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On a stochastic difference equation for the multi-dimensional weakly stationary process with discrete time

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On a stochastic difference equation for the multi-dimensional weakly stationary process with discrete time

Dedicated to Professor Mikio Sato
on his sixtieth birthday

By Yasunori Okabe*)

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§1. Introduction

The researches for the problem of linear prediction of a one-dimensional weakly stationary process X can be classified according to the difference whether the length of prediction interval is infinitely long (Kolmogorov [4], Wiener [28] and Dym-McKean [3]) or finitely limited (Krein [5] and Dym-McKean [2]). The method is to reformulate the above problem in terms of a real Hilbert space $L^2(\mathbb{R},\Delta(d\lambda))$ obtained by closing the linear hull of $\{e^{it\cdot};t\in\mathbb{R}\}$, where $\Delta(d\lambda)$ is the spectral measure of X. In particular, besides a traditional spectral theory, Krein has developed the so-called Krein's theory for the inverse spectral problem, and Dym and McKean, Jr. have reconstructed the theory of Hilbert spaces of entire functions developed by de Branges ([1]).

On the other hand, the author has studied the problem of linear prediction for X from the viewpoint of the theory of stochastic differential equations ([10],[12]~[27]). The bud of its spirit can be found in [11], where Sato's hyperfunctions play an importart role in the study of infinitely multiple Markovian property. It is important, not only from a point of view of statistical physics, but also from a probabilistic point of view, to derive a stochastic equation of motion describing the time evolution of X. In particular, the process X with reflection positivity has been investigated in detail with the project of clarifying a mathematical structure of the so-called fluctuation-dissipation theorem in statistical physics (Kubo [6] and Mori [9]). In the course of these investigations, it has been found that the time evolution of X can be described by two kinds of Langevin equations with a notable difference in character of

random forces ([18],[19]) : One is the first KMO-Langevin equation having a white noise as a random force and the other is the second KMO-Langevin equation where a colored noise named the Kubo noise is taken to be a random force. It is a key to obtain the structure theorem of the outer function of X. Since the Fourier transform of the outer function gives the canonical representation kernel for X, the author's studies are related to the case of long time prediction interval, except [16] treated by an innovation method.

Following the same spirit, Miyoshi ([7] and [8]) has derived and then characterized a stochastic differential equation for multi-dimensional weakly stationary process, by referring de Branges's theory for a multi-dimensional case and using Krein's method, which is said to be $(\alpha,\beta,\gamma,\delta)$-Langevin equation and treates the case of finite time prediction interval.

Recently, the author ([24],[25] and [26]) derived and then characterized two kinds of stochastic difference equations for one-dimensional weakly stationary time series with reflection positivity, by using the result obtained in the continuous-time case, which are called the first and second KMO-Langevin equation, related to the case of long time prediction interval.

The purpose of the present paper is to derive and then characterize a stochastic difference equation for multi-dimensional weakly stationary time series of a general type, by using the innovation method. And we call it KM_2O-Langevin equation and treat the case of finite time prediction interval. The notable point is that a random force in KM_2O-Langevin equation is not always a white noise, different from the one in $(\alpha,\beta,\gamma,\delta)$-Langevin equation.

We will state the content of this paper. Let $X = (X(n);n\in\mathbb{Z})$ be a d-dimensional weakly stationary process with mean vector zero and covariance matrix R. We define for each $n \in \mathbb{N}$ a block Toeplitz matrix $S_n(\in M(nd;\mathbb{R}))$ by

$$(1.1)\qquad S_n = \begin{bmatrix} R(0) & R(1) & \cdots & R(n-1) \\ {}^tR(1) & R(0) & R(1) & \vdots \\ \vdots & \ddots & \ddots & R(1) \\ {}^tR(n-1) & \cdots & {}^tR(1) & R(0) \end{bmatrix} .$$

We suppose that $S_n(n\in\mathbb{N})$ are invertible in what follows. In §2, we will introduce a <u>forward</u> (resp. <u>backward</u>) <u>innovation process</u> $I_+ = (I_+(n);n\in\mathbb{N})$(resp. $I_-=(I_-(n);n\in\mathbb{N})$) associated with X. It is noted that I_+(resp. I_-) is an orthogonal process. And then we derive two kinds of stochastic difference equations describing the time evolution of X :

$$(1.2)\qquad X(n) = -\sum_{k=1}^{n-1}\gamma_+(n,k)X(k) - \delta_+(n)X(0) + I_+(n) \quad (n \in \mathbb{N})$$

$$(1.3)\qquad X(-n) = -\sum_{k=1}^{n-1}\gamma_-(n,k)X(-k) - \delta_-(n)X(0) + I_-(n) \quad (n \in \mathbb{N}) ,$$

where $\gamma_+(\cdot,*)$, $\gamma_-(\cdot,*)$, $\delta_+(\cdot)$ and $\delta_-(\cdot)$ belong to $M(d;\mathbb{R})$.

We call (1.2)(resp.(1.3)) a <u>forward</u> (resp. <u>backward</u>) <u>KM_2O-Langevin equation</u> for X. It is noted that a class of non-linear Langevin equations for strongly stationary time series is derived from our approach (Remark 2.2).

We will in §3 obtain fundamental recursive relations among $\gamma_+(\cdot,*)$, $\gamma_-(\cdot,*)$, $\delta_+(\cdot)$ and $\delta_-(\cdot)$ (Theorem 3.1) : for any $n,k \in \mathbb{N}$, $n > k$,

$$(1.4)\qquad \gamma_+(n,k) = \gamma_+(n-1,k-1) + \delta_+(n)\gamma_-(n-1,n-1-k)$$

$$(1.5)\qquad \gamma_-(n,k) = \gamma_-(n-1,k-1) + \delta_-(n)\gamma_+(n-1,n-1-k) ,$$

where $\gamma_+(n,0) = \delta_+(n)$ and $\gamma_-(n,0) = \delta_-(n)$.

We denote by $V_+(n)$(resp. $V_-(n)$) the covariance matrix of $I_+(n)$(resp. $I_-(n)$)($n\in\mathbb{N}$) . By using (1.4) and (1.5), we will in §4 obtain fundamental recursive relations among $V_+(\cdot)$, $V_-(\cdot)$, $R(0)$, $\delta_+(\cdot)$ and $\delta_-(\cdot)$ (Theorem 4.1) : for any $n \in \mathbb{N}^*$,

(1.6) $$V_+(n+1) = (I-\delta_+(n+1)\delta_-(n+1))V_+(n)$$

(1.7) $$V_-(n+1) = (I-\delta_-(n+1)\delta_+(n+1))V_-(n)$$

(1.8) $$V_+(n)^t\delta_-(n+1) = \delta_+(n+1)V_-(n) ,$$

where $V_+(0) = V_-(0) = R(0)$.

By taking advantage of the innovation method, we will in §5 give a <u>forward</u> (resp. <u>backward</u>) <u>prediction formula</u> for X (Theorems 5.1 and 5.2). It will be found that prediction matrices and prediction error matrices are determined by $R(0)$, $\delta_+(\cdot)$ and $\delta_-(\cdot)$.

As a converse setting of §2~§5, we will in §6 show a reconstruction theorem (Theorem 6.1), which states that for a given system $\{V,\delta_+(n);n\in\mathbb{N}\}$ in $M(d;\mathbb{R})$ and a d-dimensional orthogonal process $I_+ = (I_+(n);n\in\mathbb{N}^*)$, the recursive relations (1.4)~(1.8) characterize the weakly stationarity property of unique solution $X_+ = (X(n);n\in\mathbb{N}^*)$ of the forward KM_2O-Langevin equation (1.2), where $I_+(0) = X(0)$, $V = R(0)$ and I_+ becomes a forward innovation process associated with X_+ .

In a forthcoming paper ([27]), we will construct an outer matrix function and then derive a KMO-Langevin equation for a multidimensional weakly stationary time series. And together with some relations between KM_2O-Langevin equation and KMO-Langevin equation, generalized fluctuation-dissipation theorems will be proved based on both Langevin equations.

§2. KM_2O-Langevin equations

Let $X = (X(n);n\in\mathbb{Z})$ be an $\mathbb{R}^d$-valued weakly stationary time series on a probability space $(\Omega,\mathcal{B},P)$ with mean vector zero and covariance matrix R :

$$R(n) = E(X(n)\,{}^tX(0)) \qquad (n \in \mathbb{Z}) . \tag{2.1}$$

Note that

$${}^tR(n) = R(-n) \qquad (n \in \mathbb{Z}) . \tag{2.2}$$

Let M be the closed subspace of $L^2(\Omega,\mathcal{B},P)$ defined by

$$M = \text{the closed linear hull of } \{X_j(n);1\le j\le d, n\in\mathbb{Z}\} . \tag{2.3}$$

We then have the unitary group $(U(m);m\in\mathbb{Z})$ acting on M such that

$$U(m)(X(n)) = X(n+m) \qquad (m,n\in\mathbb{Z}) . \tag{2.4}$$

For each $n \in \mathbb{N}$ we define a block Toeplitz matrix $S_n \in M(nd;\mathbb{R})$ by

$$S_n = \begin{pmatrix} R(0) & R(1) & \cdots & R(n-1) \\ {}^tR(1) & R(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & R(1) \\ {}^tR(n-1) & \cdots & {}^tR(1) & R(0) \end{pmatrix} . \tag{2.5}$$

Since

$$S_n = E(Y_n\,{}^tY_n) , \tag{2.6}$$

where $Y_n = {}^t(X_1(n-1),\cdots,X_d(n-1),\cdots,X_1(0),\cdots,X_d(0))$,

we can see that either of the following (2.7) and (2.8) holds :

$$S_n \in GL(nd;\mathbb{R}) \qquad \text{for any } n \in \mathbb{N} \tag{2.7}$$

$$\text{there exists } n_0 \in \mathbb{N} \text{ such that } S_n \in GL(nd;\mathbb{R}) \text{ for any } n \in \{1,\cdots,n_0\} \text{ and } S_n \notin GL(nd;\mathbb{R}) \text{ for any } n \in \{n_0+1,n_0+2,\cdots\} . \tag{2.8}$$

We treat the case where condition (2.7) holds in what follows.

Remark 2.1. If R has a spectral density matrix $\Delta = \Delta(\theta)$

$(\theta\in[-\pi,\pi))$ such that $\Delta(\theta) \in GL(d;\mathbb{C})$ for almost all $\theta \in [-\pi,\pi)$, it can be seen that condition (2.7) holds.

For each $n \in \mathbb{N}$ we define the following block Toeplitz matrix $T_n \in M(nd;\mathbb{R})$ by

$$T_n = \begin{pmatrix} R(0) & {}^tR(1) & \cdots & & {}^tR(n-1) \\ R(1) & R(0) & {}^tR(1) & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & {}^tR(1) \\ R(n-1) & \cdots & & R(1) & R(0) \end{pmatrix} . \tag{2.9}$$

Since

$$T_n = E(Z_n {}^tZ_n) , \tag{2.10}$$

where $Z_n = {}^t(X_1(0),\cdots,X_d(0),\cdots,X_1(n-1),\cdots,X_d(n-1))$, it holds from (2.6) and (2.7) that

$$T_n \in GL(nd;\mathbb{R}) \quad \text{for any } n \in \mathbb{N} . \tag{2.11}$$

For each $n \in \mathbb{N}^* = \{0,1,2,\cdots\}$ we define two closed linear subspaces $M_0^+(n)$ and $M_0^-(n)$ of M by

$$M_0^+(n) = \text{the linear hull of } \{X_j(m);1\le j\le d,0\le m\le n\} \tag{2.12}$$

$$M_0^-(n) = \text{the linear hull of } \{X_j(-m);1\le j\le d,0\le m\le n\} . \tag{2.13}$$

Then we introduce two $\mathbb{R}^d$-valued stochastic processes $I_+ = (I_+(n);n\in\mathbb{N})$ and $I_- = (I_-(n);n\in\mathbb{N})$ on $(\Omega,\mathcal{B},P)$ by

$$I_+(n) = X(n) - P_{M_0^+(n-1)} X(n) \tag{2.14}$$

$$I_-(n) = X(-n) - P_{M_0^-(n-1)} X(-n) , \tag{2.15}$$

where $P_{M_0^+(n-1)}$ (resp. $P_{M_0^-(n-1)}$) stands for the orthogonal projection on $M_0^+(n-1)$ (resp. $M_0^-(n-1)$).

