Direct Proof of the Mirror Theorem for Projective Hypersurfaces up to Degree 3 Rational Curves

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Abstract

In this paper, we directly derive a generalized mirror transformation of projective hypersurfaces of up to degree 3, genus 0 Gromov-Witten invariants by comparing the Kontsevich's localization formula with residue integral representation of the virtual structure constants. We can easily generalize our method for the rational curves of arbitrary degree, except under combinatorial complexities.

1 Introduction

In this paper, we prove a generalized mirror transformation of the genus 0 Gromov-Witten invariants of degree k hypersurface in CP^{N-1}, (we denote the hypersurface by M_N^k), up to 3 degrees. For this purpose, we introduce the virtual structure constants of M_N^k, that were first defined in our work with A. Collino [2].

Definition 1 *The virtual structure constants $\tilde{L}_n^{N,k,d}$ ($d \leq 3$, $\tilde{L}_n^{N,k,d} \neq 0$ only if $0 \leq n \leq N-1-(N-k)d$) are rational numbers defined by the following initial condition and the recursive formula.*

$$\sum_{n=0}^{k-1} \tilde{L}_n^{N,k,1} w^n = k \cdot \prod_{j=1}^{k-1} (jw + (k-j)), \tag{1.1}$$

$$\tilde{L}_n^{N,k,1} = \tilde{L}_n^{N+1,k,1}, \tag{1.2}$$

$$\frac{\tilde{L}_n^{N,k,2}}{2} = \frac{1}{2} \cdot \frac{\tilde{L}_{n-1}^{N+1,k,2}}{2} + \frac{1}{2} \cdot \frac{\tilde{L}_n^{N+1,k,2}}{2} + \frac{1}{2} \tilde{L}_n^{N+1,k,1} \cdot \tilde{L}_{n+(N-k)}^{N+1,k,1}, \tag{1.3}$$

$$\begin{aligned}
\frac{\tilde{L}_n^{N,k,3}}{3} = & \frac{2}{9} \cdot \frac{\tilde{L}_{n-2}^{N+1,k,3}}{3} + \frac{5}{9} \cdot \frac{\tilde{L}_{n-1}^{N+1,k,3}}{3} + \frac{2}{9} \cdot \frac{\tilde{L}_n^{N+1,k,3}}{3} \\
& + \frac{4}{9} \cdot \frac{\tilde{L}_{n-1}^{N+1,k,2}}{2} \cdot \tilde{L}_{n+2(N-k)}^{N+1,k,1} + \frac{1}{3} \cdot \frac{\tilde{L}_n^{N+1,k,2}}{2} \cdot \tilde{L}_{n+2(N-k)}^{N+1,k,1} \\
& + \frac{2}{9} \cdot \frac{\tilde{L}_n^{N+1,k,2}}{2} \cdot \tilde{L}_{n+1+2(N-k)}^{N+1,k,1} \\
& + \frac{2}{9} \cdot \tilde{L}_{n-1}^{N+1,k,1} \cdot \frac{\tilde{L}_{n-1+(N-k)}^{N+1,k,2}}{2} + \frac{1}{3} \cdot \tilde{L}_n^{N+1,k,1} \cdot \frac{\tilde{L}_{n-1+(N-k)}^{N+1,k,2}}{2} \\
& + \frac{4}{9} \cdot \tilde{L}_n^{N+1,k,1} \cdot \frac{\tilde{L}_{n+(N-k)}^{N+1,k,2}}{2} \\
& + \frac{1}{3} \cdot \tilde{L}_n^{N+1,k,1} \cdot \tilde{L}_{n+(N-k)}^{N+1,k,1} \cdot \tilde{L}_{n+2(N-k)}^{N+1,k,1}.
\end{aligned} \tag{1.4}$$

In the mirror computation of Gromov-Witten invariants of M_N^k, $\tilde{L}_n^{N,k,d}$ is used to denote the B-model analogue of the 3-point Gromov-Witten invariant.

$$\frac{1}{k}\langle\mathcal{O}_{h^{N-2-n}}\mathcal{O}_{h^{n-1+(N-k)d}}\mathcal{O}_h\rangle_{0,d}=\frac{d}{k}\langle\mathcal{O}_{h^{N-2-n}}\mathcal{O}_{h^{n-1+(N-k)d}}\rangle_{0,d}=$$
$$=\frac{d}{k}\int_{\overline{M}_{0,2}(CP^{N-1},d)}c_{top}(R^0(\pi_*ev_3^*(\mathcal{O}(k))))\wedge ev_1^*(h^{N-2-n})\wedge ev_2^*(h^{n-1+(N-k)d}). \tag{1.5}$$

In (1.5), h is the hyperplane class of CP^{N-1}, $\overline{M}_{0,n}(CP^{N-1},d)$ represents the moduli space comprised of degree d stable maps from the genus 0 stable curve to CP^{N-1} with n marked points, $ev_i:\overline{M}_{0,n}(CP^{N-1},d)\to CP^{N-1}$ is the evaluation map of the i-th marked point, and $\pi:\overline{M}_{0,3}(CP^{N-1},d)\to\overline{M}_{0,2}(CP^{N-1},d)$ is the forgetful map. The definition of the virtual structure constants for arbitrary degree d (≥ 1) is found in [7]. In our previous paper [9], we conjectured a residue integral representation of $\tilde{L}_n^{N,k,d}$, which can be interpreted as a result of a localization computation on the moduli space of the Gauged Sigma Model. In the following, we prepare some notations to describe the formula we have conjectured. First, we define rational functions in u,v by:

$$\begin{aligned}e(k,d;u,v) &:= \prod_{m=0}^{kd}\left(\frac{mu+(kd-m)v}{d}\right),\\ t(N,d;u,v) &:= \prod_{m=1}^{d-1}\left(\frac{mu+(d-m)v}{d}\right)^N.\end{aligned} \tag{1.6}$$

Next, we introduce an ordered partition of positive integer d.

Definition 2 *Let OP_d be the set of ordered partitions of positive integer d.*

$$OP_d=\{\sigma_d=(d_1,d_2,\cdots,d_{l(\sigma_d)})\ |\ \sum_{j=1}^{l(\sigma_d)}d_j=d\ ,\ d_j\in\mathbf{N}\}. \tag{1.7}$$

We denote an ordered partition σ_d by $(d_1,d_2,\cdots,d_{l(\sigma_d)})$. In (1.7), we denote the length of the ordered partition σ_d by $l(\sigma_d)$.

