps for a screw dislocation. On the other hand the author combine the idea of level set method and sheet structure function by [KP] or [Ko]. The author’s model includes more general situations, which the [KP]’s or [Ko]’s model include, than Smereka’s model. Moreover, the author’s model is simpler than Smereka’s model so that one can define the solution of the author’s model in viscosity solution sense. Some mathematical results for the author’s model are obtained in [O] and [GNO].

The aim of this talk is to give a brief introduction of the level set method with sheet structure function for spiral curves and some mathematical results for surface evolution of the crystal.

## 2. LEVEL SET FORMULATION

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain in  $\mathbb{R}^2$ . We denote by  $a_1, \dots, a_N \in \Omega$  locations of the center of each screw dislocations. Here we assume that  $j$ -th screw dislocation, whose center is  $a_j$ , are denoted by closed neighborhood of  $a_j$ , which is denoted by  $B_j \subset \Omega$ . We also assume that  $\partial B_j$  is smooth. We set  $W = \Omega \setminus \left(\bigcup_{j=1}^N B_j\right)$ . We assume that Burgers vector  $\vec{\gamma}_j$  at  $a_j$  is vertical to the surface, and thus we set  $\vec{\gamma}_j = (0, 0, m_j)$  where  $m_j \in \mathbb{Z} \setminus \{0\}$ . Here the orientation of  $m_j$  is such that  $m_j > 0$  (resp.  $m_j < 0$ ) if the lattice structure around  $a_j$  is anti-clockwise (resp. clockwise) spiral staircase, i.e., steps go around  $a_j$  anti-clockwise (resp. clockwise).

The spiral pattern  $\Gamma_t$  at time  $t \geq 0$  and its orientation  $\vec{\mathbf{n}}$  is given by

$$(2) \quad \Gamma_t := \{x \in \overline{W}; u(t, x) - \theta(x) \equiv 0 \pmod{2\pi\mathbb{Z}}\}, \quad \vec{\mathbf{n}} = \frac{\nabla(u - \theta)}{|\nabla(u - \theta)|}$$

with an auxiliary function  $u = u(t, x)$ , where  $\theta$  is the sheet structure function introduced by [KP] or [Ko] defined by

$$\theta(x) := \sum_{j=1}^N m_j \arg(x - a_j),$$

and  $\arg(x)$  is the function describing the argument of  $x \in \mathbb{R}^2 \setminus \{0\}$ . We note that  $\arg(x)$  is the multiple-valued function, however  $\nabla\theta$  is defined as a single-valued function of the form

$$\nabla\theta(x) = \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) \text{ for } x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}.$$

The reason why we consider  $\theta$  is a multiple-valued function is to describe a spiral curve completely. To understand the sense of the formulation (2) it is convenient to introduce a covering space of  $\overline{W}$  regarding  $\arg(x - a_j)$  as one of parameters

$$(3) \quad \mathfrak{X} := \{(x, \xi) \in \overline{W} \times \mathbb{R}^n; \arg(x - a_j) = \xi_j \text{ for } \xi = (\xi_1, \dots, \xi_N)\},$$

where the equality  $\arg(x - a_j) = \xi_j$  is in the sense of  $(\cos \xi_j, \sin \xi_j) = (x - a_j)/|x - a_j|$ . In  $\mathfrak{X}$  the inside of the crystal and the step are described by

$$(4) \quad \tilde{\Gamma}_t^u := \{(x, \xi) \in \mathfrak{X}; u(t, x) - \sum_{j=1}^N m_j \xi_j > 0\},$$

$$(5) \quad \tilde{\Gamma}_t^u := \{(x, \xi) \in \mathfrak{X}; u(t, x) - \sum_{j=1}^N m_j \xi_j = 0\},$$

respectively. The formulation (2) is derived from the projection of  $\tilde{\Gamma}_t$  onto  $\mathbb{R}^2$ . This covering space is also useful to avoid the difficulty come from the multiplicity of  $\theta$  in mathematical analysis.

To find a solution  $\Gamma_t$  to (1) we have to introduce a boundary condition on  $\partial W$ . In this talk we assume the Neumann boundary condition

$$(6) \quad \Gamma_t \perp \partial W$$

because of the mathematical reason. Then, we derive the level set equation of (1) and (6) in usual level set method because  $\theta$  is locally a smooth function on  $\overline{W}$ . Thus,  $u$  satisfies

$$(7) \quad u_t - v_\infty |\nabla(u - \theta)| \left\{ 1 + \rho_c \operatorname{div} \frac{\nabla(u - \theta)}{|\nabla(u - \theta)|} \right\} = 0 \quad \text{in } (0, T) \times W,$$

$$(8) \quad \langle \nabla(u - \theta), \vec{\nu} \rangle = 0 \quad \text{on } (0, T) \times \partial W,$$

where  $\vec{\nu}$  is the outer unit normal vector field of  $\partial W$ . The equation (7) is degenerate parabolic type and non divergence form so that we consider a solution in viscosity solution sense. One can define the viscosity solution to (7)–(8) in usual way. See [G] the details of the level set method and the definitions viscosity solution to the degenerate parabolic equations.

The comparison, existence and uniqueness of viscosity solution to (7)–(8) are obtained by [O]. The author sets  $B_j = \{x \in \mathbb{R}^2; |x - a_j| \leq \rho_j\}$  with  $\rho_j > 0$  in [O]. However, the results in [O] or [GNO] can be extended to more general  $B_j$  in our situation.

**Theorem 1** ([O, Theorem 2.1]). *Let  $u, v: [0, T) \times \overline{W} \rightarrow \mathbb{R}$  be an upper and lower semicontinuous viscosity sub- and supersolution, to (7)–(8) on  $(0, T) \times \overline{W}$ , respectively. If  $u \leq v$  on  $\{0\} \times \overline{W}$ , then  $u \leq v$  on  $[0, T) \times \overline{W}$ .*

**Theorem 2** ([O, Theorem 2.2]). *For  $u_0 \in C(\overline{W})$ , there exists a unique time-global viscosity solution  $u \in C([0, \infty) \times \overline{W})$  to (7)–(8) with  $u|_{t=0} = u_0$ .*

To describe the motion of spiral patterns on the crystal surface we have to give an initial data  $u_0 \in C(\overline{W})$  from given initial curve  $\Gamma_0$ . It is not so easy, however Goto, Nakagawa and the author obtain the existence of  $u_0$  for suitable  $\Gamma_0$ . See [GNO, §4] for the details. However, in general,  $u_0 \in C(\overline{W})$  for  $\Gamma_0$  in the level set method is not unique. Thus, it is important to investigate the uniqueness of level sets with respect to the initial curve. In [GNO] they proved the uniqueness of level sets. However, the following comparison is established by adjusting the proof in [GNO].

**Theorem 3** ([GNO]). *Let  $u, v: [0, T) \times \overline{W} \rightarrow \mathbb{R}$  be an upper and lower semicontinuous viscosity sub- and supersolution, to (7)–(8) on  $(0, T) \times \overline{W}$ , respectively. If  $\tilde{I}_0^u \subset \tilde{I}_0^v$  (resp.  $(\tilde{I}_0^u \cup \tilde{\Gamma}_0^u)^c \supset (\tilde{I}_0^v \cup \tilde{\Gamma}_0^v)^c$ ), then  $\tilde{I}_t^u \subset \tilde{I}_t^v$  (resp.  $(\tilde{I}_t^u \cup \tilde{\Gamma}_t^u)^c \supset (\tilde{I}_t^v \cup \tilde{\Gamma}_t^v)^c$ ), for  $t \in (0, T)$ , where  $\tilde{I}_t^u$  is defined as (4).*

The uniqueness of level sets follows from the above comparison.

### 3. SURFACE EVOLUTION

We derive the motion of spirals by the followings.

- (i) Construct  $u_0 \in C(\overline{W})$  from  $\Gamma_0$ .
- (ii) Solve (7)–(8) with  $u|_{t=0} = u_0$ .
- (iii) Sketch  $\{x \in \overline{W}; u(t, x) - \theta \equiv 0 \pmod{2\pi\mathbb{Z}}\}$ .

To investigate the surface evolution, we construct a surface height  $h(t, x)$  from  $u(t, x)$ . By the theory of dislocations (see [HL]),  $h$  satisfies

$$(9) \quad \Delta h = -h_0 \operatorname{div} \delta_{\Gamma_t} \mathbf{n}$$

if we have only displacement in the vertical direction and it is small enough, where  $h_0$  is the unit height of the step, and  $\delta_U$  is the Dirac's delta measure for  $U \subset \mathbb{R}^2$ . By straightforward calculation we observe that  $h(t, x) = (h_0/2\pi)\theta_{\Gamma_t}(x)$ , where  $\theta_{\Gamma_t}$  is a branch of  $\theta$  whose discontinuity is only on  $\Gamma_t$ , satisfies (9). Once we obtain a solution  $u$  to (7)–(8), we obtain

$$\theta_{\Gamma_t}(x) = \tilde{\zeta}(t, x, \Theta(x))$$

with  $\tilde{\zeta}: [0, \infty) \times \mathcal{X} \rightarrow \mathbb{R}$  defined as

$$\tilde{\zeta}(t, x, z) = z - 2\pi k \quad \text{if } (x, z) \in \{(y, \eta) \in \mathcal{X}; u(t, x) - z \in [2\pi k, 2\pi(k+1))\},$$

where  $\Theta_j(x) \in [0, 2\pi)$  is the principal value of  $\arg(x - a_j)$  and  $\Theta(x) = \sum_{j=1}^N m_j \Theta_j(x)$ . If we obtain  $h(t, x)$ , the mean growth rate of the surface in  $[t_0, t]$  is calculated by

$$H_h(t; t_0) := \frac{1}{|W|} \int_{\overline{W}} [h(t, x) - h(t_0, x)] dx,$$

where  $|W|$  is the measure of  $W$ . If  $h$  is smooth for  $t > 0$ , then we obtain the growth rate  $R(t)$  from

$$R(t) = \frac{1}{|W|} \int_{\overline{W}} h_t(t, x) dx.$$

In this talk we discuss on the surface evolution by a pair of screw dislocations with opposite orientations. In [BCF] the authors pointed out that if a pair of screw dislocations with opposite orientations is too close, then they have no influence to the surface evolution. For this claim we obtain the following results.

**Theorem 4.** *Let  $N = 2$ ,  $m_1 = -m_2 = 1$ , and  $B_j = \{x \in \overline{W}; |x - a_j| \leq \rho\}$  for some  $\rho < |a_1 - a_2|/2$ . Assume that  $|a_1 - a_2| \leq 2\rho_c$  and  $\Omega$  is large enough. Then, for any  $u_0 \in C(\overline{W})$  there exists  $M > 0$  such that  $u(t, x) < M$  for  $t > 0$ , where  $u(t, x)$  is a viscosity solution to (7)–(8).*

We recall that the inside of the crystal is denoted by  $\tilde{I}_t^u$  defined by (4). Thus, if the crystal grows up, then  $u \rightarrow \infty$  as  $t \rightarrow \infty$ . Accordingly, Theorem 4 says that there exists a bound of height of the surface in this situation.

For the motion of closed curve by (1)  $C = \{x \in \mathbb{R}^2; |x| = \rho_c\}$  is the stationary solution. and thus the curve  $\Gamma$  which is a part of circle whose radius is  $\rho_c$  and satisfies (6) only on  $\partial B_j$  ( $j = 1, 2$ ) should be a solution to (1) and (6) for our problem. However, it is not so clear because there is no continuous stationary solution, which denotes the steady curve  $C$ , to the level set equation of (1) for the closed curve. Then, to prove Theorem 4, we construct a discontinuous solution to (7)–(8). We observe that there exists  $\theta_\Gamma$  which is a lower semicontinuous branch of  $\theta$  whose discontinuity is only on  $\Gamma$ . By straightforward calculation we observe that  $\theta_\Gamma$  is a viscosity solution to (7)–(8), and thus  $v := \theta_\Gamma + M$  is also that for  $M \in \mathbb{R}$ . Thus, if we choose  $M$  large enough such that  $u_0 \leq v$  on  $\overline{W}$ , then  $u(t, \cdot) \leq v$  on  $\overline{W}$  for  $t > 0$  from Theorem 1.

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## Direct observation of elementary growth processes of ice crystals by advanced optical microscopy

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Growth and melting processes of ice crystals govern wide variety of phenomena on earth. Hence, this issue has been a subject of interest for centuries. To understand crystal growth of ice at the molecular level, one has to observe in-situ “elementary steps”, which play a key role during growth and melting processes on ice crystal surfaces. However, since observation of ice crystal surfaces by scanning probe microscopy is very difficult, so far only one group has succeeded in such observation by atomic force microscopy [1]. In this study, we adopted laser confocal microscopy combined with differential interference contrast microscopy (LCM-DIM), by which elementary steps of protein crystals (3-6 nm in height) could be visualized with sufficient contrast levels [2], and tried to visualize molecular-level surface morphologies on ice crystal surfaces.

We first attempted to observe the air-ice interface, since this interface has a much larger reflectivity than water-ice interfaces. By further improving LCM-DIM and growing ice crystals of higher quality, we finally succeeded in observing ice crystal surfaces grown by the two-dimensional (2D) nucleation growth mechanism. When steps of neighboring 2D islands coalesced with each other, the contrast of steps disappeared completely. Such disappearances of the step contrasts were commonly observed all over the crystal surface confirming that we succeeded in observing elementary steps (0.37 nm in height), for the first time, by optical microscopy. We also succeeded in observing surface melting processes at air-ice interfaces. We could visualize the appearances of two types of quasi-liquid layers (bulk-liquid like drops (BLD) and thin-liquid like layers (TLL)) and growing elementary steps simultaneously on the same crystal surface.

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# Evolution of Microstructures on Silicon Substrates by Surface Diffusion

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Recently, spontaneous shape transformation of microstructures fabricated on silicon substrates by high temperature annealing has attracted attention because of the potentiality for development of novel microstructure fabrication techniques [1]. We have studied the mechanism of the spontaneous shape transformation of high-aspect-ratio microstructures, such as one-dimensional (1D) gratings [2,3] and two-dimensional (2D) hole arrays [4], on Si(001) substrates.

The conventional fabrication technique using reactive ion etching (RIE) with SiO<sub>2</sub> mask is employed to fabricate high-aspect-ratio microstructures on n-type Si(001) substrates. High temperature annealing of the sample is conducted under hydrogen gas ambient using a ramp furnace or under ultrahigh vacuum (UHV). The structures of the samples were observed by scanning electron microscopy (SEM). For evaluation of cross-sectional profiles, the samples were cleaved parallel to (001) plane.

When 1D grating structures are annealed above ~ 1000 °C, rounding of the trench corners occurs. In Fig. 1, the evolution of a trench corner is shown with the dependence of the corner curvature on annealing time. According to the time scaling property of shape transformation presented by Herring, the characteristic length scale of a steady state profile evolves as  $t^{1/2}$  and  $t^{1/4}$  as the shape transformation occurs by evaporation-condensation and surface diffusion, respectively [5]. Thus, the decrease in the curvature as  $t^{-1/4}$  shown in Fig.1 is the evidence suggesting that the dominant mass transport mechanism responsible for the shape transformation is surface diffusion, although sublimation may be possible at such high temperature. We have performed numerical simulations of the shape transformation of one-dimensional trench structures by surface diffusion

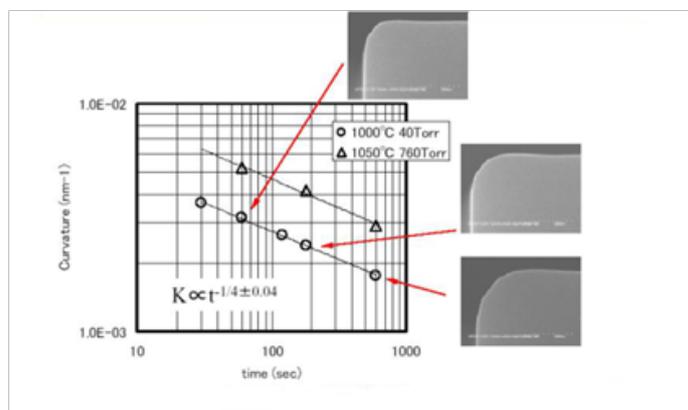


Fig. 1 — Annealing time dependence of the curvature of trench corners.

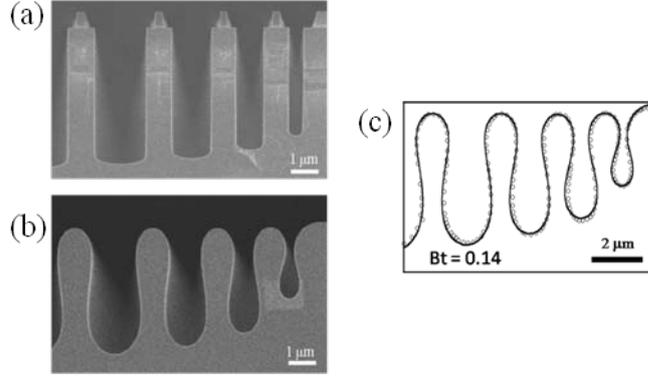


Fig. 2 Comparison of the shape transformation between the experiment and the simulation using Mullins' equation. SEM images showing (a) the initial trench array and (b) the structure after annealing. In (c), the solid line shows the profile obtained by the simulation and open circles correspond to the experimental result shown in (b).

using the Mullins' equation [6],

$$v = \frac{D_s \gamma \Omega^2 c_0}{kT} \frac{\partial^2 K}{\partial s^2}, \quad (1)$$

where  $v$  is the normal velocity of the surface,  $s$  is the arc length along the surface,  $K$  is the surface curvature,  $D_s$  is the diffusion constant,  $\gamma$  is the surface tension,  $\Omega$  is the atomic volume, and  $c_0$  is the adatom density on the surface. Figure 2 compares the profile evolution between the experiment and the simulation [7]. It is found that the simulation can reproduce the complex profile evolution of the 1D trenches although the equation (1) assumes an isotropic surface which is inadequate below the roughening temperature.

Arrays of cylindrical holes with high aspect ratios show more complex shape transformation by surface diffusion. Figure 3 shows the shape transformation of hole arrays for the two initial

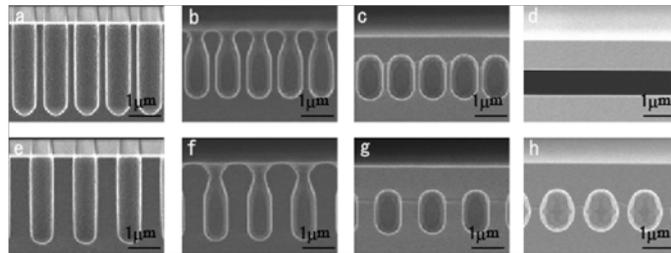


Fig. 3 Cross-sectional SEM images showing the evolution of the square arrays of cylindrical holes on Si(001) substrates during annealing in hydrogen ambient for two different periods of (a-d) 1.0  $\mu\text{m}$  and (e-h) 1.8  $\mu\text{m}$ .

structures with different periods. For both the samples, the radius and depth of the holes are 0.75 and 3.0  $\mu\text{m}$ , respectively. The periods of the patterns are 1.8 and 1.0  $\mu\text{m}$ . During the early stages of the shape transformation, the hole inlets are closed by bulging of the surface around the hole inlet. The bulging is characteristic for the shape transformation by surface diffusion. For the shape transformation of cylindrical holes the surface bulging is more enhanced than the evolution of 1D gratings as shown in Fig. 2 because of the convergence effect of the diffusion on cylindrical surfaces. After closing the inlets, vertically elongated voids remain in the bulk Si and subsequently the shape relaxation of the buried voids occurs. If the spacing between neighboring voids is sufficiently small, coalescence of voids occurs by the lateral expansion of each void, leading to the formation of a large plate-shaped void.

The shape change of the each void formed by the hole inlet closure is shown in Fig. 4. The high resolution SEM observations show that the individual voids are rather faceted. The observed facets are identified to be  $\{100\}$ ,  $\{110\}$ ,  $\{111\}$  and  $\{113\}$  facets, which are well known as thermodynamically stable facets for Si. The shape change of the void proceeds while keeping the faceted structure, involving variation of the area and shape of the each facet. In addition, it is found that the volume of the each void is kept constant during shape change, strongly suggesting that the shape change of the buried void occurs by surface diffusion.

In order to identify the mechanism of the observed shape change of the void we have performed numerical simulations of shape evolution for completely faceted voids by surface diffusion. In the simulation, instead of the curvature dependent chemical potential, the mean chemical potential [8] of each facet,

$$\mu_i = \mu_0 + \Omega K_i, \quad (2)$$

is employed. Here,  $K_i$  is the weighted mean curvature, which is given by

$$K_i = \frac{1}{S_i} \sum_{j \neq i} f_{ij} l_{ij}. \quad (3)$$

The summation is taken over all the neighboring facets of the  $i$ -th facet.  $S_i$  is the area of the  $i$ -th facet,  $l_{ij}$  is the length of the intersection between the  $i$ -th and  $j$ -th facets, and  $f_{ij}$  is a numerical factor determined by the geometry

$$f_{ij} = \frac{\gamma(\mathbf{n}_j) - c_{ij}\gamma(\mathbf{n}_i)}{\sqrt{1 - c_{ij}^2}}, \quad (4)$$

where  $\mathbf{n}$  is the unit normal vector of the surface,  $\gamma$  is the surface free energy per unit area, and  $c_{ij} = \mathbf{n}_i \cdot \mathbf{n}_j$ . The normal velocity of the  $i$ -th facet is determined by the total atom flux into the  $i$ -th facet from all the neighboring facets. Thus the normal velocity  $v_i$  of the  $i$ -th facet is given by

$$v_i = \frac{\Omega}{S_i} \sum_{j \neq i} l_{ij} J_{ij}, \quad (5)$$

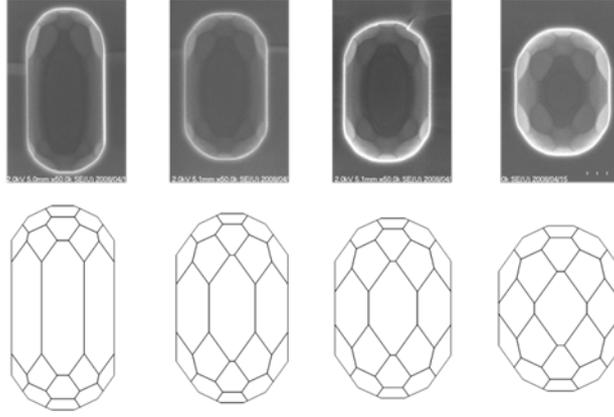


Fig. 4 SEM observations (upper panels) and simulation results (lower panels) of the evolution of a faceted void during annealing. In the experimental result, the annealing times are 5, 10, 20, and 40 min.

where  $J_{ij}$  is the flux into the  $i$ -th from  $j$ -th facets. We assume the flux as

$$J_{ij} = \frac{D_s c_0}{kT} \frac{\mu_i - \mu_j}{\Delta x_{ij}}, \quad (6)$$

where  $\Delta x_{ij}$  is the effective diffusion distance between the  $i$ th and  $j$ th facets [9]. Here, for simplicity, we define the effective diffusion distance as  $\Delta x_{ij} = S_i^{1/2} + S_j^{1/2}$ .

The simulation result of the evolution of a polyhedral void composed of  $\{100\}$ ,  $\{110\}$ ,  $\{111\}$ , and  $\{113\}$  facets in a cubic crystal is shown in Fig. 4. In the simulation, we used the surface free energies of Si reported in the literature [10]. The good agreement between the simulation and experiment implies that the dominant mass transport mechanism responsible for the shape change of the faceted voids is surface diffusion.

In conclusion, we have studied the evolution of high-aspect-ratio microstructures fabricated on Si substrates during high temperature annealing. For evolution of 1D trenches and 2D arrays of cylindrical holes on Si substrates, we have shown the dominant mass transport mechanism is surface diffusion. It has been found that the evolution of the macroscopic profile of 1D structure is well reproduced by the Mullins' equation. We have also performed numerical simulations of the shape change of a completely faceted void via only surface diffusion driven by the mean chemical potential differences between the facets. The simulation has reproduced the observed shape evolution of a single void in bulk Si during annealing.

I am grateful to all collaborators of the researches introduced in this presentation, H. Iwasaki, R. Shimizu, R. Hiruta, and H. Kuribayashi.

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# Poster Session

July 26, 2010

**Hiroki Hibino** (NTT Basic Research Laboratories)

Dynamics of Si surface morphology

**Sarah Hormozi** (Univ. of British Columbia)

Experimental studies on visco-plastic lubrication of visco-elastic fluid

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Large-time asymptotics for a class of non-coercive Hamilton-Jacobi equations appearing in crystal growth

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Curvature dependent diffusion flow on surface with thickness

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The appearances of two types of quasi-liquid layers at air-ice crystal interfaces observed by advanced optical microscopy

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Step dynamics in relaxation of gratings on Si surfaces

**Yuen Au Yeung** (TU Munich)

Crystallization in the Wulff shape via Gamma-Convergence

**Xiao-Yu Zhang** (Yamagata Univ.)

High order numerical method for nonlinear two-point boundary value problems

# ON THE LONGTIME BEHAVIOR OF SOLUTIONS TO A MODEL FOR EPITAXIAL GROWTH

MAURIZIO GRASELLI, GIANLUCA MOLA, ATSUSHI YAGI

## 1. INTRODUCTION

A well-known and relatively simple model to describe the epitaxial growth process leads to the formulation of the following fourth-order nonlinear equation

$$(1.1) \quad \partial_t u + \Delta^2 u = -\mu \nabla \cdot \left( \frac{\nabla u}{1 + |\nabla u|^2} \right) \quad \text{in } \Omega \times (0, \infty),$$

in a two-dimensional bounded domain  $\Omega \subset \mathbb{R}^2$ ,  $\mu$  being a (positive) constant called surface roughening coefficient. Here  $u$  denotes the height profile, measured in a co-moving frame, of a thin film in epitaxial growth. The biharmonic operator accounts for the surface diffusion, while the divergence type term was firstly proposed in [8] to model the behavior of adatoms (i.e., adsorbed atoms). We refer the reader to [11] and references therein for further details on equation (1.1) as well as for an analysis of its qualitative properties. We also mention that a similar equation where the divergence type term has a rather general form has been considered in [9]. However, the present nonlinearity does not satisfy the coercivity assumption [9, (H2b)] which is needed to prove the existence of a weak solution. Thus the present equation is not a particular case of the one studied in [9].

More recently, equation (1.1) has been investigated within the theory of dissipative dynamical systems in a series of papers [4, 5, 6] where further references on (1.1) can also be found. More precisely, the authors have considered the equation subject to the initial condition

$$(1.2) \quad u(0) = u_0 \quad \text{in } \Omega,$$

and to the boundary conditions

$$(1.3) \quad \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} \Delta u = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

where  $\partial_{\mathbf{n}}$  stands for the (outward) normal derivative to  $\partial\Omega$ . In [4] well-posedness and regularity results for (1.1)-(1.3) have been established. Such results lead to the definition of a suitable dynamical system which possesses the global attractor. Existence of exponential attractors and the analysis of  $\omega$ -limit sets have been the subject of [5]. Then, in [6], the stability properties of the null solution with respect to  $\mu$  has been analyzed in order to find a lower bound for the dimension of the global attractor. All these results have been obtained by assuming  $\partial\Omega$  of class  $C^4$  and working with rather smooth solutions. However, from the physical viewpoint,  $\partial\Omega$  can be nonsmooth (for instance, a polygon). Thus it seems necessary to extend the analysis of the longterm behavior to more general spatial domains and to weaker solutions. This is our first goal, namely, to provide a rather general and simple proof of the existence of a global and an exponential attractor which

allows to take nonsmooth  $\partial\Omega$ . In addition, we show that each solution converges to a single stationary state, provided that  $\partial\Omega$  is smooth enough. This is done by means of a suitable version of the Łojasiewicz-Simon inequality. An estimate of the convergence rate is also obtained.

## 2. THE DYNAMICAL SYSTEM IN $L^2(\Omega)$

Let  $H$  be the (real) Hilbert space  $L^2(\Omega)$  endowed with the usual scalar product  $\langle \cdot, \cdot \rangle$  and the related norm  $\| \cdot \|$ . Then, we consider the Hilbert triplet  $V = H^1(\Omega) \hookrightarrow H \equiv H^* \hookrightarrow V^*$  and we consider  $-\Delta : W \rightarrow H$  where

$$(2.1) \quad W = \{w \in V : \partial_{\mathbf{n}}w = 0, \quad \Delta w \in H\}$$

endowed with the graph norm  $(\|w\|^2 + \|\Delta w\|^2)^{1/2}$ . We recall that  $W \hookrightarrow H^{3/2-\epsilon}(\Omega)$  for all  $\epsilon \in (0, 1/2)$ , when  $\partial\Omega$  is only Lipschitz. Otherwise, if  $\Omega$  is a polygonal domain, then we have  $W \hookrightarrow H^{3/2}(\Omega)$ . Moreover, if  $\partial\Omega$  is of class  $C^{1,1}$ , then  $W \hookrightarrow H^2(\Omega)$ . Here and by, for the sake of convenience, we replace  $\| \cdot \|_{X^2}$  with the shorter notation  $\| \cdot \|_X$ , for any space  $X^2 = X \times X$ ,  $X$  being a Banach space. Besides  $\langle \cdot, \cdot \rangle_{X^*, X}$  denotes the duality coupling.

Let  $u_0 \in H$ . Our definition of weak solution to is the following (cf. also [10, Def.3.1])

**Definition 2.1.** *A function  $u \in C([0, \infty); H) \cap L^2_{loc}(0, +\infty; W)$  is a weak solution to (1.1)-(1.3) if*

$$(2.2) \quad \langle \partial_t w, z \rangle_{W^*, W} + \langle \Delta w, \Delta z \rangle = \mu \langle (1 + |\nabla u|^2)^{-1} \nabla u, \nabla z \rangle, \\ \forall z \in W, \text{ a.e. in } (0, \infty),$$

$$(2.3) \quad u(0) = u_0, \quad \text{a.e. in } \Omega.$$

As a consequence, the total mass of  $u(t)$  is conserved, that is,

$$(2.4) \quad \langle u(t), 1 \rangle = \langle u_0, 1 \rangle, \quad \forall t \geq 0.$$

We first prove the following continuous dependence estimate (compare with [4, Prop. 4.3])

**Theorem 2.2.** *Let  $u_0, v_0 \in H$  and denote by  $u$  and  $v$  the corresponding weak solutions to problem (1.1)-(1.3). Then, for any time  $T > 0$ , there exists a positive constant  $C$ , also depending on  $\Omega$  and  $\mu$ , such that the following continuous dependence estimate holds*

$$(2.5) \quad \|(u - v)(t)\|^2 + \int_0^t \|\Delta(u - v)(\tau)\|_H^2 d\tau \leq C e^{CT} \|u_0 - v_0\|^2,$$

for any  $t \in [0, T]$ .

It is now standard to prove the existence of a weak solution. This can be done through a Galerkin scheme (see, e.g., [10]). From now the use of such an approximation scheme will be tacitly assumed.

Then we can summarize the consequences of Theorem 2.2 with the following

**Theorem 2.3.** *Problem (1.1)-(1.3) generates a strongly continuous semigroup  $S(t)$  on the phase-space  $H$ .*

Property (2.4) lead us to define, for all  $\alpha \geq 0$ , the bounded-average (complete metric) spaces

$$H_\alpha = \{u \in H : |\langle u, 1 \rangle| \leq \alpha\}, \quad V_\alpha = V \cap H_\alpha, \quad W_\alpha = W \cap H_\alpha.$$

Accordingly, from now on we set  $\widehat{u} = u - \langle u, 1 \rangle$  ( $H_0$ -projection of  $u \in H$ ). On account of (1.3), we have  $u(t) = S(t)u_0 \in H_\alpha$  for all times  $t > 0$ , if  $u_0 \in H_\alpha$ , i.e., the metric space  $H_\alpha$  is invariant under the action of  $S(t)$ . Moreover, the dynamical system  $(H_\alpha, S(t))$  is dissipative. Indeed, recalling the proof of [4, Corollary 4.1], we have

**Theorem 2.4.** *Let  $u_0 \in H_\alpha$ . Then, for all  $R > 0$  there exists positive constants  $C_0$  and  $\kappa_0$ , depending on  $\mu, |\Omega|$  and  $\alpha$  but independent of  $R$ , such that*

$$(2.6) \quad \sup_{\|u_0\| \leq R} \|u(t)\|^2 \leq C_0 (e^{-\kappa_0 t} \|u_0\|^2 + 1),$$

and

$$(2.7) \quad \sup_{\|u_0\| \leq R} \int_t^{t+1} \|\Delta u(\tau)\|^2 d\tau \leq C_0,$$

for all  $t \geq 0$ .

Therefore the semigroup  $S(t)$  can be restricted to a *dissipative* semigroup on the phase-space  $H_\alpha$ . In addition, we have

**Theorem 2.5.** *Let  $B_{R_0} \subset H_\alpha$  a bounded absorbing set for the dynamical system  $(H_\alpha, S(t))$ . Then, there exists  $t_1 = t_1(R_0) > 1$  and  $C_1 = C_1(R_0) > 0$  such that*

$$(2.8) \quad \|u(t)\|_V \leq C_1, \quad \forall t \geq t_1,$$

Therefore,  $(H_\alpha, S(t))$  has a global attractor  $\mathcal{A}_\alpha$  bounded in  $V_\alpha$ . Moreover, there holds

$$(2.9) \quad \int_t^{t+1} \|\nabla \Delta u(\tau)\|^2 d\tau \leq C_1,$$

for all  $t \geq t_1$ .

It is also easy to prove the so-called smoothing property (see [2])

**Theorem 2.6.** *For every  $u_0, v_0 \in B_{R_0}$ , there exists  $t_2 = t_2(R_0) > 1$  and  $C_2 = C_2(R_0) > 0$  such that the following estimate holds*

$$(2.10) \quad \|S(t)u_0 - S(t)v_0\|_V \leq C_2 \|u_0 - v_0\|,$$

for any  $t \geq t_2$ .

In order to establish the existence of an exponential attractor, we also need to establish the Hölder continuity of  $(t, u_0) \mapsto S(t)u_0$ . This follows from (2.5) and

**Lemma 2.7.** *Let  $B_{R_0} \subset H_\alpha$  a bounded absorbing set for the dynamical system  $(H_\alpha, S(t))$ . Then, there exists  $C_2 = C_2(R_0) > 0$  such that*

$$(2.11) \quad \|S(t)u_0 - S(\tilde{t})u_0\| \leq C_2 |t - \tilde{t}|^{1/4},$$

for all  $t, \tilde{t} \in [t_1, t_1 + 1]$ ,  $t_1$  being given by Theorem 2.5,

Collecting the above results, on account of [2], we deduce

**Theorem 2.8.**  $(H_\alpha, S(t))$  possesses an exponential attractor  $\mathcal{E}_\alpha$  bounded in  $V_\alpha$ . As a consequence,  $\mathcal{A}_\alpha$  has finite fractal dimension.

### 3. CONVERGENCE TO EQUILIBRIUM

In this section we shall prove the convergence to equilibrium of single trajectories.

Let us set

$$Z = \{u \in H^3(\Omega) : \partial_{\mathbf{n}}u = 0 \text{ a.e. on } \partial\Omega\},$$

endowed with its (natural) norm

$$\|\cdot\|_Z^2 = \|\cdot\|_W^2 + \|\nabla\Delta\cdot\|^2.$$

We also define  $Z_\alpha = Z \cap H_\alpha$ . By using the techniques described above (see also [4]) it is not difficult to prove the following

**Proposition 3.1.** *Let  $\partial\Omega$  be of class  $C^{2,1}$ . For every  $u_0 \in H_\alpha$ , we have*

$$\bigcup_{t \geq 1} \{S(t)u_0\} \subset Z_\alpha.$$

Consider now the set  $\mathcal{S}_\alpha$  of all steady states of problem (1.1)-(1.3) with average bounded by  $\alpha$ , namely any  $u_\infty \in Z_\alpha$  such that

$$(3.1) \quad \langle \nabla\Delta u_\infty + \mu(1 + |\nabla u_\infty|^2)^{-1} \nabla u_\infty, \nabla z \rangle = 0, \quad \forall z \in V_\alpha.$$

The main result is

**Theorem 3.2.** *Let  $\partial\Omega$  be of class  $C^{2,1}$ . For every  $u_0 \in H_\alpha$  there exists  $u_\infty \in \mathcal{S}_\alpha$  such that,*

$$(3.2) \quad u(t) = S(t)u_0 \rightarrow u_\infty \quad \text{in } H^2(\Omega),$$

as  $t \rightarrow \infty$ . Moreover, there exists  $t_1 > 0$  and a positive constant  $\bar{c}$  such that

$$(3.3) \quad \|u(t) - u_\infty\|_W \leq \bar{c}(1+t)^{-\frac{\vartheta}{2(1-2\vartheta)}}, \quad \forall t \geq t_1,$$

$\vartheta \in (0, 1/2)$  being the same constant as in the Łojasiewicz-Simon inequality (see Lemma 3.4).

The key tool to prove this result is to use a suitable Łojasiewicz-Simon inequality (see, e.g., [7] and references therein). To state it, we consider the functional

$$E(u) = \frac{1}{2} \|\Delta u\|^2 - \frac{\mu}{2} \int_\Omega \ln(1 + |\nabla u|^2) d\Omega,$$

defined for all  $u \in Z$ . Clearly  $E \in C^2(W)$ , with

$$E'(u) = \Delta^2 u + \mu \nabla \cdot \left( \frac{\nabla u}{1 + |\nabla u|^2} \right) : W \rightarrow W^*,$$

and

$$E''(u)v = \Delta^2 v + \mu \nabla \cdot \left[ \frac{(1 + |\nabla u|^2)^2 \nabla v - 2(\nabla u \cdot \nabla v) \nabla u}{1 + |\nabla u|^2} \right], \quad v \in W.$$

Here and below prime denotes the Fréchet derivative. The restriction of  $E$  to  $Z$  satisfies the following basic property

**Lemma 3.3.** *The functional  $E : Z \rightarrow \mathbb{R}$  is real analytic.*

Then, the inequality we need reads

**Lemma 3.4.** *Let  $u_\infty \in Z_\alpha$  be a solution to the stationary equation (3.1). Then there exists  $\theta \in (0, 1/2]$ ,  $C > 0$  and  $\sigma > 0$  such that, for all  $u \in Z_\alpha$  satisfying  $\|u - u_\infty\|_Z \leq \sigma$ , there holds*

$$(3.4) \quad |E(u) - E(u_\infty)|^{1-\theta} \leq C \left\| \Delta^2 u + \mu \nabla \cdot \left( \frac{\nabla u}{1 + |\nabla u|^2} \right) \right\|_{Z_0^*}.$$

For all  $u \in H_\alpha$ , we define the  $\omega$ -limit as

$$\omega(u_0) = \left\{ u_\infty \in Z_\alpha : \exists t_n \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ s.t. } S(t_n)u_0 \rightarrow u_\infty \text{ in } W \right\}.$$

First notice that, by multiplying equation (1.1) by  $\partial_t u$  in  $H$ , we have

$$(3.5) \quad \frac{d}{dt} E(u) = -\|\partial_t u\|^2.$$

Note that this can be done when  $u_0 \in W$  since equation (1.1) holds almost everywhere. Therefore, we deduce the following

**Proposition 3.5.** *The functional  $E$  is a Lyapunov functional for  $(W, S(t))$ .*

Consequently, standard results entail that

**Lemma 3.6.** *For any  $u_0 \in H_\alpha$ , the set  $\omega(u_0)$  is nonempty, compact, invariant and connected in  $W$  and the following inclusion holds  $\omega(u_0) \subset \mathcal{S}_\alpha$ . Moreover,  $E$  is constant on  $\omega(u_0)$ .*

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## **Kink generation by the association of 2D clusters**

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Abstract:

The density of kinks along the growth steps of a faceted crystal scales the step velocity and hence the rate of growth of the crystal. The classical mechanism of kink generation, as a result of the thermal fluctuations of the step edge, was put forth by J. W. Gibbs; Burton, Cabrera and Frank posited that density of kinks generated by this mechanism will not increase in a supersaturated solution. In the 1970s, it was proposed that on steps of low kink density, additional kinks may be generated by the one-dimensional nucleation of new crystal rows.

We demonstrate for the crystallization of Zn-insulin a novel mechanism of kink generation, whereby 2D clusters of several insulin molecules pre-formed on the terraces between steps associate to the steps. This mechanism results in several-fold higher kink density, faster rate of crystallization, and a high sensitivity of the kinetics to small increases of the solute concentration.

Rhombohedral crystals of Zn-insulin hexamers form in the islets of Langerhans in the pancreases of many mammals. The suggested function of crystal formation is to protect the insulin from proteases and increase the degree of conversion of soluble proinsulin. To accomplish this, crystal growth should be fast and adaptable to rate fluctuations in the conversion reaction.

If the found mechanism operates during insulin crystallization in vivo, it could be a part of the biological regulation of insulin production and function. For other crystallizing materials in biological and non-biological systems, this mechanism provides an understanding of the often seen non-linear acceleration of the kinetics.

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## ATOMISTIC AND COARSE-GRAINED MODELING OF EPITAXIAL THIN FILM GROWTH

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### ABSTRACT:

Homoepitaxial thin film growth (A on A) by vapor deposition on perfectly flat single-element single-crystal surfaces corresponds to growth of a single-crystal of the element A. Here, the term “epitaxy” means that the deposited atoms reside at the natural discrete locations or adsorption sites to propagate the perfect periodic structure of the underlying crystalline substrate. Numerous experimental studies have been performed on such systems over the last two decades via scanning probe microscopy and surface-sensitive diffraction under well-controlled (impurity-free) conditions. These are the simplest possible most well-defined thin film or crystal growth systems, and thus provide an ideal venue for detailed atomistic-level modeling, or for exploration of coarse-grained or multiscale modeling strategies [1,2]. Figure 1 shows three different modeling strategies for homoepitaxial growth ranging from a fully discrete (atomistic) treatment to a fully continuum description of evolution of a film height function,  $h(\underline{x},t)$ .

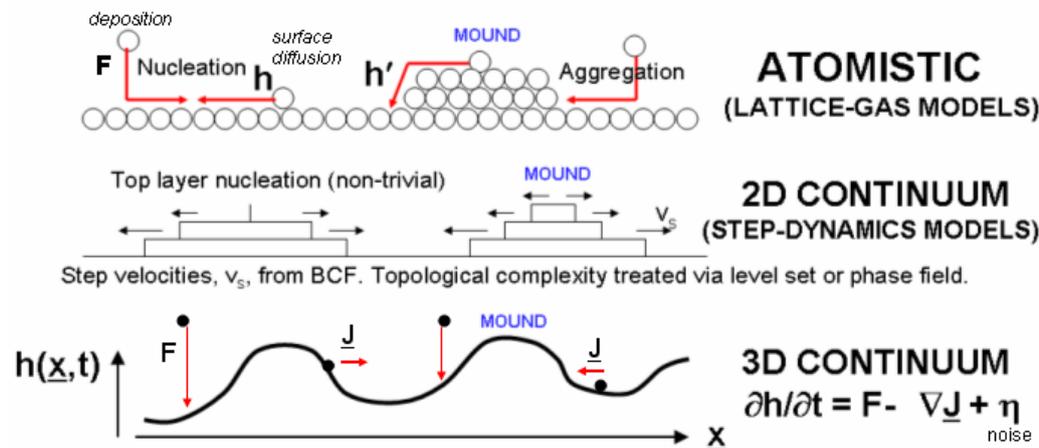


Fig.1 Schematic of different modeling strategies: atomistic LG; 2D continuum step-dynamics; 3D continuum PDE

For homoepitaxial growth, the equilibrium configuration or morphology of the film is trivial (under typical growth conditions). Loosely speaking, an equilibrated partial layer of atoms on a flat surface aggregates into a single 2D (one atom high) island [3]. Equilibrated multiple layers of deposited atoms form a flat film [4]. However, despite these simple equilibrium states, the morphologies of films deposited under typical growth conditions display a rich variety of complex morphologies. This complex behavior results from the feature that deposition drives the system very far-from-equilibrium, and the resultant non-equilibrium morphologies depend on the

details of the (surface diffusion) kinetics. For submonolayer deposition, atoms aggregate into multiple “small” 2D islands which can have complex non-equilibrium shapes (e.g., dendrites or fractals). For multilayer deposition, one typically finds “kinetic roughening” associated with the inhibited diffusion of deposited atoms from higher to lower layers. As a result, multilayer growth is “unstable” and characterized by the formation of 3D mounds (multilayer stacks of 2D islands).

There remain fundamental open issues regarding both submonolayer and multilayer growth regimes. Can one provide a precise description of the island size distribution (ISD) and spatial distribution of 2D islands on the surface during submonolayer growth? It is natural to tessellate the surface so that each island is surrounded by its own “capture zone” (CZ) and most atoms landing within that capture zone aggregate with the corresponding island. Then, one can also attempt to characterize the capture zone area distribution (CZD) [4]. For multilayer growth, one can find a delay in the mounding instability, followed by a regime of strong mound steepening, and then a subsequent regime of slope (or shape) selection and mound coarsening. These features, and especially the mound coarsening dynamics, are still incompletely characterized.

Atomistic modeling when combined with analysis via kinetic Monte Carlo (KMC) simulation, has provided a precise characterization of many aspects of evolving film morphology and also detailed insight into many of these issues [1,2]. However, coarse-grained modeling alternatives are appealing from the perspective of algorithmic efficiency, and also to provide deeper insight into fundamental issues such as development 2D island distributions or 3D mound coarsening dynamics. In addition, there is potential for analytic theory to contribute to our understanding of these systems. We provide an overview of recent progress and the current state of the field.

### SUBMONOLAYER GROWTH AND ISLAND FORMATION

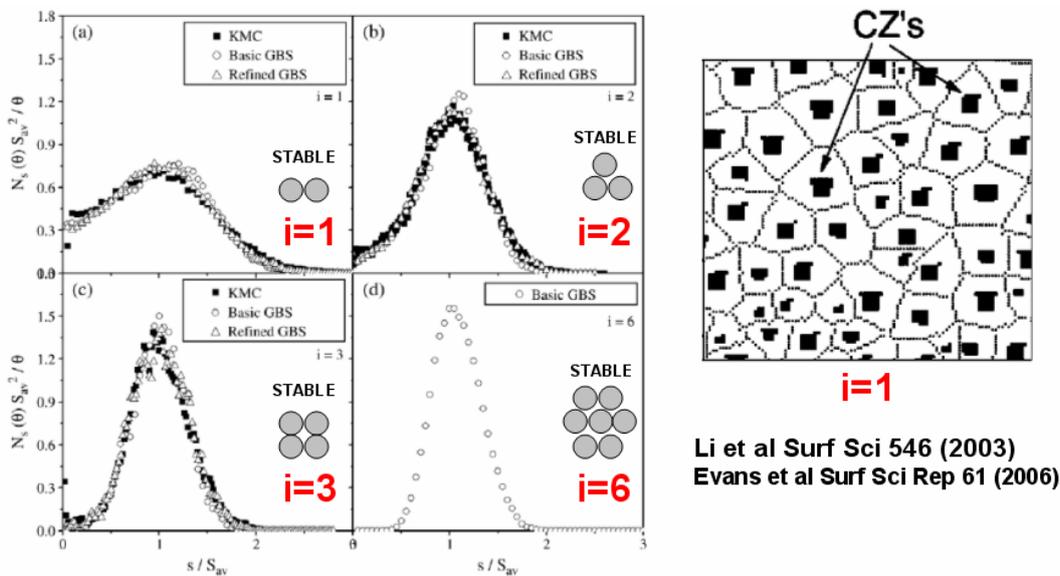


Fig.2 Left: Island size distributions versus critical size (i). Right: simulated island and CZ distribution.

For submonolayer deposition, deposited atoms diffuse across the surface and aggregate into 2D islands. The overall process involves a competition between nucleation of new islands

and growth of existing islands. Traditionally, one prescribes a critical size  $i$  such that only islands of more than  $i$  atoms are stable (but those of  $i$  or less atoms are unstable to dissociation and are in quasi-equilibrium with the diffusing adatoms). In the mid-1960's (mean-field) rate equation treatments were developed by Zinzeister, Venables, et al., to describe the island density and size distribution during submonolayer deposition. Only in the 1990's following precise analysis by KMC simulation of atomistic models was it recognized that there is a fundamental failure of mean-field nucleation theory to describe the island size distribution (ISD). These theories neglect a subtle correlation between island size and separation. More precisely, they neglect the feature that larger islands have more distant neighbors and much larger capture zones (CZ's).

One on-going goal is to develop a reliable analytic beyond-mean-field theory for the ISD. It appears that the optimum strategy is to consider simultaneously the island sizes and capture zone areas, so that the theoretical development accounts for the complex stochastic geometry of the island distribution. Perhaps most important is a reliable treatment of the spatial aspects of nucleation. Recently attention has turned to characterization of the capture zone area distribution (CZD), which like the ISD encodes important information about the island nucleation process (and specifically the critical size  $i$ ).

Another aspect of recent efforts relates to efficient simulation algorithms. Atomistic simulation becomes extremely expensive for larger critical size  $i$ . The difficulty in nucleating islands results in a high density of rapidly diffusing atoms on the surface which must all be tracked by the KMC simulation. An alternative is to replace an atomistic description of deposition and terrace diffusion with a continuum PDE description. However this must be coupled to a description of island nucleation and growth. Various step-dynamics type strategies (level-set, phase-field) have been applied, as have hybrid atomistic-continuum treatments, and a very different "geometry-based simulation" approach which is based on stochastic geometry of the CZ distribution.

#### MULTILAYER GROWTH AND MOUND FORMATION

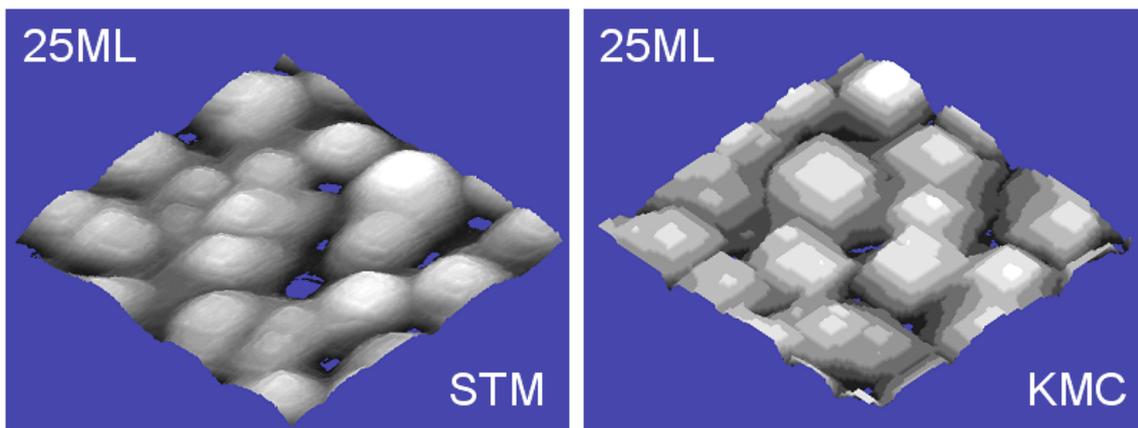


Fig.3 Multilayer deposition of 25 ML of Ag on Ag(100) at 230 K with flux  $F=0.02$  ML/s. Images ( $50 \times 50$  nm<sup>2</sup>) show well-developed square mounds. (a) STM data; (b) KMC simulation. The magnitude of the Ehrlich-Schwoebel (ES) step-edge barrier inhibiting downward transport was adjusted to  $0.07 \pm 0.01$  eV in the model in order to match experiment.

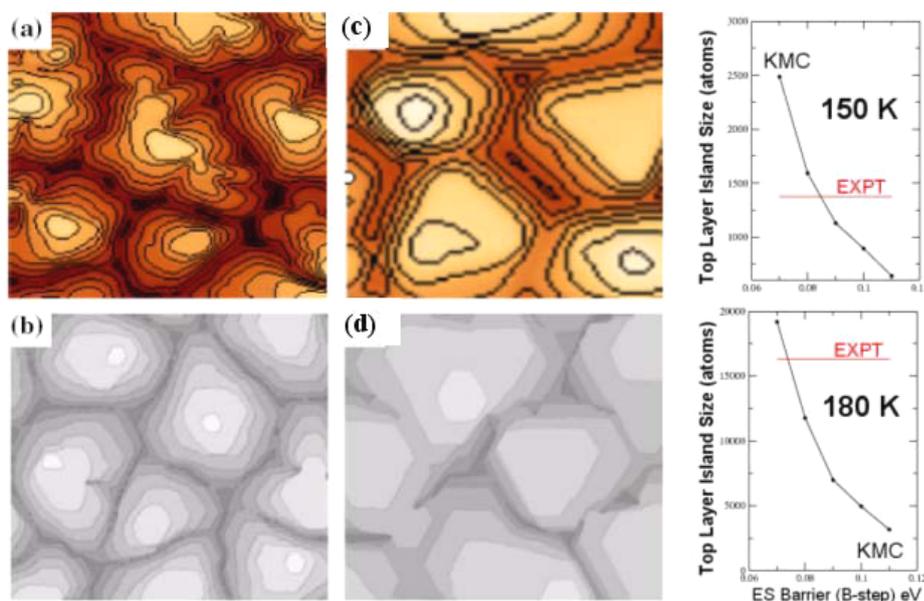


Fig.4 Bird's eye view of wedding-cake-like mound morphologies of 3ML Ag films on Ag(111): (a) STM (b) KMC at 150 K; (c) STM (d) KMC at 180 K. Simulation results for the mean size of the top terrace as a function of ES-barrier for the B-step. Experimental values are matched for B-step ES-barrier of 0.08 eV.

Figures 3 and 4 illustrate the success of atomistic modeling and KMC simulation in describing complex mound morphologies observed in growing films. However a natural goal of mathematical modeling has been to develop deeper understanding of these phenomena exploiting coarse-grained modeling. Most effort has focused on development and application of an appropriate continuum PDE (see Fig.1) for a height function,  $h(\underline{x},t)$ , which can describe this behavior. The challenge is to select the appropriate form for the surface diffusion current  $\underline{J}$  which will depend on the local surface slope, curvature, etc. There have been various phenomenological forms proposed for this current particularly in the physics literature. However, it is not clear whether these capture the correct behavior. There is also the issue of whether stochastic noise is important in morphological evolution should be included. (A simplistic analysis which indicates that evolution is effectively deterministic should be questioned.)

A more recent strategy has been to utilize step-dynamics modeling (see Fig.1) where one treats the steps at the edges of islands in each layer as continuous curves. Then, it is necessary to specify their growth velocity, and also to treat the creation of new islands and steps at the top of mounds and the annihilation of steps at the bottom of mounds. This approach has been quite effective in elucidating mound steepening as well as shape and slope selection. Another approach is to coarse-grain the step-dynamics model to rigorously obtain a 3D continuum PDE model. This approach has been attempted by several groups and certainly provides some new insights, but is still far from providing practically-usable and reliable PDE's for film evolution [6].

#### MORE COMPLEX HETEROEPITAXIAL SYSTEMS

Extensive studies of generally more complex heteroepitaxial growth (A on B) have often been motivated by the desire to create functional surface nanostructures. Here, there are additional

complications especially associated with strain due to lattice-mismatch. As a result, even the equilibrium film morphologies can be complex (e.g., arrays of self-organized 3D islands or quantum dots). Our recent work [3,4] in this direction which has explored new possibilities offered by deposition on alloy substrates (e.g., A on BC or B+C on BC). In the former case, we carefully select A and BC to avoid lattice-mismatch, and thus focus on the effect of features other than strain of film growth (specifically “quantum size effects” which can lead to selection of special heights and the development of flat-topped islands or mesas) [7].

The case B+C on BC corresponds to self-growth of an alloy for which the equilibrium state is trivial (perfect 2D alloy islands for submonolayer films; perfect flat alloy surfaces for multilayer films). However, deposition drives the system out-of-equilibrium and there has long been interest in the competition between growth and alloy ordering. We have performed realistic modeling for codeposition of Ni & Al on NiAl(110) to explore non-equilibrium behavior [8]. See Fig.5.

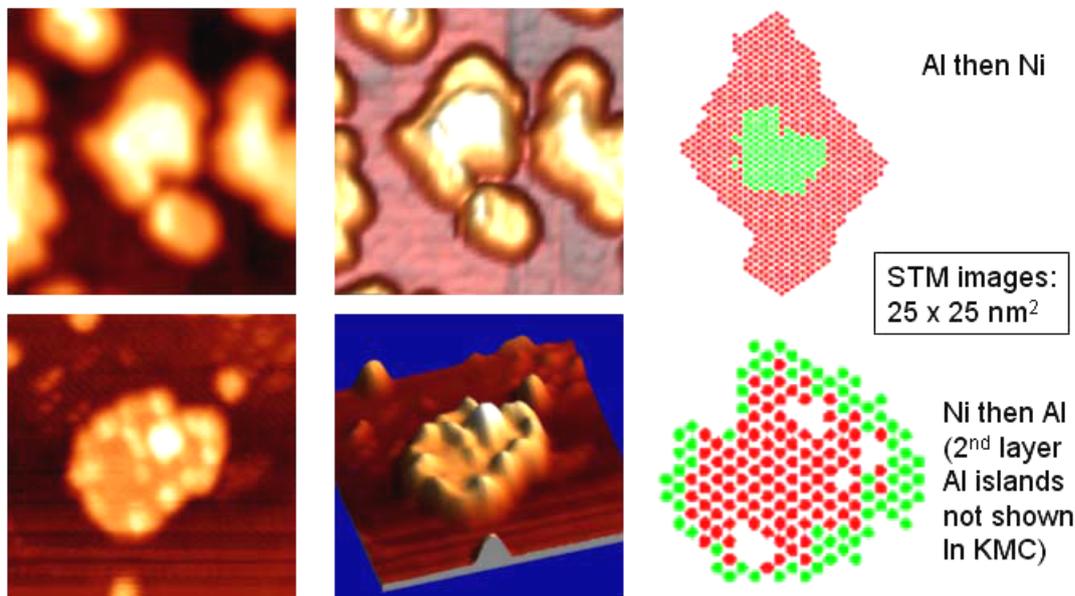


Fig. 5 STM and modeling of sequential co-deposition of Ni and Al on NiAl(110): order-dependence.

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- [4] Strictly, this applies only below a thermal roughening temperature,  $T_R$ , which is similar to  $T_c$ .
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# Dislocation dynamics

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The crystal defects called dislocations are lines whose typical length in metallic alloys is of the order of  $10^{-6}m$ , with thickness of the order of  $10^{-9}m$  (see Figure 1 for an example of observations of dislocations by electron microscopy).

In the face centered cubic structure, dislocations may move at low temperature in well defined crystallographic planes (the slip planes), at velocities of the order of  $10\text{ m}s^{-1}$ .

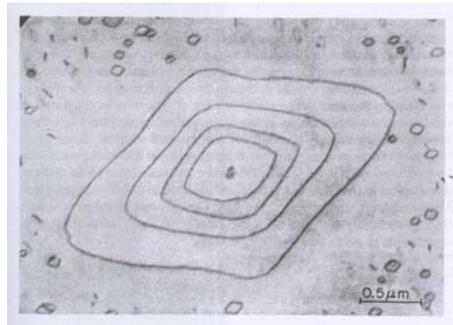


Figure 1: Dislocations in a Al-Mg alloy

The concept of dislocations has been introduced and developed in the XXth century, as the main microscopic explanation of the macroscopic plastic behaviour of metallic crystals. Since the beginning of the 90's, the research field of dislocations has enjoyed a new boom based on the increasing power of computers, allowing simulations with a large number of dislocations. This simultaneously motivated new theoretical developments for the modelling of dislocations, and mathematical analysis of these models (see for instance [1] for a review of different models of dislocation dynamics).

The plan of these lectures is the following:

- Lecture 1: Introduction to dislocation dynamics.
- Lecture 2: Mean curvature motion (MCM) as a singular limit of dislocation dynamics
- Lecture 3: Homogenization of dislocation dynamics and of particle systems

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## Lecture 1: Introduction to dislocation dynamics

We consider a dislocation line which is the boundary of an open set  $\Omega$  contained in a slip plane in a crystal (see Figure 2).

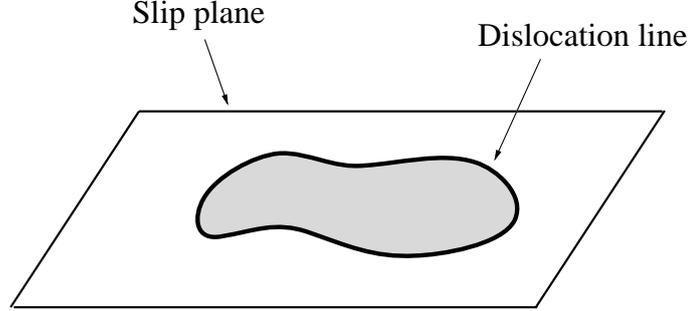


Figure 2: Example of a bounded dislocation line in a slip plane

We will show that we can associate an invariant to the dislocation line, which is called the Burgers vector  $b$ . If the mechanical behaviour of the crystal outside of the dislocation is well described by the equations of elasticity, we will explain how to compute the classical expression of the stress  $\sigma = \sigma[\Omega]$  created by the dislocation. Only one component of the stress tensor  $\sigma$  will be important to define the dynamics of the dislocation. This component is called the resolved Peach-Koehler force  $c = c[\Omega]$  and is defined as

$$c = b \cdot \sigma \cdot n$$

where  $b$  is the Burgers vector and  $n$  is the normal to the slip plane. We will show that there exists a function  $c_0$  (the kernel defined on the slip plane) such that we can write

$$c = c_0 \star 1_\Omega$$

where  $1_\Omega$  is the characteristic function of the set  $\Omega$  in the slip plane. We will explain how this kernel is related to certain Levy operators and Dirichlet to Neumann operators.

The dynamics of the dislocation is basically given by the normal velocity  $V_n$  (in the slip plane) to the curve  $\Gamma_t = \partial\Omega_t$  (where we now denote by  $\Omega_t$  the open set to show the dependence on the time  $t$ ). This normal velocity is given by

$$(0.1) \quad V_n = c[\Omega_t](x, t) \quad \text{with} \quad c[\Omega_t] = c_0 \star 1_{\Omega_t}.$$

The dynamics of a dislocation is then a system coupling the equations of elasticity for the displacement field and a geometric motion for the dislocation curve. After eliminating the displacement, the dynamics reduces to a single non-local geometric equation (0.1). It is convenient to see the open set  $\Omega_t$  as the super-level set  $\{x, u(x, t) > 0\}$  for a function  $u$  solving the following non-local Hamilton-Jacobi equation:

$$(0.2) \quad \frac{\partial u}{\partial t} = (c_0 \star 1_{\{u(\cdot, t) > 0\}}) |Du|.$$

The mathematical theory of viscosity solution and its extensions will be presented as a framework to study such kind of equations. The main mathematical difficulty is the fact that the inclusion principle can be lost for the open sets  $\Omega_t = \{x, u(x, t) > 0\}$ .

## Lecture 2:

### Mean curvature motion (MCM) as a singular limit of dislocation dynamics

This lecture will focus on the special case where we concentrate all the negative part of the kernel  $c_0$  in a Dirac mass. In that case we will show that we can mathematically recover a comparison principle for a suitable formulation. This can be formally written as

$$c_0 = J - \frac{1}{2} \left( \int_{\mathbb{R}^n} J \right) \delta_0$$

where  $n = 2$  for the plane, but we consider also higher dimensional generalizations. Here  $J$  is the following function

$$0 \leq J(-z) = J(z) = \frac{g(z/|z|)}{|z|^{n+1}} \cdot 1_{\{|z|>1\}}$$

where the smooth function  $g$  describes the anisotropy of the problem due to the elasticity of the crystal (which is natural for anisotropic dislocation curves like you can see on Figure 1). I will then consider a Slepčev reformulation of the problem (0.2) for a new level set function  $v$  which is now formally solution of the following equation:

$$\frac{\partial v}{\partial t} = (c_0 \star 1_{\{v(\cdot, t) > v(x, t)\}})(x) \cdot |Dv|$$

which is rigorously interpreted as a viscosity solution (to define precisely) of the equation

$$(0.3) \quad \frac{\partial v}{\partial t} = \left\{ -\frac{1}{2} \left( \int_{\mathbb{R}^n} J \right) + \left( \int_{\mathbb{R}^n} dz J(z) 1_{\{v(x-z, t) > v(x, t)\}} \right) \right\} \cdot |Dv|.$$

We will then try to understand the behaviour of the level sets at a large scale. This is known in physics that at large scale the dynamics of a single dislocation can be well approximated by a certain mean curvature motion. To recover mathematically this fact, we will consider the following rescaling for a small parameter  $\varepsilon > 0$

$$v^\varepsilon(x, t) = v \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2 \ln \varepsilon} \right).$$

This rescaling is almost a parabolic rescaling. We will then show that the limit as  $\varepsilon$  goes to zero of  $v^\varepsilon$  is a function  $v^0$  which solves a certain anisotropic mean curvature motion (MCM) which has a variational interpretation (see [2]). This limit motion is also connected to the Bence, Merriman, Osher algorithm proposed to compute numerically the solution to MCM.

### Lecture 3: Homogenization of dislocation dynamics and of particle systems

In a more general situation, we will consider not only a single dislocation but an infinite number of dislocations that are given by the level sets  $\Gamma_t^k = \{x, w(x, t) = k\}$ ,  $k \in \mathbb{Z}$ , for a level sets function  $w$ . Then we can write a Slepčev formulation of this problem as

$$(0.4) \quad \frac{\partial w}{\partial t} = \left\{ c_1(x) + \left( \int_{\mathbb{R}^n} dz J(z) [w(x-z, t) - w(x, t)] \right) \right\} \cdot |Dw|$$

where  $[\cdot]$  is an integer part called the floor function and  $c_1 = 0$  if there is no additional stress in the material. We will consider the particular case where there are periodic obstacles to the motion of dislocations in the crystal. This last case can be modelled assuming that the function  $c_1$  is a general  $\mathbb{Z}^n$ -periodic function. Here  $c_1$  represents the additional stress created by the obstacles. Physically the function  $w$  can be interpreted as the plastic strain in the material. A natural question is then: at large scales, what is the macroscopic dynamics corresponding to the motion of a density of dislocation curves. This is a homogenization problem. This corresponds to consider the rescaling for a small parameter  $\varepsilon$ :

$$w^\varepsilon(x, t) = \varepsilon w \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right).$$

We will show (see [4]) that  $w^\varepsilon$  converges to a function  $w^0$  as  $\varepsilon$  goes to zero, where  $w^0$  is a solution of a PDE of the form

$$(0.5) \quad \frac{\partial w^0}{\partial t} = \bar{H}(Dw^0, Lw^0)$$

where  $\bar{H}$  is an effective Hamiltonian that we can compute and  $L$  is a certain non-local operator. This limit equation has also a mechanical interpretation as the plastic law in elasto-visco-plasticity of crystals. Up to our knowledge, this is the first rigorous derivation of such a law (even if we consider a very particular situation).

This homogenization result is also naturally connected to the homogenization of the dynamics of particles with two-body interactions (see for instance [3]).

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# Behavior of solutions to an area-preserving crystalline motion

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joint work with Shigetoshi YAZAKI(University of Miyazaki)

## 1 Introduction

When a block of ice crystal is illuminated by strong beams, the ice crystal starts to melt inside of the crystal as well as the surface and each water region forms a snowflake-like-pattern which has six petals, called “Tyndall figure” (see Figure 1 (a)). This figure has a vapor bubble in water region and when this figure is refrozen, the vapor bubble remains in the ice as a hexagonal disk (see Figure 1 (b)). This hexagonal disk is a kind of negative crystals and the interior region is filled with water vapor saturated at that temperature. McConnell([6]) found these disks in the ice of Davos lake. Nakaya called this hexagonal disk “Kuuzou(空像)” in Japanese and investigated its properties [7].

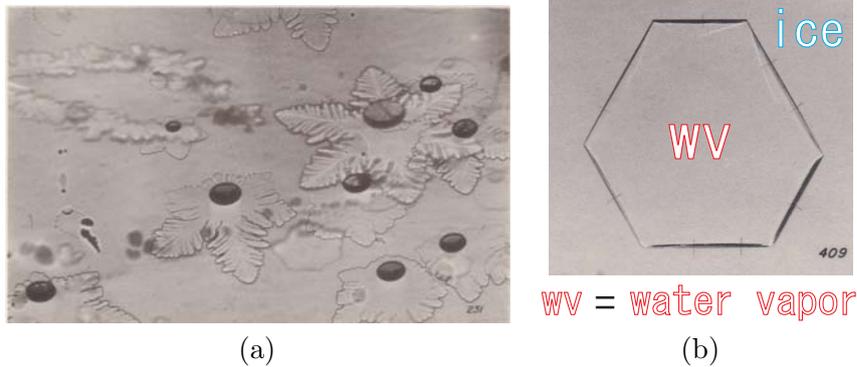


Figure 1: (a) Tyndall figures (seen from  $45^\circ$  to the  $c$ -axis) and (b) a negative crystal (by U. Nakaya).

In [5], we proposed a motion equation for a polygonal curve in the plane as a simple model of the formation process of negative crystals after the water region in a Tyndall figure is completely refrozen. This model equation is obtained by a gradient flow of total surface energy under an area-preserving constraint:

$$V_i = \bar{H} - H_i.$$

Here  $V_i$  is the outward normal velocity on the  $i$ -th facet  $\mathcal{F}_i$  of vapor region  $\Omega(t)$  (enclosed region by a polygon),  $H_i$  is the crystalline curvature of  $\mathcal{F}_i$  and  $\bar{H}$  is the average of all

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crystalline curvatures. This equation is called area-preserving crystalline motion or area-preserving crystalline curvature flow. Crystalline motion is a singular weighted curvature flow with non-smooth surface energy  $\gamma$  and J. Taylor[8] and Angenent and Gurtin [1] proposed the framework of crystalline motions. In this framework, the interfaces are restricted in the class of polygonal curves (two dimensional case) which satisfy an admissibility condition based on the equilibrium shape of the crystal. This equilibrium shape is called the *Wulff shape* and plays important roles for not only the definition of the crystalline curvature and admissibility condition, but also the asymptotic behavior of the solution polygons. The detailed formulations will be mentioned in next section.

In the case that an initial shape  $\Omega_0$  is convex, the solution polygon  $\Omega(t)$  keeps its convexity. S. Yazaki [9, Part I] show that no facets disappear globally in time and the solution polygon converges to the rescaled Wulff shape whose area is equal to that of  $\Omega_0$  in the Hausdorff metric. However, when the vapor region is surrounded by the ice region in refreezing process, many fine facets appear on the interface and the shape of the vapor region is not convex in general. Thus, in this talk, we consider the case that  $\Omega_0$  is not convex. In this case, there is a possibility that the solution has some singularities in finite time, for example, facet-extinction and self-intersection of the interface. We show the sufficient conditions on the Wulff shape and an initial polygon to keep admissibility of the solution polygons. Moreover, we also show that the solution polygon from non-convex initial polygon becomes convex in finite time.

## 2 Area-preserving crystalline motion

**Crystalline energy and the Wulff shape.** Let  $\gamma = \gamma(\mathbf{n})$  be a positive continuous function defined on  $S^1$  and describe interfacial energy density for the direction  $\mathbf{n}$ . In this note, we consider the case where the Wulff shape of  $\gamma$ ,  $\mathcal{W}_\gamma = \{x \in \mathbb{R}^2 | x \cdot \mathbf{n} \leq \gamma(\mathbf{n}) \text{ for all } \mathbf{n} \in S^1\}$ , is a convex polygon. Such  $\gamma$  is called *crystalline energy*. If  $\mathcal{W}_\gamma$  is a  $J$ -sided convex polygon ( $J \geq 3$ ), then  $\mathcal{W}_\gamma$  is expressed as

$$\mathcal{W}_\gamma = \bigcap_{i=1}^J \{x \in \mathbb{R}^2; x \cdot \nu_i \leq \gamma(\nu_i)\},$$

where  $\nu_i = \mathbf{n}(\phi_i)$  and  $\phi_i$  is the exterior normal angle of the  $i$ -th facet with  $\phi_i \in (\phi_{i-1}, \phi_{i-1} + \pi)$  for all  $i$  ( $\phi_0 = \phi_J$ ,  $\phi_{J+1} = \phi_1$ ). We define a set of normal vectors of  $\mathcal{W}_\gamma$  by  $\mathcal{N}_\gamma = \{\nu_1, \nu_2, \dots, \nu_J\}$ .

**Polygons and polygonal curves.** Let  $\Omega$  be  $N$ -sided polygon in the plane  $\mathbb{R}^2$ ,  $\mathcal{P}$  its boundary, that is,  $\mathcal{P} = \partial\Omega$  and label the position vector of vertices  $\mathbf{p}_i$  ( $i = 1, 2, \dots, N$ ) in an anticlockwise order:  $\mathcal{P} = \bigcup_{i=1}^N \mathcal{F}_i$ , where  $\mathcal{F}_i = \{(1-t)\mathbf{p}_i + t\mathbf{p}_{i+1}; t \in [0, 1]\}$  is the  $i$ -th facet ( $\mathbf{p}_0 = \mathbf{p}_N$ ,  $\mathbf{p}_{N+1} = \mathbf{p}_1$ ). The length of  $\mathcal{F}_i$  is  $d_i = |\mathbf{p}_{i+1} - \mathbf{p}_i|$ , and then the  $i$ -th unit tangent vector is  $\mathbf{t}_i = (\mathbf{p}_{i+1} - \mathbf{p}_i)/d_i$  and the  $i$ -th unit outward normal vector is  $\mathbf{n}_i = -\mathbf{t}_i^\perp$ , where  $(a, b)^\perp = (-b, a)$ . We define a set of normal vectors of  $\mathcal{P}$  by  $\mathcal{N} = \{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_N\}$ . Let  $\theta_i$  be the exterior normal angle of  $\mathcal{F}_i$ . Then  $\mathbf{n}_i = \mathbf{n}(\theta_i)$  and  $\mathbf{t}_i = \mathbf{t}(\theta_i)$  hold ( $\theta_0 = \theta_N$ ,  $\theta_{N+1} = \theta_1$ ), where  $\mathbf{t}(\theta) = (-\sin \theta, \cos \theta)$ .