It follows from condition (2.7) that there uniquely exist two systems $\{\gamma_+(n,k),\delta_+(m);k,m,n\in\mathbb{N},n>k\}$ and $\{\gamma_-(n,k),\delta_-(m);k,m,n\in\mathbb{N},n>k\}$ of members in $M(d;\mathbb{R})$ such that for any $n \in \mathbb{N}$,

$$X(n) = -\sum_{k=1}^{n-1} \gamma_+(n,k)X(k) - \delta_+(n)X(0) + I_+(n) \tag{2.16}$$

$$X(-n) = -\sum_{k=1}^{n-1} \gamma_-(n,k)X(-k) - \delta_-(n)X(0) + I_-(n) . \tag{2.17}$$

In particular, it holds that for any $n \in \mathbb{N}$,

(2.18) $\{X(0), I_+(m); m\in\mathbb{N}\}$ is orthogonal in M^d

(2.19) $\{X(0), I_-(m); m\in\mathbb{N}\}$ is orthogonal in M^d

(2.20) $M_0^+(n)$ = the linear hull of $\{X_j(0), I_{+j}(\mathfrak{L}); 1\leq j\leq d, 1\leq \mathfrak{L}\leq n\}$

(2.21) $M_0^-(n)$ = the linear hull of $\{X_j(0), I_{-j}(\mathfrak{L}); 1\leq j\leq d, 1\leq \mathfrak{L}\leq n\}$,

where $I_+(\mathfrak{L}) = {}^t(I_{+1}(\mathfrak{L}),\cdots,I_{+d}(\mathfrak{L}))$ and $I_-(\mathfrak{L}) = {}^t(I_{-1}(\mathfrak{L}),\cdots,I_{-d}(\mathfrak{L}))$.

We call I_+ (resp. I_-) a <u>forward</u> (resp. <u>backward</u>) <u>innovation process</u> associated with X . Furthermore, by regarding (2.16) (resp. (2.17)) as a stochastic difference equation describing the time evolution of X , we call (2.16) (resp.(2.17)) a <u>forward</u> (resp. <u>backward</u>) <u>KM_2O-Langevin equation</u> for X .

<u>Remark</u> 2.2. Let $Y = (Y(n); n\in\mathbb{Z})$ be a one-dimensional strongly stationary time series on $(\Omega,\mathcal{B},P)$ such that

(2.22) $Y(0) \in L^4(\Omega,\mathcal{B},P)$

(2.23) $E(Y(0)) = 0$.

We define three weakly stationary time series $X^{(j)} = (X^{(j)}(n); n\in\mathbb{Z})$ $(j=1,2,3)$ by

$$X^{(1)}(n) = \begin{pmatrix} Y(n) \\ Y(n)^2 - E(Y(0)^2) \end{pmatrix} \tag{2.24}$$

$$X^{(2)}(n) = \begin{pmatrix} Y(n) \\ Y(n)Y(n-1) - E(Y(1)Y(0)) \end{pmatrix} \tag{2.25}$$

$$X^{(3)}(n) = \begin{pmatrix} Y(n) \\ Y(n)^2 - E(Y(0)^2) \\ Y(n)Y(n-1) - E(Y(1)Y(0)) \end{pmatrix} . \tag{2.26}$$

If we suppose condition (2.7) for $X^{(j)}(j=1,2,3)$, we can derive three non-linear Langevin equations for Y through KM_2O-Langevin equations for $X^{(j)}(j=1,2,3)$. The problem of non-linear prediction for Y will be investigated based on these non-linear Langevin equations in the near future.

§3. Relations among $\gamma_+(\cdot,*)$, $\gamma_-(\cdot,*)$, $\delta_+(\cdot)$ and $\delta_-(\cdot)$

For convenience sake, we set for each $n \in \mathbb{N}$

$$\gamma_+(n,0) = \delta_+(n) \tag{3.1}$$

$$\gamma_-(n,0) = \delta_-(n) . \tag{3.2}$$

Theorem 3.1. For any $n , k \in \mathbb{N}$, $n > k$,

(i) $$\gamma_+(n,k) = \gamma_+(n-1,k-1) + \delta_+(n)\gamma_-(n-1,n-1-k)$$

(ii) $$\gamma_-(n,k) = \gamma_-(n-1,k-1) + \delta_-(n)\gamma_+(n-1,n-1-k) .$$

Proof. Fix any $n \in \{2,3,\cdots\}$. By multiplying both hand sides of equation (2.16) by ${}^tX(m)$ $(m=0,1,\cdots,n-1)$ and then taking an expectation with respect to the probability P, we see from (2.18) and (2.20) that for any $m \in \{0,1,\cdots,n-1\}$,

$$R(n-m) = -\sum_{k=1}^{n-1} \gamma_+(n,k)R(k-m) - \delta_+(n)\,{}^tR(m) . \tag{3.3}_m$$

By replacing n in $(3.3)_m$ by $n-1$, we have, for any $m \in \{0,1,\cdots,n-2\}$,

$$R(n-1-m) = -\sum_{k=0}^{n-2} \gamma_+(n-1,k)R(k-m) . \tag{3.4}_m$$

On the other hand, by multiplying both hand sides of equation (2.17) replaced n by $n-1$ by ${}^tX(-m)$ $(m=0,1,\cdots,n-2)$ and then taking an expectation with respect to P, we see from (2.19) and (2.21) that for any $m \in \{0,1,\cdots,n-2\}$,

$${}^tR(n-1-m) = -\sum_{k=0}^{n-2} \gamma_-(n-1,k)\,{}^tR(k-m) . \tag{3.5}_m$$

Now, let any $m \in \{1,2,\cdots,n-1\}$ be fixed. By combining $R(n-m)$ (resp. ${}^tR(m)$) in $(3.3)_m$ with $R(n-m)$ in $(3.4)_{m-1}$ (resp. ${}^tR(m)$ in $(3.5)_{n-1-m}$), we have

$$-\sum_{k=0}^{n-2} \gamma_+(n-1,k)R(k+1-m)$$

$$= -\sum_{k=1}^{n-1} \gamma_+(n,k)R(k-m) + \delta_+(n)\sum_{k=0}^{n-2} \gamma_-(n-1,k)\,{}^tR(k-n+1+m)$$

$$= -\sum_{k=0}^{n-2} (\gamma_+(n,k+1) - \delta_+(n)\gamma_-(n-1,n-2-k))R(k+1-m)$$

and so

$$\sum_{k=0}^{n-2} (\gamma_+(n,k+1) - \gamma_+(n-1,k) - \delta_+(n)\gamma_-(n-1,n-2-k))R(k+1-m) = 0 .$$

Therefore, it follows from condition (2.7) that (i) holds. Similarly, we can prove (ii). (Q.E.D.)

For future use in §6, for any $m \in \mathbb{N}^*$ and $n \in \mathbb{N}$, we set

(3.6) $$\eta_+(m,n) = \gamma_+(m+n,m)$$

(3.7) $$\eta_-(m,n) = \gamma_-(m+n,m) .$$

Immediately from Theorem 3.1, we have

Theorem 3.1' For any $m \in \mathbb{N}^*$ and $n \in \mathbb{N}$,

(i) $$\eta_+(m+1,n) = \eta_+(m,n) + \delta_+(m+n+1)\eta_-(n-1,m+1)$$

(ii) $$\eta_-(m+1,n) = \eta_-(m,n) + \delta_-(m+n+1)\eta_+(n-1,m+1) .$$

§4. Relations among $V_+(\cdot)$, $V_-(\cdot)$, $\delta_+(\cdot)$ and $\delta_-(\cdot)$

For convenience sake, we put

$$I_+(0) = I_-(0) = X(0) \tag{4.1}$$

and denote by $V_+(n)$ (resp. $V_-(n)$) the covariance matrix of $I_+(n)$ (resp. $I_-(n)$) ($n\in\mathbb{N}^*$) :

$$V_+(n) = E(I_+(n)\,{}^tI_+(n)) \tag{4.2}$$

$$V_-(n) = E(I_-(n)\,{}^tI_-(n)) \ . \tag{4.3}$$

Lemma 4.1 (i) $\det S_n = \prod_{k=0}^{n-1} \det V_+(k)$ for any $n \in \mathbb{N}$

(ii) $\det T_n = \prod_{k=0}^{n-1} \det V_-(k)$ for any $n \in \mathbb{N}$

(iii) $V_+(n)\ ,\ V_-(n) \in GL(nd;\mathbb{R})$ for any $n \in \mathbb{N}$.

Proof. (i) for $n = 1$ is trivial. Let any $n \in \{2,3,\cdots\}$ be fixed. By multiplying both hand sides of equation (2.16) by ${}^tX(n)$ and then taking an expectation with respect to P, we see from (2.18) and (2.20) that

$$R(0) = -\sum_{k=1}^{n-1} \gamma_+(n,k)\,{}^tR(n-k) - \delta_+(n)\,{}^tR(n) + V_+(n) \ . \tag{4.4}$$

By making a matrix representation of (4.4) and $(3.3)_m$ ($m=n-1,n-2,\cdots,1,0$), we have

$$\begin{bmatrix} I & \gamma_+(n,n-1) & \cdots & \gamma_+(n,1) & \delta_+(n) \\ & I & & & \\ & & \ddots & & \\ & & & & \\ & 0 & & & I \end{bmatrix} S_{n+1} = \begin{bmatrix} V_+(n) & 0 & \cdots & 0 \\ {}^tR(1) & & & \\ \vdots & & S_n & \\ {}^tR(n) & & & \end{bmatrix} \ .$$

which yields (i). (ii) is also proved similarly. (iii) follows from (2.7), (2.11), (i) and (ii). (Q.E.D.)

Lemma 4.2. For any $n \in \mathbb{N}^*$,

$$\text{(i)}\qquad R(n+1) = -\delta_+(n+1)V_-(n) - \sum_{k=0}^{n-1}\gamma_+(n,k)R(k+1)$$

$$\text{(ii)}\qquad {}^tR(n+1) = -\delta_-(n+1)V_+(n) - \sum_{k=0}^{n-1}\gamma_-(n,k)\,{}^tR(k+1)\ .$$

Proof. Since $V_+(n)$ is a symmetric matrix, it follows from (4.4) that for any $n \in \mathbb{N}^*$,

$$V_+(n) = R(0) + \sum_{k=0}^{n-1} R(n-k)\,{}^t\gamma_+(n,k)\ . \tag{4.5}$$

Similarly,

$$V_-(n) = {}^tR(0) + \sum_{k=0}^{n-1} {}^tR(n-k)\,{}^t\gamma_-(n,k)\ . \tag{4.6}$$

By replacing n by $n+1$ in $(3.3)_0$ and then using Theorem 3.1(i),

$$R(n+1) = -\delta_+(n+1)\Big(\sum_{k=0}^{n-1}\gamma_-(n,k)R(n-k) + R(0)\Big) - \sum_{k=0}^{n-1}\gamma_+(n,k)R(k+1)\ ,$$

which, together with (4.6), implies (i). (ii) is also proved similarly. (Q.E.D.)

Now, we will show the following fundamental

Lemma 4.3. For any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} \eta_+(k,n-k)R(k+1) = \sum_{k=0}^{n-1} R(k+1)\,{}^t\eta_-(k,n-k)\ .$$

The proof is divided into 13 steps. We denote by A_n the right hand side minus the left hand side in Lemma 4.3 :

$$A_n = \sum_{k=0}^{n-1} R(k+1)\,{}^t\eta_-(k,n-k) - \sum_{k=0}^{n-1}\eta_+(k,n-k)R(k+1)\ . \tag{4.7}$$

(Step 1) (i) $V_+(n+1) = V_+(n) - \delta_+(n+1)V_-(n)\,{}^t\delta_+(n+1) + A_n \qquad (n\in\mathbb{N})$

(ii) $V_-(n+1) = V_-(n) - \delta_-(n+1)V_+(n)\,{}^t\delta_-(n+1) - {}^tA_n \qquad (n\in\mathbb{N})$

(iii) $V_+(n)\,{}^t\delta_-(n+1) - \delta_+(n+1)V_-(n) = A_n\ . \qquad (n\in\mathbb{N})$

Proof. By applying Theorem 3.1 to (4.5),

$$V_+(n+1) = V_+(n) + \Big(\sum_{k=0}^{n-1} R(k+1)\,{}^t\eta_-(k,n-k) + R(n+1)\Big)\,{}^t\delta_+(n+1)\ ,$$

which, together with Lemma 4.2(i), implies (i). (ii) is also proved similarly. By using Lemma 4.2 again, we get (iii). (Q.E.D.)

(Step 2) (i) $R(1) = -\delta_+(1)R(0)$

(ii) ${}^tR(1) = -\delta_-(1)R(0)$

(iii) $R(0){}^t\delta_-(1) = \delta_+(1)R(0)$

(iv) $V_+(1) = (I-\delta_+(1)\delta_-(1))R(0)$

(v) $V_-(1) = (I-\delta_-(1)\delta_+(1))R(0)$.

Proof. By multiplying both hand sides of (2.16) and (2.17) for $n = 1$ by ${}^tX(0)$ and then taking an expectation with respect to P, we have (i) and (ii). (iii) follows from (i) and (ii). Next, by multiplying (2.16) for $n = 1$ by ${}^tX(1)$ and then taking an expectation, we see from (2.18), (2.20), (i) and (ii) in Step 2 that (iv) holds.(v) is also proved similarly. (Q.E.D.)

(Step 3) $A_1 = 0$.

This follows from (iii) in Step 2.

