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A factorization theorem for unfoldings of analytic
functions

Tatsuo Suwa

Abstract

Let $\tilde{f}$ and g be holomorphic function germs at 0 in $C^n \times C^q = \{(x,s)\}$. If $d_x g \wedge d_x \tilde{f} = 0$ and if $f(x) = \tilde{f}(x,0)$ is not a power or a unit, then there exists a germ λ at 0 in $C \times C^q$ such that $g(x,s) = \lambda(\tilde{f}(x,s), s)$. The result has the implication that the notion of an RL-morphism in the unfolding theory of foliation germs generalizes that of a right-left morphism in the function germ case.

The notion of an RL-morphism in the unfolding theory of foliation singularities was introduced in [5] to describe the determinacy results and in [6] the versality theorem for these morphisms is proved. This note, which should be considered as an appendix to [5] or [6], contains a factorization theorem implying that an RL-morphism is a generalization of a right-left morphism in the unfolding theory of function germs. It depends on the Mattei-Moussu factorization theorem ([1]) and is a generalization of a result of Moussu [2].

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A codim 1 foliation germ at 0 in $\mathbb{C}^n$ is a module $F = (\omega)$ over the ring of holomorphic function germs generated by a germ of an integrable 1-form ω (see Section 2). An unfolding of F with parameter space $\mathbb{C}^m = \{t\}$ is a codim 1 foliation germ $\tilde{F} = (\tilde{\omega})$ at 0 in $\mathbb{C}^n \times \mathbb{C}^m$ with a generator $\tilde{\omega}$ whose restriction to $\mathbb{C}^n \times \{0\}$ is ω . We let F_t be the foliation germ generated by the restriction ω_t of $\tilde{\omega}$ to $\mathbb{C}^n \times \{t\}$. Let $\tilde{F}'$ be another unfolding of F with parameter space $\mathbb{C}^q = \{s\}$. A morphism from $\tilde{F}'$ to $\tilde{F}$ is a holomorphic map germ $\Phi : (\mathbb{C}^n \times \mathbb{C}^q, 0) \rightarrow (\mathbb{C}^n \times \mathbb{C}^m, 0)$ such that (a) $\Phi(x, s) = (\phi(x, s), \psi(s))$ for some holomorphic map germs $\phi : (\mathbb{C}^n \times \mathbb{C}^q, 0) \rightarrow (\mathbb{C}^n, 0)$ and $\psi : (\mathbb{C}^q, 0) \rightarrow (\mathbb{C}^m, 0)$, (b) $\phi(x, 0) = x$ and (c) the pull back $\Phi^* \tilde{\omega}$ of $\tilde{\omega}$ by Φ generates $\tilde{F}'$. Thus, if we set $\phi_s(x) = \phi(x, s)$, we may think of (ϕ_s) as a family of local coordinate changes of $(\mathbb{C}^n, 0)$. For an RL-morphism, in place of (c), we only require that $\phi_s^* \omega \psi(s)$ generates F'_s for each s (see (2.1) Definition). Our previous result shows that if F has a generator of the form df for some holomorphic function germ f (strong first integral for F), then every unfolding of F admits a generator of the form $d\tilde{f}$ with $\tilde{f}$ an unfolding of f . In the unfolding theory of function germs, there are notions of a right morphism and a right-left morphism. The former involves coordinate changes in the source space $(\mathbb{C}^n, 0)$, whereas the latter involves coordinate changes in the target space $\mathbb{C}$ as well. It is not difficult to see that our morphism generalizes a right morphism in the sense that when F admits a strong first integral f , then it becomes a (strict) right morphism in the unfolding theory of f . For a foliation

without first integrals, it may not seem relevant to talk about right-left morphisms. However, as stated above, our factorization theorem shows that an RL-morphism is a natural generalization of a right-left morphism, since when $F = (df)$, an RL-morphism is exactly a right-left morphism in the unfolding theory of f . We also note that RL-morphisms are closely related to integrating factors of the foliation ((2.2) Remark 2).

1. The factorization theorem.

We denote by $\mathcal{O}_n$ the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n = \{(x_1, \dots, x_n)\}$. A germ f in $\mathcal{O}_n$ is said to be a power if $f = f_0^m$ for some positive integer m and a non-unit f_0 in $\mathcal{O}_n$. If we denote the critical set of f by $C(f)$, then $\text{codim } C(f) \geq 2$ implies that f is not a power. We quote the following factorization theorem of Mattei and Moussu.