We define the  $i$ -th height function  $h_i = \mathbf{p}_i \cdot \mathbf{n}_i = \mathbf{p}_{i+1} \cdot \mathbf{n}_i$  ( $h_0 = h_N$ ,  $h_{N+1} = h_1$ ). By using  $\{h_{i-1}, h_i, h_{i+1}\}$  and  $\{\mathbf{n}_{i-1}, \mathbf{n}_i, \mathbf{n}_{i+1}\}$ , the length of  $i$ -th facet  $d_i$  is described as follows:

$$d_i = \frac{\chi_{i-1,i}(h_{i-1} - (\mathbf{n}_{i-1} \cdot \mathbf{n}_i)h_i)}{\sqrt{1 - (\mathbf{n}_{i-1} \cdot \mathbf{n}_i)^2}} + \frac{\chi_{i,i+1}(h_{i+1} - (\mathbf{n}_i \cdot \mathbf{n}_{i+1})h_i)}{\sqrt{1 - (\mathbf{n}_i \cdot \mathbf{n}_{i+1})^2}}, \quad i = 1, 2, \dots, N,$$

where  $\chi_{i,j} = \text{sgn}(\mathbf{n}_i \wedge \mathbf{n}_j)$  and  $\mathbf{a}_1 \wedge \mathbf{a}_2 = \det(\mathbf{a}_1, \mathbf{a}_2)$  is the determinant of the  $2 \times 2$  matrix with column vectors  $\mathbf{a}_1, \mathbf{a}_2$ . Since  $\mathbf{n}_i \cdot \mathbf{n}_j = \cos(\theta_i - \theta_j)$ , we have another expression:

$$d_i = -(\cot \vartheta_i + \cot \vartheta_{i+1})h_i + h_{i-1} \text{cosec } \vartheta_i + h_{i+1} \text{cosec } \vartheta_{i+1}, \quad i = 1, 2, \dots, N, \quad (1)$$

where  $\vartheta_i = \theta_i - \theta_{i-1}$ . Note that  $0 < |\vartheta_i| < \pi$  holds for all  $i$ . Furthermore, the  $i$ -th vertex  $\mathbf{p}_i$  ( $i = 1, 2, \dots, N$ ) is described as follows:

$$\mathbf{p}_i = h_i \mathbf{n}_i + \frac{h_{i-1} - (\mathbf{n}_{i-1} \cdot \mathbf{n}_i)h_i}{\mathbf{n}_{i-1} \cdot \mathbf{t}_i} \mathbf{t}_i. \quad i = 1, 2, \dots, N. \quad (2)$$

**Admissibility and crystalline curvature.** We call  $\Omega$  and  $\mathcal{P}$  *admissible* (associated with  $\mathcal{W}_\gamma$ ) if and only if  $\mathcal{N} = \mathcal{N}_\gamma$  holds and any adjacent two normal vectors in the set  $\mathcal{N}$  are also adjacent in the set  $\mathcal{N}_\gamma$ , i.e., for any  $i$ , there exists  $j$  such that  $\{\boldsymbol{\nu}_j, \boldsymbol{\nu}_{j+1}\} = \{\mathbf{n}_i, \mathbf{n}_{i+1}\}$  holds.

Let  $\mathcal{P}$  be an admissible polygonal curve. For each facet  $\mathcal{F}_i$  a *crystalline curvature* is defined by

$$H(\mathcal{F}_i) = \chi_i \frac{l_\gamma(\mathbf{n}_i)}{d_i}, \quad i = 1, 2, \dots, N,$$

where  $\chi_i = (\chi_{i-1,i} + \chi_{i,i+1})/2$  is the transition number and it takes  $+1$  (resp.  $-1$ ) if  $\mathcal{P}$  is convex (resp. concave) around  $\mathcal{F}_i$  in the direction of  $-\mathbf{n}_i$ , otherwise  $\chi_i = 0$ ; and  $l_\gamma(\mathbf{n}_i)$  is the length of the  $j$ -th facet of  $\mathcal{W}_\gamma$  if  $\mathbf{n}_i = \boldsymbol{\nu}_j$ . If  $\Omega$  is an admissible convex polygon, then  $\mathbf{n}_i = \boldsymbol{\nu}_i$  and  $\chi_i = 1$  for all  $i = 1, 2, \dots, N = J$ ; and moreover, if  $\Omega = \mathcal{W}_\gamma$ , then the crystalline curvature is 1. In this note, we call a facet which zero transition number ‘‘inflection facet.’’

We note that the total interfacial crystalline energy on  $\mathcal{P}$  is

$$\mathcal{E}_\gamma = \sum_{i=1}^N \gamma(\mathbf{n}_i) d_i, \quad (3)$$

and the crystalline curvature  $H(\mathcal{F}_i)$  is characterized as the first variation of  $\mathcal{E}_\gamma$  on  $\mathcal{P}$  at  $\mathcal{F}_i$  with a suitable norm. Here and hereafter, we denote  $H(\mathcal{F}_i)$  by  $H_i$  for short.

**Area-preserving crystalline motion.** The normal velocity on  $\mathcal{F}_i$  in the direction  $\mathbf{n}_i$  is  $V_i = \dot{h}_i$ . Here and hereafter, we denote that the derivative of a function  $u = u(t)$  with respect to time  $t$  by  $\dot{u}$ . The area-preserving crystalline motion is the gradient flow of  $\mathcal{E}_\gamma$  along  $\mathcal{P}$  which encloses a fixed area, and it is described as follows:

$$V_i = \bar{H} - H_i, \quad i = 1, 2, \dots, N, \quad (4)$$

where

$$\bar{H} = \frac{\sum_{i=1}^N H_i d_i}{\mathcal{L}}$$

is the average of the crystalline curvature, and  $\mathcal{L} = \sum_{k=1}^N d_k$  is the total length of the curve  $\mathcal{P}$ . From (1), we have

$$\dot{d}_i = -(\cot \vartheta_i + \cot \vartheta_{i+1})V_i + V_{i-1} \text{cosec } \vartheta_i + V_{i+1} \text{cosec } \vartheta_{i+1}, \quad i = 1, 2, \dots, N. \quad (5)$$

Furthermore, by (2) we have

$$\dot{\mathbf{p}}_i = V_i \mathbf{n}_i + \frac{V_{i-1} - (\mathbf{n}_{i-1} \cdot \mathbf{n}_i)V_i}{\mathbf{n}_{i-1} \cdot \mathbf{t}_i} \mathbf{t}_i, \quad i = 1, 2, \dots, N. \quad (6)$$

Note that (4), (5) and (6) are equivalent each other. It is easy to check that the enclosed area  $\mathcal{A}(t) = \sum_{i=1}^N h_i d_i / 2$  is preserving in time:  $\dot{\mathcal{A}}(t) = \sum_{i=1}^N V_i d_i = 0$ .

### 3 Results

For any given admissible initial polygon  $\Omega_0$ , we have short time existence and uniqueness result by the standard argument since (5) is the system of ordinary differential equations.

**Known results for convex polygons.** What might happen to  $\Omega(t)$  as  $t$  tends to the maximal existence time  $T \leq \infty$ ? For this question, we have the following result.

**Theorem 1** Let the crystalline energy be  $\gamma > 0$ . Assume the initial polygon  $\Omega_0$  is an  $N$ -sided admissible convex polygon. Then the solution admissible polygon  $\Omega(t)$  exists globally in time keeping the area enclosed by the polygon constant  $\mathcal{A}$ , and  $\Omega(t)$  converges to the shape of the boundary of the Wulff shape  $\partial\mathcal{W}_{\gamma^*}$  in the Hausdorff metric as  $t$  tends to infinity, where  $\gamma_*(\mathbf{n}_i) = \gamma(\mathbf{n}_i)/W$ ,  $W = \sqrt{|\mathcal{W}_\gamma|/\mathcal{A}}$  for all  $i = 1, 2, \dots, N$  and  $|\mathcal{W}_\gamma| = \sum_{k=1}^N \gamma(\mathbf{n}_k)l_\gamma(\mathbf{n}_k)/2$  is enclosed area of  $\mathcal{W}_\gamma$ .

This theorem is proved in Yazaki [9, Part I] by using the anisoperimetric inequality or Brunn and Minkowski's inequality and the theory of dynamical systems.

#### Our results for non-convex polygons.

In the previous case, the solution polygon keeps its convexity and admissibility, that is, the length of each facet is positive globally in time and the self-intersection of  $\mathcal{P}(t)$  never occur. However, if  $\Omega_0$  is non-convex, the facet-extinction or the self-touching may occur in finite time. Indeed, we can easily construct the example of the self-intersection of  $\mathcal{P}(t)$  and  $\Omega(t)$  becomes non-admissible after the singularity. Thus, the admissibility of solution polygons may break down in finite time. To track the motion globally in time in the class of admissible polygons, we prepare the following assumptions:

(A1)  $\mathcal{W}_\gamma$  is symmetric with respect to the origin.

(A2) Transition numbers of  $\Omega_0$  are all nonnegative:  $\chi_i \geq 0$  for any  $i$ .

**Theorem 2** Assume the assumptions (A1) and (A2). Let  $\Omega_0$  be an  $N$ -sided non-convex admissible polygon. Then, there exists  $T_1 > 0$  such that the solution polygon is an  $N$ -sided admissible polygon for  $0 \leq t < T_1$  and there exists at least one inflection facet whose length tends to zero as  $t \rightarrow T_1$ . Moreover,  $\Omega(t)$  converges to an admissible polygon  $\Omega^*$  in the Hausdorff topology as  $t \rightarrow T_1$  and area of  $\Omega^*$  is equal to area of  $\Omega_0$ .

This theorem means that we can restart the motion with the initial polygon  $\Omega^*$  and obtain the solution in the class of admissible polygons beyond the singularity. If  $\Omega^*$  is non-convex, then we can apply Theorem 2 again and again. We finally have a finite sequence of facet-extinction time :  $0 < T_1 < T_2 < \dots < T_m < +\infty$ . Then, we obtain the following convexity result.

**Theorem 3** Assume that the same assumption as in Theorem 2. Then, the solution polygon becomes convex at  $t = T_m$ .

After the convexity phenomena occurs, we can apply Theorem 1. Therefore, the solution polygon exists globally in time in the class of admissible polygons and the solution polygon finally converges to the rescaled Wulff shape.

## 4 For negative crystals

For usual crystal case, enclosed region describes the crystal and then normal vector  $\mathbf{n}$  is direction from the crystal to its outside region. However, for negative crystal case, the outside region describes the crystal. Thus, applying the area-preserving crystalline motion to understand the motion of the boundary of negative crystals, we need to use  $\gamma(-\mathbf{n})$  as the interfacial energy density. Therefore, we use the figure:

$$\bigcap_{i=1}^J \{ \mathbf{x} \in \mathbb{R}^2; \mathbf{x} \cdot (-\boldsymbol{\nu}_i) \leq \gamma(\boldsymbol{\nu}_i) \},$$

as the Wulff shape for negative crystal case.

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# Short Time Uniqueness Results for Solutions of Nonlocal and Non-Monotone Geometric Equations

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## 1 Introduction

This talk are based on a joint work [6] with Professors Barles and Ley. The goal of this talk is to explain what a *non-monotone* evolution of compact hypersurfaces  $\{\Gamma_t\}_{t \geq 0} \subset \mathbb{R}^N$  moving according to the non-local law of propagation

$$V = h(x, t, \Omega_t, n(x), Dn(x)) \quad \text{on } \Gamma_t \quad (1)$$

is and to describe a method to show short time uniqueness results for the initial-value problem for *level-set* (or *geometric*) equations of (1). Here  $V$  is the normal velocity of  $\Gamma_t$  which depends, through the evolution law  $h$ , on time  $t$ , on the position of  $x \in \Gamma_t$ , on the set  $\Omega_t$  enclosed by  $\Gamma_t$ , on the unit normal  $n(x)$  to  $\Gamma_t$  at  $x$  pointing outward to  $\Omega_t$  and on its gradient  $Dn(x)$  which carries the curvature dependence of the velocity.

The main example we have in mind is the dislocation dynamics, i.e., Equation (1) with

$$h = c_0(\cdot, t) * \mathbf{1}_{\overline{\Omega}_t} + c_1(\cdot, t) - \operatorname{div}(n(x)) \quad \text{on } \Gamma_t, \quad (2)$$

where the functions  $c_0, c_1 : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$  are given functions which are bounded, Lipschitz continuous in  $x$  (uniformly with respect to  $t$ ) and  $c_0, D_x c_0 \in L^\infty([0, T]; L^1(\mathbb{R}^N))$ , “ $*$ ” denotes a convolution with respect to  $x$  variable and  $\mathbf{1}_A$  is the indicator function of a set  $A$ . Note that  $\operatorname{div}(n(x))$  is  $(n-1)$  times the mean curvature of  $\Gamma_t$  at a point  $x \in \Gamma_t$ . This example is of interest in applications and a physical assumption is that the kernel  $c_0$  may change sign. It makes this motion non-monotone as it will be explained later. This feature is the main difficulty to overcome to study the evolution. We refer to [6] for more general nonlocal and non-monotone geometric equations.

## 2 Derivation of Level-Set Equations of (2) and Level Set Approach

First of all, we present a formal derivation of the level set equation of (1) with (2). We assume that there exists a smooth function  $u : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$  such that

$$\Gamma_t = \{x \in \mathbb{R}^N \mid u(\cdot, t) = 0\},$$

$$\Omega_t = \{x \in \mathbb{R}^N \mid u(\cdot, t) > 0\} \text{ and } Du \neq 0 \text{ on } \Gamma_t.$$

A classical calculation yields

$$V = \frac{u_t}{|Du|} \text{ and } n = -\frac{Du}{|Du|}.$$

Inserting the above formulae in (1) with (2), we obtain

$$\begin{aligned} u_t &= (c[\mathbf{1}_{\{u \geq 0\}}](x, t) + \operatorname{div}\left(\frac{Du}{|Du|}\right))|Du| \\ &\text{in } \mathbb{R}^N \times (0, T), \end{aligned} \quad (3)$$

where

$$\begin{aligned} &c[\mathbf{1}_{\{u \geq 0\}}](x, t) \\ &:= \int_{\mathbb{R}^N} c_0(x - y, t) \mathbf{1}_{\{u(\cdot, t) \geq 0\}}(y) dy + c_1(x, t). \end{aligned}$$

The *level set approach* to front propagations can be described as follows generally. For a given smooth hypersurface  $\Gamma_0$  in  $\mathbb{R}^N$  (front at time  $t = 0$ ), choose  $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$  such that

$$\Gamma_0 = \{x \in \mathbb{R}^N \mid u_0(x) = 0\}$$

and solve (in some sense) the Cauchy problem for (3) with the initial value  $u_0$ , and, finally, regard

$$\{x \in \mathbb{R}^N \mid u(x, t) = 0\}.$$

as the front  $\Gamma_t$  at time  $t$ . This approach was introduced by Osher and Sethian [19] for numerical calculations and then developed, from a theoretical point of view, by Evans and Spruck [13] for the mean curvature motion and by Chen, Giga and Goto [11] for general velocities by using the theory of viscosity solution (see [12]). We refer to the monograph [16] for more details.

In order to guarantee level set approach, we need to consider the fundamental questions: (i) whether this Cauchy problem have a viscosity solution, (ii) whether viscosity solutions are unique and (iii) whether  $\Gamma_t$  depends only on  $\Gamma_0$  and not on the shape of  $u_0$  outside of  $\Gamma_0$ . We can give a positive answer for (i) and (ii) for short time. In this talk, we shall only address question (ii). For question (i), we refer to [17, 20, 5, 6]. Question (iii) have not been solved yet. Finally, it is worth mentioning that when such motion (1) is local, i.e., when  $h$  does not depend on  $\Omega_t$ , and *monotone* (see below), then it is proved by Barles and Souganidis in [9] that the motion can be defined and studied by the level set approach.

### 3 Non-Monotone Motion

In recent years, there has been much interest on the study of front propagations problems in cases when the normal velocity of the front depends on a non-local way of the enclosed region like (1). This interest was motivated by several types of applications like dislocations' theory or FitzHugh-Nagumo type systems or volume dependent velocities. It is worth pointing out that in many of the above mentioned applications, one faces *non-monotone motions*. We first recall that a motion (1) is said to be monotone when the *inclusion principle* holds, i.e., when

$$\Omega_0^1 \subset \Omega_0^2 \Rightarrow \Omega_t^1 \subset \Omega_t^2 \text{ for any } t > 0.$$

A motion is non-monotone when the inclusion principle does not hold.

We consider an easy example.

**Example 1.** Let us consider the evolution of compact hypersurfaces  $\{\Gamma_t\}_{t \geq 0} \subset \mathbb{R}^2$  moving according to the non-local law of propagation

$$V = -\text{Vol}(\Omega_t) \quad \text{on } \Gamma_t. \quad (4)$$

When we start from circles with radius  $a > 0$ , then (4) can be reduced to consider the initial-value

problem of the ordinary differential equation

$$\dot{R}(t) = -R(t)^2 \quad \text{and} \quad R(0) = a.$$

Solve the above problem and denote the solutions by  $R_a(t)$ . We have

$$R_a(t) = \frac{a}{at + 1}.$$

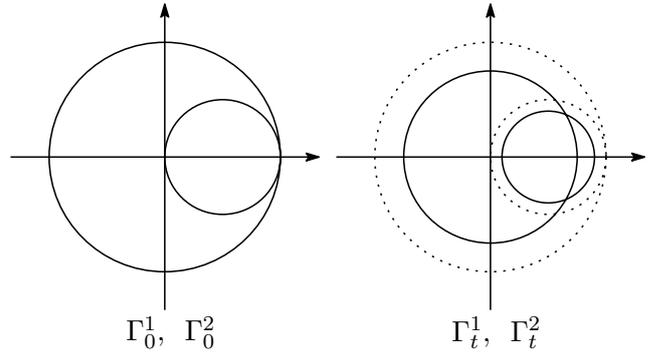
Then let us consider the initial surfaces

$$\begin{aligned} \Gamma_0^1 &:= \{x \in \mathbb{R}^2 \mid |x| = 1\}, \\ \Gamma_0^{1/2} &:= \{x \in \mathbb{R}^2 \mid |x - \frac{1}{2}| = \frac{1}{2}\}. \end{aligned}$$

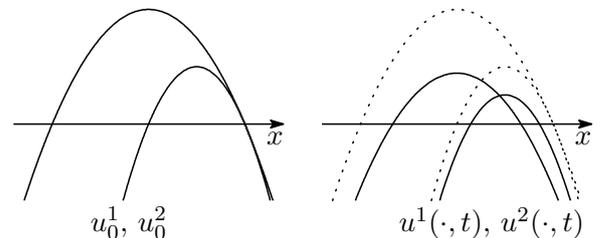
By calculating that

$$R_1(t) - (R_{\frac{1}{2}}(t) + \frac{1}{2}) = -\frac{t(t+3)}{2(t+1)(t+2)} < 0,$$

we see that non-monotone motion happens.



From viewpoint of auxiliary functions  $u^1, u^2$ , we observe the followings.



This observation tells us that we cannot expect the comparison principle for the level set equation for (4).

Finally, we note that in this example we ignore the curvature term for simplicity of calculation, but non-monotone motion still can happen even though we add the curvature term.

For such class of equations, the level set approach cannot be used directly since the classical comparison arguments of viscosity solutions' theory fail and therefore, the existence and uniqueness of viscosity solutions to these equations become an issue. Though the existence properties for such motions seem now to be well understood (see [17, 20, 5]), this is not the case for uniqueness. In particular, there are not many uniqueness results for curvature dependent velocities. While there are many results of existence and uniqueness for the motion (1) with (2) without a curvature term, as far as I know, there are only two works by Forcadel [14] and Forcadel and Monteillet [15] which investigate the motion (1) with (2), which is included by our equations with general assumptions. We describe here the study for the motion (1) with (2) without a curvature term. A short time existence and uniqueness result was first obtained in [2]. Long time existence and uniqueness results were obtained when the velocity is positive, i.e.,

$$h > 0 \quad \text{on } \Gamma_t, \quad (5)$$

by Alvarez, Cardaliaguet and Monneau in [1] and by Barles and Ley in [7] by different methods. In [3], they presented a new notion of weak solutions of the level set equation (see [3, Definition 1.1]) and gave the global existence of these weak solutions and analysed the uniqueness of them when (5) holds. A similar concept of solutions already appeared in [17, 20]. In [4], a new perimeter estimate for the evolving fronts with uniform interior cone property was proposed and by using this, the uniqueness result for dislocation dynamics equations was obtained and asymptotic equations of a FitzHugh-Nagumo type system, still under the positiveness assumption (5). Since the studies by [1, 7, 3, 4], it is now well-known that estimates on *lower gradient bound* and perimeter of 0-level sets of viscosity solutions of associated local equations are key properties to obtain existence and uniqueness results for nonlocal equations derived from dislocation dynamics model. Finally, we notice that in this talk, we *do not* use the perimeter estimate in an essential way but elementary measure estimates.

## 4 Fattening Difficulty

Let us describe the main difficulty of our problem. Considering the non-local part as a given function, we are led to the study the (local) Cauchy problem

$$\begin{cases} u_t = \left( c(x, t) + \operatorname{div} \left( \frac{Du}{|Du|} \right) \right) |Du| & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \quad (6)$$

where  $u_0 \in W^{1, \infty}(\mathbb{R}^N)$  and  $c \in C(\mathbb{R}^N \times [0, T])$  are bounded and Lipschitz continuous with respect to the  $x$  variable. One of our main results is a short time lower gradient bound estimate for the viscosity solution of (6), i.e.,

$$\begin{aligned} |Du(x, t)| &\geq \eta(t) > 0 \\ &\text{in a neighborhood of } \{u(\cdot, t) = 0\}. \end{aligned} \quad (7)$$

For first-order eikonal equations, lower gradient bound comes naturally from the Barron-Jensen's approach (see [18]). For second-order equations like (6), it is affected by the "*diffusion*" term and the *non-empty interior* difficulty and therefore we cannot expect that the property (7) holds generally and for long-time. Indeed, in [10], they consider the simple example of (6) with  $c \equiv 1$  and smooth  $u_0$  such that  $Du_0 \neq 0$  on the initial front  $\{u_0 = 0\}$ . They prove that, up to choose suitable  $u_0$ , *fattening* may occur for arbitrary  $t > 0$ , i.e., the front may develop an interior. Also, it is worth pointing out that we are dealing with the function  $h$  which is allowed to change sign in (1), contrary to [1, 7, 3, 4] where (5) is one of the main assumption to get uniqueness. It may give rise of fattening, see [8, Proposition 4.4].

It is precisely this reason which implies that there are not many results on the nonlocal second-order equations like the level sets equations (3) and we cannot expect the global uniqueness result under our general assumptions. Therefore, we can say that our short-time result is optimal in a sense.

Finally, we explain the key idea to obtain (7) for viscosity solutions of (6). In order to get it, we make the following assumption on  $u_0$ . There exist constants  $\lambda_0, \delta_0 \in (0, 1)$ ,  $\eta_0 > 0$  and  $\nu \in C(\mathbb{R}^N, \mathbb{R}^N)$  such that

$$u_0(x + \lambda\nu(x)) \geq u_0(x) + \lambda\eta_0 \quad \text{in } U_0 \quad (8)$$

for all  $\lambda \in [0, \lambda_0]$ , where  $U_0 := \{x \in \mathbb{R}^N \mid |u_0(x)| \leq \delta_0\}$ . Then we prove that such a property is preserved for the solution of (6), at least for short

time, i.e.,

$$u(x + \lambda\nu(x), t) \geq u(x, t) + \lambda\eta(t) \text{ in } U_t \quad (9)$$

for all  $\lambda \in [0, \bar{\lambda}]$ ,  $t \in [0, \bar{t} \wedge T]$  and some  $\bar{t} > 0$ ,  $\bar{\lambda} \in (0, \lambda_0]$ , where  $\eta : [0, \bar{t} \wedge T] \rightarrow [0, \infty)$  is a non-increasing continuous function such that

$$\eta(t) > 0 \text{ for all } t \in [0, \bar{t} \wedge T] \quad (10)$$

and  $U_t := \{x \in \mathbb{R}^N \mid |u(x, t)| \leq \delta_0/4\}$ . See [6, Theorem 4] for the proof.

We derive lower gradient estimate (7) from (9) formally here. We have

$$\begin{aligned} \lambda\eta(t_0) &\leq u(x_0 + \lambda\nu(x_0), t_0) - u(x_0, t_0) \\ &= \lambda \langle Du(x_0, t_0), \nu(x) \rangle + o(\lambda\|\nu\|_\infty) \\ &\leq \lambda |Du(x_0, t_0)| \|\nu\|_\infty + o(\lambda\|\nu\|_\infty) \end{aligned}$$

in a neighborhood of  $\{u(\cdot, t) = 0\}$

for all  $t \in [0, \bar{t} \wedge T]$  with  $o(r)/r \rightarrow 0$  as  $r \rightarrow 0$ . Dividing  $\lambda$  in the above and taking a sufficiently small  $\lambda \in (0, \bar{\lambda}]$ , we get the lower estimate (7).

Finally, I explain about the geometrical interpretation of (8) easily. Assume that  $\Gamma_0$  is a  $C^2$  hypersurface. Then it is well known that the signed distance function  $d_{\Gamma_0}^s$  to  $\Gamma_0$  is  $C^2$ . It follows that (8) hold with  $u_0$  such that  $u_0 = d_{\Gamma_0}^s$  in a neighborhood of  $\Gamma_0$ . Indeed, setting  $\nu(x) = Du_0(x)$ , we have

$$\begin{aligned} &u_0(x + \lambda\nu(x)) \\ &= u_0(x) + \lambda |Du_0(x)|^2 + \lambda\omega(\lambda) \\ &\geq u_0(x) + \lambda \frac{\min |Du_0(x)|^2}{2} \end{aligned}$$

where  $\omega$  is a modulus of continuity of  $Du_0$ . We refer to [6, Section 3] for details.

## 5 Short Time Uniqueness

We state a short time uniqueness result for solutions of the Cauchy problem for (3).

**Theorem 1** (Uniqueness Result of Solutions in a Short Time). *If there exist viscosity solutions of the initial-value problem (3), they are unique in  $\mathbb{R}^N \times [0, \bar{t}]$ , where  $\bar{t}$  is given by (9).*

**Remark 1.** We need to discuss existence results, but here we omit it. See [6] for details.

*Proof.* Suppose that there exist viscosity solutions  $u_1$  and  $u_2$  of (3) with the same initial value. Let  $\tau \in (0, T]$  which shall be fixed later and set

$$\delta_\tau := \max_{\mathbb{R}^N \times [0, \tau]} |(u_1 - u_2)(x, t)|.$$

Then by using a result of continuous dependence of solutions (see [6, Theorem 1]), we have

$$\delta_\tau \leq M_1 \kappa_\tau (\tau + \tau^{1/2}), \quad (11)$$

where

$$\kappa_\tau := \sup_{t \in [0, \tau]} \int_{\mathbb{R}^N} |\mathbf{1}_{\{u_1(\cdot, t) \geq 0\}}(y) - \mathbf{1}_{\{u_2(\cdot, t) \geq 0\}}(y)| dy. \quad (12)$$

Note that

$$\begin{aligned} &|\mathbf{1}_{\{u_1(\cdot, t) \geq 0\}}(y) - \mathbf{1}_{\{u_2(\cdot, t) \geq 0\}}(y)| \\ &\leq \mathbf{1}_{\{-\delta_\tau \leq u_1(\cdot, t) < 0\}}(y) + \mathbf{1}_{\{-\delta_\tau \leq u_2(\cdot, t) < 0\}}(y). \end{aligned}$$

We fix

$$t^* \in (0, \bar{t} \wedge T),$$

where  $\bar{t}$  is given by Theorem 9. Take  $\tau \leq t^*$  small enough in order that  $\delta_\tau \leq \delta_0/4$ , the lower-bound gradient estimate (7) holds on  $[0, \tau]$  and, for all  $t \in [0, \tau]$ ,  $\eta(t) \geq \eta(t^*) =: \bar{\eta} > 0$ .

We have, for  $i = 1, 2$ ,

$$\begin{aligned} &\int_{\mathbb{R}^N} \mathbf{1}_{\{-\delta_\tau \leq u_i(\cdot, t) < 0\}}(y) dy \\ &= \mathcal{L}^N(\{-\delta_\tau \leq u_i(\cdot, t)\}) - \mathcal{L}^N(\Omega_t^i), \end{aligned} \quad (13)$$

since  $\Omega_t^i := \{u_i(\cdot, t) > 0\} \subset \{-\delta_\tau \leq u_i(\cdot, t)\}$ .

We claim that

$$\{-\delta_\tau \leq u_i(\cdot, t)\} \subset (I + \frac{\delta_\tau}{\bar{\eta}} \nu)^{-1}(\Omega_t^i)$$

for  $t \in [0, \tau]$  and  $\tau$  small enough. (14)

We recall that  $\psi_\lambda = (I + \lambda\nu)$  is a  $C^1$ -diffeomorphism when  $\lambda$  is enough small. To prove the claim, let  $(x, t) \in \mathbb{R}^N \times [0, \tau]$  such that  $u_i(x, t) \geq -\delta_\tau$  and set

$$\lambda = \frac{\delta_\tau}{\bar{\eta}}.$$

We distinguish two cases. If  $u_i(x, t) \geq \delta_0/4$ , then, by the regularity of solutions (see [6, Proposition 3]),

$$\begin{aligned} &u_i(x + \lambda\nu(x), t) \geq u_i(x, t) - L|\nu(x)|\lambda \\ &\geq \delta_0/4 - L\|\nu\|_\infty\lambda \geq 0 \end{aligned}$$

for some  $L > 0$  and all  $\lambda \leq \delta_0/(4L\|\nu\|_\infty)$ . If  $-\delta_\tau \leq u_i(x, t) \leq \delta_0/4$ , then, by (9),

$$u_i(x + \lambda\nu(x), t) \geq u_i(x, t) + \lambda\bar{\eta} \geq -\delta_\tau + \lambda\bar{\eta} = 0$$

for  $\delta_\tau \leq \delta_0/4$ ,  $\lambda \leq \bar{\lambda}$  and  $t \leq \bar{t}$ . Finally, (14) holds if  $\tau$  is such that

$$\tau \leq \bar{t} \quad \text{and} \quad \delta_\tau \leq \min\left\{\frac{\delta_0}{4}, \frac{\bar{\eta}\delta_0}{4L\|\nu\|_\infty, \bar{\eta}\bar{\lambda}}\right\}.$$

By continuity of  $u_1, u_2$  which achieve the same initial condition, it is always possible to find  $\tau > 0$  small enough in order that this latter condition holds.

Note that  $\Omega_t^i \subset B(0, R_T)$  for some  $R_T > 0$ , since we are considering compact hypersurface evolution. By a change of variable, up to take  $\lambda$ , and so  $\tau$ , smaller, we have

$$\begin{aligned} & \mathcal{L}^N((I + \lambda\nu)^{-1}(\Omega_t^i)) \\ &= \int_{\Omega_t^i} \det(D(I + \lambda\nu)^{-1}) dx \\ &\leq (1 + 2N\lambda\|D\nu\|_\infty)\mathcal{L}^N(\Omega_t^i), \end{aligned}$$

since  $\det(D(I + \lambda\nu)^{-1}) = 1 - \lambda \operatorname{tr}(D\nu) + o(\lambda)$ .

From (11), (13) and (14), it follows

$$\begin{aligned} \kappa_\tau &\leq 2(\mathcal{L}^N((I + \frac{2\delta_\tau}{\eta_0}\nu)^{-1}(\Omega_t^i)) - \mathcal{L}^N(\Omega_t^i)) \\ &\leq \frac{4N\delta_\tau}{\eta_0}\mathcal{L}^N(B(0, R_T)). \end{aligned} \quad (15)$$

Therefore, we get

$$\delta_\tau \leq \frac{C}{\bar{\eta}}\delta_\tau(\tau + \sqrt{\tau})$$

for some constant  $C > 0$  which is independent of  $\tau$ . For  $\tau$  small enough, we have  $\delta_\tau = 0$ . It follows  $u_1 = u_2$  on  $\mathbb{R}^N \times [0, \tau]$ .

We consider  $\bar{\tau} = \sup\{\tau > 0 \mid u_1 = u_2 \text{ on } \mathbb{R}^N \times [0, \tau]\}$ . If  $\bar{\tau} < t^*$ , then we can repeat the above proof from time  $\bar{\tau}$  instead of 0. Finally, we have  $u_1 = u_2$  on  $\mathbb{R}^N \times [0, t^*]$  for all  $t^* < \bar{t}$ , which gives the conclusion.  $\square$

**Acknowledgements.** I am grateful to organizers for giving me an opportunity to talk in the conference “*A minisemester on evolution of interfaces*”. The preparation for this talk has been done while I was visiting Laboratoire de Mathématiques et Physique Théorique, Fédération Denis Poisson, Université de Tours. I am grateful for its hospitality.

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# Stability analysis of steady states for surface diffusion equation in a bounded domain

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## 1 Introduction

The motion of curves or surfaces under geometric evolution equations is related to the motion of phase boundaries. For example, the mean curvature flow equation

$$V = H$$

is a well-known geometric equation, which describes a motion of anti-phase boundaries and also a grain growth (see [1, 9]). Here,  $V$  is the normal velocity of the evolving surface and  $H$  is the mean curvature of surface where we use the sign convention that a sphere with the normal pointing to the inside has positive curvature.

In this talk, we study the surface diffusion equation

$$V = -\Delta H,$$

which was derived by Mullins [10] to model the motion of interfaces in the case that the motion of interfaces is governed purely by mass diffusion within the interfaces (for simplicity we set the diffusion constant to 1). Here,  $\Delta$  is the Laplace-Beltrami operator. More precisely, we consider the following problem. Given an bounded domain  $\Omega \subset \mathbb{R}^2$  we look for evolving curves  $\Gamma = \{\Gamma_t\}_{t>0}$  (for a definition, see Gurtin [5]), which lies in  $\Omega$  and satisfies  $\partial\Gamma_t \subset \partial\Omega$ , with the properties for  $t > 0$ :

$$\begin{cases} V = -\kappa_{ss} & \text{on } \Gamma_t, \\ \sphericalangle(\partial\Omega, \Gamma_t) = \pi/2 & \text{at } \partial\Omega \cap \Gamma_t, \\ \kappa_s = 0 & \text{at } \partial\Omega \cap \Gamma_t. \end{cases} \quad (1.1)$$

Here,  $\kappa$  is the curvature of  $\Gamma_t$  and a subscript  $s$  denotes the differentiation with respect to an arc-length parameter. Then we observe that (1.1) has the basic properties:

$$\frac{d}{dt}L_{\Gamma_t} \leq 0, \quad \frac{d}{dt}A_{\Gamma_t} = 0.$$

Here we denote by  $A_{\Gamma_t}$  the area enclosed by the evolving curve  $\Gamma_t$  and  $\partial\Omega$  at time  $t$  and by  $L_{\Gamma_t}$  the length of  $\Gamma_t$  at time  $t$ .

Our goal in this talk is to derive criteria of linearized stability based on the work of [4], [3], [8] which deal with the mean curvature flow. The analysis in the case of the surface diffusion is more difficult since the surface diffusion equation is fourth order nonlinear parabolic P.D.E.. For the convenience of readers, we show some differences between the mean curvature flow equation and the surface diffusion equation.

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- The curvature flow:  $V = \kappa$ 
  - The gradient flow of the length  $L_{\Gamma_t}$  with respect to the  $L^2$ -inner product.
  - Not area-preserving.
  - Steady states are the line segments.
  - Sharp interface model as a singular limit of Allen-Cahn equation.
- The surface diffusion:  $V = -\kappa_{ss}$ 
  - The gradient flow of the length  $L_{\Gamma_t}$  with respect to the  $H^{-1}$ -inner product (cf. [11]).
  - Area-preserving.
  - Steady states are the line segments and the circular arcs.
  - Sharp interface model as a singular limit of Cahn-Hilliard equation.

This talk is based on the results in [6]. In the depends, the criteria of the linearized stability of steady states for the three-phase problem with triple junction will be showed in some special cases (cf. [7]).

## 2 Linearized problem and its eigenvalue problem

To study the linearized stability of a steady state  $\Gamma_*$  with a constant curvature  $\kappa_*$ , we introduce the following linearized problem around  $\Gamma_*$ :

$$\begin{cases} \rho_t = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho & \text{for } \sigma \in (l_*^-, l_*^+), t > 0, \\ (\partial_\sigma \pm h_*^\pm)\rho = 0 & \text{at } \sigma = l_*^\pm, \\ \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0 & \text{at } \sigma = l_*^\pm. \end{cases} \quad (2.1)$$

Here,  $\sigma$  is an arc-length parameter of  $\Gamma_*$  in  $[l_*^-, l_*^+]$  and  $h_*^\pm$  are the curvature of  $\partial\Omega$  at  $\Gamma_* \cap \partial\Omega$ . Furthermore, the linearization of the area-preserving property is

$$\int_{l_*^-}^{l_*^+} \rho d\sigma = 0. \quad (2.2)$$

Since the original problem (1.1) has the area-preserving property, we need to analyze the linearized problem (2.1) for functions  $\rho$  satisfying (2.2).

Let us study the eigenvalue problem corresponding to the linearized problem (2.1). In what follows we need the duality pairing  $\langle \cdot, \cdot \rangle$  between  $(H^1(l_*^-, l_*^+))'$  and  $(H^1(l_*^-, l_*^+))$  and the following weak formulation.

**Definition 2.1** *We say that  $u_v \in H^1(l_*^-, l_*^+)$  for a given  $v \in (H^1(l_*^-, l_*^+))'$  with  $\langle v, 1 \rangle = 0$  is a weak solution of*

$$\begin{cases} -\partial_\sigma^2 u_v = v & \text{for } \sigma \in (l_*^-, l_*^+), \\ \partial_\sigma u_v = 0 & \text{at } \sigma = l_*^\pm, \end{cases} \quad (2.3)$$

if  $u_v$  satisfies

$$\langle v, \xi \rangle = \int_{l_*^-}^{l_*^+} \partial_\sigma u_v \partial_\sigma \xi d\sigma$$

for all  $\xi \in H^1(l_*^-, l_*^+)$ .

In addition we also introduce the symmetric bilinear form on  $H^1(l_*^-, l_*^+)$

$$I[\rho_1, \rho_2] := \int_{l_*^-}^{l_*^+} (\partial_\sigma \rho_1 \partial_\sigma \rho_2 - \kappa_*^2 \rho_1 \rho_2) d\sigma + h_*^+ \rho_1 \rho_2 \Big|_{\sigma=l_*^+} + h_*^- \rho_1 \rho_2 \Big|_{\sigma=l_*^-}$$

and the inner product

$$(\rho_1, \rho_2)_{-1} := \int_{l_*^-}^{l_*^+} \partial_\sigma u_{\rho_1} \partial_\sigma u_{\rho_2} d\sigma$$

where  $u_{\rho_i} \in H^1(l_*^-, l_*^+)$  for a given  $\rho_i \in (H^1(l_*^-, l_*^+))'$  with  $\langle \rho_i, 1 \rangle = 0$  is defined as the weak solution of (2.3).

By choosing an appropriate domain of definition, the linearized operator in (2.1) is given by

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow H, \quad \langle \mathcal{A}\rho, \xi \rangle := \int_{l_*^-}^{l_*^+} \partial_\sigma (\partial_\sigma^2 + \kappa_*^2) \rho \partial_\sigma \xi d\sigma$$

with

$$\begin{cases} \mathcal{D}(\mathcal{A}) = \{ \rho \in H^3(l_*^-, l_*^+) \mid (\partial_\sigma \pm h_*^\pm) \rho = 0 \text{ at } \sigma = l_*^\pm \text{ and } \int_{l_*^-}^{l_*^+} \rho d\sigma = 0 \}, \\ H = \{ \rho \in (H^1(l_*^-, l_*^+))' \mid \langle \rho, 1 \rangle = 0 \}. \end{cases}$$

Let us analyze the spectrum of  $\mathcal{A}$  in order to decide on the stability behaviour of the linearized problem (2.1). Using classical principles of the variational calculus, we can describe the spectrum of  $\mathcal{A}$  with the help of the bilinear form  $I$  and the inner product  $(\cdot, \cdot)_{-1}$ . In fact, if  $\rho$  is an eigenfunction to the eigenvalue  $\lambda$ , it holds

$$\lambda(\rho, \xi)_{-1} = (\mathcal{A}\rho, \xi)_{-1} = -I[\rho, \xi].$$

We remark that eigenvalues  $\lambda \neq 0$  always correspond to eigenfunctions which have the mean value zero. This is a natural request from (2.2). First we have the following lemma for the operator  $\mathcal{A}$ .

**Lemma 2.2** (i) *The operator  $\mathcal{A}$  is self-adjoint with respect to the inner product  $(\cdot, \cdot)_{-1}$ .*  
(ii) *The spectrum of  $\mathcal{A}$  contains a countable system of real eigenvalues.*

In addition, we have the following lemmas for the eigenvalues of  $\mathcal{A}$ .

**Lemma 2.3** *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

*be the eigenvalues of  $\mathcal{A}$  (taking the multiplicity into account).*

(i) *It holds for all  $n \in \mathbf{N}$*

$$\lambda_n = - \inf_{W \in \Sigma_n} \sup_{\rho \in W \setminus \{0\}} \frac{I[\rho, \rho]}{(\rho, \rho)_{-1}}, \quad \lambda_n = - \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^\perp \setminus \{0\}} \frac{I[\rho, \rho]}{(\rho, \rho)_{-1}}.$$

*Here  $\Sigma_n$  is the collection of  $n$ -dimensional subspaces of  $V$  and  $W^\perp$  is the orthogonal complement with respect to the inner product  $(\cdot, \cdot)_{-1}$ .*

(ii) *The eigenvalues  $\lambda_n$  depend continuously on  $h_*^+$ ,  $h_*^-$  and  $\kappa_*^2$ ; and are monotone decreasing in each of the parameters  $h_*^+$ ,  $h_*^-$  and  $(-\kappa_*^2)$ .*

**Lemma 2.4** (i) Assume  $\kappa_* \neq 0$ . Then the operator  $\mathcal{A}$  has a zero eigenvalue if and only if

$$\frac{a}{c} + \frac{b}{c}(h_*^+ + h_*^-) + h_*^+ h_*^- = 0 \quad (2.4)$$

where

$$\begin{aligned} a &= |\kappa_*|^2 L_* \sin(|\kappa_*| L_*), \\ b &= \sin(|\kappa_*| L_*) - |\kappa_*| L_* \cos(|\kappa_*| L_*), \\ c &= -L_* \sin(|\kappa_*| L_*) - \frac{2}{|\kappa_*|} \cos(|\kappa_*| L_*) + \frac{2}{|\kappa_*|} \end{aligned}$$

with  $L_* = l_*^+ - l_*^-$ . Furthermore, it holds the inequality

$$\frac{b^2}{c^2} - \frac{a}{c} > 0. \quad (2.5)$$

(ii) Assume that  $\kappa_* = 0$ . Then the operator  $\mathcal{A}$  has a zero eigenvalue if and only if

$$\frac{12}{L_*^2} + \frac{4}{L_*}(h_*^+ + h_*^-) + h_*^+ h_*^- = 0. \quad (2.6)$$

(iii) If we interpret  $a$ ,  $b$ , and  $c$  as functions of  $\kappa_*$ , we obtain

$$\frac{a}{c} \rightarrow \frac{12}{L_*^2} \quad \text{and} \quad \frac{b}{c} \rightarrow \frac{4}{L_*} \quad \text{as} \quad \kappa_* \rightarrow 0.$$

(iv) The multiplicity of a zero eigenvalue is equal to one for all  $h_*^+$ ,  $h_*^-$ , and  $\kappa_*$ .

Set

$$\mathcal{D}(h_*^+, h_*^-, \kappa_*, L_*) = \frac{a}{c} + \frac{b}{c}(h_*^+ + h_*^-) + h_*^+ h_*^-.$$

Let  $N_u$  be the number of the unstable eigenvalues and also let  $N_0$  be the number of the zero eigenvalues (counting the multiplicity). Then, we are led to the following theorem.

**Theorem 2.5** (i) If  $\mathcal{D}(h_*^-, h_*^+, \kappa_*) > 0$  and  $h_*^- > -b/c$ , then  $N_u = N_0 = 0$ .

(ii) If  $\mathcal{D}(h_*^-, h_*^+, \kappa_*) = 0$  and  $h_*^- > -b/c$ , then  $N_u = 0$  and  $N_0 = 1$ .

(iii) If  $\mathcal{D}(h_*^-, h_*^+, \kappa_*) < 0$ , then  $N_u = 1$  and  $N_0 = 0$ .

(iv) If  $\mathcal{D}(h_*^-, h_*^+, \kappa_*) = 0$  and  $h_*^- < -b/c$ , then  $N_u = 1$  and  $N_0 = 1$ .

(v) If  $\mathcal{D}(h_*^-, h_*^+, \kappa_*) > 0$  and  $h_*^- < -b/c$ , then  $N_u = 2$  and  $N_0 = 0$ .

**Remark 2.6**  $\mathcal{D}(h_*^+, h_*^-, \kappa_*, L_*) = 0$  draws the hyperbola in  $(h_*^-, h_*^+)$ -plane. Theorem 2.5 says that above the upper arc of the hyperbola we have only negative eigenvalues, which imply the stability of steady states. Underneath of it and above the lower arc of the hyperbola, we have one positive eigenvalue, which means that the number of unstable modes is one. Furthermore, underneath of it, we have two positive eigenvalues, which mean that the number of unstable modes is two.

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## Part 3

Tutorial Lectures and International Workshop  
“ Singular Diffusion and Evolving Interfaces ”

国際ワークショップと  
チュートリアルセミナー  
「特異拡散と動く界面」



# Tutorial Lectures and International Workshop “ Singular Diffusion and Evolving Interfaces ”

## 国際ワークショップとチュートリアルセミナー 「特異拡散と動く界面」

組織委員: 儀我美一, 舟木直久, 利根川吉廣, P. Rybka, 儀我美保

Organizers: T. Funaki, Y. Giga, P. Rybka, Y. Tonegawa, M.-H. Giga

Period (期間): August 2 - 6, 2010

Venue (場所): Room 309, Faculty of Science Building 3, Hokkaido University  
北海道大学理学部3号館3-309室

### August 2, 2010 (Monday)

09:00-09:50 Registration

09:50-10:00 Opening

10:00-10:50 Giovanni Bellettini (Univ. of Rome Tor Vergata)  
(Lecture I) Anisotropic perimeters: variational properties

11:10-12:00 Jose M. Mazón (Univ. of Valencia)  
(Lecture I) The total variation flow

14:00-14:50 西川貴雄 (日本大学) Takao Nishikawa (Nihon Univ.)  
Probabilistic models of interfaces and their scaling limit

15:10-16:00 山崎教昭 (神奈川大学) Noriaki Yamazaki (Kanagawa Univ.)  
Mathematical analysis of grain boundary motion models of  
Kobayashi-Warren-Carter type

16:30-17:20 Jong-Sheng Guo (Tamkang Univ.)  
Motion by curvature of planar curves with two free end points

18:00- Welcome Party

August 3, 2010 (Tuesday)

- 10:00-10:50 Giovanni Bellettini (Univ. of Rome Tor Vergata)  
(Lecture II) Anisotropic and crystalline mean curvature flow.  
Reaction-diffusion approximation.
- 11:10-12:00 Jose M. Mazón (Univ. of Valencia)  
(Lecture II) The total variation flow
- 14:00-16:00 Short Communications
- 16:20-17:10 Piotr Mucha (Univ. of Warsaw)  
The notion of an almost classical solutions to the total  
variation flow and its usefulness

August 4, 2010 (Wednesday)

- 10:00-10:50 Harald Garcke (Univ. of Regensburg)  
The Stefan problem with anisotropic Gibbs-Thomson law
- 11:10-12:00 William Allard (Duke Univ.)  
Variational problems using total variation defined by  
noneuclidean norms

August 5, 2010 (Thursday)

- 10:00-10:50 Giovanni Bellettini (Univ. of Rome Tor Vergata)  
(Lecture III) Examples of facet-breakings in crystalline mean  
curvature flow
- 11:10-12:00 Juan J. Manfredi (Univ. of Pittsburgh)  
(Lecture I) A survey of  $p$ -harmonious functions in trees  
and in Euclidean space
- 14:00-14:30 Jose S. Moll (Univ. of Valencia)  
Large solutions for some parabolic equations without absorption
- 14:40-15:10 儀我美一 (東京大学) Yoshikazu Giga (Univ. of Tokyo)  
Scale-invariant extinction time estimates for some singular  
diffusion equations
- 15:20-16:10 Yen-Hsi R. Tsai (The Univ. of Texas at Austin)  
Numerical methods for smooth and crystalline curvature flow

August 6, 2010 (Friday)

- 10:00-10:50 Giovanni Bellettini (Univ. of Rome Tor Vergata)  
(Lecture IV) Crystalline curvature flow of networks
- 11:10-12:00 Juan J. Manfredi (Univ. of Pittsburgh)  
(Lecture II) Random Tug-of-War games for the parabolic  $p$ -Laplacian
- 14:00-14:50 白川健 (神戸大学) Ken Shirakawa (Kobe Univ.)  
Phase transition models based on linear-growth interfacial energies
- 15:10-16:00 Jose M. Mazon (Univ. of Valencia)  
The Dirichlet problem for a singular elliptic equation arising  
in the level set formulation of the inverse mean curvature flow
- 16:00- Closing

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# An introduction to anisotropic and crystalline mean curvature flow

G. Bellettini\*

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## 1 Introduction

The aim of these notes is to give an elementary introduction to embedded anisotropic mean curvature flow in codimension one, with some attention to crystalline mean curvature flow. We will also discuss very briefly the generalization to a multiphase problem in the plane, namely to crystalline evolutions of planar partitions. For a better understanding of the arguments of the notes, some knowledge on motion by mean curvature in the euclidean setting would be recommended. We refer the reader to the introductory parts of the following references: [43], [94], [69], [70], [95], [12], [13], [72], [73], [74], [75], [96], [57], [86], [5], [68], [81], [100], [23].

We will mostly concentrate on the derivation of the evolution laws, rather than on detailed proofs: one reason for this is to keep the exposition within a limited number of pages. Another reason is that the proofs can be found in the original papers.

Apart from the initial section, where we often try to minimize the assumptions on the function  $\phi^\circ$ , and from the final section on partitions, the view point that we will adopt here is mostly based on the use of the anisotropic signed distance function  $d_\phi$ . As a consequence, we will not consider the evolution problem looking at the maps parametrizing the manifolds (see [105] and references therein for this parametric approach), but instead we will look only at the images of the maps. This approach is closely related to various derivations of mean curvature flow that can be found in the literature on phase transitions [65], [66], [67], [18], [56]. In this

respect, a very quick presentation of the reaction-diffusion approximation to crystalline mean curvature flow is also presented.

Mathematical and physical motivations for anisotropic mean curvature flow can be found in the large number of papers present in the literature devoted to this subject, as well as detailed reference lists. We apologize with the reader, since the bibliography in these notes is largely incomplete. We sometimes quote references weakly related to the treated argument, but that we believe to be useful for a more general point of view on that subject.

## 2 Notation

Since we will consider Finsler norms [119], [16], [108], [107], on  $\mathbb{R}^n$  and their duals, we believe that it is more clear to use a notation which distinguishes the base manifold from its tangent space, and as most as possible vectors from covectors.

Therefore we set  $M = \mathbb{R}^n$  and  $V = T_x M = \mathbb{R}^n$  the tangent space to  $M$  at any  $x \in M$ , and  $TM = M \times V$  (resp.  $T^*M = M \times V^*$ ,  $V^*$  the dual of  $V$ ) the tangent (resp. cotangent) bundle to  $M$ . We denote by  $\cdot$  and  $|\cdot|$  the scalar product and the norm in  $V$ , respectively, and by  $d(\cdot)$  the euclidean distance in  $M$ . Recall that  $V$  can be identified with  $V^*(1)$ .

$\mathcal{L}^n$  is the Lebesgue measure and  $\mathcal{H}^k$  the  $k$ -dimensional euclidean Hausdorff measure in  $M$  for  $k \in \{0, \dots, n\}$ . Recall that  $\mathcal{H}^n = \mathcal{L}^n$  [11]. If  $B \subset M$  is measurable, we often write  $\mathcal{L}^n(B) = |B|$ . We will use the words orthogonal, unit vector etc. in the euclidean sense. If  $F$  is a set, we let  $\mathcal{P}(F)$  be the class of all subsets of  $F$ .

We denote by  $\Lambda^1 V$  (resp.  $\Lambda_1 V$ ) the space of one-covectors (resp. one-vectors) of  $V$ . On these two vector spaces, we have the norm  $|\cdot|$  induced by the euclidean norm [76], [80]. We sometimes use the symbol  $\Lambda^1 V$  and sometimes  $V^*$  (which are thought of as row vectors); similarly for  $\Lambda_1 V$  and  $V$  (column vectors). We usually omit the symbol  $T$  of transposition when we write a column vector in components. The duality between  $\Lambda^1 V$  and  $\Lambda_1 V$  is denoted by  $\langle \cdot, \cdot \rangle$ .

Recall that any covector  $\xi^* \in \Lambda^1 V$  is a linear map  $V \rightarrow \mathbb{R}$ . If  $|\xi^*| = 1$  (where  $|\xi^*| := \max\{\langle \xi^*, \xi \rangle : \xi \in \Lambda_1 V, |\xi| = 1\}$ ) we can uniquely associate with  $\pm \xi^*$  its kernel, which is an hyperplane in  $V$ . Therefore there is a bijection<sup>(2)</sup> between the set of unit covectors and the set of all oriented hyperplanes of  $V$  passing through the origin. We denote by  $G^*$  (resp.  $G$ ) the set of all oriented (resp. unoriented) hyperplanes of  $V$  passing through the origin; using the euclidean scalar product, with such a hyperplane we can uniquely associate a unit vector (resp. unit vector up to its sign), orthogonal to the hyperplane. Sometimes we will identify  $G^*$  with  $\mathbb{S}^{n-1} := \{v \in V : |v| = 1\}$ . We refer the reader to [76], [80] for more details on the Grassmann algebra.

Covector fields on  $M$  have lower indices

$$\omega : x \in M \rightarrow \omega(x) = (\omega_1(x), \dots, \omega_n(x)) \in \Lambda^1 V.$$

---

<sup>1</sup>There is an isometric isomorphism  $T : V \rightarrow V^*$  defined as  $\langle \xi^*, \xi \rangle = T(\xi) \cdot \xi^*$  for any  $\xi^* \in V^*$ , and similarly, there is an isometric isomorphism  $T^* : V^* \rightarrow V$  defined as  $\langle \xi^*, \xi \rangle = T^*(\xi^*) \cdot \xi$  for any  $\xi \in V$ . Since  $V$  can be identified with  $V^{**}$ , we have  $TT^* = \text{Id}_{V^*}$ ,  $T^*T = \text{Id}_V$ .

<sup>2</sup>Sometimes it is useful to use another identification: a hyperplane  $H$  of  $V$  can be identified with the linear map  $\pi_H : V \rightarrow V$  which is the orthogonal projection of  $V$  onto  $H$ . If  $\xi$  is a unit vector which is orthogonal to  $H$ , we have  $\pi_V = \text{Id} - \xi \otimes \xi$ , where the symbol  $\xi \otimes \xi$  stands for the  $(0, 2)$ -tensor that is represented by the  $(n \times n)$  matrix having  $\xi^i \xi^j$  as its  $ij$ -th entry.

Given a function  $f : M \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$ , we denote by  $df_x \in \Lambda^1 V$  the differential of  $f$  at  $x \in M$ .

Vector fields (or contravariant vector fields) on  $M$  have upper indices,

$$X : x \in M \rightarrow X(x) = (X^1(x), \dots, X^n(x)) \in V. \quad (2.1)$$

If  $X$  is of class  $\mathcal{C}^1$ , the divergence of  $X$  is defined as  $\operatorname{div} X := \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}$ .

Given a function  $f : M \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$ ,  $\operatorname{grad} f = (\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n})^T$  is the vector field gradient of  $f$ . To simplify the notation, we write  $\nabla f$  in place of  $\operatorname{grad} f$ . If necessary,  $df_x$  will be identified with  $\nabla f(x)$ , using the euclidean scalar product. If  $f$  is of class  $\mathcal{C}^2$ , the Laplacian of  $f$  is defined as  $\Delta f := \operatorname{div} \nabla f$ .

Given open sets  $\Omega \subseteq M$ ,  $\Omega' \subseteq \mathbb{R}^m$ ,  $m \geq 1$ , and a map  $\psi : M \rightarrow \Omega'$  of class  $\mathcal{C}^1$ ,  $\psi = (\psi^1, \dots, \psi^m)$ , we denote by  $d\psi_x \in \mathcal{L}(V, W)$  the differential of  $f$  at  $x$ , where  $W$  is the tangent space to  $\Omega'$  at any of its points. The Jacobian ( $m \times n$ ) matrix representing  $d\psi_x$  is indicated by  $J\psi(x)$ . If  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$ , the  $ij$ -entry  $(J\psi(x))_{ij}$  of  $J\psi(x)$  is  $\frac{\partial \psi^i}{\partial x^j}(x)$ . Hence the  $i$ -th column of the transposed matrix  $(J\psi(x))^T$  is  $\nabla \psi^i(x)$ .

Given a smooth vector field  $X = (X^1, \dots, X^n)$ , we denote by  $\nabla X$  the matrix  $(\nabla X)_{ij} = \nabla_i X^j$ . If  $v = (v^1, \dots, v^n)$  is a column vector in  $\Lambda^1 V$  and  $A, B$  are  $(n \times n)$ -matrices, we use the notation  $Av$ ,  $v^T A$  to denote respectively the vectors of components  $(Av)_i = A_{ij} v^j$ , and  $(vA)_i = A_{ji} v^j$ .

The symbol  $E$  (resp.  $E(t)$  for  $t$  belonging to some real interval,  $E_i$  for  $i \in \mathbb{N}$ ) will denote a closed subset of  $M$  with compact boundary such that  $E = \overline{\operatorname{int}(E)}$  (resp.  $E(t) = \overline{\operatorname{int}(E(t))}$ ,  $E_i = \overline{\operatorname{int}(E_i)}$ ).

**Definition 2.1 (Lipschitz boundaries).** *We say that  $E$  is Lipschitz if the boundary  $\partial E$  of  $E$  can be written, locally, as the graph of a Lipschitz function with respect to a suitable  $(n-1)$ -dimensional orthogonal coordinate system. We will write  $\partial E \in \operatorname{Lip}(M)$ .*

Recall that if  $\partial E \in \operatorname{Lip}(M)$ , then [11] for  $\mathcal{H}^{n-1}$ -almost every  $x \in \partial E$  it is well defined the tangent plane  $T_x(\partial E)$ , which is identified with  $\pm \nu^E(x)$ , where

$$\nu^E(x) = \nu(x) \in \mathbb{S}^{n-1}$$

is the unit covector normal to  $\partial E$  at  $x$  and points toward the complement  $M \setminus E$  of  $E$ . Lipschitz and polyhedral boundaries (with a finite number of facets) will be useful in connection with crystalline anisotropies. In this context, if  $F$  is a facet of a polyhedral  $\partial E$ , we denote by  $\partial F$  (resp.  $\operatorname{int}(F)$ ) the relative boundary (resp. the relative interior) of  $F$ . We define

$$\tilde{\nu}^F \quad (2.2)$$

to be the  $\mathcal{H}^{n-2}$ -almost everywhere defined unit normal to the relative boundary  $\partial F$  of  $F$ , lying in the hyperplane  $H_F$  containing  $F$ , and pointing outside of  $F$ .

### 3 Anisotropic functionals on boundaries

Let  $M \times G^*$  be the unit cotangent bundle of  $M$  [16]. Let  $\sigma : M \times G^* \rightarrow [0, +\infty]$  be a measurable function. We shall assume that  $\sigma(x, \cdot)$  is even<sup>(3)</sup>, namely  $\sigma(x, \xi^*) = \sigma(x, -\xi^*)$ , so that we can consider  $\sigma$  as defined on  $M \times G$ . The domain of  $\sigma$  is the set  $\{(x, \xi^*) \in M \times G^* : \sigma(x, \xi^*) < +\infty\}$ , which coincides with  $M \times G$  in case that  $\sigma$  is continuous. Let  $a : M \rightarrow [0, +\infty]$  be a given function<sup>(4)</sup>. defined everywhere on  $M$ . Associated with  $\sigma$  and  $a$ , we can consider the following anisotropic functional [76], [42], [7], [8] defined on boundaries:

$$\mathcal{F}(E) := \int_{\partial E} \sigma(x, \nu(x)) a(x) d\mathcal{H}^{n-1}(x) = \int_{\partial E} \sigma(x, \nu) a d\mathcal{H}^{n-1}, \quad \partial E \in \text{Lip}(\mathbb{R}^n). \quad (3.1)$$

The functional  $\mathcal{F}$  can be extended to the class of finite perimeter sets (where now the unit normal  $\nu$  must be intended in a proper measure theoretic sense [11]): we will not need such an extension in these notes. Useful lower semicontinuity properties of this extension<sup>(5)</sup> can be found in [11].

The boundary  $\partial E$ , also called interface, divides the two sets  $E$  and  $M \setminus E$ , sometimes called phases. In case  $\sigma$  is independent of  $x$  and  $a \equiv 1$ , the quantity  $\sigma(\nu)$  can be considered as a surface tension  $v$  [1] associated with the hyperplane passing through the origin, orthogonal to the unit covector  $\nu$ .

#### 3.1 The function $\phi^o$

Define the function  $\phi^o : T^*M \rightarrow [0, +\infty]$  to be the one-homogeneous extension of  $\sigma(x, \cdot)$  on the whole space of one-covectors, i.e.,

$$\phi^o(x, \xi^*) := |\xi^*| \sigma \left( x, \frac{\xi^*}{|\xi^*|} \right), \quad (x, \xi^*) \in T^*M. \quad (3.2)$$

Then  $\phi^o(x, \cdot)$  is one-homogeneous, i.e.,

$$\phi^o(x, \lambda \xi^*) = |\lambda| \phi^o(x, \xi^*), \quad (x, \xi^*) \in T^*M, \lambda \in \mathbb{R}, \quad (3.3)$$

The function  $\phi^o(x, \cdot)$  is even, since we supposed  $\sigma(x, \cdot)$  to be even<sup>(6)</sup>. We consider the function  $\phi^o(x, \cdot)$  as acting on differentials  $df_x$  of functions  $f : M \rightarrow \mathbb{R}$  at  $x \in M$ .

We have<sup>(7)</sup>

$$\mathcal{F}_{\phi^o}(E) := \int_{\partial E} \phi^o(x, \nu) a(x) d\mathcal{H}^{n-1}(x) = \mathcal{F}(E). \quad (3.4)$$

For computational convenience, from now on we will consider the functional  $\mathcal{F}_{\phi^o}$  in place of  $\mathcal{F}$ . The gradient flow of the functional  $\mathcal{F}_{\phi^o}$  will lead to anisotropic mean curvature flow: we will mostly be concerned with the case of a function  $\phi^o$  which is independent of  $x$  and  $a \equiv 1$ .

<sup>3</sup>Various results that we will present could be extended without assuming that  $\sigma(x, \cdot)$  is even, but we prefer to keep this assumption in order to make simpler the presentation.

<sup>4</sup>We shall see that if  $\sigma$  does not depend on  $x$ , from a geometric point of view it is natural to take  $a$  to be a positive constant. For simplicity, the reader can assume  $a \equiv 1$ . We notice that, by redefining  $\sigma$ , one can also include the function  $a$  into the new  $\sigma$ . We prefer however to keep  $\sigma$  and  $a$  separate.

<sup>5</sup>Strictly related to the convexity of the function  $\phi^o(x, \cdot)$  defined in (3.2) below.

<sup>6</sup>If  $\sigma(x, \cdot)$  were not even, we should drop the absolute value on the right hand side of (3.3) and take  $\lambda > 0$ .

<sup>7</sup>We prefer to skip the dependence on  $a$  of the functional.

For all  $x \in \partial E$  for which  $\nu(x)$  is defined, we introduce [40], [39] the normalized covariant vector field<sup>(8)</sup>

$$\nu_{\phi^o}^E(x) := \frac{\nu^E(x)}{\phi^o(\nu^E(x))} = \nu_{\phi^o}(x). \quad (3.5)$$

In components  $\nu_{\phi^o} = (\nu_{\phi^o_1}, \dots, \nu_{\phi^o_n})$ .

If  $F \subset \partial E$  is a facet of a polyhedral set  $\partial E$ , we set<sup>(9)</sup>

$$\nu_{\phi^o}(F) := \frac{\nu(F)}{\phi^o(\nu(F))}, \quad (3.6)$$

where  $\nu(F)$  is the unit normal to  $\text{int}(F)$  pointing toward  $M \setminus E$ .

We define<sup>(10)</sup>

$$B_{\phi^o}(x) := \{\xi^* \in \Lambda^1 V : \phi^o(x, \xi^*) \leq 1\}, \quad x \in M. \quad (3.7)$$

The set  $B_{\phi^o}(x)$  uniquely identifies  $\phi^o(x, \cdot)$ , in view of the homogeneity property (3.3).

Notice that if  $\phi^o \in \mathcal{C}^1(M \times (\Lambda^1 V \setminus \{0\}))$ , (3.3) yields

$$\phi^o(x, \xi^*) = \xi^* \cdot \phi_{\xi^*}^o(x, \xi^*), \quad (x, \xi^*) \in M \times (\Lambda^1 V \setminus \{0\}), \quad (3.8)$$

where  $\phi_{\xi^*}^o$  denotes the gradient of  $\phi^o(x, \cdot)$  with respect to  $\xi^*$ .

**Definition 3.1 (Spatial homogeneity).** *We say that  $\sigma$  (resp.  $\phi^o$ ) is spatially homogeneous if it is independent of  $x$ .*

In this case we write  $\phi^o : \Lambda^1 V \rightarrow [0, +\infty]$ , and the right hand side of formula (3.7) is denoted by  $B_{\phi^o}$ <sup>(11)</sup>.

**Example 3.2 (Dual norms).** The first examples of spatially homogeneous  $\phi^o$  are the following:

- $\phi^o(\xi^*) = |\xi^*|$  (euclidean norm, isotropic case).
- $\phi^o(\xi^*) = \sqrt{\sum_{i,j=1}^n g^{ij} \xi_i^* \xi_j^*}$ , where  $(g^{ij})$  is a positive definite symmetric matrix (Riemannian norm). In this case  $B_{\phi^o}$  is an ellipsoid centered at the origin<sup>(12)</sup>. See Figure 2.
- Let  $p \in (1, +\infty)$  and  $\phi^o(\xi^*) := (\sum_{i=1}^n |\xi_i^*|^p)^{1/p}$  ( $l^p$  norms). If  $p > 2$  then  $\partial B_{\phi^o}$  is of class  $\mathcal{C}^2$  but there are some points of  $\partial B_{\phi^o}$  where its second fundamental form vanishes. If  $p \in (1, 2)$  then  $\partial B_{\phi^o}$  is not of class  $\mathcal{C}^2$ .
- A relevant case in these notes is when  $B_{\phi^o}$  is a (convex)  $n$ -dimensional polyhedron centered at the origin, and centrally symmetric [112], [113], [115]. See Figure 1.
- Another interesting case is when  $B_{\phi^o} = C \times [-1, 1]$ , where  $C$  is an  $(n-1)$ -dimensional centrally symmetric convex body [24].

<sup>8</sup>In the quoted references this vector field is denoted by  $\nu_{\phi}$ .

<sup>9</sup>Do not confuse this notation with the notation in (2.2).

<sup>10</sup>The set  $B_{\phi^o}(x)$  is sometimes called Frank diagram, at least under some further assumptions on  $\phi^o$ .

<sup>11</sup>The set  $\{v : v = \rho\nu, \nu \in \mathbb{S}^{n-1}, \rho = \frac{1}{\sigma(\nu)}\} = \partial B_{\phi^o}$  is sometimes called polar plot of  $\sigma$ .

<sup>12</sup>If  $g^{ij}$  would depend on  $x$ , then the ellipsoid would depend on  $x$ , and  $M$  would become the simplest example of Riemannian manifold.

**Remark 3.3 (Degenerate cases).** Let  $\sigma$  be spatially homogeneous: if there exists  $v \in \mathbb{S}^{n-1}$  such that  $\sigma(v) = 0$ , then the whole line  $\mathbb{R}v$  is contained in  $B_{\phi^o}$ . In particular,  $B_{\phi^o}$  is unbounded. On the other hand, if there exists  $v \in \mathbb{S}^{n-1}$  such that  $\sigma(v) = +\infty$ , then  $\mathbb{R}v \setminus \{0\}$  is not contained in  $B_{\phi^o}$  (hence the origin is not an interior point of  $B_{\phi^o}$ ). For example [15], [19], setting  $I := \{(\xi_0^*, \xi^*) \in \mathbb{R} \times \Lambda^1 V \simeq \Lambda^1 \mathbb{R}^{1+n} : -\xi_0^2 + |\xi^*|^2 \leq 1\}$ , we can take  $\phi^o(\xi_0^*, \xi^*) := \inf\{\lambda > 0 : (\xi_0^*, \xi^*) \in \lambda I\}$  for any  $(\xi_0^*, \xi^*) \in \Lambda^1 \mathbb{R}^{n+1}$ . Note that  $I$  is star-shaped with respect to the origin, the origin is not in the interior of  $I$ , and  $\phi^o$  takes also the value  $+\infty$ . Examples of unbounded  $B_{\phi^o}$  have been considered in [78], see also [31].

**Definition 3.4 (Convexity).** We say that  $\phi^o : T^*M \rightarrow [0, +\infty)$  is convex if  $\phi^o(x, \cdot)$  is convex for any  $x \in M$ .

**Remark 3.5.** All functions  $\phi^o$  in Example 3.2 are convex. In addition they satisfy<sup>(13)</sup>

$$\lambda|\xi| \leq \phi^o(x, \xi), \quad (x, \xi^*) \in T^*M, \quad (3.9)$$

for a suitable constant  $\lambda > 0$  (depending on  $\phi^o$ ).

**Definition 3.6 (Metrics on  $T^*M$ ).** The symbol  $\mathcal{M}(T^*M)$  denotes the class of metrics on  $T^*M$ , namely of all continuous functions  $\phi^o : T^*M \rightarrow [0, +\infty)$  which are convex, and satisfy (3.3) and (3.9).

Among convex  $\phi^o$  we are mainly interested in the crystalline ones [112].

**Definition 3.7 (Crystalline metrics).** If  $\phi^o \in \mathcal{M}(T^*M)$  is spatially homogeneous and  $B_{\phi^o}$  is a polyhedron we say that  $\phi^o$  is crystalline.

### 3.1.1 The map $T_{\phi^o}$

For a fixed  $x$ , we now define a map that will play a major role in the analysis of anisotropic mean curvature flow. In order to give the definition, we assume the validity of one of the two following hypotheses: either

$$(\phi^o)^2 \in \mathcal{C}^1(T^*M) \quad (3.10)$$

or

$$\phi^o \text{ is convex.} \quad (3.11)$$

*Notation.* If  $\phi^o$  satisfies (3.10) the symbol  $\nabla_{\xi^*}((\phi^o)^2)$  denotes the gradient vector field of  $(\phi^o(x, \cdot))^2$  with respect to  $\xi^*$ . Assumption (3.11) is equivalent to the convexity of  $(\phi^o(x, \cdot))^2$ , and in this case the same symbol  $\nabla_{\xi^*}(\phi^o)^2$  denotes the subdifferential [106] of  $(\phi^o(x, \cdot))^2$  with respect to  $\xi^*$ . Moreover, if (3.11) holds, then  $\phi_{\xi^*}^o$  denotes the subdifferential of  $\phi^o(x, \cdot)$  with respect to  $\xi^*$ .

<sup>13</sup>If we assume continuity of  $\phi^o$ , in view of (3.3) inequality (3.9) becomes equivalent to the inequality

$$\lambda|\xi| \leq \phi(x, \xi) \leq \phi^o(x, \xi^*), \quad (x, \xi^*) \in T^*M,$$

for two constants  $0 < \lambda \leq \Lambda < +\infty$ . Relevant consequences are that  $B_{\phi^o}(x)$  contains the origin in its interior, and it is star-shaped (with respect to the origin).

**Definition 3.8 (The map  $T_{\phi^o}$ ).** Let  $x \in M$ . We define  $T_{\phi^o}(x, \cdot) : \Lambda^1 V \rightarrow \mathcal{P}(\Lambda_1 V)$  as

$$T_{\phi^o}(x, \xi^*) := \frac{1}{2}(\nabla_{\xi^*}((\phi^o)^2))(x, \xi^*). \quad (3.12)$$

Under assumption (3.11),  $T_{\phi^o}(x, \cdot)$  is sometimes called duality map [45], and it is a possibly multivalued maximal monotone operator [44]. It is multivalued when  $\phi^o$  is crystalline. Note that  $T_{\phi^o}(x, \cdot)$  is one-homogeneous, namely

$$T_{\phi^o}(x, \lambda \xi^*) = |\lambda| T_{\phi^o}(x, \xi^*), \quad (x, \xi^*) \in T^* M, \lambda \in \mathbb{R}.$$

**Example 3.9 (Riemannian case).** If  $\phi^o(x, \xi) = (\sum_{i,j=1}^n g^{ij}(x) \xi_i^* \xi_j^*)^{1/2}$  is a Riemannian metric, then<sup>(14)</sup>  $(T_{\phi^o}(x, \xi^*))^i = \sum_{j=1}^n g^{ij}(x) \xi_j^*$ .