(Step 4) For any $n \in (2,3,\cdots)$,

$$A_n = \delta_+(n)I_n^{(1)}{}^t\delta_-(n) + \delta_+(n)\mathrm{II}_n^{(1)} + \mathrm{III}_n^{(1)}{}^t\delta_-(n) + \mathrm{IV}_n^{(1)} ,$$

where

$$I_n^{(1)} = V_-(n-1){}^t\delta_+(n-1) - \delta_-(n-1)V_+(n-1)$$

$$\mathrm{II}_n^{(1)} = -(R(1) + \sum_{j=1}^{n-2} n_-(n-1-j,j)R(j+1)) +$$
$$+ \delta_-(n-1)\sum_{j=0}^{n-2} R(j+1){}^tn_-(j,n-1-j) - V_-(n-1){}^tn_-(n-2,1)$$

$$\mathrm{III}_n^{(1)} = R(1) + \sum_{j=1}^{n-2} R(j+1){}^tn_+(n-1-j,j) -$$
$$- (\sum_{j=0}^{n-2} n_+(j,n-1-j)R(j+1)){}^t\delta_+(n-1) + n_+(n-2,1)V_+(n-1)$$

$$\mathrm{IV}_n^{(1)} = \sum_{j=0}^{n-3} n_+(j,n-1-j)R(j+2) + (\sum_{j=0}^{n-3} n_+(j,n-1-j)R(j+1)){}^tn_-(n-2,1) -$$

$$- \sum_{j=0}^{n-3} R(j+2)^t n_-(j,n-1-j) - n_+(n-2,1) \sum_{j=0}^{n-3} R(j+1)^t n_-(j,n-1-j) .$$

This decomposition follows from Theorem 3.1' and Lemma 4.2.

(Step 5) For any $n \in \{2,3,\cdots\}$ and any $k \in \{1,2,\cdots,n-1\}$,

$$IV_n^{(k)} = \delta_+(n-k) I_n^{(k+1)t} \delta_-(n-k) + \delta_+(n-k) II_n^{(k+1)} + III_n^{(k+1)t} \delta_-(n-k) + IV_n^{(k+1)} ,$$

where

$$I_n^{(k+1)} = - \sum_{j=0}^{k-1} n_-(j,n-k-1-j)(R(k-j) + \sum_{i=1}^{n-2k-1} R(k-j+i)^t n_+(n-k-1-i,i)) +$$
$$+ \sum_{j=0}^{k-1} (R(k-j) + \sum_{i=1}^{n-2k-1} n_-(n-k-1-i,i)R(k-j+1)) n_+(j,n-k-j) -$$
$$- n_-(k,n-2k-1)V_+(n-k-1) + V_-(n-k-1)^t n_+(k,n-2k-1)$$

$$II_n^{(k+1)} = R(k+1) + \sum_{j=1}^{n-2k-2} n_-(n-k-1-j,j)R(k+1+j) -$$
$$- \sum_{j=0}^{k} n_-(j,n-k-1-j)(\sum_{i=0}^{n-2k-2} R(k-j+1+i)^t n_-(i,n-k-1-i)) +$$
$$+ \sum_{j=1}^{k} (R(k+1-j) + \sum_{i=1}^{n-2k-2} n_-(n-k-1-i,i)R(k+1-j+i))^t n_-(n-k-1-j,j) +$$
$$+ V_-(n-k-1)^t n_-(n-2k-2,k+1)$$

$$III_n^{(k+1)} = -(R(k+1) + \sum_{j=1}^{n-2k-2} R(k+1+j)^t n_+(n-k-1-j,j)) +$$
$$+ \sum_{j=0}^{k} (\sum_{i=0}^{n-2k-2} {}^t n_+(i,n-k-1-i)R(k-j+1+i))^t n_+(j,n-k-1-j) -$$
$$- \sum_{j=1}^{k} n_+(n-k-1-j,j)(R(k+1-j) + \sum_{i=1}^{n-2k-2} R(k+1-j+i)n_+(n-k-1-i,i)) -$$
$$- n_+(n-2k-2,k+1)V_+(n-k-1)$$

$$IV_n^{(k+1)} = \sum_{j=0}^{n-2k-3} n_+(j,n-k-1-j)R(k+2+j) - \sum_{j=0}^{n-2k-3} R(k+2+j)^t n_-(j,n-k-1-j) -$$
$$- \sum_{j=1}^{k+1} n_+(n-k-1-j,j)(\sum_{i=0}^{n-2k-3} R(k+1+i)^t n_-(i,n-k-1-i)) +$$
$$+ \sum_{j=1}^{k+1} (\sum_{i=0}^{n-2k-3} n_+(i,n-k-1-i)R(k+1+i))^t n_-(n-k-1-j,j) .$$

Similarly to Step 4, this follows from Theorem 3.1' and Lemma 4.2.

(Step 6) (i) For any even $n = 2N$ $(N\in\mathbb{N})$, $IV_n^{(N)} = 0$.

(ii) For any odd $n = 2N + 1$ $(N\in\mathbb{N})$,

$$IV_n^{(N)} = \delta_+(N+1)\{-V_+(N) - \sum_{j=0}^{N-1} V_+(N-1-j)\,{}^t\delta_-(N-j)\,{}^t\delta_+(N-j) + \\ + V_-(N) + \sum_{j=0}^{N-1} \delta_-(N-j)\delta_+(N-j)V_-(N-1-j)\}\,{}^t\delta_-(N+1) .$$

Proof. Immediately from Step 5, we have (i). (ii) can be proved as follows : By Step 5,

$$IV_n^{(N)} = \delta_+(N+1)(R(N+1) + \sum_{j=1}^{N} R(N+1-j)\,{}^t\eta_-(N+1-j,j)) - \\ - (R(N+1) + \sum_{j=1}^{N} \eta_+(N+1-j,j)R(N+1-j))\,{}^t\delta_-(N+1) .$$

By Theorem 3.1' and Lemma 4.2,

$$R(N+1) + \sum_{j=1}^{N} R(N+1-j)\,{}^t\eta_-(N+1-j,j) = (-V_+(N) + \sum_{j=0}^{N-1} R(N-j)\,{}^t\eta_+(j,N-j))\,{}^t\delta_-(N+1)$$

$$R(N+1) + \sum_{j=1}^{N} \eta_+(N+1-j,j)R(N+1-j) = \delta_+(N+1)(-V_-(N) + \sum_{j=0}^{N-1} \eta_-(j,N-j)R(N-j)).$$

And so

$$IV_n^{(N)} = \delta_+(N+1)\{-V_+(N) + \sum_{j=0}^{N-1} R(N-j)\,{}^t\eta_+(j,N-j) - \\ - V_-(N) + \sum_{j=0}^{N-1} \eta_-(j,N-j)R(N-j)\}\,{}^t\delta_-(N+1) .$$

Furthermore, by using Theorem 3.1' and Lemma 4.2 again,

$$\sum_{j=0}^{N-1} R(N-j)\,{}^t\eta_+(j,N-j) = -V_+(N-1)\,{}^t\delta_-(N)\,{}^t\delta_+(N) + \sum_{j=0}^{N-2} R(N-1-j)\,{}^t\eta_+(j,N-1-j)$$

$$\sum_{j=0}^{N-1} \eta_-(j,N-j)R(N-j) = -\delta_-(N)\delta_+(N)V_-(N-1) + \sum_{j=0}^{N-2} \eta_-(j,N-1-j)R(N-1-j) ,$$

which yields

$$IV_n^{(N)} = \delta_+(N+1)\{-V_+(N) - V_+(N-1)\,{}^t\delta_-(N)\,{}^t\delta_+(N) + \\ + \sum_{j=0}^{N-2} R(N-1-j)\,{}^t\eta_+(j,N-1-j) + V_-(N) + \delta_-(N)\delta_+(N)V_-(N-1) - \\ - \sum_{j=0}^{N-2} \eta_-(j,N-1-j)R(N-1-j)\}\,{}^t\delta_-(N+1) .$$

By repeating the same procedure, we see from Step 2 that (i) holds. (ii) follows immediately from Step 5. (Q.E.D.)

(Step 7) For any fixed $n_0 \in \mathbb{N}^*$, if $V_+(n_0)^t\delta_-(n_0+1) = \delta_+(n_0+1)V_-(n_0)$, then $V_-(n_0+1)^t\delta_+(n_0+1) = \delta_-(n_0+1)V_+(n_0+1)$.

Proof. By Step 1, (ii),(iv) and (v) in Step 2,

$$\begin{aligned} V_-(n_0+1)^t\delta_+(n_0+1) &= (I-\delta_-(n_0+1)\delta_+(n_0+1))V_-(n_0)^t\delta_+(n_0+1) \\ &= \delta_-(n_0+1)(I - \delta_+(n_0+1)\delta_-(n_0+1))V_+(n_0) \\ &= \delta_-(n_0+1)V_+(n_0+1) . \end{aligned}$$

(Q.E.D.)

(Step 8) $I_n^{(1)} = II_n^{(1)} = III_n^{(1)} = IV_n^{(1)} = 0$ for any $n \in \{2,3\}$.

Proof. By (iii) in Step 2 (resp.Step 3) with Step 7, we have $I_2^{(1)} = 0$(resp. $I_3^{(1)}=0$). $II_2^{(1)}$ and $III_2^{(1)}$ follow from Step 2. A simple calculation, together with Lemma 4.2 and Step 2, yields $II_3^{(1)}$ and $III_3^{(1)}$. $IV_n^{(1)}$(n=2,3) follow from Steps 2 and 6. (Q.E.D.)

Now, by a mathematical induction, we shall show the following statement $(*)_n$ (n=2,3,$\cdots$) : for any $m \in \{2,3,\cdots,n\}$,

$$(*)_n \begin{cases} \text{(i)} & I_m^{(k)} = II_m^{(k)} = III_m^{(k)} = 0 \quad (1\le k\le M) \quad \text{if} \quad m = 2M+1 \\ & IV_m^{(M)} = 0 \\ \text{(ii)} & I_m^{(k)} = II_m^{(k)} = III_m^{(k)} = 0 \quad (1\le k\le M) \quad \text{if} \quad m = 2M . \end{cases}$$

Step 8 implies that $(*)_n$(n=2,3) hold. For any fixed $n_0 \in \{2,3,\cdots\}$, let us suppose that $(*)_{n_0}$ holds. Moreover we consider the case where n_0 is even, $n_0 = 2N_0$. The case where n_0 is odd is proved similarly. What we have to show is that

$$(*)'_{n_0,k} \qquad I_{n_0+1}^{(k)} = II_{n_0+1}^{(k)} = III_{n_0+1}^{(k)} = 0 \qquad (1\le k\le N_0) ,$$

$$(*)''_{n_0,N_0} \qquad IV_{n_0+1}^{(N_0)} = 0 .$$

Since $(*)_{n_0}$ implies that $A_m(2\le m\le n_0)$ hold, it follows from Steps 1 and 7 that for any $m \in \{2,\cdots,n_0\}$,

$$V_+(m+1) = (I - \delta_+(m+1)\delta_-(m+1))V_+(m) \tag{4.8}$$

$$V_-(m+1) = (I - \delta_-(m+1)\delta_+(m+1))V_-(m) \tag{4.9}$$

$$V_+(m)^t\delta_-(m+1) = \delta_+(m+1)V_-(m) \tag{4.10}$$

$$V_+(m+1)^t\delta_-(m+1) = \delta_+(m+1)V_-(m+1) \ . \tag{4.11}$$

(Step 9) $\quad I^{(1)}_{n_0+1} = 0 \ .$

This follows from Step 4 and (4.11).