The residue integral representation can now be given as:

$$\begin{aligned}\frac{\tilde{L}_n^{N,k,d}}{d} &= \frac{1}{k}\sum_{\sigma_d\in OP_d}\frac{1}{(2\pi\sqrt{-1})^{l(\sigma_d)+1}\prod_{j=1}^{l(\sigma_d)}d_j}\oint_{C_0}dx_{l(\sigma_d)}\cdots\oint_{C_0}dx_0(x_0)^{N-2-n}(x_{l(\sigma_d)})^{n-1+(N-k)d}\times\\ &\quad\times\prod_{j=0}^{l(\sigma_d)}\frac{1}{(x_j)^N}\prod_{j=1}^{l(\sigma_d)-1}\frac{1}{kx_j\left(\frac{x_j-x_{j-1}}{d_j}+\frac{x_j-x_{j+1}}{d_{j+1}}\right)}\prod_{j=1}^{l(\sigma_d)}\frac{e(k,d_j;x_{j-1},x_j)}{t(N,d_j;x_{j-1},x_j)}.\end{aligned} \tag{1.8}$$

In (1.8), $\frac{1}{2\pi\sqrt{-1}}\oint_{C_0}dx_j$ represents the operation of taking the residue at $x_j=0$. We must mention that the residue integral in (1.8) severely depends on the order of integration. To be more precise, we must take the residues of all x_j in descending or ascending order of subscript j. In the appendix of this paper, we prove the following theorem.

Theorem 1 *(1.8) holds true if $d\leq 3$.*

We can indeed prove that (1.8) holds true for arbitrary d, but we include only the proof for $d\leq 3$ in this paper mainly due to space concerns. The full proof will appear elsewhere. If $N\leq k$, the Gromov-Witten invariant $\frac{1}{k}\langle\mathcal{O}_{h^{N-2-n}}\mathcal{O}_{h^{n-1+(N-k)d}}\mathcal{O}_h\rangle_{0,d}$ and $\tilde{L}_n^{N,k,d}$ are different. In this case, we can write the former as the weighted homogeneous polynomial in $\tilde{L}_m^{N,k,d'}$ ($d'\leq d$). This formula is the generalized mirror transformation in the sense described in this paper. The main result of this paper is a proof of this transformation up to the case of $d=3$, which was conjectured in [8] as follows.

Theorem 2

$$\frac{1}{k}\langle\mathcal{O}_{h^{N-2-n}}\mathcal{O}_{h^{n-1+N-k}}\rangle_{0,1} = \tilde{L}_n^{N,k,1}-\tilde{L}_{1+(k-N)}^{N,k,1}, \tag{1.9}$$

$$\frac{1}{k}\langle \mathcal{O}_{h^{N-2-n}}\mathcal{O}_{h^{n-1+2(N-k)}}\rangle_{0,2} = \frac{1}{2}(\tilde{L}_n^{N,k,2}-\tilde{L}_{1+2(k-N)}^{N,k,2})-\tilde{L}_{1+(k-N)}^{N,k,1}(\sum_{j=0}^{k-N}(\tilde{L}_{n-j}^{N,k,1}-\tilde{L}_{1-2(k-N)-j}^{N,k,1})), \quad (1.10)$$

$$\begin{aligned}\frac{1}{k}\langle \mathcal{O}_{h^{N-2-n}}\mathcal{O}_{h^{n-1+3(N-k)}}\rangle_{0,3} &= \frac{1}{3}(\tilde{L}_n^{N,k,3}-\tilde{L}_{1+3(k-N)}^{N,k,3})-\tilde{L}_{1+(k-N)}^{N,k,1}(\sum_{j=0}^{k-N}(\tilde{L}_{n-j}^{N,k,2}-\tilde{L}_{1+3(k-N)-j}^{N,k,2})+C_{1,1}^{N,k,3}(n))\\ &-\frac{1}{2}\tilde{L}_{1+2(k-N)}^{N,k,2}(\sum_{j=0}^{2(k-N)}(\tilde{L}_{n-j}^{N,k,1}-\tilde{L}_{1+3(k-N)-j}^{N,k,1}))\\ &+\frac{3}{2}(\tilde{L}_{1+(k-N)}^{N,k,1})^2(\sum_{j=0}^{2(k-N)}A_j(\tilde{L}_{n-j}^{N,k,1}-\tilde{L}_{1+3(k-N)-j}^{N,k,1})), \quad (1.11)\end{aligned}$$

where

$$\begin{aligned}A_j &:= j+1, \ \textit{if} \ (0\le j\le k-N), \quad A_j := 1+2(k-N)-j, \ \textit{if} \ (k-N\le j\le 2(k-N)),\\ C_{1,1}^{N,k,3}(n) &= \sum_{j=0}^{(k-N)-1}(\sum_{m=0}^{j}\tilde{L}_{n-m}^{N,k,1}\tilde{L}_{n-2(k-N)+j-m}^{N,k,1}-\tilde{L}_{(k-N)+2+j}^{N,k,1}(\sum_{m=0}^{2(k-N)}\tilde{L}_{n-m}^{N,k,1})\\ &+\tilde{L}_{1+(k-N)}^{N,k,1}(\sum_{m=j+1}^{2(k-N)-j-1}\tilde{L}_{n-m}^{N,k,1}))\\ &-\sum_{j=0}^{(k-N)-1}(\sum_{m=0}^{j}\tilde{L}_{1+3(k-N)-m}^{N,k,1}\tilde{L}_{1+(k-N)+j-m}^{N,k,1}-\tilde{L}_{(k-N)+2+j}^{N,k,1}(\sum_{m=0}^{2(k-N)}\tilde{L}_{1+3(k-N)-m}^{N,k,1})\\ &+\tilde{L}_{1+(k-N)}^{N,k,1}(\sum_{m=j+1}^{2(k-N)-j-1}\tilde{L}_{1+3(k-N)-m}^{N,k,1})). \quad (1.12)\end{aligned}$$

Of course, the above formulas can be derived by using known methods presented in various papers, including [1],[5],[6],and [11]. In these works, the generalized mirror transformation is derived as the effect of coordinate changes of B-model deformation parameters into A-model deformation parameters. We feel that this process may be too sophisticated to capture the geometrical image of the generalized mirror transformation, that is, changing the moduli space of the Gauged Linear Sigma Model into the moduli space of stable maps. In this paper, we present an elementary and direct proof of Theorem 2 by using the result of Kontsevich [10] and Theorem 1. Our strategy is the following. First, we note the explicit formula of $\langle \mathcal{O}_{h^{N-2-n}}\mathcal{O}_{h^{n-1+(N-k)d}}\rangle_d$ that follows from Kontsevich's localization computation. This formula includes summations with combinatorial complications that have torus action characters λ_j, $(j=1,\cdots,N)$, but we can rewrite these summations into residue integrals of finite complex variables. This process is a generalization of the well-known computation from the Bott residue theorem, which is available on p.434-435 in [4]. After this operation, we take the non-equivariant limit $\lambda_j\to 0$. The resulting formula is very close to our residue integral representation of $\tilde{L}_n^{N,k,d}$ in (1.8). With this formula, to prove Theorem 2, we require an elementary combinatorial decomposition of rational functions in the integrands.