(1.1) Theorem ([1]). Let f be a germ in $\mathcal{O}_n$ which is not a power or a unit. If g is a germ in $\mathcal{O}_n$ with $dg \wedge df = 0$, then there exists a germ λ in $\mathcal{O}_1$ such that $g = \lambda \cdot f$.

The theorem is proved using the reduction theory of singularities of holomorphic 1-forms due to Seidenberg and Van den Essen. The proof is rather simple if we assume $\text{codim } C(f) \geq 2$ (see Moussu-Tougeron[3]). If $\tilde{f}$ is a germ in $\mathcal{O}_{n+q}$, we may think of $\tilde{f}$ as an unfolding of $f(x) = \tilde{f}(x, 0)$ with parameter space $\mathbb{C}^q = \{(s_1, \dots, s_q)\}$. We denote by d_x the exterior derivation with respect to x ; $d_x \tilde{f} = \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial x_i}(x, s) dx_i$.

(1.2) Theorem. Let $\tilde{f}$ be a germ in $\mathcal{O}_{n+q}$ such that $f(x) = \tilde{f}(x, 0)$ is not a power or a unit in $\mathcal{O}_n$. If g is a germ in $\mathcal{O}_{n+q}$ with $d_x g \wedge d_x \tilde{f} = 0$, then there exists a germ λ in $\mathcal{O}_{1+q}$ such that $g(x, s) = \lambda(\tilde{f}(x, s), s)$.

Proof. First we show the existence of λ as a formal power series in s . Thus we express λ as

$$\lambda(y, s) = \sum_{|v| \geq 0} \lambda^{(v)}(y) s^v, \quad \lambda^{(v)} \in \mathcal{O}_1,$$

where v denote an l -tuple $(v_1, \dots, v_l)$ of non-negative integers, $|v| = v_1 + \dots + v_l$ and $s^v = s_1^{v_1} \dots s_l^{v_l}$. In general, if $\sigma = \sum_{|v| \geq 0} \sigma^{(v)} s^v$ is a series in s with $\sigma^{(v)} \in \mathcal{O}_n^r$ for some r , we set

$$[\sigma]_p = \sum_{|v|=p} \sigma^{(v)} s^v \quad \text{and} \quad \sigma|_p = \sum_{|v|=0}^p \sigma^{(v)} s^v$$

for a non-negative integer p .

We look for λ satisfying the congruence

$$(1.3)_p \quad g(x, s) \underset{p}{=} \lambda|_p^{\sim}(f(x, s), s)$$

for $p \geq 0$, where $\underset{p}{=}$ denotes the equality mod s^v , $|v| = p + 1$.

First, $(1.3)_0$ is equivalent to

$$g(x, 0) = \lambda^{(0)}(f(x)).$$

From the condition of the theorem, we have $d(g(x, 0)) \wedge df = 0$.

Hence by (1.1), there exists a germ $\lambda^{(0)}$ in $\mathcal{O}_1$ satisfying the above. Now we suppose that we have $\lambda|_p^{\sim}$ satisfying $(1.3)_p$ and look for $[\lambda]_{p+1}$. The congruence $(1.3)_{p+1}$ reads

$$g(x, s) \underset{p+1}{=} \sum_{|v|=p+1} \lambda^{(v)}(f(x)) s^v + \lambda|_p^{\sim}(f(x, s), s).$$

Hence, for our purpose, it suffices to show that

$$(1.4) \quad d_x [g(x,s) - \lambda^{[p]}(\tilde{f}(x,s), s)]_{p+1} \wedge df = 0 .$$

By (1.3)_p , we have (1.4) if we show that

$$d_x (g(x,s) - \lambda^{[p]}(\tilde{f}(x,s), s)) \wedge d_x \tilde{f} = 0 .$$

But this follows from the condition of the theorem and

$$d_x \lambda^{[p]}(\tilde{f}(x,s), s) = \frac{\partial \lambda^{[p]}(\tilde{f}(x,s), s)}{\partial y} d_x \tilde{f} .$$

Thus we have a formal power series

$$\lambda(y,s) = \sum_{|v| \geq 0} \lambda^{(v)}(y) s^v , \quad \lambda^{(v)} \in \mathcal{O}_1$$

in s such that $g(x,s) = \lambda(\tilde{f}(x,s), s)$ as power series in (x,s) .
Since $\tilde{f}$ and g are both convergent, λ must be also convergent.

(1.5) Remarks.

1. The germ λ is determined uniquely by g (and $\tilde{f}$) . If we assume that $g(x,0) = f(x)$, then $\lambda(y,0) = y$.
2. The above theorem generalizes Corollaire 1 in [2] Ch.II,1 in the case $X = H$.