(Step 10) (i) $\quad \mathrm{II}^{(1)}_{n_0+1} = 0 \quad$ (ii) $\quad \mathrm{III}^{(1)}_{n_0+1} = 0 \ .$

Proof. We prove only (i), since (ii) is shown similarly. By Theorem 3.1', Lemma 4.2(ii), and the assumption $\mathrm{II}^{(1)}_{n_0} = 0$

$$\begin{aligned}\mathrm{II}^{(1)}_{n_0+1} &= -V_-(n_0-1)^t n_-(n_0-2,1) - \delta_-(n_0-1)V_+(n_0-1)^t\delta_-(n_0) + V_-(n_0)^t n_-(n_0-1,1) + \\ &\quad + \delta_+(n_0)\Big(\sum_{j=0}^{n_0-2} n_+(j,n_0-1-j)R(2+j) - \sum_{j=0}^{n_0-1} R(1+j)^t n_-(j,n_0-j)\Big) \ .\end{aligned}$$

Since

$$V_-(n_0-1)^t n_-(n_0-2,1) + \delta_-(n_0-1)V_+(n_0-1)^t\delta_-(n_0) = V_-(n_0-1)^t n_-(n_0-1,1) \ ,$$

we apply Theorem 3.1' again to find that

$$\begin{aligned}\mathrm{II}^{(1)}_{n_0+1} &= (V_-(n_0) - V_-(n_0-1))^t n_-(n_0-1,1) + \\ &\quad + \delta_-(n_0)\Big(\sum_{j=0}^{n_0-3} n_+(j,n_0-1-j)R(2+j) - \sum_{j=0}^{n_0-3} R(2+j)^t n_-(j,n_0-1-j) + \\ &\qquad + n_+(n_0-2,1)R(n_0) - R(n_0)^t n_-(n_0-2,1) - \\ &\qquad - \Big(R(1) + \sum_{j=0}^{n_0-2} R(n_0-j)^t n_+(j,n_0-1-j)\Big)^t\delta_-(n_0)\Big) \ .\end{aligned}$$

By using $\mathrm{IV}^{(1)}_{n_0} = 0$, coming from $(*)_{n_0}$ and Step 5, we see that

$$\mathrm{II}^{(1)}_{n_0+1} = (V_-(n_0) - V_-(n_0-1))^t n_-(n_0-1,1) +$$

$$+ \delta_-(n_0)\{n_+(n_0-2,1)(\sum_{j=0}^{n_0-3} R(1+j)\,{}^t n_-(j,n_0-1-j) + R(n_0)) -$$

$$- (\sum_{j=0}^{n_0-3} n_+(j,n_0-1-j)R(1+j) + R(n_0)) -$$

$$-(R(1) + \sum_{j=0}^{n_0-2} R(n_0-j)\,{}^t n_+(j,n_0-1-j))\,{}^t\delta_-(n_0)) \ .$$

By using Lemma 4.2(i)(ii) and $\mathrm{III}^{(1)}_{n_0} = 0$,

$$\begin{aligned}
&\mathrm{II}^{(1)}_{n_0+1} \\
&= (V_-(n_0)-V_-(n_0-1))\,{}^t n_-(n_0-1,1) + \delta_-(n_0)\delta_+(n_0)V_-(n_0-1)\,{}^t n_-(n_0-2,1) - \\
&\quad - \delta_-(n_0)(n_+(n_0-2,1)V_+(n_0-1) + R(1) + \sum_{j=0}^{n_0-2} R(n_0-j)\,{}^t n_+(j,n_0-1-j))\,{}^t\delta_-(n_0) \\
&= (V_-(n_0)-V_-(n_0-1)\,{}^t n_-(n_0-1,1) + \delta_-(n_0)\delta_+(n_0)V_-(n_0-1)\,{}^t n_-(n_0-2,1) - \\
&\quad - \delta_-(n_0)(\sum_{j=0}^{n_0-2} n_+(j,n_0-1-j)R(1+j) + R(n_0))\,{}^t\delta_+(n_0-1)\,{}^t\delta_-(n_0) \ .
\end{aligned}$$

Therefore, it follows from Theorem 3.1', Lemma 4.2(i) and (4.10) that

$$\begin{aligned}
\mathrm{II}^{(1)}_{n_0+1} &= -V_-(n_0)\,{}^t(n_-(n_0-1,1) - n_-(n_0-2,1)) - \\
&\quad - V_-(n_0-1)\,{}^t\delta_+(n_0-1)\,{}^t\delta_-(n_0) + \\
&\quad + \delta_-(n_0)\delta_+(n_0)V_-(n_0-1)\,{}^t\delta_+(n_0-1)\,{}^t\delta_-(n_0) \\
&= (V_-(n_0) - V_-(n_0-1))\,{}^t\delta_+(n_0-1)\,{}^t\delta_-(n_0) + \\
&\quad + \delta_-(n_0)\delta_+(n_0)V_-(n_0-1)\,{}^t\delta_+(n_0-1)\,{}^t\delta_-(n_0) \\
&= 0 \ . \qquad \text{(Q.E.D.)}
\end{aligned}$$

By Steps 9 and 10, we have proved that $(*)'_{n_0,1}$ holds. Moreover, for any fixed k , $1 \le k < k+1 \le N_0$, by assuming that $(*)'_{n_0,k}$ holds, we will show that $(*)'_{n_0,k+1}$ holds.

(Step 11) $\mathrm{I}^{(k+1)}_{n_0+1} = 0$.

Proof. By applying $\mathrm{II}^{(k)}_{n_0} = 0$ (resp. $\mathrm{III}^{(k)}_{n_0}=0$) to the coefficient of $\delta_+(n_0-k)$ (resp. $\delta_-(n_0-k)$), we see from Theorem 3.1', (4.8) and (4.9) that

$$I^{(k+1)}_{n_0+1} = -\delta_-(n_0-k)I' - I''\,{}^t\delta_+(n_0-k) + I''', \tag{4.12}$$

where

$$\begin{aligned}
I' = {} & \sum_{j=0}^{k-1}\Big(\sum_{i=0}^{n_0-2k} n_+(i,n_0-k-i)R(k-j+i)\Big)\,{}^t n_+(j,n_0-k-j) - \\
& - \sum_{j=1}^{k-1}(n_+(n_0-k-j,j) - n_+(n_0-k-1-j,j))\cdot \\
& \qquad \cdot\Big(R(k-j) + \sum_{i=1}^{n_0-2k} R(k-j+i)\,{}^t n_+(n_0-k-i,i)\Big) - \\
& - (n_+(n_0-2k,k) - n_+(n_0-2k-1,k))V_+(n_0-k) + \\
& + \delta_+(n_0-k)V_-(n_0-k-1)\,{}^t n_+(k-1,n_0-2k)\,, \\
I'' = {} & -\sum_{j=0}^{k-1} n_-(j,n_0-k-j)\Big(\sum_{i=0}^{n_0-2k} R(k-j+i)\,{}^t n_-(i,n_0-k-i)\Big) + \\
& + \sum_{j=1}^{k-1}\Big(R(k-j) + \sum_{i=1}^{n_0-2k} n_-(n_0-k-i,i)R(k-j+i)\Big)\cdot \\
& \qquad \cdot\,{}^t(n_-(n_0-k-j,j) - n_-(n_0-k-1-j,j)) + \\
& + V_-(n_0-k)\,{}^t(n_-(n_0-2k,k) - n_-(n_0-2k-1,k)) - \\
& - n_-(k-1,n_0-2k)V_+(n_0-k-1)\,{}^t\delta_-(n_0-k) \\
I''' = {} & -\sum_{j=0}^{k-2} n_-(j,n_0-k-1-j)\Big(R(k-1-j) + \sum_{i=1}^{n_0-2k} R(k-1-j+i)\,{}^t n_+(n_0-k-i,i)\Big) + \\
& + \sum_{j=0}^{k-2}\Big(R(k-1-j) + \sum_{i=1}^{n_0-2k} n_-(n_0-k-i,i)R(k-1-j+i)\Big)\,{}^t n_+(j,n_0-k-1-j) - \\
& - n_-(k-1,n_0-2k)V_+(n_0-k-1) + V_-(n_0-k-1)\,{}^t n_+(k-1,n_0-2k)\,.
\end{aligned}$$

By Theorem 3.1' and $I^{(k)}_{n-1} = 0$,

$$\begin{aligned}
I''' = {} & \Big(-\sum_{j=0}^{k-2} n_-(j,n_0-k-1-j)\Big(\sum_{i=0}^{n_0-2k-1} R(k-j+i)\,{}^t n_-(i,n_0-k-1-i)\Big)\Big)\,{}^t\delta_+(n_0-k) + \qquad (4.13)
\end{aligned}$$

$$+ \delta_-(n_0-k)\Big(\sum_{j=0}^{k-2}\Big(\sum_{i=0}^{n_0-2k-1} n_+(i,n_0-k-1-i)R(k-j+i)\Big)^t n_+(j,n_0-k-1-j)\Big) .$$

Next, applying Theorem 3.1' to I', we have

$$I' = \delta_+(n_0-k)\tilde{I}'^t\delta_-(n_0-k) + \delta_+(n_0-k)\tilde{II}' + \tilde{III}'^t\delta_+(n_0-k) + IV',$$

where

$$\tilde{I}' = R(k) + \sum_{i=1}^{n_0-2k} n_-(n_0-k-1-i,i)R(k+i) +$$

$$+ n_-(k-1,n_0-2k)V_+(n_0-k-1)^t\delta_-(n_0-k)+$$

$$+ \sum_{j=1}^{k-1}(R(k-j)+\sum_{i=1}^{n_0-2k} n_-(n_0-k-1-i,i)R(k-j+i))^t n_-(n_0-k-1-j,j) -$$

$$- \sum_{j=0}^{k-2} n_-(j,n_0-k-1-j)(R(k-j)+\sum_{i=0}^{n_0-2k-1} R(k-j+i)^t n_-(i,n_0-k-i))$$

$$\tilde{II}' = -n_-(k-1,n_0-2k)V_+(n_0-k-1) + V_-(n_0-k-1)^t n_+(k-1,n_0-2k) -$$

$$- \sum_{j=0}^{k-2} n_-(j,n_0-k-1-j)(R(k-1-j) + \sum_{i=1}^{n_0-2k} R(k-1-j+i)^t n_+(n_0-k-1-i,i)) +$$

$$+ \sum_{j=0}^{k-2}(R(k-1-j) + \sum_{i=1}^{n_0-2k} n_-(n_0-k-1-i,i)R(k-1-j+i))^t n_+(j,n_0-k-1-j)$$

$$\tilde{III}' = \sum_{j=0}^{n_0-2k-1} n_+(j,n_0-k-1-j)R(k+1+j) +$$

$$+ \sum_{j=1}^{k-1}\Big(\sum_{i=0}^{n_0-2k-1} n_+(i,n_0-k-1-i)R(k-j+i)\Big)^t n_+(n_0-k-1-j,j)$$

$$IV' = \sum_{j=0}^{k-2}\Big(\sum_{i=0}^{n_0-2k-1} n_+(i,n_0-k-1-i)R(k-j+i)\Big)^t n_+(j,n_0-k-1-j) .$$

By Lemma 4.2(ii) and $II^{(k)}_{n_0-1} = 0$,

$$\tilde{I}' = n_-(k-1,n_0-2k)\Big(R(n_0-k) + \sum_{j=0}^{n_0-2k-2} R(1+j)^t n_-(j,n_0-k-1-j) +$$

$$+ V_+(n_0-k-1)^t\delta_-(n_0-k)\Big) + II^{(k)}_{n_0-1} - V_+(n_0-k-1)^t n_-(n_0-2k-1,1)$$

$$= -V_+(n_0-k-1)^t n_-(n_0-2k-1,1) .$$

Furthermore, since

$$\tilde{\mathrm{II}}' = I^{(k)}_{n_0-1} = 0 ,$$

$$\text{(4.14)}\qquad I' = -\delta_+(n_0-k)V_-(n_0-k-1)\,{}^t n_-(n_0-2k-1,k)\,{}^t\delta_+(n_0-k) + \\ + \tilde{\mathrm{III}}'\,{}^t\delta_+(n_0-k) + \tilde{\mathrm{IV}}' .$$

Similarly to (4.13), we can apply $I^{(k)}_{n_0-1} = \mathrm{III}^{(k)}_{n_0-1} = 0$ to see that

$$\text{(4.15)}\qquad \mathrm{II}' = \delta_-(n_0-k)n_+(n_0-2k-1,k)V_+(n_0-k-1)\,{}^t\delta_-(n_0-k) + \\ + \delta_-(n_0-k)\,\tilde{\mathrm{III}}'' + \tilde{\mathrm{IV}}'' ,$$

where

$$\tilde{\mathrm{III}}'' = -\sum_{i=0}^{n_0-2k-1} R(k+i)\,{}^t n_-(i,n_0-k-1-i) - \\ -\sum_{j=1}^{k-1} n_+(n_0-k-1-j,j)\Big(\sum_{i=0}^{n_0-2k-1} R(k+1-j+i)\,{}^t n_-(i,n_0-k-1-i)\Big)$$

$$\tilde{\mathrm{IV}}'' = -\sum_{j=0}^{k-2} n_-(j,n_0-k-1-j)\Big(\sum_{i=0}^{n_0-2k-1} R(k-j+i)\,{}^t n_-(i,n_0-k-1-i)\Big) .$$

Thus, by (4.12), (4.13), (4.14) and (4.15),

$$\text{(4.16)}\qquad I^{(k+1)}_{n_0+1} = -\delta_-(n_0-k)\{-\delta_+(n_0-k)V_-(n_0-k-1)\,{}^t n_-(n_0-2k-1,k) + \\ + n_+(n_0-2k-1,k)V_+(n_0-k-1)\,{}^t\delta_-(n_0-k) + \tilde{\mathrm{III}}' + \tilde{\mathrm{III}}''\}\,{}^t\delta_+(n_0-k) .$$

On the other hand, by using $\mathrm{IV}^{(k)}_{n_0-1} = 0$, coming from $(*)_{n_0}$ and Step 5, we see that

$$\tilde{\mathrm{III}}' + \tilde{\mathrm{III}}'' = n_+(n_0-2k-1,k)\Big(R(n_0-k)+\sum_{j=0}^{n_0-k-2} R(1+j)\,{}^t n_-(j,n_0-k-1-j)\Big) - \\ -\Big(R(n_0-k)+\sum_{j=0}^{n_0-k-2} n_+(j,n_0-k-1-j)R(1+j)\Big)\,{}^t n_-(n_0-2k-1,k) .$$

Consequently, by substituting this into (4.16), it follows from Lemma 4.2 that $I^{(k+1)}_{n_0+1} = 0$. (Q.E.D.)