This paper is organized as follows. In Section 2, we explain the process used to reduce the combinatorial summations in Kontsevich's localization formula to residue integrals in finite variables. Then we present the residue integral representation of 2-point Gromov-Witten invariants. This representation can be directly compared with the r.h.s of (1.8) after taking the non-equivariant limit $\lambda_j\to 0$. In Section 3, we prove Theorem 2 by decomposing the rational functions in the integrands. Section 3 presents concluding remarks. In the Appendix, we prove Theorem 1, which has an important role in the proof of Theorem 2.

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2 Reduction of the Localization Formula to the Residue Integral

We start from the Kontsevich's localization formulas for 2-point genus 0 Gromov-Witten invariants of M_N^k. For the succinct presentation of these formulas, we introduce the following notation.

$$\begin{aligned} w_a(u,v) &:= \frac{u^a - v^a}{u-v} = \sum_{p+q=a-1, p,q\geq 0} u^p v^q, \\ w_a(u,v,w) &:= \sum_{p+q+r=a-2, p,q,r\geq 0} u^p v^q w^r, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} E(k,d;i,j) &:= \prod_{m=0}^{kd} \left(\frac{m\lambda_i + (kd-m)\lambda_j}{d}\right), \\ V(N;i) &:= \prod_{j\neq i, 1\leq j\leq N} (\lambda_j - \lambda_i), \\ T(N,d;i,j) &:= \prod_{k=1}^{N}\prod_{m=1}^{d-1}\left(\frac{m\lambda_i + (d-m)\lambda_j}{d} - \lambda_k\right). \end{aligned} \tag{2.14}$$

In (2.14), λ_j $(j = 1, \cdots, N)$ are characters of torus action on CP^{N-1}:

$$(X_1 : \cdots : X_N) \to (e^{\lambda_1 t} X_1 : \cdots : e^{\lambda_N t} X_N). \tag{2.15}$$

Here, we also introduce an elementary equality that will be used later in this paper:

$$w_a(x_1,x_2) + w_a(x_2,x_3) = (2x_2 - x_1 - x_3) w_a(x_1,x_2,x_3) + 2w_a(x_1,x_3). \tag{2.16}$$

With this notation, the localization formulas that represent $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d}$ $(a = N-2-n,\ b = n-1+(N-k)d)$ are described as follows:

Fact 1 (Kontsevich)

$$\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,1} = -\frac{1}{2}\sum_{i\neq j} \frac{E(k,1;i,j)(\lambda_i-\lambda_j)^2}{V(N;i)V(N;j)} \cdot w_a(\lambda_i,\lambda_j) w_b(\lambda_i,\lambda_j), \tag{2.17}$$

$$\begin{aligned} \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,2} &= -\frac{1}{4}\sum_{i\neq j} \frac{E(k,2;i,j)(\lambda_i-\lambda_j)^2}{T(N,2;i,j)V(N;i)V(N;j)} \cdot w_a(\lambda_i,\lambda_j) w_b(\lambda_i,\lambda_j) + \\ &\quad + \frac{1}{2}\sum_{i\neq j\neq l} \frac{E(k,1;i,j)E(k,1;j,l)}{V(N;i)V(N;j)V(N;l)k\lambda_j} \cdot \frac{1}{\frac{1}{\lambda_j-\lambda_i}+\frac{1}{\lambda_j-\lambda_l}} \times \\ &\quad \times \big(w_a(\lambda_i,\lambda_j) + w_a(\lambda_j,\lambda_l)\big)\big(w_b(\lambda_i,\lambda_j) + w_b(\lambda_j,\lambda_l)\big), \end{aligned} \tag{2.18}$$

$$\begin{aligned} \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,3} &= -\frac{1}{6}\sum_{i\neq j} \frac{E(k,3;i,j)(\lambda_i-\lambda_j)^2}{T(N,3;i,j)V(N;i)V(N;j)} \cdot w_a(\lambda_i,\lambda_j) w_b(\lambda_i,\lambda_j) + \\ &\quad + \frac{1}{2}\sum_{i\neq j\neq l} \frac{E(k,2;i,j)E(k,1;j,l)}{T(N,2;i,j)V(N;i)V(N;j)V(N;l)k\lambda_j} \cdot \frac{1}{\frac{2}{\lambda_j-\lambda_i}+\frac{1}{\lambda_j-\lambda_l}} \times \\ &\quad \times \big(2w_a(\lambda_i,\lambda_j) + w_a(\lambda_j,\lambda_l)\big)\big(2w_b(\lambda_i,\lambda_j) + w_b(\lambda_j,\lambda_l)\big) - \\ &\quad - \frac{1}{2}\sum_{i\neq j\neq l\neq m} \frac{E(k,1;i,j)E(k,1;j,l)E(k,1;l,m)}{V(N;i)V(N;j)V(N;l)V(N;m)k\lambda_j k\lambda_l} \times \\ &\quad \times \frac{1}{\frac{1}{\lambda_j-\lambda_i}+\frac{1}{\lambda_j-\lambda_l}} \frac{1}{\frac{1}{\lambda_l-\lambda_j}+\frac{1}{\lambda_l-\lambda_m}} \cdot \frac{1}{(\lambda_j-\lambda_l)^2} \times \end{aligned}$$

$$\times\big(w_a(\lambda_i,\lambda_j)+w_a(\lambda_j,\lambda_l)+w_a(\lambda_l,\lambda_m)\big)\times$$
$$\times\big(w_b(\lambda_i,\lambda_j)+w_b(\lambda_j,\lambda_l)+w_b(\lambda_l,\lambda_m)\big)-$$
$$-\frac{1}{6}\sum_{i\neq j,i\neq l,i\neq m}\frac{E(k,1;i,j)E(k,1;i,l)E(k,1;i,m)}{V(N;i)V(N;j)V(N;l)V(N;m)(k\lambda_i)^2}\times$$
$$\times\big(w_a(\lambda_i,\lambda_j)+w_a(\lambda_i,\lambda_l)+w_a(\lambda_i,\lambda_m)\big)\times$$
$$\times\big(w_b(\lambda_i,\lambda_j)+w_b(\lambda_i,\lambda_l)+w_b(\lambda_i,\lambda_m)\big). \qquad (2.19)$$

Remark 1 *In (2.17), (2.18) and (2.19), each summand corresponds to a tree graph that represents degeneration-type of stable maps [10]. The r.h.s. of these equalities are invariant under variations in the characters of torus action, but in (2.18) and (2.19), each summand indeed varies as the characters vary.*

Although some elementary simplification of complicated terms has been achieved, these formulas follow from the results in [10]. The above formulas include many complicated summations, but we can rewrite these summations into residue integrals in finite complex variables as follows.

