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THE FREENESS AND MINIMAL FREE RESOLUTIONS OF MODULES OF DIFFERENTIAL OPERATORS OF A GENERIC HYPERPLANE ARRANGEMENT

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ABSTRACT. Let \mathcal{A} be a generic hyperplane arrangement composed of r hyperplanes in an n -dimensional vector space, and S the polynomial ring in n variables. We consider the S -submodule $D^{(m)}(\mathcal{A})$ of the n th Weyl algebra of homogeneous differential operators of order m preserving the defining ideal of \mathcal{A} .

We prove that if $n \geq 3, r > n, m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free (Holm's conjecture). Combining this with some results by Holm, we see that $D^{(m)}(\mathcal{A})$ is free unless $n \geq 3, r > n, m < r - n + 1$. In the remaining case, we construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ by generalizing Yuzvinsky's construction for $m = 1$. In addition, we construct a minimal free resolution of the transpose of the m -jet module, which generalizes a result by Rose and Terao for $m = 1$.

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1. INTRODUCTION

In the study of a hyperplane arrangement, its derivation module plays a central character; in particular, its freeness over the polynomial ring attracts a great interest (see, e.g., Orlik-Terao [6]). Generalizing the study of the derivation module for a hyperplane arrangement to that of the modules of differential operators of higher order was initiated by Holm [4], [5]. In particular, he studied the case of generic hyperplane arrangements in detail.

Let K denote a field of characteristic zero, and \mathcal{A} a generic hyperplane arrangement in K^n composed of r hyperplanes. Let S be the polynomial ring $K[x_1, \dots, x_n]$, and $D^{(m)}(\mathcal{A})$ the S -module of homogeneous differential operators of order m of the hyperplane arrangement \mathcal{A} .

Among others, in [5], Holm gave a finite generating set of the S -module $D^{(m)}(\mathcal{A})$. As to the freeness of $D^{(m)}(\mathcal{A})$, Holm [4] (cf. [9]) proved the following:

- If $n = 2$, then $D^{(m)}(\mathcal{A})$ is free for any m .
- If $n \geq 3, r > n, m < r - n + 1$, then $D^{(m)}(\mathcal{A})$ is not free.
- If $n \geq 3, r > n, m = r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free.

Holm also conjectured that if $n \geq 3, r > n, m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free.

Snellman [9] computed the Hilbert series of $D^{(m)}(\mathcal{A})$, which supported Holm's conjecture when $n \geq 3, r > n, m > r - n + 1$, and he conjectured the Poicaré-Betti series of $D^{(m)}(\mathcal{A})$ when $n \geq 3, r > n, m < r - n + 1$.

In the derivation module case, when $n \geq 3, r > n, m < r - n + 1$ with $m = 1$, Rose-Terao [7] and Yuzvinsky [11] independently gave a minimal free resolution of $D^{(1)}(\mathcal{A})$. In the course of the proof, Rose-Terao [7] gave minimal free resolutions of all modules of

logarithmic differential forms with poles along \mathcal{A} . They also gave a minimal free resolution of S/J , where J is the Jacobian ideal of a polynomial defining \mathcal{A} . Yuzvinsky's construction [11] is more straightforward and combinatorial than [7].

In this paper, we prove Holm's conjecture, namely, we prove that if $n \geq 3, r > n, m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free. Hence, for a generic hyperplane arrangement \mathcal{A} , $D^{(m)}(\mathcal{A})$ is free unless $n \geq 3, r > n, m < r - n + 1$. In the remaining case $n \geq 3, r > n, m < r - n + 1$, we construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ by generalizing [11] and a minimal free resolution of the transpose of the m -jet module generalizing that of S/J given by [7].

After we fix notation on differential operators for a hyperplane arrangement in §2, we recall the Saito-Holm criterion in §3. It was proved by Holm, and it is a criterion for a subset of $D^{(m)}(\mathcal{A})$ to form a basis, which generalizes the Saito criterion in the case of $m = 1$.

From §4 on, we assume that $r \geq n$ and the hyperplane arrangement \mathcal{A} is generic. In §4, we recall the finite generating set of $D^{(m)}(\mathcal{A})$ given by Holm [5]. Then we recall the case $n = 2$ in §5 and the case $m = r - n + 1$ in §6 for completeness. In §7, we consider the case $m \geq r - n + 1$ and prove Holm's conjecture (Theorem 7.1).

From §8 on, we consider the case $m < r - n + 1$. In §8, we give a minimal generating set of $D^{(m)}(\mathcal{A})$ (Theorem 8.3). In §9, we generalize [11] to construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ (Theorem 9.10). In §10, we generalize the minimal free resolution of S/J given in [7] (Theorem 10.7). In §11, we prove that the S -module considered in §10 is the transpose of the m -jet module $\Omega^{[1,m]}(S/SQ)$ (Theorem 11.2), where Q is a polynomial defining \mathcal{A} .

2. THE MODULES OF DIFFERENTIAL OPERATORS FOR A HYPERPLANE ARRANGEMENT

Throughout this paper, let K denote a field of characteristic zero, \mathcal{A} a central hyperplane arrangement in K^n composed of r hyperplanes, and S the polynomial ring $K[x_1, \dots, x_n]$. We assume that $n \geq 2$.