(Step 12) (i) $\mathrm{II}^{(k+1)}_{n_0+1} = 0$ (ii) $\mathrm{III}^{(k+1)}_{n_0+1} = 0$.

Proof. We prove only (i), since (ii) is shown similarly. By applying $\mathrm{IV}^{(k)}_{n_0} = 0$ (resp. $\mathrm{II}^{(k)}_{n_0}=0$) to the coefficient of $\delta_-(n_0-k)$

(resp. ${}^t\eta_-(n_0-k-1,1)$) in $\Pi^{(k+1)}_{n_0+1}$, we see from Theorem 3.1' that

$$\Pi^{(k+1)}_{n_0+1} = V_-(n_0-k)\,{}^t(\eta_-(n_0-2k-1,k+1) - \eta_-(n_0-k-1,1)\eta_-(n_0-2k,k)) \tag{4.17}$$
$$+ (I-\delta_-(n_0-k)\delta_+(n_0-k))J_1 - \delta_-(n_0-k)J_2 - J_3\,{}^t\eta_-(n_0-k-1,1) + J_4$$

where

$$J_1 = R(k+1) + \sum_{i=1}^{n_0-2k-1} \eta_-(n_0-k-1-i,i)R(k+1+i) -$$
$$- \sum_{j=0}^{k-1} \eta_-(j,n_0-k-1-j)\Big(\sum_{i=0}^{n_0-2k-1} R(k-j+i)\,{}^t\eta_-(i,n_0-k-i)\Big)$$

$$J_2 = \sum_{j=1}^{k}\Big(\sum_{i=0}^{n_0-2k-1} \eta_+(i,n_0-k-i)R(k+1-j+i)\Big)\,{}^t\eta_-(n_0-k-j,j)$$

$$J_3 = -\sum_{j=0}^{k-1} \eta_-(j,n_0-k-j)\Big(\sum_{i=0}^{n_0-2k-1} R(k-j+i)\,{}^t\eta_-(i,n_0-k-i)\Big) -$$
$$+ \sum_{j=1}^{k-1}\Big(R(k-j)+\sum_{i=1}^{n_0-2k} \eta_-(n_0-k-i,i)R(k+1+i)\Big)\,{}^t\eta_-(n_0-k-j,j) +$$
$$+ \eta_-(k,n_0-2k)R(n_0-k)$$

$$J_4 = \sum_{j=2}^{k}\Big(R(k+1-j) + \sum_{i=1}^{n_0-2k-1} \eta_-(n_0-k-i,i)R(k+1-j+i)\Big)\,{}^t\eta_-(n_0-k-j,j)\ .$$

Since

$$J_3 = \Pi^{(k)}_{n_0} - \Big(R(k) + \sum_{i=1}^{n_0-2k} \eta_-(n_0-k-i,i)R(k+i)\Big) -$$
$$- V_-(n_0-k)\,{}^t\eta_-(n_0-2k,k) + \eta_-(k,n_0-k)R(n_0-k)\ ,$$

it follows from Theorem 3.1' that

$$J_3 = -\delta_-(n_0-k)\Big(\sum_{i=0}^{n_0-2k-2} \eta_+(i,n_0-k-1-i)R(k+1+i)\Big) -$$
$$- \Big(R(k) + \sum_{i=1}^{n_0-2k-1} \eta_-(n_0-k-1-i,i)R(k+i)\Big) - V_-(n_0-k)\,{}^t\eta_-(n_0-2k,k)$$

and so

$$\delta_-(n_0-k)J_2 + J_3\,{}^t\eta_-(n_0-k-1,1) \tag{4.18}$$
$$= -(I-\delta_-(n_0-k)\delta_+(n_0-k))(R(k) +$$

$$+ \sum_{i=1}^{n_0-2k-1} n_-(n_0-k-1-i,i)R(k+i))\,{}^t n_-(n_0-k-1,1) +$$

$$+ \delta_-(n_0-k)\delta_+(n_0-k) \sum_{j=2}^{k} (R(k+1-j) +$$

$$+ \sum_{i=1}^{n_0-2k-1} n_-(n_0-k-1-i,i)R(k+1-j+i))\,{}^t n_-(n_0-k-j,j) +$$

$$+ \delta_-(n_0-k) \sum_{j=2}^{k} \Big(\sum_{i=0}^{n_0-2k-2} n_+(n_0-k-1-i,i)R(k+2-j+i))\,{}^t n_-(n_0-k-j,j) -$$

$$- V_-(n_0-k)\,{}^t n_-(n_0-2k,k)\,{}^t n_-(n_0-k-1,1) .$$

Forthermore, we note from Theorem 3.1' that

(4.19) J_4

$$= \sum_{j=2}^{k} (R(k+1-j) + \sum_{i=1}^{n_0-2k-1} n_-(n_0-k-1-i,i)R(k+1-j+i))\,{}^t n_-(n_0-k-j,j) +$$

$$+ \delta_-(n_0-k) \sum_{j=2}^{k} \Big(\sum_{i=0}^{n_0-2k-2} n_+(n_0-k-1-i,i)R(k+2-j+i))\,{}^t n_-(n_0-k-j,j) .$$

Therefore, by using Theorem 3.1' and (4.9), we see from (4.17), (4.18) and (4.19) that

$$(4.20) \qquad \mathrm{II}_{n_0+1}^{(k+1)} = (1-\delta_-(n_0-k)\delta_+(n_0-k))J_5 ,$$

where

$$J_5 = R(k+1) + \sum_{i=1}^{n_0-2k-2} n_-(n_0-k-1-i,i)R(k+1+i) +$$

$$+ \sum_{j=1}^{k} (R(k+1-j) + \sum_{i=1}^{n_0-2k-2} n_-(n_0-k-1-i,i)R(k+1-j+i))\,{}^t n_-(n_0-k-j,j) -$$

$$- \sum_{j=0}^{k-1} n_-(j,n_0-k-1-j)(R(k-j)\,{}^t\delta_-(n_0-k) + \sum_{i=1}^{n_0-2k-1} R(k-j+i)\,{}^t n_-(i,n_0-k-i)) +$$

$$+ n_-(k,n_0-2k-1)(R(n_0-k) + \sum_{i=1}^{k} R(n_0-k-i)\,{}^t n_-(n_0-k-i,i)) +$$

$$+ V_-(n_0-k-1)\,{}^t n_-(n_0-2k-1,k+1) .$$

We note from Lemma 4.2(ii) that

$$R(n_0-k) + \sum_{i=1}^{k} R(n_0-k-i)\,{}^t n_-(n_0-k-1-i,i)$$

$$= -V_+(n_0-k-1)\,{}^t\delta_+(n_0-k) - \sum_{i=0}^{n_0-2k-2} R(1+i)\,{}^t n_-(i,n_0-k-1-i) .$$

Therefore, by using Theorem 3.1', we get

(4.21) J_5

$$= R(k+1) + \sum_{i=1}^{n_0-2k-2} n_-(n_0-k-1-i,i)R(k+1+i) - V_-(n_0-k-1)\,{}^t n_-(n_0-2k-2,k+1) +$$

$$+ \sum_{j=1}^{k}\Big(R(k+1-j)+\sum_{i=1}^{n_0-2k-2} n_-(n_0-k-1-i,i)R(k+1-j+i)\Big)\,{}^t n_-(n_0-k-1-j,j) -$$

$$- \sum_{j=0}^{k} n_-(j,n_0-k-1-j)\Big(\sum_{i=0}^{n_0-2k-2} R(k-j+1+i)\,{}^t n_-(i,n_0-k-1-i)\Big) +$$

$$+ \Big(\sum_{j=1}^{k}\Big(R(k+1-j) + \sum_{i=1}^{n_0-2k-2} n_-(n_0-k-1-i,i)R(k+1-j+i)\Big)\,{}^t n_+(j-1,n_0-k-j) -$$

$$- \sum_{j=0}^{k-1} n_-(j,n_0-k-1-j)\Big(R(k-j) + \sum_{i=1}^{n_0-2k-1} R(k-j+i)\,{}^t n_+(n_0-k-1-i,i)\Big) +$$

$$+ n_-(k,n_0-2k-1)\Big(-V_+(n_0-k-1) + \sum_{i=1}^{k} R(n_0-k-i)\,{}^t n_+(i-1,n_0-k-i)\Big) +$$

$$+ V_-(n_0-k-1)\,{}^t n_+(k,n_0-2k-1)\Big)\,{}^t\delta_-(n_0-k)$$

$$= \mathrm{II}^{(k+1)}_{n_0} + I^{(k+1)}_{n_0}\,{}^t\delta_-(n_0-k)$$

$$= 0 ,$$

noting that $k+1 \le N_0$. Consequently, by (4.20) and (4.21), we see that $\mathrm{II}^{(k+1)}_{n_0+1} = 0$. (Q.E.D.)

Finally, we will show $(*)''_{n_0,N_0}$, that is,

(Step 13) $\quad IV^{(N_0)}_{n_0+1} = 0 .$

Proof. Since it follows from (4.8) and (4.9) that

$$V_+(N_0) + \sum_{j=0}^{N_0-1} V_+(N_0-1-j)\,{}^t\delta_-(N_0-j)\,{}^t\delta_+(N_0-j)$$

$$= V_-(N_0) + \sum_{j=0}^{N_0-1} \delta_-(N_0-j)\delta_+(N_0-j)V_-(N_0-1-j)$$

$$= R(0) ,$$

we see from (ii) in Step 6 that Step 13 holds. (Q.E.D.)

Thus we have completed the proof of Lemma 4.3. By noting Steps 1, 2 and 3 in the proof of Lemma 4.3, we have

Theorem 4.1. For any $n \in \mathbb{N}^*$,

(i) $V_+(n+1) = V_+(n) - \delta_+(n+1)V_-(n)^t\delta_+(n+1)$

(ii) $V_+(n+1) = (I-\delta_+(n+1)\delta_-(n+1))V_+(n)$

(iii) $V_-(n+1) = V_-(n) - \delta_-(n+1)V_+(n)^t\delta_-(n+1)$

(iv) $V_-(n+1) = (I-\delta_-(n+1)\delta_+(n+1))V_-(n)$

(v) $V_+(n)^t\delta_-(n+1) = \delta_+(n+1)V_-(n)$

(vi) $V_+(n+1)^t\delta_-(n+1) = \delta_+(n+1)V_-(n+1)$.

By Theorems 3.1 and 4.1, we find that the parameters $\gamma_+(\cdot,*)$, $\gamma_-(\cdot,*)$, $V_+(\cdot)$ and $V_-(\cdot)$ in KM_2O-Langevin equations (2.16) and (2.17) can be uniquely determined by $R(0)$, $\delta_+(\cdot)$ and $\delta_-(\cdot)$. We call the system $\{\gamma_+(n,k),\delta_+(m),V_+(m)$; $m,n,k\in\mathbb{N}$, $n>k\}$ (resp.$\{\gamma_-(n,k)$, $\delta_-(m),V_-(m)$; $m,n,k\in\mathbb{N}$, $n>k\}$) the forward (resp. backward) KM_2O-Langevin data associated with $\mathbf{X}$.

For future use in a modelling of time series, we will rewrite Lemma 4.2 into

Theorem 4.2. For any $n \in \mathbb{N}^*$,

(i) $\delta_+(n+1) = -(R(n+1) + \sum_{k=0}^{n-1} \gamma_+(n,k)R(k+1))V_-(n)^{-1}$

(ii) $\delta_-(n+1) = -({}^tR(n+1) + \sum_{k=0}^{n-1} \gamma_-(n,k)^tR(k+1))V_+(n)^{-1}$.

§5. The prediction formula and prediction error

For each $m,n \in \mathbb{N}$, $m \geq n$, we define a forward (resp. backward) prediction matrix $P_+(m,n)$ (resp. $P_-(m,n)$) $\in M(d;\mathbb{R})$ and a forward (resp. backward) prediction error matrix $e_+(m,n)$ (resp. $e_-(m,n)$) $\in M(d;\mathbb{R})$ by

$$P_+(m,n) = E(X(m)\,{}^t I_+(n))V_+(n)^{-1/2} \tag{5.1}$$

$$P_-(m,n) = E(X(-m)\,{}^t I_-(n))V_-(n)^{-1/2} \tag{5.2}$$

$$e_+(m,n) = E((X(m)-P_{M_0^+(n)}X(m))\,{}^t(X(m)-P_{M_0^+(n)}X(m))) \tag{5.3}$$

$$e_-(m,n) = E((X(-m)-P_{M_0^-(n)}X(-m))\,{}^t(X(-m)-P_{M_0^-(n)}X(-m))) \,. \tag{5.4}$$

Theorem 5.1. For any $m,n \in \mathbb{N}$, $m \geq n$,

$$\text{(i)} \qquad X(m) = R(m)R(0)^{-1}X(0) + \sum_{k=1}^{m} P_+(m,k)V_+(k)^{-1/2}I_+(k)$$

$$\text{(ii)} \qquad P_{M_0^+(n)}X(m) = R(m)R(0)^{-1}X(0) + \sum_{k=1}^{n} P_+(m,k)V_+(k)^{-1/2}I_+(k)$$

$$\text{(iii)} \qquad e_+(m,n) = \sum_{k=n+1}^{m} P_+(m,k)\,{}^t P_+(m,k) \,.$$

Proof. We put

W = the left hand side in (i) - the right hand side in (i).

