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Algorithms for Finding a Minimum Repetition Representation of a String

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Abstract. A string with many repetitions can be written compactly by replacing h -fold contiguous repetitions of substring r with $(r)^h$. We refer to such a compact representation as a *repetition representation string* or RRS, by which a set of disjoint or nested tandem arrays can be compacted. In this paper, we study the problem of finding a *minimum RRS* or MRRS, where the size of an RRS is defined to be the sum of its component letter sizes and the sizes needed to describe the repetitions $(\cdot)^h$ which are defined as $w_R(h)$ using a repetition weight function w_R . We develop two dynamic programming algorithms to solve the problem. One is CMR that works for any repetition weight function, and the other is CMR-C that is faster but can be applied only when the repetition weight function is constant. CMR-C is an $O(w(n+z))$ -time algorithm using $O(n+z)$ space for a given string with length n , where w and z are the number of distinct primitive tandem repeats and the number of their occurrences, respectively. Since $w = O(n)$ and $z = O(n \log n)$ in the worst case, CMR-C is an $O(n^2 \log n)$ -time $O(n \log n)$ -space algorithm, which is faster than CMR by $((\log n)/n)$ -factor.

Keywords: tandem repeat, string algorithm

1 Introduction

A contiguous repeat of a substring embedded in a string is called a *tandem array*, or a *tandem repeat* (or a *square*) for a two-fold repetition. The problem of finding tandem arrays and repeats has been studied for more than two decades in the fields of computer science, mathematics and biology [4]. Efficient algorithms for finding *all* or *primitive* tandem repeats, namely, repeats whose repeated units themselves are not tandem arrays, have been developed so far [2–4].

In this paper, we consider *repetition representation strings*, or RRSs, which allows to use notations $(r)^h$ instead of contiguous sequences $rr \cdots r$ of h identical strings r . An RRS can represent many tandem arrays simultaneously but cannot always represent all tandem arrays in a string. An RRS is a string representation for a set of *disjoint* or *nested* tandem arrays in a string, where two substrings of

a string are disjoint if they have no intersection, and $(r)^h$ and $(r')^{h'}$ are nested if r contains $(r')^{h'}$. Different RRSs are possible for a string depending on which set of tandem arrays is represented. As an evaluation measure for RRSs, we use their sizes, that is, we consider the problem of finding a *minimum RRS* (MRRS) for a given string. Our expectation that an MRRS represents an essential repetition structure of a string is supported by the well-known MDL principle: “the success in finding regularities can be measured by the length with which the data can be described” (www.mdl-research.org).

We define the size of an RRS as the sum of its component sizes, namely, the sum of its component letter sizes and the sizes needed to describe the repetitions $(\cdot)^h$ which are defined as $w_R(h)$ using a repetition weight function w_R .

We developed two algorithms: CMR and CMR-C. CMR works for any repetition weight function w_R , though it is slow; it runs in $O(n^3)$ time and $O(n^2)$ space. On the other hand, CMR-C works only when the repetition weight function is constant, but it is faster; it is an $O(w(n+z))$ -time $O(n+z)$ -space algorithm using a sophisticated technique of finding tandem repeats. Here, n is the length of a given string, z is the number of (occurrences of) primitive tandem repeats in the string, and w is the number of distinct strings among them. Considering the worst case setting [1, 2]: $w = O(n)$ and $z = O(n \log n)$, these complexities correspond to $O(n^2 \log n)$ time and $O(n \log n)$ space, which are smaller than those of CMR by $((\log n)/n)$ -factor. We further note that CMR-C is more effective for smaller w and z .

2 Problem Setting

Let Σ be a finite *alphabet*, whose elements are called *letters*. A *string* s is a sequence of letters with finite length. The length of string s is the number of letters in s which is denoted by $|s|$. More general notion called a *repetition representation string*, or an RRS, is defined as follows. First, all strings are RRSs themselves. If r_1 and r_2 are RRSs, then $r_1 r_2$, concatenation of r_1 and r_2 , is also an RRS. If r is an RRS, then $(r)^h$ ($h \geq 2$) is also an RRS that is another representation of a concatenation $rr \cdots r$ of h identical RRSs r . Note that parentheses ‘(’, ‘)’ and codes for numbers are special symbols not contained in Σ . A substring represented by RRS $(r)^h$ is called a *tandem array*, and it is called a *tandem repeat* when $h = 2$. We say that $(r)^h$ is *expanded* to the concatenation $rr \cdots r$ of h identical RRSs r and reversely $rr \cdots r$ is *reduced* to $(r)^h$. String s without $(\cdot)^h$ notation is said to be *represented by an RRS* r if s is obtained by expanding all the $(\cdot)^h$ in r for any h .

