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Author(s)	Nakazi, Takahiko
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Weighted Norm Inequalities And Uniform Algebras

Takahiko Nakazi

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Weighted Norm Inequalities
And
Uniform Algebras

By

Takahiko Nakazi*

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan

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Abstract. Generalizations of the classical conjugation operator can be defined on a general uniform algebra. In this paper the L^2 weighted problems for the conjugation operators are considered and it is shown that the weights have forms similar to the classical case. The results in this paper have applications to concrete uniform algebras, for example, a polydisc algebra and a uniform algebra which consists of rational functions.

§1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Fix a nonzero complex homomorphism τ on A and let m be the representing measure for τ on X. Put $A_0 = \{f \in A ; \tau(f) = 0\}$ and $C_0 = \{f \in C(X) ; \int_X f dm = 0\}$. A uniformly closed subspace is called A-invariant if it is invariant under multiplications by the functions in A. Suppose I is a closed invariant subspace in C_0. For example, A_0 is such a subspace. It is clear that there exists a largest subspace K_0 contained in such a closed invariant subspaces, and, in fact, $K_0 = \{f \in C_0 ; \int_X fg dm = 0 \text{ for all } g \in A\}$.

The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by A is defined to be the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak*-closure of A in $L^\infty = L^\infty(m)$ when

$p = \infty$. Let v be a real function in L^∞ ; $\tilde{v}$ denotes the conjugate function, that is, $v + i\tilde{v}$ is in H^1 and $\int_X \tilde{v} dm = 0$. Of course, it is possible that v need not have the conjugate function $\tilde{v}$.

In this paper, we are interested in the nonnegative weight functions w for which there is a positive constant C such that

$$(1) \qquad \int_X |f|^2 w dm \leq C \int_X |f + \bar{g}|^2 w dm$$

for all f in A and all g in I . In the case A is the disc algebra and m is the normalized Lebesgue measure, such weight functions w are well known when $I = A_0$ (cf. [4],[6]) . It is difficult to give a characterization analogous to the disc algebra setting even if in the case of an annulus algebra. However in previous papers [7, Section 3], [8, Section 4] and [9, Section 7], we determined weight functions w when w satisfies a stronger weighted norm inequality than (1), if $I = K_0$, for a general uniform algebra .

That is,

(2) there exists a constant ρ in $[0,1)$ and a function k in $K_0^\perp \cap L^1$ such that $|w - k| \leq \rho w$.

When A is the disc algebra, the Helson-Szegö theorem, which was proved in [1, Proposition 2], shows that w satisfies (2) if and only if

$$w = e^{u+\tilde{v}}$$

where u and v are in L^∞ and $\| v \|_\infty < \pi/2$. The main result in this paper shows this condition persists for a polydisc algebra and a uniform algebra which consists of rational functions.

§2. Weighted norm inequalities

Let P^v be the orthogonal projection from L^2 to vH^2 where v is in $\mathcal{L}$, the set of all measurable positive functions on X which are both bounded and bounded away from zero. Fix I a closed invariant subspace in C_0. ρ_I^v denotes P^v restricted to $vA + v^{-1}\bar{I}$. The following theorem is known in the case when $I = A_0$ or $I = K_0$ [9, Theorem 14 and 14'] . For a general I we can show a lifting theorem on $A \times \bar{I}$ as in the proof of [9, Theorem 2]. Then the following theorem follows as in the proof of [9, Theorem 14]. However we will sketch of a direct proof, not using a lifting theorem. We need the following lemma which is probably known.

Lemma 1. Let M be a subspace of L^1. If for any h in M

$$\left|\int_X hwdm\right| \leq C \int_X |h|wdm ,$$

then there exists a function k in $M^{\perp} \cap L^1$ such that $|w - k| \leq Cw$.

The proof is due to the Hahn-Banach theorem, that is, $(L^1)^* = L^{\infty}$. For the proof, m may be any probability measure.

Theorem 1. Let w be a nonnegative function in L^1. Then, ρ_I^v is uniformly bounded in $L^2(w)$ with respect to v in $\mathcal{L}$ if and only if there exists a constant ρ in $[0,1)$ and a function k in $I^{\perp} \cap L^1$ such that

$$|w - k| \leq \rho w$$

Proof. It is easy to see that $\| P_I^v \| \leq C$ for all $v \in \mathcal{L}$ if and only if

$$|\int_X fgwdm|^2 \leq (1 - C^{-1}) \int_X |vf|^2 wdm \int_X |v^{-1}|^2 wdm$$

for all $f \in A$, all $g \in I$ and all $v \in \mathcal{L}$. Hence the proof of "if" in the theorem is easy to prove.