By (2.20), we see that each component of W belongs to the linear hull of $\{I_{+j}(\ell);1\leq j\leq d,0\leq \ell\leq m\}$. It follows from (2.18) and (5.1) that $E(W\,{}^t I_+(\ell)) = 0 (0\leq \ell\leq m)$. Hence, we find that $W = 0$, implying (i). (ii) follows from (2.18), (2.20) and (i). By noting (2.18) again, we see that (iii) follows from (i) and (ii). (Q.E.D.)

Similarly, we obtain

Theorem 5.2. For any $m,n \in \mathbb{N}$, $m \geq n$,

(i) $$X(-m) = {}^{t}R(m)R(0)^{-1}X(0) + \sum_{k=1}^{m} P_{-}(m,k)V_{-}(k)^{-1/2}I_{-}(k)$$

(ii) $$P_{M_0^{-}(n)} X(-m) = {}^{t}R(m)R(0)^{-1}X(0) + \sum_{k=1}^{n} P_{-}(m,k)V_{-}(k)^{-1/2}I_{-}(k)$$

(iii) $$e_{-}(m,n) = \sum_{k=n+1}^{m} P_{-}(m,k)\,{}^{t}P_{-}(m,k) \ .$$

We will show that the prediction matrices $P_{+}(\cdot,*)$ and $P_{-}(\cdot,*)$ can be also determined by $R(0)$, $\delta_{+}(\cdot)$ and $\delta_{-}(\cdot)$ through the following

Theorem 5.3. For any $m,n \in \mathbb{N}$, $m > n$,

(i) $$P_{+}(n,n) = V_{+}(n)^{1/2}$$
$$P_{+}(m,n) = -\sum_{k=n}^{m-1} \gamma_{+}(m,k)P_{+}(k,n)$$

(ii) $$P_{-}(n,n) = V_{-}(n)^{1/2}$$
$$P_{-}(m,n) = -\sum_{k=n}^{m-1} \gamma_{-}(m,k)P_{-}(k,n) \ .$$

Proof. By (2.16), (2.18), (2.20) and (5.1), $P_{+}(n,n) = V_{+}(n)^{1/2}$. By substituting (2.16) replaced n by m into the right hand side of (5.1), we find from (2.18) and (2.20) that the second statement in (i) holds. (ii) is also proved similarly. (Q.E.D.)

By using Theorem 4.1(ii)(iv), we see from Theorems 5.1(iii), 5.2 (iii) and 5.3 that the one-step prediction error matrices $e_{+}(n+1,n)$ and $e_{-}(n+1,n)$ can be explicitly represented by $R(0), \delta_{+}(\mathfrak{L})$ and $\delta_{-}(\mathfrak{L})(1 \leq \mathfrak{L} \leq n+1)$ through the following

Theorem 5.4. For any $n \in \mathbb{N}$,

(i) $$e_{+}(n+1,n) = (I-\delta_{+}(n+1)\delta_{-}(n+1))\cdots(I-\delta_{+}(1)\delta_{-}(1))R(0)$$

(ii) $$e_{-}(n+1,n) = (I-\delta_{-}(n+1)\delta_{+}(n+1))\cdots(I-\delta_{-}(1)\delta_{+}(1))R(0) \ .$$

§6. A construction theorem

As a converse setting of §2~§5, we are given a system $\{V,\delta_+(n);n\in\mathbb{N}\}$ of members in $M(d;\mathbb{R})$ such that

(6.1) V is a symmetric positive definite

(6.2) $\delta_+(n) \in M(d;\mathbb{R}) \quad (n \in \mathbb{N})$.

Then we construct a triple $(V_+(1),\delta_-(1),V_-(1))$ by

$$(6.3)\quad \begin{cases} V_+(1) = V - \delta_+(1)V^t\delta_+(1) \\ \delta_-(1) = V^t\delta_+(1)V^{-1} \\ V_-(1) = V - \delta_-(1)V^t\delta_-(1) \ . \end{cases}$$

In order to continue the following construction of $(V_+(n),\delta_-(n),V_-(n))$ from $(V_+(n-1),\delta_-(n-1),V_-(n-1))(n \in \mathbb{N})$:

$$(6.4)\quad \begin{cases} V_+(n) = V_+(n-1) - \delta_+(n)V_-(n-1)^t\delta_+(n) \\ \delta_-(n)V_+(n-1) = V_-(n-1)^t\delta_+(n) \\ V_-(n) = V_-(n-1) - \delta_-(n-1)V_+(n-1)^t\delta_-(n-1) \ , \end{cases}$$

we suppose that

(6.5) $V_+(n-1) \in GL(d;\mathbb{R}) \quad (n \in \mathbb{N})$,

where $V_+(0) = V_-(0) = V$.

Next, we construct a system $\{\gamma_+(m,n),\gamma_-(m,n);m,n\in\mathbb{N}^*,m>n\}$ of members in $M(d;\mathbb{R})$ according to the algorithm in Theorem 3.1.

Furthermore, we prepare any $\mathbb{R}^d$-valued stochastic process $I_+ = (I_+(n);n\in\mathbb{N}^*)$ on a probability space $(\Omega,\mathcal{B},P)$ such that

(6.6) $E(I_+(n)) = 0 \quad (n \in \mathbb{N}^*)$

(6.7) $E(I_+(n)^tI_+(m)) = \delta_{nm}V_+(n) \quad (n,m \in \mathbb{N}^*)$

and then construct an $\mathbb{R}^d$-valued stochastic process $X_+ = (X(n);n\in\mathbb{N}^*)$ by

(6.8) $X(0) = I_+(0)$

$$(6.9)\qquad X(n) = -\sum_{k=0}^{n-1} \gamma_+(n,k)X(k) + I_+(n) \qquad (n \in \mathbb{N}) .$$

Now we will show

Theorem 6.1. X_+ is weakly stationary.

For any $m,n \in \mathbb{N}^*$, we put

$$(6.10)\qquad R(m,n) = E(X(m)\,{}^tX(n)) .$$

Since

$$(6.11)\qquad {}^tR(m,n) = R(n,m) ,$$

for the proof of Theorem 6.1, it suffices to show

$$(6.12)\qquad R(m+\ell,n+\ell) = R(m,n) \quad (m \geq n \geq 0 , \ \ell \geq 1) .$$

The proof is divided into 22 steps.

(Step 1) $E(X(m)\,{}^tI_+(n)) = \delta_{mn}V_+(n) \qquad (0 \leq m \leq n) .$

This follows immediately from (6.7), (6.8) and (6.9).

(Step 2) (i) $R(0,0) = R(1,1) = R(2,2) = V$

(ii) $R(1,0) = R(2,1) = -\delta_+(1)V$

(iii) $R(2,0) = \delta_+(1)\delta_+(1)V - \delta_+(2)V_-(1) .$

These are shown by a simple calculation.

(Step 3) (i) $\delta_-(n+1)V_+(n) = V_-(n)\,{}^t\delta_+(n+1) \qquad (n \in \mathbb{N}^*)$

(ii) $\delta_-(n)V_+(n) = V_-(n)\,{}^t\delta_-(n) \qquad (n \in \mathbb{N}) .$

Proof. (i) is included in (6.4) and (ii) is proved similarly to Step 7 in the proof of Lemma 4.3. (Q.E.D.)

(Step 4) (i) $$R(n,\ell-1) = -\sum_{k=0}^{n-1} \eta_+(k,n-k)R(k,\ell-1) \qquad (n \geq \ell \geq 1)$$

(ii) $$R(n,n) = -\sum_{k=0}^{n-1} \eta_+(k,n-k)R(k,n) + V_+(n) \qquad (n \geq 1)$$

(iii) $$V_+(n) = R(n,n) + \sum_{k=1}^{n-1} \eta_+(k-1,n-k)R(k,n) + \delta_+(n)\Big(\sum_{k=1}^{n-1} \eta_-(n-k-1,k)R(k,n) + R(0,n)\Big) \qquad (n \geq 2) ,$$

where $(\eta_+(m,n),\ \eta_-(m,n)\ ;\ m \in \mathbb{N}^*,\ n \in \mathbb{N})$ is defined by (3.6) and (3.7).

Proof. By multiplying both hand sides of (6.9) by ${}^tX(\ell-1)$ (resp. ${}^tX(n)$) and then taking an expectation with respect to the probability P, we have (i) (resp. (ii)). (iii) follows from (ii) and the algorithm in Theorem 3.1'. (Q.E.D.)

(Step 5) (i) $R(n,0) = -\delta_+(n)V_-(n-1) - \sum_{k=1}^{n-1} \eta_+(k-1,n-k)R(k,0)$ $(n \in \mathbb{N})$

(ii) $V_-(n-1) = R(0,0) + \sum_{k=1}^{n-1} R(0,k)\,{}^t\eta_-(n-1-k,k)$ $(n \in \mathbb{N})$

(iii) $R(n,0) = -\delta_+(n)V - \sum_{k=1}^{n-1} \eta_+(k,n-k)R(k,0)$ $(n \in \mathbb{N})$.

Proof. By a mathematical induction about n, we show (i) and (ii). By Step 2, (i) and (ii) for $n = 1$ hold. Let us assume that (i) and (ii) hold for $n = n_0$. Since $V_-(n_0)$ is symmetric, by (6.4) (6.11) and (i) for $n = n_0$,

$$V_-(n_0) = V_-(n_0-1) - V_-(n_0-1)\,{}^t\delta_+(n_0)\,{}^t\delta_-(n_0)$$

$$= V_-(n_0-1) + \Big(R(0,n_0) + \sum_{k=1}^{n_0-1} R(0,k)\,{}^t\eta_+(k-1,n_0-k)\Big)\,{}^t\delta_-(n_0) .$$

On the other hand, by Theorem 3.1',

$$R(0,0) + \sum_{k=1}^{n_0} R(0,k)\,{}^t\eta_-(n_0-k,k)$$

$$= R(0,0) + \sum_{k=1}^{n_0-1} R(0,k)\,{}^t\eta_-(n_0-1-k,k) +$$

$$+ \Big(R(0,n_0) + \sum_{k=1}^{n_0-1} R(0,k)\,{}^t\eta_+(k-1,n_0-k)\Big)\,{}^t\delta_-(n_0) .$$

Hence, we have (ii) for $n = n_0+1$. Next, by (6.9) and Theorem 3.1',

$$R(n_0+1,0) = -\sum_{k=0}^{n_0} \eta_+(k,n_0+1-k)R(k,0)$$

$$= -\sum_{k=1}^{n_0} n_+(k-1,n_0+1-k)R(k,0) -$$

$$- \delta_+(n_0+1)(R(0,0) + \sum_{k=1}^{n_0} n_-(n_0-k,k)R(k,0)) ,$$

which, together with (ii) for $n = n_0$, implies (i) for $n = n_0+1$. By substituting (ii) into (i), we see from Theorem 3.1' that (iii) holds. (Q.E.D.)

Before proceeding to the next step, we put the following statements : for each $n \in \mathbb{N}^*$,

$$(6.13)_n \qquad \sum_{k=0}^{n-1} n_+(k,n-k)R(k+1,0) = \sum_{k=0}^{n-1} R(k+1,0)\,{}^t n_-(k,n-k)$$

$$(6.14)_n \qquad V_+(n) = R(0,0) + \sum_{k=1}^{n} R(k,0)\,{}^t n_+(n-k,k)$$

$$(6.15)_n \qquad R(0,n) = -\delta_-(n)V_+(n-1) - \sum_{k=1}^{n-1} n_-(k-1,n-k)R(0,k)$$

$$(6.16)_n \qquad R(0,n) = -\delta_-(n)V - \sum_{k=1}^{n-1} n_-(k,n-k)R(0,k) .$$

(Step 6) For any fixed $n \in \mathbb{N}$,

(i) If $(6.13)_n$ holds, then $(6.15)_{n+1}$ does.

(ii) If $(6.14)_{n-1}$ and $(6.15)_n$ hold, then $(6.14)_n$ does.

(iii) If $(6.14)_n$ and $(6.15)_n$ hold, then $(6.16)_n$ does.