In the case considered in Remark 3.3, where  $(\phi^o(\xi^*))^2 = -(\xi_0^*)^2 + (\xi_1^*)^2 + \dots + (\xi_n^*)^2$ , the map  $T_{\phi^o}$  takes  $\xi^* = (\xi_0^*, \xi_1^*, \dots, \xi_n^*)$  into  $(-\xi_0^*, \xi_1^*, \dots, \xi_n^*)$ , exchanging the sign of the zeroth component.

**Remark 3.10.** If  $\phi^o$  is spatially homogeneous and  $\xi^* \in \partial B_{\phi^o}$ , then  $T_{\phi^o}(\xi^*)$  is a suitable normalization<sup>(15)</sup> of the exterior normal cone orthogonal to  $\partial B_{\phi^o}$  at  $\xi^*$ .

**Definition 3.11.** Let  $\partial E$  be Lipschitz and let  $x \in \partial E$  be a point where  $\nu(x)$  is defined. If (3.10) holds we define [40], [39] the contravariant vector field  $n_\phi^E = n_\phi$  at  $x$  as

$$n_\phi(x) := T_{\phi^o}(x, \nu_{\phi^o}(x)) = \phi_{\xi^*}^o(x, \nu_{\phi^o}(x)).$$

In components<sup>(16)</sup>  $n_\phi = (n_\phi^1, \dots, n_\phi^n)$ . If  $\phi^o$  is convex,  $n_\phi$  is sometimes called the Cahn-Hoffman vector field.

Notice that, using (3.3), it follows that

$$\langle \nu_{\phi^o}(x), n_\phi(x) \rangle = 1. \quad (3.13)$$

**Remark 3.12 (Cahn-Hoffman selections).** Under the sole assumption (3.11), and supposing also for simplicity that  $\phi^o$  is spatially homogeneous, there are several possible choices of vector fields  $\eta : \partial E \rightarrow V$  which satisfy  $\eta(x) \in T_{\phi^o}(\nu_{\phi^o}(x))$  for  $\mathcal{H}^{n-1}$ -almost every  $x \in \partial E$ , since in this case  $T_{\phi^o}(\nu_{\phi^o}(x))$  is a (compact) convex set. In Section 4 we will impose further regularity on  $\eta$  in order to define what we will call  $\phi$ -regular sets.

If  $F \subset \partial E$  is a facet of a polyhedral  $\partial E$  and  $\phi^o$  is crystalline, we set

$$\tilde{B}_\phi^F := T_{\phi^o}(\nu_{\phi^o}(F)), \quad (3.14)$$

see Figure 5. Note the presence of the symbol  $\phi$  (that we are going to define in the next section) on the left hand side of (3.14).

<sup>14</sup>The map  $T_{\phi^o}$  is used to exchange the indices from lower to upper.

<sup>15</sup>See the fifth item of Remark 3.18 below.

<sup>16</sup>Pay attention to the notation:  $n$  is the dimension of  $V$ ,  $n_\phi$  is the vector field.

### 3.2 The convex function $\phi$

Under one of the two assumptions (3.10), (3.11), given  $x \in M$  we can consider the image  $\Sigma(x)$  of the boundary of the star-shaped set  $B_{\phi^o}(x)$  via the map  $T_{\phi^o}(x, \cdot)$ ,

$$\Sigma(x) := T_{\phi^o}(x, \partial B_{\phi^o}(x)).$$

If  $B_{\phi^o}(x)$  is not convex then it may happen, for instance in  $n = 2$  dimensions and if  $\partial B_{\phi^o}$  is a smooth simple closed curve having the origin in its interior, that  $\Sigma(x)$  is a curve with cusps and self-intersections [78]. These kind of singularities cannot occur if  $\phi^o$  is convex<sup>(17)</sup>, and in this case it is possible to define a function  $\phi : TM \rightarrow [0, +\infty]$  as follows:

$$\phi(x, \xi) := \inf \{ \lambda > 0 : (x, \xi) \in \lambda T_{\phi^o}(x, B_{\phi^o}(x)) \}, \quad (x, \xi) \in TM.$$

Then  $\phi(x, \cdot)$  is one-homogeneous, namely

$$\phi(x, \lambda \xi) = |\lambda| \phi(x, \xi), \quad (x, \xi) \in TM, \lambda \in \mathbb{R}. \quad (3.15)$$

Moreover  $\Sigma(x) = \partial T_{\phi^o}(x, B_{\phi^o}(x))$ . Finally, if we define

$$B_\phi(x) := T_{\phi^o}(x, B_{\phi^o}(x)),$$

then

$$B_\phi(x) = \{ (x, \xi) \in TM : \phi(x, \xi) \leq 1 \},$$

and  $\phi$  is convex (i.e.,  $\phi(x, \cdot)$  is convex for any  $x \in M$ ).

**Remark 3.13.** As it follows from the above presentation, when writing the symbol  $\phi$  we assume that  $\phi^o$  is convex (and, as a consequence, so is  $\phi$ ).

**Definition 3.14 (Metrics on  $TM$ ).** *The symbol  $\mathcal{M}(TM)$  denotes the class of metrics on  $TM$ , namely of all continuous functions  $\phi$  which are convex and satisfy (3.15) and*

$$\phi(x, \xi) \geq \mu |\xi|, \quad (x, \xi) \in TM,$$

for a suitable constant  $\mu > 0$  (depending on  $\phi$ ).

**Definition 3.15 (Regular metrics on  $TM$ ).** *Let  $\phi \in \mathcal{M}(TM)$ . We say that  $\phi$  is regular if for any  $x \in M$  the set  $B_\phi(x)$  has boundary of class  $\mathcal{C}^\infty$  and each principal curvature of  $\partial B_\phi(x)$  is strictly positive at each point of  $\partial B_\phi(x)$ . We denote by  $\mathcal{M}_{\text{reg}}(TM)$  the class of all regular metrics in  $TM$ .*

It is possible to prove that if  $\phi \in \mathcal{M}_{\text{reg}}(TM)$ , then  $B_{\phi^o}(x)$  has boundary of class  $\mathcal{C}^\infty$  and each principal curvature of  $\partial B_{\phi^o}(x)$  is strictly positive at each point of  $\partial B_{\phi^o}(x)$ . Namely,  $\phi^o \in \mathcal{M}_{\text{reg}}(T^*M)$ . See also [105] for a list of related properties.

**Example 3.16 (Minkowski space).** If a metric  $\phi$  on  $TM$  is spatially homogeneous, it is a norm on  $\Lambda_1 V$ , called a Minkowski norm (or Minkowski metric). The normed vector space  $(\Lambda_1 V, \phi)$  is called Minkowski space [119] and is the simplest example of a Finsler manifold [16].

---

<sup>17</sup>It is not the aim of these notes to investigate the interesting case of a nonconvex  $B_{\phi^o}$ .

The symbol  $\nabla_\xi(\phi^2)$  (resp.  $\phi_\xi$ ) denotes the subdifferential of  $(\phi(x, \cdot))^2$  (resp. of  $\phi(x, \cdot)$ ) with respect to  $\xi$ .

**Definition 3.17 (The map  $T_\phi$ ).** Let  $x \in M$ . We define  $T_\phi(x, \cdot) : \Lambda_1 V \rightarrow \mathcal{P}(\Lambda^1 V)$  as

$$T_\phi(x, \xi) := \frac{1}{2}(\nabla_\xi(\phi^2))(x, \xi), \quad (x, \xi) \in TM. \quad (3.16)$$

$T_\phi(x, \cdot)$  is a one-homogeneous maximal monotone map.

**Remark 3.18 (Duality).** Assume  $\phi^o \in \mathcal{M}(T^*M)$  and  $\phi \in \mathcal{M}(TM)$ . The following properties hold [107], [119].

- $\phi(x, \xi) = \sup \{ \langle \xi^*, \xi \rangle : \xi^* \in \Lambda^1 V, \phi^o(x, \xi^*) \leq 1 \}$  for any  $(x, \xi^*) \in TM$ <sup>(18)</sup>;
- $\phi^{oo} = \phi$  (the dual of  $\Lambda^1 V$  can be identified with  $\Lambda_1 V$ );
- if  $\phi^o$  is crystalline then  $\phi$  is crystalline;
- if  $T_{\phi^o}(x, \cdot)$  and  $T_\phi(x, \cdot)$  are single valued, then [39]
  - for any  $x \in M$ ,  $\xi \in \Lambda_1 V \setminus \{0\}$  and  $\xi^* \in \Lambda^1 V \setminus \{0\}$  we have  $\phi^o(x, \phi_\xi(x, \xi)) = \phi(x, \phi^o(x, \xi^*)) = 1$ , and  $\phi^o(x, \xi^*)\phi_\xi(x, \phi_{\xi^*}^o(x, \xi^*)) = \xi^*$ ,  $\phi(x, \xi)\phi_{\xi^*}^o(x, \phi_\xi(x, \xi)) = \xi$ ;
  - $T_\phi(x, \cdot)T_{\phi^o}(x, \cdot) = \text{Id}_{\Lambda^1 V}$ ,  $T_{\phi^o}(x, \cdot)T_\phi(x, \cdot) = \text{Id}_{\Lambda_1 V}$ .
- Assume for simplicity that  $\phi^o$  is spatial homogeneous. Then  $T_\phi$  (resp.  $T_{\phi^o}$ ) takes  $\partial B_\phi$  (resp.  $\partial B_{\phi^o}$ ) onto  $\partial B_{\phi^o}$  (resp. onto  $\partial B_\phi$ ). If  $\xi \in \partial B_\phi$ ,  $T_\phi(\xi)$  is the intersection of the closed outward normal cone to  $\partial B_\phi$  with  $\partial B_{\phi^o}$ .

**Remark 3.19.** Assuming  $\phi$  to be convex, it is equivalent<sup>(19)</sup> to develop the theory starting with  $\phi$  and then defining  $\phi^o$  by duality (replace  $\phi$  by  $\phi^o$  and  $\Lambda^1 V$  with  $\Lambda_1 V$  in the first item of Remark 3.18).

**Example 3.20 (Polyhedral dual bodies).** In Figure 1 for a crystalline  $\phi^o$ , we show  $B_{\phi^o}$  and its dual body  $B_\phi$ . If  $\xi \in \partial B_\phi$  is a point in the relative interior of a facet, then the normal cone  $T_\phi(\xi)$  to  $\partial B_\phi$  at  $\xi$  is a vertex in  $\partial B_{\phi^o}$ ; if  $\xi \in \partial B_\phi$  is a point in the relative interior of an edge, then  $T_\phi(\xi)$  is a closed edge in  $\partial B_{\phi^o}$ ; if  $\xi \in \partial B_\phi$  is a vertex, then  $T_\phi(\xi)$  is a closed facet in  $\partial B_{\phi^o}$ .

When  $\phi$  (resp.  $\phi^o$ ) is regular and spatially homogeneous, we simply write  $\phi \in \mathcal{M}(\Lambda_1 V)$  (resp.  $\phi^o \in \mathcal{M}(\Lambda^1 V)$ ).

**Remark 3.21.** Let  $\phi \in \mathcal{M}(\Lambda_1 V)$  be spatially homogeneous. We give here a recipe to construct the dual body  $B_{\phi^o}$  of  $B_\phi$ , see Figure 2. Assume for simplicity that  $\phi \in \mathcal{C}^1(\Lambda_1 V \setminus \{0\})$ . Take a point  $\xi \in \partial B_\phi$ . Then  $\frac{T_\phi(\xi)}{|T_\phi(\xi)|}$  is orthogonal to  $\partial B_\phi$  at  $\xi$ , and points out of  $B_\phi$ . Moreover

$$|T_\phi(\xi)| = (\text{dist}(T_\xi(\partial B_\phi), 0))^{-1}.$$

Indeed, setting  $\xi^* := T_\phi(\xi)$ , we have that  $\xi^*$  realizes the maximum in the first item of Remark 3.18, so that  $1 = \phi(\xi) = \langle \xi^*, \xi \rangle$ . Therefore  $1 = \phi(\xi) = |\xi^*| \langle \nu^{B_\phi}, \xi \rangle$ , and hence

<sup>18</sup>The function  $\phi(x, \cdot)$  is sometimes called the support function of  $B_{\phi^o}(x)$ , and  $B_{\phi^o}(x)$  is called the polar reciprocal of  $B_\phi(x)$ , [119, pag. 50].  $B_{\phi^o}(x)$  is called the dual body of  $B_\phi(x)$ . Once we assume  $\phi^o$  to be convex, then the right hand side of the first item in Remark 3.18 can be taken as the definition of  $\phi$ .

<sup>19</sup>There could be, however, geometrical or physical reasons to prefer  $B_\phi$  instead of  $B_{\phi^o}$  as the starting point of the theory.

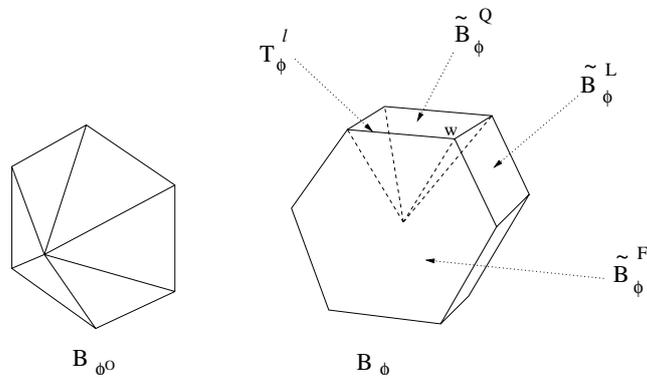


Figure 1: two convex polyhedral bodies one dual of the other. Compare Figure 5 and (3.14):  $F$ ,  $L$  and  $Q$  are facets of  $E$  in Figure 5, and  $\tilde{B}_{\phi}^F$ ,  $\tilde{B}_{\phi}^L$  and  $\tilde{B}_{\phi}^Q$  are the corresponding facets of  $B_{\phi}$ .

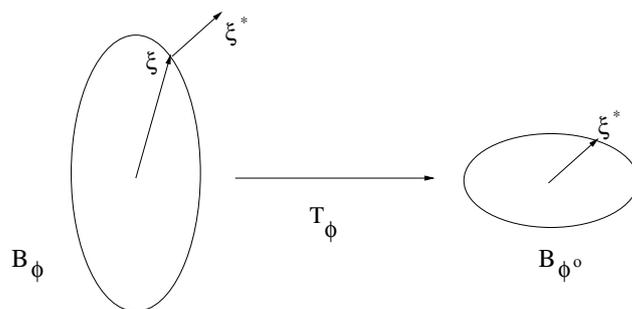


Figure 2: Pick  $\xi \in \partial B_{\phi}$ . The covector  $T_{\phi}(\xi) \in \partial B_{\phi^0}$  is orthogonal to  $\partial B_{\phi}$ , and  $|T_{\phi}(\xi)| = \langle \nu^{B_{\phi}}(\xi), \xi \rangle^{-1}$ .

$|\xi^*| = \langle \nu^{B_\phi}, \xi \rangle^{-1}$ . It is then enough to observe that the euclidean distance  $\text{dist}(T_\xi(\partial B_\phi), 0)$  between the tangent space  $T_\xi(\partial B_\phi)$  to  $\partial B_\phi$  at  $\xi$  and the origin equals  $\langle \nu^{B_\phi}, \xi \rangle$ . In this way we construct  $B_{\phi^o}$ , starting from  $B_\phi$ , since  $\partial B_{\phi^o}$  consists of all points of the form  $T_\phi(\xi)$ , with  $\xi \in \partial B_\phi$ .

**Remark 3.22 (The Legendre transform).** Some of the above concepts, as it can be seen from formula (3.17) below, can be given in terms of the Legendre transform, that for completeness we recall here. Let  $f : TM \rightarrow [0, +\infty)$  be a continuous function, such that for any  $x \in M$  the map  $\xi \rightarrow f(x, \xi)$  is convex and<sup>(20)</sup> of class  $\mathcal{C}^1$ . Define  $f^* : T^*M \rightarrow (-\infty, +\infty]$  as

$$f^*(x, \xi^*) := \sup\{\langle \xi^*, \xi \rangle - f(x, \xi) : \xi \in \Lambda_1 V\}.$$

Let  $\mathcal{E} := \{(\xi, \tau) \in \Lambda_1 V \times \mathbb{R} : \tau > f(x, \xi)\}$  be the epigraph of  $f(x, \cdot)$ , which is a convex set. Given  $\xi^* \in \Lambda^1 V \setminus \{0\}$ , consider in  $V \times \mathbb{R}$  the set  $\mathcal{P}^{\xi^*}$  of all parallel hyperplanes orthogonal to  $(\xi^*, -1)$ . If there is does not exist any point in  $\partial \mathcal{E}$  for which the tangent space to  $\partial \mathcal{E}$  belongs to  $\mathcal{P}^{\xi^*}$ , then  $f^*(x, \xi^*) = +\infty$ . Otherwise, if there exists one point  $z \in \partial \mathcal{E}$  having tangent space belonging to  $\mathcal{P}^{\xi^*}$ , we take the unique  $\pi^{\xi^*} \in \mathcal{P}^{\xi^*}$  containing  $z$ . Then we consider the intersection of  $\pi^{\xi^*}$  with the vertical axis  $\{0\} \times \mathbb{R}$ , and we define  $f^*(x, \xi^*)$  as minus the vertical component of such an intersection, namely

$$f^*(x, \xi^*) = -t, \quad (0, t) = \pi^{\xi^*} \cap \{(\xi, \tau) \in \Lambda_1 V \times \mathbb{R} : \xi = 0\}.$$

For example,

$$\nu \in \Lambda^1 V, c \in \mathbb{R}, f(x, \xi) = \langle \xi, \nu \rangle + c \Rightarrow f^*(\xi^*) = \begin{cases} -c & \text{if } \xi^* = \nu, \\ +\infty & \text{if } \xi^* \neq \nu, \end{cases}$$

$$\alpha \in \mathbb{R}, f(x, \xi) = \alpha |\xi|^2 \Rightarrow f^*(\xi^*) = \frac{1}{4\alpha} |\xi^*|^2,$$

and for a non everywhere differentiable function a similar construction gives

$$f(x, \xi) = \phi(\xi) \Rightarrow f^*(\xi^*) = \begin{cases} 0 & \text{if } \phi^o(\xi^*) \leq 1, \\ +\infty & \text{if } \phi^o(\xi^*) > 1. \end{cases} \quad (3.17)$$

### 3.3 The distance function $\text{dist}_\phi$

We shall assume from now on that  $\phi : TM \rightarrow [0, +\infty)$  is continuous<sup>(21)</sup>.

**Definition 3.23.** Given  $x, y \in M$  we set

$$\text{dist}_\phi(x, y) := \inf \left\{ \int_0^1 \phi(\gamma, \dot{\gamma}) dt : \gamma \in AC([0, 1]; M), \gamma(0) = x, \gamma(1) = y \right\}, \quad (3.18)$$

<sup>20</sup>Even if  $f(x, \cdot)$  is  $\mathcal{C}^1$  and not convex (or convex but not of class  $\mathcal{C}^1$ ), still  $f^*(x, \cdot)$  is defined and it is convex.

<sup>21</sup>Discontinuous  $\phi(\cdot, \xi)$  have been considered in [3], [4], see also the references in these papers.

where  $AC([0, 1]; M)$  denotes the class of all absolutely continuous [11] curves  $\gamma : [0, 1] \rightarrow M$ . Notice that if  $\phi$  is spatially homogeneous and convex, then  $\text{dist}_\phi(x, y) = \phi(y - x)$ . Recall that if  $\phi(x, \xi) = |\xi|$ , we set  $\text{dist}_\phi = d$ .

For any  $F \subseteq M$  we denote

$$\text{dist}_\phi(x, F) := \inf_{y \in F} \text{dist}_\phi(x, y), \quad x \in M.$$

The next definition will be used only for rather regular sets.

**Definition 3.24 (Signed  $\phi$ -distance).** *Assume that  $\partial E \in \text{Lip}(M)$ . We define the signed  $\phi$ -distance function from  $\partial E$  negative in  $E$  and positive in  $M \setminus E$  as*

$$d_\phi(x) := \text{dist}_\phi(x, E) - \text{dist}_\phi(x, M \setminus E). \quad x \in M. \quad (3.19)$$

### 3.3.1 $\phi$ -Volume

Once we have the distance  $\text{dist}_\phi$  at our disposal, we can define the  $n$ -dimensional Hausdorff measure  $\mathcal{H}_\phi^n$  with respect to the distance  $\text{dist}_\phi$  [76], i.e., for  $S \subseteq \mathbb{R}^n$

$$\mathcal{H}_\phi^n(S) := \frac{\omega_n}{2^n} \lim_{\rho \rightarrow 0^+} \inf \left\{ \sum_{i=1}^{+\infty} (\text{diam}_{\text{dist}_\phi}(S_i))^n : S \subseteq \bigcup_{i=1}^{+\infty} S_i, \text{diam}_{\text{dist}_\phi}(S_i) < \rho \right\}, \quad (3.20)$$

where, if  $F \subseteq \mathbb{R}^n$ ,  $\text{diam}_{\text{dist}_\phi}(F) := \sup\{d_\phi(x, y) : x, y \in F\}$ , and  $\omega_n := \mathcal{L}^n(\{\xi \in M : |\xi| < 1\})$ . Notice that if  $\phi$  is spatially homogeneous, then  $\mathcal{H}_\phi^n(B_\phi) = \omega_n$ , since  $\text{diam}_{\text{dist}_\phi}(B_\phi) = 2$ .

**Example 3.25.** Assume that  $\phi$  is spatially homogeneous and riemannian, i.e.,  $\phi(\xi) = |\sqrt{g}\xi|$  for any  $\xi \in V$ , where  $g = (g_{ij})$  is a symmetric positive definite  $(n \times n)$ -matrix, and we write  $g = \sqrt{g^T} \sqrt{g}$ . Then

$$\begin{aligned} \mathcal{L}^n(B_\phi) &= \frac{\omega_n}{\det \sqrt{g}} \\ \mathcal{H}_\phi^n(S) &= \det \sqrt{g} \mathcal{L}^n(S) = \frac{\omega_n}{\mathcal{L}^n(B_\phi)} \mathcal{L}^n(S) = \mathcal{H}^n(T_\phi(S)). \end{aligned}$$

We recall the following representation result for the Hausdorff measure [46].

Define

$$\text{vol}_\phi(x) := \frac{\omega_n}{\mathcal{L}^n(B_\phi(x))}, \quad x \in M. \quad (3.21)$$

**Theorem 3.26 (Representation of  $\phi$ -volume).** *If  $B \subseteq \Omega$  is a Borel set, then*

$$\mathcal{H}_\phi^n(B) = \int_B \text{vol}_\phi \, dx. \quad (3.22)$$

The distance function  $d_\phi$  is useful for various reasons; one of them is that it gives a natural extension of  $\nu_{\phi^\circ}$  out of  $\partial E$ .

### 3.4 Eikonal equation and extensions

Let  $\phi \in \mathcal{M}_{\text{reg}}(TM)$  and  $\partial E$  be compact and of class  $\mathcal{C}^\infty$ . It is possible to prove that there exists a tubular neighbourhood of  $\partial E$  where the signed  $\phi$ -distance  $d_\phi$  in (3.19) is smooth, see also [85].

The proof of the following theorem can be found for instance in [40]. See [17], [52] for related results.

**Theorem 3.27 (Eikonal equation).** *Let  $\partial E$  be compact and let  $U \subset \mathbb{R}^n$  be a tubular neighbourhood of  $\partial E$  such that  $d_\phi \in \mathcal{C}^\infty(U)$ . Then  $d_\phi$  satisfies the eikonal equation in  $U$ :*

$$(\phi^o(x, \nabla d_\phi(x)))^2 = 1, \quad x \in U, \quad (3.23)$$

so that in particular

$$\nabla d_\phi = \nu_{\phi^o} \quad \text{on } \partial E.$$

**Definition 3.28 (Extension of  $n_\phi$ ).** *Under the assumptions at the beginning of the section, we can extend the Cahn-Hoffman vector field  $n_\phi$  on the whole of  $U$  as follows:*

$$N_\phi := T_{\phi^o}(x, \nabla d_\phi(x)), \quad x \in U. \quad (3.24)$$

Note that

$$\phi^o(x, N_\phi(x)) = 1, \quad \langle N_\phi(x), \nabla d_\phi(x) \rangle = 1, \quad x \in U.$$

### 3.5 Appendix: definitions of $\nabla_\phi$ , $\text{div}_\phi$ , $\Delta_\phi$ . $\phi$ -Distributional perimeter

Assume that  $\phi^o \in \mathcal{M}_{\text{reg}}(T^*M)$ . For completeness, we define here various operators<sup>(22)</sup> naturally related to  $\phi^o$ . If  $u \in \mathcal{C}^2(M)$  we define the vector field

$$\nabla_\phi u(x) := T_{\phi^o}(x, \nabla u(x)), \quad x \in M. \quad (3.25)$$

Note that if  $\phi(x, \xi) = (\sum_{i,j=1}^n g_{ij}(x) \xi^i \xi^j)^{1/2}$  is a riemannian metric in  $M$ , then the  $i$ -th component of  $\nabla_\phi u(x)$  equals  $\sum_{j=1}^n g^{ij}(x) \nabla_j u(x)$  where  $(g^{ij})$  is the inverse of  $(g_{ij})$ .

If  $\eta \in \mathcal{C}^1(M; V)$  we set

$$\begin{aligned} \text{div}_\phi \eta &:= \text{div} \eta + \nabla(\log(\text{vol}_\phi)) \cdot \eta, \\ \Delta_\phi u &:= \text{div}_\phi \nabla_\phi u. \end{aligned} \quad (3.26)$$

With the above definitions we have the following Gauss-Green type formula.

**Proposition 3.29 (Divergence Theorem).** *If  $\Omega \subset M$  is a bounded open set of class  $\mathcal{C}^1$ ,  $u \in \mathcal{C}^1(\Omega)$  and  $g \in \mathcal{C}^1(\Omega; \Lambda_1 V) \cap \mathcal{C}(\overline{\Omega}; \Lambda_1 V)$ , then*

$$\int_\Omega u \text{div}_\phi g \, d\mathcal{H}_\phi^n + \int_\Omega \nabla u \cdot g \, d\mathcal{H}_\phi^n = - \int_{\partial\Omega} u \nu_{\phi^o}^\Omega \cdot g \, \phi^o(x, \nu^\Omega) \text{vol}_\phi \, d\mathcal{H}^{n-1}. \quad (3.27)$$

---

<sup>22</sup>For simplicity, we give the definitions independently of the function  $a$  (see (3.1)), or more precisely assuming the validity of (3.29) below. See also [29].

*Proof.* Definition (3.26) of  $\operatorname{div}_\phi$  implies

$$\begin{aligned} u \operatorname{div}_\phi g \operatorname{vol}_\phi &= u \operatorname{div} g \operatorname{vol}_\phi + u \nabla(\log(\operatorname{vol}_\phi)) \cdot g \operatorname{vol}_\phi \\ &= \operatorname{div}(u g \operatorname{vol}_\phi) - \nabla u \cdot g \operatorname{vol}_\phi. \end{aligned}$$

Hence, using the Gauss–Green Theorem and recalling (3.22), we get

$$\begin{aligned} \int_\Omega u \operatorname{div}_\phi g \, d\mathcal{H}_\phi^n &= \int_{\partial\Omega} u \nu^\Omega \cdot g \frac{\omega_n}{\mathcal{L}^n(B_\phi)} \, d\mathcal{H}^{n-1} - \int_\Omega \nabla u \cdot g \, d\mathcal{H}_\phi^n \\ &= \int_{\partial\Omega} u \nu_{\phi^\circ}^\Omega \cdot g \phi^\circ(x, \nu^\Omega) \operatorname{vol}_\phi \, d\mathcal{H}^{n-1} - \int_\Omega \nabla u \cdot g \, d\mathcal{H}_\phi^n. \end{aligned}$$

□

In view of (3.27) it is natural to introduce the surface measure

$$d\mathcal{P}_\phi(B) := \int_{B \cap \partial E} \phi^\circ(x, \nu(x)) \operatorname{vol}_\phi(x) \, d\mathcal{H}^{n-1}(x), \quad B \subseteq M. \quad (3.28)$$

**Remark 3.30.** A rather natural choice of the function  $a$  in (3.1) and (3.4) is

$$a = \operatorname{vol}_\phi. \quad (3.29)$$

With this choice we have that  $\mathcal{F}_{\phi^\circ}$  equals the functional in (3.28) when  $B = M$ . This functional turns out to be the  $\phi$ -perimeter, defined in the distributional sense [3], [40], and also the  $\phi$ -Minkowski content [76], [11], [40], [21], defined as

$$\mathcal{M}_\phi^{n-1}(\partial E) := \lim_{\rho \rightarrow 0^+} \frac{\mathcal{H}_\phi^n(\{x \in M : \operatorname{dist}_\phi(x, \partial E) < \rho\})}{2\rho}, \quad (3.30)$$

but not<sup>(23)</sup> the  $(n-1)$ -dimensional Hausdorff measure with respect to  $\operatorname{dist}_\phi$ ,

$$\mathcal{H}_\phi^{n-1}(S) := \frac{\omega_{n-1}}{2^{n-1}} \lim_{\rho \rightarrow 0^+} \inf \left\{ \sum_{i=1}^{+\infty} (\operatorname{diam}_{\operatorname{dist}_\phi}(S_i))^k : S \subseteq \bigcup_{i=1}^{+\infty} S_i, \operatorname{diam}_{\operatorname{dist}_\phi}(S_i) < \rho \right\}, \quad (3.31)$$

where  $\omega_{n-1} := \mathcal{L}^{n-1}(\xi \in \mathbb{R}^{n-1} : |\xi| < 1)$ . It is interesting to observe that, adopting (3.28) as the definition of  $(n-1)$ -dimensional  $\phi$ -measure, it turns out that  $B_\phi$  satisfies the isoperimetric property [40]. Eventually, other geometric measures could be considered [119], [10], for instance the Benson area [41], [10].<sup>(24)</sup>

<sup>23</sup>Even for a spatially homogeneous  $\phi$  [40].

<sup>24</sup>If  $\xi^* \in \Lambda^1 V$ , it is possible to prove that  $|\xi^*| = \sup \{\det[\xi^*, \nu_1, \dots, \nu_{n-1}]\}$ , where the supremum is taken over all unit covectors  $\nu_1, \dots, \nu_{n-1} \in \Lambda^1 V$ , and  $[\xi^*, \nu_1, \dots, \nu_n]$  denotes the matrix having  $\xi^*, \nu_1, \dots, \nu_n$  as columns. Such an inequality can be checked using the Hadamard inequality  $|\det A| \leq \prod_{j=1}^n (\sum_{i=1}^n a_{ij}^2)^{1/2}$ , where  $A = (a_{ij})$  is a  $n \times n$  matrix. The Benson area of  $\partial E$  is then defined as  $\int_{\partial E} \mathfrak{b}(x, \nu) \, d\mathcal{H}^{n-1}$ , where  $\mathfrak{b}(x, \cdot) : \Lambda^1 V \rightarrow \max \{\det[\xi^*, \nu_1, \dots, \nu_{n-1}] : \nu_i \in \Lambda^1 V, \phi^\circ(x, \nu_i) \leq 1\}$ . This surface measure is strictly related to the De Giorgi mass [64], [10], which turns out to be, for instance for a spatially homogeneous  $\phi^\circ$ ,  $\int_{\partial E} \lambda(T) \, d\mathcal{H}^{n-1}$ , where  $\lambda(T) := \sup \{\mathcal{H}^{n-1}(\eta(B_\phi \cap T)) : \eta = (\eta_1, \dots, \eta_{n-1}) \in GL(T, T), \phi^\circ(\eta_i) \leq 1 \forall i \in \{1, \dots, n-1\}\}$  for any  $T \in G$ .

## 4 $\phi$ -regular sets

Assume in this section that  $\phi^\circ$  is convex and spatially homogeneous<sup>(25)</sup>. The next definitions become interesting when  $T_{\phi^\circ}$  is multivalued hence, roughly speaking, when  $B_{\phi^\circ}$  has corners, edges, etc. In these notes, we will apply these definitions in the crystalline case.

Let  $\partial E$  be Lipschitz. In order to look for a solution of an anisotropic (and in particular crystalline) mean curvature flow starting from  $\partial E$ , it is necessary to devise a certain class of regularity for the flowing hypersurfaces.

We will give different definitions, depending on whether we want to consider a whole neighbourhood of  $\partial E$  or not. All definitions have advantages and disadvantages. One motivation for considering the neighbourhoods comes from phase transitions (in particular the reaction-diffusion approximation considered in Section 9), where the interface is diffuse.

Let us begin with the definitions using the neighbourhoods, and with the most stringent one. Recall that  $\nabla d_\phi$  naturally extends the covector field  $\nu_{\phi^\circ}$  out of  $\partial E$ , and  $N_\phi$  extends the vector field  $n_\phi$ .

**Definition 4.1 (Neighbourhood-Lipschitz  $\phi$ -regular sets).** *We say that  $E$  is neighbourhood-Lipschitz  $\phi$ -regular if there exists a tubular neighbourhood  $U$  of  $\partial E$  and a vector field  $\eta \in \text{Lip}(U; \Lambda_1 V)$  such that  $\eta(x) \in T_{\phi^\circ}(\nabla d_\phi(x))$  for almost every  $x \in U$ .*

If  $T_{\phi^\circ}$  is single-valued then  $T_{\phi^\circ}(\nabla d_\phi(x))$  is a singleton and it reduces to the vector field  $N_\phi$ . Lipschitz regularity seems to be the strongest regularity one can require. Nevertheless, a difficulty related to Definition 4.1 is that the divergence<sup>(26)</sup> of  $\eta$  belongs just to  $L^\infty(U)$ , hence has not, a priori, a well defined trace on  $\partial E$ . This difficulty remains in the following definition<sup>(27)</sup>.

**Definition 4.2 (Neighbourhood- $L^\infty$   $\phi$ -regular sets).** *We say that  $E$  is neighbourhood- $L^\infty$   $\phi$ -regular if there exists a tubular neighbourhood  $U$  of  $\partial E$  and a bounded vector field  $\eta$  such that  $\text{div} \eta \in L^\infty(U)$  and  $\eta(x) \in T_{\phi^\circ}(\nabla d_\phi(x))$  for almost every  $x \in U$ .*

Let us now pass to a (rather intrinsic) definition. Define

$$\text{Nor}_\phi(\partial E; M) := \{N : \partial E \rightarrow M : N(x) \in T_{\phi^\circ}(\nu_\phi^E(x)) \text{ for } \mathcal{H}^{n-1} \text{ a.e. } x \in \partial E\}.$$

**Definition 4.3 (Lipschitz  $\phi$ -regular sets).** *We say that  $E$  is Lipschitz  $\phi$ -regular if there exists a vector field  $\eta \in \text{Nor}_\phi(\partial E; M) \cap \text{Lip}(\partial E; M)$ . We say that  $E$  is polyhedral Lipschitz  $\phi$ -regular if  $E$  is Lipschitz  $\phi$ -regular and it is polyhedral<sup>(28)</sup>.*

The difficulties related to constructing a vector field with Lipschitz regularity on  $\partial E$  in explicit examples are essentially the same as the ones in Definition 4.1; in addition, when talking about the divergence of  $\eta$ , we are forced now to speak about a tangential divergence. On facets, the tangential divergence we will consider will be the euclidean tangential divergence [109]  $\text{div}_\tau$ . Again, one could relax the regularity of  $\eta$  in Definition 4.3, for instance by requiring  $\eta$  to be bounded with tangential divergence in  $L^2(\partial E)$  or in  $L^\infty(\partial E)$ .

<sup>25</sup>Various definitions could be generalized for  $\phi^\circ$  depending on  $x$ , at least when  $\phi^\circ \in \mathcal{M}_{\text{reg}}(TM)$ .

<sup>26</sup>One advantage: this divergence is taken in the ambient space  $M$ .

<sup>27</sup>Definition 4.2 could be in turn relaxed by requiring  $\text{div} \eta \in L^2(U)$ .

<sup>28</sup>All polyhedral sets considered in these notes are assumed to have a finite number of facets.

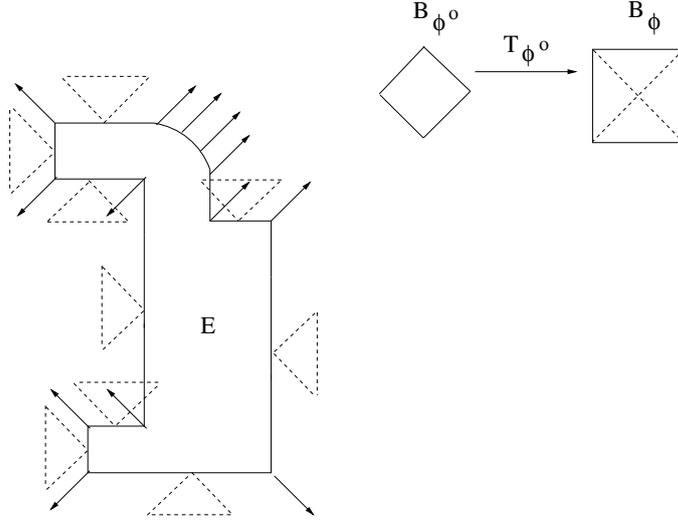


Figure 3: A Lipschitz  $\phi$ -regular set  $E$  when  $B_\phi$  is the square  $[-1, 1]^2$ . Curved portions of  $\partial E$  may be present: we will see that there the crystalline curvature must vanish.

**Definition 4.4** ( *$L^\infty$ - $\phi$ -regular sets*). We say that a polyhedral set  $E$  is  $L^\infty$ - $\phi$ -regular if there exists a vector field  $\eta \in \text{Nor}_\phi(\partial E; M)$  having tangential divergence  $\text{div}_\tau \eta$  in  $L^\infty(\partial E)$ .

Finally, we point out another notion that has been considered in [27].

**Definition 4.5** ( *$rB_\phi$ -condition*). Let  $r > 0$ . We say that  $E$  satisfies the  $rB_\phi$ -condition if, for any  $x \in \partial E$ , there exists  $y \in M$  such that

$$rB_\phi + y \subseteq E \quad \text{and} \quad x \in \partial(rB_\phi + y).$$

It turns out that if  $E$  is neighbourhood-Lipschitz  $\phi$ -regular then there exists  $r > 0$  such that  $E$  and  $\overline{M \setminus E}$  satisfy the  $rB_\phi$ -condition. Moreover, if  $E$  is convex then  $E$  is neighbourhood- $L^\infty$   $\phi$ -regular if and only if  $E$  and  $\overline{M \setminus E}$  satisfy the  $rB_\phi$ -condition for some  $r > 0$ .

## 4.1 Examples

If  $n = 2$ , the structure of a Lipschitz  $\phi$ -regular set  $E$  is, roughly speaking, the following:  $\partial E$  is a closed simple Lipschitz curve which is a sequence (with a precise order) of segments parallel to some edge of  $\partial B_\phi$  and of segments or arcs corresponding to vertices of  $\partial B_\phi$ .

**Example 4.6** (*A Lipschitz  $\phi$ -regular curve*). Let  $\phi(\xi) := \max\{|\xi_1|, |\xi_2|\}$ , so that  $B_\phi = [-1, 1]^2$ , and let  $E$  be as in Figure 3. At the vertices of  $\partial E$  the vector  $\nu_\phi^E$  is not defined. However, let  $v$  be a vertex of  $\partial E$ , and let  $F_1$  and  $F_2$  be the two arcs or segments of  $\partial E$  having  $v$  as a vertex. For any  $x$  in the relative interior of  $F_i$ , the closed convex set  $T_{\phi^o}(\nu_\phi^E(x))$  is either a segment or a singleton, independent of  $x$  and depending only on  $F_i$ . Let us denote it by  $K_i$ . What makes  $E$  Lipschitz  $\phi$ -regular is the fact that  $K_1 \cap K_2$  is a singleton. This produces a unique vector at each vertex of  $\partial E$ ; then we can construct infinitely many vector fields  $\eta \in \text{Nor}_\phi(\partial E; \mathbb{R}^2) \cap \text{Lip}(\partial E; \mathbb{R}^2)$  lying inside the dotted triangles with the assigned values at the vertices.

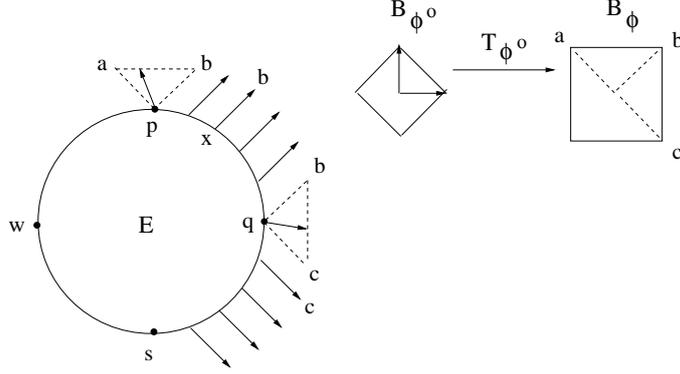


Figure 4:  $E$  is not  $L^\infty$ - $\phi$ -regular. Any Cahn-Hoffman selection is forced to jump at the points  $p, q, s, w$  of  $\partial E$ .

On the other hand, for the same  $\phi$  as in Example 4.6, the euclidean unit ball is not Lipschitz  $\phi$ -regular, and not even  $L^\infty$ - $\phi$ -regular. Its regularity is analogous to the regularity of the square in the euclidean geometry.

**Example 4.7 (The circle is not  $L^\infty$ - $\phi$ -regular).** Let  $n = 2$  and  $\phi$  be as in Example 4.6. Let  $E := \{z \in \mathbb{R}^2 : |z| \leq 1\}$ , see Figure 4. Then  $E$  is not Lipschitz  $\phi$ -regular. Indeed,  $T_{\phi^o}(\nu_\phi^E(p))$  is the upper horizontal segment  $[a, b]$  of  $\partial B_\phi$  (we depict a corresponding dotted triangle at  $p$ ). Similarly,  $T_{\phi^o}(\nu_\phi^E(q))$  is the right vertical segment  $[b, c]$  of  $\partial B_\phi$ . On the other hand, any point  $x$  on  $\partial E$  lying in the (relatively) open arc  $A$  between  $p$  and  $q$  is such that  $T_{\phi^o}(\nu_\phi^E(x)) = b$ . We deduce that any vector field  $\eta \in \text{Nor}_\phi(\partial E; \mathbb{R}^2)$  must fulfill  $\eta \equiv b$  on  $A$ , and  $\eta \equiv c$  on the open arc on  $\partial E$  between  $q$  and  $\omega$ . Hence, any vector we choose inside the dotted triangles (for instance, the triangle at  $q$ ) will produce a discontinuity in the vector field  $\eta$  (at  $q$ ). We conclude that  $E$  is not  $L^\infty$ - $\phi$ -regular.

**Example 4.8 (A Lipschitz  $\phi$ -regular polyhedral surface).** Let  $B_\phi$  be as in Figure 1, and  $E$  as in Figure 5. If  $x \in \text{int}(Q)$  then  $\nu_\phi^E(x)$  coincides with the top vertex of  $\partial B_{\phi^o}$ , and  $T_{\phi^o}(\nu_\phi^E(x))$  is the top facet  $\tilde{B}_\phi^Q$  of  $\partial B_\phi$ . We depict  $T_{\phi^o}(\nu_\phi^E(x))$  as a pyramid. Therefore  $\eta(x)$  is constrained to lie in  $\tilde{B}_\phi^Q$ . If  $x \in \partial E$  is in the interior of an edge (say the edge  $l$ ) of  $\partial Q$ , then  $\nu_\phi^E$  is not defined at  $x$ . However the intersection  $T_\phi^l$  of  $\tilde{B}_\phi^Q$  with  $\tilde{B}_\phi^F$  is defined, and it is the top edge of the frontal facet of  $\partial B_\phi$ . We have depicted this set as a triangle. Therefore  $\eta(x)$  is constrained to lie in  $T_\phi^l$ . If  $x \in \partial E$  is a vertex (say the vertex  $p$ ) of  $\partial Q$ , then  $\nu_\phi^E$  is not defined at  $p$ . What is defined is the intersection  $w$  of  $\tilde{B}_\phi^Q \cap \tilde{B}_\phi^F \cap \tilde{B}_\phi^L$ , and we have depicted this point at  $p$  as a segment. Therefore  $\eta(p)$  must coincide with  $w$ , see also Figure 1. A choice of a vector field  $\eta \in \text{Nor}_\phi(\partial E; \mathbb{R}^3) \cap \text{Lip}(\partial E; \mathbb{R}^3)$  can be made by hand.

## 4.2 Normal traces

We give here some notions that will be useful in the definition of the crystalline mean curvature. Recall the definition of  $\tilde{\nu}^F$  given in (2.2)

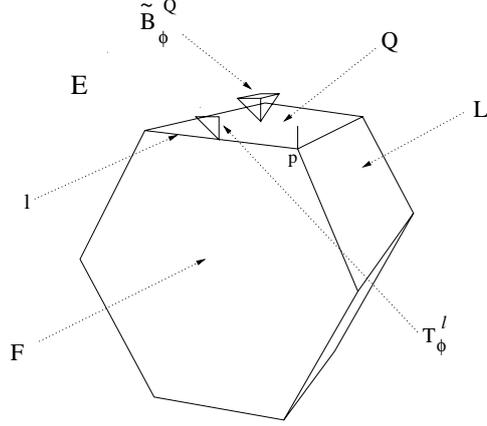


Figure 5:  $B_\phi$  is as in Figure 1.

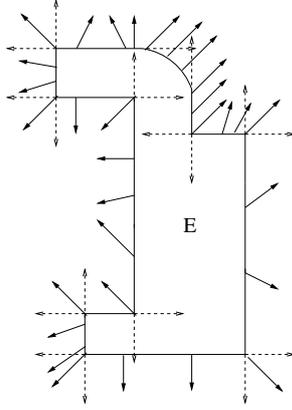


Figure 6:  $c_F$  is independent of the choice of  $\eta$  among all vector fields making  $E$  Lipschitz  $\phi$ -regular.

**Definition 4.9 (The normal traces  $c_F$ ).** Let  $E$  be a polyhedral Lipschitz  $\phi$ -regular set, let  $\eta \in \text{Nor}_\phi(\partial E; M) \cap \text{Lip}(\partial E; M)$ , and let  $F \subset \partial E$  be a facet of  $\partial E$ . We define the normal trace function  $c_F \in L^\infty(\partial F)$  as

$$c_F := \tilde{\nu}^F \cdot \eta. \quad (4.1)$$

**Example 4.10.** Let  $n = 2$ ,  $B_\phi$  and  $E$  be as in Figure 3. In Figure 6 we depict a vector field  $\eta$  which makes  $\partial E$  Lipschitz  $\phi$ -regular. The constants  $c_F$  do not depend on the particular choice of  $\eta$ . The dotted vectors at the vertices indicate the unit normals (in the line containing the facet  $F$ ) pointing outward  $F$  (i.e.,  $\tilde{\nu}^F$ ).

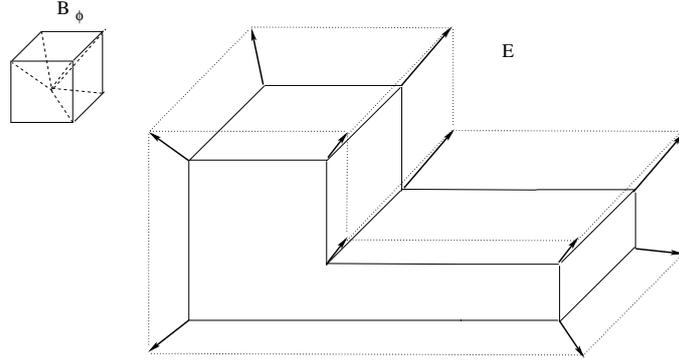


Figure 7: on the edges of  $\partial E$  the vector  $\eta$  must lie in the dotted regions

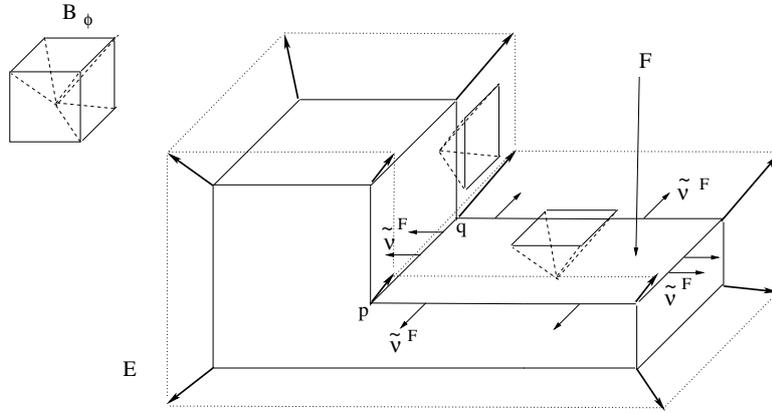


Figure 8: A Lipschitz  $\phi$ -regular set  $E$  when  $B_\phi$  is the cube. On the relative interior of  $[p, q]$  the function  $c_F$  is negative (and constant).

**Example 4.11.** Let  $n = 3$ ,  $B_\phi = [-1, 1]^3$ , and  $E$  be the set of Figure 8.  $E$  is a polyhedral Lipschitz  $\phi$ -regular set, since it is possible to construct a vector field  $\eta \in \text{Nor}_\phi(\partial E; \mathbb{R}^3) \cap \text{Lip}(\partial E; \mathbb{R}^3)$ . Indeed, first we identify  $\eta$  on the vertices of  $\partial E$ . If  $v$  is a vertex of  $\partial E$ , the intersection of  $\tilde{B}_\phi^Q$  over all facets  $Q$  of  $\partial E$  containing  $v$  is a singleton: we define this singleton to be the value of  $\eta$  at  $v$  (see the bold vectors in Figure 7). Next, on a facet  $Q \subset \partial E$ , it is enough to take suitable convex combinations of the values of  $\eta$  at the vertices of  $Q$  (possibly first subdividing  $Q$  into two or more rectangles if  $Q$  itself is not a rectangle) to obtain the required properties on  $\eta$ .

The bold vectors at the vertices of  $\partial E$  are the unique possible values for  $\eta$ . The vector field  $\tilde{\nu}^F$  points outside of  $F$ , and on  $]p, q[$  points toward  $E$ . The pyramids with vertex on the relative interior of the two facets having  $[p, q]$  in common represent the corresponding facets of  $\partial B_\phi$  (for instance,  $T_{\phi^\circ}(\nu_{\phi^\circ}(F))$  for the facet  $F$ ), i.e. the range of admissibility of  $\eta$ . It follows that  $c_F$  is negative on  $]p, q[$ , while  $c_F$  is positive on the remaining relatively open edges of  $\partial F$ .

Given a Lipschitz  $\phi$ -regular set  $E$ , in general it is possible to prove (see for instance [34],

[35]) that  $c_F$  does not depend on the choice of  $\eta$  in  $\text{Nor}_\phi(\partial E; M) \cap \text{Lip}(\partial E; M)$ , and for  $\mathcal{H}^{n-1}$ -almost every  $x \in \partial F$

$$c_F(x) = \begin{cases} \max\{\langle \tilde{\nu}^F(x), \xi \rangle : \xi \in T_{\phi^\circ}(\nu_{\phi^\circ}(F))\} & \text{if } \tilde{\nu}^F(x) \text{ points outside } E, \\ \min\{\langle \tilde{\nu}^F(x), \xi \rangle : \xi \in T_{\phi^\circ}(\nu_{\phi^\circ}(F))\} & \text{if } \tilde{\nu}^F(x) \text{ points inside } E. \end{cases} \quad (4.2)$$

**Remark 4.12.** For a polyhedral Lipschitz  $\phi$ -regular set, it is possible to extend the notion of normal trace also to vector fields  $N \in \text{Nor}_\phi(\partial E; M)$  with  $\text{div}_\tau N \in L^\infty(\partial E)$ : such a normal trace turns out to coincide with the right hand side of (4.2).

## 5 First variations: functionals on boundaries

In this section we discuss the first variation of the functional  $\mathcal{F}_{\phi^o}$ , in order to devise a possible notion of  $\phi$ -mean curvature. In the computations of this section it appears to be useful to have at our disposal quantities (such as the Cahn-Hoffman vector field) defined on a tubular neighbourhood of the interface  $\partial E$ .

### 5.1 Spatially homogeneous smooth $\phi^o$

Let us assume that  $\phi^o$  is spatially homogeneous and of class  $\mathcal{C}^1(\Lambda^1 V \setminus \{0\})$ . Let us also assume that  $\partial E$  is of class  $\mathcal{C}^2$ , and that there are no  $x \in \partial E$  where  $\phi^o(\nu^E(x)) = 0$ . This is in particular satisfied if  $\phi^o$  is in addition a metric on  $\Lambda^1 V$ , in view of (3.9).

Let us introduce a class of admissible variations. Let  $\Psi \in \mathcal{C}_c^\infty(M \times \mathbb{R}; M)$ , and set  $\Psi_\lambda(x) := \Psi(x, \lambda)$  for any  $x \in M$  and  $\lambda \in \mathbb{R}$ . Assume that  $\Psi_0 = \text{Id}$ , and  $\Psi_\lambda - \text{Id}$  has compact support in  $M$  for any  $\lambda \in \mathbb{R}$ . We can write

$$\Psi_\lambda(x) := x + \lambda X(x) + o(\lambda),$$

where  $X := \frac{\partial \Psi_\lambda}{\partial \lambda}|_{\lambda=0}$ . The vector field  $X = (X^1, \dots, X^n)$  can be considered as the initial velocity field of the deformation.

A direct computation shows that

$$\det(\nabla \Psi_\lambda) = 1 + \lambda \text{tr}(\nabla X) + o(\lambda). \quad (5.1)$$

In particular

$$\frac{d}{d\lambda} |\det(\nabla \Psi_\lambda)|_{\lambda=0} = \text{div} X. \quad (5.2)$$

Set

$$E_\lambda := \Psi_\lambda(E).$$

The next result was proved essentially in [39] (see also [31]), in the case of a convex regular metric  $\phi^o$ . The proof that we present here does not require the convexity of  $\phi^o$ , and is slightly different.

**Theorem 5.1 (First variation I).** *We have*

$$\frac{d}{d\lambda} \mathcal{F}_{\phi^o}(E_\lambda)|_{\lambda=0} = c_n \int_{\partial E} \left( \text{div} X - n_\phi^i \nu_{\phi^o_j} \nabla_i X^j \right) \phi^o(\nu) d\mathcal{H}^{n-1}, \quad (5.3)$$

where  $c_n := \frac{\omega_n}{\mathcal{L}^n(B_\phi)}$ .

*Proof.* Let  $u \in \mathcal{C}^2(M)$  be such that  $E = \{u \leq 0\}$ ,  $\partial E = \{u = 0\}$ , and  $\nabla u \neq 0$  on  $\partial E$ . Then<sup>(29)</sup>  $\nu^E = \frac{\nabla u^T}{|\nabla u|}$  on  $\partial E$ . Define  $v_\lambda : M \rightarrow \mathbb{R}$  as

$$v_\lambda(\Psi_\lambda(x)) := u(x), \quad x \in M. \quad (5.4)$$

---

<sup>29</sup>  $\nu^E$  is considered as a covector field (rows), while  $\nabla u$  as a vector field (columns). Sometimes in the sequel of these notes we will omit the transposition symbol, identifying  $\nu^E$  with  $\frac{\nabla u}{|\nabla u|}$ .

If  $|\lambda|$  is small enough, we have

$$E_\lambda = \{\Psi_\lambda(x) : u(x) \leq 0\} = \{y : u(\Psi_\lambda^{-1}(y)) \leq 0\} = \{v_\lambda \leq 0\},$$

$\partial E_\lambda = \{v_\lambda = 0\}$ ,  $\nabla v_\lambda \neq 0$  on  $\partial E_\lambda$ , hence

$$\nu^{E_\lambda} = \frac{\nabla v_\lambda^T}{|\nabla v_\lambda|} \quad \text{on } \partial E_\lambda.$$

In order to proceed in the proof, we recall the area and coarea<sup>(30)</sup> formulas [76].

- The area formula: if  $g : M \rightarrow \mathbb{R}$  is integrable,  $f : M \rightarrow M$  is an injective Lipschitz map, and  $\Omega \subseteq M$ , then

$$\int_{f(\Omega)} g dy = \int_\Omega g(f) |\det(\nabla f)| dx.$$

- The coarea formula: if  $w \in \text{Lip}(M)$  satisfies  $\text{ess} - \inf |\nabla w| > 0$ ,  $g : M \rightarrow \mathbb{R}$  is integrable, and  $\mu \in \mathbb{R}$ , then

$$\int_{\{w > \mu\}} g d\mathcal{H}^{n-1} = \int_\mu^{+\infty} \left( \int_{\{w=s\}} \frac{g}{|\nabla w|} d\mathcal{H}^{n-1} \right) ds. \quad (5.5)$$

It is also useful to make the following observation: we have

$$\mathcal{H}^{n-1}(\partial E_\lambda) = \int_{\partial E} \frac{|\nabla v_\lambda(\Psi_\lambda)| |\det(\nabla \Psi_\lambda)|}{|\nabla u|} d\mathcal{H}^{n-1}. \quad (5.6)$$

Indeed, if  $\rho > 0$ , the area formula with the choice  $f = \Psi_\lambda$ ,  $\Omega = \{|u| < \rho\}$  (so that  $f(\Omega) = \{|v_\lambda| < \rho\}$ ) and  $g(x) = |\nabla v_\lambda|$ , gives

$$\int_{\{|v_\lambda| < \rho\}} |\nabla v_\lambda(y)| dy = \int_{\{|u| < \rho\}} |\nabla v_\lambda(\Psi_\lambda(x))| |\det(\nabla \Psi_\lambda(x))| dx. \quad (5.7)$$

Hence, by the coarea formula applied to the left hand side of (5.7) with the choice  $w = v_\lambda$ , and by the smoothness of  $\partial E_\lambda$  it follows

$$\begin{aligned} & \lim_{\rho \rightarrow 0^+} \frac{1}{2\rho} \int_{\{|u| < \rho\}} |\nabla v_\lambda(\Psi_\lambda)| |\det(\nabla \Psi_\lambda)| dx \\ &= \lim_{\rho \rightarrow 0^+} \frac{1}{2\rho} \int_{-\rho}^{\rho} \mathcal{H}^{n-1}(\{v_\lambda = s\}) ds = \mathcal{H}^{n-1}(\partial E_\lambda). \end{aligned} \quad (5.8)$$

On the other hand, using again the coarea formula with the choice  $w = u$  and the smoothness of  $u$  it follows

$$\lim_{\rho \rightarrow 0^+} \frac{1}{2\rho} \int_{\{|u| < \rho\}} |\nabla v_\lambda(\Psi_\lambda)| |\det(\nabla \Psi_\lambda)| dx = \int_{\partial E} \frac{|\nabla v_\lambda(\Psi_\lambda)| |\det(\nabla \Psi_\lambda)|}{|\nabla u|} d\mathcal{H}^{n-1}. \quad (5.9)$$

---

<sup>30</sup>The coarea formula appears implicitly in [61].

Then (5.6) follows from (5.8) and (5.9).

We now pass to the proof of (5.3). Using the area formula, and arguing as in the proof of (5.6), we have

$$\frac{1}{c_n} \mathcal{F}_{\phi^\circ}(E_\lambda) = \int_{\partial E_\lambda} \phi^\circ \left( \frac{\nabla v_\lambda}{|\nabla v_\lambda|} \right) d\mathcal{H}^{n-1} = \int_{\partial E} \phi^\circ \left( \frac{\nabla v_\lambda}{|\nabla v_\lambda|}(\Psi_\lambda) \right) |\det \nabla \Psi_\lambda| \frac{|\nabla v_\lambda|}{|\nabla u|} d\mathcal{H}^{n-1} \quad (5.10)$$

Differentiating (5.4) with respect to  $x^j$  it follows  $\frac{\partial u}{\partial x^j} = \frac{\partial v_\lambda}{\partial y^i} (\delta_{ij} + \lambda \frac{\partial X^i}{\partial y^j})$ , hence if we set

$$J(x) := (\nabla X(x))^T, \quad x \in U,$$

we have<sup>(31)</sup>

$$\nabla v_\lambda(\Psi_\lambda(x))^T = \nabla u(x)^T (\text{Id} + \lambda J(x))^{-1}, \quad x \in U. \quad (5.11)$$

In particular

$$\nabla v_\lambda(\Psi_\lambda) = \nabla u \quad \text{if } \lambda = 0, \quad \text{that is on } \partial E.$$

From (5.11) it follows

$$\frac{d}{d\lambda} (\nabla v_\lambda(\Psi_\lambda(x))^T)_{|\lambda=0} = -\nabla u(x)^T J(x), \quad x \in U. \quad (5.12)$$

Using (5.12) and (5.2) it follows

$$\frac{d}{d\lambda} \left( |\det \nabla \Psi_\lambda| \frac{|\nabla v_\lambda|}{|\nabla u|} \right)_{|\lambda=0} = \text{tr} \left( (\text{Id} - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}) \nabla X \right). \quad (5.13)$$

As a consequence of (5.11) and (5.12), at any point  $x \in U$  we have

$$\frac{d}{d\lambda} \left[ \frac{\nabla v_\lambda(\Psi_\lambda)^T}{|\nabla v_\lambda(\Psi_\lambda)|} \right]_{|\lambda=0} = -\frac{\nabla u^T}{|\nabla u|} J + \left\langle \frac{\nabla u^T}{|\nabla u|} J, \frac{\nabla u}{|\nabla u|} \right\rangle \frac{\nabla u^T}{|\nabla u|}. \quad (5.14)$$

Using (3.5), (5.10), (5.14), (5.13) we have

$$\begin{aligned} \frac{1}{c_n} \frac{d}{d\lambda} \mathcal{F}_{\phi^\circ}(E_\lambda)_{|\lambda=0} &= \int_{\partial E} \langle -\nu \nabla X^T + \langle \nu \nabla X^T, \nu^T \rangle \nu, n_\phi \rangle d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial E} (\text{div} X - \langle \nu^T \nabla X, \nu^T \rangle) \phi^\circ(\nu) d\mathcal{H}^{n-1}. \end{aligned} \quad (5.15)$$

Recalling (3.13) we have that the second addendum  $\langle n_\phi, \langle \nu^T \nabla X, \nu^T \rangle \nu \rangle$  on the right hand side of (5.15) can be written as

$$\langle \nu^T \nabla X, \nu^T \rangle \langle \nu_{\phi^\circ}, n_\phi \rangle \phi^\circ(\nu) = \langle \nu^T \nabla X, \nu^T \rangle \phi^\circ(\nu),$$

(where we have used (3.13)), and therefore cancels with the the fourth addendum. Then (5.3) follows.  $\square$

**Remark 5.2.** We note once more that Theorem 5.1 is valid without assuming that  $\phi^\circ$  is convex.

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<sup>31</sup>Recall that with our conventions the gradient vector field  $\nabla u(x)$  is a column.

**Remark 5.3.** The previous computation holds also for a function  $\sigma$  (resp.  $\phi^\circ$ ) defined on a relatively open subset  $S$  of  $\mathbb{S}^{n-1}$  (resp. on  $\{\lambda\xi^* : \xi^* \in S, \lambda \in \mathbb{R}\}$ ), provided  $\frac{\nabla v_\lambda}{|\nabla v_\lambda|}(x)$  still belongs to  $S$ .

**Definition 5.4.** *We set*

$$\operatorname{div}_{\tau,\phi} X := \operatorname{tr}((\operatorname{Id} - n_\phi \otimes \nu_{\phi^\circ}) \nabla X) = \operatorname{div} X - n_\phi^i \nu_{\phi^\circ}^j \nabla_i X^j.$$

Notice that the matrix<sup>32</sup>  $\operatorname{Id} - n_\phi \otimes \nu_{\phi^\circ}$  is not symmetric.

**Definition 5.5** ( $\phi$ -mean curvature). *We define*

$$\kappa_\phi := \operatorname{div}_{\tau,\phi} n_\phi, \quad \mathbf{H}_\phi := \kappa_\phi \nu_{\phi^\circ} \quad \text{on } \partial E.$$

The following result shows that we can equivalently define the  $\phi$ -mean curvature using the divergence in  $M$ , provided we use the natural extension of  $n_\phi$ .

**Lemma 5.6.** Let  $\phi^\circ \in \mathcal{M}_{\operatorname{reg}}(T^*M)$  and let  $N_\phi : U \rightarrow M$  be the extension of  $n_\phi$  as defined in (3.24). Then

$$\kappa_\phi = \operatorname{div} N_\phi = \phi_{\xi_i^* \xi_j^*}^{\circ} (\nabla d_\phi) \nabla_{ij}^2 d_\phi = \Delta_\phi d_\phi.$$

*Proof.* Define  $f(z) := \langle \nu^E(x), N_\phi(z) \rangle$  for any  $z \in U$ . Then  $f$  has a maximum at  $x$  (with value  $\phi^\circ(\nu^E(x))$ ). Therefore  $\nabla f(x) = 0$ , i.e.,  $\nu_j^E(x) \nabla_i N_\phi^j(x) = 0$ . Hence  $\nu_{\phi^\circ}^j(x) \nabla_i N_\phi^j(x) = 0$ .  $\square$

We now assume in particular that  $\phi^\circ$  is convex.

**Corollary 5.7 (First variation II).** *Let  $\phi^\circ \in \mathcal{M}_{\operatorname{reg}}(T^*M)$ . We have*

$$\frac{d}{d\lambda} \mathcal{F}_{\phi^\circ}(E_\lambda)|_{\lambda=0} = c_n \int_{\partial E} \langle \mathbf{H}_\phi, X \rangle \phi^\circ(\nu) d\mathcal{H}^{n-1}. \quad (5.16)$$

*Proof.* Split  $X$  as

$$X = X_{\perp,\phi} + X_\tau, \quad X_{\perp,\phi} := \langle X, \nabla d_\phi \rangle N_\phi =: \psi N_\phi, \quad X_\tau := X - X_{\perp,\phi}.$$

Note that  $\langle X_{\perp,\phi}, \nabla d_\phi \rangle = \langle X, \nabla d_\phi \rangle$ , and that  $\langle X_\tau, \nabla d_\phi \rangle = 0$ , namely  $X_\tau$  is a tangent vector field to  $\partial E$ . From (5.3) it follows that the function

$$X \rightarrow \frac{d}{d\lambda} \mathcal{F}_{\phi^\circ}(E_\lambda)|_{\lambda=0}$$

is linear with respect to  $X$ . Moreover, it is possible to show that the contribution of  $X_\tau$  to  $\mathcal{F}_{\phi^\circ}(E_\lambda)$  is of order  $o(\lambda)$ . Therefore we can neglect  $X_\tau$  in the first variation, and consider only  $X_{\perp,\phi}$ . We have

$$\begin{aligned} \frac{1}{c_n} \frac{d}{d\lambda} \mathcal{F}_{\phi^\circ}(E_\lambda)|_{\lambda=0} &= \int_{\partial E} \left( \operatorname{div}(\psi N_\phi) - N_\phi^i \nu_{\phi^\circ}^j \nabla_i(\psi N_\phi^j) \right) \phi^\circ(\nu) d\mathcal{H}^{n-1} \\ &= \int_{\partial E} \left( \psi \operatorname{div} N_\phi - \psi N_\phi^i \nu_{\phi^\circ}^j \nabla_i N_\phi^j \right) \phi^\circ(\nu) d\mathcal{H}^{n-1}, \end{aligned}$$

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<sup>32</sup>To be consistent with the indices, here  $\operatorname{Id}$  has one lower index and one upper index, and  $n_\phi \otimes \nu_{\phi^\circ}$  is a  $(1, 1)$  tensor.

where, recalling (3.13), we have used  $N_\phi^i \nu_{\phi^o, j} \nabla_i (\psi N_\phi^j) = \langle \nabla \psi, N_\phi \rangle + \psi N_\phi^i \nu_{\phi^o, j} \nabla_i N_\phi^j$ . Therefore

$$\frac{d}{d\lambda} \mathcal{F}_{\phi^o}(E_\lambda)|_{\lambda=0} = c_n \int_{\partial E} \psi \left( \operatorname{div} N_\phi - N_\phi^i \nu_{\phi^o, j} \nabla_i N_\phi^j \right) \phi^o(\nu) \, d\mathcal{H}^{n-1}$$

which is (5.16).  $\square$

**Corollary 5.8.** *We have the integration by parts formula*

$$c_n \int_{\partial E} \operatorname{div}_{\tau, \phi} X \, \phi^o(\nu) \, d\mathcal{H}^{n-1} = c_n \int_{\partial E} \langle \mathbf{H}_\phi, X \rangle \phi^o(\nu) \, d\mathcal{H}^{n-1}, \quad X \in \mathcal{C}_c^1(M; M).$$

**Example 5.9.** Let  $\phi^o \in \mathcal{M}_{\text{reg}}(T^*M)$  be spatially homogeneous. Then

$$\kappa_\phi = n - 1 \quad \text{on } \partial B_\phi. \quad (5.17)$$

Take  $u(\zeta) = 1 - \phi(\zeta)$  for  $\zeta \in M$ ; then  $\nabla u(\zeta) = -\phi_\xi(\zeta)$ . Hence  $\phi_\xi^o(\nabla u(\zeta)) = -\zeta/\phi(\zeta)$  on  $M$ . Consequently  $\kappa_\phi = -\operatorname{div}_\phi(\phi_\xi^o(\nabla u(\zeta))) = \operatorname{div}(\zeta/\phi(\zeta))$ . Then, as  $\zeta \cdot \phi_\xi(\zeta) = \phi(\zeta)$ , we have

$$\operatorname{div} \left( \frac{\zeta}{\phi(\zeta)} \right) = \frac{\operatorname{div} \zeta}{\phi(\zeta)} - \frac{\zeta \cdot \phi_\xi(\zeta)}{\phi^2(\zeta)} = \frac{n}{\phi(\zeta)} - \frac{1}{\phi(\zeta)} = \frac{n-1}{\phi(\zeta)}.$$

**Example 5.10.** Let  $n = 2$ , and assume that  $\phi^o(\xi^*) = \phi^o(\xi^*) = \rho\gamma(\theta)$ , where  $(\rho, \theta)$  are polar coordinates in the  $\xi^*$ -plane, i.e.,  $\xi_1^* = \rho \cos \theta$ ,  $\xi_2^* = \rho \sin \theta$ . Then the curvature  $\kappa_\phi$  of a smooth curve  $\partial E$  is (see for instance [39] and the next section)

$$\kappa_\phi = \kappa(\gamma + \gamma_{\theta\theta}), \quad (5.18)$$

where  $\gamma_{\theta\theta}$  denotes the second derivative of  $\gamma$  with respect to  $\theta$ .

### 5.1.1 Curves: parametric computation

Let us compute the first variation of  $\mathcal{F}_{\phi^o}$  in the special case  $n = 2$ , using a parametric approach. Write  $\nu = \nu(\theta) = -(\cos \theta, \sin \theta) = \tau(\theta)^\perp$ , where  $^\perp$  denotes the counterclock-wise rotation of  $\pi/2$ , and define  $\gamma : [0, 2\pi[ \rightarrow \mathbb{R}^2$  as

$$\gamma(\theta) := \sigma(\nu)$$

In this section  $M = \mathbb{R}^2$ .

**Theorem 5.11 (First variation: curves).** *Let  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  be a regular parametrization of  $\partial E$ . Let  $\beta \in \mathcal{C}_c^2([0, 1]; \mathbb{R}^2)$ ,  $\lambda \in \mathbb{R}$ , and  $\alpha_\lambda := \alpha + \lambda\beta$ . Then*

$$\frac{d}{d\lambda} \mathcal{F}_\sigma(E_\lambda)|_{\lambda=0} = \int_0^1 \langle (\gamma(\theta) + \gamma_{\theta\theta}(\theta)) \kappa \nu, \beta \rangle \, dt, \quad (5.19)$$

where  $\alpha_\lambda$  is a regular parametrization of  $\partial E_\lambda$ , and  $\kappa = \frac{1}{|\alpha'|^2} (\alpha'' - \langle \alpha'', \frac{\alpha'}{|\alpha'|} \rangle \frac{\alpha'}{|\alpha'|})$  is the euclidean curvature of  $\partial E$ , where  $'$  denotes the derivative with respect to  $t \in [0, 1]$ .

*Proof.* Set  $\tau_\lambda = \tau_\lambda(t) := \frac{\alpha'_\lambda(t)}{|\alpha'_\lambda(t)|} = (-\sin \theta_\lambda(t), \cos \theta_\lambda(t))$ , and set  $-\nu_\lambda := \tau_\lambda^\perp$ . We have

$$\begin{aligned} \frac{d}{d\lambda} \mathcal{F}(E_\lambda) &= \frac{d}{d\lambda} \int_0^1 \gamma(\theta_\lambda(t)) |\alpha'_\lambda(t)| dt \\ &= \int_0^1 \gamma_\theta(\theta_\lambda) \frac{d\theta_\lambda}{d\lambda} |\alpha'_\lambda| dt + \int_0^1 \gamma(\theta_\lambda) \tau_\lambda \cdot \beta' dt =: I_\lambda + II_\lambda \end{aligned}$$

We have, integrating by parts and using  $\frac{d}{dt} \tau_\lambda|_{\lambda=0} = -\kappa\nu$ ,

$$\begin{aligned} II_{|\lambda=0} &= - \int_0^1 \gamma_\theta(\theta) \frac{d\theta_\lambda}{dt} \Big|_{\lambda=0} \tau \cdot \beta dt + \int_0^1 \gamma(\theta) \kappa\nu \cdot \beta dt \\ &= - \int_0^1 \gamma_\theta(\theta) \theta' \tau \cdot \beta dt + \int_0^1 \gamma(\theta) \kappa\nu \cdot \beta dt. \end{aligned} \tag{5.20}$$

To compute  $\frac{d\theta_\lambda}{d\lambda}|_{\lambda=0}$  we differentiate  $\alpha' + \lambda\beta' = |\alpha' + \lambda\beta'|(-\sin \theta_\lambda, \cos \theta_\lambda)$  with respect to  $\lambda$ . We have

$$\beta' = \tau \cdot \beta' \tau + |\alpha'| \nu \frac{d\theta_\lambda}{d\lambda} \Big|_{\lambda=0}$$

which implies

$$\frac{d\theta_\lambda}{d\lambda} \Big|_{\lambda=0} = \nu \cdot \frac{\beta'}{|\gamma'|}.$$

Substituting in (5.20), integrating by parts, using  $\frac{d\nu}{d\theta} = -\tau$  and  $\frac{d\theta}{dt} = \kappa$ , gives

$$I_{|\lambda=0} = \int_0^1 \gamma_\theta(\theta) \nu \cdot \beta' dt = \int_0^1 \gamma_{\theta\theta}(\theta) \kappa\nu \cdot \beta dt + \int_0^1 \gamma_\theta(\theta) \theta' \tau \cdot \beta dt,$$

and (5.19) follows.  $\square$

## 5.2 Inhomogeneous $\phi^\circ$

We define

$$\kappa_\phi := \operatorname{div}_\phi n_\phi = \operatorname{div} n_\phi + \nabla(\log(\operatorname{vol}_\phi)) \cdot n_\phi, \tag{5.21}$$

and the vector mean curvature  $\kappa_\phi$  to  $\partial E$  as  $\mathbf{H}_\phi := \kappa_\phi \nu_{\phi^\circ}$ .