The *size* of an RRS can be calculated using given two non-negative weight functions, *alphabet weight function* w_Σ on Σ and *repetition weight function* w_R on the set of natural numbers at least 2. Given w_Σ and w_R , the size $l(r)$ of an RRS r is recursively defined as follows:

$$\begin{aligned} l(\lambda) &= 0 \text{ for empty string } \lambda, \quad l(a) = w_\Sigma(a) \text{ for } a \in \Sigma, \\ l(r_1 r_2) &= l(r_1) + l(r_2) \text{ for all RRSs } r_1 \text{ and } r_2, \text{ and} \\ l((r)^h) &= l(r) + w_R(h). \end{aligned}$$

Remark 1. In most cases, we use constant functions as w_Σ and w_R for simplicity. A constant function w_R in the uniform cost model and $w_R(h) = \log h + c$ in the logarithmic cost model seem natural, where c is a some constant. In practice, w_Σ and w_R should be decided depending on applications.

The problem we deal with in this paper is the following one.

Problem 1. Given a string s , an alphabet weight function w_Σ and a repetition weight function w_R , find a minimum(-sized) RRS r that represents s .

3 Algorithms

In this section, we show algorithms for problem 1. We assume that s is an arbitrary string $a_1a_2\cdots a_n$ with length n . For $1 \leq i \leq j \leq n$, we let $s[i..j]$ denote the substring $a_i a_{i+1} \cdots a_j$ of s , and let $r_*[i..j]$ denote a minimum RRS (MRRS) representing $s[i..j]$ in the following subsections.

3.1 General Algorithm

First, we show algorithm CMR that works for any repetition weight function w_R . CMR calculates an MRRS representing a given string $s[1..n]$ by calculating MRRSs representing $s[i..j]$ for all the substrings $s[i..j]$ of $s[1..n]$ using dynamic programming. Its dynamic programming is based on the following proposition, which says that an MRRS representing a string is either a repetition of an MRRS representing its some prefix or a concatenation of two MRRSs representing its two substrings that are made by cutting the string at some position.

Proposition 1. *For all $1 \leq i < j \leq n$, $r_*[i..j]$ is one of the followings:*

- (1) $(r_*[i..i+d-1])^h$ for some $d \geq 1$ and $h \geq 2$ with $hd = j - i + 1$,
- (2) $r_*[i..i+d]r_*[i+d+1, j]$ for some $0 \leq d < j - i$.

For given $1 \leq i < j \leq n$, assume that $r_*[i'..j']$ is already known for all $1 \leq i' < j' \leq n$ with $j' - i' < j - i$. By the proposition, $r_*[i..j]$ is obtainable by searching an MRRS among at most $\lfloor (j - i + 1)/2 \rfloor$ possibilities of (1) and $j - i$ possibilities of (2). If $l(r_*[i'..j'])$ for all $1 \leq i' < j' \leq n$ with $j' - i' < j - i$ is already calculated, the size for each possible RRS can be calculated in constant time. Thus, such search can be finished in $O(j - i)$ time. This means that an MRRS $r_*[1..n]$ for a given string $s[1..n]$ is obtainable in $O(n^3)$ time and $O(n^2)$ space by dynamic programming using the fact that $r_*[i..i] = s[i..i] = a_i$ for all $1 \leq i \leq n$.

3.2 Algorithm for a Constant Repetition Weight Function

In this subsection, we describe an algorithm faster than CMR in the case with a constant repetition weight function. In this subsection, w_R is always supposed to be a constant function.

First, note that

$$l(\underbrace{(rr \cdots r)}_{h \text{ times}}^g) = hl(r) + w_R(g) \geq l(r) + w_R(hg) = l((r)^{hg})$$

because w_R is supposed to be a constant function. This means that we only have to check combinations of *primitive* tandem arrays, where a tandem array $(r)^h$ is said to be primitive if r cannot be represented by RRS $(r')^{h'}$ for any string r' and $h' \geq 2$. Such a tandem array is called a *primitive* tandem repeat when $h = 2$. It is known that there are at most $O(n \log n)$ occurrences of primitive tandem repeats in a string of length n .