Proof. By (6.9) and Theorem 3.1',

$$R(0,n+1) = -\sum_{k=0}^{n} R(0,k)\,{}^t n_+(k,n+1-k)$$

$$= -\sum_{k=1}^{n} R(0,k)\,{}^t n_+(k-1,n+1-k) -$$

$$- (R(0,0) + \sum_{k=1}^{n} R(0,k)\,{}^t n_-(n-k,k))\,{}^t\delta_+(n+1) .$$

And so by (ii) in Step 5,

$$R(0,n+1) = -{}^t(\sum_{k=0}^{n-1} n_+(k,n-k)R(k+1,0)) - V_-(n)\,{}^t\delta_+(n+1) .$$

Therefore, we have (i). (ii) and (iii) are proved similarly as in the proof of (ii) and (iii) in Step 5. (Q.E.D.)

We will show $(6.13)_n$ and $(6.14)_n$ $(n \in \mathbb{N})$ by a mathematical induction about n in Step 7~Step 17. Immediately from (i) and (ii) in Step 2, $(6.13)_1$ and $(6.14)_1$ hold. For any fixed $n_0 \in \{2,3,\cdots\}$, let us assume that $(6.13)_\ell$ and $(6.14)_\ell$ hold for any $\ell \in \{1,2,\cdots,n_0-1\}$.

(Step 7) If $(6.13)_{n_0}$ holds, then $(6.14)_{n_0}$ does.

This follows from (i) and (ii) in Step 6.

(Step 8) (i) the left hand side of $(6.13)_{n_0} = -\sum_{k=1}^{n_0} C_{n_0}(k)\,{}^t\delta_-(k)$

(ii) the right hand side of $(6.13)_{n_0} = \sum_{k=1}^{n_0} D_{n_0}(k)\,{}^t\delta_-(k)$,

where

$$(6.17)\quad C_{n_0}(k) = \sum_{\ell=k}^{n_0-1} n_+(\ell,n_0-\ell)\Big(R(\ell+1-k,0) + \sum_{m=0}^{k-2} R(\ell-m,0)\,{}^t n_+(m,k-1-m)\Big) + n_+(k-1,n_0-k+1)V_+(k-1)$$

$$(6.18)\quad D_{n_0}(k) = R(n+1-k) + \sum_{m=0}^{k-2} R(n_0-m,0)\,{}^t n_+(m,k-1-m) .$$

Proof. Since it follows from (i) in Step 6 that $(6.15)_\ell$ $(1 \le \ell \le n_0)$ hold, we get

$$(6.19)\quad \text{the left hand side of } (6.13)_{n_0} = -\sum_{k=1}^{n_0-1} n_+(k,n_0-k)\Big(V_+(k)\,{}^t\delta_-(k+1) + \sum_{\ell=1}^{k} R(\ell,0)\,{}^t n_-(\ell-1,k+1-\ell)\Big) - \delta_+(n_0)V\,{}^t\delta_-(1) .$$

By Theorem 3.1',

$$(6.20)\quad n_-(\ell-1,k+1-\ell) = \delta_-(k+1-\ell) + \sum_{j=1}^{\ell-1} \delta_-(k+1-\ell+j)\,n_+(k-\ell,j)$$

$$(6.21)\quad n_-(k,n_0-k) = \delta_-(n_0-k) + \sum_{j=1}^{k} \delta_-(n_0-k+j)\,n_+(n_0-k-1,j) .$$

By substituting (6.20) (resp. (6.21)) into (6.19) (resp. the right hand side of $(6.13)_{n_0}$), we have (i)(resp. (ii)) in Step 8. (Q.E.D.)

(Step 9) For any $k \in \{1,2,\cdots,n_0\}$,

$$C_{n_0}(k) + D_n(k) = \sum_{\ell=0}^{k-1} \delta_+(n_0-\ell) E_{n_0}(\ell;k),$$

where for $\ell \in \{0,\cdots,k-3\}$,

(6.22) $E_{n_0}(\ell;k)$

$$= \sum_{i=0}^{n_0-k-1} n_-(i,n_0-\ell-1-i)R(n_0-k-i,0) + n_-(n_0-k,k-\ell-1)V_+(k-1) -$$
$$- \sum_{j=0}^{\ell} (R(\ell-j,0) + \sum_{i=1}^{k-\ell-1} n_-(n_0-\ell-1-i,i)R(\ell-j+i,0))\,{}^t n_+(j,k-1-j) +$$
$$+ \sum_{j=\ell+1}^{k-2} \Big(\sum_{i=0}^{n_0-k-1} n_-(i,n_0-\ell-1-i)R(n_0-j-1-i,0)\Big)\,{}^t n_+(j,k-1-j)$$

(6.23) $E_{n_0}(k-2;k)$

$$= \sum_{i=0}^{n_0-k-1} n_-(i,n_0-k+1-i)R(n_0-k-i,0) + n_-(n_0-k,1)V_+(k-1) -$$
$$- \sum_{j=0}^{k-2} (R(k-2-j,0) + n_-(n_0-k,1)R(k-1-j,0))\,{}^t n_+(j,k-1-j)$$

(6.24) $$E_{n_0}(k-1;k) = V_+(k-1) - (V + \sum_{j=0}^{k-2} R(k-1-j,0)\,{}^t n_+(j,k-1-j)).$$

By (6.17),

(6.25) $$C_{n_0}(k) = n_+(k-1,n_0-k+1)V_+(k-1) + \sum_{\ell=k}^{n_0-1} n_+(\ell,n_0-\ell)R(\ell+1-k,0) +$$
$$+ \sum_{m=0}^{k-2} \Big(\sum_{\ell=k}^{n_0-1} n_+(\ell,n_0-\ell)R(\ell-m,0)\Big)\,{}^t n_+(m,k-1-m).$$

Moreover, it follows from Theorem 3.1' and (iii) in Step 5 that for any $m \in \{0,1,\cdots,k-1\}$,

(6.26) $$\sum_{\ell=k}^{n_0-1} n_+(\ell,n_0-\ell)R(\ell-m,0)$$

$$= \sum_{j=0}^{m-1} \delta_+(n_0-j)\Big(\sum_{i=0}^{n_0-k-1} n_-(i,n_0-1-j-i)R(n_0-1-m-i,0)\Big) -$$

$$- R(n_0-m,0) - \sum_{j=0}^{k-1-m} n_+(j,n_0-m-j)R(j,0) .$$

By substituting (6.26) into (6.25), we find from (6.18) that

$$(6.27)\quad C_{n_0}(k) = -D_{n_0}(k) + n_+(k-1,n_0-k+1)V_+(k-1) - \delta_+(n_0-k+1)V +$$

$$+ \sum_{m=0}^{k-2}\Big\{ \sum_{j=0}^{m-1} \delta_+(n_0-j)\Big(\sum_{i=0}^{n_0-k-1} n_-(i,n_0-1-j-i)R(n_0-1-m-i,0)\Big) -$$

$$- \sum_{j=0}^{k-1-m} n_+(j,n_0-m-j)R(j,0)\Big\}\, {}^t n_+(m,k-1-m) +$$

$$+ \sum_{j=0}^{m-1} \delta_+(n_0-j)\Big(\sum_{i=0}^{n_0-k-1} n_-(i,n_0-1-j-i)R(n_0-k-i,0)\Big) .$$

Further, it follows from Theorem 3.1' that for any $m \in \{0,1,\cdots,k-2\}$,

$$(6.28)\quad n_+(k-1,n_0-k+1) = \delta_+(n_0-k+1) + \sum_{j=1}^{k-1} \delta_+(n_0-k+1+j)n_-(n_0-k,j)$$

$$(6.29)\quad \sum_{j=0}^{k-1-m} n_+(j,n_0-m-j)R(j,0)$$

$$= \sum_{j=0}^{k-1-m} \delta_+(n_0-m-j)\Big(R(j,0) + \sum_{i=j+1}^{k-1-m} n_-(n_0-m-i-1,i-j)R(i,0)\Big) .$$

Thus, by substituting (6.28) and (6.29) into (6.27), we can see that Step 9 holds. (Q.E.D.)

(Step 10) $\quad E_{n_0}(k-1;k) = 0 \quad$ for any $k \in \{1,\cdots,n_0\}$.

This follows from the assumption of mathematical induction that $(6.14)_{k-1}$ holds.

(Step 11) $\quad E_{n_0}(k-2;k) = 0 \quad$ for any $k \in \{2,\cdots,n_0\}$.

Proof. By (6.23) and $(6.14)_{k-1}$,

$$(6.30)\quad E_{n_0}(k-2;k) = \sum_{i=0}^{n_0-k-1} n_-(i,n_0-k+1-i)R(n_0-k-i,0) + n_-(n_0-k,1)V -$$

$$- \sum_{j=0}^{k-2} R(k-2-j,0)\, {}^t n_+(j,k-1-j) .$$

By Theorem 3.1' and (iii) in Step 5,

$$\sum_{i=0}^{n_0-k-1} n_-(i,n_0-k+1-i)R(n_0-k-i,0)$$

$$= -\delta_-(n_0-k+1)\delta_+(n_0-k)V + \sum_{i=0}^{n_0-k-3} n_-(i,n_0-k-1)R(n_0-k-2-i,0) +$$

$$+ \delta_-(n_0-k)(R(n_0-k-1,0) + \sum_{i=1}^{n_0-k-2} n_+(n_0-k-1-i,i)R(n_0-k-1-i,0)) .$$

By repeating the same procedure, we have

$$\sum_{i=0}^{n_0-k-1} n_-(i,n_0-k+1-i)R(n_0-k-i,0) = -(\sum_{i=1}^{n_0-k} \delta_-(i+1)\delta_+(i))V \tag{6.31}$$

$$= (-n_-(n_0-k,1) + \delta_-(1))V .$$

On the other hand, by Theorem 3.1',

$$\sum_{j=0}^{k-2} R(k-2-j,0)\,{}^t n_+(j,k-1-j)$$

$$= (R(k-2,0) + V\,{}^t\delta_-(k-2) + \sum_{j=1}^{k-3} R(j,0)\,{}^t n_-(j,k-2-j))\,{}^t\delta_+(k-1) +$$

$$+ \sum_{j=0}^{k-3} R(k-3-j,0)\,{}^t n_+(j,k-2-j)$$

and so by (6.16)$_{k-2}$

$$\sum_{j=0}^{k-2} R(k-2-j,0)\,{}^t n_+(j,k-1-j) = \sum_{j=0}^{k-3} R(k-3-j,0)\,{}^t n_+(j,k-2-j) .$$

By repeating the same procedure,

$$\sum_{j=0}^{k-2} R(k-2-j,0)\,{}^t n_+(j,k-1-j) = \delta_-(1)V . \tag{6.32}$$

Thus, by (6.30), (6.31) and (6.32), we have Step 11. (Q.E.D.)

(Step 12) For any $\ell \in \{0,1,\cdots,k-2\}$,

$$E_{n_0}(\ell;k) = \sum_{i=0}^{n_0-k} n_-(i,n_0-\ell-i)R(n_0-k-i,0) +$$

$$- \sum_{j=0}^{\ell} (R(\ell-j,0) + \sum_{i=1}^{k-\ell-2} n_-(n_0-\ell-1-i,i)R(\ell-j+i,0))\,{}^t n_+(j,k-1-j) +$$

$$- \sum_{j=\ell+1}^{k-2} (\sum_{i=j-\ell}^{k-\ell-2} n_-(n_0-\ell-1-i,i)R(i-j+\ell,0))\,{}^t n_+(j,k-1-j) +$$

$$+\sum_{j=\ell+1}^{k-2}\Big(\sum_{i=j-\ell}^{n_0-\ell-1} n_-(n_0-\ell-1-i,i)R(i-j+\ell,0)\Big)\,{}^t n_+(j,k-1-j)\ .$$

Proof. By substituting $(6.14)_{k-1}$ into (6.22), we have

$$E_{n_0}(\ell;k) = \sum_{i=0}^{n_0-k} n_-(i,n_0-\ell-1-i)R(n_0-k-i,0) - \tag{6.33}$$
$$-\sum_{j=0}^{\ell}\Big(R(\ell-j,0)+\sum_{i=1}^{k-\ell-2} n_-(n_0-\ell-1-i,i)R(\ell-j+i,0)\Big)\,{}^t n_+(j,k-1-j) +$$
$$+\sum_{j=\ell+1}^{k-2}\Big(\sum_{i=0}^{n_0-k} n_-(i,n_0-\ell-1-i)R(n_0-j-i-1,0)\Big)\,{}^t n_+(j,k-1-j)\ .$$

Since

$$\sum_{i=j-\ell}^{k-\ell-2} n_-(n_0-\ell-1-i,i)R(i-j+\ell,0) + \sum_{i=0}^{n_0-k} n_-(i,n_0-\ell-1-i)R(n_0-j-i-1,0)$$
$$= \sum_{i=j-\ell}^{n_0-\ell-1} n_-(n_0-\ell-1-i,i)R(i-j+\ell,0)\ ,$$

it follows from (6.33) that Step 12 holds. (Q.E.D.)