2. Some types of morphisms in the unfolding theory of foliation germs.

We denote by Ω_n the $\mathcal{O}_n$ -modules of germs of holomorphic 1-forms at 0 in $\mathbb{C}^n$. We recall (Cf.[4],[5]) that a codim 1 foliation germ at 0 in $\mathbb{C}^n$ is a rank 1 free sub- $\mathcal{O}_n$ -module $F = (\omega)$ of Ω_n with a generator satisfying the integrability condition $d\omega \wedge \omega = 0$. The singular set $S(F)$ of F is defined to be the singular set $\{x | \omega(x)=0\}$ of ω . We always assume that $\text{codim } S(F) \geq 2$. An unfolding of $F = (\omega)$ is a codim 1 foliation germ $\mathcal{F} = (\tilde{\omega})$ at 0 in $\mathbb{C}^n \times \mathbb{C}^m = \{(x,t)\}$, for some m , with a generator $\tilde{\omega}$ satisfying $\iota^* \tilde{\omega} = \omega$, where ι denotes the embedding of $\mathbb{C}^n$ into $\mathbb{C}^n \times \mathbb{C}^m$ given by $\iota(x) = (x,0)$. We call $\mathbb{C}^m$ the parameter space of $\mathcal{F}$. We recall the following definition ([5](2.1),[6](1.1)).

(2.1) Definition. Let $\mathcal{F}$ and $\mathcal{F}'$ be two unfoldings of F with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^q = \{(s_1, \dots, s_q)\}$, respectively.

(I) An RL-morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a pair (Φ, ψ) satisfying the following conditions :

(a) Φ and ψ are holomorphic map germs making the diagram

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^q, 0) & \xrightarrow{\Phi} & (\mathbb{C}^n \times \mathbb{C}^m, 0) \\ \downarrow & & \downarrow \\ (\mathbb{C}^q, 0) & \xrightarrow{\psi} & (\mathbb{C}^m, 0) \end{array}$$

commutative, where the vertical maps are the projections.

(b) $\Phi(x,0) = (x,0)$.

(c) For any generator $\tilde{\omega}$ of $\tilde{F}$, there is a germ $\alpha = (\alpha_1, \dots, \alpha_q)$ in O_{n+q}^q such that the germ

$$\Phi^* \tilde{\omega} + \sum_{k=1}^q \alpha_k ds_k$$

generates $\tilde{F}'$.

(II) A morphism from $\tilde{F}'$ to $\tilde{F}$ is an RL-morphism such that for any generator $\tilde{\omega}$ of $\tilde{F}$, we may choose $\alpha = 0$ in (c) .

(2.2) Remarks.

1. In the both cases, we may replace "any" by "some" .
2. From the integrability condition we see that, for α in (c) , each $\alpha_k(x,0)$ is an integrating factor of $\omega = i^* \tilde{\omega}$, i.e., $\alpha_k(x,0)d\omega = d(\alpha_k(x,0)) \wedge \omega$.
3. We have a "versality theorem" for each type of morphisms ([4],[6]) .

If a germ $\tilde{f}$ in O_{n+m} is an unfolding of f , i.e., if $i^* \tilde{f} = f$, then $\tilde{F} = (d\tilde{f})$ is an unfolding of $F = (df)$ with parameter space $\mathbb{C}^m$ and conversely, any unfolding of $F = (df)$ has a generator of the form $d\tilde{f}$ with $\tilde{f}$ an unfolding of f ([4] p.47) . We recall the following definition (cf.[7] Definition 3.2) .

(2.4) Definition. Let $\tilde{f}$ and g be two unfoldings of f with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^q = \{(s_1, \dots, s_q)\}$, respectively.

(I) A right-left morphism from g to $\tilde{f}$ is a pair (Φ, ψ) satisfying (I)(a) and (b) in (2.1) Definition and

$$(c) \quad g(x,s) = \lambda(\Phi^* \tilde{f}(x,s), s)$$

for some λ in $\mathcal{O}_{1+Q}$ with $\lambda(y,0) = y$.

(II) A strict right morphism from g to $\tilde{f}$ is a right-left morphism such that $\lambda(y,s) = y$ in (c).

The following is a direct consequence of (1.2) Theorem.

(2.4) Proposition. Let $\tilde{f}$ and g be unfoldings of f . A pair (Φ, ψ) is, respectively, a right-left morphism or a strict right morphism from g to $\tilde{f}$ if and only if it is an RL-morphism or a morphism from $\tilde{f}' = (dg)$ to $\tilde{f} = (d\tilde{f})$.

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