Proposition 1

$$\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_1 = -\frac{1}{2}\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,1;x_1,x_2)(x_1-x_2)^2}{(x_1)^N(x_2)^N}\cdot w_a(x_1,x_2)w_b(x_1,x_2), \qquad (2.20)$$

$$\begin{aligned}\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_2 = & -\frac{1}{4}\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,2;x_1,x_2)(x_1-x_2)^2}{(\frac{x_1+x_2}{2})^N(x_1)^N(x_2)^N}\cdot w_a(x_1,x_2)w_b(x_1,x_2)+\\ & +\frac{1}{2}\frac{1}{(2\pi\sqrt{-1})^3}\oint_{C_0}dx_3\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,1;x_1,x_2)e(k,1;x_2,x_3)}{(x_1)^N(x_2)^N(x_3)^Nkx_2}\cdot\frac{1}{\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3}}\times\\ & \times\big(w_a(x_1,x_2)+w_a(x_2,x_3)\big)\big(w_b(x_1,x_2)+w_b(x_2,x_3)\big), \end{aligned} \qquad (2.21)$$

$$\begin{aligned}\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_3 = & -\frac{1}{6}\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,3;x_1,x_2)(x_1-x_2)^2}{(\frac{2x_1+x_2}{3})^N(\frac{x_1+2x_2}{3})^N(x_1)^N(x_2)^N}\cdot w_a(x_1,x_2)w_b(x_1,x_2)+\\ & +\frac{1}{4}\frac{1}{(2\pi\sqrt{-1})^3}\oint_{C_0}dx_3\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,2;x_1,x_2)e(k,1;x_2,x_3)}{(\frac{x_1+x_2}{2})^N(x_1)^N(x_2)^N(x_3)^Nkx_2}\cdot\frac{1}{\frac{2}{x_2-x_1}+\frac{1}{x_2-x_3}}\times\\ & \times\big(2w_a(x_1,x_2)+w_a(x_2,x_3)\big)\big(2w_b(x_1,x_2)+w_b(x_2,x_3)\big)+\\ & +\frac{1}{4}\frac{1}{(2\pi\sqrt{-1})^3}\oint_{C_0}dx_3\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,1;x_1,x_2)e(k,2;x_2,x_3)}{(\frac{x_2+x_3}{2})^N(x_1)^N(x_2)^N(x_3)^Nkx_2}\cdot\frac{1}{\frac{1}{x_2-x_1}+\frac{2}{x_2-x_3}}\times\\ & \times\big(w_a(x_1,x_2)+2w_a(x_2,x_3)\big)\big(w_b(x_1,x_2)+2w_b(x_2,x_3)\big)-\\ & -\frac{1}{2}\frac{1}{(2\pi\sqrt{-1})^4}\oint_{C_0}dx_4\oint_{C_0}dx_3\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,1;x_1,x_2)e(k,1;x_2,x_3)e(k,1;x_3,x_4)}{(x_1)^N(x_2)^N(x_3)^N(x_4)^Nkx_2kx_3}\times\\ & \times\frac{1}{\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3}}\frac{1}{\frac{1}{x_3-x_2}+\frac{1}{x_3-x_4}}\cdot\frac{1}{(x_2-x_3)^2}\times\\ & \times\big(w_a(x_1,x_2)+w_a(x_2,x_3)+w_a(x_3,x_4)\big)\big(w_b(x_1,x_2)+w_b(x_2,x_3)+w_b(x_3,x_4)\big)-\\ & -\frac{1}{6}\frac{1}{(2\pi\sqrt{-1})^4}\oint_{C_0}dx_4\oint_{C_0}dx_3\oint_{C_0}dx_2\oint_{C_0}dx_1\frac{e(k,1;x_1,x_2)e(k,1;x_1,x_3)e(k,1;x_1,x_4)}{(x_1)^N(x_2)^N(x_3)^N(x_4)^N(kx_1)^2}\times\\ & \times\big(w_a(x_1,x_2)+w_a(x_1,x_3)+w_a(x_1,x_4)\big)\big(w_b(x_1,x_2)+w_b(x_1,x_3)+w_b(x_1,x_4)\big). \end{aligned} \qquad (2.22)$$

proof) Due to space concerns, we include the proof for $d=1,2$. The proof for the case of $d=3$ proceeds in a similar fashion. We start from the case of $d=1$. By the elementary residue theorem, we can rewrite the r.h.s. of (2.17) into the following residue integral.

$$\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_1 = -\frac{1}{2}\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_{(0,5R)}}dx_2\oint_{C_{(0,R)}}dx_1\frac{e(k,1;x_1,x_2)(x_1-x_2)^2}{\prod_{j=1}^N\big((x_1-\lambda_j)(x_2-\lambda_j)\big)}\cdot w_a(x_1,x_2)w_b(x_1,x_2). \qquad (2.23)$$

In (2.23), $C(0,a)$ denotes the circle centered at 0 and with radius a, and R is a sufficiently large positive real number greater than $\max.\{3|\lambda_j| \mid j=1,\cdots,N\}$. In the case of $d=2$, we also have similar equalities:

$$-\frac{1}{4}\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_{(0,5R)}}dx_2\oint_{C_{(0,R)}}dx_1\frac{e(k,2;x_1,x_2)(x_1-x_2)^2}{\prod_{j=1}^{N}\left(\left(\frac{x_1+x_2}{2}-\lambda_j\right)(x_1-\lambda_j)(x_2-\lambda_j)\right)}\cdot w_a(x_1,x_2)w_b(x_1,x_2)=$$
$$=-\frac{1}{4}\sum_{i\neq j}\frac{E(k,2;i,j)(\lambda_i-\lambda_j)^2}{T(N,2;i,j)V(N;i)V(N;j)}\cdot w_a(\lambda_i,\lambda_j)w_b(\lambda_i,\lambda_j)-$$
$$-2\sum_{i\neq j}\frac{e(k,2;\lambda_i,2\lambda_j-\lambda_i)(\lambda_i-\lambda_j)^2}{\left(\prod_{l=1}^{N}(2\lambda_j-\lambda_i-\lambda_l)\right)V(N;i)V(N;j)}\cdot w_a(\lambda_i,2\lambda_j-\lambda_i)w_b(\lambda_i,2\lambda_j-\lambda_i), \tag{2.24}$$