For each $\ell,m \in \mathbb{N}$, put

$$F(m,\ell) = \sum_{i=0}^{m} n_-(m-i,\ell+i)R(i,0)\ . \tag{6.34}$$

(Step 13) (i) $F(m,1) = \delta_-(1)V$

(ii) $F(1,\ell) = (n_-(1,\ell) - \delta_-(\ell+1)\delta_+(1))V$

(iii) $F(m,\ell) = F(m-1,\ell) + \delta_-(m+\ell)\sum_{j=1}^{\ell-1}\delta_+(m+j)F(m-1,j)\ .$

Proof. (i) and (ii) follow from (6.31) and Step 2, respectively. By using Theorem 3.1' repeatedly,

$$F(m,\ell) = F(m-1,\ell) + \delta_-(m+\ell)\Big(\sum_{i=0}^{m-1} n_+(\ell+i-1,m-i)R(i,0) + R(m,0)\Big)$$
$$= F(m-1,\ell) + \delta_-(m+\ell)\sum_{j=1}^{\ell-1}\delta_+(m+j)F(m-1,j) +$$
$$+\delta_-(m+\ell)\Big(\sum_{j=0}^{m-1} n_+(j,m-j)R(j,0) + R(m,0)\Big)\ ,$$

which, together with (iii) in Step 5, implies (iii). (Q.E.D.)

For each $\ell \in \{0,\cdots,k-2\}$, put

$$(6.35)\qquad G(\ell;k) = \sum_{i=0}^{\ell} R(\ell-i,0)\,{}^t n_+(i,k-1-i) .$$

(Step 14) For any $\ell \in \{0,\cdots,k-2\}$,

$$E_{n_0}(\ell;k) = -G(\ell;k) - \sum_{i=1}^{k-\ell-2} n_-(n_0-\ell-1-i,i)G(\ell+i;k) +$$
$$+ F(n_0-k,k-\ell-1) + \sum_{i=1}^{k-\ell-2} F(n_0-k+i,k-\ell-1-i)\,{}^t n_+(k-1-i,i) .$$

This follows from Step 12, (6.34) and (6.35).

(Step 15) For any $k \in \{2,\cdots,n_0\}$ and any $\ell \in \{0,\cdots,k-2\}$,

$$E_{n_0}(\ell;k) = E_{n_0-1}(\ell;k) + \delta_-(n_0-\ell-1)\sum_{j=2}^{k-\ell-1} \delta_+(n_0-\ell-j)E_{n_0-1}(\ell-1+j;k) .$$

Proof. By (iii) in Step 13,

$$(6.36)\qquad F(n_0-k,k-\ell-1) + \sum_{i=1}^{k-\ell-2} F(n_0-k+i,k-\ell-1-i)\,{}^t n_+(k-1-i,i)$$
$$= F(n_0-k-1,k-\ell-1) + \delta_-(1)V\,{}^t n_+(\ell+1,k-\ell-2) +$$
$$+ \sum_{i=1}^{k-\ell-3} F(n_0-k+i-1,k-\ell-1-i)\,{}^t n_+(k-1-i,i) +$$
$$+ \delta_-(n_0-\ell-1)\sum_{j=2}^{k-\ell-2} \delta_+(n_0-\ell-j)(F(n_0-k-1,k-\ell-j) +$$
$$+ \sum_{i=1}^{k-\ell-1-j} F(n_0-k+i-1,k-\ell-j-i)\,{}^t n_+(k-1-i,i)) +$$
$$+ \delta_-(n_0-\ell-1)\delta_+(n_0-k+1)F(n_0-k-1,1) .$$

On the other hand, by Theorem 3.1'.

$$n_-(n_0-\ell-1-i,i) = n_-(n_0-\ell-2-i,i) + \delta_-(n_0-\ell-1)\delta_+(n_0-\ell-1-i) +$$
$$+ \delta_-(n_0-\ell-1)\sum_{j=1}^{i-1} \delta_+(n_0-\ell-1-j)n_-(n_0-\ell-2-i,j)$$

and so

$$(6.37)\qquad \sum_{i=1}^{k-\ell-2} n_-(n_0-\ell-1-i,i)G(\ell+i;k)$$

$$= \sum_{i=1}^{k-\ell-2} n_-(n_0-\ell-2-i,i)G(\ell+i;k) + \delta_-(n_0-\ell-1)\delta_+(n_0-k+1)G(k-2;k) +$$
$$+ \delta_-(n_0-\ell-1) \sum_{j=1}^{k-\ell-3} \delta_+(n_0-\ell-1-j)(G(\ell+j;k) +$$
$$+ \sum_{i=j+1}^{k-\ell-2} n_-(n_0-\ell-2-i,j)G(\ell+i;k)) .$$

Hence, by combining Step 14 with (6.36) and (6.37), we have Step 15.

(Q.E.D.)

(Step 16) For any $k \in \{2,\cdots,n_0\}$ and any $\ell \in \{0,\cdots,k-2\}$,

$$E_{n_0}(\ell ; k) = 0 .$$

Proof. By Step 15, it suffices to show that $E_2(0;2) = 0$, which is proved in Step 11. (Q.E.D.)

(Step 17) For any $n \in \mathbb{N}$, $(6.13)_n$ and $(6.14)_n$ hold.

This follows from Step 8~Step 16.

Finally we will show (6.12) by a mathematical induction such that for any fixed $n_0 \in \{2,3,\cdots\}$,

if

(6.38) $R(j,k) = R(j-k,0)$ for any $j,k \in \mathbb{N}^*$, $0 \le k \le j \le n_0$,

then

(6.39) $R(n_0+1,k) = R(n_0+1-k,0)$ for any $k \in \{0,\cdots,n_0+1\}$.

(Step 18) For any $k \in \{1,\cdots,n_0\}$,

$$R(n_0+1,k) = R(n_0,k-1) - \delta_+(n_0+1)H(k;n_0) ,$$

where

(6.40) $$H(k;n) = R(0;k) + \sum_{j=0}^{n-1} n_-(j,n-j)R(n-k,j) \quad (1 \le k \le n) .$$

Proof. By multiplying both hand sides of (6.9) replaced n by n_0+1 by ${}^tX(k)$ and then taking an expectation with respect to P , it follows from Step 1 and Theorem 3.1' that

$$R(n_0+1,k) = -\delta_+(n_0+1)(R(0,k) + \sum_{j=1}^{n_0} n_-(n_0-j,j)R(j,k)) -$$
$$- \delta_+(n_0)R(1,k) - \sum_{j=1}^{n_0-1} n_+(j,n_0-j)R(j+1,k) .$$

And so by (6.11) and the assumption (6.38) ,

$$R(n_0+1,k) = -\delta_+(n_0+1)(R(0,k) + \sum_{j=1}^{n_0} n_-(n_0-j,j)R(n_0-k,n_0-j)) -$$
$$- \delta_+(n_0)R(0,k-1) - \sum_{j=1}^{n_0-1} n_+(j,n_0-j)R(j,k-1)$$
$$= -\delta_+(n_0+1)H(k;n_0) - \delta_+(n_0)R(0,k-1) -$$
$$- \sum_{j=1}^{n_0-1} n_+(j,n-j)R(j,k-1) ,$$

which, together with (i) in Step 4, yields Step 18. (Q.E.D.)

(Step 19) $H(m;m) = 0$ for any $m \in \mathbb{N}$.

Proof. By Steps 6 and 17, we can apply $(6.16)_m$ to get Step 19.

(Q.E.D.)

(Step 20) For any $\ell \in \{1,\cdots,n_0\}$ and any $k \in \{1,\cdots,\ell-1\}$,

$$H(k;\ell) = H(k;\ell-1) + \delta_-(\ell) \sum_{j=1}^{k-1} \delta_+(\ell-j)H(k-j,\ell-1-j) .$$

Proof. By the assumption (6.38),

$$H(k;\ell) = R(0,k) + \sum_{j=0}^{\ell-k} n_-(j,\ell-j)R(\ell-k,j) + \sum_{j=\ell-k+1}^{\ell-1} n_-(j,\ell-j)R(\ell-k,j)$$
$$= R(0,k) + \sum_{j=0}^{\ell-k} n_-(j,\ell-j)R(\ell-k-j,0) + \sum_{j=\ell-k+1}^{\ell-1} n_-(j,\ell-j)R(0,j-\ell+k) .$$

By Theorem 3.1',

$$H(k;\ell) = R(0,k) + \sum_{j=0}^{\ell-k-1} n_-(j,\ell-j-1)R(\ell-1-j,0) +$$
$$+ \sum_{j=\ell-k}^{\ell-2} n_-(j,\ell-1-j)R(0,j-\ell+1+k) +$$
$$+ \delta_-(\ell)(R(\ell-k,0) + \sum_{j=1}^{\ell-k} n_+(\ell-j-1,j)R(\ell-k-j,0) +$$

$$+ \sum_{j=\ell-k+1}^{\ell-1} \eta_+(\ell-j-1,j)R(0,j-\ell+k))$$

$$= H(k;\ell-1) + \delta_-(\ell)(R(\ell-k,0) + \sum_{j=1}^{\ell-k} \eta_+(\ell-j-1,j)R(\ell-k-j,0) +$$

$$+ \sum_{j=\ell-k+1}^{\ell-1} \eta_+(\ell-j-1,j)R(0,j-\ell+k)) \ .$$

By using Theorem 3.1' again,

$$H(k;\ell) = H(k;\ell-1) + \delta_-(\ell)\delta_+(\ell-1)H(k-1;\ell-1) +$$

$$+ \delta_-(\ell)(R(\ell-k,0) + \sum_{j=1}^{\ell-k} \eta_+(\ell-j-2,j)R(\ell-k-j,0) +$$

$$+ \sum_{j=\ell-k+1}^{\ell-2} \eta_+(\ell-j-2,j)R(0,j-\ell+k)) \ .$$

The same repetition yields that

$$H(k;\ell) = H(k,\ell-1) + \delta_-(\ell) \sum_{j=1}^{k-1} \delta_+(\ell-j)H(k-j;\ell-1-j) +$$

$$+ \delta_-(\ell)(R(\ell-k,0) + \sum_{j=1}^{\ell-k} \eta_+(\ell-k-j,j)R(\ell-k-j,0)) \ ,$$

which, together with (iii) in Step 5, implies Step 20. (Q.E.D.)

(Step 21) $R(n_0+1,k) = R(n_0,k-1)$ for any $k \in (1,\cdots,n_0)$.

Proof. By Steps 2 and 19, we can apply a mathematical induction to see that $H(k;\ell) = 0$ for any $\ell \in (1,\cdots,n_0)$ and any $k \in (1,\cdots,\ell)$. Hence Step 21 follows from Step 18. (Q.E.D.)

(Step 22) $R(n_0+1,n_0+1) = R(0,0)$.

Proof. By multiplying both hand sides of (6.9) replaced n by n_0+1 by ${}^tX(n_0+1)$ and then taking an expectation with respect to P, it follows from Step 1 and the assumption (6.38) that

$$R(n_0+1,n_0+1) = -\delta_+(n_0+1)R(0,n_0+1) + V_+(n_0+1) +$$

$$+ \sum_{j=1}^{n_0} \eta_+(j,n_0+1-j)R(0,n_0+1-j) \ .$$

By Steps 6 and 17, we can substitute $(6.16)_{n_0+1}$ into the above to see that

$$R(n_0+1,n_0+1) = V_+(n_0+1) + \delta_+(n_0+1)\delta_-(n_0+1)V_+(n_0) -$$
$$- \sum_{j=1}^{n_0} (n_+(j,n_0+1-j) - \delta_+(n_0+1)n_-(n_0-j,j))R(0,n_0+1-j)$$

and so by Theorems 3.1' and 4.1(ii),

$$R(n_0+1,n_0+1) = V_+(n_0) - \sum_{j=1}^{n_0} n_+(j-1,n_0+1-j)R(0,n_0+1-j)$$
$$= V_+(n_0) - \sum_{j=1}^{n_0} n_+(n_0-j,j)R(0,j) .$$

Therefore, by Step 17, we can apply $(6.14)_{n_0+1}$ to the above to see that $R(n_0+1,n_0+1) = R(0,0)$. (Q.E.D.)

Thus, we have completed the proof of Theorem 6.1.

Added in proof After this paper was prepared, the author learned from Doctor of Engineering H. Sakai that the algorithm (i), (ii) in Theorem 3.1, (ii), (iv) in Theorem 4.1 and (i), (ii) in Theorem 4.2 has been derived for the fitting of a multi-dimensional autoregressive model [N. Levinson, J. Math. Phys, 25(1947), 261-278 ; J. Durbin, Biometrika, 46(1959), 306-316 ; P. Whittle, Biometrika, 50 (1963), 129-134 ; R. A. Wiggins and E. A. Robinson, J. Geophys. Res., 70(1965), 1885-1891]. This algorithm is called Levinson-Durbin algorithm for one-dimensional case and Levinson-Whittle-Wiggins-Robinson algorithm for multi-dimensional case. In particular, the fundamental relation in Lemma 4.3, and parameters $\delta_+(\cdot)$ and $\delta_-(\cdot)$ are called Burg's relation and partial autocorrelation coefficients, respectively. The role that autoregressive processes have in our theory of KM_2O-Langevin equations will be discussed in [27]. The author would like to thank Dr. Prof. H. Sakai for his communication.

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