The proof of the next theorem can be found in [39].

**Theorem 5.12 (First variation).** *Let  $\phi^\circ \in \mathcal{M}(T^*M)$ . Adopting the same notation of Theorem 5.1, we have*

$$\frac{d}{d\lambda} \mathcal{F}_{\phi^\circ}(E_\lambda)|_{\lambda=0} = \int_{\partial E} \langle \mathbf{H}_\phi, X \rangle \phi^\circ(x, \nu) \operatorname{vol}_\phi d\mathcal{H}^{n-1}. \tag{5.22}$$

### 5.3 The crystalline case

The computation of the first variation of  $\mathcal{F}_{\phi^o}$  is much more complicated in the crystalline case, because of the nondifferentiability of both the surface and the integrand. We report here some results from [34], [35], which indicate how to define the crystalline mean curvature. Let  $\phi^o$  be crystalline. Let  $E$  be a polyhedral neighbourhood-Lipschitz  $\phi$ -regular set, let  $U \supset \partial E$  be an open set of  $M$  and  $\eta \in \text{Lip}(U; V)$  such that  $\eta \in T_{\phi^o}(\nabla d_{\phi})$  almost everywhere in  $U$ . Let  $\Psi \in \text{Lip}(U \times \mathbb{R}; M)$ , with  $\Psi(x, \lambda) := x + \lambda X(x)$ , for a given initial velocity field  $X \in \text{Lip}(U; M)$ . In the computation of the first variation of  $\mathcal{F}_{\phi^o}$  we now find some technical difficulties: for instance we have to be able to  $\mathcal{H}^{n-1}$ -a.e. take the divergence of  $X$  on  $\partial E$ . This is not immediately guaranteed from the regularity of  $X$ , since  $\text{div} X$  is in  $L^\infty(U)$ . We therefore prefer to slightly change our point of view. Assume then  $E$  to be polyhedral Lipschitz  $\phi$ -regular, and define

$$\mathcal{H}_{\text{div}} := \{N \in \text{Nor}_{\phi}(\partial E; M) : \text{div}_{\tau} N \in L^2(\partial E)\}.$$

Let  $X \in \text{Lip}(\partial E; V)$ . As in the smooth case,  $\mathcal{F}_{\phi^o}$  does not change under infinitesimal tangential variations. Therefore we restrict ourselves to consider  $\phi$ -normal fields, hence we assume that  $X$  can be written as  $X = \psi\eta$ , where  $\psi \in \text{Lip}(\partial E)$  and  $\eta \in \text{Nor}_{\phi}(\partial E; M) \cap \text{Lip}(\partial E; M)$ . In order to continue, we have to extend  $\psi$  and  $\eta$  in a suitable neighbourhood of  $\partial E$ . One can show that there exist  $\varepsilon > 0$  and an open set  $U$  containing  $\partial E$  such that the map  $(x, \lambda) \in \partial E \times (-\varepsilon, \varepsilon) \rightarrow x + \lambda\eta(x) \in U$  is bilipschitz. We write  $(\pi_{\eta}(\cdot), \lambda_{\eta}(\cdot)) \in \partial E \times (-\varepsilon, \varepsilon)$  on  $U$  the inverse of this map. Define  $\psi^e \in \text{Lip}(U)$ ,  $\eta^e \in \text{Lip}(U; \mathbb{R}^n)$  as  $\psi^e(z) := \psi(\pi_{\eta}(z))$ ,  $\eta^e(z) := \eta(\pi_{\eta}(z))$ , and set  $X^e := \psi^e\eta^e$ . For  $\lambda \in \mathbb{R}$  with  $|\lambda|$  small enough, define  $\Psi(z, \lambda) := z + \lambda\psi^e(z)\eta^e(z)$ , and let  $\Psi_{\lambda}$  and  $E_{\lambda}$  be as Section 5.1.

**Theorem 5.13.** *We have*

$$\inf_{\psi \in \text{Lip}(\partial E), c_n \int_{\partial E} \psi^2 \phi^o(\nu) d\mathcal{H}^{n-1} \leq 1} \lim_{\lambda \rightarrow 0^+} \frac{\mathcal{F}_{\phi^o}(E_{\lambda}) - \mathcal{F}_{\phi^o}(E)}{\lambda} = - \min_{N \in \mathcal{H}_{\text{div}}} \mathcal{K}(N), \quad (5.23)$$

where

$$\mathcal{K}(N) := c_n \int_{\partial E} (\text{div}_{\tau} N)^2 \phi^o(\nu) d\mathcal{H}^{n-1}. \quad (5.24)$$

The minimization problem in (5.23) in general may admit more than one solution, and two minimizers have the same divergence. In the following we denote by  $N_{\min} \in \mathcal{H}_{\text{div}}$  a minimizer.

**Definition 5.14 (Crystalline mean curvature).** *We define the  $\phi$ -mean curvature  $\kappa_{\phi}^E$  of  $E$  as  $\kappa_{\phi}^E = \kappa_{\phi} := \text{div}_{\tau} N_{\min} \in L^2(\partial E)$ .*

For simplicity of notation, we will sometimes write  $\kappa_{\phi}$  in place of  $\kappa_{\phi}^E$ .

It turns out that the  $\phi$ -mean curvature of  $\partial B_{\phi}$  is constantly equal to  $n - 1$ .

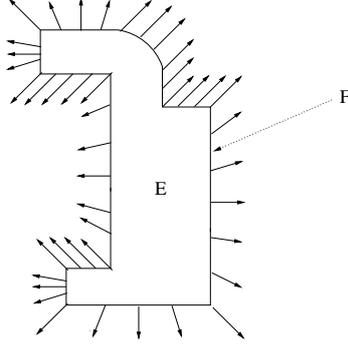


Figure 9: the vector field  $N_{\min} : \partial E \rightarrow \mathbb{R}^2$  is, on facets and arcs of  $\partial E$ , the linear combination of the values of  $\eta$  at the vertices.

**Example 5.15 (Polygonal curves).** Let  $n = 2$ . Let us compute explicitly the  $\phi$ -curvature of a two-dimensional Lipschitz  $\phi$ -regular set  $E$ , letting  $\eta \in \text{Nor}_\phi(\partial E; \mathbb{R}^2) \cap \text{Lip}(\partial E; \mathbb{R}^2)$ . Given a facet  $F \subset \partial E$  (in this case  $F$  equals a segment  $[z, w]$ ), the minimum problem (5.27) becomes

$$\inf \left\{ \int_{]z, w[} (N'(s))^2 d\mathcal{H}^1(s) : N \in L^2(]z, w[; \Pi_{[z, w]}), N' \in L^2(]z, w[), \right. \\ \left. N(x) \in T_{\phi^\circ}(\nu_{\phi^\circ}(x)) \text{ for a.e. } x \in ]z, w[, N(z) = c_z, N(w) = c_w \right\},$$

where  $c_z$  (resp.  $c_w$ ) is the orthogonal projection of  $\eta(z)$  (resp. of  $\eta(w)$ ) on the line  $\Pi_{[z, w]}$ , with the correct sign, and  $]z, w[$  is the relative interior of  $[z, w]$ .

We now observe that the above minimum problem has a unique solution  $N_{\min}^F$ , which is simply the linear function connecting  $c_z$  at  $z$  with  $c_w$  at  $w$ . Hence, when  $n = 2$ , not only the divergence of a minimizer is unique, but also the minimizer itself. If we now repeat this procedure for any facet, and on each facet we add to  $N_{\min}^F$  the proper (constant) normal component to  $F$ , we end up with the vector field  $N_{\min} : \partial E \rightarrow \mathbb{R}^2$  whose divergence is the  $\phi$ -curvature of  $\partial E$ . An example of this vector field is depicted in Figure 9. Curved regions in  $\partial E$  have zero  $\phi$ -curvature. On the other hand, if  $F$  is a facet of  $\partial E \subset \mathbb{R}^2$  and  $B_F \subset \partial B_\phi$  is the corresponding facet in  $\partial B_\phi$ ,  $\kappa_\phi^F$  is constant on  $F$  and

$$\kappa_\phi^F = \delta_F \frac{|B_F|}{|F|} \quad \text{on } \text{int}(F), \quad (5.25)$$

where  $\delta_F \in \{0, \pm 1\}$  is a convexity factor:  $\delta_F = 1$  (resp.  $\delta_F = -1$ ,  $\delta_F = 0$ ) if  $E$  is locally convex (resp. if  $E$  is locally concave,  $E$  is neither locally convex nor locally concave) at  $F$ .

Lipschitz  $\phi$ -regular sets have  $\phi$ -curvature which is more regular than being only square integrable [35].

**Theorem 5.16 (Regularity).** *We have  $\kappa_\phi \in L^\infty(\partial E)$ . Moreover,  $\kappa_\phi$  has bounded variation on all facets of  $\partial E$  corresponding to facets of  $\partial B_\phi$ .*

**Remark 5.17.** Assume that  $\partial E$  is a polyhedral Lipschitz  $\phi$ -regular set. We do not know under which further conditions on  $\partial E$  (if any) the functional  $\mathcal{K}$  in (5.24) admits a minimizer in  $\mathcal{H}_{\text{div}} \cap \text{Lip}(\partial E; M)$  or not. See also formula (8.5) (and Remark 5.18) below: in that case a discontinuous minimizing vector field with bounded divergence is constructed on the facet  $F$ .

### 5.3.1 A minimum problem on $F$ : $\phi$ -mean curvature on $F$

In this section we can assume for simplicity that  $n = 3$  and that  $c_n = 1$ . The case  $n = 2$  is trivial.  $E$  is a polyhedral Lipschitz  $\phi$ -regular set. We recall some notation that we have already occasionally used. The symbol  $F$  will always denote a (polyhedral) facet of  $\partial E$  such that  $\tilde{B}_\phi^F$  is a facet of  $B_\phi$ . If  $[p, q]$  is a closed edge of a polyhedral set, by  $]p, q[$  we denote the relative interior of  $[p, q]$ .

$\Pi_F$  is the affine plane spanned by the facet  $F$ . Whenever necessary, we identify  $\Pi_F$  with the plane parallel to  $\Pi_F$  and passing through the origin, and  $F$  with its orthogonal projection on this latter plane. We will assume for simplicity that  $\tilde{B}_\phi^F$  contains the origin of  $\Pi_F$  in its interior, and is symmetric with respect to the origin itself.

We let  $\tilde{\phi}_F : \Pi_F \rightarrow [0, +\infty[$  be the convex and one-homogeneous function on  $\Pi_F$  such that  $\{\tilde{\phi}_F \leq 1\} = \tilde{B}_\phi^F$ . We denote by  $\tilde{\phi}_F^\circ$  the dual of  $\tilde{\phi}_F$  (recall the first item of Remark 3.18). If no confusion is possible, we omit the dependence on  $F$  of  $\tilde{\phi}_F$ , thus writing  $\tilde{\phi}$  in place of  $\tilde{\phi}_F$ . We indicate by  $\kappa_\phi^B$  the  $\phi$ -curvature of a Lipschitz  $\phi$ -regular set  $B \subset \Pi_F$ . We also set

$$P_{\tilde{\phi}}(F) := \int_{\partial F} \tilde{\phi}^\circ(\tilde{\nu}^F) d\mathcal{H}^1.$$

We want to recall another way to define the crystalline mean curvature  $\kappa_\phi^E$  on a facet  $F$  of  $\partial E$ , using a localized minimum problem on  $F$ . Set

$$\text{Nor}_\phi(F; \Pi_F) := \{N \in L^\infty(F; \Pi_F) : N(x) \in T_{\phi^\circ}(\nu_{\phi^\circ}(F)) \text{ for } \mathcal{H}^2 \text{ a.e. } x \in F\}.$$

Any  $N \in \text{Nor}_\phi(F; \Pi_F)$  with  $\text{div} N \in L^2(\text{int}(F))$  admits a normal trace  $\langle \tilde{\nu}^F, N \rangle$  on  $\partial F$ . Set

$$\mathcal{H}_{\text{div}}(F; \Pi_F) := \{N \in \text{Nor}_\phi(F; \Pi_F) : \text{div} N \in L^2(F), \langle \tilde{\nu}^F, N \rangle = c_F \text{ } \mathcal{H}^1 \text{ a.e. on } \partial F\}.$$

We define the functional  $\mathcal{K}(\cdot, F) : \mathcal{H}_{\text{div}}(F; \Pi_F) \rightarrow [0, +\infty)$  as

$$\mathcal{K}(N, F) := \int_F (\text{div} N)^2 \phi^\circ(\nu^E) d\mathcal{H}^2 = \phi^\circ(\nu(F)) \int_{\text{int}(F)} (\text{div} N)^2 d\mathcal{H}^2. \quad (5.26)$$

The minimum problem

$$\inf \{\mathcal{K}(N, F) : N \in \mathcal{H}_{\text{div}}(F; \Pi_F)\} \quad (5.27)$$

admits a solution, and two minimizers have the same divergence. Let us denote by  $N_{\text{min}}^F$  a solution of problem (5.27). It turns out that

$$\kappa_\phi^E = \text{div} N_{\text{min}}^F, \quad \mathcal{H}^2 \text{ a.e. in } F.$$

Notice once more that the above equality says that the crystalline  $\phi$ -mean curvature of  $\partial E$  can be obtained, on the facet  $F$ , as the divergence of a vector field which minimizes a problem

localized on  $F$ . However this minimum problem depends on the shape of  $\partial E$  around  $F$ : indeed, we are assigning the normal trace of  $N_{\min}^F$  on  $\partial F$  via the functions  $c_F$ . The following remark is useful in concrete situations, and is a consequence of the strict convexity of the functional  $\mathcal{K}$  in the divergence.

**Remark 5.18 (Minimality criterion).** Let  $f = \operatorname{div} \bar{N}$  where  $\bar{N}$  is a vector field belonging to  $\mathcal{H}_{\operatorname{div}}(F; \Pi_F)$ . Assume that  $f$  satisfies the Euler-Lagrange inequality

$$\int_F f \operatorname{div}(\bar{N} - N) \, d\mathcal{H}^2 \leq 0, \quad N \in \mathcal{H}_{\operatorname{div}}(F; \Pi_F). \quad (5.28)$$

Then  $\bar{N}$  is a solution of (5.27).

As a corollary of this minimality criterion it follows that if  $f$  is constant on  $F$  then (5.28) is satisfied (with the equality in place of the inequality).

## 6 $\phi$ -calibrability

Theorem 5.16 makes possible to speak of the jump set of  $\kappa_\phi$  on the facets of  $\partial E$  corresponding to facets of  $\partial B_\phi$ . If  $F \subset \partial E$  is such a facet, it may be of interest finding necessary and sufficient conditions on  $E$  and  $F$  ensuring that the jump set of  $\kappa_\phi$  on  $F$  is empty: that is, to prove that  $\kappa_\phi$  is continuous on  $F$ . Assume that this is the case: then for small times in the crystalline mean curvature flow,  $F$  is expected to translate parallelly to itself if  $\kappa_\phi$  is constant on  $F$  or to bend if  $\kappa_\phi$  is continuous but not constant on  $F$ .

**Definition 6.1 (Calibrability).** *We say that  $F$  is  $\phi$ -calibrable if  $\kappa_\phi^E$  is constant on  $F$ .*

Recalling Definition 6.1 and the results of Section 5.3.1, we deduce, for instance in  $n = 3$  dimensions, that a facet  $F$  is  $\phi$ -calibrable if and only if there exists a vector field  $N : F \rightarrow \Pi_F$  which is a solution to:

$$\left\{ \begin{array}{l} N \in L^\infty(F; \Pi_F), \\ N(x) \in T_{\phi^\circ}(\nu_{\phi^\circ}(F)) \text{ for } \mathcal{H}^2 \text{ a.e. } x \in F, \\ \operatorname{div} N = \frac{1}{|F|} \int_{\partial F} c_F d\mathcal{H}^2, \\ \langle \tilde{\nu}^F, N \rangle = c_F \quad \mathcal{H}^1 \text{ a.e. on } \partial F. \end{array} \right. \quad (6.1)$$

The quantity

$$\frac{1}{|F|} \int_{\partial F} c_F d\mathcal{H}^2 =: v_F$$

can be interpreted as the mean velocity of  $F$ , and is sometimes called weighted mean curvature; in case of a convex  $F$  with  $E$  convex at  $F$  (see Definition 6.3 below) this velocity is positive. Hence

$$-v_F \nu_{\phi^\circ}(F)$$

represents the normal velocity vector of  $F$ .

To construct examples of facet breaking in crystalline mean curvature flow, the first step is exactly to find facets which are not  $\phi$ -calibrable. Therefore, we are led to look for criteria that allow to decide whether a facet is  $\phi$ -calibrable or not [34], [35].

Given a finite perimeter set [11]  $B$  in the hyperplane  $H_F$  containing  $F$ , we denote by  $\partial^* B$  the reduced boundary of  $B$ . We also let

$$c_B(x) := \begin{cases} \max\{\tilde{\nu}^B \cdot p : p \in \tilde{B}_\phi^F\} & \text{if } x \in \partial^* B \setminus \partial F, \\ c_B(x) = c_F(x) & \text{if } x \in \partial^* B \cap \operatorname{int}(F). \end{cases} \quad (6.2)$$

The following result is proved in [36].

**Theorem 6.2 (Characterization).**  *$F$  is  $\phi$ -calibrable if and only if for any  $B \subseteq F$  of finite perimeter we have*

$$v_B := \frac{1}{|B|} \int_{\partial^* B} c_B d\mathcal{H}^2 \geq \frac{1}{|F|} \int_{\partial F} c_F d\mathcal{H}^2. \quad (6.3)$$

Before sketching the proof of Theorem 6.2, let us recall [14] that given a function  $u$  of bounded variation in  $F$  and a vector field  $X \in L^\infty(F; H_F)$ , the following generalized Gauss-Green formula holds:

$$\int_F u \operatorname{div} X \, dx + \int_F \theta(X, Du) |Du| = \int_{\partial F} [X \cdot \tilde{\nu}^F] u \, d\mathcal{H}^1.$$

Here  $Du$  is the distributional derivative of  $u$ , which is measure; moreover, the density  $\theta(X, Du)$ , the total variation measure  $|Du|$  and the normal trace  $[X, \tilde{\nu}^F]$  are suitably weakly defined.

*Sketch of proof of Theorem 6.2.* The implication  $F$   $\phi$ -calibrable  $\Rightarrow v_B \geq v_F$  can be proved as follows. We know that  $\operatorname{div} N = v_F$  on  $F$ . Therefore, integrating  $\operatorname{div} N$  on  $F$  and using the Gauss-Green Theorem we get

$$|B| \operatorname{div} N = \int_B \operatorname{div} N \, dx = \int_{\partial^* B} \tilde{\nu}^B \cdot N \, d\mathcal{H}^1 \leq \int_{\partial^* B} c_B \, d\mathcal{H}^1,$$

where in the last equality we use also the definition (6.2) of  $c_B$ .

The opposite implication can be proved as follows. Assume by contradiction that  $F$  is not  $\phi$ -calibrable. Given any  $\lambda \in \mathbb{R}$  define  $\Omega_\lambda := \{x \in F : \operatorname{div} N_{\min}(x) < \lambda\}$ . Using Theorem 5.16 it follows that there exists  $\lambda < v_F$  such that  $\Omega_\lambda \neq \emptyset$  has finite perimeter. We have, using the properties of functions of bounded variations [11] and the Gauss-Green theorem,

$$\begin{aligned} \int_{\Omega_\lambda} \operatorname{div} N_{\min} \, dx &= - \int_{\operatorname{int}(F) \cap \partial^* \Omega_\lambda} \theta(N_{\min}, D1_{\Omega_\lambda}) \, d\mathcal{H}^1 + \int_{\partial F} [N_{\min} \cdot \tilde{\nu}^F] 1_{\Omega_\lambda} \, d\mathcal{H}^1 \\ &= - \int_{\operatorname{int}(F) \cap \partial^* \Omega_\lambda} \theta(N_{\min}, D1_{\Omega_\lambda}) \, d\mathcal{H}^1 + \int_{\partial F \cap \partial^* \Omega_\lambda} [N_{\min} \cdot \tilde{\nu}^F] \, d\mathcal{H}^1. \end{aligned}$$

It is now possible to prove the following property:

$$-\theta(N_{\min}, D1_{\Omega_\lambda})(x) = \max\{\tilde{\nu}^{\Omega_\lambda}(x) \cdot p : p \in \tilde{B}_\phi^F\} \quad \mathcal{H}^1 - \text{a.e. } x \in \operatorname{int}(F) \cap \partial^* \Omega_\lambda,$$

and also the property  $[N_{\min} \cdot \tilde{\nu}^F] = c_F = c_{\Omega_\lambda}$  on  $\partial F \cap \partial^* \Omega_\lambda$ . Therefore  $-\theta(N_{\min}, D1_{\Omega_\lambda}) = c_{\Omega_\lambda}$  on  $\operatorname{int}(F) \cap \partial^* \Omega_\lambda$ , and hence

$$\int_{\Omega_\lambda} \operatorname{div} N_{\min} \, dx = \int_{\partial^* \Omega_\lambda} c_{\Omega_\lambda} \, d\mathcal{H}^1.$$

It follows

$$v_F > \lambda > \frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} \operatorname{div} N_{\min} \, dx = \frac{1}{|\Omega_\lambda|} \int_{\partial^* \Omega_\lambda} c_{\Omega_\lambda} \, d\mathcal{H}^1 \geq v_F,$$

which is a contradiction.  $\square$

Heuristically, proving that a facet instantly breaks during the subsequent crystalline mean curvature flow means to find a subset  $B \subset F$  such that  $v_B < v_F$ .

## 6.1 The case of convex facets

**Definition 6.3 (Convexity at a facet).** *We say that  $E$  is convex at  $F$  if  $E$  lies, locally around  $F$ , from one side of the hyperplane  $H_F$  containing  $F$ .*

It is possible to prove that if the Lipschitz  $\phi$ -regular set  $E$  is convex at  $F$ , then  $F$  is Lipschitz  $\tilde{\phi}_F$ -regular. Under this convexity assumption, we have that  $v_F = \frac{1}{|F|} \int_{\partial F} \tilde{\phi}_F^o(\tilde{\nu}^F) d\mathcal{H}^1$ . In addition  $\kappa_\phi$  turns out to be convex in  $F$ .

The following result is proved in [36].

**Theorem 6.4.** *Assume that  $E$  is convex at  $F$  and that  $F$  is convex. Then  $F$  is  $\phi$ -calibrable if and only if*

$$\sup_{\partial F} \kappa_{\tilde{\phi}}^F \leq \frac{1}{|F|} \int_{\partial F} \tilde{\phi}_F^o(\tilde{\nu}^F) d\mathcal{H}^1. \quad (6.4)$$

The sup in (6.4) is the essential supremum, since  $\kappa_{\tilde{\phi}}^F$  is a function in  $L^\infty(\partial F)$ . Recall that  $\tilde{\kappa}_\phi^F$  is the  $\tilde{\phi}_F$ -curvature of  $\partial F$  (as a subset of  $\Pi_F$ ).

Hence, under the assumptions of Theorem 6.4, problem (6.1) is solvable if and only if the  $\tilde{\phi}$ -curvature of  $\partial F$  is bounded above by the constant on the right hand side of (6.4); this means, roughly speaking, that the edges of  $\partial F$  cannot be too “short”.

## 7 Anisotropic mean curvature flow

In this section we quickly recall the definition of anisotropic mean curvature flow, and we present the main example of evolution. We will not consider the case of unbounded hypersurfaces (such as graphs on the whole of  $\mathbb{R}^{n-1}$ , for instance).

### 7.1 Regular case

We assume in this subsection that  $\phi \in \mathcal{M}_{\text{reg}}(TM)$ .

**Definition 7.1** ( *$\phi$ -mean curvature flow*). *Let  $T > 0$  and, for any  $t \in [0, T]$ , let  $E(t) \subset M$  be a set with compact boundary. We say that  $(E(t))_{t \in [0, T]}$  is a smooth  $\phi$ -mean curvature flow in  $[0, T]$  starting from  $E = E(0)$  if the following conditions hold:*

(i) *there exists an open set  $A \subset M \times [0, +\infty)$  such that  $\cup_{t \in [0, T]} (\partial E(t) \times \{t\}) \subset A$  and, if we define*

$$d_\phi(z, t) := \text{dist}_\phi(z, E(t)) - \text{dist}_\phi(z, M \setminus E(t)), \quad z \in M, \quad t \in [0, T],$$

*we have  $d_\phi \in \mathcal{C}^\infty(A)$ ;*

(ii)

$$\frac{\partial}{\partial t} d_\phi(x, t) = \text{div}_\phi^{E(t)}(x), \quad x \in \partial E(t), \quad t \in [0, T]. \quad (7.1)$$

Observe that  $\frac{\partial}{\partial t} d_\phi$  is positive for an expanding set.

**Example 7.2.** Given  $R_0 > 0$ , let us show that  $\{\xi \in M : \phi(\xi) < R_0\}$  has an evolution shrinking self-similarly under the flow (7.1). Looking for a solution of the form  $\{\xi \in M : \phi(\xi) < R(t)\}$ , we have  $d_\phi(z, t) = \phi(z) - R(t)$ , and (7.1) becomes  $\dot{R} = -\frac{n-1}{R}$  (recall Example 5.9). Hence  $R(t) = \sqrt{R_0^2 - 2(n-1)t}$  for  $t \in [0, \frac{R_0^2}{2(n-1)})$ , which disappears for times larger than  $\frac{R_0^2}{2(n-1)}$ .

The evolution law (7.1) is the gradient flow of  $\mathcal{F}_{\phi^\circ}$  [2], [63], [6]. We refer for instance to the papers [39] and references therein for more.

### 7.2 Crystalline case

Assume now that  $\phi$  is crystalline. Unless  $n = 2$ , the definition of crystalline mean curvature flow is much more involved. Let us begin with the two-dimensional case ( $M = \mathbb{R}^2$ ).

#### 7.2.1 Curves

Let  $\partial E \subset \mathbb{R}^2$  be a closed simple polygonal curve,  $S_j \subset \partial E$  an edge of length  $L_j > 0$  and  $\nu_j$  the exterior euclidean unit normal to  $S_j$ . We define  $\delta_{S_j}$  to be 1 (resp. -1) if  $S_j$  and its two adjacent edges form a convex (resp. concave) curve, and 0 otherwise. Let  $L_{B_\phi}(\nu_j)$  be the length of the edge of  $\partial B_\phi$  having  $\nu_j$  as exterior normal: we will restrict here to consider polygonal curves  $\partial E$  (and  $\partial E(t)$ ) which consist of a sequence of segments having the same ordered set of normal orientations as  $\partial B_\phi$ . Such a  $\partial E$  is Lipschitz  $\phi$ -regular.

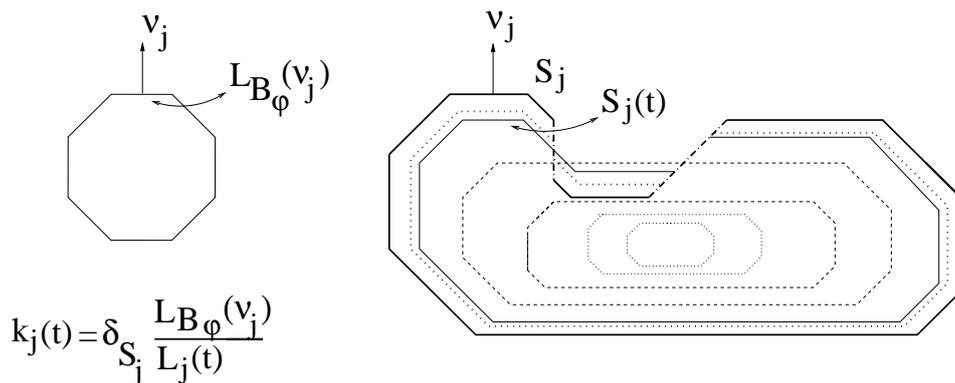


Figure 10: The motion by crystalline curvature starting from the curve in bold when  $B_\phi$  is an octagone. We depict the motion also after the disappearing of some edge.

Recall that the crystalline curvature of  $S_j$  equals

$$\kappa_\phi^j := \delta_{S_j} \frac{L_{B_\phi}(v_j)}{L_j},$$

see (5.25).

Given two parallel segments  $S_1, S_2$ , we call the distance vector of  $S_2$  from  $S_1$  the vector having norm  $\text{dist}(S_1, S_2)$  pointing from  $S_1$  to  $S_2$ .

Let us define the local in time crystalline curvature flow of a polygonal Lipschitz  $\phi$ -regular curve (with a finite number of sides), supposing that no side disappears.

**Definition 7.3.** *Let  $\partial E(t)$  be a family of time-parametrized polygonal Lipschitz  $\phi$ -regular curves. We say that  $\partial E(t)$  moves by crystalline curvature in  $[0, T]$ ,  $T > 0$ , if each side either translates parallel to itself or stays still (and does not disappear) for any  $j$ , the distance vector  $h_j(t)$  between the edge  $S_j(t)$  and  $S_j(0)$  is of class  $\mathcal{C}^1([0, T])$ , and<sup>(33)</sup>*

$$\frac{\dot{h}_j(t)}{\phi^\circ(v_j)} = -\kappa_\phi^j(t)v_j, \quad t \in [0, T].$$

Convex portions of the curve contract in the direction of their inner normal, while concave portions expand in the direction of the outer normal (see Figure 10). See [114], [116], [98], [79], [82], [87], [88], [83], [84], [98], [110], [111], [120] for various qualitative properties.

## 7.2.2 Hypersurfaces

In this section we recall a definition of crystalline mean curvature flow (for which the set  $B_\phi$  shrinks self-similarly). Recall that  $\partial E(t)$  is always assumed to be compact and Lipschitz. The next definition has been used in [32] (see also [30] in two dimensions) to prove a comparison principle for crystalline mean curvature flow.

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<sup>33</sup>See also Remark 10.14 below.

**Definition 7.4 (Neighbourhood- $L^\infty$   $\phi$ -regular flow).** Let  $T > 0$ . A neighbourhood- $L^\infty$   $\phi$ -regular flow on  $[0, T]$  is a map  $t \in [0, T] \rightarrow E(t) \subset M$  satisfying the following properties:

- (i) there exists an open set  $A \subset M \times [0, +\infty)$  such that  $\cup_{t \in [0, T]} (\partial E(t) \times \{t\}) \subset A$  and  $d_\phi(z, t) := \text{dist}_\phi(z, E(t)) - \text{dist}_\phi(z, M \setminus E(t))$  is Lipschitz in  $A$ ;
- (ii) there exists a bounded vector field  $n : A \rightarrow V$  such that  $n \in T_{\phi^\circ}(\nabla d_\phi)$  almost everywhere in  $A$ ,  $\text{div} n \in L^\infty(A)$ , and there exists  $\lambda > 0$  such that

$$\left| \frac{\partial d_\phi}{\partial t}(z, t) - \text{div} n(z, t) \right| \leq \lambda |d_\phi(z, t)| \quad \text{for a.e. } (z, t) \in A. \quad (7.2)$$

In (7.2) the divergence of  $n$  is taken in  $M$ , hence we avoid to restrict it on a specific boundary. Notice also that the left hand side of (7.2) tends to zero as  $d_\phi(z, t)$  tends to zero<sup>(34)</sup>. Other definitions could be given by imposing for instance in addition that  $E(t)$  and  $M \setminus E(t)$  satisfy the  $rB_\phi$ -ball condition (see [27]), or by imposing the evolution law only on the flowing manifold, possibly using the vector field  $N_{\min}^{E(t)}$ . We do not give any detail here. We refer to the already mentioned papers, to [20] and to Section 8.

**Remark 7.5.** We are not aware of a local existence theorem of the flow even starting from a polyhedral set  $E$  in  $n \geq 3$  dimensions, one of the reasons being the presence of the facet-breaking phenomenon. A short time existence and uniqueness result has been proved in [26], provided  $E$  is convex. In view of the poor knowledge on existence and uniqueness of crystalline mean curvature flow in three dimensions, it is not yet completely clear that Definition 7.4 is the most natural one for this kind of geometric flows.

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<sup>34</sup>For euclidean motion by mean curvature ( $\phi(\cdot) = |\cdot|$ ), the left hand side of (7.2) can be controlled by  $|d_\phi|$  times the  $L^\infty$ -norm squared of the length of the second fundamental form of the flowing manifolds.



We notice however that inequalities (8.1) express exactly condition (6.4). Indeed  $\kappa_{\tilde{\phi}}^F = 0$  on the edges  $]S, T[$  and  $]T, U[$ . Moreover if  $|L|$  stands for the length of one of the four remaining edges  $d = [R, Z]$ ,  $a = [Z, V]$ ,  $b = [U, V]$  and  $c = [R, S]$ , which we generically denote by  $L$ , we have, recalling (5.25),

$$\kappa_{\tilde{\phi}}^F = \frac{2}{|L|} \quad \text{on int}(L).$$

Therefore the supremum of  $\kappa_{\tilde{\phi}}^F$  is reached either at the edge  $b$  or at the edge  $c$ . We distinguish two cases. The first case is when  $b \leq c$ , so that the supremum of  $\kappa_{\tilde{\phi}}^F$  is reached at the edge  $b$ . Then the second inequality in (8.1) is automatically satisfied, since  $a/(a+b) < 1$ . A direct computation gives  $c_F = 1$  on  $\partial F$  (see (4.1) and Example (4.11)); therefore

$$\int_{\partial F} c_F d\mathcal{H}^1 = 2(a+d).$$

Since  $|F| = cd + b(a-c)$ , the inequality (6.4) reads as  $\frac{2}{b} \leq \frac{2(a+d)}{cd+ab-bc}$ , which is equivalent to  $b \geq \frac{cd}{c+d}$ , and gives the first inequality in (8.1).

If  $b \geq c$ , the supremum of  $\kappa_{\tilde{\phi}}^F$  is reached at the edge  $c$ . Then the first inequality in (8.1) is automatically satisfied. Moreover inequality (6.4) reads as  $\frac{2}{c} \leq \frac{2(a+d)}{cd+ab-bc}$ , which is equivalent to  $c \geq \frac{ab}{a+b}$ .

#### 8.0.4 On $\phi$ -calibrability of the other facets of $\partial E$

All facets of  $\partial E$  different from  $F$  and its opposite one are  $\phi$ -calibrable, since they are rectangles. For rectangular facets  $F$  such that  $E$  is convex at  $F$  this is a direct consequence of Theorem 6.4. For instance, consider the right lateral facet  $F_2$  of  $E$ : the edges of  $F_2$  are  $b$  and  $e$ . Assume  $b \leq e$ . Theorem 6.4 reads as  $\frac{2}{b} \leq \frac{2(b+e)}{be}$ , which is always satisfied (with the strict inequality).

However there are rectangular facets  $Q \subset \partial E$  such that  $E$  is not convex at  $Q$ . The  $\phi$ -calibrability of those facets follows from the following result.

**Remark 8.3.** Let  $R \subset \mathbb{R}^2$  be a rectangle with edges  $l_1, \dots, l_4$  parallel to the coordinate axes, let  $\tilde{\nu}_i$  be the exterior unit normal to  $R$  at  $\text{int}(l_i)$  and let  $|l_i|$  be the length of  $l_i$ . Let  $l_1$  and  $l_3$  be the edges parallel to the  $x$ -axis,  $l_1$  the lower one, and  $l_2$  be the right edge. Fix for simplicity the origin at the intersection between  $l_4$  and  $l_1$ . Let  $\alpha_i \in [-1, 1]$  for  $i = 1, \dots, 4$ . Consider the vector field  $n := (n_1, n_2)$  defined, for  $(x, y) \in R$ , as

$$n_1(x, y) := \frac{\alpha_2 x}{|l_1|} - \alpha_4 \left(1 - \frac{x}{|l_1|}\right) = n_1(x), \quad n_2(x, y) := \frac{\alpha_3 y}{|l_4|} - \alpha_1 \left(1 - \frac{y}{|l_4|}\right) = n_2(y).$$

Notice that  $n_1$  (resp.  $n_2$ ) depends only on  $x$  (resp. on  $y$ ). Then

$$\text{div} n = \frac{\alpha_2 + \alpha_4}{|l_1|} + \frac{\alpha_3 + \alpha_1}{|l_4|} = \frac{|l_2|(\alpha_2 + \alpha_4)}{|R|} + \frac{|l_3|(\alpha_3 + \alpha_1)}{|R|} = |R|^{-1} \sum_{i=1}^4 \alpha_i |l_i|.$$

Moreover,  $\langle \tilde{\nu}_i, n \rangle = \alpha_i$  for  $i = 1, \dots, 4$ . Indeed, for instance on  $l_3$  (resp. on  $l_4$ ) we have  $\langle \tilde{\nu}_3, n(x, y) \rangle = n_2(x, |l_4|) = \alpha_3$  (resp.  $\langle \tilde{\nu}_4, n(x, y) \rangle = -n_1(0, y) = \alpha_4$ ).

The vector field  $n$  satisfies also  $|n_1|, |n_2| \leq 1$ . Summarizing

$$\begin{cases} \max\{|n_1|, |n_2|\} \leq 1 & \text{in } \text{int}(R), \\ \text{div } n = |R|^{-1} \sum_{i=1}^4 \alpha_i |l_i| & \text{in } \text{int}(R), \\ \langle n, \tilde{\nu}_i \rangle = \alpha_i, & \text{in } \text{int}(l_i). \end{cases} \quad (8.2)$$

Fix now

$$a = 2, \quad b = 1/4, \quad c = 1, \quad d = 1, \quad e = 1/2. \quad (8.3)$$

With these choices,

the facet  $F$  is not  $\phi$ -calibrable,

in view of Proposition 8.1.

### 8.0.5 On $\phi$ -calibrability of facets of $\partial E(t)$ , $t > 0$

Let us consider a set  $E(t)$  of the form depicted in Figure 13, for a fixed  $t > 0$  small enough, hence in particular the edge  $[\alpha(t), \beta(t)]$  is short enough.

Let us consider facets  $F_2(t)$ ,  $F_5(t)$ ,  $F_6(t)$ ,  $P(t)$  and its opposite one: these are rectangular facets where the set  $E(t)$  is convex. These facets are  $\phi$ -calibrable as a consequence of Theorem 6.4.

Concerning facets  $F_3(t)$ ,  $(F \setminus P)(t)$  and its opposite one: these are rectangular facets, and they are  $\phi$ -calibrable thanks to Remark 8.3.

Facets  $F_1(t)$  and  $F_4(t)$ : these are non rectangular facets. Notice that  $F_1 = F_1(0)$  satisfies  $\text{ess} - \sup_{\partial F_1} \kappa_{\tilde{\phi}}^{F_1} < P_{\tilde{\phi}}(F_1)/|F_1|$ , which implies that  $\text{ess} - \sup_{\partial F_1(t)} \kappa_{\tilde{\phi}}^{F_1(t)} < P_{\tilde{\phi}}(F_1(t))/|F_1(t)|$  for short times.  $\phi$ -calibrability of  $F_1(t)$  follows from Theorem 6.2.

The most delicate analysis requires the facet  $F_4(t)$ , since  $E(t)$  is neither convex nor concave at  $F_4(t)$ . It is possible to prove that  $F_4(t)$  is  $\phi$ -calibrable under the assumptions (8.3), for  $t > 0$  small enough.

### 8.0.6 Construction of the flow

We now want to show that  $E(t)$  is a crystalline mean curvature flow, in the sense of Definition 7.4. Each set  $E(t)$  is Lipschitz  $\phi$ -regular, since a vector field in  $\text{Nor}_{\phi}(\partial E(t); \mathbb{R}^3) \cap \text{Lip}(\partial E(t); \mathbb{R}^3)$  can be constructed by hand.

*Step 1. Construction of the velocity field  $\text{div}N(\cdot, 0)$  on  $\partial E$ .*

We construct a vector field  $N(\cdot, 0) \in \mathcal{H}_{\text{div}}(\partial E; \mathbb{R}^3)$  at time 0 as follows. Let  $Q$  be a facet of  $\partial E$ , consider  $N_{\min}^Q$ , and define, for  $\mathcal{H}^2$ -almost every  $x \in \text{int}(Q)$ , the two components of  $N(x, 0)$  lying in the plane  $\Pi_Q$  as  $N_{\min}^Q(x)$ . Add the proper constant third component on each  $\text{int}(Q)$  in such a way that the three-dimensional vector field (still denoted by  $N(\cdot, 0)$ ) belongs to  $\mathcal{H}_{\text{div}}(\partial E; \mathbb{R}^3)$ . The initial normal velocity of  $\partial E$  is then  $\text{div}N(\cdot, 0)$  on  $\text{int}(Q)$ . For the moment, this definition of velocity is not explicit.

*Step 2. Identification of  $\text{div}N(\cdot, 0)$  on facets different from  $F$  and its opposite one.*

Each facet  $Q$  of  $\partial E$  different from  $F$  and its opposite is  $\phi$ -calibrable. It follows that, on  $Q$ , the initial normal velocity equals the constant appearing on the right hand side of the partial

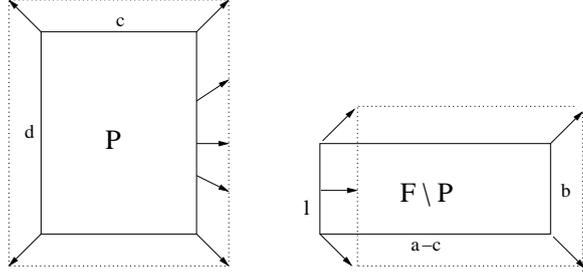


Figure 12:  $P$  and  $F \setminus P$  is a partition of the frontal facet  $F$  of  $\partial E$ . We depict the vector field  $\overline{N}$  on  $\partial P$  and on  $\partial(F \setminus P)$ . The construction is based on Remark 8.3.

differential equation in (8.2) expressing the divergence of the vector field, namely  $v_Q$ . This is a consequence of Remark 5.18 and the sentence after it.

To determine this constant we have to find the values of  $\alpha_i$ , i.e. the value of  $c_Q$  on each facet  $Q$ . We have

- (i)  $c_{F_6} = 1$  on  $\partial F_6$ ,  $c_{F_1} = 1$  on  $\partial F_1$ ,  $c_{F_2} = 1$  on  $\partial F_2$ ,  $c_{F_5} = 1$  on  $\partial F_5$ . Hence

$$\begin{aligned} \operatorname{div} N_{\min}^E &= \frac{2(d+e)}{de} \text{ on } \operatorname{int}(F_6) \\ \operatorname{div} N_{\min}^E &= \frac{2(a+e)}{ae} \text{ on } \operatorname{int}(F_1), \\ \operatorname{div} N_{\min}^E &= \frac{2(b+e)}{be} \text{ on } \operatorname{int}(F_2), \\ \operatorname{div} N_{\min}^E &= \frac{2(c+e)}{ce} \text{ on } \operatorname{int}(F_5). \end{aligned}$$

- (ii)  $c_{F_3} = 1$  on  $\partial F_3$  and  $c_{F_4} = 1$  on  $\partial F_4$  except that on  $]T, J[$ , where  $c_{F_3} = c_{F_4} = -1$ , see Figures 11 and 7.

*Step 3. Identification of  $\operatorname{div} N(\cdot, 0)$  on  $F$  and on its opposite facet.*

Let us consider the facet  $F$  (the arguments for the facet opposite to  $F$  are the same). We have  $c_F = 1$  on  $\partial F$ . We know that there does not exist a vector field defined on  $\operatorname{int}(F)$  having constant divergence, whose normal trace on  $\partial F$  is one and lying in  $T_{\phi^\circ}(\nu_{\phi^\circ}(F))$ .

Let us subdivide  $F$  into two rectangles  $P$  and  $F \setminus P$  as in Figure 11; in Figure 12 the two rectangles are depicted disjoint. We use the explicit construction of Remark 8.3 separately on  $P$  and  $F \setminus P$ , taking the constants  $\alpha_i$  as follows.

- On  $\partial P$  all  $\alpha_i$  are equal to one;
- on  $\partial(F \setminus P)$  the  $\alpha_i$  are equal to one except that on the dotted segment  $l$ , where the corresponding  $\alpha_j$  is equal to  $-1$ .

Remark 8.3 provides two vector fields

$$M_P : P \rightarrow \mathbb{R}^2, \quad M_{F \setminus P} : F \setminus P \rightarrow \mathbb{R}^2,$$

with the following properties:

- (a)  $M_P \in \mathcal{H}_{\operatorname{div}}(P; \Pi_F)$ ,  $\operatorname{div} M_P \equiv \frac{2(d+e)}{dc}$  on  $\operatorname{int}(P)$ ;

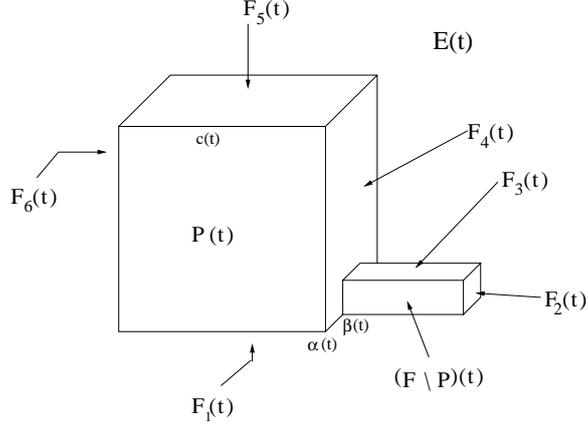


Figure 13: Picture of the set  $E(t)$  for small positive  $t$

$$(b) \ M_{F \setminus P} \in \mathcal{H}_{\text{div}}(F \setminus P; \Pi_F), \quad \text{div} M_{F \setminus P} \equiv \frac{2(a-c)}{b(a-c)} = \frac{2}{b}.$$

The vector field  $\overline{N}$  is explicit, since the construction in Remark 8.3 is explicit. Observe that

$$\text{div} M_P < \text{div} M_{F \setminus P}. \quad (8.4)$$

We let

$$\overline{N} := \begin{cases} M_P & \text{on } \text{int}(P), \\ M_{F \setminus P} & \text{on } \text{int}(F \setminus P). \end{cases}$$

It is interesting to observe that the component of  $\overline{N}$  in  $\Pi_F$  orthogonal to  $l$  is continuous along  $l$ , see Figure 12. On the other hand, the component of  $\overline{N}$  in  $\Pi_F$  tangent to  $l$  is discontinuous along  $l$ . It follows that

$$\overline{N} \text{ is discontinuous on } \text{int}(F),$$

and

$$\text{div} \overline{N} \in L^\infty(F).$$

In particular,  $\overline{N} \in \mathcal{H}_{\text{div}}(F; \Pi_F)$ .

Let us now check that  $\overline{N}$  satisfies the Euler-Lagrange inequality (5.28). The divergence of  $\overline{N}$  is constant on the interior of  $P$  and  $F \setminus P$ , and therefore, to check that (5.28) holds, we have to prove that

$$\frac{2(d+c)}{dc} \int_{\text{int}(P)} \text{div}(\overline{N} - N) \, d\mathcal{H}^2 + \frac{2}{b} \int_{\text{int}(F \setminus P)} \text{div}(\overline{N} - N) \, d\mathcal{H}^2 \leq 0 \quad \forall N \in \mathcal{H}_{\text{div}}(F; \Pi_F). \quad (8.5)$$

We have

$$\begin{aligned} & \frac{2(d+c)}{dc} \int_{\text{int}(P)} \text{div}(\overline{N} - N) \, d\mathcal{H}^2 + \frac{2}{b} \int_{\text{int}(F \setminus P)} \text{div}(\overline{N} - N) \, d\mathcal{H}^2 \\ &= \frac{2(d+c)}{dc} \int_{\partial P} \langle \tilde{\nu}^P, \overline{N} - N \rangle \, d\mathcal{H}^1 + \frac{2}{b} \int_{\partial(F \setminus P)} \langle \tilde{\nu}^{F \setminus P}, \overline{N} - N \rangle \, d\mathcal{H}^1 \\ &= \left( \frac{2(d+c)}{dc} - \frac{2}{b} \right) \int_l \langle \tilde{\nu}^P, \overline{N} - N \rangle \, d\mathcal{H}^1 \leq 0, \end{aligned} \quad (8.6)$$

since  $\frac{2(d+c)}{dc} - \frac{2}{b} < 0$  by (8.4), and since, by construction, the normal trace of  $\overline{N}$  on  $\text{int}(l)$  is maximal (in the direction of  $\tilde{\nu}^P$ ) among all vector fields satisfying the same constraints (see Figure 12), so that  $\langle \overline{N}(x) - N(x), \tilde{\nu}^P(x) \rangle \geq 0$  for  $\mathcal{H}^1$ -almost every  $x \in l$ .

Using Remark 5.18, we conclude that  $\overline{N}$  is a solution of (5.27), and therefore  $\text{div} \overline{N}$  is the  $\phi$ -mean curvature of  $\partial E$  on  $\text{int}(F)$ , and  $\text{div} \overline{N} = \text{div} N(\cdot, 0)$  on  $\text{int}(F)$ .

*Step 4. Construction of the normal velocity of  $\partial E(t)$ .*

Let us now consider the set  $E(t)$  for small positive times, constructed by flowing (shrinking) a generic facet  $L(t)$  of  $\partial E(t)$  with constant normal velocity equals to  $\frac{1}{|L(t)|} \int_{\partial L(t)} c_{L(t)} d\mathcal{H}^1$ .

Observe that all facets of  $\partial E(t)$  are  $\phi$ -calibrable, so they do not further subdivide. In addition, on each  $\text{int}(L(t))$  the normal velocity equals the divergence of a solution of (5.27) (where  $F$  is replaced by  $L(t)$ ).

Through *steps 1-4* we have constructed a flow starting from  $E$ . Actually, this is the unique crystalline mean curvature flow of  $E$  in a reasonably large class of flows.

**Remark 8.4.** The vector field  $N$  previously defined admits an extension (by lines) in  $U \times [0, T]$ , where  $U$  is a suitable open set containing  $\partial E(t)$ ,  $t \in [0, T]$ . More precisely, let  $y \in U$  and let  $x \in \partial E(t)$  be the unique point with the property that  $y$  belongs to the straight line  $\{x + sN(x, t)\}_{s \in \mathbb{R}}$  (this property is fulfilled if  $U$  is sufficiently thin, and for those points  $x$  where  $N(x, t)$  is continuous, hence if  $t > 0$  for all points, while if  $t = 0$  excluding points on the segment  $l$ ). Then we define  $N(y, t) := N(x, t)$ . With this definition the evolution that we have constructed is in the sense of Definition 7.4. It turns out that this evolution is unique in that class.

## 9 The reaction-diffusion approximation

Motion by mean curvature can be approximated by the zero-level sets of solutions of a singularly perturbed parabolic equation of Ginzburg-Landau type [67], [56]. This approximation result can be generalized to anisotropic and crystalline mean curvature flow, and several authors contributed to the final results, which are sometimes valid even after the onset of singularities (excluding fattening). A partial list of references can be found for instance in the papers [38], [29]. In this section we briefly recall the main statement in the crystalline case, and one of its consequences, namely the comparison principle, which implies a uniqueness result.

Assume that  $\phi^\circ$  is crystalline. Let us introduce the relaxed evolution law. Let  $\Omega \subset M$  be a smooth bounded open set. For  $s \in [-1, 1]$  let  $W(s) := (1 - s^2)^2$  and  $\psi := W'/2$ . We denote by  $\gamma$  the unique smooth strictly increasing function<sup>(35)</sup> exponentially asymptotic, at  $\pm\infty$ , to the two stable zeroes  $\pm 1$  of  $\psi$ , satisfying

$$-\gamma'' + \psi(\gamma) = 0, \quad \gamma(0) = 0. \quad (9.1)$$

Let  $\delta \geq 3$  be a fixed natural number such that, if for any  $\varepsilon \in (0, 1]$  we let  $\xi_\varepsilon := \delta |\log \varepsilon|$ , then  $\gamma(\pm\xi_\varepsilon) = \pm 1 + O(\varepsilon^{2\delta})$ ,  $\gamma'(\pm\xi_\varepsilon) = O(\varepsilon^{2\delta})$ . Denote by  $\gamma_\varepsilon$  a smooth increasing function which coincides with  $\gamma$  on  $[-\xi_\varepsilon, \xi_\varepsilon]$  and assumes the corresponding asymptotic values  $\pm 1$  outside the interval  $(-2\xi_\varepsilon, 2\xi_\varepsilon)$ .

Let  $\varepsilon \in (0, 1]$ ,  $T > 0$  and let  $u_0$  belong to the Sobolev space  $H^1(\Omega)$ , and suppose also that

$$\mathcal{E}_\phi(u_0) := \int_\Omega \phi^\circ(\nabla u_0)^2 + W(u_0) \, dx < +\infty.$$

Let us consider the problem

$$\begin{cases} \varepsilon u_t - \varepsilon \operatorname{div}(T_{\phi^\circ}(\nabla u)) + \frac{1}{\varepsilon} \psi(u) \ni 0 & \text{in } \Omega \times (0, T), \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega, \\ T_{\phi^\circ}(\nabla u) \cdot \nu^\Omega = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (9.2)$$

Let us define what is a solution to (9.2). For the definitions of parabolic spaces, we refer for instance to [71]. For an introduction to parabolic partial differential equations we refer for instance to [99].

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<sup>35</sup>An hyperbolic tangent.

**Definition 9.1 (Sub/super solutions).** A pair  $(u, \zeta)$  is a subsolution of (9.2) if, for any  $T > 0$ , the following properties hold:

(i)  $u \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$  and  $\zeta \in (L^2(\Omega \times (0, T)))^n$ ;

(ii) for any  $\varphi \in H^1(\Omega; [0, +\infty))$  and a.e.  $t \in (0, T)$

$$\int_{\Omega} \left( \varepsilon u_t \varphi + \varepsilon \zeta \cdot \nabla \varphi + \frac{1}{\varepsilon} \psi(u) \varphi \right) dx \leq 0; \quad (9.3)$$

(iii)  $u(x, 0) \leq u_0(x)$  for a.e.  $x \in \Omega$ ;

(iv) for a.e.  $(x, t) \in \Omega \times (0, T)$

$$\zeta(x, t) \in T_{\phi^o}(\nabla u(x, t)). \quad (9.4)$$

The pair  $(u, \zeta)$  is a supersolution of (9.2) if (i) and (iv) hold, and conditions (ii) and (iii) hold with  $\geq$  in place of  $\leq$ . The couple  $(u, \zeta)$  is a solution of (9.2) if it is both a subsolution and a supersolution.

By (i), (iv) and the one-homogeneity of  $T_{\phi^o}$ , it follows that  $\zeta \in L^\infty(0, T; (L^2(\Omega))^n)$ . The following results hold.

**Lemma 9.2 (Comparison).** Let  $(u^-, \zeta^-)$  and  $(u^+, \zeta^+)$  be respectively a subsolution and a supersolution of (9.2). Then  $u^- \leq u^+$  a.e. in  $\Omega \times (0, T)$ .

**Theorem 9.3 (Existence and uniqueness).** Problem (9.2) admits a solution  $(u, \zeta)$ . Moreover, if  $(u_1, \zeta_1)$  and  $(u_2, \zeta_2)$  are two solutions of (9.2), then  $u_1 = u_2$  a.e. in  $\Omega \times (0, T)$ .

## 9.1 Approximation and comparison principle

Following [30] and [32] we recall the convergence and comparison results.

**Theorem 9.4 (Convergence).** Let  $E(t)$  be a neighbourhood- $L^\infty$   $\phi$ -regular flow on  $[0, T]$ . For any  $\varepsilon > 0$  let  $u_\varepsilon$  be the solution of problem (9.2) with the  $\varepsilon$ -dependent initial datum

$$u_\varepsilon(x, 0) = u_\varepsilon^0(x) := \gamma_\varepsilon \left( \frac{d_\phi(x, 0)}{\varepsilon} \right), \quad (9.5)$$

where as usual  $d_\phi(x, 0) := \text{dist}_\phi(x, E(0)) - \text{dist}_\phi(x, M \setminus E(0))$ . Let  $\Sigma_\varepsilon(t)$  denote the zero level set of  $u_\varepsilon(\cdot, t)$ <sup>36</sup>. Then there exist  $\varepsilon_0 \in ]0, 1]$  and a constant  $C$  depending on  $(E(t))_{t \in [0, T]}$ , and independent of  $\varepsilon \in ]0, \varepsilon_0]$ , such that for any  $\varepsilon \in ]0, \varepsilon_0]$

$$\begin{aligned} \Sigma_\varepsilon(t) &\subset \{x \in \Omega : \text{dist}(x, \partial E(t)) \leq C\varepsilon |\log \varepsilon|^2\}, \\ \partial E(t) &\subset \{x \in \Omega : \text{dist}(x, \Sigma_\varepsilon(t)) \leq C\varepsilon |\log \varepsilon|^2\}, \end{aligned} \quad t \in [0, T]. \quad (9.6)$$

Using Lemma 9.2 and Theorem 9.4 it is possible to deduce the following result.

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<sup>36</sup>Since  $u_\varepsilon(\cdot, t)$  is not a priori a continuous function, this zero level set must be properly defined.

**Corollary 9.5 (Uniqueness).** *Let  $E_1(t)$  and  $E_2(t)$  be two neighbourhood- $L^\infty$   $\phi$ -regular flows on  $[0, T]$ . Then*

$$E_1(0) \subseteq E_2(0) \Rightarrow E_1(t) \subseteq E_2(t), \quad t \in [0, T].$$

*Hence  $E_1(0) = E_2(0) \Rightarrow E_1(t) = E_2(t)$  for any  $t \in [0, T]$ .*

As a consequence, a  $\phi$ -regular flow depends only on  $E(0)$ , hence it does not depend on the choice of the vector field which makes it neighbourhood- $L^\infty$   $\phi$ -regular.

**Remark 9.6.** We are not aware of a direct proof of the comparison principle for crystalline mean curvature flow in  $n \geq 3$  dimensions, without using the reaction-diffusion approximation.

## 10 Anisotropic functionals on partitions and crystalline flow of planar triods

Functionals defined on boundaries have a rather natural extension as functionals defined on bounded variation functions taking a finite number of values (sometimes called functionals on partitions)<sup>(37)</sup>. As in the two-phases case, we will not study such functionals in full generality<sup>(38)</sup>, and we will confine ourselves to the following particular situation: only one anisotropy  $\phi^o$  will be used, that will be assumed convex and spatially homogeneous. Only special partitions will be considered, consisting of a finite number of Lipschitz phases.

Let  $\phi \in \mathcal{M}(TM)$  be spatially homogeneous. By a Lipschitz hypersurface with Lipschitz boundary we mean a  $(n - 1)$ -dimensional set  $\Sigma \subset M$  which can be written locally as the graph of a Lipschitz function defined on an open subset of  $\mathbb{R}^{n-1}$ , and such that each point of its relative boundary  $\partial\Sigma$  can be written locally as the graph of a Lipschitz function defined on an open Lipschitz subset of  $\mathbb{R}^{n-2}$ . If  $x \in \Sigma$  (resp.  $x \in \partial\Sigma$ ) we denote by  $T_x(\Sigma)$  (resp.  $T_x(\partial\Sigma)$ ) the tangent space to  $\Sigma$  (resp. to  $\partial\Sigma$ ) at  $x$ . We also denote by  $\Pi_{T_x(\Sigma)}$  (resp.  $\Pi_{T_x(\partial\Sigma)}$ ) the orthogonal projection on  $T_x(\Sigma)$  (resp. on  $T_x(\partial\Sigma)$ ). Any Lipschitz function or vector field defined on  $\Sigma$  will be considered as defined up to  $\partial\Sigma$ .

Given a Lipschitz hypersurface  $\Sigma \subset M$  with boundary, we define

$$\mathcal{M}_\phi(\Sigma) := \int_\Sigma \phi^o(\nu) d\mathcal{H}^{n-1}, \quad (10.1)$$

where  $\nu(x)$  is a euclidean unit normal vector to  $\Sigma$  at  $\mathcal{H}^{n-1}$ -almost every  $x \in \Sigma$ .

**Definition 10.1 (Lipschitz partitions).** *A Lipschitz (resp. smooth) partition of  $M$  is a finite family  $\{E_i\}_i$  of subsets<sup>(39)</sup> of  $M$  such that  $\cup_i \bar{E}_i = \mathbb{R}^n$ ,  $E_i \cap E_j = \emptyset$  for  $i \neq j$ , and  $\partial E_i \cap \partial E_j$ , when it is nonempty, is a Lipschitz (resp. smooth) hypersurface with Lipschitz (resp. smooth) boundary, called interface. If  $n = 2$ , by a  $m$ -multiple junction of  $\{E_i\}$  ( $m \geq 3$  a natural number) we mean a point  $q$  belonging to  $m$  distinct interfaces. If in addition  $m = 3$  we say that  $q$  is a triple junction of  $\{E_i\}$ .*

Given a Lipschitz partition  $\{E_i\}$  of  $M$ , we set

$$\Sigma_{ij} := \partial E_i \cap \partial E_j, \quad i \neq j, \quad \Gamma := \bigcup_{i,j} \Sigma_{ij}, \quad J := \bigcup_{i,j} \partial \Sigma_{ij}, \quad (10.2)$$

where  $\partial \Sigma_{ij}$  is the relative boundary of  $\Sigma_{ij}$ , and

$$\mathcal{M}_\phi(\Gamma) := \sum_{i,j} \mathcal{M}_\phi(\Sigma_{ij}). \quad (10.3)$$

We denote by  $\nu^{ij}$  a  $\mathcal{H}^{n-1}$ -a.e. defined euclidean unit normal to  $\Sigma_{ij}$  and we set  $\nu_{\phi^o}^{ij} := \nu^{ij} / \phi^o(\nu^{ij})$ . For notational simplicity, when  $n = 2$  the sets  $\partial E_i \cap \partial E_j$  are also denoted by  $\Sigma_k$ ,

<sup>37</sup>These latter functionals are in turn generalized by functionals defined on special functions of bounded variation, such as the Mumford-Shah functional [11].

<sup>38</sup>See [11], [53], [54], [55] and references therein.

<sup>39</sup>Called phases.

using one index only, and  $\nu_{\phi^o}^{ij}$  will be denoted by  $\nu_{\phi^o}^k$ . When  $n = 2$  the set  $\Gamma$  is sometimes called network.

When  $\{E_1, E_2, E_3\}$  is a partition of  $\mathbb{R}^2$  into three sets having only one triple junction (denoted by  $q$ ) the set  $\Gamma$  defined in (10.2) will be called triod, and denoted by  $\Pi$ . If the partition is Lipschitz  $\phi$ -regular in the sense of Definition 10.8 below, the triod is said to be Lipschitz  $\phi$ -regular. We call angles of  $\Pi$  the three angles at  $q$  between  $\Sigma_1, \Sigma_2, \Sigma_3$ .

## 10.1 First variation

Assume  $\phi \in \mathcal{M}_{\text{reg}}(TM)$  is spatially homogeneous. We assume that  $\Sigma$  is a  $(n-1)$ -dimensional smooth bounded embedded orientable manifold with (smooth) boundary.  $\nu$  is a smooth euclidean unit normal vector field to  $\Sigma$ , smoothly defined up to  $\partial\Sigma$ . We define, at each point of  $\bar{\Sigma}$ ,  $\nu_{\phi^o} := \nu/\phi^o(\nu)$ ,  $n_\phi := T_{\phi^o}(\nu_{\phi^o})$ , and on  $\Sigma$  the  $\phi$ -mean curvature  $\kappa_\phi$  of  $\Sigma$  as  $\kappa_\phi := \text{div}_\tau n_\phi$ .

**Definition 10.2 ( $\phi$ -conormal vector).** *We denote by  $n_\phi^{\partial\Sigma} : \partial\Sigma \rightarrow M$  the vector field defined as follows: if  $x \in \partial\Sigma$  then*

$$(i) \quad n_\phi^{\partial\Sigma}(x) \in \left\{ \text{span}\left(T_x(\partial\Sigma), n_\phi(x)\right) \right\}^\perp;$$

$$(ii) \quad |n_\phi^{\partial\Sigma}(x)| = |n_\phi(x) - \Pi_{T_x(\partial\Sigma)} n_\phi(x)|;$$

$$(iii) \quad n_\phi^{\partial\Sigma}(x) \text{ points out of } \Sigma.$$

Observe that  $\dim \left\{ \text{span}\left(T_x(\partial\Sigma), n_\phi(x)\right) \right\}^\perp = 1$ , since  $n_\phi(x)$  and  $T_x(\partial\Sigma)$  are linearly independent, as a consequence of  $\langle \nu_{\phi^o}(x), n_\phi(x) \rangle = 1$ .

If  $\phi(\xi) = |\xi|$ , then  $n_\phi^{\partial\Sigma}$  is the usual conormal unit euclidean vector pointing out of  $\Sigma$ . Note also that in  $n = 2$  dimensions condition (i) reduces to  $n_\phi^{\partial\Sigma}(x) \cdot n_\phi(x) = 0$ , and condition (ii) reduces to  $|n_\phi^{\partial\Sigma}(x)| = |n_\phi(x)|$ .

### 10.1.1 The smooth 2-dimensional case

In this subsection we assume  $n = 2$  (hence  $M = \mathbb{R}^2$ ) and we compute the first variation of  $\mathcal{M}_\phi$  using a parametric approach [37], for  $\phi \in \mathcal{M}_{\text{reg}}(TM)$ .

**Theorem 10.3 (Curves with boundary).** *Let  $\Sigma \subset \mathbb{R}^2$  be a smooth simple curve with boundary  $\partial\Sigma = \{p, q\}$ . Let  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  be a regular parametrization of  $\Sigma$  with  $\alpha(0) = p$  and  $\alpha(1) = q$ . Let  $\beta \in \mathcal{C}^2([0, 1]; \mathbb{R}^2)$ ,  $\lambda \in \mathbb{R}$ , and let  $\Sigma_\lambda$  be the curve parametrized by  $\alpha + \lambda\beta$ . Then*

$$\frac{d}{d\lambda} \mathcal{M}_\phi(\Sigma_\lambda)|_{\lambda=0} = \int_\Sigma \kappa_\phi \nu_{\phi^o} \cdot \beta \phi^o(\nu) d\mathcal{H}^1 + n_\phi^{\partial\Sigma}(q) \cdot \beta(1) + n_\phi^{\partial\Sigma}(p) \cdot \beta(0). \quad (10.4)$$

*Proof.* Set  $\tau := \frac{\alpha'}{|\alpha'|}$  and  $\nu := \tau^\perp$ , where  $\perp$  is the counterclockwise rotation of  $\pi/2$ . Recalling (10.1) we have

$$\begin{aligned} \frac{d}{d\lambda} \mathcal{M}_\phi(\Sigma_\lambda)|_{\lambda=0} &= \frac{d}{d\lambda} \int_0^1 \phi^\circ \left( (\alpha' + \lambda\beta')^\perp \right) dt|_{\lambda=0} \\ &= \int_0^1 \phi_\xi^\circ(\nu) \cdot (\beta^\perp)' dt = - \int_0^1 \frac{d}{dt}(\phi_\xi^\circ(\nu)) \cdot \beta^\perp dt - \phi_\xi^\circ(\nu(q))^\perp \cdot \beta(1) + \phi_\xi^\circ(\nu(p))^\perp \cdot \beta(0). \end{aligned} \quad (10.5)$$

We now observe that  $\beta^\perp = -\beta \cdot \nu \tau + \beta \cdot \tau \nu$ . Moreover,  $\phi_\xi^\circ(\nu) = n_\phi$  by definition, and from [39] we have  $\phi_{\xi\xi}^\circ(\nu) \tau \cdot \nu = 0$  and  $\kappa_\phi = \kappa \phi_{\xi\xi}^\circ(\nu) \tau \cdot \tau$ , where  $\kappa$  is the euclidean curvature. Therefore, using  $\frac{d\nu}{dt} = \frac{d\nu}{ds} \frac{ds}{dt} = \kappa \nu |\alpha'|$  where  $s$  is the arclength parameter, we have

$$\int_0^1 \frac{d}{dt}(\phi_\xi^\circ(\nu)) \cdot \beta^\perp dt = - \int_0^1 \kappa \phi_{\xi\xi}^\circ(\nu) \tau \cdot \tau \nu \cdot \beta |\alpha'| dt = - \int_\Sigma \kappa_\phi \nu_{\phi^\circ} \cdot \beta \phi^\circ(\nu) d\mathcal{H}^1. \quad (10.6)$$

Then (10.4) follows from (10.5) and (10.6).  $\square$

**Corollary 10.4 (Networks).** *Let  $\{E_i\}$  be a smooth partition of  $\mathbb{R}^2$  and let  $q$  be a  $m$ -multiple junction of  $\{E_i\}$ ,  $m \geq 3$ . Let  $\Sigma_1, \dots, \Sigma_m$  be the  $m$  arcs of the partitions meeting at  $q$ . Let  $\alpha_i : [0, 1] \rightarrow \mathbb{R}^2$  be a regular parametrization of  $\Sigma_i$  such that  $\alpha_i(1) = q$  for any  $i = 1, \dots, m$ . Let  $\beta_i \in \mathcal{C}^2([0, 1]; \mathbb{R}^2)$  be such that  $\beta_i(0) = 0$  and  $\beta_i(1) = \beta_j(1) =: \beta(1)$  for every  $i, j \in \{1, \dots, m\}$ , let  $\lambda \in \mathbb{R}$  and  $\Sigma_\lambda^i$  be the curve parametrized by  $\alpha_i + \lambda\beta_i$  and  $\Gamma_\lambda := \bigcup_{i=1}^m \Sigma_\lambda^i$ . Then*

$$\frac{d}{d\lambda} \mathcal{M}_\phi(\Gamma_\lambda)|_{\lambda=0} = \int_\Gamma \kappa_\phi \nu_\phi \cdot \beta \phi^\circ(\nu) d\mathcal{H}^1 + \beta(1) \cdot \sum_{i=1}^m n_\phi^{\partial\Sigma_i}(q). \quad (10.7)$$

*In particular, if for any  $\beta_i$  as above we have  $\frac{d}{d\lambda} \mathcal{M}_\phi(\Gamma_\lambda)|_{\lambda=0} = 0$ , then each  $\Sigma_i$  has zero  $\phi$ -mean curvature, and*

$$\sum_{i=1}^m n_\phi^{\partial\Sigma_i}(q) = 0. \quad (10.8)$$

### 10.1.2 The smooth $n$ -dimensional case

In this subsection we assume  $n \geq 2$  and we state the first variation of  $\mathcal{M}_\phi$  [37]. Let  $\Psi_\lambda$  and  $X$  be as in Section 5.1.

**Theorem 10.5 (First variation: manifolds with boundary).** *Let  $\Sigma \subset M$  be a smooth hypersurface with boundary. Set  $\Sigma_\lambda := \Psi_\lambda(\Sigma)$ . Then*

$$\frac{d}{d\lambda} \mathcal{M}_\phi(\Sigma_\lambda)|_{\lambda=0} = \int_\Sigma \kappa_\phi \nu_{\phi^\circ} \cdot X \phi^\circ(\nu) d\mathcal{H}^{n-1} + \int_{\partial\Sigma} n_\phi^{\partial\Sigma} \cdot X d\mathcal{H}^{n-2}. \quad (10.9)$$

**Remark 10.6.** If  $n = 2$ , the right hand side of (10.9) reduces to right hand side of (10.4).

**Corollary 10.7 (Partitions).** *Let  $\{E_i\}$  be a smooth partition of  $M$ . Set  $\Sigma_\lambda^{ij} := \Psi_\lambda(\Sigma_{ij})$  and  $\Gamma_\lambda := \bigcup_{i=1}^m \Sigma_\lambda^{ij}$ . Then*

$$\frac{d}{d\lambda} \mathcal{M}_\phi(\Gamma_\lambda)|_{\lambda=0} = \int_\Gamma \kappa_\phi \nu_\phi \cdot X \phi^\circ(\nu) d\mathcal{H}^{n-1} + \int_\Gamma \left( \sum_{i,j} n_\phi^{\partial\Sigma_{ij}} \right) \cdot X d\mathcal{H}^{n-2}. \quad (10.10)$$

*In particular, if  $\frac{d}{d\lambda} \mathcal{M}_\phi(\Gamma_\lambda)|_{\lambda=0} = 0$ , then each  $\Sigma_{ij}$  has zero  $\phi$ -mean curvature and the balance condition holds:*

$$\sum_{i,j} n_\phi^{\partial\Sigma_{ij}} = 0 \quad \text{on } \Gamma. \quad (10.11)$$

From now on, up to the end of the notes, we will assume  $n = 2$  (so that  $M = \mathbb{R}^2$ ) and  $\phi$  crystalline.