and

$$\frac{1}{2}\frac{1}{(2\pi\sqrt{-1})^3}\oint_{C_{(0,25R)}}dx_3\oint_{C_{(0,5R)}}dx_2\oint_{C_{(0,R)}}dx_1\frac{e(k,1;x_1,x_2)e(k,1;x_2,x_3)}{kx_2\prod_{j=1}^{N}\left((x_1-\lambda_j)(x_2-\lambda_j)(x_3-\lambda_j)\right)}\times$$
$$\times\frac{(x_2-x_1)(x_2-x_3)}{2x_2-x_1-x_3}\big(w_a(x_1,x_2)+w_a(x_2,x_3)\big)\big(w_b(x_1,x_2)+w_b(x_2,x_3)\big)=$$
$$=\frac{1}{2}\sum_{i\neq j,j\neq l}\frac{E(k,1;i,j)E(k,1;j,l)}{k\lambda_jV(N;i)V(N;j)V(N;l)}\frac{(\lambda_j-\lambda_i)(\lambda_j-\lambda_l)}{2\lambda_j-\lambda_i-\lambda_l}\big(w_a(\lambda_i,\lambda_j)+w_a(\lambda_j,\lambda_l)\big)\big(w_b(\lambda_i,\lambda_j)+w_b(\lambda_j,\lambda_l)\big)+$$
$$+\frac{1}{2}\sum_{i\neq j}\frac{e(k;1;\lambda_i,\lambda_j)e(k,1;\lambda_j,2\lambda_j-\lambda_i)(\lambda_i-\lambda_j)^2}{k\lambda_jV(N;i)V(N;j)V(N;l)\left(\prod_{l=1}^{N}(2\lambda_j-\lambda_i-\lambda_l)\right)}\big(w_a(\lambda_i,\lambda_j)+w_a(\lambda_j,2\lambda_j-\lambda_i)\big)\times$$
$$\times\big(w_b(\lambda_i,\lambda_j)+w_b(\lambda_j,2\lambda_j-\lambda_i)\big). \tag{2.25}$$

In contrast, the following relations easily follow.

$$e(k,2;\lambda_i,2\lambda_j-\lambda_i)=\frac{e(k;1;\lambda_i,\lambda_j)e(k,1;\lambda_j,2\lambda_j-\lambda_i)}{k\lambda_j},$$
$$w_a(\lambda_i,\lambda_j)+w_a(\lambda_j,2\lambda_j-\lambda_i)=2w_a(\lambda_i,2\lambda_j-\lambda_i). \tag{2.26}$$

The second equality follows from (2.16). With these relations, the second terms of the r.h.s. of (2.24) and (2.25) cancel one another. As such, we obtain the following:

$$\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_2=$$
$$=-\frac{1}{4}\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_{(0,5R)}}dx_2\oint_{C_{(0,R)}}dx_1\frac{e(k,2;x_1,x_2)(x_1-x_2)^2}{\prod_{j=1}^{N}\left(\left(\frac{x_1+x_2}{2}-\lambda_j\right)(x_1-\lambda_j)(x_2-\lambda_j)\right)}\cdot w_a(x_1,x_2)w_b(x_1,x_2)+$$
$$+\frac{1}{2}\frac{1}{(2\pi\sqrt{-1})^3}\oint_{C_{(0,25R)}}dx_3\oint_{C_{(0,5R)}}dx_2\oint_{C_{(0,R)}}dx_1\frac{e(k,1;x_1,x_2)e(k,1;x_2,x_3)}{kx_2\prod_{j=1}^{N}\left((x_1-\lambda_j)(x_2-\lambda_j)(x_3-\lambda_j)\right)}\times$$
$$\times\frac{(x_2-x_1)(x_2-x_3)}{2x_2-x_1-x_3}\big(w_a(x_1,x_2)+w_a(x_2,x_3)\big)\big(w_b(x_1,x_2)+w_b(x_2,x_3)\big). \tag{2.27}$$

If we look at the r.h.s. of (2.23) and (2.27), we can easily see from the coordinate change $x_j=\frac{1}{z_j}$ that each summand is invariant under variations in λ_j. Therefore, we can take the non-equivariant limit $\lambda_j\to 0$; we can take the $R\to 0$ limit as well. This operation leads us to the equalities of the proposition.

Remark 2 *In Remark 1, we noticed that each summand in (2.18) is not invariant under variations in the characters. But as can be seen in (2.24) and (2.25), we can make it invariant by adding suitable rational functions of characters. These additional rational functions cancel out after adding up summands that correspond to tree graphs. The same mechanism also works in the case of $d=3$.*

3 Proof of Theorem 2

Before moving on to the proof of Theorem 2, we note the following equality.

$$\tilde{L}_n^{N,k,d}=\tilde{L}_{N-1-(N-k)d-n}^{N,k,d}. \tag{3.28}$$

This naturally follows from Theorem 1.

Let us start from the case of $d=1$. In this case, we apply a trivial equality to the r.h.s of (2.20):

$$(x_1-x_2)^2 w_a(x_1,x_2)w_b(x_1,x_2)=(x_1^a-x_2^a)(x_2^b-x_1^b)=x_1^a x_2^b+x_1^b x_2^a-x_1^{a+b}-x_2^{a+b}. \tag{3.29}$$

Then the theorem follows from Theorem 1 and (3.28). In the case of $d=2$, we apply (3.29) to the first summand of the r.h.s. of (2.21). To the second summand of (2.21), we apply the following decomposition of the rational function in the integrand.

$$\begin{aligned}
&\frac{(w_a(x_1,x_2)+w_a(x_2,x_3))(w_b(x_1,x_2)+w_b(x_2,x_3))}{\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3}}=\\
&=\frac{(x_2-x_1)(x_2-x_3)(w_a(x_1,x_2)+w_a(x_2,x_3))(w_b(x_1,x_2)+w_b(x_2,x_3))}{2x_2-x_1-x_3}=\\
&=\frac{((x_1)^a-(x_3)^a)((x_3)^b-(x_1)^b)}{2x_2-x_1-x_3}+(x_2-x_1)w_a(x_1,x_2)w_b(x_1,x_2)+(x_2-x_3)w_a(x_2,x_3)w_b(x_2,x_3).
\end{aligned} \tag{3.30}$$

Then the first summand in the r.h.s. of (2.21) and the first term in the decomposition (3.30) sum to the following:

$$\frac{1}{2}k(\tilde{L}_n^{N,k,2}-\tilde{L}_{1+2(k-N)}^{N,k,2}), \tag{3.31}$$

using (3.28) and Theorem 1. The second and the third terms in the decomposition (3.30) result in the following:

$$k\sum_{j=0}^{b-1}\tilde{L}_{1+k-N}^{N,k,1}(\tilde{L}_{1+j+k-N}^{N,k,1}-\tilde{L}_{1+a+j+k-N}^{N,k,1}), \tag{3.32}$$

using $(x_i-x_j)w_a(x_i,x_j)=x_i^a-x_j^a$, (3.28) and Theorem 1. But if we note $a=N-2-n$, $b=n-1-2(k-N)$ and (3.28), (3.32) becomes the following:

$$-k\sum_{j=0}^{k-N}\tilde{L}_{1+k-N}^{N,k,1}(\tilde{L}_{n-j}^{N,k,1}-\tilde{L}_{1+2(k-N)-j}^{N,k,1}). \tag{3.33}$$

This completes the proof for the case of $d=2$.