We shall prove the "only if" part

Let $h \in I$. Put $E_n = \{x \in X ; 0 < |h(x)| < 1/n\}$, $F_0 = \{x \in X ; h(x) = 0\}$ and $F_n = \{x \in X ; |h(x)| \geq 1/n\}$. Define v_n by the formula

$$v_n(x) = \begin{cases} 1 & , x \in E_n \\ 1/n & , x \in F_0 \\ |h(x)|^{1/2} & , x \in F_n . \end{cases}$$

Then $v_n \in \mathcal{L}$ and

$$|\int_X hwdm|^2 \leq (1 - C^{-1}) \int_X |v_n|^2 wdm \int_X |v_n^{-1} h|^2 wdm$$

Since $\lim_n \int_X |v_n|^2 wdm = \int_X |h| wdm$ and

$\lim_n \int_X |v_n^{-1} h|^2 wdm \leq \int_X |h| wdm$, for any $h \in I$

$$|\int_X hwdm| \leq (1 - C^{-1}) \int_X |h| wdm$$

This proof is similar to that of the caluculation of a norm of a Hankel operator in [7, Theorem 1]. Now the lemma above implies the "only if" part.

Let N_τ denote the set of representing measures for τ on

X . If N_τ is finite dimensional and m is a core point of N_τ , in Theorem 1 we can use $\mathcal{E}$ instead of $\mathcal{L}$ where $\mathcal{E}$ is a finite subgroup of $\mathcal{L}$ (cf. [7, Proposition 5], [9, Theorem 14]). Moreover if m is a unique logmodular measure we can show that if ρ_I^v is bounded for each v in $\mathcal{E}$, then ρ_I^v is uniformly bounded with respect to $\mathcal{E}$ (cf.[7, Theorem 6], [9, Theorem 15]). If A is the disc algebra and m is the normalized Lebesgue measure, then $\mathcal{E}$ consists of nonzero positive constants and hence the Helson-Szegö theorem follows if $I = A_0$.

It is an interesting to point out that Theorem 1 is completely genral. However we wish to find the weight functions w for which $\rho_I = \rho_I^1$ is bounded (only when $v = 1$). For this purpose we introduce a constant γ_0 which was first used in [8, Section 1]. Denoting the coset of an f in $(L^\infty)^{-1}$ in $(L^\infty)^{-1}/(H^\infty)^{-1}$ by (f), define $\|(f)\| = \inf \{\| g \|_\infty \| g^{-1} \|_\infty ; g \in (f)\}$ and $\gamma_0 = \sup \{\|(f)\| ; (f) \in (L^\infty)^{-1}/(H^\infty)^{-1}\}$. If A is the disc algebra then $\gamma_0 = 1$ and if H^∞ is the algebra of bounded analytic functions on a multiply connected demain, then γ_0 is finite [8, Section 2]. The following theorem is known in the case of $I = A_0$ or $I = K_0$ (see [8, Corollary 1], [9, Theorem 16]). The proof is almost the same if we use Theorem 1.

Theorem 2. Let w be a nonnegative function in L^1. If ρ_I is bounded in $L^2(w)$ and if $\| \rho_I \| = C$ then there exists a function k in $I^{\perp} \cap L^1$ such that

$$|w - k| \leq \rho w$$

and $\rho = \gamma_0 \sqrt{C^2 - 1}/C$. Hence if $C < \gamma_0 / \sqrt{\gamma_0^2 - 1}$ then $\rho < 1$.

§3. Characterizations of weights

By Theorems 1 and 2 we are interested in the weights w which satisfy (2). $K_0^{\perp} \cap L^1$ coincides usually with H^1 and then (2) is the following.

(I) There exists a constant ρ in $[0,1)$ and a function k in H^1 such that $|w - k| \leq \rho w$.

We will consider the equivalence of (I) and the following property.

(II) $w = e^{u+\tilde{v}}$ where u and v are in L^∞ and $\| v \|_\infty < \pi/2$.

For many examles (I) is equivalent to (II) (see §4).

Lemma 2. Let w be a nonnegative function and k a function in L^1 . For a constant ρ in $[0,1)$, $|w - k| \leq \rho w$ if and only if

$$w = e^u |k|$$

where $|\arg k(x)| \leq \pi/2 - \varepsilon$ if $k(x) \neq 0$ and $\varepsilon > 0$, and u is in L^∞ .