### 10.1.3 The crystalline case in $n = 2$ dimensions

We denote by  $\text{Lip}_{\nu,\phi}(\Gamma; \mathbb{R}^2)$  the space of vector fields  $N : \Gamma \rightarrow \mathbb{R}^2$  such that  $N|_{\Sigma_{ij}} \in \text{Lip}(\Sigma_{ij}; \mathbb{R}^2)$  and  $N|_{\Sigma_{ij}}(x) \in T_{\phi^\circ}(\nu_{\phi^\circ}^{ij}(x))$  for  $\mathcal{H}^1$ -almost every  $x \in \Sigma_{ij}$ . Set

$$\mathcal{N} := \left\{ N \in \text{Lip}_{\nu,\phi}(\Gamma, \mathbb{R}^2) : \sum_{i,j} (N|_{\Sigma_{ij}})^{\partial\Sigma_{ij}} = 0 \text{ on } J \right\}. \quad (10.12)$$

See the appendix for more on the balance condition.

**Definition 10.8 (Lipschitz  $\phi$ -regular partitions).** *If  $\mathcal{N} \neq \emptyset$ , the partition  $\{E_i\}$  is said to be Lipschitz  $\phi$ -regular.*

We now want to define the  $\phi$ -mean curvature. if  $\{E_i\}_i$  is a Lipschitz  $\phi$ -regular partition of  $M$  then the minimum problem

$$\min \left\{ \int_\Gamma (\text{div}_\tau N)^2 \phi^\circ(\nu) d\mathcal{H}^1 : N \in \mathcal{N} \right\} \quad (10.13)$$

admits a unique<sup>(40)</sup> solution which identifies the direction along which the functional (10.3) decreases most quickly. Let  $N_{\min} : \Gamma \rightarrow \mathbb{R}^2$  be the solution of problem (10.13).

**Definition 10.9 (Crystalline curvature of a network).** *Let  $\{E_i\}$  be a Lipschitz  $\phi$ -regular partition. We define the  $\phi$ -curvature  $\kappa_\phi$  of  $\Gamma$  as*

$$\kappa_\phi := \text{div}_\tau N_{\min}, \quad \text{a.e. on } \Gamma.$$

## 10.2 Triods

In this section we report some results on triods from [28]. We denote by  $\mathbf{n}$  a positive integer and we assume that  $B_\phi = P_{\mathbf{n}}$ , where  $P_{\mathbf{n}}$  denotes the regular polygon of  $\mathbf{n}$  ( $\mathbf{n}$  even) sides of length  $L$  inscribed in the unit circle centered at the origin of  $\mathbb{R}^2$ , having two horizontal sides and oriented in clockwise sense.

<sup>40</sup>Remember that we are considering partitions in the plane.

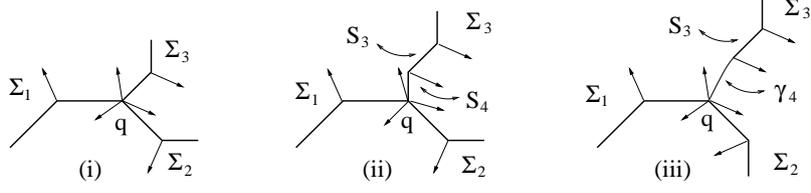


Figure 14: (i) Elementary, (ii) quasi-elementary, (iii) non-polygonal triod ( $B_\phi = P_8$ ). Note that  $\kappa_\phi = 0$  on  $S_3$  in (i) and (ii),  $\kappa_\phi < 0$  on  $S_4$  in (ii), and  $\kappa_\phi = 0$  on  $\gamma_4$  in (iii).

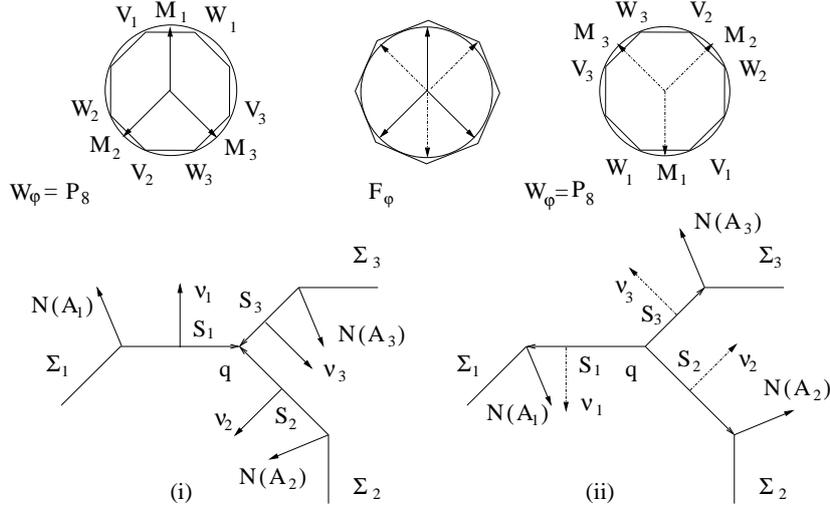


Figure 15: These triods have the same evolution according to system (10.18). Our convention is to take the orientation as in (i).

**Definition 10.10 (Elementary, quasi-elementary and non-polygonal triods).** Let  $\Pi = \cup_{j=1}^3 \Sigma_j$  be a Lipschitz  $\phi$ -regular triod. We say that  $\Pi$  is elementary if

- ( $\mathcal{E}$ ) each interface  $\Sigma_j$  is the union of a segment  $S_j$  of finite length  $L_j > 0$  and a half-line  $R_j$  such that  $S_j$  and  $R_j$  reproduce two consecutive sides of  $B_\phi$ , see Figure 14 (i).

We say that  $\Pi$  is degenerate if two interfaces satisfy ( $\mathcal{E}$ ) and the remaining one is a half-line. We say that  $\Pi$  is quasi-elementary if two interfaces satisfy ( $\mathcal{E}$ ) and the remaining one  $\Sigma_k$  is the union of two segments  $S_4$  and  $S_k$  of finite lengths,  $L_4 > 0$  and  $L_k > 0$  respectively, and a half-line  $R_k$  such that  $S_4$  and  $S_k$ , and  $S_k$  and  $R_k$ , reproduce two consecutive sides of  $B_\phi$ , see Figure 14 (ii).

We say that  $\Pi$  is non-polygonal if two interfaces satisfy ( $\mathcal{E}$ ) and the remaining one  $\Sigma_k$  is the union of a curve  $\gamma_4$ , a segment  $S_k$  of finite length  $L_k > 0$  and a half-line  $R_k$  such that  $S_k$  and  $R_k$  reproduce two consecutive sides of  $B_\phi$ , see Figure 14 (iii).

Given a triod  $\Pi$  and  $N \in \mathcal{N}$ , we set  $A_j := \overline{S_j} \cap \overline{R_j}$  for any  $j = 1, 2, 3$  such that  $R_j \neq \emptyset$ ,  $A_4 := \overline{S_4} \cap \overline{S_k}$  if  $\Pi$  is quasi-elementary, and  $A_4 := \overline{\gamma_4} \cap \overline{S_k}$  if  $\Pi$  is non-polygonal.

*Conventions:* let  $\nu$  be the  $\mathcal{H}^1$ -almost everywhere defined euclidean unit normal to  $\Pi$  oriented in such a way that  $\nu_{|\text{int}(S_j)} \cdot N(A_j) > 0$ . We set  $\nu_j := \nu_{|\text{int}(S_j)}$ ,  $\tau_j := -\nu_j^\perp$  and  $l_j := L_j \tau_j$ , for any  $j = 1, 2, 3$ , and also  $j = 4$  if  $\Pi$  is quasi-elementary. Thus  $\{\tau_j, \nu_j\}$  is a positively oriented basis of  $\mathbb{R}^2$  and, without loss of generality, we assume that each  $l_j$  points towards  $q$ . We denote by  $\kappa_\phi(l_j)$  the  $\phi$ -curvature of  $S_j$ .

For an elementary triod, we assume that  $S_1$  is horizontal and  $\Sigma_2$  and  $\Sigma_3$  are given in counterclockwise sense as in Figure 15. We denote by  $V_j, W_j$  the vertices of the side of  $P_n$  (in clockwise sense) having  $\nu_j$  as outer normal and by  $M_j$  the middle point of the segment  $[V_j, W_j]$ . Note that

$$\tau_1 \cdot \nu_3 = -\tau_1 \cdot \nu_2, \quad \nu_1 \cdot \tau_3 = -\nu_1 \cdot \tau_2, \quad \tau_1 \cdot \nu_3 = -\nu_1 \cdot \tau_3. \quad (10.14)$$

We recall the notion of stability [37].

**Definition 10.11 (Stable triods).** *Let  $\Pi$  be a  $\phi$ -regular triod. We say that  $\Pi$  is stable if  $(N_{\min})_{|\Sigma_j}(q)$  is not a vertex of  $B_\phi$  for any  $j = 1, 2, 3$ . We say that  $\Pi$  is unstable if it is not stable.*

Non-polygonal triods are always unstable, while elementary, degenerate and quasi-elementary triods can be either stable or unstable.

### 10.3 Crystalline flows of triods

As usual, given two parallel (possibly infinite) segments  $S_1, S_2$ , we call the distance vector of  $S_2$  from  $S_1$  the vector having norm  $\text{dist}(S_1, S_2)$  pointing from  $S_1$  to  $S_2$ .

**Definition 10.12.** *Let  $T > 0$  and  $\Pi$  be an elementary triod (resp. degenerate). For any  $t \in [0, T]$ , let  $\Pi(t)$  be a Lipschitz  $\phi$ -regular triod and  $q(t)$  its triple junction. We say that  $t \in [0, T] \mapsto \Pi(t)$  is a  $\phi$ -curvature flow starting from  $\Pi = \Pi(0)$  if for any  $t \in (0, T)$*

- (i)  $\Pi(t)$  is either elementary or quasi-elementary or non-polygonal (resp. degenerate);
- (ii) for any  $j = 1, 2, 3$ , each  $R_j(t)$  has zero normal velocity and each  $S_j(t)$  is parallel to  $S_j(0) = S_j$ ;
- (iii) for each  $j = 1, 2, 3$ , and also  $j = 4$  if  $\Pi(t)$  is quasi-elementary, denoting by  $h_j(t)$  the distance vector of the segment  $S_j(t)$  from  $S_j(0) = S_j$ , then  $h_j \in C^1([0, T]; \nu_j \mathbb{R})$  and

$$\begin{cases} \dot{h}_j(t) \\ \phi^\circ(\nu_j) \end{cases} = -\kappa_\phi(l_j(t)) \nu_j \quad (10.15)$$

$$h_j(0) = 0.$$

**Remark 10.13.** Since  $\phi^\circ(\nu_j)$  is a constant independent of  $j \in \{1, 2, 3, 4\}$ , the system in (10.16) is equivalent, up to a rescaling in time, to

$$\dot{h}_j(t) = -\kappa_\phi(l_j(t)) \nu_j. \quad (10.16)$$

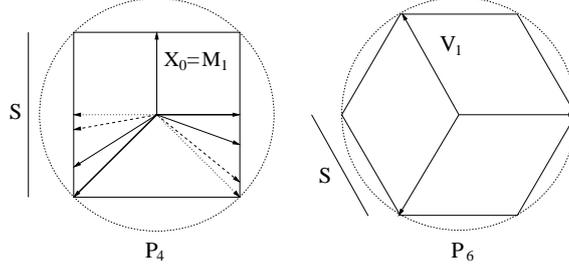


Figure 16:  $P_4$  admits infinitely many unordered pairs  $\{Y, Z\}$  satisfying  $X_0 + Y + Z = 0$  in correspondence of  $X_0 = M_1$ .  $P_6$  has a unique pair in correspondence of all  $X \in \partial P_6$ .

Let

$$h_j^\nu(t) := h_j(t) \cdot \nu_j, \quad \text{for } j = 1, 2, 3, 4. \quad (10.17)$$

Then  $h_j(t) = h_j^\nu(t) \nu_j$  and, with this notation, system (10.16) becomes

$$\begin{cases} \dot{h}_j^\nu(t) = -\kappa_\phi(l_j(t)) = -\frac{1}{L_j(t)} \left[ N_{\min|\Sigma_j(t)}(q(t)) - N_{\min}(A_j(t)) \right] \cdot \tau_j \\ h_j^\nu(0) = 0. \end{cases} \quad (10.18)$$

**Remark 10.14.**  $S_j(t)$  moves in the same direction of  $\nu_j$  if and only if  $\kappa_\phi(l_j(t)) < 0$ . Furthermore, system (10.18) is invariant under the change of the orientation of  $\Pi(t)$  (see Figure 15).

Finally, it is possible to prove the following short time existence and uniqueness theorem for the  $\phi$ -curvature flow of a triod.

**Theorem 10.15.** *Let  $\Pi$  be elementary and stable. Then there exist  $T > 0$  and a unique stable  $\phi$ -curvature flow  $t \in [0, T] \mapsto \Pi(t)$  starting from  $\Pi$  for any  $t \in [0, T]$ .*

## 10.4 Appendix

The angles of an elementary triod are given by the angles between the vectors  $\nu_j$ 's and are determined by the balance condition at  $q$  (see (10.12)) that, in turn, is related to the existence of admissible triplets.

**Definition 10.16 (Admissible triplets).** *We call admissible triplet any triplet of vectors  $(X, Y, Z) \in (\partial B_\phi)^3$  satisfying*

$$X + Y + Z = 0. \quad (10.19)$$

It is possible to prove the following result.

**Lemma 10.17 (Geometry of admissible triplets).** *Let  $\psi : \mathbb{R}^2 \rightarrow [0, +\infty)$  be a convex norm on  $\mathbb{R}^2$ . Let  $X \in \partial B_\psi$ . Then there exist two distinct vectors  $Y, Z$  in  $\partial B_\psi$  such that  $(X, Y, Z)$  is an admissible triplet. Moreover, if either  $B_\psi$  is strictly convex or for any segment  $S \subset \partial B_\psi$  parallel to  $X \in \partial B_\psi$  we have  $|S| \leq |X|$ , then the unordered pair  $\{Y, Z\}$  is unique. Finally, if there exist  $X_0 \in \partial B_\psi$  and a segment  $S \subset \partial B_\psi$  parallel to  $X_0$  with  $|S| > |X_0|$ , then there are infinitely many unordered pairs  $\{Y, Z\}$  of distinct vectors in  $\partial B_\psi$  such that  $(X_0, Y, Z)$  is an admissible triplet.*

**Example 10.18.** If  $B_\psi = P_4$  and  $X_0 = M_1$  (see Figure 16), then  $|S| = 2|X_0|$ ; hence there are infinitely many pairs  $\{Y, Z\}$  of distinct vectors in  $\partial P_4$  satisfying  $X_0 + Y + Z = 0$ . Moreover, any elementary triod has always two angles of  $\pi/2$ . If  $B_\psi = P_6$  and  $X = V_1$  (see Figure 16), then  $|S| = |V_1|$ ; hence for any  $X \in B_\psi$  there exists a unique unordered pair  $\{Y, Z\}$  satisfying (10.19).

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# THE TOTAL VARIATION FLOW\*

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## Abstract

We summarize in this lectures some of our results about the Minimizing Total Variation Flow, which have been mainly motivated by problems arising in Image Processing. First, we recall the role played by the Total Variation in Image Processing, in particular the variational formulation of the restoration problem. Next we outline some of the tools we need: functions of bounded variation (Section 2), pairing between measures and bounded functions (Section 3) and gradient flows in Hilbert spaces (Section 4). Section 5 is devoted to the Neumann problem for the Total variation Flow. Finally, in Section 6 we study the Cauchy problem for the Total Variation Flow.

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# 1 The Total Variation Flow in Image Processing

We suppose that our image (or data)  $u_d$  is a function defined on a bounded and piecewise smooth open set  $D$  of  $\mathbb{R}^N$  - typically a rectangle in  $\mathbb{R}^2$ . Generally, the degradation of the image occurs during image acquisition and can be modeled by a linear and translation invariant blur and additive noise. The equation relating  $u$ , the real image, to  $u_d$  can be written as

$$u_d = Ku + n, \quad (1)$$

where  $K$  is a convolution operator with impulse response  $k$ , i.e.,  $Ku = k * u$ , and  $n$  is an additive white noise of standard deviation  $\sigma$ . In practice, the noise can be considered as Gaussian.

The problem of recovering  $u$  from  $u_d$  is ill-posed. First, the blurring operator need not be invertible. Second, if the inverse operator  $K^{-1}$  exists, applying it to both sides of (1) we obtain

$$K^{-1}u_d = u + K^{-1}n. \quad (2)$$

Writing  $K^{-1}n$  in the Fourier domain, we have

$$K^{-1}n = \left( \frac{\hat{n}}{\hat{k}} \right)^\vee$$

where  $\hat{f}$  denotes the Fourier transform of  $f$  and  $f^\vee$  denotes the inverse Fourier transform. From this equation, we see that the noise might blow up at the frequencies for which  $\hat{k}$  vanishes or it becomes small.

The typical strategy to solve this ill-conditioning is regularization. Then the solution of (1) is estimated by minimizing a functional

$$J_\gamma(u) = \| Ku - u_d \|_2^2 + \gamma \| Qu \|_2^2, \quad (3)$$

which yields the estimate

$$u_\gamma = (K^t K + \gamma Q^t Q)^{-1} K^t u_d, \quad (4)$$

$Q$  being a regularization operator.

The first regularization method consisted in choosing between all possible solutions of (2) the one which minimized the Sobolev (semi) norm of  $u$

$$\int_D |Du|^2 dx,$$

which corresponds to the case  $Qu = \nabla u$ . Then the solution of (3) given by (4) in the Fourier domain is given by

$$\hat{u} = \frac{\bar{\hat{k}}}{|\hat{k}|^2 + 4\gamma\pi^2|\xi|^2} \hat{u}_d.$$

From the above formula we see that high frequencies of  $u_d$  (hence, the noise) are attenuated by the smoothness constraint. This was an important step, but the results were not satisfactory, mainly due to the inability of the previous functional to resolve discontinuities (edges) and oscillatory textured patterns. The smoothness constraint is too restrictive. Indeed, functions in  $W^{1,2}(D)$  cannot have discontinuities along rectifiable curves. These observations motivated the introduction of Total Variation in image restoration models by L. Rudin, S. Osher and E. Fatemi in their seminal work [23]. The a priori hypothesis is that functions of bounded variation (the *BV* model) [2],[13],[24]) are a reasonable functional model for many problems in image processing, in particular, for restoration problems ([22],[23]). Typically, functions of bounded variation have discontinuities along rectifiable curves, being continuous in some sense (in the measure theoretic sense) away from discontinuities. The discontinuities could be identified with edges.

On the basis of the *BV*-model, Rudin-Osher-Fatemi [23] proposed to solve the following constrained minimization problem

$$\begin{aligned} & \text{Minimize } \int_D |Du| \, dx \\ & \text{with } \int_D Ku = \int_D u_d, \quad \int_D |Ku - u_d|^2 \, dx = \sigma^2 |D|. \end{aligned} \tag{5}$$

The first constraint corresponds to the assumption that the noise has zero mean, and the second that its standard deviation is  $\sigma$ . The constraints are a way to incorporate the image acquisition model given in terms of equation (1). Under some assumption

$$\left\| u_d - \int_{\Omega} u_d \right\| \geq \sigma^2,$$

the constraint

$$\int_D |Ku - u_d|^2 \, dx = \sigma^2 |D| \tag{6}$$

is equivalent to the constraint

$$\int_D |Ku - u_d|^2 \, dx \leq \sigma^2 |D|,$$

which amounts to say that  $\sigma$  is an upper bound of the standard deviation of  $n$ . Moreover, assuming that  $K1 = 1$ , the constraint  $\int_D Ku = \int_D u_d$  is automatically satisfied [10].

In practice, the above problem is solved via the following unconstrained minimization problem

$$\text{Minimize } \int_{\Omega} |Du| \, dx + \frac{\lambda}{2} \int_{\Omega} |Ku - u_d|^2 \, dx \tag{7}$$

for some Lagrange multiplier  $\lambda$ .

The most successful analysis of the connections between (5) and (7) was given by A. Chambolle and P.L. Lions in [10]. Indeed, they proved that both problems are equivalent for some positive value of the Lagrange multiplier  $\lambda$ .

Let us define the functional  $\Phi : L^2(\Omega) \rightarrow (-\infty, +\infty]$  by

$$\Phi(u) = \begin{cases} \int_{\Omega} \|Du\| & \text{if } u \in BV(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \quad (8)$$

**Proposition 1** *If  $u$  is a solution of (5), then there is some  $\lambda \geq 0$  such that*

$$-\lambda K^t(Ku - u_d) \in \partial\Phi(u). \quad (9)$$

In particular, the Euler-Lagrange equation associated with the denoising problem, that is, for problem (5) with  $K = I$ , is the equation

$$-\lambda(u - u_d) \in \partial\Phi(u). \quad (10)$$

Formally,

$$\partial\Phi(u) = -\operatorname{div} \left( \frac{Du}{|Du|} \right).$$

Now, the problem is to give a sense to (10) as a partial differential equation, describing the subdifferential of  $\Phi$  in a distributional sense.

Motivated by the image restoration problem we initiated in [3] the study of the minimizing total variation flow  $u_t = \operatorname{div}(\frac{Du}{|Du|})$ . Indeed, this PDE is the gradient descent associated to the energy

$$\int_{\Omega} |Du|.$$

Observe that we are not considering the constraints given by the image acquisition model in this simplified energy. Thus our conclusions will not directly inform us about the complete model (5). Instead, our purpose was to understand how the minimizing total variation flow minimizes the total variation of a function. There are many flows which minimize the total variation of a function. Let us mention in particular the mean curvature motion ([21])

$$\frac{\partial u}{\partial t} = |Du| \operatorname{div} \left( \frac{Du}{|Du|} \right). \quad (11)$$

Indeed, this flow corresponds to the motion of curves in  $\mathbb{R}^2$  or hypersurfaces  $S(t)$  in  $\mathbb{R}^N$  by mean curvature, i.e.,

$$X_t = H\vec{N} \quad (12)$$

where  $X$  denotes a parametrization of  $S(t)$ ,  $H$  denotes its mean curvature and  $\vec{N}$  the outer unit normal. The classical motion given by (12) corresponds to the gradient descent of the area functional  $\int_S dS$ . Both flows, the classical mean curvature motion (12), and its viscosity solution (11) formulation have been studied by many authors, we refer in

particular to the work by L.C. Evans and J. Spruck [14]. They proved, in particular, that the total variation of the (viscosity) solution of (11) decreases during the evolution, as it should happen since the flow decreases the  $(N - 1)$  Hausdorff measure of the level set surfaces of the solution  $u$  and the total variation corresponds to the integral of the  $(N-1)$  Hausdorff measure of the boundaries of the level sets. Let us compare the behaviour of the minimizing total variation flow with respect to the mean curvature motion flow. The viscosity solution formulation on the classical mean curvature motion has to be interpreted as follows. If  $S(t)$  is a surface moving by mean curvature with initial condition  $S(0)$ , and  $u(0, x)$  is the signed distance to  $S(0)$ , i.e., if  $u(0, x) = d(x, S(0))$  when  $x$  is outside  $S(0)$ , and  $u(0, x) = -d(x, S(0))$  if  $x$  is inside  $S(0)$ , then  $S(t) = \{x : u(t, x) = 0\}$  for any  $t \geq 0$ , where  $u(t, x)$  is the viscosity solution of (11). This is the level set formulation of the classical motion by mean curvature, initially proposed by S. Osher and J. Sethian in [21] and whose mathematical analysis was given in [14] and was followed by many other works. In particular, as it was shown by G. Barles, H.M. Soner and P. Souganidis [7], if instead of embedding  $S(0)$  as the zero level set of a continuous function we just set  $u(0, x) = \chi_{C(0)}$  where  $C(0)$  is the region inside  $S(0)$ , and we assume that  $S(0)$  is a smooth surface, then  $u(t, x) = \chi_{C(t)}$  where  $C(t)$  is the region inside  $S(t)$ . Thus, the mean curvature motion flow decreases the total variation of  $\chi_{C(0)}$  by decreasing the  $(N - 1)$ -Hausdorff measure of the boundary  $S(t)$  of  $C(t)$  [15]. Now, since the total variation of any function  $u_0(x) = h\chi_C$  is

$$TV(h\chi_C) = hPer(C)$$

we see that two basic ways of minimizing the total variation of such a function are: either we decrease the height of  $u_0(x)$  or we decrease the perimeter of its boundary. Our purpose was to explain which strategy was followed by the minimizing total variation flow. As we shall see below, under some geometric conditions for the sets  $C(0)$ , the strategy of the minimizing total variation flow consists in decreasing the height of the function without distortion of its boundary, while a distortion of the boundary will occur when these conditions are not satisfied, in particular, this will happen at points with a strong curvature. Thus the strategy followed by the minimizing total variation flow, compared to the one followed by the mean curvature motion is quite different. This gives an idea of the behaviour of (5), at least what are the infinitesimal effects of (5) on the initial datum  $u(0, x)$ . The methods and results obtained can also be used to produce particular explicit solutions of the denoising problem which corresponds to the kernel  $K$  in (7) being the identity, i.e.,  $K = I$ .

## 2 Functions of Bounded Variation

Due to the linear growth condition on the Lagrangians associated with the total variation, the natural energy space to study them is the space of functions of bounded variation. In

this section we collect some basic results of the theory of functions of bounded variation. For more information we refer the reader to [2], [13], [17], [24].

## 2.1 Definitions

Throughout this section,  $\Omega$  denotes an open subset of  $\mathbb{R}^N$ .

**Definition 1** A function  $u \in L^1(\Omega)$  whose partial derivatives in the sense of distributions are measures with finite total variation in  $\Omega$  is called a *function of bounded variation*. The vector space of functions of bounded variation in  $\Omega$  is denoted by  $BV(\Omega)$ . Thus  $u \in BV(\Omega)$  if and only if  $u \in L^1(\Omega)$  and there are Radon measures  $\mu_1, \dots, \mu_N$  with finite total mass in  $\Omega$  such that

$$\int_{\Omega} u \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} \varphi d\mu_i \quad \forall \varphi \in C_0^\infty(\Omega), i = 1, \dots, N.$$

If  $u \in BV(\Omega)$ , the total variation of the measure  $Du$  is

$$\|Du\| = \sup \left\{ \int_{\Omega} u \operatorname{div}(\phi) dx : \phi \in C_0^\infty(\Omega, \mathbb{R}^N), |\phi(x)| \leq 1 \text{ for } x \in \Omega \right\}.$$

The space  $BV(\Omega)$ , endowed with the norm

$$\|u\|_{BV} = \|u\|_1 + \|Du\|,$$

is a Banach space. If  $u \in BV(\Omega)$ , the total variation  $\|Du\|$  may be regarded as a measure, whose value on an open set  $U \subseteq \Omega$  is

$$\|Du\|(U) = \sup \left\{ \int_U u \operatorname{div}(\phi) dx : \phi \in C_0^\infty(U, \mathbb{R}^N), |\phi(x)| \leq 1 \text{ for } x \in U \right\}.$$

We also use

$$\int_U \|Du\|$$

to denote  $\|Du\|(U)$ .

For  $u \in BV(\Omega)$ , the gradient  $Du$  is a Radon measure that decomposes into its absolutely continuous and singular parts

$$Du = D^a u + D^s u.$$

Then  $D^a u = \nabla u \mathcal{L}^N$  where  $\nabla u$  is the Radon-Nikodym derivative of the measure  $Du$  with respect to the Lebesgue measure  $\mathcal{L}^N$ . There is also the polar decomposition  $D^s u = \overline{D^s u} |D^s u|$  where  $|D^s u|$  is the total variation measure of  $D^s u$ .

The total variation is lower semi-continuous. More concretely, we have the following result.

**Theorem 1** Suppose that  $u_i \in BV(\Omega)$ ,  $i = 1, 2, \dots$ , and  $u_i \rightarrow u$  in  $L^1_{loc}(\Omega)$ . Then

$$\|Du\|(\Omega) \leq \liminf_{i \rightarrow \infty} \|Du_i\|(\Omega).$$

We say that  $u \in L^1_{loc}(\Omega)$  is *locally of bounded variation* if  $\varphi u \in BV(\Omega)$  for any  $\varphi \in C_0^\infty(\Omega)$ . We denote by  $BV_{loc}(\Omega)$  the space of functions which are locally of bounded variation.

Here and in what follows we shall denote by  $\mathcal{H}^\alpha$  the Hausdorff measure of dimension  $\alpha$  in  $\mathbb{R}^N$ . In particular,  $\mathcal{H}^{N-1}$  denotes the  $(N-1)$ -dimensional Hausdorff measure and  $\mathcal{H}^N$ , the  $N$ -dimensional Hausdorff measure, coincides with the (outer) Lebesgue measure in  $\mathbb{R}^N$ .

## 2.2 Approximation by Smooth Functions

**Theorem 2** Assume that  $u \in BV(\Omega)$ . There exists a sequence of functions  $u_i \in C^\infty(\Omega) \cap BV(\Omega)$  such that

- (i)  $u_i \rightarrow u$  in  $L^1(\Omega)$ ;
- (ii)  $\|Du_i\|(\Omega) \rightarrow \|Du\|(\Omega)$  as  $i \rightarrow \infty$ .

Moreover,

- (iii) if  $u \in BV(\Omega) \cap L^q(\Omega)$ ,  $q < \infty$ , we can find the functions  $u_i$  such that  $u_i \in L^q(\Omega)$  and  $u_i \rightarrow u$  in  $L^q(\Omega)$ ;
- (iv) if  $u \in BV(\Omega) \cap L^\infty(\Omega)$ , we can find the  $u_i$  such that  $\|u_i\|_\infty \leq \|u\|_\infty$  and  $u_i \rightarrow u$  in  $L^\infty(\Omega)$ -weakly\*.

Finally,

- (v) if  $\partial\Omega$  is Lipschitz continuous one can find the  $u_i$  such that

$$u_i|_{\partial\Omega} = u|_{\partial\Omega} \quad \text{for all } i.$$

**Theorem 3** Assume that  $u \in BV(\Omega)$ . There exists a sequence of functions  $u_i \in C^\infty(\Omega) \cap BV(\Omega)$  such that

- (i)  $u_i \rightarrow u$  in  $L^1(\Omega)$ ;
- (ii) if  $U \subset\subset \Omega$  is such that  $\|Du\|(\partial U) = 0$ , then

$$\lim_{i \rightarrow \infty} \|Du_i\|(U) = \|Du\|(U).$$

Moreover, if  $u \in L^q(\Omega)$ ,  $1 \leq q < \infty$  or  $u \in L^\infty(\Omega)$ , one can find the  $u_i$  satisfying (iii) or (iv), respectively, of the above result.

**Definition 2** Let  $u_i, u \in BV(\Omega)$ ,  $i = 1, 2, \dots$ . We say that  $u_i$  strictly converges to  $u$  in  $BV(\Omega)$  if both conditions (i), (ii) of Theorem 2 hold.

**Definition 3** Let  $u_i, u \in BV(\Omega)$ ,  $i = 1, 2, \dots$ . We say that  $u_i$  weakly\* converges to  $u$  in  $BV(\Omega)$  if  $u_i \rightarrow u$  in  $L^1_{loc}(\Omega)$  and  $Du_i$  weakly\* converges to  $Du$  as measures in  $\Omega$ .

**Proposition 2** If  $u_i, u \in BV(\Omega)$ . Then  $u_i \rightarrow u$  weakly\* in  $BV(\Omega)$  if and only if  $\{u_i\}$  is bounded in  $BV(\Omega)$  and converges to  $u$  in  $L^1_{loc}(\Omega)$ . Moreover, if

$$\|Du_i\|(\Omega) \rightarrow \|Du\|(\Omega) \quad \text{as } i \rightarrow \infty,$$

and we consider the measures

$$\mu_i(B) = \int_{B \cap \Omega} Du_i, \quad \mu(B) = \int_{B \cap \Omega} Du,$$

for all Borel set  $B \subset \mathbb{R}^N$ . Then  $\mu_i \rightarrow \mu$  weakly\* as (vector valued) Radon measures in  $\mathbb{R}^N$ .

**Theorem 4** If  $(u_k) \subseteq BV(\Omega)$  strictly converges to  $u$  and  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is continuous and 1-positively homogeneous, we have

$$\lim_{k \rightarrow \infty} \int_{\Omega} \phi f \left( \frac{Du_k}{\|Du_k\|} \right) d\|Du_k\| = \int_{\Omega} \phi f \left( \frac{Du}{\|Du\|} \right) d\|Du\|$$

for any bounded continuous function  $\phi : \Omega \rightarrow \mathbb{R}$ . As a consequence

$$f \left( \frac{Du_k}{\|Du_k\|} \right) \|Du_k\| \quad \text{weakly* converge in } \Omega \text{ to } f \left( \frac{Du}{\|Du\|} \right) \|Du\|.$$

In particular,  $\|Du_k\| \rightarrow \|Du\|$  weakly\* in  $\Omega$ .

## 2.3 Traces and Extensions

Assume that  $\Omega$  is open and bounded with  $\partial\Omega$  Lipschitz. We observe that since  $\partial\Omega$  is Lipschitz, the outer unit normal  $\nu$  exists  $\mathcal{H}^{N-1}$  a.e. on  $\partial\Omega$ .

**Theorem 5** Assume that  $\Omega$  is open and bounded, with  $\partial\Omega$  Lipschitz. There exists a bounded linear mapping

$$T : BV(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^{N-1})$$

such that

$$\int_{\Omega} u \operatorname{div}(\varphi) dx = - \int_{\Omega} \varphi \cdot dDu + \int_{\partial\Omega} \varphi \cdot \nu Tu d\mathcal{H}^{N-1}$$

for all  $u \in BV(\Omega)$  and  $\varphi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ . Moreover, for any  $u \in BV(\Omega)$  and for  $\mathcal{H}^{N-1}$  a.e.  $x \in \partial\Omega$ , we have

$$\lim_{r \rightarrow 0^+} r^{-N} \int_{B(x,r) \cap \Omega} |u - Tu(x)| dy = 0.$$

**Theorem 6** *Let  $\Omega$  be an open bounded set, with  $\partial\Omega$  Lipschitz. Then the trace operator  $u \rightarrow Tu$  is continuous between  $BV(\Omega)$ , endowed with the topology induced by the strict convergence, and  $L^1(\partial\Omega, \mathcal{H}^{N-1} \llcorner \partial\Omega)$ .*

**Theorem 7** *Assume that  $\Omega$  is open and bounded, with  $\partial\Omega$  Lipschitz. Let  $u_1 \in BV(\Omega)$ ,  $u_2 \in BV(\mathbb{R}^N \setminus \bar{\Omega})$ . We define*

$$v(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega \\ u_2(x) & \text{if } x \in \mathbb{R}^N \setminus \bar{\Omega}. \end{cases}$$

Then  $v \in BV(\mathbb{R}^N)$  and

$$\|Dv\|(\mathbb{R}^N) = \|Du_1\|(\Omega) + \|Du_2\|(\mathbb{R}^N \setminus \bar{\Omega}) + \int_{\partial\Omega} |Tu_1 - Tu_2| d\mathcal{H}^{N-1}.$$

In particular, if

$$Eu = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \bar{\Omega}. \end{cases}$$

then  $Eu \in BV(\mathbb{R}^N)$  provided  $u \in BV(\Omega)$ .

## 2.4 Sets of Finite Perimeter and the Coarea Formula

**Definition 4** An  $\mathcal{L}^N$  measurable subset  $E$  of  $\mathbb{R}^N$  has *finite perimeter* in  $\Omega$  if  $\chi_E \in BV(\Omega)$ . The perimeter of  $E$  in  $\Omega$  is  $P(E, \Omega) = \|D\chi_E\|(\Omega)$ .

We shall denote the measure  $\|D\chi_E\|$  by  $\|\partial E\|$  and  $P(E, \mathbb{R}^N)$  by  $\operatorname{Per}(E)$ .

**Theorem 8** *Let  $E$  be a set of finite perimeter in  $\Omega$  and let  $D\chi_E = \nu_E \|D\chi_E\|$  be the polar decomposition of  $D\chi_E$ . Then the generalized Gauss-Green formula holds*

$$\int_E \operatorname{div}(\varphi) dx = - \int_{\Omega} \langle \nu_E, \varphi \rangle d\|D\chi_E\|$$

for all  $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$ .

**Theorem 9 (Coarea formula for BV-functions)**

Let  $u \in BV(\Omega)$ . Then

(i)  $E_{u,t} := \{x \in \Omega : u(x) > t\}$  has finite perimeter for  $\mathcal{L}^1$  a.e.  $t \in \mathbb{R}$  and

(ii)  $\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(E_{u,t}, \Omega) dt.$

(iii) Conversely, if  $u \in L^1(\Omega)$  and

$$\int_{-\infty}^{\infty} P(E_{u,t}, \Omega) dt < \infty,$$

then  $u \in BV(\Omega)$ .

**2.5 Isoperimetric Inequality****Theorem 10 (Sobolev inequality)**

There exists a constant  $C > 0$  such that

$$\|u\|_{L^{N/N-1}(\mathbb{R}^N)} \leq C \|Du\|(\mathbb{R}^N)$$

for all  $u \in BV(\mathbb{R}^N)$ .

If  $u \in L^1(\Omega)$ , the mean value of  $u$  in  $\Omega$  is

$$u_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_{\Omega} u(x) dx.$$

**Theorem 11 (Poincaré's inequality)**

Let  $\Omega$  be open and bounded with  $\partial\Omega$  Lipschitz. Suppose that  $\Omega$  is connected. Then

$$\int_{\Omega} |u - u_\Omega| dx \leq C \|Du\|(\Omega) \quad \forall u \in BV(\Omega)$$

for some constant  $C$  depending only on  $\Omega$ .

**Theorem 12** Let  $N > 1$ . For any set  $E$  of finite perimeter in  $\mathbb{R}^N$  either  $E$  or  $\mathbb{R}^N \setminus E$  has finite Lebesgue measure and

$$\min \left\{ \mathcal{L}^N(E), \mathcal{L}^N(\mathbb{R}^N \setminus E) \right\} \leq C [\text{Per}(E)]^{\frac{N}{N-1}}$$

for some dimensional constant  $C$ .

**Theorem 13 (Embedding Theorem)**

Let  $\Omega$  be open and bounded, with  $\partial\Omega$  Lipschitz. Then the embedding  $BV(\Omega) \rightarrow L^{N/N-1}(\Omega)$  is continuous and  $BV(\Omega) \rightarrow L^p(\Omega)$  is compact for all  $1 \leq p < \frac{N}{N-1}$ .

The continuity of the embedding of Theorem 13 and Theorem 11 imply the following Sobolev-Poincaré inequality

$$\|u - u_\Omega\|_p \leq C \|Du\|(\Omega) \quad \forall u \in BV(\Omega), 1 \leq p \leq \frac{N}{N-1} \quad (13)$$

for some constant  $C$  depending only on  $\Omega$ .

### 3 Pairings Between Measures and Bounded Functions

In this section we give some of the main points of the results about pairing between measures and bounded functions given by G. Anzellotti in [6] (see also [18]).

#### 3.1 Trace of the Normal Component of Certain Vector Fields

It is well known that summability conditions on the divergence of a vector field  $z$  in  $\Omega$  yield trace properties for the normal component of  $z$  on  $\partial\Omega$ . In this section we define a function  $[z, \nu] \in L^\infty(\partial\Omega)$  which is associated to any vector field  $z \in L^\infty(\Omega, \mathbb{R}^N)$  such that  $\operatorname{div}(z)$  is a bounded measure in  $\Omega$ .

Let  $\Omega$  be an open set in  $\mathbb{R}^N$ ,  $N \geq 2$ , and  $1 \leq p \leq N$ ,  $\frac{N}{N-1} \leq q \leq \infty$ . We shall consider the following spaces:

$$BV(\Omega)_q := BV(\Omega) \cap L^q(\Omega)$$

$$BV(\Omega)_c := BV(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$$

$$X(\Omega)_p := \{z \in L^\infty(\Omega, \mathbb{R}^N) : \operatorname{div}(z) \in L^p(\Omega)\}$$

$$X(\Omega)_\mu := \{z \in L^\infty(\Omega, \mathbb{R}^N) : \operatorname{div}(z) \text{ is a bounded measure in } \Omega\}.$$

In the next theorem we define a pairing  $\langle z, u \rangle_{\partial\Omega}$ , for  $z \in X(\Omega)_\mu$  and  $u \in BV(\Omega)_c$ . We need the following result, which can be easily obtained by the same technique that Gagliardo uses in [16] in proving his extension theorem  $L^1(\partial\Omega) \rightarrow W^{1,1}(\Omega)$ .

**Lemma 1** *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$  with Lipschitz boundary. Then, for any given function  $u \in L^1(\partial\Omega)$  and for any given  $\epsilon > 0$  there exists a function  $w \in W^{1,1}(\Omega) \cap C(\Omega)$  such that*

$$w|_{\partial\Omega} = u$$

$$\int_{\Omega} |\nabla w| dx \leq \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} + \epsilon$$

$$w(x) = 0 \quad \text{if } \operatorname{dist}(x, \partial\Omega) > \epsilon.$$

Moreover, for any fixed  $1 \leq q < \infty$ , one can find the function  $w$  such that

$$\|w\|_q \leq \epsilon.$$

Finally, if one has also  $u \in L^\infty(\partial\Omega)$ , one can find  $w$  such that

$$\|w\|_\infty \leq \|u\|_\infty.$$

**Theorem 14** Assume that  $\Omega \subset \mathbb{R}^N$  is an open bounded set with Lipschitz boundary  $\partial\Omega$ . Denote by  $\nu(x)$  the outward unit normal to  $\partial\Omega$ . Then there exists a bilinear map  $\langle z, u \rangle_{\partial\Omega} : X(\Omega)_\mu \times BV(\Omega)_c \rightarrow \mathbb{R}$  such that

$$\langle z, u \rangle_{\partial\Omega} = \int_{\partial\Omega} u(x)z(x) \cdot \nu(x) d\mathcal{H}^{N-1} \quad \text{if } z \in C^1(\Omega, \mathbb{R}^N) \quad (14)$$

$$|\langle z, u \rangle_{\partial\Omega}| \leq \|z\|_\infty \int_{\partial\Omega} |u(x)| d\mathcal{H}^{N-1} \quad \text{for all } z, u. \quad (15)$$

**Proof.** For  $u \in BV(\Omega)_c \cap W^{1,1}(\Omega)$  and  $z \in X(\Omega)_\mu$ , we define

$$\langle z, u \rangle_{\partial\Omega} := \int_{\Omega} u \operatorname{div}(z) dx + \int_{\Omega} z \cdot \nabla u dx.$$

We remark that if  $u, v \in BV(\Omega)_c \cap W^{1,1}(\Omega)$  and  $u = v$  on  $\partial\Omega$  then one has

$$\langle z, u \rangle_{\partial\Omega} = \langle z, v \rangle_{\partial\Omega} \quad \text{for all } z \in X(\Omega)_\mu.$$

In fact, by standard techniques in Sobolev spaces theory, we can find a sequence of functions  $g_i \in \mathcal{D}(\Omega)$  such that, for all  $z \in X(\Omega)_\mu$ , one has

$$\begin{aligned} \langle z, u - v \rangle_{\partial\Omega} &= \int_{\Omega} (u - v) \operatorname{div}(z) dx + \int_{\Omega} z \cdot \nabla(u - v) dx \\ &= \lim_{i \rightarrow \infty} \left( \int_{\Omega} g_i \operatorname{div}(z) dx + \int_{\Omega} z \cdot \nabla g_i dx \right) = 0. \end{aligned}$$

Now, we define  $\langle z, u \rangle_{\partial\Omega}$  for all  $u \in BV(\Omega)_c$  by setting

$$\langle z, u \rangle_{\partial\Omega} = \langle z, w \rangle_{\partial\Omega},$$

where  $w$  is any function in  $BV(\Omega)_c \cap W^{1,1}(\Omega)$  such that  $u = w$  on  $\partial\Omega$ . This is a valid definition, in view of the preceding remark and because of the Lemma 1.

To prove (15), we take a sequence  $u_n \in BV(\Omega)_c \cap C^\infty(\Omega)$  converging to  $u$  as in Theorem 2 and we get

$$|\langle z, u \rangle_{\partial\Omega}| = |\langle z, u_n \rangle_{\partial\Omega}| \leq \left| \int_{\Omega} u_n \operatorname{div}(z) dx \right| + \|z\|_\infty \int_{\Omega} |\nabla u_n| dx$$

for all  $z$  and for all  $n$ . Hence, taking limit when  $n \rightarrow \infty$  we have

$$|\langle z, u \rangle_{\partial\Omega}| \leq \left| \int_{\Omega} u \operatorname{div}(z) dx \right| + \|z\|_\infty \int_{\Omega} \|Du\|.$$

Now, for a fixed  $\epsilon > 0$  we consider a function  $w$  as in Lemma 1. Then

$$|\langle z, u \rangle_{\partial\Omega}| = |\langle z, w \rangle_{\partial\Omega}| \leq \|w\|_\infty \int_{\Omega \setminus \Omega_\epsilon} |\operatorname{div}(z)| + \|z\|_\infty \left( \int_{\partial\Omega} |u| dx + \epsilon \right),$$

where  $\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$ . Since  $\operatorname{div}(z)$  is a measure of bounded total variation in  $\Omega$ ,

$$\lim_{\epsilon \rightarrow 0^+} \int_{\Omega \setminus \Omega_\epsilon} |\operatorname{div}(z)| dx = 0.$$

Consequently, (15) holds.  $\square$

**Theorem 15** *Let  $\Omega$  be as in Theorem 14. Then there exists a linear operator  $\gamma : X(\Omega)_\mu \rightarrow L^\infty(\partial\Omega)$  such that*

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty \quad (16)$$

$$\langle z, u \rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma(z)(x)u(x) d\mathcal{H}^{N-1} \quad \text{for all } u \in BV(\Omega)_c \quad (17)$$

$$\gamma(z)(x) = z(x) \cdot \nu(x) \quad \text{for all } x \in \partial\Omega \text{ if } z \in C^1(\overline{\Omega}, \mathbb{R}^N). \quad (18)$$

*The function  $\gamma(z)$  is a weakly defined trace on  $\partial\Omega$  of the normal component of  $z$ . We shall denote  $\gamma(z)$  by  $[z, \nu]$ .*

**Proof.** Take a fix  $z \in X(\Omega)_\mu$ . Consider the functional  $F : L^\infty(\partial\Omega) \rightarrow \mathbb{R}$  defined by

$$F(u) := \langle z, w \rangle_{\partial\Omega},$$

where  $w \in BV(\Omega)_c$  is such that  $w|_{\partial\Omega} = u$ . By estimate (15),

$$|F(u)| \leq \|z\|_\infty \|u\|_1.$$

Hence there exists a function  $\gamma(z) \in L^\infty(\partial\Omega)$  such that

$$F(u) = \int_{\partial\Omega} \gamma(z)(x)u(x) d\mathcal{H}^{N-1}$$

and the result follows.  $\square$

Obviously,  $X(\Omega)_p \subset X(\Omega)_\mu$  for all  $p \geq 1$  and the trace  $[z, \nu]$  is defined for all  $z \in X(\Omega)_p$ .

### 3.2 The Measure $(z, Du)$

Approximating by smooth functions and applying Green's formula, the following result can be deduced easily.

**Proposition 3** *Let  $\Omega$  be as in Theorem 14 and  $1 \leq p \leq \infty$ . Then, for all  $z \in X(\Omega)_p$  and  $u \in W^{1,1}(\Omega) \cap L^{p'}(\Omega)$ , one has*

$$\int_{\Omega} u \operatorname{div}(z) dx + \int_{\Omega} z \cdot \nabla u dx = \int_{\partial\Omega} [z, \nu] u d\mathcal{H}^{N-1}. \quad (19)$$

In the sequel we shall consider pairs  $(z, u)$  such that one of the following conditions holds

$$\left\{ \begin{array}{l} a) u \in BV(\Omega)_{p'}, z \in X(\Omega)_p \quad \text{and } 1 < p \leq N; \\ b) u \in BV(\Omega)_\infty, z \in X(\Omega)_1; \\ c) u \in BV(\Omega)_c, z \in X(\Omega)_\mu. \end{array} \right. \quad (20)$$

**Definition 5** Let  $z, u$  be such that one of the conditions (20) holds. Then we define a functional  $(z, Du) : \mathcal{D}(\Omega) \rightarrow \mathbb{R}$  as

$$\langle (z, Du), \varphi \rangle := - \int_{\Omega} u \varphi \operatorname{div}(z) dx - \int_{\Omega} uz \cdot \nabla \varphi dx.$$

**Theorem 16** For all open set  $U \subset \Omega$  and for all function  $\varphi \in \mathcal{D}(U)$ , one has

$$|\langle (z, Du), \varphi \rangle| \leq \|\varphi\|_{\infty} \|z\|_{L^{\infty}(U)} \int_U \|Du\|, \quad (21)$$

hence  $(z, Du)$  is a Radon measure in  $\Omega$ .

**Proof.** Take a sequence  $u_n \in C^{\infty}(\Omega)$  converging to  $u$  as in Theorem 3. Take  $\varphi \in \mathcal{D}(U)$  and consider an open set  $V$  such that  $\operatorname{supp}(\varphi) \subset V \subset\subset U$ . Then

$$|\langle (z, Du_n), \varphi \rangle| \leq \|\varphi\|_{\infty} \|z\|_{L^{\infty}(U)} \int_V \|Du_n\| \quad \text{for all } n \in \mathbb{N}.$$

From here, taking limit as  $n \rightarrow \infty$ , the result follows.  $\square$

We shall denote by  $|(z, Du)|$  the measure total variation of  $(z, Du)$  and by  $\int_B |(z, Du)|$ ,  $\int_B (z, Du)$  the values of these measures on every Borel set  $B \subset \Omega$ .

As a consequence of the above theorem, the following result holds.

**Corollary 1** The measures  $(z, Du)$ ,  $|(z, Du)|$  are absolutely continuous with respect to the measure  $\|Du\|$  and

$$\left| \int_B (z, Du) \right| \leq \int_B |(z, Du)| \leq \|z\|_{L^{\infty}(U)} \int_B \|Du\|$$

for all Borel sets  $B$  and for all open sets  $U$  such that  $B \subset U \subset \Omega$ . Moreover, by the Radon-Nikodym Theorem, there exists a  $\|Du\|$ -measurable function

$$\theta(z, Du, \cdot) : \Omega \rightarrow \mathbb{R}$$

such that

$$\int_B (z, Du) = \int_B \theta(z, Du, x) \|Du\| \quad \text{for all Borel sets } B \subset \Omega$$

and

$$\|\theta(z, Du, \cdot)\|_{L^{\infty}(\Omega, \|Du\|)} \leq \|z\|_{\infty}.$$

Assume  $u, z$  satisfy one of the conditions (20). By writing

$$z \cdot D^s u := (z, Du) - (z \cdot \nabla u) d\mathcal{L}^N,$$

we have that  $z \cdot D^s u$  is a bounded measure. Furthermore, with an approximation argument to the one used in the proof of Theorem 16, we have that  $z \cdot D^s u$  is absolutely continuous with respect to  $\|D^s u\|$  (and, thus, it is a singular measure respect to  $\mathcal{L}^N$ ), and

$$|z \cdot D^s u| \leq \|z\|_{\infty} \|D^s u\|. \quad (22)$$

### 3.3 The Green formula

**Lemma 2** Assume  $u, z$  satisfy one of the conditions (20). Let  $u_n \in C^\infty(\Omega) \cap BV(\Omega)$  converging to  $u$  as in Theorem 2. Then we have

$$\int_{\Omega} z \cdot \nabla u_n dx \rightarrow \int_{\Omega} (z, Du).$$

**Proof.** For a given  $\epsilon > 0$ , we take an open set  $U \subset\subset \Omega$  such that

$$\int_{\Omega \setminus U} \|Du\| < \epsilon.$$

Let  $\varphi \in \mathcal{D}(\Omega)$  be such that  $\varphi(x) = 1$  in  $U$  and  $0 \leq \varphi \leq 1$  in  $\Omega$ . Then

$$\begin{aligned} & \left| \int_{\Omega} (z, Du_n) - \int_{\Omega} (z, Du) \right| \leq \\ & | \langle (z, Du_n), \varphi \rangle - \langle (z, Du), \varphi \rangle | + \int_{\Omega} |(z, Du_n)|(1 - \varphi) + \int_{\Omega} |(z, Du)|(1 - \varphi). \end{aligned}$$

Since

$$\begin{aligned} & \lim_{n \rightarrow \infty} \langle (z, Du_n), \varphi \rangle = \langle (z, Du), \varphi \rangle, \\ & \limsup_{n \rightarrow \infty} \int_{\Omega} |(z, Du_n)|(1 - \varphi) \leq \|z\|_{\infty} \limsup_{n \rightarrow \infty} \int_{\Omega \setminus U} \|Du_n\| < \epsilon \|z\|_{\infty}, \\ & \int_{\Omega} |(z, Du)|(1 - \varphi) \leq \epsilon \|z\|_{\infty} \end{aligned}$$

and  $\epsilon$  is arbitrary, the lemma follows.  $\square$

We give now the expected *Green's formula* relating the function  $[z, \nu]$  and the measure  $(z, Du)$ .

**Theorem 17** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$  with Lipschitz boundary and let  $z, u$  be such that one of the conditions (20) holds, then we have

$$\int_{\Omega} u \operatorname{div}(z) dx + \int_{\Omega} (z, Du) = \int_{\partial\Omega} [z, \nu] u d\mathcal{H}^{N-1}. \quad (23)$$

**Proof.** We assume that (20) (a) holds, in the general case an extension of Proposition 3 is needed. Take a sequence of functions  $u_n \in C^\infty(\Omega) \cap BV(\Omega)$  converging to  $u$  as in Theorem 2. Then, by Lemma 2 and Proposition 3, we have

$$\begin{aligned} \int_{\Omega} u \operatorname{div}(z) dx + \int_{\Omega} (z, Du) &= \lim_{n \rightarrow \infty} \left( \int_{\Omega} u_n \operatorname{div}(z) dx + \int_{\Omega} z \cdot \nabla u_n dx \right) \\ &= \lim_{n \rightarrow \infty} \int_{\partial\Omega} [z, \nu] u_n d\mathcal{H}^{N-1} = \int_{\partial\Omega} [z, \nu] u d\mathcal{H}^{N-1}. \end{aligned}$$

$\square$

**Remark 1** Observe that with a similar proof to the one the above theorem, in the case  $\Omega = \mathbb{R}^N$ , the following integration by parts formula, for  $z$  and  $w$  satisfying one of the conditions (20), holds:

$$\int_{\mathbb{R}^N} w \operatorname{div}(z) dx + \int_{\mathbb{R}^N} (z, Dw) = 0. \quad (24)$$

In particular, if  $\Omega$  is bounded and has finite perimeter in  $\mathbb{R}^N$ , from (24) it follows

$$\int_{\Omega} \operatorname{div}(z) dx = \int_{\mathbb{R}^N} (z, -D\chi_{\Omega}) = \int_{\partial^* \Omega} \theta(z, -D\chi_{\Omega}, x) d\mathcal{H}^{N-1}. \quad (25)$$

Notice also that as a consequence of Corollary 1, if  $z_1, z_2 \in X(\mathbb{R}^N)_p$  and  $z_1 = z_2$  almost everywhere on  $\Omega$ , then  $\theta(z_1, -D\chi_{\Omega}, x) = \theta(z_2, -D\chi_{\Omega}, x)$  for  $\mathcal{H}^{N-1}$ -almost every  $x \in \partial^* \Omega$ .

If  $\Omega$  is a bounded open set with Lipschitz boundary, then (25) has a meaning also if  $z$  is defined only on  $\Omega$  and not on the whole of  $\mathbb{R}^N$ , precisely when  $z \in L^\infty(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(z) \in L^N(\Omega)$ . In this case we mean that  $\theta(z, -D\chi_{\Omega}, \cdot)$  coincides with  $[z, \nu]$ .

**Remark 2** Let  $\Omega \subset \mathbb{R}^2$  be a bounded open set with Lipschitz boundary, and let  $z_{\text{inn}} \in L^\infty(\Omega; \mathbb{R}^2)$  with  $\operatorname{div}(z_{\text{inn}}) \in L^2_{\text{loc}}(\Omega)$ , and  $z_{\text{out}} \in L^\infty(\mathbb{R}^2 \setminus \bar{\Omega}; \mathbb{R}^2)$  with  $\operatorname{div}(z_{\text{out}}) \in L^2_{\text{loc}}(\mathbb{R}^2 \setminus \bar{\Omega})$ . Assume that

$$\theta(z_{\text{inn}}, -D\chi_{\Omega}, x) = -\theta(z_{\text{out}}, -D\chi_{\mathbb{R}^2 \setminus \bar{\Omega}}, x) \quad \text{for } \mathcal{H}^1 - \text{a.e } x \in \partial\Omega.$$

Then if we define  $z := z_{\text{inn}}$  on  $\Omega$  and  $z := z_{\text{out}}$  on  $\mathbb{R}^2 \setminus \bar{\Omega}$ , we have  $z \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$  and  $\operatorname{div}(z) \in L^2_{\text{loc}}(\mathbb{R}^2)$ .

## 4 Gradient Flows

One of the more important examples of maximal monotone operator in Hilbert spaces comes from the optimization theory, they are the subdifferentials of convex functions which we introduce next.

Hereafter  $H$  will denote a real Hilbert space, with inner product  $(/)$  and norm  $\| \cdot \|$ .

### 4.1 Convex functions in Hilbert spaces

A function  $\varphi : H \rightarrow ]-\infty, +\infty]$  is *convex* provided

$$\varphi(\alpha u + (1 - \alpha)v) \leq \alpha \varphi(u) + (1 - \alpha)\varphi(v)$$

for all  $\alpha \in [0, 1]$  and  $u, v \in H$ .

We denote

$$D(\varphi) = \{u \in H : \varphi(u) \neq +\infty\} \quad (\text{effective domain}).$$

We say that  $\varphi$  is *proper* if  $D(\varphi) \neq \emptyset$ .

We say  $\varphi$  is *lower semi-continuous* (l.s.c) if  $u_n \rightarrow u$  in  $H$  implies  $\varphi(u) \leq \liminf_{n \rightarrow \infty} \varphi(u_n)$ .

Some of the properties of  $\varphi$  are reflected in its epigraph:

$$\text{epi}(\varphi) := \{(u, r) \in H \times \mathbb{R} : r \geq \varphi(u)\}.$$

For instance, it is easy to see that  $\varphi$  is convex if and only if  $\text{epi}(\varphi)$  is a convex subset of  $H$ ; and  $\varphi$  is lower semi-continuous if and only if  $\text{epi}(\varphi)$  is closed.

The *subdifferential*  $\partial\varphi$  of  $\varphi$  is the operator defined by

$$w \in \partial\varphi(z) \iff \varphi(u) \geq \varphi(z) + (w/u - z) \quad \forall u \in H.$$

We say  $u \in D(\partial\varphi)$ , the domain of  $\partial\varphi$ , provided  $\partial\varphi(u) \neq \emptyset$ .

Observe that  $0 \in \partial\varphi(z) \iff \varphi(u) \geq \varphi(z) \quad \forall u \in H \iff$

$$\varphi(z) = \min_{u \in D(\varphi)} \varphi(u).$$

Therefore, we have that  $0 \in \partial\varphi(z)$  is the *Euler equation* of the variational problem

$$\varphi(z) = \min_{u \in D(\varphi)} \varphi(u).$$

If  $(z, w), (\hat{z}, \hat{w}) \in \partial\varphi$ , then  $\varphi(z) \geq \varphi(\hat{z}) + (\hat{w}/z - \hat{z})$  and  $\varphi(\hat{z}) \geq \varphi(z) + (w/\hat{z} - z)$ . Adding these inequalities we get

$$(w - \hat{w}/z - \hat{z}) \geq 0.$$

Thus,  $\partial\varphi$  is a monotone operator.

Next we will discuss the relation between subdifferentials, directional derivatives and the Gâteaux derivative. Let  $\varphi : H \rightarrow ]-\infty, +\infty]$ . The *directional derivative*  $D_v\varphi(u)$  of  $\varphi$  at the point  $u \in D(\varphi)$  in the direction  $v \in H$  is defined by

$$D_v\varphi(u) = \lim_{\lambda \downarrow 0} \frac{\varphi(u + \lambda v) - \varphi(u)}{\lambda}$$

whenever the limit exists. If there exists  $w \in H$  such that  $D_v\varphi(u) = (v/w)$  for all  $v \in H$ , then  $\varphi$  is called *Gâteaux differentiable* at  $u$ , and  $w$  is called the *Gâteaux derivative* of  $\varphi$  at  $u$ , which will be denoted by  $\varphi'(u)$ .

**Proposition 4** Let  $\varphi : H \rightarrow ]-\infty, +\infty]$  be convex and proper. If  $\varphi$  is Gâteaux differentiable at  $u$ , then

$$\partial\varphi(u) = \{\varphi'(u)\}.$$

**Proof.** Given  $w \in H$ , since  $\varphi$  is convex, we have

$$\begin{aligned} (\varphi'(u)/w - u) &= D_{w-u}\varphi(u) = \lim_{\lambda \downarrow 0} \frac{\varphi(u + \lambda(w - u)) - \varphi(u)}{\lambda} \\ &= \lim_{\lambda \downarrow 0} \frac{\varphi(\lambda w + (1 - \lambda)u) - \varphi(u)}{\lambda} \leq \varphi(w) - \varphi(u). \end{aligned}$$

Hence,  $\varphi'(u) \in \partial\varphi(u)$ .

On the other hand, if  $v \in \partial\varphi(u)$ , given  $w \in H$  and  $\lambda > 0$ , we have

$$\frac{\varphi(u + \lambda w) - \varphi(u)}{\lambda} \geq \frac{1}{\lambda}(v/u + \lambda w - u) = (v/w),$$

from where it follows that

$$D_w\varphi(u) \geq (v/w) \quad \forall w \in H.$$

Moreover, taking  $w = -w$ , we have

$$-D_{-w}\varphi(u) \leq (v/w) \leq D_w\varphi(u).$$

Therefore, since  $\varphi$  is Gâteaux differentiable at  $u$ , we get

$$(\varphi'(u)/w) = -(\varphi'(u)/-w) \leq (v/w) \leq (\varphi'(u)/w) \quad \forall w \in H,$$

and consequently,  $v = \varphi'(u)$ . □

**Remark 3** In the case  $\varphi$  is continuous at  $u$ , also the reciprocal is true (see [11]). That is, in this case we have

$$\varphi \text{ is Gateaux differentiable at } u \Leftrightarrow \partial\varphi(u) = \{v\},$$

and in this case  $v = \varphi'(u)$ .

**Example 1** It is easy to see that if  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$  is defined by  $\varphi(x) := \|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ ,  $x = (x_1, \dots, x_n) \in \mathbb{R}^N$ , then

$$\partial\varphi(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } x \neq 0 \\ \overline{B_1(0)} & \text{if } x = 0. \end{cases}$$

**Example 2** Let  $\Omega \subset \mathbb{R}^N$  an open bounded set with smooth boundary. Consider the function  $\varphi : L^2(\Omega) \rightarrow ]-\infty, +\infty]$  defined by

$$\varphi(u) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 & \text{if } u \in W_0^{1,2}(\Omega) \\ +\infty & u \in L^2(\Omega) \setminus W_0^{1,2}(\Omega). \end{cases}$$

Then, it is well known (see for instance [8]) that

$$D(\partial\varphi) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$$

and

$$v \in \partial\varphi(u) \Leftrightarrow v = -\Delta u.$$

Hence, the following are equivalent:

(i)  $u$  is a solution of the variational problem

$$\varphi(u) = \min_{w \in L^2(\Omega)} \varphi(w).$$

(ii)  $u$  is a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

**Theorem 18** Let  $\varphi : H \rightarrow ]-\infty, +\infty]$  be convex, proper and lower semi-continuous. Then, for each  $w \in H$  and  $\lambda > 0$ , the problem

$$u + \lambda\partial\varphi(u) \ni w$$

has a unique solution  $u \in D(\partial\varphi)$ .

**Proof.** Given  $w \in H$  and  $\lambda > 0$ , consider the functional  $J : H \rightarrow ]-\infty, +\infty]$  defined by

$$J(u) := \frac{1}{2}\|u\|^2 + \lambda\varphi(u) - (u/w). \quad (26)$$

We intend to show that  $J$  attains its minimum over  $H$ . Let us first claim that  $J$  is weakly lower semi-continuous, that is,

$$u_n \rightharpoonup u \text{ weakly in } H \Rightarrow J(u) \leq \liminf_{n \rightarrow \infty} J(u_n). \quad (27)$$

Obviously, it is enough to show (27) for  $\varphi$ . Let  $u_{n_k}$  such that

$$l = \liminf_{n \rightarrow \infty} \varphi(u_n) = \lim_{k \rightarrow \infty} \varphi(u_{n_k}).$$

For each  $\epsilon > 0$  the set  $K_\epsilon := \{w \in H : \varphi(w) \leq l + \epsilon\}$  is closed and convex, and consequently is weakly closed. Since all but finitely many points  $\{u_{n_k}\}$  lie in  $K_\epsilon$ ,  $u \in K_\epsilon$ , and consequently

$$\varphi(u) \leq l + \epsilon = \liminf_{n \rightarrow \infty} \varphi(u_n) + \epsilon.$$

Since the above inequality is true for all  $\epsilon > 0$ , (27) follows.

Next we assert that

$$\varphi(u) \geq -C - C\|u\| \quad \forall u \in H \quad (28)$$

for some constant  $C > 0$ . To verify this claim we suppose to the contrary that for each  $n \in \mathbb{N}$  there exists  $u_n \in H$  such that

$$\varphi(u_n) \leq -n - n\|u_n\| \quad \forall n \in \mathbb{N}. \quad (29)$$

If the sequence  $\{u_n\}$  is bounded in  $H$ , there exists a weakly convergent subsequence  $u_{n_k} \rightharpoonup u$ . But then, (27) and (29) imply the contradiction  $\varphi(u) = -\infty$ . Thus we may assume, passing if necessary to a subsequence, that  $\|u_n\| \rightarrow \infty$ . Select  $u_0 \in H$  so that  $\varphi(u_0) < \infty$ . Set

$$v_n := \frac{u_n}{\|u_n\|} + \left(1 - \frac{1}{\|u_n\|}\right) u_0.$$

Then, by the convexity of  $\varphi$ , we have

$$\begin{aligned} \varphi(v_n) &\leq \frac{1}{\|u_n\|} \varphi(u_n) + \left(1 - \frac{1}{\|u_n\|}\right) \varphi(u_0) \\ &\leq \frac{1}{\|u_n\|} (-n - n\|u_n\|) + |\varphi(u_0)| \leq -n + |\varphi(u_0)|. \end{aligned}$$

As  $\{v_n\}$  is bounded, we can extract a weakly convergent subsequence  $v_{n_k} \rightharpoonup v$ , and again derive the contradiction  $\varphi(v) = -\infty$ . Therefore, we establish the claim (28).

Choose a minimizing sequence  $\{u_n\}$  so that

$$\lim_{n \rightarrow \infty} J(u_n) = \inf_{v \in H} J(v) = m.$$

By (28), it is not difficult to see that  $m \in \mathbb{R}$ . Then, having in mind (28), there exists  $M > 0$ , such that

$$M \geq J(u_n) \geq \frac{1}{2} \|u_n\|^2 - (\lambda C + \|w\|) \|u_n\| - \lambda C$$

$$= \frac{1}{2}(\|u_n\| - (\lambda C + \|w\|))^2 - \lambda C - \frac{1}{2}(\lambda C + \|w\|)^2.$$

Thus, we have  $\{u_n\}$  is bounded. We may then extract a weakly convergent subsequence  $u_{n_k} \rightharpoonup u$ . Then, by (27)  $J$  has a minimum at  $u$ . Therefore,  $0 \in \partial J$ . Now, it is easy to see that  $\partial J(u) = u - w + \lambda \partial \varphi(u)$ , and so

$$u + \lambda \partial \varphi(u) \ni w.$$

Finally, to see the uniqueness, suppose as well

$$\bar{u} + \lambda \partial \varphi(\bar{u}) \ni w.$$

Then,  $u + \lambda v = w$ ,  $\bar{u} + \lambda \bar{v} = w$  for  $v \in \partial \varphi(u)$ ,  $\bar{v} \in \partial \varphi(\bar{u})$ . Hence, by the monotony of  $\partial \varphi$ , we have

$$0 \leq (u - \bar{u}/v - \bar{v}) = \left(u - \bar{u}/\frac{\bar{u}}{\lambda} - \frac{u}{\lambda}\right) = -\frac{1}{\lambda}\|u - \bar{u}\|^2.$$

Since  $\lambda > 0$ ,  $u = \bar{u}$ . □

**Definition 6** Let  $\varphi : H \rightarrow ]-\infty, +\infty]$  be convex, proper and lower semi-continuous. For each  $\lambda > 0$  define the *resolvent*  $J_\lambda^\varphi$  of  $\partial \varphi$  as the operator  $J_\lambda^\varphi : H \rightarrow D(\partial \varphi)$  defined by  $J_\lambda^\varphi(w) := u$ , where  $u$  is the unique solution of

$$u + \lambda \partial \varphi(u) \ni w.$$

The *Yosida approximation* is the operator  $A_\lambda^\varphi : H \rightarrow H$  defined by

$$A_\lambda^\varphi(w) := \frac{1}{\lambda}(w - J_\lambda^\varphi(w)).$$

In the next result we collect some of the properties of the resolvent operator and the Yosida approximation.

**Theorem 19** Let  $\varphi : H \rightarrow ]-\infty, +\infty]$  be convex, proper and lower semi-continuous. For  $\lambda > 0$ , let  $J_\lambda = J_\lambda^\varphi$  and  $A_\lambda = A_\lambda^\varphi$ . The following statements hold

- (i)  $\|J_\lambda(w) - J_\lambda(\bar{w})\| \leq \|w - \bar{w}\|$  for all  $w, \bar{w} \in H$ .
- (ii)  $\|A_\lambda(w) - A_\lambda(\bar{w})\| \leq \frac{2}{\lambda}\|w - \bar{w}\|$  for all  $w, \bar{w} \in H$ .
- (iii)  $0 \leq (w - \bar{w}/A_\lambda(w) - A_\lambda)$ , i.e.,  $A_\lambda$  is a monotone operator.
- (iv)  $A_\lambda(w) \in \partial \varphi(J_\lambda(w))$  for all  $w \in H$ .
- (v) If  $w \in D(\partial \varphi)$ , then

$$\sup_{\lambda > 0} \|A_\lambda(w)\| \leq |(\partial \varphi)^0(w)| := \min\{\|u\| : u \in \partial \varphi(w)\}.$$

(vi) For each  $w \in \overline{D(\partial\varphi)}$ ,

$$\lim_{\lambda \downarrow 0} J_\lambda(w) = w.$$

**Proof.** (i) Let  $u = J_\lambda(w)$ ,  $\bar{u} = J_\lambda(\bar{w})$ . Then  $u + \lambda v = w$ ,  $\bar{u} + \lambda \bar{v} = \bar{w}$  for some  $v \in \partial\varphi(u)$ ,  $\bar{v} \in \partial\varphi(\bar{u})$ . Therefore

$$\begin{aligned} \|w - \bar{w}\|^2 &= \|u - \bar{u} + \lambda(v - \bar{v})\|^2 \\ &= \|u - \bar{u}\|^2 + 2\lambda(u - \bar{u}/v - \bar{v}) + \lambda^2\|v - \bar{v}\|^2 \geq \|u - \bar{u}\|^2. \end{aligned}$$

This prove assertion (i). Assertion (ii) follows from (i) and the definition of Yosida approximation.

(iii) We have

$$\begin{aligned} (w - \bar{w}/A_\lambda(w) - A_\lambda) &= \frac{1}{\lambda}(w - \bar{w}/w - \bar{w} - (J_\lambda(w) - J_\lambda(\bar{w}))) \\ &= \frac{1}{\lambda}(\|w - \bar{w}\|^2 - (w - \bar{w}/ - (J_\lambda(w) - J_\lambda(\bar{w})))) \\ &\geq \frac{1}{\lambda}(\|w - \bar{w}\|^2 - \|w - \bar{w}\|\|J_\lambda(w) - J_\lambda(\bar{w})\|) \geq 0, \end{aligned}$$

according to (i).

(iv) Note that  $u = J_\lambda(w)$  if and only if  $u + \lambda v = w$  for some  $v \in \partial\varphi(u) = \partial\varphi(J_\lambda(w))$ .

But

$$v = \frac{1}{\lambda}(w - u) = \frac{1}{\lambda}(w - J_\lambda(w)) = A_\lambda(w).$$

(v) Assume  $w \in D(\partial\varphi)$ ,  $u \in \partial\varphi(w)$ . Let  $z = J_\lambda(w)$ , so that  $z + \lambda v = w$ , where  $v \in \partial\varphi(z)$ . The, by monotonicity, we have

$$0 \leq (w - z/u - v) = \left( w - J_\lambda(w)/u - \frac{1}{\lambda}(w - J_\lambda(w)) \right) = (\lambda A_\lambda(w)/u - A_\lambda(w)).$$

Consequently

$$\lambda\|A_\lambda(w)\|^2 \leq (\lambda A_\lambda(w)/u) \leq \lambda\|A_\lambda(w)\| \|u\|,$$

and so

$$\|A_\lambda(w)\| \leq \|u\|.$$

Since this estimate is valid for all  $\lambda > 0$  and  $u \in \partial\varphi(w)$ , assertion (v) follows.