Now, we turn to the case of $d=3$. As in the case of $d=1,2$, we apply (3.29) to the first summand of the r.h.s. of (2.22). To the second and the third summands, we apply the following decompositions.

$$\begin{aligned}
&\frac{2(2w_a(x_1,x_2)+w_a(x_2,x_3))(2w_b(x_1,x_2)+w_b(x_2,x_3))}{\frac{2}{x_2-x_1}+\frac{1}{x_2-x_3}}=\\
&=\frac{(x_2-x_1)(x_2-x_3)(2w_a(x_1,x_2)+w_a(x_2,x_3))(2w_b(x_1,x_2)+w_b(x_2,x_3))}{\frac{x_2-x_1}{2}+x_2-x_3}=\\
&=2\frac{((x_1)^a-(x_3)^a)((x_3)^b-(x_1)^b)}{\frac{x_2-x_1}{2}+x_2-x_3}+4(x_2-x_1)w_a(x_1,x_2)w_b(x_1,x_2)+2(x_2-x_3)w_a(x_2,x_3)w_b(x_2,x_3).
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
&\frac{2(w_a(x_1,x_2)+2w_a(x_2,x_3))(w_b(x_1,x_2)+2w_b(x_2,x_3))}{\frac{1}{x_2-x_1}+\frac{2}{x_2-x_3}}=\\
&=\frac{(x_2-x_1)(x_2-x_3)(w_a(x_1,x_2)+2w_a(x_2,x_3))(w_b(x_1,x_2)+2w_b(x_2,x_3))}{x_2-x_1+\frac{x_2-x_3}{2}}=\\
&=2\frac{((x_1)^a-(x_3)^a)((x_3)^b-(x_1)^b)}{x_2-x_1+\frac{x_2-x_3}{2}}+2(x_2-x_1)w_a(x_1,x_2)w_b(x_1,x_2)+4(x_2-x_3)w_a(x_2,x_3)w_b(x_2,x_3).
\end{aligned} \tag{3.35}$$

Lastly, we apply the following decomposition to the fourth summand.

$$
\begin{aligned}
&-\frac{(w_a(x_1,x_2)+w_a(x_2,x_3)+w_a(x_3,x_4))(w_b(x_1,x_2)+w_b(x_2,x_3)+w_b(x_3,x_4))}{(\frac{1}{x_2-x_1}+\frac{1}{x_2-x_3})(\frac{1}{x_3-x_2}+\frac{1}{x_3-x_4})(x_2-x_3)^2}\\
&=\frac{(x_2-x_1)(x_3-x_4)(w_a(x_1,x_2)+w_a(x_2,x_3)+w_a(x_3,x_4))(w_b(x_1,x_2)+w_b(x_2,x_3)+w_b(x_3,x_4))}{r_1r_2}\\
&=\frac{((x_1)^a-(x_4)^a)((x_4)^b-(x_1)^b)}{r_1r_2}+\\
&+\frac{2(x_3-x_1)w_a(x_1,x_3)w_b(x_1,x_3)+(x_3-x_4)w_a(x_3,x_4)w_b(x_3,x_4)}{r_1}+\\
&+\frac{2(x_2-x_4)w_a(x_2,x_4)w_b(x_2,x_4)+(x_2-x_1)w_a(x_1,x_2)w_b(x_1,x_2)}{r_2}+\\
&+\frac{1}{2}\big(w_a(x_1,x_2)w_b(x_1,x_2)+w_a(x_3,x_4)w_b(x_3,x_4)+w_a(x_1,x_2)w_b(x_2,x_3)+w_a(x_2,x_3)w_b(x_1,x_2)+\\
&+w_a(x_2,x_3)w_b(x_3,x_4)+w_a(x_3,x_4)w_b(x_2,x_3)\big)+\\
&+(x_3-x_1)\big(w_a(x_1,x_3)w_b(x_1,x_2,x_3)+w_a(x_1,x_2,x_3)w_b(x_1,x_3)+\frac{1}{2}r_1w_a(x_1,x_2,x_3)w_b(x_1,x_2,x_3)\big)+\\
&+(x_2-x_4)\big(w_a(x_2,x_4)w_b(x_2,x_3,x_4)+w_a(x_2,x_3,x_4)w_b(x_2,x_4)+\frac{1}{2}r_2w_a(x_2,x_3,x_4)w_b(x_2,x_3,x_4)\big), \quad (3.36)
\end{aligned}
$$

where $r_1=2x_2-x_1-x_3$, $r_2=2x_3-x_2-x_4$. By (3.28) and Theorem 1, the first summand of the r.h.s. of (2.22), the first terms of (3.34) and (3.35) and the term with denominator r_1r_2 in (3.36) together sum to $\frac{1}{3}k(\tilde{L}_n^{N,k,3}-\tilde{L}_{1+3(k-N)}^{N,k,3})$. The second term of (3.34), the third term of (3.35) and

$$\frac{2(x_3-x_1)w_a(x_1,x_3)w_b(x_1,x_3)}{r_1},\quad \frac{2(x_2-x_4)w_a(x_2,x_4)w_b(x_2,x_4)}{r_2}, \tag{3.37}$$

in (3.36) sum to the following:

$$-k\sum_{j=0}^{k-N}\tilde{L}_{1+k-N}^{N,k,1}(\tilde{L}_{n-j}^{N,k,2}-\tilde{L}_{1+3(k-N)-j}^{N,k,2}), \tag{3.38}$$

by Theorem 1 and the same approach used to derive (3.33). The third term of (3.34), the second term of (3.35) and

$$\frac{(x_3-x_4)w_a(x_3,x_4)w_b(x_3,x_4)}{r_1},\quad \frac{(x_2-x_1)w_a(x_1,x_2)w_b(x_1,x_2)}{r_2}, \tag{3.39}$$

in (3.36) similarly sum to the following:

$$-k\cdot\frac{1}{2}\sum_{j=0}^{2(k-N)}\tilde{L}_{1+2(k-N)}^{N,k,2}(\tilde{L}_{n-j}^{N,k,1}-\tilde{L}_{1+3(k-N)-j}^{N,k,1}). \tag{3.40}$$