Let $E(i)$ denote the set of primitive tandem repeats ending at the i th letter of s . For each tandem repeat $s[i - 2j + 1..i]$ in $E(i)$, namely, for each tandem repeat with width j and ending at i , $h_*(i, j)$ denotes an optimal number of repetitions of the last j letters as an RRS-representation of $s[1..i]$:

$$h_*(i, j) \stackrel{\text{def}}{=} \arg \min_g \{l(r(g)) : \mu(r(g)) = s[1..i], \\ r(g) = r_*[1..i - gj](r_*[i - j + 1..i])^g\},$$

where $\mu(r)$ denotes an expanded string of r . Then, $r_*[1..i - h_*(i, j)j](r_*[i - j + 1..i])^{h_*(i, j)}$ is a minimum RRS representing $s[1..i]$ among the RRSs of the form $r_*[1..i - gj](r_*[i - j + 1..i])^g$ ($g \geq 2$) by the definition, and also a minimum RRS even among all the RRSs of the form $r[1..i - gj](r[i - j + 1..i])^g$ ($g \geq 2$) for any RRSs $r[1..i - gj]$ and $r[i - j + 1..i]$ representing $s[1..i - gj]$ and $s[i - j + 1..i]$, respectively. Fortunately, $h_*(i, j)$ can be calculated efficiently using dynamic programming.

The following proposition indicates that candidate RRSs for $r_*[1..i]$ can be narrowed to only $|E(i)| + 1$ RRSs.

Proposition 2. *One of the RRSs of the following form (1) or (2) is an MRRS representing $s[1..i]$.*

- (1) $r_*[1..i - 1]a_i$,
- (2) $r_*[1..i - h_*(i, j)j](r_*[i - j + 1..i])^{h_*(i, j)}$ for $s[i - 2j + 1..i] \in E(i)$.

By the above propositions, $r_*[1..n]$ can be calculated using dynamic programming if an appropriate representation of $E(i)$ is prepared. We call such a dynamic programming algorithm FindBestComb. As an input of FindBestComb, $E(i)$ is assumed to be given by a linked list, where the list entry for tandem repeat $s[i - 2j + 1..i]$ has the repeat width j , an MRRS $r_*[i - j + 1..i]$ and its size, and the pointer to the entry for the tandem repeat $s[i - 3j + 1..i - j]$. FindBestComb calculates $l(r_*[1..i])$ in the increasing order of i . For each i , it searches an MRRS $r_*[1..i]$ among $|E(i)| + 1$ candidates shown in Proposition 2 using already calculated $l(r_*[1..i'])$ for $i' < i$. Note that $h_*(i, j)$ can be calculated using already calculated $h_*(i - j, j)$, which can be accessed through the pointer to the entry for the repeat $s[i - 3j + 1..i - j]$. An MRRS $r_*[1..n]$ can be constructed by traceback if which one among $|E(i)| + 1$ candidates is an MRRS $r_*[1..i]$ is memorized for

CMR-C %[Calculate an MRRS of a string for a Constant repetition weight function]

Input: $s[1..n]$: string,
 w_Σ : alphabet weight function, w_R : constant repetition weight function

Output: $r_*[1..n]$: an MRRS representing $s[1..n]$

Step 1 Make the suffix tree T decorated with the endpoints of all primitive tandem repeats in the vocabulary for the reversed string of s using the algorithm developed by Gusfield and Stoye [2].

Step 2 Prepare linked list $E(i)$ for $i = 1, \dots, n$ by executing Prepare_E(T 's root node).

Step 3 For each entry of tandem repeat $s[i - 2j + 1..i]$ in $E(i)$ ($i = 1, 2, \dots, n$), set an MRRS and its size of $s[i - j + 1..i]$ to the entry by executing Set_Min_Size($s[1..n], w_\Sigma, w_R, E$).

Step 4 Calculate $r_*[1..n]$ by executing FindBestComb($s[1..n], w_\Sigma, w_R, E$).