(vi) If  $w \in D(\partial\varphi)$ , by (v), we have

$$\|J_\lambda(w) - w\| = \lambda\|A_\lambda(w)\| \leq \lambda|(\partial\varphi)^0(w)|,$$

and hence

$$\lim_{\lambda \downarrow 0} J_\lambda(w) = w.$$

Let  $w \in \overline{D(\partial\varphi)}$ . Given  $\epsilon > 0$  there exists  $\bar{w} \in D(\partial\varphi)$  such that  $\|w - \bar{w}\| \leq \frac{\epsilon}{4}$ . Now, since  $\bar{w} \in D(\partial\varphi)$ , there exists  $\lambda_0 > 0$  such that  $\|J_{\lambda_0}(\bar{w}) - \bar{w}\| \leq \frac{\epsilon}{2}$ . Then,

$$\begin{aligned} \|J_{\lambda_0}(w) - w\| &\leq \|J_{\lambda_0}(w) - J_{\lambda_0}(\bar{w})\| + \|J_{\lambda_0}(\bar{w}) - \bar{w}\| + \|w - \bar{w}\| \\ &\leq 2\|w - \bar{w}\| + \|J_{\lambda_0}(\bar{w}) - \bar{w}\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

## 4.2 Gradient Flows in Hilbert spaces

Many problems in Physics and Mechanics can be written as a gradient system, that is a system of ordinary differential equation of the form

$$\begin{cases} u'(t) = -\nabla V(u(t)) & 0 < t < T \\ u(0) = u_0 \in \mathbb{R}^N, \end{cases}$$

where  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a potential. In this section we are going to consider the generalization infinite dimensional (in the context of Hilbert spaces) of the gradient system. We propose now to study differential equation of the form

$$\begin{cases} u'(t) + \partial\varphi(u(t)) \ni 0 & t \geq 0 \\ u(0) = u_0 \in H, \end{cases} \quad (30)$$

where  $\varphi : H \rightarrow ]-\infty, +\infty]$  is a convex, proper and lower semi-continuous function. A problem of the form (63) is called a *gradient flow*. Many partial differential equation can be rewritten as a gradient flow in an appropriate Hilbert space of functions. For example, as we see in the Example 2, if  $\Omega \subset \mathbb{R}^N$  is an open bounded set with smooth boundary, and we consider the function  $\varphi : L^2(\Omega) \rightarrow ]-\infty, +\infty]$  defined by

$$\varphi(u) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 & \text{if } u \in W_0^{1,2}(\Omega) \\ +\infty & u \in L^2(\Omega) \setminus W_0^{1,2}(\Omega). \end{cases}$$

Then,

$$D(\partial\varphi) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$$

and

$$v \in \partial\varphi(u) \Leftrightarrow v = -\Delta u.$$

Therefore, the initial valued problem for the heat equation

$$\begin{cases} u_t = \Delta u & \text{in } (0, \infty) \times \Omega \\ u = 0 & \text{on } (0, \infty) \times \partial\Omega \\ u(0, x) = u_0(x) & x \in \Omega \end{cases}$$

can be rewritten as a gradient flow in  $L^2(\Omega)$ .

We have the following existence and uniqueness result for solutions of the gradient flows.

**Theorem 20** *Let  $\varphi : H \rightarrow ]-\infty, +\infty]$  be convex, proper and lower semi-continuous. For each  $u_0 \in D(\partial\varphi)$  there exists a unique function  $u \in C([0, \infty[, H)$ , with  $u' \in L^\infty(0, \infty; H)$  such that  $u(0) = u_0$ ,  $u(t) \in D(\partial\varphi)$  for each  $t > 0$  and*

$$u'(t) + \partial\varphi(u(t)) \ni 0, \quad \text{for a.e. } t \geq 0.$$

**Proof.** For  $\lambda > 0$ , let  $J_\lambda = J_\lambda^\varphi$  the resolvent of  $\partial\varphi$  and  $A_\lambda = A_\lambda^\varphi$  its Yosida approximation. By Theorem 19,  $A_\lambda : H \rightarrow H$  is Lipschitz continuous mapping, and thus, by the classical Picard-Lindelöf Theorem there exists a unique solution  $u_\lambda \in C^1([0, \infty[; H)$  of the problem

$$\begin{cases} u'_\lambda(t) + A_\lambda(u_\lambda(t)) = 0 & t \geq 0 \\ u_\lambda(0) = u_0. \end{cases} \quad (31)$$

Our plan is to show that as  $\lambda \rightarrow 0^+$  the functions  $u_\lambda$  converge to a solutions of our problem. We divide the proof in several steps.

*Step 1.* Given  $v \in H$ , let  $v_\lambda$  the solution of the problem

$$\begin{cases} v'_\lambda(t) + A_\lambda(v_\lambda(t)) = 0 & t \geq 0 \\ v_\lambda(0) = v. \end{cases} \quad (32)$$

Then, by the monotony of  $A_\lambda$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_\lambda(t) - v_\lambda(t)\|^2 \\ &= (u'_\lambda(t) - v'_\lambda(t)/u_\lambda(t) - v_\lambda(t)) = (-A_\lambda(u_\lambda(t)) + A_\lambda(v_\lambda(t)))/u_\lambda(t) - v_\lambda(t) \leq 0. \end{aligned}$$

Hence, integrating we get

$$\|u_\lambda(t) - v_\lambda(t)\| \leq \|u_0 - v\| \quad \forall t \geq 0. \quad (33)$$

In particular, if  $h > 0$  and  $v = u_\lambda(h)$ , then by uniqueness  $v_\lambda(t) = u_\lambda(t+h)$ . Consequently, (33) implies

$$\|u_\lambda(t+h) - u_\lambda(t)\| \leq \|u_\lambda(h) - u_0\|.$$

Dividing by  $h$ , letting  $h \rightarrow 0$ , and having in mind Theorem 19 (v), we obtain that

$$\|u'_\lambda(t)\| \leq \|u'_\lambda(0)\| = \|A_\lambda(u_0)\| \leq |(\partial\varphi)^0(u_0)|. \quad (34)$$

*Step 2.* We next take  $\lambda, \mu > 0$  and compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\lambda(t) - u_\mu(t)\|^2 &= (u'_\lambda(t) - u'_\mu(t)/u_\lambda(t) - u_\mu(t)) \\ &= (-A_\lambda(u_\lambda(t)) + A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)). \end{aligned} \quad (35)$$

Now

$$\begin{aligned} u_\lambda(t) - u_\mu(t) &= (u_\lambda(t) - J_\lambda(u_\lambda(t))) + (J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t))) + (J_\mu(u_\mu(t)) - u_\mu(t)) \\ &= \lambda A_\lambda(u_\lambda(t)) + J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t)) - \mu A_\mu(u_\mu(t)). \end{aligned}$$

Consequently

$$\begin{aligned} &(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)) \\ &= (A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/J_\lambda(u_\lambda(t)) - J_\mu(u_\mu(t))) \\ &+ (A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/\lambda A_\lambda(u_\lambda(t))) - \mu A_\mu(u_\mu(t)). \end{aligned} \quad (36)$$

Since  $A_\lambda(u_\lambda(t)) \in \partial\varphi(J_\lambda(u_\lambda(t)))$  and  $A_\mu(u_\mu(t)) \in \partial\varphi(J_\mu(u_\mu(t)))$ , the monotonicity of  $\partial\varphi$  implies that the first term of the right hand side of (36) is nonnegative. Thus

$$\begin{aligned} &(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)) \geq \\ &\lambda \|A_\lambda(u_\lambda(t))\|^2 + \mu \|A_\mu(u_\mu(t))\|^2 - (\lambda + \mu) \|A_\lambda(u_\lambda(t))\| \|A_\mu(u_\mu(t))\|. \end{aligned}$$

Since

$$\begin{aligned} &(\lambda + \mu) \|A_\lambda(u_\lambda(t))\| \|A_\mu(u_\mu(t))\| \leq \\ &\lambda \left( \|A_\lambda(u_\lambda(t))\|^2 + \frac{1}{4} \|A_\mu(u_\mu(t))\|^2 \right) + \mu \left( \|A_\mu(u_\mu(t))\|^2 + \frac{1}{4} \|A_\lambda(u_\lambda(t))\|^2 \right) \end{aligned}$$

we deduce

$$(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t))/u_\lambda(t) - u_\mu(t)) \geq -\frac{\lambda}{4} \|A_\mu(u_\mu(t))\|^2 - \frac{\mu}{4} \|A_\lambda(u_\lambda(t))\|^2.$$

Then, by (34), we get

$$(A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t)))/u_\lambda(t) - u_\mu(t) \geq -\frac{\lambda + \mu}{4}|(\partial\varphi)^0(u_0)|.$$

Therefore, by (35) and (36), we obtain the inequality

$$\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 \leq \frac{\lambda + \mu}{2}|(\partial\varphi)^0(u_0)|$$

and hence

$$\|u_\lambda(t) - u_\mu(t)\|^2 \leq \frac{\lambda + \mu}{2}t|(\partial\varphi)^0(u_0)| \quad \forall t \geq 0. \quad (37)$$

In view of the estimate (37) there exists a function  $u \in C([0, \infty[, H)$  such that

$$u_\lambda \rightarrow u \quad \text{uniformly in } C([0, T], H)$$

as  $\lambda \downarrow 0$ , for each time  $T > 0$ . Furthermore estimate (34) implies

$$u'_\lambda \rightharpoonup u' \quad \text{weakly in } L^2(0, T; H) \quad (38)$$

for each  $T > 0$ , and

$$\|u'(t)\| \leq |(\partial\varphi)^0(u_0)| \quad \text{a.e. } t. \quad (39)$$

*Step 3.* We must show  $u(t) \in D(\partial\varphi)$  for each  $t \geq 0$  and

$$u'(t) + \partial\varphi(u(t)) \ni 0, \quad \text{for a.e. } t \geq 0.$$

Now, by (34)

$$\|J_\lambda(u_\lambda(t)) - u_\lambda(t)\| = \lambda\|A_\lambda u_\lambda(t)\| = \lambda\|u'_\lambda(t)\| \leq \lambda|(\partial\varphi)^0(u_0)|.$$

Hence

$$J_\lambda(u_\lambda) \rightarrow u \quad \text{uniformly in } C([0, T], H) \quad (40)$$

for each  $T > 0$ .

On the other hand, for each  $t \geq 0$ ,

$$-u'_\lambda(t) = A_\lambda(u_\lambda(t)) \in \partial\varphi(J_\lambda(u_\lambda(t))).$$

Thus, given  $w \in H$ , we have

$$\varphi(w) \geq \varphi(J_\lambda(u_\lambda(t))) - (u'_\lambda(t)/w - J_\lambda(u_\lambda(t))).$$

Consequently if  $0 \leq s \leq t$ ,

$$(t-s)\varphi(w) \geq \int_s^t \varphi(J_\lambda(u_\lambda(\tau))) d\tau - \int_s^t (u'_\lambda(\tau)/w - J_\lambda(u_\lambda(\tau))) d\tau.$$

In view of (40), the lower semi-continuity of  $\varphi$ , and Fatou's Lemma, we conclude upon sending  $\lambda \downarrow 0$  that

$$(t-s)\varphi(w) \geq \int_s^t \varphi(u(\tau)) d\tau - \int_s^t (u'(\tau)/w - u(\tau)) d\tau.$$

Therefore

$$\varphi(w) \geq \varphi(u(t)) - (u'(t)/w - u(t))$$

if  $t$  is a Lebesgue point of  $u'$ ,  $\varphi(u)$ . Hence, for almost all  $t \geq 0$

$$\varphi(w) \geq \varphi(u(t)) - (u'(t)/w - u(t))$$

for all  $w \in H$ . Thus  $u(t) \in D(\partial\varphi)$ , with

$$u'(t) + \partial\varphi(u(t)) \ni 0, \quad \text{for a.e. } t \geq 0.$$

Finally, we prove that  $u(t) \in D(\partial\varphi)$  for each  $t \geq 0$ . To see this, fix  $t \geq 0$  and choose  $t_n \rightarrow t$  such that  $u(t_n) \in D(\partial\varphi)$ ,  $-u'(t_n) \in \partial\varphi(u(t_n))$ . In view of (39) we may assume, upon passing to a subsequence, that

$$u'(t_n) \rightharpoonup v \quad \text{weakly in } H.$$

Fix  $w \in H$ . Then

$$\varphi(w) \geq \varphi(u(t_n)) - (u'(t_n)/w - u(t_n)).$$

Let  $t_n \rightarrow t$  and recall that  $u \in C([0, \infty[, H)$  and  $\varphi$  is lower semi-continuous. We obtain that

$$\varphi(w) \geq \varphi(u(t)) - (v/w - u(t)).$$

Hence  $u(t) \in D(\partial\varphi)$  and  $-v \in \partial\varphi(u(t))$ .

*Step 4.* To prove uniqueness assume  $\bar{u}$  is another solution and compute

$$\frac{1}{2} \frac{d}{dt} \|u(t) - \bar{u}(t)\|^2 = (u'(t) - \bar{u}'(t)/u(t) - \bar{u}(t)) \leq 0 \quad \text{for a.e. } t \geq 0,$$

since  $-u'(t) \in \partial\varphi(u(t))$  and  $-\bar{u}'(t) \in \partial\varphi(\bar{u}(t))$ . Then, integrating we obtain that

$$\|u(t) - \bar{u}(t)\|^2 \leq \|u(0) - \bar{u}(0)\|^2.$$

□

Under the assumptions of the above theorem if for each  $u_0 \in D(\partial\varphi)$  we define

$$S(t)u_0 := u(t) \quad \forall t \geq 0,$$

$u(t)$  being the unique solution of problem

$$\begin{cases} u'(t) + \partial\varphi(u(t)) \ni 0, & \text{for a.e. } t \geq 0 \\ u(0) = u_0, \end{cases} \quad (41)$$

we have the family of operator  $(S(t))_{t \geq 0}$  satisfying

- (i)  $S(0) = I$ ,
- (ii)  $S(t+s) = S(t)S(s)$  for all  $s, t \geq 0$ ,
- (iii) the mapping  $t \mapsto S(t)u_0$  is continuous from  $[0, \infty[$  into  $H$ .

A family of operators  $(S(t))_{t \geq 0}$  satisfying the conditions (i)-(iii) is called a *nonlinear semigroup of operators*.

Observe that as a consequence of the above theorem we have

$$\|S(t)u_0 - S(t)\bar{u}_0\| \leq \|u_0 - \bar{u}_0\|, \quad \forall t \geq 0, \text{ and } u_0, \bar{u}_0 \in D(\partial\varphi). \quad (42)$$

Using this inequality the semigroup of nonlinear operators  $(S(t))_{t \geq 0}$  can be extended to  $\overline{D(\partial\varphi)}$ . In the case  $D(\partial\varphi)$  is dense in  $H$ , which happens in many applications, we have  $(S(t))_{t \geq 0}$  is a nonlinear semigroup in  $H$ .

Theorem 20 is a particular case of the following general situation. Let  $A \subset H \times H$  an operator (possible multivaluate) in the real Hilbert space  $H$ . We say that  $A$  is *monotone* if

$$(u - \bar{u}/v - \bar{v}) \geq 0 \quad \forall (u, \bar{u}), (v, \bar{v}) \in A.$$

Recall we have showed that  $\partial\varphi$  is a monotone operator. Now, if  $\varphi$  is convex, lower semi-continuous and proper, it can be proved that  $\partial\varphi$  is maximal monotone (see, [9], [8]), i.e., every monotone extension of  $\partial\varphi$  coincides with  $\partial\varphi$ . The following theorem is a classical result due to G. Minty [20].

**Theorem 21 (Minty Theorem)** *Let  $A$  a monotone operator in the real Hilbert space  $H$ . Then,  $A$  is maximal monotone if and only if  $\text{Ran}(I + \lambda A) = H$  for all  $\lambda > 0$ .*

Given an operator  $A \subset H \times H$ , consider the abstract Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni 0, & \text{for a.e. } t \in (0, T) \\ u(0) = u_0. \end{cases} \quad (43)$$

We say that a function  $u \in C([0, T]; H)$  is a *strong solution* of problem (43) if  $u(0) = u_0$ ,  $u$  is derivable a.e.  $t \in (0, T)$ ,  $u(t) \in D(A)$  and satisfies (43) for almost all  $t \in (0, T)$ .

Theorem 20 states that for every  $u_0 \in D(\partial\varphi)$ ,  $u(t) = S(t)u_0$  is a strong solution of the abstract Cauchy problem associated with  $\partial\varphi$ . Now, this result is also true in the general case in which  $A$  is a maximal monotone operator (see [9], [8]). Moreover, in the case  $A = \partial\varphi$ , with  $\varphi : H \rightarrow ]-\infty, +\infty]$  a convex, proper and lower semi-continuous, we also have (see [9]) that for all  $u_0 \in \overline{D(\partial\varphi)}$ ,  $u(t) = S(t)u_0$  is a strong solution.

## 5 The Neumann Problem for the Total Variation Flow

This section is devoted to prove existence and uniqueness of solutions for the Minimizing Total Variation Flow with Neumann boundary conditions, namely

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div} \left( \frac{Du}{|Du|} \right) & \text{in } Q = (0, \infty) \times \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } S = (0, \infty) \times \partial\Omega \\ u(0, x) = u_0(x) & \text{in } x \in \Omega, \end{cases} \quad (44)$$

where  $\Omega$  is a bounded set in  $\mathbb{R}^N$  with Lipschitz continuous boundary  $\partial\Omega$  and  $u_0 \in L^1(\Omega)$ . As we saw in the previous section, this partial differential equation appears when one uses the steepest descent method to minimize the Total Variation, a method introduced by L. Rudin, S. Osher and E. Fatemi ([23]) in the context of image denoising and reconstruction. Then solving (44) amounts to regularize or, in other words, to filter the initial datum  $u_0$ . This filtering process has less destructive effect on the edges than filtering with a Gaussian, i.e., than solving the heat equation with initial condition  $u_0$ . In this context the given *image*  $u_0$  is a function defined on a bounded, smooth or piecewise smooth open subset  $\Omega$  of  $\mathbb{R}^N$ , typically,  $\Omega$  will be a rectangle in  $\mathbb{R}^2$ . As argued in [1], the choice of Neumann boundary conditions is a natural choice in image processing. It corresponds to the reflection of the picture across the boundary and has the advantage of not imposing any value on the boundary and not creating edges on it. When dealing with the deconvolution or reconstruction problem one minimizes the Total Variation Functional, i.e., the functional

$$\int_{\Omega} |Du| \quad (45)$$

under some constraints which model the process of image acquisition, including blur and noise.

## 5.1 Strong Solutions in $L^2(\Omega)$

Consider the energy functional  $\Phi : L^2(\Omega) \rightarrow (-\infty, +\infty]$  defined by

$$\Phi(u) = \begin{cases} \int_{\Omega} \|Du\| & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \quad (46)$$

Since the functional  $\Phi$  is convex, lower semi-continuous and proper, then  $\partial\Phi$  is a maximal monotone operator with dense domain, generating a contraction semigroup in  $L^2(\Omega)$  (see subsection 4.2 or [9]). Therefore, we have the following result.

**Theorem 22** *Let  $u_0 \in L^2(\Omega)$ . Then there exists a unique strong solution in the semi-group sense  $u$  of (44) in  $[0, T]$  for every  $T > 0$ , i.e.,  $u \in C([0, T]; L^2(\mathbb{R}^N)) \cap W_{\text{loc}}^{1,2}(0, T; L^2(\Omega))$ ,  $u(t) \in D(\partial\Phi)$  a.e. in  $t \in [0, T]$  and*

$$-u'(t) \in \partial\Phi(u(t)) \quad \text{a.e. in } t \in [0, T]. \quad (47)$$

*Moreover, if  $u$  and  $v$  are the strong solutions of (44) corresponding to the initial conditions  $u_0, v_0 \in L^2(\Omega)$ , then*

$$\|u(t) - v(t)\|_2 \leq \|u_0 - v_0\|_2 \quad \text{for any } t > 0. \quad (48)$$

Our task will be to give a sense to (47) as a partial differential equation, describing the subdifferential of  $\Phi$  in a distributional sense. To be precise we should not say distributional sense since the test functions will be functions in  $BV(\Omega)$ . To do that we need to recall first some results inspired in the duality theory of the Convex Analysis.

Let  $H$  be a real Hilbert space, with inner product  $(\ / )$ . Let  $\Psi : H \rightarrow [0, \infty]$  be any function. Let us define  $\tilde{\Psi} : H \rightarrow [0, \infty]$  by

$$\tilde{\Psi}(x) = \sup \left\{ \frac{(x/y)}{\Psi(y)} : y \in H \right\} \quad (49)$$

with the convention that  $\frac{0}{0} = 0$ ,  $\frac{0}{\infty} = 0$ . Note that  $\tilde{\Psi}(x) \geq 0$ , for any  $x \in H$ . Note also that the supremum is attained on the set of  $y \in H$  such that  $(x/y) \geq 0$ . Note also that we have the following Cauchy-Schwartz inequality

$$(x/y) \leq \tilde{\Psi}(x)\Psi(y) \quad \text{if } \Psi(y) > 0.$$

The following Lemma is a simple consequence of the above definition.

**Lemma 3** *Let  $\Psi_1, \Psi_2 : H \rightarrow [0, \infty]$ . If  $\Psi_1 \leq \Psi_2$ , then  $\tilde{\Psi}_2 \leq \tilde{\Psi}_1$ .*

**Proposition 5** *If  $\Psi$  is convex, lower semi-continuous and positive homogeneous of degree 1, then  $\tilde{\Psi} = \Psi$ .*

**Proof.** Since  $\frac{(y/x)}{\Psi(x)} \leq \tilde{\Psi}(y)$  for any  $x, y \in H$ , we also have that  $\frac{(y/x)}{\tilde{\Psi}(y)} \leq \Psi(x)$  for any  $x, y \in H$ . This implies that  $\tilde{\Psi}(x) \leq \Psi(x)$  for any  $x \in H$ . Assume that there is some  $x \in H$  and  $\epsilon > 0$  such that  $\tilde{\Psi}(x) + \epsilon < \Psi(x)$ , hence, in particular,  $\Psi(x) > 0$  and  $\tilde{\Psi}(x) < \infty$ . Using Hahn-Banach's Theorem there is  $y \in H$  separating  $x$  from the closed convex set  $C := \{z \in H : \Psi(z) \leq \tilde{\Psi}(x) + \epsilon\}$ . Since  $0 \in C$  we may even assume that  $(y/x) = 1$  and  $(y/z) \leq \alpha < 1$  for any  $z \in C$ . Note that, from the definition of  $\tilde{\Psi}$ , we have

$$\tilde{\Psi}(x) \geq \frac{1}{\tilde{\Psi}(y)}. \quad (50)$$

Let us prove that  $\tilde{\Psi}(y) \leq \frac{1}{\tilde{\Psi}(x) + \epsilon}$ . For that it will be sufficient to prove that

$$\frac{(y/z)}{\Psi(z)} \leq \frac{1}{\tilde{\Psi}(x) + \epsilon} \quad (51)$$

for any  $z \in H$  such that  $(y/z) \geq 0$ . Let  $z \in H$ ,  $(y/z) \geq 0$ . If  $\Psi(z) = \infty$ , then (51) holds. If  $\Psi(z) = 0$ , then also  $\Psi(tz) = 0$  for any  $t \geq 0$ . Hence  $tz \in C$  for all  $t \geq 0$ , and we have that  $0 \leq (y/tz) \leq 1$  for all  $t \geq 0$ . Thus  $(y/z) = 0$ , and, therefore, (51) holds. Finally, assume that  $0 < \Psi(z) < \infty$ . Let  $t > 0$  be such that  $\Psi(tz) = \tilde{\Psi}(x) + \epsilon$ . Using that  $tz \in C$ , we have

$$\frac{(y/z)}{\Psi(z)} = \frac{(y/tz)}{\Psi(tz)} \leq \frac{1}{\tilde{\Psi}(x) + \epsilon}.$$

Both (50) and (51) give a contradiction. We conclude that  $\tilde{\Psi}(x) = \Psi(x)$  for any  $x \in H$ .  $\square$

**Lemma 4** *Assume that  $\Psi$  is convex, lower semi-continuous and positive homogeneous of degree 1. If  $u \in D(\partial\Psi)$  and  $v \in \partial\Psi(u)$ , then  $(v/u) = \Psi(u)$ .*

**Proof.** Indeed, if  $v \in \partial\Psi(u)$ , then

$$(v/w - u) \leq \Psi(w) - \Psi(u), \quad \text{for all } w \in H.$$

To obtain the result it suffices to take  $w = 0$  and  $w = 2u$  in the above inequality.  $\square$

**Theorem 23** *Assume that  $\Psi$  is convex, lower semi-continuous and positive homogeneous of degree 1. Then  $v \in \partial\Psi(u)$  if and only if  $\tilde{\Psi}(v) \leq 1$  and  $(v/u) = \Psi(u)$  (hence,  $\tilde{\Psi}(v) = 1$  if  $\Psi(u) > 0$ ).*

**Proof.** When  $(v/u) = \Psi(u)$ , condition  $v \in \partial\Psi(u)$  may be written as  $(v/x) \leq \Psi(x)$  for all  $x \in H$ , which is equivalent to  $\tilde{\Psi}(v) \leq 1$ .  $\square$

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary. Let us consider the space (see Section 3)

$$X(\Omega)_2 := \left\{ z \in L^\infty(\Omega, \mathbb{R}^N) : \operatorname{div}(z) \in L^2(\Omega) \right\}.$$

Let us define for  $v \in L^2(\Omega)$

$$\Psi(v) = \inf \left\{ \|z\|_\infty : z \in X(\Omega)_2, v = -\operatorname{div}(z) \text{ in } \mathcal{D}'(\Omega), [z, \nu] = 0 \right\}, \quad (52)$$

where  $\nu$  denotes the outward unit normal to  $\partial\Omega$  and  $[z, \nu]$  is the trace of the normal component of  $z$  (see Section 3). We define  $\Psi(v) = +\infty$  if does not exists  $z \in X(\Omega)_2$  satisfying  $v = -\operatorname{div}(z)$  in  $\mathcal{D}'(\Omega)$ ,  $[z, \nu] = 0$ .

Observe that  $\Psi$  is convex, lower semi-continuous and positive homogeneous of degree 1. Moreover, it is easy to see that, if  $\Psi(v) < \infty$ , the infimum in (52) is attained, i.e., there is some  $z \in X(\Omega)_2$  such that  $v = -\operatorname{div}(z)$  in  $\mathcal{D}'(\Omega)$ ,  $[z, \nu] = 0$  and  $\Psi(v) = \|z\|_\infty$ .

**Proposition 6** *We have that  $\Psi = \tilde{\Phi}$ .*

**Proof.** Let  $v \in L^2(\Omega)$ . If  $\Psi(v) = \infty$ , then we have  $\tilde{\Phi}(v) \leq \Psi(v)$ . Thus, we may assume that  $\Psi(v) < \infty$ . Let  $z \in X(\Omega)_2$  be such that  $v = -\operatorname{div}(z)$  and  $[z, \nu] = 0$ . Then

$$\int_{\Omega} v u \, dx = \int_{\Omega} (z, Du) \leq \|z\|_\infty \Phi(u) \quad \text{for all } u \in BV(\Omega) \cap L^2(\Omega).$$

Taking supremums in  $u$  we obtain  $\tilde{\Phi}(v) \leq \|z\|_\infty$ . Now, taking infimums in  $z$ , we obtain  $\tilde{\Phi}(v) \leq \Psi(v)$ .

To prove the opposite inequality, let us denote

$$D = \left\{ \operatorname{div}(z) : z \in C_0^\infty(\Omega, \mathbb{R}^N) \right\}.$$

Then

$$\begin{aligned} \sup_{v \in L^2} \frac{\int_{\Omega} v u \, dx}{\Psi(v)} &\geq \sup_{v \in D} \frac{\int_{\Omega} v u \, dx}{\Psi(v)} \geq \sup_{v \in D, \Psi(v) < \infty} \frac{\int_{\Omega} v u \, dx}{\Psi(v)} \\ &\geq \sup_{z \in C_0^\infty(\Omega, \mathbb{R}^N)} \frac{-\int_{\Omega} u \operatorname{div}(z) \, dx}{\|z\|_\infty} = \Phi(u). \end{aligned}$$

Thus,  $\Phi \leq \tilde{\Psi}$ . This implies that  $\tilde{\Psi} \leq \tilde{\Phi}$ , and, using Proposition 5, we obtain that  $\Psi \leq \tilde{\Phi}$ .  $\square$

We have the following characterization of the subdifferential  $\partial\Phi$ .

**Theorem 24** *The following assertions are equivalent:*

(a)  $v \in \partial\Phi(u)$ ;

(b)

$$u \in L^2(\Omega) \cap BV(\Omega), \quad v \in L^2(\Omega), \quad (53)$$

$$\exists z \in X(\Omega)_2, \quad \|z\|_\infty \leq 1, \quad \text{such that } v = -\operatorname{div}(z) \quad \text{in } \mathcal{D}'(\Omega), \quad (54)$$

and

$$\int_\Omega (z, Du) = \int_\Omega \|Du\|, \quad (55)$$

$$[z, \nu] = 0 \quad \text{on } \partial\Omega; \quad (56)$$

(c) (53) and (54) hold, and

$$\int_\Omega (w - u)v \, dx \leq \int_\Omega z \cdot \nabla w \, dx - \int_\Omega \|Du\|, \quad \forall w \in W^{1,1}(\Omega) \cap L^2(\Omega); \quad (57)$$

(d) (53) and (54) hold, and

$$\int_\Omega (w - u)v \, dx \leq \int_\Omega (z, Dw) - \int_\Omega \|Du\| \quad \forall w \in L^2(\Omega) \cap BV(\Omega); \quad (58)$$

(e) (53) and (54) hold, and (58) holds with the equality instead of the inequality.

**Proof.** By Theorem 23, we have that  $v \in \partial\Phi(u)$  if and only if  $\tilde{\Phi}(v) \leq 1$  and  $\int_\Omega vu \, dx = \Phi(u)$ . Since  $\tilde{\Phi} = \Psi$ , from the definition of  $\Psi$  and the observation following it, it follows that there is some  $z \in X(\Omega)_2$  such that  $v = -\operatorname{div}(z)$  in  $\mathcal{D}'(\Omega)$ ,  $[z, \nu] = 0$  and  $\tilde{\Phi}(v) = \|z\|_\infty$ . Hence, we have  $v \in \partial\Phi(u)$  if and only if there is some  $z \in X(\Omega)_2$ , with  $\|z\|_\infty \leq 1$ , such that  $v = -\operatorname{div}(z)$  in  $\mathcal{D}'(\Omega)$ ,  $[z, \nu] = 0$  and  $\int_\Omega vu \, dx = \Phi(u)$ . Then, applying Green's formula (23) the equivalence of (a) and (b) follows.

To obtain (e) from (b) it suffices to multiply both terms of the equation  $v = -\operatorname{div}(z)$  by  $w - u$ , for  $w \in L^2(\Omega) \cap BV(\Omega)$  and to integrate by parts using Green's formula (23). It is clear that (e) implies (d), and (d) implies (c). To prove that (b) follows from (d) we choose  $w = u$  in (58) and we obtain that

$$\int_\Omega \|Du\| \leq \int_\Omega (z, Du) \leq \|z\|_\infty \int_\Omega \|Du\| \leq \int_\Omega \|Du\|.$$

To obtain (56) we choose  $w = u \pm \varphi$  in (58) with  $\varphi \in C^\infty(\overline{\Omega})$  and we obtain

$$\pm \int_\Omega v\varphi \, dx \leq \pm \int_\Omega z \cdot D\varphi = -\pm \int_\Omega \operatorname{div}(z) \varphi \, dx + \pm \int_{\partial\Omega} [z, \nu] \varphi \, d\mathcal{H}^{N-1},$$

which implies (56). In order to prove that (c) implies (d), let  $w \in BV(\Omega) \cap L^2(\Omega)$ . Using Theorem 2 we know that there exists a sequence  $w_n \in C^\infty(\Omega) \cap BV(\Omega) \cap L^2(\Omega)$  such that

$$w_n \rightarrow w \text{ in } L^2(\Omega) \text{ and } \int_{\Omega} |\nabla w_n| dx \rightarrow \int_{\Omega} \|Dw\|.$$

Then

$$\begin{aligned} \int_{\Omega} z \cdot \nabla w_n dx &= - \int_{\Omega} \operatorname{div}(z) w_n dx + \int_{\partial\Omega} [z, \nu] w_n d\mathcal{H}^{N-1} \\ &\rightarrow - \int_{\Omega} \operatorname{div}(z) w dx + \int_{\partial\Omega} [z, \nu] w d\mathcal{H}^{N-1} = \int_{\Omega} (z, Dw). \end{aligned}$$

Now, we use  $w_n$  as test function in (57) and let  $n \rightarrow \infty$  to obtain (58).  $\square$

**Definition 7** *We say that  $u \in C([0, T]; L^2(\Omega))$  is a strong solution of (44) if*

$$u \in W_{\text{loc}}^{1,2}(0, T; L^2(\Omega)) \cap L_w^1([0, T]; BV(\Omega)),$$

$u(0) = u_0$ , and there exists  $z \in L^\infty([0, T] \times \Omega; \mathbb{R}^N)$  such that  $\|z\|_\infty \leq 1$ ,

$$[z(t), \nu] = 0 \text{ in } \partial\Omega, \text{ a.e. } t \in [0, T]$$

satisfying

$$u_t = \operatorname{div}(z) \text{ in } \mathcal{D}'([0, T] \times \Omega)$$

and

$$\begin{aligned} \int_{\Omega} (u(t) - w) u_t(t) dx &= \int_{\Omega} (z(t), Dw) - \int_{\Omega} \|Du(t)\| \\ &\forall w \in L^2(\Omega) \cap BV(\Omega), \text{ a.e. } t \in [0, T]. \end{aligned} \tag{59}$$

Obviously, using Theorem 24, a strong solution of (44) is a strong solution in the sense of semigroups. The converse implication follows along the same lines, except for the measurability of  $z(t, x)$ . To ensure the joint measurability of  $z$  one takes into account that, by Theorem 20, semigroup solutions can be approximated by implicit in time discretizations of (47), and one constructs a function  $z(t, x) \in L^\infty((0, T) \times \Omega)$  satisfying the requirements contained in Definition 7. We do not give the details of this proof here. We have obtained the following result.

**Theorem 25** *Let  $u_0 \in L^2(\Omega)$ . Then there exists a unique strong solution  $u$  of (44) in  $[0, T] \times \Omega$  for every  $T > 0$ . Moreover, if  $u$  and  $v$  are the strong solutions of (44) corresponding to the initial conditions  $u_0, v_0 \in L^2(\Omega)$ , then*

$$\|u(t) - v(t)\|_2 \leq \|u_0 - v_0\|_2 \text{ for any } t > 0. \tag{60}$$

**Remark 4** It is possible to obtain existence and uniqueness of solutions for any initial datum in  $L^1(\Omega)$ . In this case we need to use truncation functions of type  $T_k$ :  $T_k(r) = [k - (k - |r|)^+] \text{sign}_0(r)$ ,  $k \geq 0$ ,  $r \in \mathbb{R}$ , and the concept of solution is the following

**Definition 8** A measurable function  $u : (0, T) \times \Omega \rightarrow \mathbb{R}$  is a *weak solution* of (44) in  $(0, T) \times \Omega$  if  $u \in C([0, T], L^1(\Omega)) \cap W_{loc}^{1,1}(0, T; L^1(\Omega))$ ,  $T_k(u) \in L_w^1(0, T; BV(\Omega))$  for all  $k > 0$  and there exists  $z \in L^\infty((0, T) \times \Omega)$  with  $\|z\|_\infty \leq 1$ ,  $u_t = \text{div}(z)$  in  $\mathcal{D}'((0, T) \times \Omega)$  such that

$$\int_{\Omega} (T_k(u(t)) - w)u_t(t) dx \leq \int_{\Omega} z(t) \cdot \nabla w dx - \int_{\Omega} \|DT_k(u(t))\| \quad (61)$$

for every  $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$  and a.e. on  $[0, T]$ .

In [3] (see also [5]) we prove the following existence and uniqueness result.

**Theorem 26** *Let  $u_0 \in L^1(\Omega)$ . Then there exists a unique weak solution of (44) in  $(0, T) \times \Omega$  for every  $T > 0$  such that  $u(0) = u_0$ . Moreover, if  $u(t), \hat{u}(t)$  are weak solutions corresponding to initial data  $u_0, \hat{u}_0$ , respectively, then*

$$\|(u(t) - \hat{u}(t))^+\|_1 \leq \|(u_0 - \hat{u}_0)^+\|_1 \quad \text{and} \quad \|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1, \quad (62)$$

for all  $t \geq 0$ .

To prove Theorem 26 we shall use the techniques of completely accretive operators and the Crandall-Liggett's semigroup generation Theorem ([12]). For that, we shall associate a completely accretive operator  $\mathcal{A}$  to the formal differential expression  $-\text{div}(\frac{Du}{|Du|})$  together with Neumann boundary conditions. Then, using Crandall-Liggett's semigroup generation Theorem we conclude that the abstract Cauchy problem in  $L^1(\Omega)$

$$\begin{cases} \frac{du}{dt} + \mathcal{A}u \ni 0, \\ u(0) = u_0 \end{cases} \quad (63)$$

has a unique strong solution  $u \in C([0, T], L^1(\Omega)) \cap W_{loc}^{1,1}(0, T; L^1(\Omega))$  ( $\forall T > 0$ ) with initial datum  $u(0) = u_0$ , and we shall prove that strong solutions of (63) coincide with weak solutions of (44).

## 5.2 Asymptotic Behaviour of Solutions

To see that our concept of solution is useful we are going to compute explicitly the evolution of the characteristic function of a ball.

**Theorem 27** Let  $\Omega = B(0, R)$  be the ball in  $\mathbb{R}^N$  centered at 0 with radius  $R$ , and  $u_0(x) = k\chi_{B(0,r)}$ , where  $0 < r < R$  and  $k > 0$ . Then, the strong solution of (44) for the initial datum  $u_0$  is given by

$$u(t) = \begin{cases} \left(k - \frac{N}{r}t\right) \chi_{B(0,r)} + \frac{Nr^{N-1}}{R^N - r^N}t \chi_{B(0,R) \setminus B(0,r)} & \text{if } 0 \leq t \leq T \\ \left(k - \frac{N}{r}T\right) \chi_{B(0,R)} = \frac{Nr^{N-1}}{R^N - r^N}T \chi_{B(0,R)} & \text{if } t \geq T, \end{cases} \quad (64)$$

where  $T$  is given by

$$T \left( \frac{N}{r} + N \frac{r^{N-1}}{R^N - r^N} \right) = k. \quad (65)$$

**Proof.** We look for a solution of (44) of the form  $u(t) = \alpha(t)\chi_{B(0,r)} + \beta(t)\chi_{B(0,R) \setminus B(0,r)}$  on some time interval  $(0, T)$  defined by the inequalities  $\alpha(t) > \beta(t)$  for all  $t \in (0, T)$ , and  $\alpha(0) = k$ ,  $\beta(0) = 0$ . Then, we shall look for some  $z \in L^\infty((0, T) \times B(0, R))$  with  $\|z\|_\infty \leq 1$ , such that

$$\alpha'(t) = \operatorname{div}(z(t)) \quad \text{in } (0, T) \times B(0, r) \quad (66)$$

$$z(t, x) = -\frac{x}{|x|} \quad \text{on } (0, T) \times \partial B(0, r),$$

$$\beta'(t) = \operatorname{div}(z(t)) \quad \text{in } (0, T) \times (B(0, R) \setminus B(0, r))$$

$$z(t, x) = -\frac{x}{|x|} \quad \text{on } (0, T) \times \partial B(0, r) \quad (67)$$

$$z(t) \cdot n = 0 \quad \text{on } (0, T) \times \partial B(0, R)$$

and

$$\int_{B(0,R)} z(t) \cdot Du(t) = \int_{B(0,R)} |Du(t)| \quad \text{for all } t \in (0, T). \quad (68)$$

Integrating equation (66) in  $B(0, r)$  we obtain

$$\alpha'(t)|B(0, r)| = \int_{B(0,r)} \operatorname{div}(z(t))dx = \int_{\partial B(0,r)} z(t) \cdot n = -\mathcal{H}^{N-1}(\partial B(0, r)).$$

Thus

$$\alpha'(t) = -\frac{N}{r}.$$

and, therefore,

$$\alpha(t) = k - \frac{N}{r}t.$$

In this case we take  $z = -\frac{x}{r}$  and (66) holds. Similarly, we deduce that

$$\beta'(t) = \mu := N \frac{r^{N-1}}{R^N - r^N},$$

hence,

$$\beta(t) = N \frac{r^{N-1}}{R^N - r^N} t.$$

Our first observation is that  $T$  is given by

$$T \left( \frac{N}{r} + N \frac{r^{N-1}}{R^N - r^N} \right) = k. \quad (69)$$

To construct  $z$  in  $(0, T) \times (B(0, R) \setminus B(0, r))$  we shall look for  $z$  of the form  $z(t, x) = \rho(|x|) \frac{x}{|x|}$  such that  $\operatorname{div}(z(t)) = \beta'(t)$ ,  $\rho(r) = -1$ ,  $\rho(R) = 0$ . Since

$$\operatorname{div}(z(t)) = \nabla \rho(|x|) \cdot \frac{x}{|x|} + \rho(|x|) \operatorname{div}\left(\frac{x}{|x|}\right) = \rho'(|x|) + \rho(|x|) \frac{N-1}{|x|},$$

we must have

$$\rho'(s) + \rho(s) \frac{N-1}{s} = N \frac{r^{N-1}}{R^N - r^N} \quad s \in (r, R). \quad (70)$$

The solution of (89) such that  $\rho(R) = 0$  is

$$\rho(s) = \frac{\mu s}{N} - \frac{\mu R^N}{N s^{N-1}}$$

which also satisfies  $\rho(r) = -1$ . Thus, in  $B(0, R) \setminus B(0, r)$ ,

$$z(t, x) = \frac{\mu x}{N} - \frac{\mu R^N x}{N |x|^N}.$$

It is easy to check that (68) holds. Thus

$$u(t) = \left(k - \frac{N}{r} t\right) \chi_{B(0, r)} + \frac{N r^{N-1}}{R^N - r^N} t \chi_{B(0, R) \setminus B(0, r)},$$

in  $(0, T) \times B(0, R)$  where  $T$  is given by (69). On the other hand, we take

$$u(t) = \left(k - \frac{N}{r} T\right) \chi_{B(0, R)} = \frac{N r^{N-1}}{R^N - r^N} T \chi_{B(0, R)},$$

and  $z(t, x) = 0$  in  $(T, \infty) \times B(0, R)$ . It is easy to check that  $u(t)$  is the solution of (44) in  $(0, \infty) \times B(0, R)$  with initial datum  $u_0(x)$ .  $\square$

**Remark 5** The above result show that there is no spatial smoothing effect, for  $t > 0$ , similar to the linear heat equation and many other quasi-linear parabolic equations. In our case, the solution is discontinuous and has the minimal required spatial regularity:  $u(t) \in BV(\Omega) \setminus W^{1,1}(\Omega)$ .

Respect to the asymptotic behaviour of solutions obtained in Theorem 26, using Lyapunov functional methods we have proved in [3] (see also [5]) the following result.

**Theorem 28** *Let  $u_0 \in L^2(\Omega)$  and  $u(t)$  the unique weak solution of (44) such that  $u(0) = u_0$ . Then*

$$\|u(t) - (u_0)_\Omega\|_1 \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where

$$(u_0)_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_{\Omega} u_0(x) \, dx.$$

Moreover, if  $u_0 \in L^\infty(\Omega)$  there exists a constant  $C$ , independent of  $u_0$ , such that

$$\|u(t) - (u_0)_\Omega\|_p \leq \frac{C \|u_0\|_2^2}{t} \quad \text{for all } t > 0, \quad \text{and } 1 \leq p \leq \frac{N}{N-1}.$$

Now, we are going to prove, by energy methods that in the two dimensional case, in fact, this asymptotic state is reached in finite time.

**Theorem 29** *Suppose  $N = 2$ . Let  $u_0 \in L^2(\Omega)$  and  $u(t, x)$  the unique strong solution of problem (44). Then there exists a finite time  $T_0$  such that*

$$u(t) = (u_0)_\Omega = \frac{1}{\mathcal{L}^N(\Omega)} \int_{\Omega} u_0(x) \, dx \quad \forall t \geq T_0.$$

**Proof:** Since  $u$  is a strong solution of problem (44), there exists  $z \in L^\infty(Q)$  with  $\|z\|_\infty \leq 1$ ,  $u_t = \text{div}(z)$  in  $\mathcal{D}'(Q)$  such that

$$\int_{\Omega} (u(t) - w) u_t(t) \, dx = \int_{\Omega} (z(t), Dw) - \int_{\Omega} \|Du(t)\| \quad (71)$$

for all  $w \in BV(\Omega) \cap L^\infty(\Omega)$ . Hence, taking  $w = (u_0)_\Omega$  as test function in (71), it yields

$$\int_{\Omega} (u(t) - (u_0)_\Omega) u_t(t) \, dx = - \int_{\Omega} \|Du(t)\|.$$

Now, by Sobolev-Poincaré inequality for  $BV$  functions (13) and having in mind that we have conservation of mass, we obtain

$$\|u(t) - (u_0)_\Omega\|_2 \leq C \int_{\Omega} \|Du(t)\|.$$

Thus, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u(t) - (u_0)_{\Omega})^2 dx + \frac{1}{C} \|u(t) - (u_0)_{\Omega}\|_2 \leq 0. \quad (72)$$

Therefore, the function

$$y(t) := \int_{\Omega} (u(t) - (u_0)_{\Omega})^2 dx$$

satisfies the inequality

$$y'(t) + My(t)^{1/2} \leq 0,$$

from where it follows that there exists  $T_0 > 0$  such that  $y(t) = 0$  for all  $t \geq T_0$ .  $\square$

By Theorem 29, given  $u_0 \in L^2(\Omega)$ , if  $u(t, x)$  is the unique strong solution of problem (44), then

$$T^*(u_0) := \inf\{t > 0 : u(t) = (u_0)_{\Omega}\} < \infty.$$

In [4] (see also [5]) we study of the behaviour of  $u(t)$  near  $T^*(u_0)$  establishing the following result.

**Theorem 30** *Suppose  $N = 2$ . Let  $u_0 \in L^2(\Omega)$  and let  $u(t, x)$  be the unique strong solution of problem (44). Let*

$$w(t, x) := \begin{cases} \frac{u(t, x) - (u_0)_{\Omega}}{T^*(u_0) - t} & \text{if } 0 \leq t < T^*(u_0), \\ 0 & \text{if } t \geq T^*(u_0). \end{cases}$$

*Then, there exists an increasing sequence  $t_n \rightarrow T^*(u_0)$ , and a solution  $v^* \neq 0$  of the stationary problem*

$$(S_N) \begin{cases} -\operatorname{div} \left( \frac{Dv}{|Dv|} \right) = v & \text{in } \Omega \\ \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases}$$

*such that*

$$\lim_{n \rightarrow \infty} w(t_n) = v^* \quad \text{in } L^p(\Omega)$$

*for all  $1 \leq p < \infty$ .*

## 6 The Cauchy Problem for the Total Variation Flow

### 6.1 Initial Conditions in $L^2(\mathbb{R}^N)$

The purpose of this Subsection is to prove existence and uniqueness of the Minimizing Total Variation Flow in  $\mathbb{R}^N$

$$\frac{\partial u}{\partial t} = \operatorname{div} \left( \frac{Du}{|Du|} \right) \quad \text{in } ]0, \infty[ \times \mathbb{R}^N, \quad (73)$$

coupled with the initial condition

$$u(0, x) = u_0(x) \quad x \in \mathbb{R}^N, \quad (74)$$

when  $u_0 \in L^2(\mathbb{R}^N)$ .

**Definition 9** A function  $u \in C([0, T]; L^2(\mathbb{R}^N))$  is called a *strong solution* of (73) if

$$u \in W_{\text{loc}}^{1,2}(0, T; L^2(\mathbb{R}^N)) \cap L_w^1(0, T; BV(\mathbb{R}^N))$$

and there exists  $z \in L^\infty(]0, T[ \times \mathbb{R}^N; \mathbb{R}^N)$  with  $\|z\|_\infty \leq 1$  such that

$$u_t = \operatorname{div}(z) \quad \text{in } \mathcal{D}'(]0, T[ \times \mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} (u(t) - w) u_t(t) dx = \int_{\mathbb{R}^N} (z(t), Dw) - \int_{\mathbb{R}^N} \|Du(t)\| \quad (75)$$

for all  $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ , a.e.  $t \in [0, T]$ .

The main result of this subsection is the following existence and uniqueness theorem.

**Theorem 31** *Let  $u_0 \in L^2(\mathbb{R}^N)$ . Then there exists a unique strong solution  $u$  of (73), (74) in  $[0, T] \times \mathbb{R}^N$  for every  $T > 0$ . Moreover, if  $u$  and  $v$  are the strong solutions of (73) corresponding to the initial conditions  $u_0, v_0 \in L^2(\mathbb{R}^N)$ , then*

$$\|u(t) - v(t)\|_2 \leq \|u_0 - v_0\|_2 \quad \text{for any } t > 0. \quad (76)$$

**Proof.** Let us introduce the following multivalued operator  $\mathcal{B}$  in  $L^2(\mathbb{R}^N)$ : a pair of functions  $(u, v)$  belongs to the graph of  $\mathcal{B}$  if and only if

$$u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \quad v \in L^2(\mathbb{R}^N), \quad (77)$$

$$\text{there exists } z \in X(\mathbb{R}^N)_2 \text{ with } \|z\|_\infty \leq 1, \text{ such that } v = -\operatorname{div}(z) \quad (78)$$

and

$$\int_{\mathbb{R}^N} (w - u)v \, dx \leq \int_{\mathbb{R}^N} z \cdot \nabla w \, dx - \int_{\mathbb{R}^N} \|Du\|, \quad \forall w \in L^2(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N).$$

Let also  $\Psi : L^2(\mathbb{R}^N) \rightarrow ]-\infty, +\infty]$  be the functional defined by

$$\Psi(u) := \begin{cases} \int_{\mathbb{R}^N} \|Du\| & \text{if } u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N) \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N). \end{cases} \quad (79)$$

Since  $\Psi$  is convex and lower semi-continuous in  $L^2(\mathbb{R}^N)$ , its subdifferential  $\partial\Psi$  is a maximal monotone operator in  $L^2(\mathbb{R}^N)$ .

We divide the proof of the theorem into two steps.

*Step 1.* The following assertions are equivalent:

- (a)  $(u, v) \in \mathcal{B}$ ;
- (b) (77) and (78) hold,

and

$$\int_{\mathbb{R}^N} (w - u)v \, dx \leq \int_{\mathbb{R}^N} (z, Dw) - \int_{\mathbb{R}^N} \|Du\| \quad (80)$$

for all  $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ ;

- (c) (77) and (78) hold, and (80) holds with the equality instead of the inequality;
- (d) (77) and (78) hold, and

$$\int_{\mathbb{R}^N} (z, Du) = \int_{\mathbb{R}^N} \|Du\|. \quad (81)$$

It is clear that (c) implies (b), and (b) implies (a), while (d) follows from (b) taking  $w = u$  in (80) and using (21). In order to prove that (a) implies (b) it is enough to use Theorem 2 and Lemma 2 as in the proof of Theorem 24. To obtain (c) from (d) it suffices to multiply both terms of the equation  $v = -\operatorname{div}(z)$  by  $w - u$ , for  $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ , and to integrate by parts using (24).

*Step 2.* We also have  $\mathcal{B} = \partial\Psi$ . The proof is similar to the one given in Section 5.1 for the Neumann problem and we omit the details.

As a consequence, the semigroup generated by  $\mathcal{B}$  coincides with the semigroup generated by  $\partial\Psi$  and therefore  $u(t, x) = e^{-t\mathcal{B}}u_0(x)$  is a strong solution of

$$u_t + \mathcal{B}u \ni 0,$$

i.e.,  $u \in W_{\text{loc}}^{1,2}(]0, T[; L^2(\mathbb{R}^N))$  and  $-u_t(t) \in \mathcal{B}u(t)$  for almost all  $t \in ]0, T[$ . Then, according to the equivalence proved in Step 1, we have that

$$\int_{\mathbb{R}^N} (u(t) - w)u_t(t) \, dx = \int_{\mathbb{R}^N} (z(t), Dw) - \int_{\mathbb{R}^N} \|Du(t)\| \quad (82)$$

for all  $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$  and for almost all  $t \in ]0, T[$ . Now, choosing  $w = u - \varphi$ ,  $\varphi \in C_0^\infty(\mathbb{R}^N)$ , we see that  $u_t(t) = \text{div}(z(t))$  in  $\mathcal{D}'(\mathbb{R}^N)$  for almost every  $t \in ]0, T[$ . We deduce that  $u_t = \text{div}(z)$  in  $\mathcal{D}'(]0, T[ \times \mathbb{R}^N)$ . We have proved that  $u$  is a strong solution of (73) in the sense of Definition 9.

The contractivity estimate (76) of Theorem 31 follows as in Theorem 26. This concludes the proof of the theorem.  $\square$

Given a function  $g \in L^2(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$ , we define

$$\|g\|_* := \sup \left\{ \left| \int_{\mathbb{R}^N} g(x)u(x) \, dx \right| : u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \int_{\mathbb{R}^N} \|Du\| \leq 1 \right\}.$$

Part (b) of the next Lemma gives a characterization of  $\mathcal{B}(0)$ . This was proved by Y. Meyer in [19] in the context of the analysis of the Rudin-Osher-Fatemi model for image denoising.

**Lemma 5** *Let  $f \in L^2(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$  and  $\lambda > 0$ . The following assertions hold.*

(a) *The function  $u$  is the solution of*

$$\min_{w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)} D(w), \quad D(w) := \int_{\mathbb{R}^N} \|Dw\| + \frac{1}{2\lambda} \int_{\mathbb{R}^N} (w - f)^2 \, dx \quad (83)$$

*if and only if there exists  $z \in X(\mathbb{R}^N)_2$  satisfying (81) with  $\|z\|_\infty \leq 1$  and  $-\lambda \text{div}(z) = f - u$ .*

(b) *The function  $u \equiv 0$  is the solution of (83) if and only if  $\|f\|_* \leq \lambda$ .*

(c) *If  $N = 2$ ,  $\mathcal{B}(0) = \{f \in L^2(\mathbb{R}^2) : \|f\|_* \leq 1\}$ .*

**Proof.** (a). Thanks to the strict convexity of  $D$ ,  $u$  is the solution of (83) if and only if  $0 \in \partial D(u) = \partial \Psi(u) + \frac{1}{\lambda}(u - f) = \mathcal{B}(u) + \frac{1}{\lambda}(u - f)$ , where  $\Psi$  is defined in (79) and the last equality follows from Step 2 in the proof of Theorem 31. This means, recalling the definition of  $\mathcal{B}$  in the proof of Theorem 31, that there exists  $z \in X(\mathbb{R}^N)_2$  satisfying (81) with  $\|z\|_\infty \leq 1$  and  $-\lambda \text{div}(z) = f - u$ .

(b). The function  $u \equiv 0$  is the solution of (83) if and only if

$$\int_{\mathbb{R}^N} \|Dv\| + \frac{1}{2\lambda} \int_{\mathbb{R}^N} (v - f)^2 \, dx \geq \frac{1}{2\lambda} \int_{\mathbb{R}^N} f^2 \, dx \quad \forall v \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N). \quad (84)$$

Replacing  $v$  by  $\epsilon v$  (where  $\epsilon > 0$ ), expanding the  $L^2$ -norm, dividing by  $\epsilon > 0$ , and letting  $\epsilon \rightarrow 0+$  we have

$$\left| \int_{\mathbb{R}^N} f(x)v(x) dx \right| \leq \lambda \int_{\mathbb{R}^N} \|Dv\| \quad \forall v \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N). \quad (85)$$

Since (85) implies (84), we have that (84) and (85) are equivalent. The assertion follows by observing that (85) is equivalent to  $\|f\|_* \leq \lambda$ .

(c) Let  $N = 2$ . We have

$$\mathcal{B}(0) = \left\{ f \in L^2(\mathbb{R}^2) : \exists z \in X(\mathbb{R}^2)_2, \|z\|_\infty \leq 1, -\operatorname{div}(z) = f \right\}.$$

On the other hand, from (a) and (b) it follows that  $\|f\|_* \leq 1$  if and only if there exists  $z \in X(\mathbb{R}^2)_2$  with  $\|z\|_\infty \leq 1$  and such that  $f = -\operatorname{div}(z)$ . Then the assertion follows.  $\square$

Let us give a heuristic explanation of what the vector field  $z$  represents. Condition (81) essentially means that  $z$  has unit norm and is orthogonal to the level sets of  $u$ . In some sense,  $z$  is invariant under local contrast changes. To be more precise, we observe that if  $u = \sum_{i=1}^p c_i \chi_{B_i}$  where  $B_i$  are sets of finite perimeter such that  $\mathcal{H}^{N-1}((B_i \cup \partial^* B_i) \cap (B_j \cup \partial^* B_j)) = 0$  for  $i \neq j$ ,  $c_i \in \mathbb{R}$ , and

$$-\operatorname{div} \left( \frac{Du}{|Du|} \right) = f \in L^2(\mathbb{R}^N), \quad (86)$$

then also  $-\operatorname{div} \left( \frac{Dv}{|Dv|} \right) = f$  for any  $v = \sum_{i=1}^p d_i \chi_{B_i}$  where  $d_i \in \mathbb{R}$  and  $\operatorname{sign}(d_i) = \operatorname{sign}(c_i)$ . Indeed, there is a vector field  $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$  such that  $\|z\|_\infty \leq 1$ ,  $-\operatorname{div}(z) = f$  and (81) holds. Then one can check that

$$\|D\chi_{B_i}\| = \operatorname{sign}(c_i)(z, D\chi_{B_i})$$

as measures in  $\mathbb{R}^N$  and, as a consequence,  $(z, Dv) = \|Dv\|$  as measures in  $\mathbb{R}^N$ .

Let us also observe that the solutions of (86) are not unique. Indeed, if  $u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$  is a solution of (86) and  $g \in C^1(\mathbb{R})$  with  $g'(r) > 0$  for all  $r \in \mathbb{R}$ , then  $w = g(u)$  is also a solution of (86). In other words, a global contrast change of  $u$  produces a new solution of (86). In an informal way, the previous remark can be rephrased by saying that also local contrast changes of a given solution of (86) produce new solutions of it. To express this non-uniqueness in a more general way we suppose that  $(u_1, v), (u_2, v) \in \mathcal{B}$ , i.e., there are vector fields  $z_i \in X(\mathbb{R}^N)_2$  with  $\|z_i\|_\infty \leq 1$ , such that

$$-\operatorname{div}(z_i) = v, \quad \int_{\mathbb{R}^N} (z_i, Du_i) = \int_{\mathbb{R}^N} \|Du_i\|, \quad i = 1, 2.$$

Then

$$\begin{aligned} 0 &= - \int_{\mathbb{R}^N} (\operatorname{div}(z_1) - \operatorname{div}(z_2))(u_1 - u_2) dx = \int_{\mathbb{R}^N} (z_1 - z_2, Du_1 - Du_2) \\ &= \int_{\mathbb{R}^N} \|Du_1\| - (z_2, Du_1) + \int_{\mathbb{R}^N} \|Du_2\| - (z_1, Du_2). \end{aligned}$$

Hence

$$\int_{\mathbb{R}^N} \|Du_1\| = \int_{\mathbb{R}^N} (z_2, Du_1) \quad \text{and} \quad \int_{\mathbb{R}^N} \|Du_2\| = \int_{\mathbb{R}^N} (z_1, Du_2).$$

In other words,  $z_1$  is in some sense a unit vector field of normals to the level sets of  $u_2$  and a similar thing can be said of  $z_2$  with respect to  $u_1$ . Any two solutions of (86) should be related in this way.

## 6.2 Explicit Solutions

We are going to compute explicitly the evolution of the characteristic function of a ball and an annulus.

**Lemma 6** *Let  $u_0 = k\chi_{B_r(0)}$ . Then the unique solution  $u(t, x)$  of problem (73) with initial datum  $u_0$  is given by*

$$u(t, x) = \text{sign}(k) \frac{N}{r} \left( \frac{|k|r}{N} - t \right)^+ \chi_{B_r(0)}(x).$$

Observe that we may write

$$u(t, x) = \text{sign}(k) \left( |k| - \frac{\mathcal{H}^{N-1}(\partial B_r(0))}{\mathcal{L}^N(B_r(0))} t \right)^+ \chi_{B_r(0)}(x).$$

**Proof.** Suppose that  $k > 0$ , the solution for  $k < 0$  being constructed in a similar way. We look for a solution of (73) of the form  $u(t, x) = \alpha(t)\chi_{B_r(0)}(x)$  on some time interval  $(0, T)$ . Then, we shall look for some  $z(t) \in X(\mathbb{R}^N)_2$  with  $\|z\|_\infty \leq 1$ , such that

$$u'(t) = \text{div}(z(t)) \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad (87)$$

$$\int_{\mathbb{R}^N} (z(t), Du(t)) = \int_{\mathbb{R}^N} \|Du(t)\|. \quad (88)$$

If we take  $z(t)(x) = -\frac{x}{r}$  for  $x \in \partial B_r(0)$ , integrating equation (87) in  $B_r(0)$  we obtain

$$\begin{aligned} & \alpha'(t) \mathcal{L}^N(B_r(0)) \\ &= \int_{B_r(0)} \text{div}(z(t)) \, dx = \int_{\partial B_r(0)} z(t) \cdot \nu \, d\mathcal{H}^{N-1} = -\mathcal{H}^{N-1}(\partial B_r(0)). \end{aligned}$$

Thus

$$\alpha'(t) = -\frac{N}{r},$$

and, therefore,

$$\alpha(t) = k - \frac{N}{r}t.$$

In that case,  $T$  must be given by  $T = \frac{kr}{N}$ .

To construct  $z$  in  $(0, T) \times (\mathbb{R}^N \setminus B_r(0))$  we shall look for  $z$  of the form  $z = \rho(\|x\|) \frac{x}{\|x\|}$  such that  $\operatorname{div}(z(t)) = 0$ ,  $\rho(r) = -1$ . Since

$$\operatorname{div}(z(t)) = \nabla \rho(\|x\|) \cdot \frac{x}{\|x\|} + \rho(\|x\|) \operatorname{div} \left( \frac{x}{\|x\|} \right) = \rho'(\|x\|) + \rho(\|x\|) \frac{N-1}{\|x\|},$$

we must have

$$\rho'(s) + \rho(s) \frac{N-1}{s} = 0 \quad \text{for } s > r. \quad (89)$$

The solution of (89) such that  $\rho(r) = -1$  is

$$\rho(s) = -r^{N-1} s^{1-N}.$$

Thus, in  $\mathbb{R}^N \setminus B_r(0)$ ,

$$z(t) = -r^{N-1} \frac{x}{\|x\|^N}.$$

Consequently, the candidate for  $z(t)$  is the vector field

$$z(t, x) := \begin{cases} -\frac{x}{r} & \text{if } x \in B_r(0) \text{ and } 0 \leq t \leq T \\ -r^{N-1} \frac{x}{\|x\|^N} & \text{if } x \in \mathbb{R}^N \setminus \overline{B_r(0)}, \text{ and } 0 \leq t \leq T \\ 0 & \text{if } x \in \mathbb{R}^N \text{ and } t > T, \end{cases}$$

and the corresponding function  $u(t, x)$  is

$$u(t, x) = \left( k - \frac{N}{r} t \right) \chi_{B_r(0)}(x) \chi_{[0, T]}(t),$$

where  $T = \frac{kr}{N}$ . Let us check that  $u(t, x)$  satisfies (87), (88). If  $\varphi \in \mathcal{D}(\mathbb{R}^N)$  and  $0 \leq t \leq T$ , we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\partial z_i(t)}{\partial x_i} \varphi \, dx &= -\frac{1}{r} \int_{B_r(0)} \varphi \, dx + \int_{\partial B_r(0)} \frac{x_i}{r} \frac{x_i}{r} \varphi \, d\mathcal{H}^{N-1} \\ &- \int_{\mathbb{R}^N \setminus B_r(0)} \frac{\partial}{\partial x_i} \left( \frac{r^{N-1} x_i}{\|x\|^N} \right) \varphi \, dx - \int_{\partial B_r(0)} \frac{r^{N-1}}{r^N} x_i \frac{x_i}{r} \varphi \, d\mathcal{H}^{N-1}. \end{aligned}$$

Hence

$$\int_{\mathbb{R}^N} \operatorname{div}(z(t)) \varphi \, dx = -\frac{N}{r} \int_{B_r(0)} \varphi \, dx,$$

and consequently, (87) holds. Finally, if  $0 \leq t \leq T$ , by Green's formula, we have

$$\begin{aligned} \int_{\mathbb{R}^N} (z(t), Du(t)) &= - \int_{\mathbb{R}^N} \operatorname{div}(z(t))u(t) \, dx = \\ &- \int_{B_r(0)} \left(k - \frac{N}{r}t\right) \operatorname{div}(z(t)) \, dx = \int_{B_r(0)} \left(k - \frac{N}{r}t\right) \frac{N}{r} \, dx = \\ &\left(k - \frac{N}{r}t\right) \frac{N}{r} \mathcal{L}^N(B_r(0)) = \left(k - \frac{N}{r}t\right) \mathcal{H}^{N-1}(\partial B_r(0)) = \int_{\mathbb{R}^N} \|Du(t)\|. \end{aligned}$$

Therefore (88) holds, and consequently  $u(t, x)$  is the solution of (73) with initial datum  $u_0 = k\chi_{B_r(0)}$ .  $\square$

**Lemma 7** *Let  $\Omega = B_R(0) \setminus \overline{B_r(0)}$ ,  $0 < r < R$  and  $u_0 = k\chi_\Omega$ . Then the unique solution  $u(t, x)$  of problem (73) with initial datum  $u_0$  is*

$$u(t, x) = \operatorname{sign}(k) \left( |k| - \frac{\operatorname{Per}(\Omega)}{\mathcal{L}^N(\Omega)} t \right) \chi_\Omega(x) + \frac{\operatorname{Per}(B_r(0))}{\mathcal{L}^N(B_r(0))} t \chi_{B_r(0)}(x) \quad (90)$$

$t \in [0, T_1]$ ,  $x \in \mathbb{R}^N$ , where  $T_1$  is such that

$$T_1 \cdot \left( \frac{\operatorname{Per}(\Omega)}{\mathcal{L}^N(\Omega)} + \frac{\operatorname{Per}(B_r(0))}{\mathcal{L}^N(B_r(0))} \right) = |k|$$

and  $u(t, x)$  evolves as the solution given in Lemma 6 until its extinction.

**Proof.** Let  $\xi : \mathbb{R}^N \rightarrow \mathbb{R}^N$  be the vector field defined as

$$\xi(x) := \begin{cases} \frac{x}{r} & \text{for } x \in B_r(0), \\ \left( (Rr)^{N-1} \frac{R+r}{\|x\|^N} - (R^{N-1} + r^{N-1}) \right) \frac{x}{R^N - r^N}, & x \in B_R(0) \setminus \overline{B_r(0)}, \\ -\frac{R^{N-1}}{\|x\|^N} x & \text{for } x \in \mathbb{R}^N \setminus \overline{B_R(0)}. \end{cases}$$

Then  $\|\xi\|_\infty \leq 1$ ,  $\operatorname{div}(\xi) = \frac{N}{r} = \frac{\operatorname{Per}(B_r(0))}{\mathcal{L}^N(B_r(0))}$  on  $B_r(0)$ ,  $\operatorname{div}(\xi) = -\frac{\operatorname{Per}(\Omega)}{\mathcal{L}^N(\Omega)}$  on  $B_R(0) \setminus \overline{B_r(0)}$ ,  $\operatorname{div}(\xi) = 0$  on  $\mathbb{R}^N \setminus \overline{B_R(0)}$ , and  $\xi \cdot \nu^{B_r(0)} = 1$  on  $\partial B_r(0)$ ,  $\xi \cdot \nu^{B_R(0)} = -1$  on  $\partial B_R(0)$ . Therefore, one can check that the solution  $u$  of (73) with initial condition  $u_0 = \chi_\Omega$  in  $[0, T_1]$  is given by (90). At  $t = T_1$ , the two evolving sets reach the same height and  $u(T_1, x) = \alpha \chi_{B_R(0)}$  for some  $\alpha > 0$ . For  $t > T_1$  the solution  $u$  is equal to the solution starting from  $\alpha \chi_{B_R(0)}$  (at time  $T_1$ ) as it is described in Lemma 6.  $\square$

**Remark 6** The above results show that there is no spatial smoothing effect, for  $t > 0$ , similar to the case of the linear heat equation and many other quasilinear parabolic equations. In our case, the solution is discontinuous and has the minimal required spatial regularity:  $u(t, \cdot) \in BV(\mathbb{R}^N) \setminus W^{1,1}(\mathbb{R}^N)$ .

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# PROBABILISTIC MODELS OF INTERFACES AND THEIR SCALING LIMIT

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## 1. INTRODUCTION

We discuss the large scale hydrodynamic behavior of interfaces separating two distinct phases. The problem of the phase separation was first investigated from microscopic aspect in mathematically rigorous way by [2] for the Ising model in the low temperature regime. An interface at macroscopic level called the Wulff shape was derived from  $\pm$  spin systems described by the finite volume canonical Gibbs measure with  $-$  spins' boundary condition given the number of sites occupied by  $+$  spins inside the domain. The Wulff shape is characterized by a variational problem minimizing the total surface tension. The results of [2] have been generalized into several directions afterward. However, there exist very few dynamic results, for instance, for the Glauber or the Kawasaki dynamics in the low temperature regime, because of serious analytical difficulties. In order to explain microscopic motions of interfaces, [8] introduced several simplified models including the Ginzburg-Landau  $\nabla\phi$  interface model which is main object in this talk.

## 2. GINZBURG-LANDAU $\nabla\phi$ INTERFACE MODEL

The Ginzburg-Landau  $\nabla\phi$  interface model determines the stochastic dynamics of a discretized hypersurface separating two phases. The position of the hypersurface is described by height variables  $\phi = \{\phi(x); x \in \Gamma\}$  measured from a fixed  $d$ -dimensional discrete hyperplane  $\Gamma$ . We then admit an energy (Hamiltonian) to the interface  $\phi$  by:

$$(2.1) \quad H(\phi) = \frac{1}{2} \sum_{\substack{x, y \in \Gamma, \\ |x-y|=1}} V(\phi(x) - \phi(y)).$$

The potential  $V$  in the Hamiltonian  $H$  is assumed to satisfy the conditions as follows:

- (i)  $V \in C^2(\mathbb{R}^d)$ .
- (ii)  $V$  is symmetric, i.e.  $V(\eta) = V(-\eta)$  for all  $\eta \in \mathbb{R}$ .
- (iii) There exist constants  $c_+, c_- > 0$  such that

$$c_- \leq V''(\eta) \leq c_+, \quad \eta \in \mathbb{R}$$

holds.

Once the energy  $H$  is introduced, the dynamics of the interfaces can be determined by means of the Langevin equation

$$(2.2) \quad d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t) dt + \sqrt{2}dw_t(x), \quad x \in \Gamma,$$

where  $\{w_t(x); x \in \Gamma\}$  is a family of independent one dimensional Brownian motions. Here, we take  $\Gamma = (\mathbb{Z}/N\mathbb{Z})^d$  or take  $\Gamma = ND \cap \mathbb{Z}^d$  with a bounded domain  $D \subset \mathbb{R}^d$  with a smooth

boundary. For the former, we consider (2.2) under the periodic boundary condition. For the latter, we consider (2.2) under the Dirichlet boundary condition.

We regard the time evolution (2.2) as the motion of the microscopic interface. Let us consider the macroscopic interface by changing the scale for time and space. We adopt the diffusive scaling, that is, the scaling  $N^2$  for time while  $N$  for space. More precisely, we consider we consider the macroscopic interface  $h^N$  defined by

$$h^N(t, x/N) = \phi_{N^2t}(x), \quad x \in \Gamma$$

and with suitable interpolation. We then have the following:

**Theorem 2.1** ([4] for periodic b.c., [7] for Dirichlet b.c.). *We assume that  $h^N(0)$  converges to some function  $h_0$  in  $L^2$ -space. We then have that  $h^N(t)$  converges to  $h$  in  $L^2$ -space for every  $t \geq 0$ , where  $h$  is the unique weak solution for the nonlinear partial differential equation*

$$(2.3) \quad \frac{\partial h}{\partial t} = \operatorname{div} [(\nabla\sigma)(\nabla h)].$$

with initial data  $h_0$ , where  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$  is a function called “surface tension.”

Theorem 2.1 can be regarded as the law of the large numbers. The “central limit theorem” is also established by [5], and the “large deviation principle” is established by [3]. Note that the rate functional appearing at the latter has the representation

$$I(h) = \int_0^t \left\| \frac{\partial h}{\partial t} - \operatorname{div} [(\nabla\sigma)(\nabla h)] \right\|_{L^2}^2 dt$$

if  $h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is smooth enough. Roughly saying, the relationship

$$P(h^N \doteq h) \doteq \exp(-N^d I(h))$$

holds asymptotically as  $N \rightarrow \infty$ .

The function  $\sigma(u)$  appearing in (2.3) describes the energy of the macroscopic interface with tilt  $u \in \mathbb{R}^d$ . We note that the equation (2.3) is the gradient flow with respect to the energy functional

$$(2.4) \quad \Sigma(h) = \int \sigma(\nabla h(\theta)) d\theta$$

in  $L^2$ -space. The functional  $\Sigma$  is called “total surface tension,” which gives the total energy of the interface  $h$ . These quantities are highly related to the equilibrium states for (2.2). Let us consider the Gibbs measure on  $\mathbb{R}^\Gamma$  defined by

$$\mu_\Gamma(d\phi) = \frac{1}{Z_\Gamma} \exp(-H(\phi)) \prod_{x \in \Gamma} d\phi(x),$$

where  $d\phi(x)$  is the Lebesgue measure on  $\mathbb{R}$  and  $Z_\Gamma$  is the normalizing constant. We note that  $\mu_\Gamma$  is the reversible measure for the stochastic dynamics (2.2). The surface tension  $\sigma$  appearing in (2.3) is defined via the limit of Gibbs measures as  $\Gamma \uparrow \mathbb{Z}^d$ . The total surface tension  $\Sigma$  also appears in the static situation, the large deviation principle for the Gibbs measure. Taking  $\Gamma = D_N$  defined above and scaling only for space, the large deviation principle with the speed  $N^d$  and the rate functional  $\Sigma$  of (2.4) is established in [1]. Roughly speaking, the relationship

$$\mu_{D_N}(h^N \doteq h) \doteq \exp(-N^d \Sigma(h))$$

holds asymptotically as  $N \rightarrow \infty$ .