The remaining terms of the decomposition (3.36) and the fifth summand of the r.h.s. of (2.22) are reorganized as follows.

$$
\begin{aligned}
&k\tilde{L}_{1+k-N}^{N,k,1}\Big(\frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\tilde{L}_{1+i+j+k-N}^{N,k,1}\tilde{L}_{2+2(k-N)}^{N,k,1}+\\
&+\frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\tilde{L}_{1+i+k-N}^{N,k,1}\tilde{L}_{2-j+2(k-N)}^{N,k,1}+\frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\tilde{L}_{1+j+k-N}^{N,k,1}\tilde{L}_{2+i+2(k-N)}^{N,k,1}+\\
&+\frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-2}(\tilde{L}_{1+i+k-N}^{N,k,1}\tilde{L}_{3+j+2(k-N)}^{N,k,1}-\tilde{L}_{2+i+j+k-N}^{N,k,1}\tilde{L}_{2+2(k-N)}^{N,k,1})+\\
&+\frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-2}(\tilde{L}_{1+k-N}^{N,k,1}\tilde{L}_{3+i+j+2(k-N)}^{N,k,1}-\tilde{L}_{2+j+k-N}^{N,k,1}\tilde{L}_{2+i+2(k-N)}^{N,k,1})+
\end{aligned}
$$

$$+\sum_{i=0}^{a-2}\sum_{j=0}^{i}(\tilde{L}^{N,k,1}_{1+j+k-N}\tilde{L}^{N,k,1}_{a+b-i+2(k-N)}-\tilde{L}^{N,k,1}_{1+b+j+k-N}\tilde{L}^{N,k,1}_{a-i+2(k-N)})-$$

$$-\frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\tilde{L}^{N,k,1}_{1+k-N}\tilde{L}^{N,k,1}_{1+i-j+k-N}-\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\tilde{L}^{N,k,1}_{1+i+k-N}\tilde{L}^{N,k,1}_{1+j+k-N}\Big)=$$

$$=k\tilde{L}^{N,k,1}_{1+k-N}\Big(-\sum_{i=0}^{a-1}\sum_{j=0}^{i}(\tilde{L}^{N,k,1}_{n+j-2(k-N)}\tilde{L}^{N,k,1}_{n+i+1-(k-N)}-\tilde{L}^{N,k,1}_{1+j+(k-N)}\tilde{L}^{N,k,1}_{2+i+2(k-N)})+$$

$$+\sum_{i=0}^{a-1}\sum_{j=0}^{k-N}\tilde{L}^{N,k,1}_{n+1+i}(\tilde{L}^{N,k,1}_{n+j-2(k-N)}-\tilde{L}^{N,k,1}_{1+j+(k-N)})+$$

$$+\sum_{j=0}^{a-1}(\tilde{L}^{N,k,1}_{1+2(k-N)}\tilde{L}^{N,k,1}_{n+j-2(k-N)}-\tilde{L}^{N,k,1}_{n-(k-N)}\tilde{L}^{N,k,1}_{1+j+(k-N)})+$$

$$+\frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{k-N}\tilde{L}^{N,k,1}_{1+k-N}(\tilde{L}^{N,k,1}_{2+i+j+2(k-N)}-\tilde{L}^{N,k,1}_{1+i+j+(k-N)})\Big)=$$

$$=k\tilde{L}^{N,k,1}_{1+k-N}\Big(-\sum_{i=0}^{(k-N)-1}\sum_{j=0}^{i}(\tilde{L}^{N,k,1}_{n+j-2(k-N)}\tilde{L}^{N,k,1}_{n+i+1-(k-N)}-\tilde{L}^{N,k,1}_{1+j+(k-N)}\tilde{L}^{N,k,1}_{2+i+2(k-N)})+$$

$$+\sum_{i=0}^{2(k-N)}\sum_{j=0}^{k-N}\tilde{L}^{N,k,1}_{1+j+(k-N)}(\tilde{L}^{N,k,1}_{n-i}-\tilde{L}^{N,k,1}_{1+i+(k-N)})+$$

$$+\frac{1}{2}\sum_{i=0}^{k-N}\sum_{j=0}^{k-N}\tilde{L}^{N,k,1}_{1+k-N}(\tilde{L}^{N,k,1}_{n-i-j}-\tilde{L}^{N,k,1}_{1+i+j+(k-N)})\Big)=$$

$$=-k\tilde{L}^{N,k,1}_{1+k-N}C^{N,k,3}_{1,1}(n)+k\cdot\frac{3}{2}(\tilde{L}^{N,k,1}_{1+k-N})^2(\sum_{j=0}^{2(k-N)}A_j(\tilde{L}^{N,k,1}_{n-j}-\tilde{L}^{N,k,1}_{1+3(k-N)-j})). \tag{3.41}$$

In this derivation, we only use the following conditions.

$$a=N-2-n,\ \ b=n-1-3(k-N),\ \ \tilde{L}^{N,k,1}_{n}=\tilde{L}^{N,k,1}_{k-1-n}. \tag{3.42}$$

This derivation requires the careful treatment of summations. The final formula completes the proof of Theorem 2.

4 Conclusion

Our motivation in this paper is to explicitly understand the difference between the moduli space of the Gauged Linear Sigma Model and the moduli space of stable maps. Through the proof presented in this paper, we can see the detailed process of changing the moduli space of the non-linear sigma model into the moduli space of the linear sigma model for lower instanton numbers. Moreover, we can indeed extend this paper's method to rational curves of higher degrees because we do not use the geometrical simplicity of the moduli space of rational curves of lower degrees. As can be seen in [9], the generalized mirror transformation for rational curves with $d=4,5$ has a quite complicated structure, but if we combine the scheme of the generalized mirror transformation proposed in [6] with our residue integral representation of the virtual structure constants, we can expect the derivation of a general proof of the mirror theorem.

One of the main features of this paper involve the translation of combinatorial summations in Kontsevich's localization formula into residue integrals, which enables us to directly compare the Gromov-Witten invariants with the virtual structure constants. This translation can be applied to various examples. At the very least, we can use this residue integral to prove the mirror theorem of $\mathcal{O}(1)\oplus\mathcal{O}(-3)\rightarrow\mathbf{P}^1$ [3]. We also can apply the residue integral representation to prove the mirror theorem at higher genus. Regardless, we must enhance the combinatorial sophistication of our method.