Proof. (see [1, Proposition 2] for the disc algebra). Suppose $|w - k| \leq \rho w$ and put

$$n(x) = \begin{cases} (w(x) - k(x)/\rho w(x) & \text{if } w(x) > 0 \\ 1 & \text{if } w(x) = 0 , \end{cases}$$

then $|n(x)| \leq 1$, $k(x) = \rho w(x)(1/\rho - n(x))$ if $w(x) > 0$ and $k(x) = 0$ if $w(x) = 0$. Since $1/\rho > 1$, $\mathrm{Re}\, k(x) \geq 0$ and $|\arg k(x)| \leq \pi/2 - \varepsilon$ for some $\varepsilon > 0$ if $k(x) \neq 0$. Put

$$u(x) = \begin{cases} -\log\rho |1/\rho - n(x)| & \text{if } w(x) > 0 \\ 0 & \text{if } w(x) = 0 , \end{cases}$$

then $w = e^u|k|$ and this implies the part of "only if". Conversely suppose $w = e^u|k|$ and $|\arg k| \leq \pi/2 - \varepsilon$. We may assume $e^u \geq 1$. Hence if $w(x) > 0$, then $|1 - e^{-u}k(x)/|k(x)|| \leq \rho_2 < 1$ for some $\rho_2 \in [0,1)$ because $|1 - k(x)/|k(x)|| \leq \rho_1 < 1$. Thus the "if" part follows.

Let $L = L(m)$ be the set of measurable functions f, such that $\log^+|f|$ is integrable. We define a metric in L by setting

$$d(f, g) = \inf_{t>0} [t + m(\{x \in X ; |f(x) - g(x)| \geq t\})] + \int_X |\log^+|f| - \log^+|g||dm .$$

With the metric d, L becomes a complete metric space. The Hardy algebra $H = H(m)$ is the closure of A in $L(m)$. H contains each of H^p, $0 < p \leq \infty$. In many examples (see §4) $H \cap L^p = H^p$ and the following is valid:

(3) If f is a function in H^1 such that $\mathrm{Re} f \geq 0$ then $1/f$ belongs to H.

Lemma 3. Suppose H^1 has the property (3). If f is a function in H^1 and $\mathrm{Re} f \geq 0$, then $\mathrm{Log} f$ belongs to $H \cap L^1$ where $\mathrm{Log} z$ is the principal branch of the logarithm.

Proof. (see [2, Theorem 12] and [10, Proposition 5] for $N_\tau = \{m\}$) . By the property (3) $1/(f + \varepsilon) \in H \cap L^\infty$ and $\mathrm{Re}(1/f + \varepsilon) \geq 0$ for any $\varepsilon > 0$. Using a Taylor series expansion

$$\mathrm{Log}(1/f + \varepsilon) = \log|1/f + \varepsilon| + i \arg(1/f + \varepsilon)$$

belongs to $H \cap L^\infty$. Now, for every x , $\log(1/f(x) + \varepsilon) \to \mathrm{Log}1/f(x)$ as $\varepsilon \to 0$. Further $|\arg(1/f(x) + \varepsilon)| \leq \pi/2$ and as $\varepsilon \to 0$, $\log|1/f(x) + \varepsilon| \to \log|1/f(x)|$. $\log|1/f(x) + \varepsilon| \in L^1$ and by the property (3) $\log|1/f(x)| \in L^1$. Given $\varepsilon_0 > 0$, for $0 < \varepsilon < \varepsilon_0$, we have

$$|\log|1/f(x) + \varepsilon|| \leq |\log|1/f(x)|| + |\log|1/f(x) + \varepsilon_0|| \ .$$

Applying Lebesgue's dominated convergence theorem we see that $\mathrm{Log}(1/f + \varepsilon) \to \mathrm{Log}1/f$ in L^1 as $\varepsilon \to 0$ and hence $\mathrm{Log}1/f$ is in $H \cap L^1$.

Theorem 3. Suppose $H \cap L^1 = H^1$. Let w be a nonnegative function in L^1 , then the following are valid.

(i) If w satisfies (II) , then w satisfies (I).

(II) Suppose H^1 has the property (3). If w satisfies (I), then w satisfies (II).

Proof. (i) Since v in (II) is in L^∞ , $\tilde{v} - iv \in H^1$ and there exist $\tilde{v}_n - iv_n \in A$ such that $v_n \to v$ in L^1 and $\tilde{v}_n \to \tilde{v}$ a.e.. By Lemma 2.4 in [3, p123], $e^{\tilde{v}-iv} \in H \cap L^1$. Set $k =$

$e^{\tilde{v}-iv}$, then by the hypothesis $k \in H^1$, $w = e^u|k|$ and $|\arg k| \leq \pi/2 - \varepsilon$ for some $\varepsilon > 0$. Now Lemma 2 implies (I).