### 3. GINZBURG-LANDAU $\nabla\phi$ INTERFACE MODEL WITH A CONSERVATION LAW

Another dynamics can be associated with the Hamiltonian  $H$  by considering the equation

$$(3.1) \quad d\phi_t(x) = \Delta \frac{\partial H}{\partial \phi(x)}(\phi_t) dt + \sqrt{-2\Delta} dw_t(x), \quad x \in \Gamma,$$

where  $\Delta$  is the discrete Laplacian on  $\Gamma$ . The time evolution (3.1) can be regarded as the Langevin equation on the space with different spatial structures, that is, the  $H^{-1}$  metric. We emphasize that the dynamics (3.1) has a conservation law, that is, we have

$$\sum_{x \in \Gamma} \phi_t(x) \equiv \sum_{x \in \Gamma} \phi_0(x)$$

by using Itô's formula. In this sense, we can regard the time evolution (2.2) as the dynamics corresponding to Glauber dynamics of the particle system (with no conservation law) or the dynamics of type A in [6], and regard the time evolution (3.1) as the dynamics corresponding to Kawasaki dynamics of the particle system (the total number of particle is conserved) or the dynamics of type B in [6].

Let us consider the macroscopic interface via scaling. We remark that the suitable scaling for the time evolution (3.1) is different from the diffusive one, that is, the scaling  $N^4$  for time while  $N$  for space. More precisely, we consider the macroscopic interface  $h^N$  defined by

$$h^N(t, x/N) = \phi_{N^4 t}(x), \quad x \in \Gamma$$

and suitable interpolation. We then have the following:

**Theorem 3.1** ([7] for periodic b.c.). *We assume that  $h^N(0)$  converges to some  $h_0$  in  $H^{-1}$ -space. We then have that  $h^N(t)$  converges to  $h$  in  $H^{-1}$ -space, where  $h$  is the unique weak solution for the nonlinear partial differential equation*

$$(3.2) \quad \frac{\partial h}{\partial t} = -\Delta \operatorname{div} [(\nabla \sigma)(\nabla h)].$$

with initial data  $h_0$ .

We note that the equation (3.2) is also the gradient flow with respect to the total surface tension defined by (2.4) on the  $H^{-1}$ -space.

The corresponding "large deviation" also holds, though some additional assumption is required.

**Theorem 3.2.** *We assume that  $h^N(0)$  converges to some  $h_0$  in  $H^{-1}$ -space. We also assume that the spatial dimension is smaller or equal to 3. We then have that  $h^N(t)$  satisfies the large deviation principle with the speed  $N^d$ . Its rate functional has the representation*

$$I(h) = \int_0^t \left\| \frac{\partial h}{\partial t} + \Delta \operatorname{div} [(\nabla \sigma)(\nabla h)] \right\|_{H^{-1}}^2 dt$$

if  $h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is smooth enough.

*Remark 3.1.* To describe the asymptotics like Theorems 3.1, 3.2, we need the information for the equilibrium states for the stochastic dynamics (3.1) on the infinite lattice  $\mathbb{Z}^d$ . The information which we already have is not sufficient to derive the result corresponding to Theorem 3.1 with Dirichlet boundary condition, and the result corresponding to Theorem 3.2 with the arbitrary spatial dimension. At present, they remain still open.

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# Mathematical Analysis of Grain Boundary Motion Models of Kobayashi-Warren-Carter Type

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## 1 Introduction

This is a joint work with Akio Ito<sup>1</sup> and Nobuyuki Kenmochi<sup>2</sup>.

In this talk we consider the following phase-field model of grain boundaries with constraint, denoted by (P):

$$(P) \begin{cases} \eta_t - \kappa \Delta \eta + g(\eta) + \alpha'(\eta) |\nabla \theta| = 0 & \text{a.e. in } Q_T := \Omega \times (0, T), \\ \alpha_0(\eta) \theta_t - \nu \Delta \theta - \operatorname{div} \left( \alpha(\eta) \frac{\nabla \theta}{|\nabla \theta|} \right) + \partial I_{[-\theta^*, \theta^*]}(\theta) \ni 0 & \text{a.e. in } Q_T, \\ \frac{\partial \eta}{\partial n} = 0, \quad \theta = 0 & \text{a.e. on } \Sigma_T := \Gamma \times (0, T), \\ \eta(x, 0) = \eta_0(x), \quad \theta(x, 0) = \theta_0(x) & \text{for a.e. } x \in \Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^N$  ( $N \geq 1$ ) with regular boundary  $\Gamma := \partial\Omega$ ,  $T > 0$  is a fixed finite time,  $\kappa > 0$  and  $\nu > 0$  are given small constants,  $g(\cdot)$ ,  $\alpha(\cdot)$  and  $\alpha_0(\cdot)$  are given functions on  $\mathbf{R}$ ,  $\partial I_{[-\theta^*, \theta^*]}(\cdot)$  is the subdifferential of the indicator function  $I_{[-\theta^*, \theta^*]}(\cdot)$  on the closed interval  $[-\theta^*, \theta^*]$  with some constant  $\theta^* > 0$ ,  $\partial/\partial n$  is the outward normal derivative on  $\Gamma$ , and  $\eta_0(x)$ ,  $\theta_0(x)$  are given initial data.

The system (P) is called a grain boundary motion model of Kobayashi-Warren-Carter type [12, 13]. In the dynamics of grain structure in various materials, the variable  $\theta$  is an indicator of the mean orientation of crystallines and the variable  $\eta$  is an order parameter for the degree of crystalline orientational order;  $\eta = 1$  implies the completely oriented state and  $\eta = 0$  is the state where no meaningful value of orientation exists. There are many mathematical models of grain boundary formation. For some related works, we refer to [3, 4, 15, 16].

In connection with this subject, the singular diffusion equations,

$$u_t = \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right), \quad \text{more generally, } u_t = \frac{1}{b(x)} \operatorname{div} \left( a(x) \frac{\nabla u}{|\nabla u|} \right),$$

kindred to the second equation of (P), have been studied by a lot of mathematicians from various view-points (cf. [1, 2, 5, 11, 14, 17]).

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Kobayashi et al. [12] considered  $\eta$  and  $\theta$  as a polar coordinate system  $(\eta, \theta)$  in two dimensional space, and they proposed a grain boundary motion model (P) without constraint  $\partial I_{[-\theta^*, \theta^*]}(\cdot)$ . Moreover, in [12, 13], some numerical experiments were obtained when  $\hat{g}(\eta) := \frac{1}{2}(1 - \eta)^2$ ,  $\alpha_0(\eta) = \alpha(\eta) = \eta^2$  and  $\Omega$  is a bounded domain in  $\mathbf{R}^2$ .

Recently assuming that  $\{\eta_0, \theta_0\}$  is a pair of good initial data in  $H^1(\Omega) \times H_0^1(\Omega)$ , system (P) without constraint  $\partial I_{[-\theta^*, \theta^*]}(\cdot)$  was studied in [6, 7, 8] from the theoretical point of view. In the case when  $\alpha_0 \geq \delta (> 0)$  on  $\mathbf{R}$  for a positive constant  $\delta$ , Ito et al. [6] showed the existence-uniqueness of solutions to the one-dimensional model (P) without  $\partial I_{[-\theta^*, \theta^*]}(\cdot)$  and with  $-\kappa \Delta \eta$  replaced by  $-(\sigma \eta_t + \kappa \eta)_{xx}$ ,  $0 < \sigma < \infty$ , in the first equation. Also in the case when  $\alpha_0 \geq \delta (> 0)$  on  $\mathbf{R}$ , the authors [7] showed the existence of a global solution to (P) without  $\partial I_{[-\theta^*, \theta^*]}(\cdot)$  in higher dimensional spaces and the uniqueness in one dimensional space. Furthermore the authors [8] constructed global weak solutions to (P) without  $\partial I_{[-\theta^*, \theta^*]}(\cdot)$  in the case when  $\alpha_0 \geq 0$  on  $\mathbf{R}$  (namely,  $\alpha_0$  is possibly degenerate) and  $\Omega$  is a bounded domain in  $\mathbf{R}^N$  ( $1 \leq N \leq 3$ ).

In this talk we consider the problem (P) in the physical situation that the whole region is already solidified and filled with some grains, so that we may assume that the orientation angle  $\theta$  has two threshold values  $-\theta^*$  and  $\theta^*$ , where  $\theta^*$  is a prescribed positive constant. Hence we take account of  $\partial I_{[-\theta^*, \theta^*]}(\cdot)$  in the second equation of (P).

The main object of this talk is to show the global existence of a weak solution to (P) in the case when  $[\eta_0, \theta_0]$  is the initial data in  $L^2(\Omega) \times L^2(\Omega)$ . Moreover we establish a result on the large-time behavior of solutions to (P), which was suggested by numerical experiments in [12, 13].

## 2 Existence-uniqueness of solutions for (P)

Here we assume the following conditions:

- (A1)  $\alpha_0$  is a function in  $C^2(\mathbf{R})$  such that  $\alpha_0 \geq \delta_0$  on  $\mathbf{R}$  for a positive constant  $\delta_0$ .
- (A2)  $\alpha$  is a non-negative function in  $C^1(\mathbf{R})$ , whose derivative  $\alpha'$  is non-decreasing and bounded on  $\mathbf{R}$  such that  $\alpha'(0) = 0$ .
- (A3)  $g$  is a Lipschitz continuous function on  $\mathbf{R}$ . Suppose that  $g \leq 0$  on  $(-\infty, 0]$  and  $g \geq 0$  on  $[1, \infty)$ . Also we denote by  $\hat{g}$  a primitive of  $g$ , and assume that  $\hat{g}$  is non-negative on  $\mathbf{R}$ .
- (A4)  $\kappa, \nu$  and  $\theta^*$  are real positive constants.
- (A5)  $\eta_0 \in L^2(\Omega)$  with  $0 \leq \eta_0 \leq 1$  a.e. on  $\Omega$ , and  $\theta_0 \in L^2(\Omega)$  with  $|\theta_0| \leq \theta^*$  a.e. on  $\Omega$ .

Next we give a weak formulation for (P) in the variational sense.

**Definition 2.1.** Let  $0 < T < \infty$ . Then, given initial data  $\{\eta_0, \theta_0\} \in L^2(\Omega) \times L^2(\Omega)$ , a pair  $\{\eta, \theta\}$  of functions  $\eta : [0, T] \rightarrow L^2(\Omega)$  and  $\theta : [0, T] \rightarrow L^2(\Omega)$  is called a solution of (P) on  $[0, T]$ , if the following conditions are satisfied:

- (i)  $\eta \in C([0, T]; L^2(\Omega)) \cap W_{loc}^{1,2}((0, T]; L^2(\Omega)) \cap L_{loc}^\infty((0, T]; H^1(\Omega)) \cap L_{loc}^2((0, T]; H^2(\Omega))$ .

(ii)  $\theta \in C([0, T]; L^2(\Omega)) \cap W_{loc}^{1,2}((0, T]; L^2(\Omega)) \cap L_{loc}^\infty((0, T]; H_0^1(\Omega))$ , and  $|\theta| \leq \theta^*$  a.e. on  $Q_T$ .

(iii) The following parabolic equation holds:

$$\eta'(t) - \kappa \Delta_N \eta(t) + g(\eta(t)) + \alpha'(\eta(t)) |\nabla \theta(t)| = 0 \quad \text{in } L^2(\Omega) \quad \text{for a.e. } t \in (0, T),$$

where  $\eta' := \frac{d\eta}{dt}$  and  $\Delta_N : D(\Delta_N) := \{z \in H^2(\Omega); \frac{\partial z}{\partial n} = 0 \text{ a.e. on } \Gamma\} \longrightarrow L^2(\Omega)$  is the Laplacian with homogeneous Neumann boundary condition.

(iv) For a.e.  $t \in (0, T)$  the following variational inequality holds:

$$\begin{aligned} & (\alpha_0(\eta(t))\theta'(t), \theta(t) - z) + \nu (\nabla \theta(t), \nabla \theta(t) - \nabla z) \\ & + \int_{\Omega} \alpha(\eta(x, t)) |\nabla \theta(x, t)| dx \leq \int_{\Omega} \alpha(\eta(x, t)) |\nabla z(x)| dx, \\ & \forall z \in H_0^1(\Omega) \text{ with } |z| \leq \theta^* \text{ a.e. in } \Omega, \end{aligned}$$

where  $\theta' := \frac{\partial \theta}{\partial t}$  and  $(\cdot, \cdot)$  is the standard inner product in  $L^2(\Omega)$ .

(v)  $\eta(0) = \eta_0$  and  $\theta(0) = \theta_0$  in  $L^2(\Omega)$ .

A pair  $\{\eta, \theta\}$  of functions  $\eta : [0, \infty) \rightarrow L^2(\Omega)$  and  $\theta : [0, \infty) \rightarrow L^2(\Omega)$  is called a solution of (P) on  $[0, \infty)$  or a global (in time) solution of (P), if it is a solution of (P) on  $[0, T]$  for every finite  $T > 0$ .

The first main result of this talk is concerned with an existence of solutions for (P).

**Theorem 2.2** (cf. [9]). *Assume (A1)–(A5) hold, and let  $T$  be any finite positive real number. Then there is at least one solution  $\{\eta, \theta\}$  of (P) on  $[0, T]$  in the sense of Definition 2.1, and  $\eta$  satisfies  $0 \leq \eta \leq 1$  a.e. on  $Q_T$ .*

Also the next main result is concerned with a uniqueness of solutions for (P).

**Theorem 2.3** (cf. [9]). *Assume (A1)–(A4),  $\eta_0 \in H^1(\Omega)$  with  $0 \leq \eta_0 \leq 1$  a.e. on  $\Omega$ ,  $\theta_0 \in H_0^1(\Omega)$  with  $|\theta_0| \leq \theta^*$  a.e. on  $\Omega$ , and the space dimension of  $\Omega$  is one, i.e.,  $\Omega = (-L, L)$  for a positive number  $L$ . Then the solution  $\{\eta, \theta\}$  obtained by Theorem 2.2 is unique.*

### 3 Large-time behavior of solutions to (P)

In this section we discuss the large-time behavior of solutions to (P) as  $t \rightarrow \infty$ .

Now we consider the steady-state system for (P), which is of the form:

$$(S) \begin{cases} -\kappa \Delta \eta + g(\eta) + \alpha'(\eta) |\nabla \theta| = 0 & \text{in } \Omega \\ -\nu \Delta \theta - \operatorname{div} \left( \alpha(\eta) \frac{\nabla \theta}{|\nabla \theta|} \right) + \partial_{[-\theta^*, \theta^*]}(\theta) \ni 0 & \text{in } \Omega \\ \frac{\partial \eta}{\partial n} = 0, \quad \theta = 0 & \text{on } \Gamma \end{cases}$$

A pair of functions  $\{\eta, \theta\}$  is a solution of (S), called a steady-state solution of (P), if and only if  $\theta = 0$  in  $L^2(\Omega)$  and  $-\kappa\Delta_N\eta + g(\eta) = 0$  in  $L^2(\Omega)$ . In fact, let  $\{\eta, \theta\}$  be any solution of (S). Then it follows from the second equation of (S) that

$$\frac{\nu}{2}\|\nabla\theta\|_{L^2(\Omega)}^2 + \int_{\Omega} \alpha(\eta)|\nabla\theta|dx = \min_{z \in H_0^1} \left\{ \frac{\nu}{2}\|\nabla z\|_{L^2(\Omega)}^2 + \int_{\Omega} \alpha(\eta)|\nabla z|dx + \int_{\Omega} I_{[-\theta^*, \theta^*]}(z)dx \right\},$$

where  $\|\cdot\|_{L^2(\Omega)}$  is the inner product in  $L^2(\Omega)$ . The above minimum is 0 and is taken at  $z = 0$ . Hence  $\theta = 0$  and the first equation of (S) is  $-\kappa\Delta_N\eta + g(\eta) = 0$  in  $L^2(\Omega)$ . Also it is clear that any pair of functions  $\theta = 0$  and  $\eta$  satisfying  $-\kappa\Delta_N\eta + g(\eta) = 0$  in  $L^2(\Omega)$  is a solution of (S).

Here, for simplicity, we denote by  $S_0$  the set of all solutions of (S), namely

$$S_0 := \{\{\eta, 0\}; \eta \in D(\Delta_N), -\kappa\Delta_N\eta + g(\eta) = 0 \text{ in } L^2(\Omega)\}.$$

Then we have the following third main result of this talk, which is concerned with the large-time behavior of solutions to (P) as  $t \rightarrow \infty$ .

**Theorem 3.1** (cf. [10]). *Assume (A1)–(A5) hold, and let  $\{\eta, \theta\}$  be a solution of (P) on  $[0, \infty)$ . Denote by  $\omega(\eta, \theta)$  the  $\omega$ -limit set of  $\{\eta(t), \theta(t)\}$  as  $t \rightarrow \infty$ , namely*

$$\omega(\eta, \theta) := \left\{ \{\xi, z\} \in L^2(\Omega) \times L^2(\Omega) \mid \begin{array}{l} \eta(t_n) \rightarrow \xi \text{ in } L^2(\Omega), \theta(t_n) \rightarrow z \text{ in } L^2(\Omega) \\ \text{for some } t_n \text{ with } t_n \uparrow \infty \end{array} \right\}.$$

Then  $\omega(\eta, \theta) \subset S_0$ .

Note that the solution of (S) is not unique, namely, the set  $S_0$  is not a singleton in general, because of the term  $g(\eta)$ . So, we assume the additional condition for  $g$ . Then we have the following main result, which is concerned with the asymptotic convergence of all solutions of (P) as  $t \rightarrow \infty$  in a special case of  $g$ .

**Theorem 3.2** (cf. [10]). *In addition to (A1)–(A5), suppose that  $g < 0$  on  $[0, 1)$  and  $g(1) = \hat{g}(1) = 0$ . Let  $\{\eta, \theta\}$  be any solution of (P) on  $[0, \infty)$ . Then*

$$\eta(t) \longrightarrow 1 \text{ in } H^1(\Omega) \quad \text{and} \quad \theta(t) \longrightarrow 0 \text{ in } H_0^1(\Omega) \quad \text{as } t \rightarrow \infty, \quad (3.1)$$

and the convergence (3.1) is uniform with respect to all the initial data  $\{\eta_0, \theta_0\}$ , and  $\{1, 0\}$  is a unique steady-state solution of (P).

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# MOTION BY CURVATURE OF PLANAR CURVES WITH TWO FREE END POINTS

JONG-SHENQ GUO

In this talk, we shall present a recent joint work with Xinfu Chen on the following problem on the evolution of planar curves.

**Problem (P):** Given an initial curve  $\Gamma(0)$ , find a family of curves  $\{\Gamma(t)\}_{0 < t < T}$  that lie on the upper-half plane, have end points on the  $x$ -axis with contact angle  $\psi_-$  on the left and  $\psi_+$  on the right, and evolve according to the motion by curvature; see Figure 1 (a).

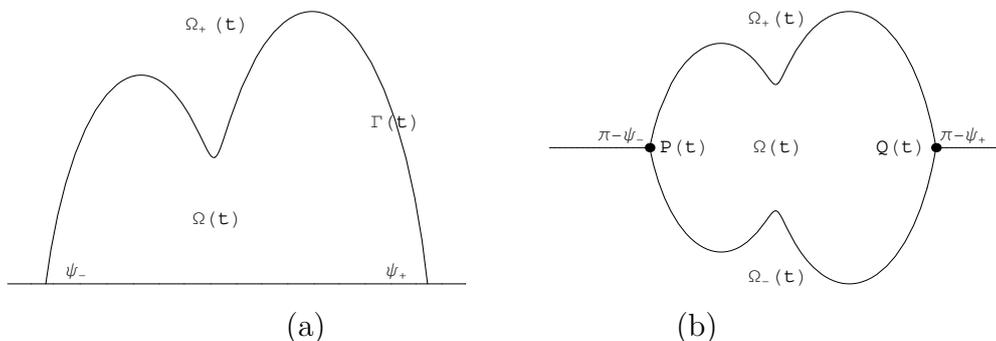


FIGURE 1. Figure (b) is a schematic snap shot of a diminishing grain domain  $\Omega(t)$  surrounded by two other grain domains  $\Omega_+(t)$  and  $\Omega_-(t)$ ; the dots  $P(t)$  and  $Q(t)$  are the so-called triple junctions of three grain domains. When  $\Omega(t)$  is symmetric about the  $x$ -axis, figure (a), modelled by problem (P), is the upper-half part of figure (b)

One motivation of our investigation of problem (P) originates from the study of evolution of grain domains in polycrystals. Here by a grain it refers to a periodic lattice structure of composite particles of a crystal; see Angenent and Gurtin [7, 27], Herring [28, 29], Mullins [38, 39, 40], Sutton and Baluffi [45], Woodruff [46], as well as Kobayashi, Warren, and Carter [34, 35, 36] for more physical background. In such a sense, all grains are physically and chemically identical, except their orientations. A grain boundary is the intersection of two grain domains at which orientations of different lattices do not match. Similarly, a triple junction is the meeting place of three grain domains. It is commonly believed that at a triple

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*Keywords:* Motion by curvature, contact angle, triple junction, self-similar solution.

junction, the intersection angles are fixed, a principal quite often called the Herring condition [28, 29] (under such a principal, we indeed should have  $\psi_+ = \psi_-$ ). Grain boundaries are often modelled by the (mean) curvature flows; see the theoretical and laboratorial studies of the group of Adams, Ta'asan, Kinderlehrer, Livshits, Manolache, Mason, Wu, Mullins, Rother, Rollett and Saylor [2, 3, 32], and also mathematical oriented studies of Bronsard and Retich [11], Kinderlehrer and Liu [31], Mantegazza, Novaga and Tortorelli [37].

It is observed that the evolution of grain boundaries makes a network of grains topologically simpler and simpler. This is achieved by diminishing of grains; creation of grains is very rare, except at very early stage of the formation of polycrystals. Here in this paper we consider a mostly observed scenario depicted in Figure 1 (b). Due to the mathematical challenge, here we shall focus only on a situation where  $\Omega(t)$  is symmetric about the  $x$ -axis. Then the evolution of the grain boundary between  $\Omega(t)$  and  $\Omega_+(t)$  is described in the Problem **(P)**.

When no triple junctions are involved, mathematically one studies the curvature flow of a simple closed curve (the boundary of a bounded smooth domain). A fundamental result in this direction is that of Grayson [25] who proved that the curvature evolution of a simple smooth curve remains simple and smooth until it shrinks to a single point; in addition, in its final stage the curve, after an appropriate magnification, becomes closer and closer to a circle. Here we shall prove a similar result:  $\Omega(t)$  shrinks to a single point in an asymptotically self-similar manner.

In the literature, there have been many studies on the (mean) curvature flow of non-simple curves (or hypersurfaces in higher spatial dimension), notably the work of Brakke [9], Evans and Spruck [18, 19, 20, 21], Chen, Giga and Goto [15]. In these studies, either there is non-uniqueness, such as the varifold solution [9, 30], or there is uniqueness, such as the viscosity solution [18, 15], but the uniqueness is obtained in a sense by taking the union of all Brakke's varifold solutions [30]. There is also an approach by regarding the curvature flow as the limit of a scalar Allen-Cahn equation [4, 10, 13, 16, 23, 22, 30, 41, 42]; however, the scalar Allen-Cahn equation [4] can model only two grains.

Thus, in the study of (mean) curvature flow, the existence theory established in [9, 18, 15, 30, 44] on the one hand are beautiful and complete in modelling two phase problems such as the phase transition between liquid and solid; on the other hand, the uniqueness for multiple ( $\geq 3$ ) phase problems has to be reconsidered. For evolution of grains in polycrystals, for example, one has to take into account conditions at triple junctions [11, 37, 31, 42]. Indeed, this is another motivation of this work. In addition to an earlier work [14], we intend to address relevant problems in resolving non-uniqueness in the classical curvature flow.

In this talk, we shall first review some background of problem **(P)** and some known results related to problem **(P)**. Then, we establish, using an elementary scalar PDE approach, local in time existence of a unique solution for any  $C^{1+\alpha}$  ( $\alpha > 0$ ) initial curve  $\Gamma(0)$ ; such a result is indeed established earlier in [12] using a sophisticated semi-group theory and in [11, 37] using a system of parabolic equations. Next, we show that there exists a unique self-similar solution, following the discussion of our earlier paper [14] and also Abresch and Langer [1]; as a byproduct we supply an analytic proof for the monotonicity of a period function originally proven by Abresch and Langer [1] with the help of numerical verifications of certain quantities. Finally, we show that  $\Gamma(t)$  shrinks to a point in a self-similar manner. Due to technical difficulties, we assume that  $\psi_{\pm} \in (0, \pi/2)$  and initially  $\Gamma(0)$  is a graph  $y = u^0(x)$ . We expect the same conclusion holds for a generic simple initial curve and positive  $\psi_{\pm}$  satisfying  $\psi_+ + \psi_- < \pi$ . We leave this important extension as an open problem.

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# Short Communications

August 3, 2010

**Marina Chugunova** (Univ. of Toronto)

Finite speed propagation of the interface and blow-up solutions in long-wave unstable thin-film equations

**Sarah Hormozi** (Univ. of British Columbia)

Exotic flows in visco-plastic lubrication

**Masato Kimura** (Kyushu Univ.)

On some generalizations of polygonal motions

**Hiroshi Watanabe** (Chuo Univ.)

BV-entropy solutions to nonlinear strongly degenerate parabolic equations

The notion of an almost classical solutions to the total variation flow  
and its usefulness

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The equation which is the topic of my talk

$$u_t - \frac{d}{dx} (\operatorname{sgn}(u_x)) = 0, \quad u(a) = A, \quad u(b) = B. \quad (1)$$

is a one-dimensional example of the total-variation flow. The motivations to study this problem is twofold: a) image analysis; b) crystal growth problems. There are different physically relevant models where, a similar to ours surface energy appears, but the corresponding evolutionary problem is not necessarily set up, see e.g. [ABCM, AMM, BL].

The mentioned above formal interpretation of (1) as a steepest descent of the total variation permits to write equation (1) as a gradient flow  $u_t \in -\partial E(u)$  for a functional  $E$ . This is why we can apply the abstract nonlinear semigroup theory of Komura to obtain existence of solutions. However, the generality of this tool does not permit to study fine points of solutions to (1).

Solutions to (1) enjoy interesting properties, Fukui and Giga, [FG], have noticed that facets, i.e. flat parts of the solution, persist provided that they have zero slope. Zero is exactly the point of singularity of function  $|\cdot|$ . This is why the problem of facet evolution is not only nonlocal but highly anisotropic. Our equation (1) is at least formally parabolic of the second order. The above behavior of solutions we call the *sudden directional diffusion*.

As we have already mentioned some properties of facets were established in [FG], e.g. their finite speed of propagation was calculated. What is missing is the description of the process how they merge and how they are created. In [MR1] we studied a problem similar to (1). There we worked with a simplification of the flow of a closed curve by mean weighted curvature. The main difference is that here we deal with Dirichlet boundary conditions which lead to a very big set of stationary solutions, more exactly every sufficiently smooth monotone function satisfying the boundary condition is an equilibrium. In [MR1] we had one attracting equilibrium.

In [MR1] we have shown existence of a global in time solution to a related problem. We also studied their asymptotics. More precisely we have shown existence of so-called *almost classical solutions*, i.e. there is a finite number of time instances when the time derivative does not exist, however, the right time derivative is always well-defined. In addition, for all except those special time instances the solution at  $t$  is differentiable in space except for a finite number of points. Furthermore, in [MR1] we studied a discretization of the flow, we used the implicit Euler scheme in time. As a result we had a sequence of elliptic problems. We also showed that facets naturally arise in the elliptic problem, we also suggested that this leads to appearance of facets in the dynamics problem, but we did not present a rigorous proof. We do this here for a slightly different problem. Our approach is as

follows. We notice that the implicit discretization leads to a series of Yosida approximations to the operation on the RHS of (1). We can study them quite precisely, because we can consider variable time steps. As a result we can capture the moment when two facets merge. We do not perform any further special considerations.

We want to see how the regularity is transported from the semidiscretized problem to the solution of the evolution equation. We exploit the close relationship of the semi-discretization and Yosida approximation.

Our main goal is monitoring the evolution of the facets and a precise description of the regularity of solutions to (1), which we construct here. For this purpose we apply methods, which are distinctively different from those in the literature. We develop ideas which appeared in our earlier works.

Our approach applies the implicit Euler scheme to (1), which results in the problem

$$u^{n+1} = u^n + h \frac{d}{dx} (\operatorname{sgn}(u_x^{n+1})). \quad (2)$$

This process resembles looking for a good notion of a weak solutions to a PDE. Since we came up with an integral equation we will call its solutions *mild* ones. We are able to show that the mild solutions are unique.

Secondly, (2) may be interpreted as an Euler-Lagrange equation for a non-standard variational functional. Namely we set

$$\mathcal{J}(u) = \begin{cases} \int_a^b |Du| & \text{if } u \in D(\mathcal{J}) \equiv \{u \in BV(a, b), u(a) = A, u(b) = B\}, \\ +\infty & L^2(a, b) \setminus D(\mathcal{J}), \end{cases}$$

where  $\int_a^b |Du|$  is the total variation of measure  $Du$ . Then, (2) may be seen as

$$u^n \in u^{n+1} + h \partial \mathcal{J}(u^{n+1}), \quad (3)$$

where  $\partial \mathcal{J}$  is the subdifferential of  $\mathcal{J}$ . It is not difficult to see that the well-established convex analysis will yield existence of a unique solution to inclusion (3). This solution will be called *variational*. Since variational solutions are stronger (we shall see this), thus both solutions coincide.

Thus, no matter which point of view we adopted,  $u^{n+1}$  is given as the action of the nonlinear resolvent operator  $R(1/h, A)$  on  $u^n$ , i.e.

$$u^{n+1} = R(1/h, A)(u^n) \equiv (Id + hA)^{-1}(u^n),$$

where  $A = -\frac{\partial}{\partial x} \operatorname{sgn} \frac{\partial}{\partial x}$ . However, the notion of a mild solution does not permits us to interpret (2) easily. On the other hand, by convex analysis we can see (2) as an inclusion.

The definition of the nonlinear resolvent operator leads to a detailed study of  $\mathcal{J}$ . One of our results is a characterization of solutions to (2). The advantage of (2) is that it permits to monitor closely the behavior of the facets. It says that the regularity propagates. That is, if  $u^n$  is such that  $u_x^n$  belongs to the  $BV$  space and the number of connected components of the properly understood set  $\{x : u_x^n(x) = 0\}$  is finite, then  $u_x^{n+1}$  has the same property.

It is well-know that the nonlinear resolvent leads to Yosida approximation, which is the key object in the construction of the nonlinear semigroup in the Komura theory. Namely, we set

$$A_k u = k(u - R(k, A)u). \quad (4)$$

Our observation is that a maximal monotone multivalued operator like  $\text{sgn}$  taking values in  $[0, 1]$  may be composed with a multifunction properly generalizing a function of bounded total variation. We shall describe here this composition denoted by  $\bar{\circ}$ . We introduced such an operation in [MR1], see also [MR2]. We also point to an essential difficulty here, which is the problem of composition of two multidimensional operator. Even if both of them are maximal monotone the result need not be monotone nor single valued. If the outer on the two operators we compose is a subdifferential, then we expect that the result is closely related to the minimal section of the subdifferential.

One of our main results says that  $A_k u$  defined by (4) indeed converges to  $-\frac{\partial}{\partial x} \text{sgn} \bar{\circ} u_x$ . In this way we justify correctness of the new notion. Due to the “explicit” nature of  $\bar{\circ}$  we may describe better the regularity of solutions to (2).

Once we constructed the Yosida approximation we show existence on short time intervals of solution to the approximating problem  $u_t^n = A_k(u^n)$ , where  $u^n(t_0)$  is given. In fact, the methods is close in spirit to the construction of the nonlinear semigroup.

Next, we show convergence of the approximate solutions. Here, we use the full power of the Yosida approximation to capture the finite number of time instances when the solution  $u(t)$  is just right differentiable with respect to time, otherwise the derivative exists. The point is that we can control the discretization parameter  $h$  in (2), so that we can monitor the time instances when facets merge.

Finally, we present numerical simulations. They are based upon the semidiscretization. Since they present a series of time snapshots, these pictures contain only the round-off error. At each time step there is no discretization error.

The talk is based on joint results [KMR] with Karolina Kielak and Piotr Rybka.

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## Variational problems using total variation defined by noneuclidean norms.

William K. Allard

### Abstract

Let  $\Phi$  be a norm on  $\mathbf{R}^n$ . It is clear that one can extend the classical notion of a function of bounded variation on an open subset  $\Omega$  of  $\mathbf{R}^n$ , defined using the euclidean norm, by replacing this euclidean norm with  $\Phi$ . There are a number of reasons for doing this; for example, if one uses graph cut methods to compute classical total variation regularization one must replace the euclidean norm by a polygonal approximation. In this talk I will describe recent work with Kevin Vixie wherein we prove natural analogues of most of the results of the my papers *Total variation regularization for image denoising: I, II, III*, which recently appeared in **SIAM Journal on Mathematical Analysis** and **SIAM Journal on Imaging Sciences**, at least when  $n = 2$ , in the case where the euclidean norm is replaced by  $\Phi$ . Our methods break down when  $n > 2$ . It is would be interesting to know what happens when  $n > 2$ .

**$p$ -HARMONIOUS FUNCTIONS, ASYMPTOTIC MEAN  
VALUE PROPERTIES,  
AND TUG-OF-WAR GAMES WITH NOISE**

TUTORIAL LECTURES AND INTERNATIONAL WORKSHOP “SINGULAR  
DIFFUSION AND EVOLVING INTERFACES”  
HOKKAIDO UNIVERSITY  
BY  
JUAN J. MANFREDI

The fundamental works of Kolmogorov, Ito, Kakutani, Doob, Hunt, Lévy, and many others have shown the profound and powerful connection between classical linear potential theory and the corresponding probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common thread in mean value properties. In these two lectures, we will see how this approach turns out to be very useful in the non-linear theory as well.

In these notes we present an extended abstract of the lectures and a bibliography of relevant papers. These notes are based on the joint papers [MPR], [MPR2], [MPR3], and [MPR4] with Mikko Parviainen (Helsinki) and Julio Rossi (Buenos Aires), and on the 2010 doctoral thesis of Alexander Sviridov [S].

**Part 1. T**

This is the first draft of these notes. All errors are the exclusive responsibility of the author.

LECTURE 1: A SURVEY OF  $p$ -HARMONIOUS FUNCTIONS IN TREES AND IN  
EUCLIDEAN SPACE

1. **The  $p$ -Laplacian Gambling House.** Start with a set  $\mathfrak{X}$  endowed with a  $\sigma$ -algebra  $\mathcal{B}$ . Decompose

$$\mathfrak{X} = X \cup Y$$

as a disjoint union of two non-empty sets  $X$  and  $Y$ . We shall call  $X$  the interior and  $Y$  the boundary. For each point  $x \in \mathfrak{X}$  we have a nonempty set  $S(x) \subset \mathfrak{X}$  of successors of  $x$ . For points  $y \in Y$  we require that  $S(y) = \{y\}$ . Moreover, the set  $S(x)$  comes equipped with a probability measure supported in  $S(x)$  denoted by  $\mu(x)$ . For points  $y \in Y$  on the boundary we have that  $\mu(y) = \delta_y$ .

We are given non-negative numbers  $\alpha$  and  $\beta$  so that  $\alpha + \beta = 1$  and a pay-off function  $F: Y \mapsto \mathbb{R}$ .

At every point  $x \in \mathfrak{X}$  we have a family of probability measures  $\Gamma(x)$  in  $(\mathfrak{X}, \mathcal{B})$  given by

$$(1.1) \quad \Gamma(x) = \left\{ \frac{\alpha}{2} (\delta_{x_I} + \delta_{x_{II}}) + \beta \mu(x) : x_I, x_{II} \in S(x) \right\}$$

To play a Tug-of-War game with noise starting at a point  $x_0 \in \mathfrak{X}$ , choose a probability  $\gamma_0[x_0] \in \Gamma(x_0)$ . The next position  $x_1 \in S(x_0)$  is selected according to  $\gamma_0[x_0]$ . Once  $x_0$  and  $x_1$  are chosen, we pick a probability  $\gamma_1[x_0, x_1] \in \Gamma(x_1)$  to determine the next game position  $x_2 \in S(x_1)$ . In this manner we determine a particular history

$$x = (x_0, x_1, x_2, \dots) \in \mathfrak{X} \times \mathfrak{X} \times \dots \times \mathfrak{X} \times \dots = \mathfrak{X}^\infty.$$

The game ends when we reach the boundary  $Y$  since once  $x_j \in Y$  we have  $x_{j+1} \in S(x_j) = \{x_j\}$ . We write

$$\tau(x) = \inf\{k : x_k \in Y\}$$

for the first time we hit the boundary with the understanding that  $\tau(x) = \infty$  if the boundary is never reached. If the game ends at a point  $y \in Y$  the pay-off value is  $F(y)$ .

Let us denote by  $\mathcal{B}^j$  the product  $\sigma$ -algebra in  $\mathfrak{X}^j$  and by  $\mathcal{B}^\infty$  the  $\sigma$ -algebra in  $\mathfrak{X}^\infty$  generated by the cylinder sets

$$A_0 \times A_1 \times \dots \times A_j \times \mathfrak{X} \times \mathfrak{X} \times \dots,$$

where  $A_k \in \mathcal{B}_k$  for  $k = 0, 1, \dots, j$ .

Applying the Kolmogorov-Tulcea construction, it follows that there exists a unique probability measure  $\mathbb{P}_\sigma^{x_0}$  in  $(\mathfrak{X}^\infty, \mathcal{B}^\infty)$  with transition probabilities

$$(1.2) \quad \mathbb{P}_\sigma^{x_0}(\{x_{j+1} \in A\} | \mathcal{B}_{j+1}) = \gamma_j[x_0, x_1, \dots, x_j]$$

We call the collection of probability measures

$$\sigma = (\gamma_0[x_0], \gamma_1[x_0, x_1], \dots, \gamma_k[x_0, x_1, \dots, x_k], \dots)$$

a *strategy*.

This formalism, coming from [MS], is equivalent to the presentation in [PSSW]. In this paper a strategy  $S$  is a collection of mappings  $\sigma_j: \mathfrak{X}^{j+1} \mapsto \mathfrak{X}$  indicating the next move  $x_{j+1} = \sigma_j(x_0, x_1, \dots, x_j)$  given the partial history  $(x_0, x_1, \dots, x_j)$ . A pair of strategies  $S_I$  and  $S_{II}$  and a starting point determine a family of measures

$$\{\mathbb{P}_{S_I, S_{II}}^{x_0}\}_{x_0 \in \mathfrak{X}}$$

that describe the game played under this pair of strategies. That is, the players choose either  $x_I$  or  $x_{II}$  to move there in case they win the coin toss. Their choices determine the probability measures  $\gamma[x_0, x_1, \dots, x_k]$  given  $(x_0, x_1, \dots, x_{k-1})$  and vice versa. Player I will try to choose points  $x_I$  to maximize the pay-off while player II will try to choose points  $x_{II}$  to

minimize the pay-off. Each pair of strategies  $(S_I, S_{II})$ ,  $S_I$  for player I and  $S_{II}$  for player II as in [PSSW], determine a strategy in this sense and vice versa. We write

$$\sigma = (S_I, S_{II})$$

Having fixed a strategy  $\sigma$  and assuming, as we do from now on, that the game ends a. s.

$$(1.3) \quad \mathbb{P}_\sigma^{x_0}(\tau(\mathfrak{X}) < \infty) = 1,$$

we average with respect to  $\mathbb{P}_\sigma^{x_0}$  to obtain the expected pay-off for the Tug-of-War game starting at  $x_0$

$$(1.4) \quad u_\sigma(x_0) = \mathbb{E}_\sigma^{x_0}[F(x_\tau)].$$

To write down the mean value property satisfied by  $u_\sigma$  we condition on the first move using (1.2) with  $j = 0$ .

**Lemma 1.** ([MS], Chapter 2) *The value function  $u_\sigma(x)$  satisfies the mean value property*

$$(1.5) \quad u_\sigma(x) = \frac{\alpha}{2} (u_{\sigma[x_I]}(x_I) + u_{\sigma[x_{II}]}(x_{II})) + \beta \int_{S(x_0)} u_{\sigma[y]}(y) d\mu(y)$$

Here the conditional strategy  $\sigma[y_0]$  is defined as follows for  $y_0 \in S(x_0)$

$$\sigma[y_0] = (\gamma_1[x_0, y_0], \gamma_2[x_0, y_0, y_1] \dots, \gamma_k[x_0, y_0, y_1 \dots y_k], \dots)$$

so that  $\mathbb{P}_{\sigma[y_0]}^{y_0}$  is the conditional distribution of  $(x_2, x_3, \dots)$  given that  $x_1 = y_0$ .

Let us stop and consider the particular case when  $\alpha = 0$  and  $\beta = 1$ . In this case –the linear case– the strategies are irrelevant since  $\Gamma(x)$  is always  $\mu(x)$  so that there is only one family of measures  $\{\mathbb{P}^{x_0}\}_{x_0 \in \mathfrak{X}}$ . We recover the classical mean value formula

$$u(x) = \int_{S(x)} u(y) d\mu(y).$$

But the case of interest to us is when we have  $\alpha \neq 0$ . In this case the value function for player I is

$$u_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_\sigma^x[F(x_\tau)]$$

and for player II

$$u_{II}(x) = \sup_{S_{II}} \inf_{S_I} \mathbb{E}_\sigma^x[F(x_\tau)].$$

Player I lets Player II choose a strategy, presumably to decrease  $\mathbb{E}_\sigma^{x_0}[F(x_\tau)]$ , and then do as best a possible. Notice that we always have

$$u_I(x) \leq u_{II}(x) \quad \text{for all } x \in \mathfrak{X}.$$

It turns out that in many cases the game has a value; that is

$$(1.6) \quad u_I(x) = u_{II}(x) \quad \text{for all } x \in \mathfrak{X},$$

and that this function satisfies a version of the Mean Value Property (1) given by

$$(1.7) \quad u(x) = \frac{\alpha}{2} \left( \sup_{y \in S(x)} u(y) + \inf_{y \in S(x)} u(y) \right) + \beta \int_{S(x)} u(y) d\mu(y).$$

Equation (1.7) is the Dynamic Programming Principle or DPP for short. Next, we will present two scenarios in which all the details above have been worked out.

**2.  $p$ -harmonious functions.** Consider a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^n$  and fix  $\varepsilon > 0$ . To prescribe boundary values, let us denote the compact boundary strip of width  $\varepsilon$  by

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega : \text{dist}(x, \partial\Omega) \leq \varepsilon\}.$$

Let  $\mathfrak{X} = \overline{\Omega}$  with the Borel  $\sigma$ -algebra,  $X = \Omega \setminus \Gamma_\varepsilon$  and  $Y = \Gamma_\varepsilon$ . The successors of  $x$  are  $S(x) = \overline{B}_\varepsilon(x) = \{y \in \mathbb{R}^n : |y - x| \leq \varepsilon\}$  and the measure  $\mu(x)$  is just the Lebesgue measure restricted to  $S(x)$  and normalized so that  $\mu(x)(S(x)) = 1$ . As it will be clear later on, we take  $\alpha$  and  $\beta$  to be

$$(1.8) \quad \alpha = \frac{p-2}{p+n}, \quad \text{and} \quad \beta = \frac{2+n}{p+n}.$$

Notice that since  $\alpha \geq 0$  we necessarily have  $p \geq 2$ .

We are given a bounded Borel pay-off function  $F : \Gamma_\varepsilon \rightarrow \mathbb{R}$  and play the Tug-of-War game with parameters  $\alpha$  and  $\beta$  and obtain value functions  $u_I^\varepsilon$  and  $u_{II}^\varepsilon$ , where we have chosen to emphasize the dependence on the step size  $\varepsilon$ . The following results are from [MPR2]:

**Theorem 2.** *The value functions  $u_I^\varepsilon$  and  $u_{II}^\varepsilon$  are  $p$ -harmonious in  $\Omega$  with boundary values  $F : \Gamma_\varepsilon \rightarrow \mathbb{R}$ ; that is, they both satisfy*

$$(1.9) \quad u_\varepsilon(x) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \int_{B_\varepsilon(x)} u_\varepsilon dy \quad \text{for every } x \in \Omega,$$

and

$$u_\varepsilon(x) = F(x), \quad \text{for every } x \in \Gamma_\varepsilon.$$

The **existence** of  $p$ -harmonious functions with given boundary values is obtained by playing the Tug-of-War games with noise. **Uniqueness** follows by using martingales, although the equation is not linear. This was first proved to the best of my knowledge in [PSSW] for  $p = \infty$ .

For finite  $p$  whether the original Tug-of-War game with noise described in [PS] has a value is an open problem. For our modified version of the  $p$ -game **we do have a value**. The key is to judiciously choose strategies so that we can bring martingales into play.

**Lemma 3.** *Let  $v_\varepsilon$  be  $p$ -harmonic such that  $F \leq v_\varepsilon$  on  $\Gamma_\varepsilon$ . Player I chooses an arbitrary strategy  $S_I$  and player II chooses a strategy  $S_{II}^0$  that almost minimizes  $v_\varepsilon$ ,*

$$v_\varepsilon(x_k) \leq \inf_{y \in \bar{B}_\varepsilon(x_{k-1})} v_\varepsilon(y) + \eta 2^{-k}.$$

*Then  $M_k = v_\varepsilon(x_k) + \eta 2^{-k}$  is a supermartingale for any  $\eta > 0$  and  $u_I^\varepsilon \leq v_\varepsilon$ .*

We can now see how the inequality at the boundary literally walks into the interior by using Doob's optional stopping theorem for martingales

$$\begin{aligned} u_I^\varepsilon(x_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0} [F(x_\tau)] \\ &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0} [v_\varepsilon(x_\tau) + \eta 2^{-\tau}] \\ &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0} [M_\tau] \\ &\leq \sup_{S_I} M_0 = v^\varepsilon(x_0) + \eta \end{aligned}$$

An extension of the above technique gives the uniqueness of the value function.

**Theorem 4.** [MPR2] *The game has a value. That is  $u_I^\varepsilon = u_{II}^\varepsilon$ .*

Most importantly for our purposes is the fact the  $p$ -harmonic functions satisfy the Strong Comparison Principle:

**Theorem 5.** [MPR2] *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. and let  $u_\varepsilon$  and  $v_\varepsilon$  be  $p$ -harmonic with boundary data  $F_u \geq F_v$  in  $\Gamma_\varepsilon$ . Then if there exists a point  $x_0 \in \Omega$  such that  $u_\varepsilon(x_0) = v_\varepsilon(x_0)$ , it follows that  $u_\varepsilon = v_\varepsilon$  in  $\Omega$ , and, moreover, the boundary values satisfy  $F_u = F_v$  in  $\Gamma_\varepsilon$ .*

To prove that  $p$ -harmonic functions converge to the unique solution of the Dirichlet problem for the  $p$ -Laplacian in  $\Omega$  with fixed continuous boundary values, we assume that  $\Omega$  is bounded and satisfies the exterior cone condition.

**Theorem 6.** [MPR2] *Consider the unique viscosity solution  $u$  to*

$$(1.10) \quad \begin{cases} \operatorname{div}(|\nabla u|^{p-2} \nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases}$$

*and let  $u_\varepsilon$  be the unique  $p$ -harmonic function with boundary values  $F$ . Then*

$$u_\varepsilon \rightarrow u \quad \text{uniformly in } \bar{\Omega} \quad \text{as } \varepsilon \rightarrow 0.$$

The above limit only depends on the values of  $F$  on  $\partial\Omega$ , and therefore any continuous extension of  $F|_{\partial\Omega}$  to  $\Gamma_{\varepsilon_0}$  gives the same limit.

The key to prove this theorem is to pass from the discrete setting of  $p$ -harmonious functions to the continuous case of  $p$ -harmonic functions. This is done by means of a characterization of  $p$ -harmonic functions in terms of asymptotic mean value properties.

**Theorem 7.** [MPR] *Let  $u \in C(\Omega)$  such that for all  $x \in \Omega$  we have*

$$\frac{\alpha}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \int_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \quad \text{as } \epsilon \rightarrow 0.$$

*Then  $u$  is  $p$ -harmonic in  $\Omega$ . Here  $\alpha$  and  $\beta$  are chosen as in (1.8).*

The converse of this theorem holds if we weaken the asymptotic expansion to hold only in the viscosity sense. See [MPR] for details. Another approach to pass from the discrete to the continuous for fully-nonlinear equations has been given by Kohn and Serfaty [KS] by using a deterministic control theory approach.

**3. Directed Trees.** Consider the simplest case of ternary trees. We follow the formalism developed in [KLW]. A directed tree  $T$  with regular 3-branching consists of the empty set  $\emptyset$  as the top vertex, 3 sequences of length 1 with terms chosen from the set  $\{0, 1, 2\}$ , 9 sequences of length 2 with terms chosen from the set  $\{0, 1, 2\}, \dots, 3^r$  sequences of length  $r$  with terms chosen from the set  $\{0, 1, 2\}$  and so on. The elements of  $T$  are the *vertices*. Each vertex  $v$  at level  $r$  has three successors

$$S(v) = \{v_0, v_1, v_2\}.$$

Let  $u: T \mapsto \mathbb{R}$  be a real valued function. The *gradient* of  $u$  at the vertex  $v$  is the vector in  $\mathbb{R}^3$

$$\nabla u(v) = (u(v_0) - u(v), u(v_1) - u(v), u(v_2) - u(v)).$$

The *divergence* of a vector  $X = (x, y, z) \in \mathbb{R}^3$  is

$$\text{div}(X) = x + y + z.$$

A function  $u$  is *harmonic* if it satisfies the Laplace equation

$$(1.11) \quad \text{div}(\nabla u) = 0.$$

Observe that a function  $u$  is harmonic if and only if it satisfies the mean value property

$$u(v) = \frac{1}{3}(u(v_0) + u(v_1) + u(v_2))$$

A *branch* of  $T$  is an infinite sequence of vertices, each followed by one of its immediate successors. We denote a branch  $b$  starting at the vertex  $b_1$  as follows  $b = (b_1, b_2, \dots, b_r, \dots)$ . The collection of all branches forms the boundary of the tree  $T$  is denoted by  $\partial T$ . By using an expansion in base three we see that we can identify  $\partial T$  with the interval  $[0, 1]$ . Note that the *classical Cantor set*  $\mathcal{C}$  is the subset of  $\partial T$  formed by branches that don't go through any vertex labeled 1.

Set  $\mathfrak{X} = T \cup \partial T$ ,  $X = T$  and  $Y = \partial T$ . The measure  $\mu(v)$  is the normalized counting measure in  $S(v)$

$$\mu(v) = \frac{1}{3} (\delta_{v_0} + \delta_{v_1} + \delta_{v_2}).$$

The pay-off function  $F: \partial T \mapsto \mathbb{R}$  is defined on the unit interval  $[0, 1]$ . We are ready to play games in  $T$ .

Think of a random walk started at the top vertex  $\emptyset$  and move downward by choosing successors at random with uniform probability. When you get at  $\partial T$  at the branch point  $b$  determined by the random walk, you get paid  $f(b)$  dollars. Every time we run the game we get a sequence of vertices  $v_1, v_2, \dots, v_k, \dots$  that determine a point on  $b$  the boundary  $\partial T$ . The set of all boundary points that start at a give vertex  $v$  is a ternary interval of length  $3^{-r}$  if  $v$  is at level  $r$  that we denote by  $I_v$ . Averaging out over all possible plays that start at  $v_1$  we obtain the value function

$$(1.12) \quad \mathbb{E}^{v_1}[f(t)] = u(v_1) = \frac{1}{|I_{v_1}|} \int_{I_{v_1}} f(b) db,$$

which is indeed harmonic in  $T$ . Therefore we have the well-known

**Lemma 8.** *Dirichlet Problem in Trees ( $p = 2$ ): Given a continuous (indeed in  $L^1([0, 1])$ ) function  $f: [0, 1] \mapsto \mathbb{R}$  the unique harmonic function  $u: T \mapsto \mathbb{R}$  such that*

$$\lim_{r \rightarrow \infty} u(b_r) = f(b)$$

for every branch  $b = (b_r) \in \partial T$  is given by (1.12).

Let us now play a Tug-of-War game with noise. Choose  $\alpha \geq 0$ ,  $\beta \geq 0$  such that  $\alpha + \beta = 1$ . Start at  $\emptyset$ . With probability  $\alpha$  the players play Tug-of-War. With probability  $\beta$  move downward by choosing successors at random. When you get at  $\partial T$  at the point  $b$  player II pays  $f(b)$  dollars to player I. The value function  $u$  verifies the **dynamic programming principle** or **mean value property**

$$(1.13) \quad u(v) = \frac{\alpha}{2} \left( \max_i \{u(v_i)\} + \min_i \{u(v_i)\} \right) + \beta \left( \frac{u(v_0) + u(v_1) + u(v_2)}{3} \right)$$

that we can interpret as a PDE on the tree by using the following formula for a generalized divergence depending on the parameters  $\alpha$  and  $\beta$ .

**Definition 9.** Let  $X = (x, y, z)$  be a vector in  $\mathbb{R}^3$ . The  $(\alpha, \beta)$ -divergence of  $X$  is given by

$$\operatorname{div}_{\alpha, \beta}(X) = \frac{\alpha}{2} (\max\{x, y, z\} + \min\{x, y, z\}) + \beta \left( \frac{x + y + z}{3} \right).$$

**Theorem 10.** [S] We have the equivalence

$$DPP \approx MVP \approx PDE$$

in the sense that the function  $u$  satisfies the equation (1.13) in the tree  $T$  if and only if

$$(1.14) \quad \operatorname{div}_{\alpha,\beta}(\nabla u) = 0$$

Some particular cases are:

**(i) The Linear Case:**  $\alpha = 0, \beta = 1$  that corresponds to the linear case  $p = 2$  of harmonic functions (1.11).

**(ii) The Discrete  $\infty$ -Laplacian:**  $\alpha = 1, \beta = 0$  that corresponds to the case  $p = \infty$ . In this case the divergence is

$$\operatorname{div}_{\infty}(X) = \operatorname{div}_{1,0}(X) = \frac{1}{2}(\max\{x, y, z\} + \min\{x, y, z\})$$

and the equation is the the discrete  $\infty$ -Laplacian  $\operatorname{div}_{\infty}(\nabla u) = 0$ .

**(iii) The Discrete  $p$ -Laplacian:** For  $\alpha \neq 0$  and  $\beta \neq 0$  we can select  $p$  as in (1.8), but the role of  $n$  is not intrinsically defined, to obtain the discrete  $p$ -Laplacian We remark that this is the non-divergence form of the  $p$ -Laplacian (1.14). A discrete version of the  $p$ -Laplacian in divergence form can be found in [KLW].

While the formula (1.12) for the solution to the Dirichlet problem for  $p = 2$  is explicit, there are not such formulas to my knowledge for the case  $p \neq 2$ . However, the game theoretic interpretation allows us to find explicit formulas in some special, but interesting cases.

Suppose that  $f$  is monotonically increasing. In this case the best strategy  $S_I^*$  for player I is always to move right and the best strategy  $S_{II}^*$  for player II always to move left. Starting at the vertex  $v$  at level  $k$

$$v = 0.b_1b_2 \dots b_k, \quad b_j \in \{0, 1, 2\}$$

we always move either left (adding a 0) or right (adding a 1). In this case  $I_v$  is a Cantor-like set  $I_v = \{0.b_1b_2 \dots b_kd_1d_2 \dots\}$ ,  $d_j \in \{0, 2\}$ .

**Theorem 11.** [S] *The  $(\alpha, \beta)$ -harmonic function with boundary values  $f$  in the tree  $T$  is given by*

$$u(v) = \int_{I_v} f(b) d\mathbb{P}_v^{\alpha,\beta} db,$$

where  $d\mathbb{P}_v^{\alpha,\beta}$  is a probability in  $[0, 1]$ .

Moreover in the case  $\alpha = 0, \beta = 1$ , which corresponds to  $p = 2$  the measure  $d\mathbb{P}_v^{0,1}$  is just the Lebesgue measure, and in the case  $\alpha = 1, \beta = 0$ , which corresponds to the case  $p = \infty$ , the measure  $d\mathbb{P}_v^{1,0}$  is a Cantor measure supported in  $I_v$ .

To see why this theorem might be true observe that

$$u(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^v [f(b)] = \mathbb{E}_{S_I^*, S_{II}^*}^v [f(b)].$$

Since the strategies used are always the same, we are indeed in a linear situation. All we need to do is to compute the probability  $\mathbb{P}_{S_I^*, S_{II}^*}^v$ .

LECTURE 2: RANDOM TUG-OF-WAR GAMES FOR THE PARABOLIC *p*-LAPLACIAN.

Consider the heat equation. By using Taylor expansions, we observe that a function  $u$  solves

$$u_t(x, t) = \Delta u(x, t)$$

if and only if

$$u(x, t) = \int_{t-\varepsilon^2/(n+2)}^t \int_{B_\varepsilon(x)} u(y, s) dy ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.$$

In the case  $p \neq 2$  our results are easier to state if we rescale the time variable so that we consider viscosity solutions  $u$  to the equation,

$$(2.15) \quad (n + p)u_t(x, t) = |\nabla u|^{2-p} \Delta_p u(x, t).$$

These are characterized by the asymptotic mean value formula

$$u(x, t) = \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} u(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} u(y, s) \right\} ds + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} u(y, s) dy ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0,$$

that should hold in the viscosity sense.

These mean value formulas are related to the Dynamic Programming Principle (DPP) satisfied by the value functions of parabolic tug-of-war games with noise. The DPP is precisely the mean value formula without the correction term  $o(\varepsilon^2)$ . We call functions that satisfy the DPP  $(p, \varepsilon)$ -parabolic. For elliptic counterparts see [LG], [LGA], and [MPR2]. It turns out that  $(p, \varepsilon)$ -parabolic equations have interesting properties making them interesting on their own, but in addition, they approximate solutions to the corresponding parabolic equation.

Le Gruyer and Archer [LGA, LG] used a mean value approach to solve the infinity Laplace equation and related problems. Oberman [O] implemented various convergent difference schemes for infinity harmonic functions using mean values. Kohn and Serfaty [KS] studied a deterministic game theoretic approach to general parabolic equations. They consider a large class of fully nonlinear parabolic equations including the mean curvature flow. Barron, Evans, and Jensen [BEJ] considered various generalizations of  $L^\infty$ -variational problems. In particular, they obtained a version of our results in the case  $p = \infty$ , see Theorem 30 below.

**4. An asymptotic mean value characterization for parabolic equations.** Recall that for  $1 < p < \infty$  we have

$$(2.16) \quad |\nabla u|^{2-p} \Delta_p u = (p-2)\Delta_\infty u + \Delta u,$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

denotes the  $p$ -Laplacian and

$$\Delta_\infty u = |\nabla u|^{-2} \langle D^2 u \nabla u, \nabla u \rangle = |\nabla u|^{-2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

the 1-homogeneous infinity Laplacian. Observe that in equation (2.15) we get

$$u_t = \Delta_\infty u$$

when  $p \rightarrow \infty$ , and

$$(n+2)u_t = \Delta u$$

when  $p = 2$ .

Let  $T > 0$ , and  $\Omega \subset \mathbb{R}^n$  be an open set, and let  $\Omega_T = \Omega \times (0, T)$  be a space-time cylinder with the parabolic boundary

$$\partial_p \Omega_T = \{\partial \Omega \times [0, T]\} \cup \{\Omega \times \{0\}\}.$$

We denote the mean value integral with the usual notation

$$\int_B f(y) dy = \frac{1}{|B|} \int_B f(y) dy.$$

The parabolic equation (2.15) is singular when the gradient vanishes. We recall the definition of viscosity solution based on semicontinuous extensions of the operator, and refer the reader to Chen-Giga-Goto [CGG], Evans-Spruck [ES], and Giga's monograph [G]. Below we denote by  $\lambda_{\max}((p-2)D^2\phi(x, t))$ , and  $\lambda_{\min}((p-2)D^2\phi(x, t))$  the largest, and the smallest of the eigenvalues to the symmetric matrix  $(p-2)D^2\phi(x, t) \in \mathbb{R}^{n \times n}$  for a smooth test function  $\phi$ . We write  $\lambda_{\max}((p-2)D^2\phi(x, t))$  instead of  $(p-2)\lambda_{\max}(D^2\phi(x, t))$  to give a unified treatment for the cases  $p \geq 2$  and  $1 < p < 2$ .

**Definition 12.** A function  $u : \Omega_T \rightarrow \mathbb{R}$  is a viscosity solution to (2.15) if  $u$  is continuous and whenever  $(x_0, t_0) \in \Omega_T$  and  $\phi \in C^2(\Omega_T)$  is such that

- i)  $u(x_0, t_0) = \phi(x_0, t_0)$ ,
- ii)  $u(x, t) > \phi(x, t)$  for  $(x, t) \in \Omega_T$ ,  $(x, t) \neq (x_0, t_0)$ ,

then we have at the point  $(x_0, t_0)$

$$\begin{cases} (n+p)\phi_t \geq (p-2)\Delta_\infty \phi + \Delta \phi, & \text{if } \nabla \phi(x_0, t_0) \neq 0, \\ (n+p)\phi_t \geq \lambda_{\min}((p-2)D^2\phi) + \Delta \phi, & \text{if } \nabla \phi(x_0, t_0) = 0. \end{cases}$$

Moreover, we require that when touching  $u$  with a test function from above all the inequalities are reversed and  $\lambda_{\min}((p-2)D^2\phi)$  is replaced by  $\lambda_{\max}((p-2)D^2\phi)$ .

It will become useful to observe that we can further reduce the number of test functions in the definition of a viscosity solution. Indeed, if the gradient of a test function vanishes we may assume that  $D^2\phi = 0$ , and thus  $\lambda_{\max} = \lambda_{\min} = 0$ . Nothing is required if  $\nabla\phi = 0$  and  $D^2\phi \neq 0$ . The proof follows the ideas in [ES], see also [CGG] and Lemma 3.2. in [JK] for  $p = \infty$ . For the convenience of the reader we provide the details.

**Lemma 13.** *A function  $u : \Omega_T \rightarrow \mathbb{R}$  is a viscosity solution to (2.15) if  $u$  is continuous and whenever  $(x_0, t_0) \in \Omega_T$  and  $\phi \in C^2(\Omega_T)$  is such that*

- i)  $u(x_0, t_0) = \phi(x_0, t_0)$ ,*
- ii)  $u(x, t) > \phi(x, t)$  for  $(x, t) \in \Omega_T$ ,  $(x, t) \neq (x_0, t_0)$ ,*

*then at the point  $(x_0, t_0)$  we have*

$$\begin{cases} (n+p)\phi_t \geq (p-2)\Delta_\infty\phi + \Delta\phi, & \text{if } \nabla\phi(x_0, t_0) \neq 0, \\ \phi_t(x_0, t_0) \geq 0, & \text{if } \nabla\phi(x_0, t_0) = 0, \text{ and } D^2\phi(x_0, t_0) = 0. \end{cases}$$

*Moreover, we require that when testing from above all the inequalities are reversed.*

Similarly to in the elliptic case in [MPR], the asymptotic mean value formulas hold in a *viscosity sense*. We test the mean value formulas for  $u$  with a test function touching  $u$  from above or below.

**Definition 14.** *A continuous function  $u$  satisfies the asymptotic mean value formula*

$$(2.17) \quad \begin{aligned} u(x, t) &= \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B_\varepsilon}(x)} u(y, s) + \min_{y \in \overline{B_\varepsilon}(x)} u(y, s) \right\} ds \\ &\quad + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} u(y, s) dy ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

*in the viscosity sense at  $(x, t) \in \Omega_T$  if for every  $\phi$  as in Lemma 13, we have*

$$(2.18) \quad \begin{aligned} \phi(x, t) &\geq \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B_\varepsilon}(x)} \phi(y, s) + \min_{y \in \overline{B_\varepsilon}(x)} \phi(y, s) \right\} ds \\ &\quad + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \phi(y, s) dy ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

*and analogously when testing from below.*

Observe that the asymptotic mean value formula is free of gradients, and, in particular, that the case  $\nabla\phi(x, t) = 0$ ,  $D^2\phi(x, t) = 0$  is included. Next we characterize viscosity solutions to  $(n+p)u_t = |\nabla u|^{2-p} \Delta_p u$ .

**Theorem 15.** *Let  $1 < p \leq \infty$  and let  $u$  be a continuous function in  $\Omega_T$ . The asymptotic mean value formula*

$$u(x, t) = \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} u(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} u(y, s) \right\} ds + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} u(y, s) dy ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0,$$

*holds for every  $(x, t) \in \Omega_T$  in the viscosity sense if and only if  $u$  is a viscosity solution to*

$$(n + p)u_t(x, t) = |\nabla u|^{2-p} \Delta_p u(x, t).$$

*Above*

$$\alpha = \frac{p-2}{p+n}, \quad \beta = \frac{2+n}{p+n}.$$

Observe that  $\alpha \geq 0, \beta \geq 0, \alpha + \beta = 1$ , and that if  $p = 2$ , then  $\alpha = 0$ , and  $\beta = 1$  and if  $p = \infty$ , then  $\alpha = 1$  and  $\beta = 0$ . Thus, as a special case of the above theorem, we obtain an asymptotic mean value formula for the parabolic infinity Laplacian. This equation was recently studied in [JK] and [J].

**Theorem 16.** *Let  $u$  be a continuous function in  $\Omega_T$ . The asymptotic mean value formula*

$$u(x, t) = \frac{1}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} u(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} u(y, s) \right\} ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0,$$

*holds for every  $(x, t) \in \Omega_T$  in the viscosity sense if and only if  $u$  is a viscosity solution to*

$$u_t(x, t) = \Delta_\infty u(x, t).$$

**5. Proof of Theorem 15.** We divide the proof in three parts: First, we consider the cases  $p = 2$  and  $p = \infty$  separately, and then combine the results to obtain Theorem 15 for any  $1 < p \leq +\infty$ .

THE HEAT EQUATION: Let us first consider the smooth case.

**Proposition 17.** *Let  $u$  be a smooth function in  $\Omega_T$ . The asymptotic mean value formula*

$$u(x, t) = \int_{t-\varepsilon^2/(n+2)}^t \int_{B_\varepsilon(x)} u(y, s) dy ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0,$$

*holds for all  $(x, t) \in \Omega_T$  if and only if*

$$u_t(x, t) = \Delta u(x, t)$$

*in  $\Omega_T$ .*

*Proof.* Let  $(x, t) \in \Omega_T$  and let  $u$  be a smooth function. We use the Taylor expansion

$$\begin{aligned}
 (2.19) \quad u(y, s) &= u(x, t) + \nabla u(x, t) \cdot (y - x) + \frac{1}{2} \langle D^2 u(x, t)(y - x), (y - x) \rangle \\
 &\quad + u_t(x, t)(s - t) + o(|y - x|^2 + |s - t|) \\
 &= u(x, t) + \sum_{i=1}^n \frac{\partial u}{\partial x_i} (y - x)_i \\
 &\quad + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} (y - x)_i (y - x)_j \\
 &\quad + u_t(x, t)(s - t) + o(|y - x|^2 + |s - t|).
 \end{aligned}$$

Averaging both sides, we get

$$\begin{aligned}
 (2.20) \quad &\int_{t-\varepsilon^2/(n+2)}^t \int_{B_\varepsilon(x)} u(y, s) dy ds \\
 &= u(x, t) + \int_{B_\varepsilon(x)} \nabla u(x, t) \cdot (y - x) dy \\
 &\quad + \frac{1}{2} \int_{B_\varepsilon(x)} \langle D^2 u(x, t)(y - x), (y - x) \rangle dy \\
 &\quad + u_t(x, t) \int_{t-\varepsilon^2/(n+2)}^t (s - t) ds + o(\varepsilon^2).
 \end{aligned}$$

Because of symmetry, the first integral on the right hand side vanishes and the second can be simplified as in [MPR] to get

$$\frac{1}{2} \int_{B_\varepsilon(x)} \langle D^2 u(x, t)(y - x), (y - x) \rangle dy = \frac{\varepsilon^2}{2(n+2)} \Delta u(x, t).$$

Finally,

$$\int_{t-\varepsilon^2/(n+2)}^t (s - t) ds = -\frac{\varepsilon^2}{2(n+2)},$$

and thus (2.20) implies

$$\begin{aligned}
 (2.21) \quad &\int_{t-\varepsilon^2/(n+2)}^t \int_{B_\varepsilon(x)} u(y, s) dy ds \\
 &= u(x, t) + \frac{\varepsilon^2}{2(n+2)} (\Delta u(x, t) - u_t(x, t)) + o(\varepsilon^2).
 \end{aligned}$$

This holds for any smooth function.

If  $u$  is a solution to the heat equation, then (2.21) immediately implies that  $u$  satisfies the asymptotic mean value property. According to classical results, a solution to the heat equation is smooth and thus smoothness assumption is not restrictive here.

Next we assume that a smooth  $u$  satisfies the asymptotic mean value formula and show that then  $u$  is a solution to the heat equation. According to the assumption and (2.21), we have

$$\begin{aligned} u(x, t) &= \int_{t-\varepsilon^2/(n+2)}^t \int_{B_\varepsilon(x)} u(y, s) dy ds + o(\varepsilon^2) \\ &= u(x, t) + \frac{\varepsilon^2}{2(n+2)} (\Delta u(x, t) - u_t(x, t)) + o(\varepsilon^2). \end{aligned}$$

Dividing by  $\varepsilon$  and passing to the limit  $\varepsilon \rightarrow 0$  implies

$$0 = \Delta u(x, t) - u_t(x, t).$$

This finishes the proof.  $\square$

In the space-time cylinders  $B_\varepsilon(x) \times (t - \varepsilon^2, t)$ , the asymptotic mean value formula characterizes solutions to the rescaled heat equation

$$(n+2)u_t(x, t) = \Delta u(x, t).$$

In this case, (2.21) takes the form

$$(2.22) \quad \begin{aligned} &\int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} u(y, s) dy ds \\ &= u(x, t) + \frac{\varepsilon^2}{2(n+2)} (\Delta u(x, t) - (n+2)u_t(x, t)) + o(\varepsilon^2). \end{aligned}$$

Alternatively, the same argument shows that solutions to the heat equation are also characterized by asymptotic mean value formula

$$u(x, t) = \int_{B_\varepsilon(x)} u\left(y, t - \frac{\varepsilon^2}{2(n+2)}\right) dy + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.$$

**THE PARABOLIC INFINITY LAPLACIAN:** Next we turn our attention to the homogeneous parabolic infinity Laplacian. We show that the asymptotic mean value formula

$$u(x, t) = \frac{1}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} u(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} u(y, s) \right\} ds + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0,$$

characterizes the viscosity solutions to

$$u_t = \Delta_\infty u.$$

The proof employs the Taylor expansion (2.19) and uses the fact that the minimum and maximum of the test function  $\phi$  over the ball  $\overline{B}_\varepsilon(x)$  at a fixed time is approximately obtained at the points

$$x - \varepsilon \frac{\nabla \phi}{|\nabla \phi|} \quad \text{and} \quad x + \varepsilon \frac{\nabla \phi}{|\nabla \phi|}.$$

The integration over a time interval takes care of the term that involves time derivatives.

*Proof of Theorem 16.* To begin with, choose a point  $(x, t) \in \Omega_T$ ,  $\varepsilon > 0$ ,  $s \in (t - \varepsilon^2, t)$  and any smooth  $\phi$ . Denote by  $x_1^{\varepsilon, s}$  a point in which  $\phi$  attains its minimum over a ball  $\overline{B}_\varepsilon(x)$  at time  $s$ , that is,

$$\phi(x_1^{\varepsilon, s}, s) = \min_{y \in \overline{B}_\varepsilon(x)} \phi(y, s).$$

Evaluating the Taylor expansion (2.19) for  $\phi$  at  $y = x_1^{\varepsilon, s}$ , we get

$$\begin{aligned} \phi(x_1^{\varepsilon, s}, s) &= \phi(x, t) + \nabla\phi(x, t) \cdot (x_1^{\varepsilon, s} - x) \\ &\quad + \frac{1}{2} \langle D^2\phi(x, t)(x_1^{\varepsilon, s} - x), (x_1^{\varepsilon, s} - x) \rangle \\ &\quad + \phi_t(x, t)(s - t) + o(\varepsilon^2 + |s - t|), \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . Evaluating the Taylor expansion at  $y = \tilde{x}_1^{\varepsilon, s}$ , where  $\tilde{x}_1^{\varepsilon, s}$  is the symmetric point of  $x_1^{\varepsilon, s}$  with respect to  $x$ , given by

$$\tilde{x}_1^{\varepsilon, s} = 2x - x_1^{\varepsilon, s},$$

we obtain

$$\begin{aligned} \phi(\tilde{x}_1^{\varepsilon, s}, s) &= \phi(x, t) - \nabla\phi(x, t) \cdot (x_1^{\varepsilon, s} - x) \\ &\quad + \frac{1}{2} \langle D^2\phi(x, t)(x_1^{\varepsilon, s} - x), (x_1^{\varepsilon, s} - x) \rangle \\ &\quad + \phi_t(x, t)(s - t) + o(\varepsilon^2 + |s - t|). \end{aligned}$$

Adding the expressions, we get

$$\begin{aligned} \phi(\tilde{x}_1^{\varepsilon, s}, s) + \phi(x_1^{\varepsilon, s}, s) - 2\phi(x, t) &= \langle D^2\phi(x, t)(x_1^{\varepsilon, s} - x), (x_1^{\varepsilon, s} - x) \rangle \\ &\quad + 2\phi_t(x, t)(s - t) + o(\varepsilon^2 + |s - t|). \end{aligned}$$

As  $x_1^{\varepsilon, s}$  is the point where the minimum of  $\phi(\cdot, s)$  on  $\overline{B}_\varepsilon(x)$  is attained, it follows that

$$\phi(\tilde{x}_1^{\varepsilon, s}, s) + \phi(x_1^{\varepsilon, s}, s) - 2\phi(x, t) \leq \max_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) - 2\phi(x, t),$$

and thus

$$\begin{aligned} \max_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) - 2\phi(x, t) \\ \geq \langle D^2\phi(x, t)(x_1^{\varepsilon, s} - x), (x_1^{\varepsilon, s} - x) \rangle + 2\phi_t(x, t)(s - t) + o(\varepsilon^2 + |s - t|). \end{aligned}$$

Integration over the time interval implies

$$\begin{aligned} (2.23) \quad &\frac{1}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) \right\} ds - \phi(x, t) \\ &\geq \frac{\varepsilon^2}{2} \left( \int_{t-\varepsilon^2}^t \left\langle D^2\phi(x, t) \frac{x_1^{\varepsilon, s} - x}{\varepsilon}, \frac{x_1^{\varepsilon, s} - x}{\varepsilon} \right\rangle ds - \phi_t(x, t) \right) + o(\varepsilon^2). \end{aligned}$$

This inequality holds for any smooth function. By considering a point where  $\phi$  attains its maximum, we could derive a reverse inequality.