Fig. 1. Algorithm CMR-C

each $i = 1, 2, \dots, n$. The time and space complexity of FindBestComb is $O(n \log n)$ because the number of occurrences of primitive tandem repeats is $O(n \log n)$.

Linked lists $E(i)$ for all $i = 1, \dots, n$ and the pointers to the entries for tandem repeats $s[i - 3j + 1..i - j]$ which are set for all entries of primitive tandem repeat $s[i - 2j + 1..i]$ in $E(i)$ ($i = 1, \dots, n$) are calculated in $O(n \log n)$ time using the suffix tree of s decorated with the endpoints of all primitive tandem repeats in the vocabulary, which can be constructed in $O(n)$ time by the algorithm developed by Gusfield and Stoye [2]. The calculation can be done by postorder traversal of the decorated suffix tree for the *reversed string* of s . We call such a recursive algorithm Prepare_E. We also let each entry in $E(i)$ have a pointer to the entry for the leftmost occurrence of the same type tandem repeat, which is also set in algorithm Prepare_E. This pointer is used to save computational cost for occurrences of the same type tandem repeat as described below. Since the number of occurrences of primitive tandem repeats are $O(n \log n)$, the above procedure is done in $O(n \log n)$ time and $O(n \log n)$ space.

There is still one thing we have to do before executing FindBestComb for the whole string. An MRRS $r_*[i - j + 1..i]$ and its size $l(r_*[i - j + 1..i])$ in each entry for tandem repeat $s[i - 2j + 1..i]$ must be calculated. There are $O(n \log n)$ occurrences of primitive tandem repeats but MRRSs and their size have to be calculated only for distinct tandem repeats, namely, tandem repeats in the *vocabulary* (the set of different repeat types) [2], the number of which is $O(n)$. Applying FindBestComb to $s[i - j + 1..i]$ for each primitive tandem repeat $s[i - 2j + 1..i]$ in the vocabulary, an MRRS $r_*[i - j + 1..i]$ and its size $l(r_*[i - j + 1..i])$ can be calculated in $O(n \log n)$ time and $O(n \log n)$ space, so the minimum size for all tandem repeats in the vocabulary can be obtained in $O(n^2 \log n)$ time and $O(n \log n)$ space. We call the algorithm for this task Set_Min_Size.

Putting all together, algorithm CMR-C shown in Fig. 1, an algorithm finding an MRRS representing $s[1..n]$, is obtained.

By the argument so far, we have proved the following theorem.

Theorem 1. $r_*[1..n]$ can be calculated in $O(n^2 \log n)$ time and $O(n \log n)$ space.

If the number of occurrences of primitive tandem repeats is z , FindBestComb and Prepare_E can be executed in $O(n+z)$ time and $O(n+z)$ space. Furthermore, if the number of primitive tandem repeats in the vocabulary is w , Set_Min_Len can be executed in $O(w(n+z))$ time and $O(n+z)$ space. Thus, the following corollary holds.

Corollary 1. $r_*[1..n]$ can be calculated in $O(w(n+z))$ time and $O(n+z)$ space when $w > 0$, where w and z is the number of primitive tandem repeats in the vocabulary and the number of their occurrences, respectively.

4 Concluding Remarks

In this paper, we considered representations called an MRRS for a string, in which a set of disjoint or nested contiguous repeats are compacted, and proposed dynamic programming algorithms CMR and CMR-C to calculate it for a given string. CMR-C was theoretically proved to run in $O(w(n+z))$ time and $O(n+z)$ space, which indicates that it is fast if the number of distinct tandem repeat w and the number of its occurrences z are not large compared to the string length n . According to our experiments on DELL Precision T7500 (cpu: Intel(R) Xeon(R) E5520 [2.27GHz], memory: 2GB), CMR-C is fast enough for large-scale, synthetic datasets on strings; for a random binary string of 1,638,400 letters, 8.2 seconds were enough for CMR-C to find an MRRS. The size of MRRS can be a measure of how well a string is organized in terms of repeated structures. From our experimental result of such a structural complexity analysis applied to DNA sequences or chromosomes of the most major nine species (*A. thaliana*, *C. elegans*, zebrafish, fruit fly, chicken, human, mouse, yeast and rat), we found that the MRRS size was unique for each species.

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