(ii) By Lemma 2 $w = e^u|k|$ where $|\arg k(x)| \leq \pi/2 - \varepsilon$ if $k(x) \neq 0$ and $\varepsilon > 0$, and u is in L^∞ . By Lemma 3 $\text{Log}k = \tilde{v} - iv \in H^1$, $v(x) = \arg k(x)$ and so $|k| = e^{\tilde{v}}$. This implies (II).

§4. The finite dimensional case

Suppose N_τ is finite dimensional n and m is a core point of N_τ. Let N^∞ be the real anihilator of A in L^∞_R and $N^\infty_C = N^\infty + iN^\infty$ (cf. [3,p109]). Suppose $N^\infty \subset C(X)$. Then $K_0 = A_0 + N^\infty_C$ and hence $K_0^\perp \cap L^1 = H^1$ by [3, p106].

Theorem 4. Let w be a nonnegative function in L^1. Then ρ^v_I is uniformly bounded in $L^2(w)$ with respect to v in $\mathcal{E} = \exp N^\infty$ if and only if

$$w = e^{u+\tilde{v}}$$

where u and v are in L^∞ and $\| v \|_\infty < \pi/2$.

Proof. By [3, V, Theorem 4.2] $H \cap L^1 = H^1$. By the argument of A. Devinatz in [2] H^1 has the property (3). For if $f \in H^1$ and $\mathrm{Re} f \geq 0$ then by [3, V, Lemma 2.4] $e^{tf} \in H \cap L^\infty = H^\infty$ for $t \leq 0$. By the proof of [2, Lemma 4], $1/(f + \varepsilon) \in H^\infty$ for any $\varepsilon > 0$. By the proof of [2, Theorem 12] we can show that $\log|f| \in L^1$ and

$$-\infty < \log\left|\int f dm\right| = \sup_{\mu \leftarrow S} \int \log | f | d\mu$$

where S is the set of logmodular measures in N_τ. By [3, V, Theorem 5.2] f^{-1} belongs to H. By Theorem 3 (I) and (II) are

equivalent. Now Theorem 2 and the remark above implies the theorem.

If dim $N_\tau = 0$, Theorem 4 is due to Y. hno [10], A. Devinatz [2], and I.I.Jr. Hirschman and R. Rochberg [5].

§5. Concrete examples which (I) and (II) are equivalent

In the disc algebra $K_0^{\perp} \cap L^1 = H^1$, and (I) and (II) are equivalent. We shall give examples in which these are valid.

(i) Let Y be a compact subset of the plane whose complement has a finite number of components and let $R(Y)$ be the uniform closure of the rational functions in $C(Y)$. Let A be the restriction of $R(Y)$ to its Shilov boundary X ; then $M_A = Y$. If $\tau \in M_A$ is in the interior of Y (we assume it is non empty) and m is harmonic measure, then m is a unique logmodular measure of N_τ and dim $N_\tau = n < \infty$ [3, p116]. Then $N^\infty \subset C(X)$. Theorem 4 applies to this situation.

(ii) Let A be a subalgebra of the disc algebra which contains the constants and which has finite codimension. If $\tau(f) = f(0)$ for f in A and m is the normalized Lebesgue measure on the unit circle T , then N_τ is finite dimensional, m is a core point of N_τ , $N^\infty \subset C(X)$. Theorem 4 applies to this situation.

(iii) The unit polydisc U^n and the torus T^n are cartesian products of n copies of the unit disc U and of the unit circle T , respectively. $A(U^n)$ is the class of all continuous complex functions on the closure $\overline{U}^n$ of U^n with holomorphic restrictions to U^n is holomorphic there. Let $A = A(U^n)|X$ and $X = T^n$. This is the so-called polydisc algebra. For simplicity we assume $n = 2$. Let m be the normalized Lebesgue measure ;

then m is a representing measure for τ on T^2 where $\tau(f) = f(0)$ and $0 \in U^2$. Then $K_0 = \{f \in C(T^2) ; \hat{f}(n,m) = 0$ for $(-n, -m)$ in $Z_+^2\}$ where Z_+^2 denotes the set of all pairs (n,m) of nonnegative integers n and m. $K_0^\perp \cap L^1 = H^1$ and H^1 has the property (3). For if f is a function in H^1 such that $\mathrm{Re} f \geq 0$ then by [11, Theorem 4.4.6] and [11, Theorem 4.4.9] f is outer, that is, $\log|f(0)| = \int_{T^2} \log|f| dm$. Then $1/f$ is analytic on U^n and $1/f$ belongs to H (see[11, Theorem 3.3.5])

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