Because  $\phi$  is smooth, if  $\nabla\phi(x, t) \neq 0$ , so is  $\nabla\phi(x, s)$  for  $t - \varepsilon^2 \leq s \leq t$  and for small enough  $\varepsilon > 0$  and thus  $x_1^{\varepsilon, s} \in \partial B_\varepsilon(x)$  for small  $\varepsilon$ . We deduce

$$\lim_{\varepsilon \rightarrow 0} \frac{x_1^{\varepsilon, s} - x}{\varepsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x, t).$$

Moreover, we get the limit

$$(2.24) \quad \lim_{\varepsilon \rightarrow 0} \int_{t-\varepsilon^2}^t \left\langle D^2\phi(x, t) \frac{x_1^{\varepsilon, s} - x}{\varepsilon}, \frac{x_1^{\varepsilon, s} - x}{\varepsilon} \right\rangle ds = \left\langle D^2\phi(x, t) \frac{\nabla\phi}{|\nabla\phi|}(x, t), \frac{\nabla\phi}{|\nabla\phi|}(x, t) \right\rangle = \Delta_\infty\phi(x, t).$$

Next we assume that  $u$  satisfies the asymptotic mean value formula in the viscosity sense and show that then  $u$  satisfies the definition of a viscosity solution whenever  $\nabla\phi \neq 0$ . In particular, we have

$$0 \geq -\phi(x, t) + \frac{1}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) \right\} ds + o(\varepsilon^2),$$

for any smooth  $\phi$  touching  $u$  at  $(x, t) \in \Omega_T$  from below. By the previous inequality, the left hand side of (2.23) is bounded above by  $o(\varepsilon^2)$ . It follows from this fact dividing (2.23) by  $\varepsilon^2$ , passing to a limit, and using (2.24) that

$$0 \geq \Delta_\infty\phi(x, t) - \phi_t(x, t).$$

To prove a reverse inequality, we first derive a reverse inequality to (2.23) by considering the maximum point of  $\phi$ , and then choose a function  $\phi$  that touches  $u$  from above.

To prove the reverse implication, assume that  $u$  is a viscosity solution. Let  $\phi$ ,  $\nabla\phi \neq 0$ , be a smooth test function touching  $u$  from above at  $(x, t) \in \Omega_T$ . We have

$$(2.25) \quad \Delta_\infty\phi(x, t) - \phi_t(x, t) \geq 0.$$

It suffices to prove

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left( -\phi(x, t) + \frac{1}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) \right\} ds \right) \geq 0.$$

This again follows from (2.23). Indeed, divide (2.23) by  $\varepsilon^2$ , use (2.24), and deduce from (2.25) that the limit on the right hand side is bounded from below by zero. The argument for the reverse inequality is analogous.

Finally, let  $\nabla\phi(x, t) = 0$ , and suppose that  $\phi$  touches  $u$  at  $(x, t)$  from below. According to Lemma 13, we may also assume that  $D^2\phi(x, t) = 0$ ,

and thus the Taylor expansion implies

$$\phi(y, s) - \phi(x, t) = \phi_t(x, t)(s - t) + o(\varepsilon^2)$$

in the space-time cylinder. Thus supposing that the asymptotic mean value formula holds at  $(x, t)$ , we deduce

$$\begin{aligned} 0 &\geq \frac{1}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} \left( \phi(y, s) - \phi(x, t) \right) + \min_{y \in \overline{B}_\varepsilon(x)} \left( \phi(y, s) - \phi(x, t) \right) \right\} ds \\ &\quad + o(\varepsilon^2) \\ &= \int_{t-\varepsilon^2}^t \phi_t(x, t)(s - t) ds + o(\varepsilon^2) \\ &= -\frac{\varepsilon^2}{2} \phi_t(x, t) + o(\varepsilon^2). \end{aligned}$$

Dividing by  $\varepsilon^2$ , and passing to a limit, we get  $0 \leq \phi_t(x, t)$ . Lemma 13 and an analogous calculation when testing from above shows that  $u$  is a viscosity solution.

Suppose then that  $u$  is a viscosity solution and  $\phi$  is a test function with  $\nabla \phi(x, t) = 0$ ,  $D^2 \phi(x, t) = 0$  that touches  $u$  at  $(x, t)$  from below. Then a similar calculation as above implies

$$\begin{aligned} &\int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(y, s) \right\} ds - 2\phi(x, t) \\ &= -\varepsilon^2 \phi_t(x, t) + o(\varepsilon^2). \end{aligned}$$

By Lemma 13,  $\phi_t(x, t) \geq 0$ . Thus, dividing the above equality by  $\varepsilon^2$  and passing to the limit shows that the asymptotic expansion holds.  $\square$

A similar proof also shows that  $u$  is a viscosity solution to

$$u_t(x, t) = \Delta_\infty u(x, t)$$

if and only if

$$u(x, t) = \frac{1}{2} \left\{ \max_{y \in \overline{B}_\varepsilon(x)} u \left( y, t - \frac{\varepsilon^2}{2} \right) + \min_{y \in \overline{B}_\varepsilon(x)} u \left( y, t - \frac{\varepsilon^2}{2} \right) \right\} + o(\varepsilon^2) \text{ as } \varepsilon \rightarrow 0$$

in the viscosity sense.

**THE  $p$ -LAPLACIAN:** Next we combine the asymptotic mean value formulas from the previous sections. The main point is that, formally, adding the equations

$$(n + 2)u_t = \Delta, u$$

and

$$(p - 2)u_t = (p - 2)\Delta_\infty u$$

we obtain

$$(n+p)u_t = \Delta u + (p-2)\Delta_\infty u;$$

that is,

$$(n+p)u_t = |\nabla u|^{2-p} \Delta_p u.$$

*Proof of Theorem 15.* Assume first that  $p \geq 2$  so that  $\alpha \geq 0$ . Multiplying (2.22) by  $\beta$  and (2.23) by  $\alpha$ , and adding, we obtain

$$\begin{aligned} & \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left\{ \max_{y \in \overline{B_\varepsilon(x)}} \phi(y, s) + \min_{y \in \overline{B_\varepsilon(x)}} \phi(y, s) \right\} ds \\ & \quad + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \phi(y, s) dy ds - \phi(x, t) \\ (2.26) \quad & \geq \frac{\alpha \varepsilon^2}{2} \left( \int_{t-\varepsilon^2}^t \left\langle D^2 \phi(x, t) \frac{x_1^{\varepsilon, s} - x}{\varepsilon}, \frac{x_1^{\varepsilon, s} - x}{\varepsilon} \right\rangle ds - \phi_t(x, t) \right) \\ & \quad + \frac{\beta \varepsilon^2}{2(n+2)} (\Delta \phi(x, t) - (n+2)\phi_t(x, t)) + o(\varepsilon^2) \\ & = \frac{\beta \varepsilon^2}{2(n+2)} \left( (p-2) \int_{t-\varepsilon^2}^t \left\langle D^2 \phi(x, t) \frac{x_1^{\varepsilon, s} - x}{\varepsilon}, \frac{x_1^{\varepsilon, s} - x}{\varepsilon} \right\rangle ds \right. \\ & \quad \left. + \Delta \phi(x, t) - (n+p)\phi_t(x, t) \right) + o(\varepsilon^2). \end{aligned}$$

Notice that this again holds for any smooth function, and (2.24) still holds whenever  $\nabla \phi \neq 0$ . The rest of the proof follows closely the proof of Theorem 16. Further, by considering the maximum point instead of the minimum point  $x_1^{\varepsilon, s}$ , we can derive a reverse inequality to (2.26).

If  $p < 2$ , it follows that  $\alpha < 0$  and the inequality (2.26) is reversed. On the other hand, so is the reverse inequality that can be obtained by considering the maximum point instead of the minimum point  $x_1^{\varepsilon, s}$ . Thus we still have the both inequalities, and we can repeat the same argument.  $\square$

An analogous proof also shows that  $u$  is a solution to

$$(n+p)u_t(x, t) = |\nabla u|^{2-p} \Delta_p u(x, t)$$

in the viscosity sense if and only if

$$\begin{aligned} (2.27) \quad u(x, t) &= \frac{\alpha}{2} \left\{ \max_{y \in \overline{B_\varepsilon(x)}} u\left(y, t - \frac{\varepsilon^2}{2}\right) + \min_{y \in \overline{B_\varepsilon(x)}} u\left(y, t - \frac{\varepsilon^2}{2}\right) \right\} \\ & \quad + \beta \int_{B_\varepsilon(x)} u\left(y, t - \frac{\varepsilon^2}{2}\right) dy + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

We will take this formulation as a starting point when studying the tug-of-war games with limited number of rounds in the next section.

6.  **$(p, \varepsilon)$ -parabolic functions and Tug-of-war games.** Motivated by the asymptotic mean value theorems, we next study the functions satisfying the mean value property (2.27) without the correction term  $o(\varepsilon^2)$  for  $p \geq 2$ . We call these functions  $(p, \varepsilon)$ -parabolic. It turns out that  $(p, \varepsilon)$ -parabolic functions have interesting properties to be studied in their own right, but in addition they approximate solutions to (2.15), and are value functions of a tug-of-war-game with noise when the number of rounds is limited.

Recall that  $\Omega_T \subset \mathbb{R}^{n+1}$  is an open set. To prescribe boundary values, we denote the boundary strip of width  $\varepsilon$  by

$$\Gamma_\varepsilon = \left( S^\varepsilon \times \left(-\frac{\varepsilon^2}{2}, T\right] \right) \cup \left( \Omega \times \left(-\frac{\varepsilon^2}{2}, 0\right] \right),$$

where

$$S^\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega : \text{dist}(x, \partial\Omega) \leq \varepsilon\}.$$

Below  $F : \Gamma_\varepsilon \rightarrow \mathbb{R}$  denotes a bounded Borel function.

**Definition 18.** *The function  $u_\varepsilon$  is  $(p, \varepsilon)$ -parabolic,  $2 \leq p \leq \infty$ , in  $\Omega_T$  with boundary values  $F$  if*

$$\begin{aligned} u_\varepsilon(x, t) &= \frac{\alpha}{2} \left\{ \sup_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon\left(y, t - \frac{\varepsilon^2}{2}\right) + \inf_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon\left(y, t - \frac{\varepsilon^2}{2}\right) \right\} \\ &\quad + \beta \int_{B_\varepsilon(x)} u_\varepsilon\left(y, t - \frac{\varepsilon^2}{2}\right) dy \quad \text{for every } (x, t) \in \Omega_T \\ u_\varepsilon(x, t) &= F(x, t), \quad \text{for every } (x, t) \in \Gamma_\varepsilon, \end{aligned}$$

where

$$\alpha = \frac{p-2}{p+n}, \quad \beta = \frac{n+2}{p+n}.$$

The reason for using the boundary strip  $\Gamma_\varepsilon$  instead of simply using the parabolic boundary  $\partial_p \Omega_T$  is the fact that  $\overline{B}_\varepsilon(x) \times \{t - \frac{\varepsilon^2}{2}\}$  is not necessarily contained in  $\Omega_T$ .

Next we study the tug-of-war game with noise studied in [MPR2], and in a different form in Peres-Sheffield [PS]. See also Peres-Schramm-Sheffield-Wilson [PSSW]. It is a zero-sum-game between two players, Player I and Player II. In this paper, there are two key differences: the game has a preset maximum number of rounds and boundary values may change with time.

To be more precise, at the beginning we fix the maximum number of rounds to be  $N$  and place a token at a point  $x_0 \in \Omega$ . The players toss a biased coin with probabilities  $\alpha$  and  $\beta$ ,  $\alpha + \beta = 1$ . If they get heads (probability  $\alpha$ ), they play a tug-of-war game, that is, a fair coin is tossed and the winner of the toss decides a new game position  $x_1 \in \overline{B}_\varepsilon(x_0)$ . On the other hand, if they get tails (probability  $\beta$ ), the game state moves according to the uniform probability density to a random point in the ball  $B_\varepsilon(x_0)$ . They

continue playing the game until either the token hits the boundary strip  $S^\varepsilon$  or the number of rounds reaches  $N$ . We denote by  $\tau_N \in \{0, 1, \dots, N\}$  the hitting time of  $S^\varepsilon$  or  $N$ , whichever comes first, and by  $x_{\tau_N} \in \Omega \cup S^\varepsilon$  the end point of the game. When no confusion arises, we simply write  $\tau$ . At the end of the game Player I earns  $\mathcal{F}(x_{\tau_N}, \tau_N)$  while Player II earns  $-\mathcal{F}(x_{\tau_N}, \tau_N)$ . Here

$$\mathcal{F} : (S^\varepsilon \times \{0, \dots, N\}) \cup (\Omega \times \{N\}) \rightarrow \mathbb{R}$$

is a given payoff function.

Denote by  $H = \Omega \cup S^\varepsilon$ . A run of the game is a sequence

$$\omega = (\omega_0, \omega_1, \dots, \omega_N) \in H^{N+1}.$$

We define random variables

$$x_k(\omega) = \omega_k, \quad x_k : H^{N+1} \rightarrow \mathbb{R}^n, \quad k = 0, 1, \dots, N,$$

and

$$\tau_N(\omega) = \min\{N, \inf\{k : x_k(\omega) \in S^\varepsilon, k = 0, 1, \dots, N\}\}.$$

A strategy  $S_I$  for Player I is a function which gives the next game position

$$S_I(x_0, x_1, \dots, x_k) = x_{k+1} \in \overline{B}_\varepsilon(x_k)$$

if Player I wins the coin toss. Similarly, Player II plays according to a strategy  $S_{II}$ .

The fixed starting point  $x_0$ , the number of rounds  $N$ , the domain  $\Omega$  and the strategies  $S_I$  and  $S_{II}$  determine a unique probability measure  $\mathbb{P}_{S_I, S_{II}}^{x_0, N}$  in  $H^{N+1}$ . This measure is built by using the initial distribution  $\delta_{x_0}(A)$ , and the family of transition probabilities

$$\begin{aligned} \pi_{S_I, S_{II}}(x_0(\omega), \dots, x_k(\omega), A) &= \pi_{S_I, S_{II}}(\omega_0, \dots, \omega_k, A) \\ &= \beta \frac{|A \cap B_\varepsilon(\omega_k)|}{|B_\varepsilon(\omega_k)|} + \frac{\alpha}{2} \delta_{S_I(\omega_0, \dots, \omega_k)}(A) + \frac{\alpha}{2} \delta_{S_{II}(\omega_0, \dots, \omega_k)}(A). \end{aligned}$$

For more details, we refer to [MPR2, MPR3, PSSW].

The expected payoff, when starting from  $x_0$  with the maximum number of rounds  $N$ , and using the strategies  $S_I, S_{II}$ , is

$$\mathbb{E}_{S_I, S_{II}}^{x_0, N}[\mathcal{F}(x_{\tau_N}, \tau_N)] = \int_{H^{N+1}} \mathcal{F}(x_{\tau_N}(\omega), \tau_N(\omega)) d\mathbb{P}_{S_I, S_{II}}^{x_0, N}(\omega).$$

The *value of the game for Player I* when starting at  $x_0$  with the maximum number of rounds  $N$  is given by

$$u_I^{\varepsilon, N}(x_0, 0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0, N}[\mathcal{F}(x_{\tau_N}, \tau_N)]$$

while the *value of the game for Player II* is given by

$$u_{II}^{\varepsilon, N}(x_0, 0) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^{x_0, N}[\mathcal{F}(x_{\tau_N}, \tau_N)].$$

More generally, we define the value of the game when starting at  $x$  and playing for  $h = N - k$  rounds to be

$$u_I^{\varepsilon, N}(x, k) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x, h}[\mathcal{F}(x_{\tau_h}, k + \tau_h)]$$

while the *value of the game for Player II* is given by

$$u_{II}^{\varepsilon, N}(x, k) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^{x, h}[\mathcal{F}(x_{\tau_h}, k + \tau_h)].$$

Here  $\tau_h \in \{0, 1, \dots, h\}$  is the hitting time of the boundary  $(S^\varepsilon \times \{0, \dots, N\}) \cup (\Omega \times \{N\})$ . In order to accommodate for time dependent boundary values, we need to keep track of the number  $k$  of rounds played. The values  $u_I^{\varepsilon, N}(x, k)$  and  $u_{II}^{\varepsilon, N}(x, k)$  are the expected outcomes the each player can guarantee when the game starts at  $x$  with maximum number of rounds  $N - k$ .

The next lemma states the *Dynamic Programming Principle* (DPP) for the tug-of-war game with a maximum number of rounds. For a detailed proof in the elliptic case see [MPR3]. The parabolic case turns out to be easier since backtracking can be directly implemented. See Chapter 3 in [MS2] and [MS].

**Lemma 19** (DPP). *The value function for Player I satisfies*

$$\begin{aligned} u_I^{\varepsilon, N}(x, k) &= \frac{\alpha}{2} \left\{ \sup_{\overline{B_\varepsilon}(x)} u_I^{\varepsilon, N}(y, k + 1) + \inf_{\overline{B_\varepsilon}(x)} u_I^{\varepsilon, N}(y, k + 1) \right\} \\ &\quad + \beta \int_{B_\varepsilon(x)} u_I^{\varepsilon, N}(y, k + 1) dy, \quad \text{if } x \in \Omega \text{ and } k < N, \\ u_I^{\varepsilon, N}(x, k) &= \mathcal{F}(x, k), \quad \text{if } x \in S^\varepsilon \text{ or } k = N. \end{aligned}$$

*The value function for Player II,  $u_{II}^{\varepsilon, N}$ , satisfies the same equation.*

The expectation is obtained by summing up the expectations of three possible outcomes for the next step with the corresponding probabilities, Player I chooses the next position (probability  $\alpha/2$ ), Player II chooses (probability  $\alpha/2$ ) and the next position is random (probability  $\beta$ ). This is the heuristic background for the DPP.

Next we describe the change of time scale that relates values of the tug-of-war games with noise and  $(p, \varepsilon)$ -parabolic functions. The definition of  $(p, \varepsilon)$ -parabolic function  $u_\varepsilon$ , Definition 18, refers to a forward-in-time parabolic equation. The values  $u_\varepsilon(\cdot, t)$  at time  $t$  are determined by the values  $u_\varepsilon(\cdot, t - \frac{\varepsilon^2}{2})$ . In contrast, in Lemma 19 above, the values at step  $k$  are determined by the values at step  $k + 1$ .

For  $0 < t < T$  let  $N(t)$  be the integer defined by

$$\frac{2t}{\varepsilon^2} \leq N(t) < \frac{2t}{\varepsilon^2} + 1.$$

We use the shorthand notation  $N(t) = \lceil 2t/\varepsilon^2 \rceil$ . Set  $t_0 = t$  and  $t_{k+1} = t_k - \varepsilon^2/2$  for  $k = 0, 1, \dots, N(t) - 1$ ; that is,

$$t_k = \varepsilon^2 \frac{N(t) - k}{2} + t_{N(t)}.$$

Observe that  $t_{N(t)} \in (-\frac{\varepsilon^2}{2}, 0]$ . When no confusion arises, we simply write  $N$  for  $N(t)$ .

Given  $F : \Gamma_\varepsilon \rightarrow \mathbb{R}$  a boundary value function, define a payoff function  $\mathcal{F}_t : \{S^\varepsilon \times \{0, \dots, N\}\} \cup \{\Omega \times \{N\}\} \rightarrow \mathbb{R}$  by

$$(2.28) \quad \mathcal{F}_t(x_\tau, \tau) = F(x_\tau, \varepsilon^2(N - \tau)/2 + t_N) = F(x_\tau, t_\tau).$$

It might be instructive to think of a parabolic cylinder  $\Omega \times (0, t)$  when  $t$  and  $\varepsilon$  are given determining  $N$  and  $t_N$ . The game begins at  $k = 0$  corresponding to  $t_0 = t$  in the time scale. When we play one round  $k \rightarrow k + 1$ , the clock steps  $\varepsilon^2/2$  backwards,  $t_{k+1} = t_k - \varepsilon^2/2$ , and we play until we get outside the cylinder when  $k = \tau$  corresponding to  $t_\tau$  in the time scale.

Next we define

$$(2.29) \quad u_1^\varepsilon(x, t) = u_1^{\varepsilon, N(t)}(x, 0).$$

This defines values for every instant.

For these functions, the DPP takes the form

$$\begin{aligned} u_1^\varepsilon(x, t) &= \frac{\alpha}{2} \left\{ \sup_{y \in \overline{B}_\varepsilon(x)} u_1^\varepsilon\left(y, t - \frac{\varepsilon^2}{2}\right) + \inf_{y \in \overline{B}_\varepsilon(x)} u_1^\varepsilon\left(y, t - \frac{\varepsilon^2}{2}\right) \right\} \\ &\quad + \beta \int_{B_\varepsilon(x)} u_1^\varepsilon\left(y, t - \frac{\varepsilon^2}{2}\right) dy \quad \text{for every } (x, t) \in \Omega_T \\ u_1^\varepsilon(x, t) &= F(x, t), \quad \text{for every } (x, t) \in \Gamma_\varepsilon, \end{aligned}$$

which agrees with Definition 18.

**COMPARISON AND CONVERGENCE:** The  $(p, \varepsilon)$ -parabolic functions satisfy comparison principle and are unique. The proofs are based on martingale arguments similar to those in [MPR2] recalling (2.29) and the fact that the relevant stopping time is now bounded.

We start with a comparison principle for the value functions. The connection of boundary values in different formulations is given in (2.28) and to simplify the notation we will use  $F$  in both formulations.

**Theorem 20.** *If  $v_\varepsilon$  is a  $(p, \varepsilon)$ -parabolic function in  $\Omega_T$  with boundary values  $F_{v_\varepsilon}$  in  $\Gamma_\varepsilon$  such that  $F_{v_\varepsilon} \geq F_{u_1^\varepsilon}$ , then  $v_\varepsilon \geq u_1^\varepsilon$ .*

*Proof.* Player I follows any strategy and Player II follows a strategy  $S_{II}^0$  such that at  $x_{k-1} \in \Omega$  he chooses to step to a point that almost minimizes  $v_\varepsilon(\cdot, t_k)$ , that is, to a point  $x_k \in \overline{B}_\varepsilon(x_{k-1})$  such that

$$v_\varepsilon(x_k, t_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k) + \eta 2^{-k}$$

for some fixed  $\eta > 0$ .

Choose  $(x_0, t_0) \in \Omega_T$ , and set  $N = \lceil 2t_0/\varepsilon^2 \rceil$ . It follows that

$$\begin{aligned} & \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [v_\varepsilon(x_k, t_k) + \eta 2^{-k} \mid x_0, \dots, x_{k-1}] \\ & \leq \frac{\alpha}{2} \left\{ \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k) + \eta 2^{-k} + \sup_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k) \right\} \\ & \quad + \beta \int_{B_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k) dy + \eta 2^{-k} \\ & \leq v_\varepsilon(x_{k-1}, t_{k-1}) + \eta 2^{-(k-1)}, \end{aligned}$$

where we have estimated the strategy of Player I by sup and used the fact that  $v_\varepsilon$  is  $(p, \varepsilon)$ -parabolic. Thus

$$M_k = v_\varepsilon(x_k, t_k) + \eta 2^{-k}$$

is a supermartingale. Since  $F_{v_\varepsilon} \geq F_{u_I^\varepsilon}$  at  $\Gamma_\varepsilon$ , we deduce

$$\begin{aligned} u_I^\varepsilon(x_0, t_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0, N} [F_{u_I^\varepsilon}(x_\tau, t_\tau)] \leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [F_{v_\varepsilon}(x_\tau, t_\tau) + \eta 2^{-\tau}] \\ &= \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [v_\varepsilon(x_\tau, t_\tau) + \eta 2^{-\tau}] \\ &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [M_0] = v_\varepsilon(x_0, t_0) + \eta, \end{aligned}$$

where the fact that  $\tau$  is a bounded stopping time allowed us to use the optional stopping theorem for  $M_k$ . Since  $\eta$  was arbitrary this proves the claim.  $\square$

Similarly, we can prove that  $u_{II}^\varepsilon$  is the largest  $(p, \varepsilon)$ -parabolic function: Player II follows any strategy and Player I always chooses to step to the point where  $v_\varepsilon$  is almost maximized. This implies that  $v_\varepsilon(x_k) - \eta 2^{-k}$  is a submartingale.

Next we show that the game has a value. This together with the previous comparison principle proves the uniqueness of  $(p, \varepsilon)$ -parabolic functions with given boundary values.

**Theorem 21.** *With a given payoff function, the game has a value; that is, we have the equality*

$$u_I^\varepsilon = u_{II}^\varepsilon.$$

*Proof.* It always holds that  $u_I^\varepsilon \leq u_{II}^\varepsilon$  so it remains to show  $u_{II}^\varepsilon \leq u_I^\varepsilon$ . To see this we use the same argument as in the previous theorem: Player II follows a strategy  $S_{II}^0$  such that at  $x_{k-1} \in \Omega$ , he always chooses to step to a point that almost minimizes  $u_I^\varepsilon$ , that is, to a point  $x_k$  such that

$$u_I^\varepsilon(x_k, t_k) \leq \inf_{y \in \overline{B_\varepsilon}(x_{k-1})} u_I^\varepsilon(y, t_k) + \eta 2^{-k},$$

for a fixed  $\eta > 0$ . We start from the point  $(x_0, t_0)$  so that  $N = \lceil 2t_0/\varepsilon^2 \rceil$ . It follows that from the choice of strategies and the dynamic programming principle for  $u_I^\varepsilon$  that

$$\begin{aligned} & \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [u_I^\varepsilon(x_k, t_k) + \eta 2^{-k} \mid x_0, \dots, x_{k-1}] \\ & \leq \frac{\alpha}{2} \left\{ \inf_{y \in \overline{B_\varepsilon}(x_{k-1})} u_I^\varepsilon(y, t_k) + \eta 2^{-k} + \sup_{y \in \overline{B_\varepsilon}(x_{k-1})} u_I^\varepsilon(y, t_k) \right\} \\ & \quad + \beta \int_{B_\varepsilon(x_{k-1})} u_I^\varepsilon(y, t_k) dy + \eta 2^{-k} \\ & \leq u_I^\varepsilon(x_{k-1}, t_{k-1}) + \eta 2^{-(k-1)}. \end{aligned}$$

Thus

$$M_k = u_I^\varepsilon(x_k, t_k) + \eta 2^{-k}$$

is a supermartingale. According to the optional stopping theorem

$$\begin{aligned} u_{II}^\varepsilon(x_0, t_0) &= \inf_{S_{II}} \sup_{S_I} \mathbb{E}_{S_I, S_{II}}^{x_0, N} [F(x_\tau, t_\tau)] \leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [F(x_\tau, t_\tau) + \eta 2^{-\tau}] \\ &= \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [u_I^\varepsilon(x_\tau, t_\tau) + \eta 2^{-\tau}] \\ &\leq \sup_{S_I} \mathbb{E}_{S_I, S_{II}^0}^{x_0, N} [u_I^\varepsilon(x_0, t_0) + \eta] = u_I^\varepsilon(x_0, t_0) + \eta. \end{aligned}$$

□

Theorems 20 and 21 imply uniqueness for  $(p, \varepsilon)$ -parabolic functions.

**Theorem 22.** *There exists a unique  $(p, \varepsilon)$ -parabolic function with given boundary values  $F$ , and it coincides with the value of the game by virtue of (2.29).*

*Proof.* Due to the dynamic programming principle, the values of the games are  $(p, \varepsilon)$ -parabolic functions. This proves the existence part of the theorem. Theorems 20 and 21 together with the remark after Theorem 20 imply the uniqueness. □

This theorem together with Theorem 20 gives the comparison principle for  $(p, \varepsilon)$ -parabolic functions.

**Theorem 23.** *If  $v_\varepsilon$  and  $u_\varepsilon$  are  $(p, \varepsilon)$ -parabolic functions with boundary values  $F_{v_\varepsilon} \geq F_{u_\varepsilon}$ , then  $v_\varepsilon \geq u_\varepsilon$  in  $\Omega_T$ .*

Next, we show that  $(p, \varepsilon)$ -parabolic functions approximate solutions to

$$(n + p)u_t(x, t) = |\nabla u|^{2-p} \Delta_p u(x, t).$$

To prove the convergence, we use the Arzela-Ascoli type compactness lemma. Note that  $(p, \varepsilon)$ -parabolic functions are, in general, discontinuous. Nevertheless, their oscillation is controlled at scale  $\varepsilon$ . Therefore, the Arzela-Ascoli lemma has to be modified accordingly. For the proof of the lemma below, the reader can consult for example [MPR2].

**Lemma 24.** *Let  $\{u_\varepsilon : \overline{\Omega}_T \rightarrow \mathbb{R}, \varepsilon > 0\}$  be a set of functions such that*

- (1) *there exists  $C > 0$  so that  $|u_\varepsilon(x, t)| < C$  for every  $\varepsilon > 0$  and every  $(x, t) \in \overline{\Omega}_T$ ,*
- (2) *given  $\eta > 0$  there are constants  $r_0$  and  $\varepsilon_0$  such that for every  $\varepsilon < \varepsilon_0$  and any  $(x, t), (y, s) \in \overline{\Omega}$  with  $|x - y| + |t - s| < r_0$  it holds*

$$|u_\varepsilon(x, t) - u_\varepsilon(y, s)| < \eta.$$

*Then, there exists a uniformly continuous function  $u : \overline{\Omega}_T \rightarrow \mathbb{R}$  and a subsequence still denoted by  $\{u_\varepsilon\}$  such that*

$$u_\varepsilon \rightarrow u \quad \text{uniformly in } \overline{\Omega}_T,$$

*as  $\varepsilon \rightarrow 0$ .*

First we recall the estimate for the stopping time of a random walk from [MPR2]. In this lemma, there is no bound for the maximum number of rounds.

**Lemma 25.** *Let us consider an annular domain  $B_R(z) \setminus \overline{B}_\delta(z)$  and a random walk such that when at  $x_{k-1}$ , the next point  $x_k$  is chosen according to a uniform probability distribution at  $B_\varepsilon(x_{k-1}) \cap B_R(z)$ . Let*

$$\tau^* = \inf\{k : x_k \in \overline{B}_\delta(z)\}.$$

*Then*

$$\mathbb{E}^{x_0}(\tau^*) \leq \frac{C(R/\delta) \operatorname{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2},$$

*for  $x_0 \in B_R(z) \setminus \overline{B}_\delta(z)$ . Above  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .*

Next we derive an estimate for the asymptotic uniform continuity of a family  $\{u_\varepsilon\}$  of  $(p, \varepsilon)$ -parabolic functions with fixed boundary values.

We assume that  $\Omega$  satisfies an exterior sphere condition: For each  $y \in \partial\Omega$ , there exists  $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$  with  $\delta > 0$  such that  $y \in \partial B_\delta(z)$ . Below  $\delta$  is always chosen small enough according to this condition.

We also assume that  $F$  satisfies

$$(2.30) \quad |F(x, t_x) - F(y, t_y)| \leq L \left( |x - y| + |t_x - t_y|^{1/2} \right)$$

in  $\Gamma_\varepsilon$ . First, we consider the case where  $(y, t_y)$  is a point at the lateral boundary strip.

**Lemma 26.** *Let  $F$  and  $\Omega$  be as above. The  $(p, \varepsilon)$ -parabolic function  $u_\varepsilon$  with the boundary data  $F$  satisfies*

$$(2.31) \quad \begin{aligned} & |u_\varepsilon(x, t_x) - u_\varepsilon(y, t_y)| \\ & \leq C \min \left\{ |x - y|^{1/2} + o(1), t_x^{1/2} + \varepsilon \right\} + L |t_x - t_y|^{1/2} + 2L\delta \end{aligned}$$

for every  $(x, t_x) \in \Omega$ , and  $y \in S^\varepsilon$ . The constant  $C$  depends on  $\delta, n, L$  and the diameter of  $\Omega$ . In the above inequality  $o(1)$  is taken relative to  $\varepsilon$ .

*Proof.* Suppose for the moment that  $t_x = t_y$ , denote  $t_0 = t_x = t_y$ , and set  $x_0 = x$  as well as  $N = \lceil 2t_x/\varepsilon^2 \rceil$ . By the exterior sphere condition, there exists  $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$  such that  $y \in \partial B_\delta(z)$ . Player I chooses a strategy of pulling towards  $z$ , denoted by  $S_I^z$ . Then the calculation

$$(2.32) \quad \begin{aligned} & \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [|x_k - z| | x_0, \dots, x_{k-1}] \\ & \leq \frac{\alpha}{2} \{|x_{k-1} - z| + \varepsilon + |x_{k-1} - z| - \varepsilon\} + \beta \int_{B_\varepsilon(x_{k-1})} |x - z| dx \\ & \leq |x_{k-1} - z| + C\varepsilon^2 \end{aligned}$$

implies that

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for some  $C$  independent of  $\varepsilon$ . The first inequality follows from the choice of the strategy, and the second from the estimate

$$\int_{B_\varepsilon(x_{k-1})} |x - z| dx \leq |x_{k-1} - z| + C\varepsilon^2.$$

The optional stopping theorem and Jensen's inequality then gives

$$(2.33) \quad \begin{aligned} \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [|x_\tau - z| + |t_\tau - t_0|^{1/2}] &= \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} \left[ |x_\tau - z| + \varepsilon \left(\frac{\tau}{2}\right)^{1/2} \right] \\ &\leq |x_0 - z| + C\varepsilon \left( \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [\tau] \right)^{1/2}. \end{aligned}$$

In formula (2.32), the expected distance of the pure tug-of-war is bounded by  $|x_{k-1} - z|$  whereas the expected distance of the pure random walk is slightly larger. Therefore, we can bound from above the stopping time of our process by a stopping time of the random walk in the setting of Lemma 25 by choosing  $R > 0$  such that  $\Omega \subset B_R(z)$ . Thus, we obtain

$$\begin{aligned} \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [\tau] &\leq \min \left\{ \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [\tau^*], N \right\} \\ &\leq \min \left\{ C(R/\delta)(\text{dist}(\partial B_\delta(z), x_0) + o(1))/\varepsilon^2, N \right\}. \end{aligned}$$

Since  $y \in \partial B_\delta(z)$ , we have

$$\text{dist}(\partial B_\delta(z), x_0) \leq |y - x_0|,$$

and together with (2.33) this gives

$$\mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [|x_\tau - z| + |t_\tau - t_0|^{1/2}] \leq \min \{C(R/\delta)(|x_0 - y| + o(1)), C\varepsilon^2 N\}^{1/2} + |x_0 - z|.$$

Thus, we end up with

$$\begin{aligned} F(z, t_0) - L \left( \min \{C(R/\delta)(|x_0 - y| + o(1)), C\varepsilon^2 N\}^{1/2} + |x_0 - z| \right) \\ \leq \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [F(x_\tau, t_\tau)] \\ \leq F(z, t_0) + L \left( \min \{C(R/\delta)(|x_0 - y| + o(1)), C\varepsilon^2 N\}^{1/2} + |x_0 - z| \right), \end{aligned}$$

which implies

$$\begin{aligned} \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0, N} [F(x_\tau, t_\tau)] \\ \geq \inf_{S_{II}} \mathbb{E}_{S_I^z, S_{II}}^{x_0, N} [F(x_\tau, t_\tau)] \\ \geq F(z, t_0) - L \left( \min \{C(R/\delta)(|x_0 - y| + o(1)), C\varepsilon^2 N\}^{1/2} + |x_0 - z| \right) \\ \geq F(y, t_0) - 2L\delta - L \min \{C(R/\delta)(|x_0 - y| + o(1)), C\varepsilon^2 N\}^{1/2}. \end{aligned}$$

The upper bound can be obtained by choosing for Player II a strategy where he points to  $z$ , and thus (2.31) follows.

Finally, if  $t_x \neq t_y$ , then we utilize the above estimate and obtain

$$\begin{aligned} |u_\varepsilon(x, t_x) - u_\varepsilon(y, t_y)| &\leq |u_\varepsilon(x, t_x) - u_\varepsilon(y, t_x)| + |u_\varepsilon(y, t_x) - u_\varepsilon(y, t_y)| \\ &\leq 2L\delta + \min \{C(R/\delta)(|x - y| + o(1)), C\varepsilon^2 N\}^{1/2} + L|t_x - t_y|^{1/2}, \end{aligned}$$

and the proof is completed by recalling that  $N = \lceil 2t_x/\varepsilon^2 \rceil$ .  $\square$

Next we consider the case when the boundary point  $(y, t_y)$  lies at the initial boundary strip.

**Lemma 27.** *Let  $F$  and  $\Omega$  be as in Lemma 26. The  $(p, \varepsilon)$ -parabolic function  $u_\varepsilon$  with the boundary data  $F$  satisfies*

$$(2.34) \quad |u_\varepsilon(x, t_x) - u_\varepsilon(y, t_y)| \leq C \left( |x - y| + t_x^{1/2} + \varepsilon \right),$$

and for every  $(x, t_x) \in \Omega_T$  and  $(y, t_y) \in \Omega \times (-\varepsilon^2/2, 0]$ .

*Proof.* Set  $x_0 = x$ , and  $N = \lceil 2t_x/\varepsilon^2 \rceil$ . Player I pulls to  $y$ . Then

$$M_k = |x_k - y|^2 - Ck\varepsilon^2$$

is a supermartingale. Indeed,

$$\begin{aligned} & \mathbb{E}_{S_I^y, S_{II}}^{x_0, N} [|x_k - y|^2 \mid x_0, \dots, x_{k-1}] \\ & \leq \frac{\alpha}{2} \{ (|x_{k-1} - y| + \varepsilon)^2 + (|x_{k-1} - y| - \varepsilon)^2 \} + \beta \int_{B_\varepsilon(x_{k-1})} |x - y|^2 dx \\ & \leq \alpha \{ |x_{k-1} - y|^2 + \varepsilon^2 \} + \beta (|x_{k-1} - y|^2 + C\varepsilon^2) \leq |x_{k-1} - y|^2 + C\varepsilon^2. \end{aligned}$$

According to optional stopping theorem,

$$\mathbb{E}_{S_I^y, S_{II}}^{x_0, N} [|x_\tau - y|^2] \leq |x_0 - y|^2 + C\varepsilon^2 \mathbb{E}_{S_I^y, S_{II}}^{x_0, N} [\tau],$$

and since the stopping time is bounded by  $\lceil 2t_x/\varepsilon^2 \rceil$ , this implies

$$\mathbb{E}_{S_I^y, S_{II}}^{x_0, N} [|x_\tau - y|^2] \leq |x_0 - y|^2 + C(t_x + \varepsilon^2).$$

Finally, Jensen's inequality gives

$$\begin{aligned} \mathbb{E}_{S_I^y, S_{II}}^{x_0, N} [|x_\tau - y|] & \leq (|x_0 - y|^2 + C(t_x + \varepsilon^2))^{1/2} \\ & \leq |x_0 - y| + C(t_x^{1/2} + \varepsilon). \end{aligned}$$

The rest of the argument is similar to the one used in the previous proof. In particular, we obtain the upper bound by choosing for Player II a strategy where he points to  $y$ . We end up with

$$|u_\varepsilon(x, t_x) - u_\varepsilon(y, t_y)| \leq C \left( |x - y| + t_x^{1/2} + \varepsilon \right). \quad \square$$

Next we will show that  $(p, \varepsilon)$ -parabolic functions are asymptotically uniformly continuous.

**Lemma 28.** *Let  $F$  and  $\Omega$  be as in Lemma 26. Let  $\{u_\varepsilon\}$  be a family of  $(p, \varepsilon)$ -parabolic functions. Then this family satisfies the conditions in Lemma 24.*

*Proof.* It follows from the definition of  $(p, \varepsilon)$ -parabolic function that

$$|u_\varepsilon| \leq \sup_{\Gamma_\varepsilon} F$$

and we can thus concentrate on the second condition of Lemma 24. Observe that the case  $x, y \in \Gamma_\varepsilon$  readily follows from the uniform continuity of  $F$ , and thus we can concentrate on the cases  $x \in \Omega$ ,  $y \in S^\varepsilon$ , and  $x, y \in \Omega$ .

Choose any  $\eta > 0$ . By (2.31) and (2.34), there exists  $\varepsilon_0 > 0$ ,  $\delta > 0$ , and  $r_0 > 0$  so that

$$|u_\varepsilon(x, t_x) - u_\varepsilon(y, t_y)| < \eta$$

for all  $\varepsilon < \varepsilon_0$  and for any  $(x, t_x) \in \Omega_T$ ,  $(y, t_y) \in \Gamma_\varepsilon$  such that  $|x - y|^{1/2} + |t_x - t_y|^{1/2} \leq r_0$ .

Next we consider a slightly smaller domain

$$\tilde{\Omega}_T = \{(z, t) \in \Omega_T : d((z, t), \partial_p \Omega_T) > r_0/3\}$$

with

$$d((z, t), \partial_p \Omega_T) = \inf\{|z - y|^{1/2} + |t - s|^{1/2} : (y, s) \in \partial_p \Omega\},$$

and the boundary strip

$$\tilde{\Gamma} = \{(z, t) \in \bar{\Omega}_T : d((z, t), \partial_p \Omega_T) \leq r_0/3\}.$$

Suppose then that  $x, y \in \Omega_T$  with  $|x - y|^{1/2} + |t_x - t_y|^{1/2} < r_0/3$ . First, if  $x, y \in \tilde{\Gamma}$ , then we can estimate

$$|u_\varepsilon(x, t_x) - u_\varepsilon(y, t_y)| \leq 3\eta$$

for  $\varepsilon < \varepsilon_0$  by comparing the values at  $x$  and  $y$  to the nearby boundary values and using the previous step. Finally, a translation argument finishes the proof. Let  $(x, t_x), (y, t_y) \in \tilde{\Omega}_T$ . Without loss of generality we may assume that  $t_x > t_y$ . Define

$$\tilde{F}(z, t_z) = u_\varepsilon(z - x + y, t_z + t_y - t_x) + 3\eta \quad \text{for } (z, t_z) \in \tilde{\Gamma}.$$

We have

$$\tilde{F}(z, t_z) \geq u_\varepsilon(z, t_z) \quad \text{in } \tilde{\Gamma}$$

by the reasoning above. Solve the  $(p, \varepsilon)$ -parabolic function  $\tilde{u}_\varepsilon$  in  $\tilde{\Omega}_T$  with the boundary values  $\tilde{F}$  in  $\tilde{\Gamma}$ . By the comparison principle Theorem 23, and the uniqueness Theorem 22, we deduce

$$u_\varepsilon(x, t_x) \leq \tilde{u}_\varepsilon(x, t_x) = u_\varepsilon(x - x + y, t_x - t_x + t_y) + 3\eta = u_\varepsilon(y, t_y) + 3\eta \quad \text{in } \tilde{\Omega}_T.$$

The lower bound follows by a similar argument.  $\square$

**Corollary 29.** *Let  $F$  satisfy the continuity condition (2.30) and  $\Omega$  satisfy the exterior sphere condition. Let  $\{u_\varepsilon\}$  be a family of  $(p, \varepsilon)$ -parabolic functions with boundary values  $F$ . Then there exists a uniformly continuous  $u$  and a subsequence still denoted by  $\{u_\varepsilon\}$  such that*

$$u_\varepsilon \rightarrow u \quad \text{uniformly in } \bar{\Omega}$$

as  $\varepsilon \rightarrow 0$ .

**Theorem 30.** *Let  $F$  satisfy the continuity condition (2.30) and  $\Omega$  satisfy the exterior sphere condition. Then, the uniform limit*

$$u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon$$

*of  $(p, \varepsilon)$ -parabolic functions obtained in Corollary 29 is a viscosity solution to the equation*

$$(n + p)u_t(x, t) = |\nabla u|^{2-p} \Delta_p u(x, t)$$

*with boundary values  $F$ .*

*Proof.* First, clearly  $u = F$  on  $\partial\Omega$ , and we can focus attention on showing that  $u$  is a viscosity solution. Similarly as in (2.26), we can derive for any  $\phi \in C^2$  an estimate

$$\begin{aligned}
 (2.35) \quad & \frac{\alpha}{2} \left\{ \max_{y \in \overline{B_\varepsilon}(x)} \phi\left(y, t - \frac{\varepsilon^2}{2}\right) + \min_{y \in \overline{B_\varepsilon}(x)} \phi\left(y, t - \frac{\varepsilon^2}{2}\right) \right\} \\
 & + \beta \int_{B_\varepsilon(x)} \phi\left(y, t - \frac{\varepsilon^2}{2}\right) dy - \phi(x, t) \\
 & \geq \frac{\beta\varepsilon^2}{2(n+2)} \left( (p-2) \left\langle D^2\phi(x, t) \frac{x_1^{\varepsilon, t-\varepsilon^2/2} - x}{\varepsilon}, \frac{x_1^{\varepsilon, t-\varepsilon^2/2} - x}{\varepsilon} \right\rangle \right. \\
 & \quad \left. + \Delta\phi(x, t) - (n+p)\phi_t(x, t) \right) + o(\varepsilon^2),
 \end{aligned}$$

where

$$\phi\left(x_1^{\varepsilon, t-\varepsilon^2/2}, t - \frac{\varepsilon^2}{2}\right) = \min_{y \in \overline{B_\varepsilon}(x)} \phi\left(y, t - \frac{\varepsilon^2}{2}\right).$$

Suppose then that  $\phi$  touches  $u$  at  $(x, t)$  from below. By the uniform convergence, there exists sequence  $\{(x_\varepsilon, t_\varepsilon)\}$  converging to  $(x, t)$  such that  $u_\varepsilon - \phi$  has an approximate minimum at  $(x_\varepsilon, t_\varepsilon)$ , that is, for  $\eta_\varepsilon > 0$ , there exists  $(x_\varepsilon, t_\varepsilon)$  such that

$$u_\varepsilon(y, s) - \phi(y, s) \geq u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon) - \eta_\varepsilon,$$

in the neighborhood of  $(x_\varepsilon, t_\varepsilon)$ . Further, set  $\tilde{\phi} = \phi + u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon)$ , so that

$$u_\varepsilon(x_\varepsilon, t_\varepsilon) = \tilde{\phi}(x_\varepsilon, t_\varepsilon) \quad \text{and} \quad u_\varepsilon(y, s) \geq \tilde{\phi}(y, s) - \eta_\varepsilon.$$

Thus, by recalling the fact that  $u_\varepsilon$  is  $(p, \varepsilon)$ -parabolic, we obtain

$$\begin{aligned}
 (2.36) \quad \eta_\varepsilon & \geq -\tilde{\phi}(x_\varepsilon, t_\varepsilon) + \beta \int_{B_\varepsilon(x_\varepsilon)} \tilde{\phi}\left(y, t_\varepsilon - \frac{\varepsilon^2}{2}\right) dy \\
 & + \frac{\alpha}{2} \left\{ \sup_{y \in \overline{B_\varepsilon}(x_\varepsilon)} \tilde{\phi}\left(y, t_\varepsilon - \frac{\varepsilon^2}{2}\right) + \inf_{y \in \overline{B_\varepsilon}(x_\varepsilon)} \tilde{\phi}\left(y, t_\varepsilon - \frac{\varepsilon^2}{2}\right) \right\}.
 \end{aligned}$$

According to (2.35), choosing  $\eta_\varepsilon = o(\varepsilon^2)$ , and observing  $\nabla\phi = \nabla\tilde{\phi}$ ,  $D^2\tilde{\phi} = D^2\phi$ , we have

$$\begin{aligned}
 0 & \geq \frac{\beta\varepsilon^2}{2(n+2)} \left( (p-2) \left\langle D^2\phi(x_\varepsilon, t_\varepsilon) \frac{x_1^{\varepsilon, t-\varepsilon^2/2} - x_\varepsilon}{\varepsilon}, \frac{x_1^{\varepsilon, t-\varepsilon^2/2} - x_\varepsilon}{\varepsilon} \right\rangle \right. \\
 & \quad \left. + \Delta\phi(x_\varepsilon, t_\varepsilon) - (n+p)\phi_t(x_\varepsilon, t_\varepsilon) \right) + o(\varepsilon^2).
 \end{aligned}$$

Suppose that  $\nabla\phi(x, t) \neq 0$ . Dividing by  $\varepsilon^2$  and letting  $\varepsilon \rightarrow 0$ , we get

$$0 \geq \frac{\beta}{2(n+2)} \left( (p-2)\Delta_\infty\phi(x) + \Delta\phi(x) - (n+p)\phi_t(x, t) \right).$$

To verify the other half of the definition of a viscosity solution, we derive a reverse inequality to (2.35) by considering the maximum point of the test function and choose a function  $\phi$  which touches  $u$  from above. The rest of the argument is analogous.

Now we consider the case  $\nabla\phi(x, t) = 0$ . By Lemma 13, we can also assume that  $D^2\phi(x, t) = 0$  and it suffices to show that

$$\phi_t(x, t) \geq 0.$$

In this case, (2.35) takes the form

$$\begin{aligned} & \frac{\alpha}{2} \left\{ \max_{y \in \overline{B}_\varepsilon(x)} \phi\left(y, t - \frac{\varepsilon^2}{2}\right) + \min_{y \in \overline{B}_\varepsilon(x)} \phi\left(y, t - \frac{\varepsilon^2}{2}\right) \right\} \\ & \quad + \beta \int_{B_\varepsilon(x)} \phi\left(y, t - \frac{\varepsilon^2}{2}\right) dy - \phi(x, t) \\ & \geq -\frac{\beta\varepsilon^2(n+p)}{2(n+2)}\phi_t(x, t) + o(\varepsilon^2). \end{aligned}$$

Since (2.36) still holds, we can repeat the argument above.

Finally, we conclude that also the original sequence converges to a unique viscosity solution. To this end, observe that by above any sequence  $\{u_\varepsilon\}$  contains a subsequence that converges uniformly to some viscosity solution  $u$ . By [CGG] (see also [ES] and [GGIS]), viscosity solutions to (2.15) are uniquely determined by their boundary values. Hence we conclude that the whole original sequence converges.  $\square$

Observe that the above theorem also gives a proof of the existence of viscosity solutions to (2.15) using probabilistic arguments.

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**$p$ -HARMONIOUS FUNCTIONS, ASYMPTOTIC MEAN VALUE PROPERTIES, AND TUG-OF-WAR GAMES WITH NOISE**

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## LARGE SOLUTIONS FOR SOME PARABOLIC EQUATIONS WITHOUT ABSORPTION

SALVADOR MOLL

In this talk I will present some new results obtained in a joint work with F. Petitta about existence and uniqueness of entropy/renormalized large solutions for the parabolic p-laplacian problem without absorption for the case  $1 < p < 2$ ; i.e.

$$(P)_p \begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u) & \text{in } \Omega \times [0, T) \\ u = +\infty & \text{in } \partial\Omega \times [0, T) \end{cases}$$

as well as existence and uniqueness of entropy solutions of large solutions for the total variation flow:

$$(P)_1 \begin{cases} u_t = \operatorname{div} \left( \frac{Du}{|Du|} \right) & \text{in } \Omega \times [0, T) \\ u = +\infty & \text{in } \partial\Omega \times [0, T) \end{cases}$$

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# Scale-invariant extinction time estimates for some singular diffusion equations

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A total variation flow is a gradient flow of the total variation and is often used in image analysis for denoising and restoring image. It is often used in materials science to describe evolution of a crystal surface.

We consider two models. One is an  $L^2$ -gradient flow of the total variation, i.e.,

$$u_t = \operatorname{div} (\nabla u / |\nabla u|) \quad (1)$$

and the other is an  $H^{-1}$ -gradient flow, i.e.,

$$u_t = -\Delta \operatorname{div} (\nabla u / |\nabla u|). \quad (2)$$

Both equations have a strong diffusivity effect for the surface with slope zero and the solution becomes flat in finite time. For (1) it is known but for (2) it was not yet proved rigorously. We are interested in estimating extinction time both for (1) and (2). To fix idea we impose for example a periodic boundary condition with zero average condition. In this talk for a given initial data  $u_0$  we derive an upper bound for the extinction time  $T^*(u_0)$ , the first time when the solution  $u$  vanishes identically zero. Such a time is important since it is the time that all pattern disappears.

Our goal is to derive a scale-independent estimate for  $T^*(u_0)$  from above both (1) and (2). The estimate for (1) is more or less known. The extinction time is estimated by

$$T^*(u_0) \leq S_n \|u_0\|_{L^n},$$

where  $S_n$  is the best Sobolev constant when the space dimension is  $n$ . For (2) the estimate is more involved and we are only successful for lower dimensions  $1 \leq n \leq 4$ . Even  $T^*(u_0) < \infty$  is a new result. Our estimate for  $n = 4$  is

$$T^*(u_0) \leq C \|u_0\|_{\dot{H}^{-1}}$$

while for  $1 \leq n \leq 3$

$$T^*(u_0) \leq C \|u_0\|_{\dot{W}^{-3,p}}^{1/\theta-1} \|u_0\|_{\dot{H}^{-1}}^{2-1/\theta}$$

with  $(4 - n + n/p)\theta = 2 - n/2 + n/p$ ,  $1/2 < \theta \leq 1$ ,  $1 \leq p < \infty$ . Here the norm  $\|\cdot\|_{\dot{W}^{-m,p}}$  denotes the dual norm of homogeneous  $W^{m,p'}$  space with  $1/p + 1/p' = 1$  so that  $\dot{H}^{-1} = \dot{W}^{-1,2}$ . The constant  $C$  is of course independent of  $u_0$  and moreover, it is dilation invariant. The constant  $C$  depends only on  $\theta$  (and blows up as  $\theta \downarrow 1/2$ ). The exponents  $\theta$  and  $p$  are chosen so that the estimate is invariant under all scaling transformations which makes the equation (2) invariant.

A key observation is a new interpolation inequality

$$\|u\|_{\dot{H}^{-1}} \leq C_* \|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}^{1-\theta} \left( \int |\nabla u| \right)^\theta$$

and the growth estimate of the solution

$$\frac{d}{dt} \|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}} \leq (\text{volume of periodic cell})^{1/p}$$

together with an energy estimate

$$\frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^{-1}}^2 = - \int |\nabla u|,$$

where  $\int |\nabla u|$  denotes the total variation of the measure  $\nabla u$ .

This is a joint work with Robert V. Kohn of Courant Institute.

# NUMERICAL METHODS FOR SMOOTH AND CRYSTALLINE CURVATURE FLOW

**Yen-Hsi Richard Tsai**

Abstract.

We present two numerical methods for planar anisotropic mean curvature flow. The methods are based on the variational approach of Almgren, Taylor and Wang, and Chambolle. Our approach uses the Split-Bregman method for total variation minimization. In the crystalline anisotropy case, we derive an algorithm for a corresponding crystalline shrinkage (or soft thresholding) problem. In the smooth anisotropy case, we show that the Split-Bregman method yields an algorithm related to the inverse scale space flow of Burger, et al.

This is a joint work with A. Oberman, S. Osheer, and R. Takei.

# Phase transition models based on linear-growth interfacial energies

Shirakawa, Ken (Kobe Univ., Japan)

## 1 Introduction.

Let  $\Omega \subset \mathbb{R}^2$  be a two-dimensional bounded domain with a smooth boundary  $\Gamma := \partial\Omega$ , and let  $Q := (0, +\infty) \times \Omega$  be the product space of the time-interval  $(0, +\infty)$  and the spatial domain  $\Omega$ . Also, for any open set  $D \subset \Omega$ , let  $D^{\text{ex}}$  be the external part  $\Omega \setminus \overline{D}$  of  $D$ .

In this talk, we take an origin-symmetric compact and convex set  $W \subset \mathbb{R}^2$ , to consider a coupled system of a PDE and a constrained total variation flow, denoted by  $(S)_W$ .

**System  $(S)_W$ .** Find a pair  $[\theta, w]$  of functions  $\theta \in W_{\text{loc}}^{1,2}([0, +\infty); H^2(\Omega) \cap H_0^1(\Omega))$  and  $w \in W_{\text{loc}}^{1,2}([0, +\infty); L^2(\Omega)) \cap L^\infty(0, +\infty; BV(\Omega))$ , such that:

$$\begin{cases} (\theta + w)_t - \Delta(\theta + \mu\theta_t) = 0, & \text{in } Q, \\ \text{subject to the initial-boundary conditions;} \end{cases} \quad (1)$$

$$\begin{cases} w_t - \kappa \operatorname{div}(\partial f_W^\circ(Dw)) + \partial I_{[-1,1]}(w) \ni w + \theta, & \text{in } Q, \\ \text{subject to the initial-boundary conditions.} \end{cases} \quad (2)$$

System  $(S)_W$  is a mathematical model to represent the dynamics of a solid-liquid phase transition. In the context,  $\theta = \theta(t, x)$  is the relative temperature, assuming the critical temperature to be 0, and  $w = w(t, x)$  is the order parameter that indicates the physical state (phase) of material by the value on  $[-1, 1]$ .

The PDE as in (1) is the heat equation, including an additional relaxation term  $\mu\theta_t$  with a small and positive constant  $\mu$ , and this equation is treated under the homogeneous Dirichlet boundary condition.

The evolution inclusion as in (2) is the kinetic equation of the phase dynamics, and the boundary condition is provided in the form of the homogeneous Neumann type. This inclusion is derived as a  $L^2$ -gradient flow of a modified version of the free-energy, proposed by Visintin [16, Chapter VI]:

$$w \in L^2(\Omega) \mapsto \mathcal{F}_W(w; \theta) := \kappa \int_{\Omega} f_W^\circ(Dw) + \int_{\Omega} \left\{ I_{[-1,1]}(w) - \frac{1}{2} w^2 - \theta w \right\} dx, \quad (3)$$

prescribed with a function (temperature)  $\theta \in L^2(\Omega)$ .

Here,  $\kappa$  is a positive and small constant.  $f_W^\circ$  as in (3) is the dual norm (polar function) of a two-dimensional norm  $f_W \in C(\mathbb{R}^2)$ , having the compact convex set  $W$  as its closed unit ball, and  $\partial f_W^\circ$  as in (2) denotes the subdifferential of the norm  $f_W^\circ$ . Besides, the notation  $\int_{\Omega} f_W^\circ(Dw)$  denotes the so-called anisotropic total variation with respect to  $f_W$ , that is defined as follows:

$$w \in L^1(\Omega) \mapsto \int_{\Omega} f_W^\circ(Dw) := \sup \left\{ \int_{\Omega} w \operatorname{div} \varphi \, dx \mid \varphi \in C_c^1(\Omega; \mathbb{R}^2), \right. \\ \left. f_W(\varphi) \leq 1 \text{ on } \Omega \right\}. \quad (4)$$

In view of this, the inclusion (2) can be regarded as a generalized version of the total variation flow. Then, the compact convex set  $W$  is called ‘‘Wulff shape’’, and its shape

is supposed to correspond to the structural unit of crystal. Additionally,  $I_{[-1,1]}$  as in (3) is the indicator function on the compact interval  $[-1, 1]$ , and  $\partial I_{[-1,1]}$  as in (2) is its subdifferential.

In (3), the indicator function  $I_{[-1,1]}$  is built in to constrain the range of the parameter  $w$  onto the supposed one  $[-1, 1]$ . However, this indicator function also makes the integrand:

$$\omega \in \mathbb{R} \mapsto I_{[-1,1]}(\omega) - \frac{1}{2}\omega^2 - \vartheta\omega \quad (\vartheta \in \mathbb{R});$$

be the so-called double-well type function that characterizes the phase bi-stability in the observing solid-liquid phase transition.

As a mathematical model, the system  $(S)_W$  is described in a simplified form, and hence it is not so difficult to check the basic properties, such as the well-posedness and the large-time behavior. In fact, referring to [11], it will be seen that the  $\omega$ -limit points for the orbits of  $(S)_W$  are expressed as a pair  $[0, w_\infty]$  of the constant equilibrium temperature 0 (the critical value), and the solution  $w_\infty$  of the following inclusion, denoted by  $(S_\infty)_W$ .

**Inclusion  $(S_\infty)_W$ .** Find a function  $w_\infty \in BV(\Omega) \cap L^\infty(\Omega)$ , such that:

$$\begin{cases} -\kappa \operatorname{div}(\partial f_W^\circ(Dw_\infty)) + \partial I_{[-1,1]}(w_\infty) \ni w_\infty, & \text{in } \Omega, \\ \text{subject to the boundary condition (inherited from (2)).} \end{cases}$$

In this sense, the inclusion  $(S_\infty)_W$  can be said as the steady-state problem for  $(S)_W$ , and each pair  $[0, w_\infty]$  of the  $\omega$ -limit point can be said as the steady-state solution of  $(S)_W$ .

In this talk, we will focus on special steady-state solutions, to see some geometric association between represented interfaces and Wulff shapes. To this end, the Wulff shape  $W \subset \mathbb{R}^2$  will be supposed to belong to one of the following two cases.

(Case 0) (Isotropic case) the case when  $W = \mathbb{D}^2 := \overline{\operatorname{conv}}(\mathbb{S}^1)$ .

(Case 1) (Anisotropic case of crystalline type) the case when:

$$W \in \mathcal{P} := \left\{ P \subset \mathbb{R}^2 \mid \begin{array}{l} P \text{ is a origin-symmetric compact convex polygon,} \\ \text{such that } \partial P \text{ is circumscribed to } \mathbb{S}^1 \end{array} \right\}.$$

On that basis, let us set the following three items, as the discussion points in this talk:

- (a) to see the geometric structure of the interfacial patterns in steady-state (steady-state patterns), in (Case 0) and (Case 1);
- (b) to study the stability for the steady-state patterns, in (Case 0) and (Case 1);
- (c) to see some continuous dependence of stable steady-state patterns with respect to Wulff shapes, when (Case 0) is regarded as a limiting situation of (Case 1).

## 2 Main Theorems

In this talk, four theorems are presented as the main conclusions, and they are respectively stated as follows.

**Main Theorem 1.** (Structural observation in (Case 0), cf. [12, 15]) Let us denote by  $\mathcal{X}_0$  the solution class of the steady-state problem  $(S_\infty)_{\mathbb{D}^2}$  in (Case 0). Then, the class  $\mathcal{S}(\mathbb{D}^2)$ , defined blow, is a subclass of  $\mathcal{X}_0$ .

$$\mathcal{S}(\mathbb{D}^2) := \left\{ w_D := \chi_D - \chi_{D^{\operatorname{ex}}} \mid \begin{array}{l} D \subset \Omega \text{ is a domain satisfying the marginated} \\ \text{conditions, labeled as (A1)}_0\text{-(A4)}_0 \text{ (see Fig. 1)} \\ \text{to get the general idea).} \end{array} \right\}.$$

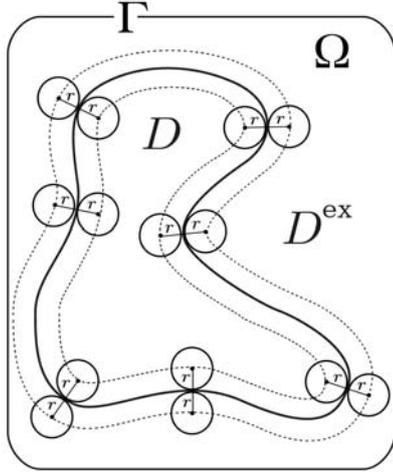


Fig. 1

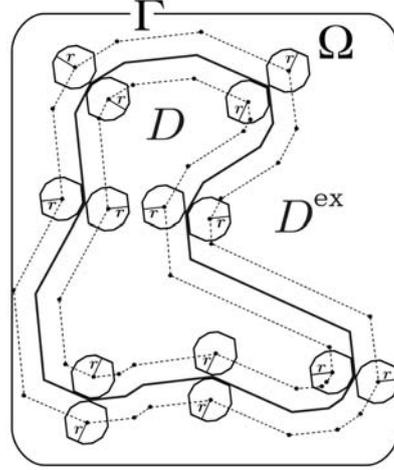


Fig. 2

(A1)<sub>0</sub>  $\Gamma_D := \partial D \cap \Omega$  is a Jordan curve.

(A2)<sub>0</sub>  $\exists r > 2\kappa$ , s.t.

$$D = \bigcup_{x \in D, W_0(x; r) \subset D} W_0(x; r) \quad \text{and} \quad D^{\text{ex}} = \bigcup_{x \in D^{\text{ex}}, W_0(x; r) \subset D^{\text{ex}}} W_0(x; r), \quad (5)$$

where  $W_0(y; \rho)$  denotes the interior of  $(y + \rho\mathbb{D}^2) \cap \Omega$ ,  $\forall y \in \mathbb{R}^2$ ,  $\forall \rho > 0$ .

(A3)<sub>0</sub> The tubular domain  $\left\{ x \in \Omega \mid \inf_{y \in \Gamma_D} |y - x| < r \right\}$  is  $C^2$ -diffeomorphic with the rectangle  $[0, 1] \times (-1, 1)$ .

(A4)<sub>0</sub>  $\Gamma_D$  has at most a finite number of inflection points.

**Main Theorem 2.** (Structural observation in (Case 1), cf. [13, 14]) Let us fix any  $P \in \mathcal{P}$ , and let us denote by  $\mathcal{X}_P$  the solution class of the steady-state problem  $(S_\infty)_P$  in (Case 1) when  $W = P$ . Then, the class  $\mathcal{S}(P)$ , defined below, is a subclass of  $\mathcal{X}_P$ .

$$\mathcal{S}(P) := \left\{ w_D := \chi_D - \chi_{D^{\text{ex}}} \mid \begin{array}{l} D \subset \Omega \text{ is a domain satisfying the marginated} \\ \text{conditions, labeled as } (A1)_P\text{-}(A3)_P \text{ (see} \\ \text{Fig. 2 to get the general idea).} \end{array} \right\}.$$

(A1)<sub>P</sub>  $\Gamma_D := \partial D \cap \Omega$  is a polygonal (piecewise linear) Jordan curve, such that its any edge (the part of segment) is parallel to one of those of the Wulff shape  $P$ .

(A2)<sub>P</sub>  $\exists r > 2\kappa$ , s.t.  $D = \bigcup_{x \in D, W_P(x; r) \subset D} W_P(x; r)$  and  $D^{\text{ex}} = \bigcup_{x \in D^{\text{ex}}, W_P(x; r) \subset D^{\text{ex}}} W_P(x; r)$ ,

where  $W_P(y; \rho)$  denotes the interior of  $(y + \rho P) \cap \Omega$ ,  $\forall y \in \mathbb{R}^2$ ,  $\forall \rho > 0$ .

(A3)<sub>P</sub>  $0 < \forall \rho < r$ , two compact sets:

$$\left\{ x \in D \mid \inf_{y \in \Gamma_D} f_P(y - x) = \rho \right\} \quad \text{and} \quad \left\{ x \in D^{\text{ex}} \mid \inf_{y \in \Gamma_D} f_P(y - x) = \rho \right\};$$

are both Jordan curves, included in  $\Omega$ .

**Main Theorem 3.** (Stability analysis, cf. [14]) Let us fix any Wulff shape  $W$  belonging to one of (Case 0)-(Case 1), and let us take any  $w_D \in \mathcal{S}(W)$  with the characteristic domain  $D = w_D^{-1}(1)$ . Here, let us set:

$$\Gamma_D(\rho)_W := \left\{ x \in \Omega \mid \inf_{y \in \Gamma_D} f_W(y - x) \leq \rho \right\}, \quad 0 < \forall \rho < 2\kappa.$$

Also, for the steady-state solution  $[0, w_D]$ , let us set:

$$U_W \left( \begin{bmatrix} 0 \\ w_D \end{bmatrix}; \delta, \rho \right) := \left\{ \begin{bmatrix} \tilde{\theta} \\ \tilde{w} \end{bmatrix} \in \begin{array}{c} H^2(\Omega) \cap H_0^1(\Omega) \\ \times \\ BV(\Omega) \cap L^\infty(\Omega) \end{array} \left| \begin{array}{l} |\tilde{\theta}|_{H^2(\Omega)} \leq 1, \quad |\tilde{\theta}|_{H_0^1(\Omega)} \leq \delta, \\ |\tilde{w} - w_D|_{L^\infty(\Omega \setminus \Gamma_D(\rho)_W)} \leq \delta, \\ \int_{\Omega} f_W^\circ(D\tilde{w}) \leq \int_{\Omega} f_W^\circ(Dw_D) + \delta \end{array} \right. \right\};$$

for any  $\delta, \rho > 0$ . Then, we find positive and small constants  $\delta_*, \rho_*$ , which characterize the stability of the steady-state solution  $[0, w_D]$ , as follows.

(\*) If  $0 < \delta < \delta_*$ ,  $0 < \rho < \rho_*$ , and if a solution  $[\theta, w]$  of the system  $(S)_W$  satisfies  $\begin{bmatrix} \theta(0) \\ w(0) \end{bmatrix} \in U_W \left( \begin{bmatrix} 0 \\ w_D \end{bmatrix}; \delta, \rho \right)$ , then there exists a finite time  $t_*(\delta, \rho)$ , depending on  $\delta$  and  $\rho$ , such that:

$$w(t) = w_D \text{ a.e. in } \Omega \setminus \Gamma_D(\rho)_W, \text{ for any } t \geq t_*(\delta, \rho).$$

Namely, the oscillation, given in the range of  $U_W \left( \begin{bmatrix} 0 \\ w_D \end{bmatrix}; \delta, \rho \right)$ , is recovered at a finite time  $t_*(\delta, \rho)$ , except on the  $\rho$ -neighborhood  $\Gamma_D(\rho)_W$  of the interface  $\Gamma_D$ .

**Main Theorem 4.** (Continuous dependence from (Case 1) to (Case 0)) Let us define:

$$\omega\text{-}\mathcal{S}(\mathcal{P}) := \left\{ \bar{w} \in BV(\Omega) \left| \begin{array}{l} \exists \{P_n\} \subset \mathcal{P}, \quad \exists \{\bar{w}_n \mid \bar{w}_n \in \mathcal{S}_{P_n}, n \in \mathbb{N}\}, \quad \text{s.t.} \\ \bullet \text{ dist}_{\mathbb{R}^2}(\partial P_n, \mathbb{S}^1) \rightarrow 0 \\ \bullet \bar{w}_n \rightarrow \bar{w} \text{ in } L^2(\Omega) \end{array} \right. \right\} \text{ as } n \rightarrow +\infty.$$

where  $\text{dist}_{\mathbb{R}^2}(\cdot, \cdot)$  is the Hausdorff distance between subsets in  $\mathbb{R}^2$ . Then, the following three statements hold.

(I) (Upper-bound)  $\omega\text{-}\mathcal{S}(\mathcal{P}) \subset \mathcal{X}_0$ , and furthermore:

$$\omega\text{-}\mathcal{S}(\mathcal{P}) \subset \mathcal{S}^* := \left\{ w_D := \chi_D - \chi_{D^{\text{ex}}} \left| \begin{array}{l} D \subset \Omega \text{ is a domain which satisfies the} \\ \text{condition (A1)}_0, \text{ and satisfies (5), as in} \\ \text{(A2)}_0, \text{ for some } r \geq 2\kappa. \end{array} \right. \right\}.$$

(II) (Lower bound)

$$\omega\text{-}\mathcal{S}(\mathcal{P}) \supset \mathcal{S}_* := \left\{ w_D := \chi_D - \chi_{D^{\text{ex}}} \left| \begin{array}{l} D \subset \Omega \text{ is a domain, which satisfies just} \\ \text{two conditions (A1)}_0 \text{ and (A2)}_0 \end{array} \right. \right\}.$$

Namely,  $\mathcal{S}_*$  is also a subclass of  $\mathcal{X}_0$ , which is strictly wider than  $\mathcal{S}(\mathbb{D}^2)$ , as in Main Theorem 1.

(III) (Stability) For any  $w_D \in \mathcal{S}_*$ , the steady-state solution  $[0, w_D]$  shows the stability, just mentioned in (\*) of Main Theorem 3. To conclude, the conditions  $(A3)_0$ - $(A4)_0$  are eventually unnecessary for the stability analysis in the isotropic case.

The above Main Theorems will be proved, with helps from a lot of mathematical theories, obtained by Amar-Bellettini [1], Ambrosio-Fusco-Pallara [2], Andreu-Caselles-Mazón [3], Attouch [4], Bellettini-Caselles-Chambolle-Novaga [5], Caselles-Chambolle-Moll-Novaga [6], Giga-Giga [7], Kenmochi [8], Moll [9], Mosco [10], and so on.

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# The Dirichlet problem for a singular elliptic equation arising in the level set formulation of the inverse mean curvature flow

José M. Mazón\*

In this lecture we consider the Dirichlet problem associated with a nonlinear singular elliptic equation arising in the level set formulation of the inverse mean curvature flow; namely,

$$-\operatorname{div} \left( \frac{Du}{|Du|} \right) + |Du| = f.$$

We introduce a suitable concept of weak solution, for which we prove existence and uniqueness of the homogeneous Dirichlet problem in a bounded open set of  $\mathbb{R}^N$ , in the case  $0 \leq f \in L^q(\Omega)$ ,  $q > N$ . Moreover, examples of explicit solutions are shown.

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