A Appendix: Proof of Theorem 1

We prove Theorem 1 by showing that the r.h.s. of (1.8) satisfies the initial condition and the recursive formulas (1.1), (1.2), (1.3) and (1.4). For this purpose, we note here the following relation between the rational functions that appear in the residue integrals.

$$\prod_{j=0}^{l(\sigma_d)}\frac{1}{(x_j)^N}\prod_{j=1}^{l(\sigma_d)-1}\frac{1}{kx_j\left(\frac{x_j-x_{j-1}}{d_j}+\frac{x_j-x_{j+1}}{d_{j+1}}\right)}\prod_{j=1}^{l(\sigma_d)}\frac{e(k,d_j;x_{j-1},x_j)}{t(N,d_j;x_{j-1},x_j)}=$$
$$=\prod_{j=0}^{l(\sigma_d)}\frac{1}{(x_j)^{N+1}}\prod_{j=1}^{l(\sigma_d)-1}\frac{1}{kx_j\left(\frac{x_j-x_{j-1}}{d_j}+\frac{x_j-x_{j+1}}{d_{j+1}}\right)}\prod_{j=1}^{l(\sigma_d)}\frac{e(k,d_j;x_{j-1},x_j)}{t(N+1,d_j;x_{j-1},x_j)}\times$$
$$\times(x_0x_1\cdots x_{l(\sigma_d)})\prod_{j=1}^{l(\sigma_d)}\prod_{i=1}^{d_j-1}(\frac{ix_{j-1}+(d_j-i)x_j}{d_j}). \quad (A.43)$$

For the case of $d=1$, the r.h.s of (1.8) becomes the following:

$$\frac{1}{k}\cdot\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_0}dx_1\oint_{C_0}dx_0\; x_0^{N-2-n}x_1^{n-1+N-k}\frac{e(k,1;x_0,x_1)}{x_0^Nx_1^N}=$$
$$=k\frac{1}{(2\pi\sqrt{-1})^2}\oint_{C_0}dx_1\oint_{C_0}dx_0\frac{\prod_{j=1}^{k-1}(jx_0+(k-j)x_1)}{x_0^{n+1}x_1^{k-n}}. \quad (A.44)$$

Hence, (1.1) and (1.2) hold true.

(A.43) tells us that the recursive formulas for $d=2,3$ follow from the adequate decomposition of

$$(x_0x_1\cdots x_{l(\sigma_d)})\prod_{j=1}^{l(\sigma_d)}\prod_{i=1}^{d_j-1}(\frac{ix_{j-1}+(d_j-i)x_j}{d_j}).$$

The decompositions are explicitly given as follows.

d=2 case

$$\sigma_2=(2):\quad x_0x_1\frac{x_0+x_1}{2},$$
$$\sigma_2=(1,1):\quad x_0x_1x_2=x_0x_2\frac{x_0+x_2}{2}+\frac{1}{2}rx_0x_2,\ \ (r=2x_1-x_0-x_2). \quad (A.45)$$

d=3 case

$$\sigma_3=(3):\quad x_0x_1\frac{2x_0+x_1}{3}\frac{x_0+2x_1}{3}=x_0x_1(\frac{2}{9}x_0^2+\frac{5}{9}x_0x_1+\frac{2}{9}x_1^2),$$
$$\sigma_3=(2,1):\quad x_0x_1x_2\frac{x_0+x_1}{2}=x_0x_2(\frac{2}{9}x_0^2+\frac{5}{9}x_0x_2+\frac{2}{9}x_2^2+r_1(\frac{4}{9}x_0+\frac{1}{3}x_1+\frac{2}{9}x_2)),$$
$$(r_1=\frac{x_1-x_0}{2}+x_1-x_2),$$
$$\sigma_3=(1,2):\quad x_0x_1x_2\frac{x_1+x_2}{2}=x_0x_2(\frac{2}{9}x_0^2+\frac{5}{9}x_0x_2+\frac{2}{9}x_2^2+r_2(\frac{2}{9}x_0+\frac{1}{3}x_1+\frac{4}{9}x_2)),$$
$$(r_2=x_1-x_0+\frac{x_1-x_2}{2}),$$
$$\sigma_3=(1,1,1):\quad x_0x_1x_2x_3=x_0x_3(\frac{2}{9}x_0^2+\frac{5}{9}x_0x_3+\frac{2}{9}x_3^2+r_3(\frac{2}{9}x_0+\frac{1}{3}x_1+\frac{4}{9}x_3)+$$
$$+r_4(\frac{4}{9}x_0+\frac{1}{3}x_2+\frac{2}{9}x_3)+\frac{1}{3}r_3r_4),$$
$$(r_3=2x_1-x_0-x_2,\ r_4=2x_2-x_1-x_3). \quad (A.46)$$

With these decompositions, the same argument with respect to residue integrals as that used in the proof of Theorem 2 leads us to the desired recursive formulas. We can prove the recursive formulas for higher degrees by extending this approach.

References

[1] T.Coates, A.B.Givental. *Quantum Riemann-Roch, Lefschetz and Serre* Ann. of Math. (2) 165 (2007), no. 1, 15–53.

[2] A. Collino, M.Jinzenji. *On the Structure of Small Quantum Cohomology Rings for Projective Hypersurfaces* Commun.Math.Phys.206:157-183,1999.

[3] B.Forbes, M.Jinzenji *J functions, non-nef toric varieties and equivariant local mirror symmetry of curves* Int. J. Mod. Phys. A 22 (2007), no. 13, 2327–2360.

[4] P. Griffiths, J. Harris. *Principles of Algebraic Geometry* Wiley-Interscience Series of Texts, Monographs and Tracts, 1978.

[5] H.Iritani. *Quantum D-modules and Generalized Mirror Transformations* Topology 47 (2008), no. 4, 225–276.

[6] M.Jinzenji. *Coordinate change of Gauss-Manin system and generalized mirror transformation* Internat. J. Modern Phys. A 20 (2005), no. 10, 2131–2156.

[7] M.Jinzenji. *Gauss-Manin System and the Virtual Structure Constants* Int.J.Math. 13 (2002) 445-478.

[8] M.Jinzenji. *On the Quantum Cohomology Rings of General Type Projective Hypersurfaces and Generalized Mirror Transformation* Int.J.Mod.Phys. A15 (2000) 1557-1596

[9] M.Jinzenji. *Virtual Structure Constants as Intersection Numbers of Moduli Space of Polynomial Maps with Two Marked Points* Letters in Mathematical Physics, Vol.86, No.2-3, 99-114 (2008)

[10] M.Kontsevich. *Enumeration of Rational Curves via Torus Actions* The moduli space of curves, R.Dijkgraaf, C.Faber, G.van der Geer (Eds.), Progress in Math., v.129, Birkhäuser, 1995, 335-368.

[11] B.Lian, K.Liu and S.T.Yau. *Mirror Principle III* Asian J. Math. **3** (1999), no.4, 771-800.