For a hyperplane $H \in \mathcal{A}$, we fix a linear form $p_H \in S$ defining H . Set

$$(2.1) \quad Q := Q_{\mathcal{A}} := \prod_{H \in \mathcal{A}} p_H.$$

Let $D(S) = S\langle \partial_1, \dots, \partial_n \rangle$ denote the n th Weyl algebra, where $\partial_j = \frac{\partial}{\partial x_j}$. For a nonzero differential operator $P = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha}(x) \partial^{\alpha} \in D(S)$, the maximum of $|\alpha|$ with $f_{\alpha} \neq 0$ is called the *order* of P , where

$$\partial^{\alpha} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

for $\alpha = (\alpha_1, \dots, \alpha_n)$. If P has no nonzero f_{α} with $|\alpha| \neq m$, it is said to be *homogeneous of order m* . We denote by $D^{(m)}(S)$ the S -submodule of $D(S)$ of differential operators homogeneous of order m .

We denote by $*$ the action of $D(S)$ on S . For an ideal I of S ,

$$(2.2) \quad D(I) := \{\theta \in D(S) \mid \theta * I \subseteq I\}$$

is called the *idealizer* of I .

We set

$$(2.3) \quad D(\mathcal{A}) := D(\langle Q \rangle).$$

Holm [5, Theorem 2.4] proved

$$(2.4) \quad D(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D(\langle p_H \rangle).$$

We denote by $D^{(m)}(\mathcal{A})$ the S -submodule of $D(\mathcal{A})$ of differential operators homogeneous of order m . Then Holm [5, Proposition 4.3] proved

$$D(\mathcal{A}) = \bigoplus_{m=0}^{\infty} D^{(m)}(\mathcal{A}).$$

A differential operator homogeneous of order 1 is nothing but a derivation. Hence $D^{(1)}(\mathcal{A})$ is the module of logarithmic derivations along \mathcal{A} .

The polynomial ring $S = \bigoplus_{p=0}^{\infty} S_p$ is a graded algebra, where S_p is the K -vector subspace spanned by the monomials of degree p . The n th Weyl algebra $D(S)$ is a graded S -module with $\deg(x^\alpha \partial^\beta) = |\alpha| - |\beta|$. Each $D^{(m)}(\mathcal{A})$ is a graded S -submodule of $D(S)$. An element $P = \sum_{\alpha \in \mathbb{N}^n} f_\alpha(x) \partial^\alpha \in D^{(m)}(\mathcal{A})$ is said to be *homogeneous of polynomial degree* p , and denoted by $\text{pdeg} P = p$, if $f_\alpha \in S_p$ for all α with nonzero f_α .

3. SAITO-HOLM CRITERION

To prove that $D^{(1)}(\mathcal{A})$ is a free S -module, the Saito criterion ([8, Theorem 1.8 (ii)], see also [6, Theorem 4.19]) is very useful. Holm [4] generalized the Saito criterion to the one for $D^{(m)}(\mathcal{A})$. In this section, we briefly review Holm's generalization.

Set

$$s_m := \binom{n+m-1}{m}, \quad t_m := \binom{n+m-2}{m-1}.$$

Let

$$\{x^{\alpha^{(1)}}, x^{\alpha^{(2)}}, \dots, x^{\alpha^{(s_m)}}\}$$

be the set of monomials of degree m . For operators $\theta_1, \dots, \theta_{s_m}$, define an $s_m \times s_m$ coefficient matrix $M_m(\theta_1, \dots, \theta_{s_m})$ by

$$M_m(\theta_1, \dots, \theta_{s_m}) := \begin{bmatrix} \theta_1 * \frac{x^{\alpha^{(1)}}}{\alpha^{(1)}!} & \cdots & \theta_{s_m} * \frac{x^{\alpha^{(1)}}}{\alpha^{(1)}!} \\ \vdots & \ddots & \vdots \\ \theta_1 * \frac{x^{\alpha^{(s_m)}}}{\alpha^{(s_m)}!} & \cdots & \theta_{s_m} * \frac{x^{\alpha^{(s_m)}}}{\alpha^{(s_m)}!} \end{bmatrix},$$

where $\alpha! = (\alpha_1!)(\alpha_2!) \cdots (\alpha_n!)$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

The proofs of the following two propositions go similarly to those of [6, Proposition 4.12] and [6, Proposition 4.18].

Proposition 3.1 (III Proposition 5.2 in [4] (cf. Proposition 4.12 in [6])). *If $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$, then*

$$\det M_m(\theta_1, \dots, \theta_{s_m}) \in \langle Q^{t_m} \rangle.$$

Proposition 3.2 (III Proposition 5.7 in [4] (cf. Proposition 4.18 in [6])). *Suppose that $D^{(m)}(\mathcal{A})$ is a free S -module. Then the rank of $D^{(m)}(\mathcal{A})$ is s_m .*

The following is a generalization of the Saito criterion. This was proved by Holm [4, III Theorem 5.8].

Theorem 3.3 (Saito-Holm criterion). *Given $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$, the following two conditions are equivalent:*

- (1) $\det M_m(\theta_1, \dots, \theta_{s_m}) = cQ_{\mathcal{A}}^{t_m}$ for some $c \in K^\times$,
- (2) $\theta_1, \dots, \theta_{s_m}$ form a basis for $D^{(m)}(\mathcal{A})$ over S .

The following is an easy consequence of Theorem 3.3.

Theorem 3.4 (III Theorem 5.9 in [4] (cf. Theorem 4.23 in [6])). *Let $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$ be linearly independent over S . Then $\theta_1, \dots, \theta_{s_m}$ form a basis for $D^{(m)}(\mathcal{A})$ over S if and only if*

$$\sum_{j=1}^{s_m} \text{pdeg } \theta_j = rt_m.$$

Suppose that $D^{(m)}(\mathcal{A})$ is free over S . We denote by $\exp D^{(m)}(\mathcal{A})$ the multi-set of polynomial degrees of a basis for $D^{(m)}(\mathcal{A})$. The expression

$$\exp D^{(m)}(\mathcal{A}) = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \dots\}$$

means that $\exp D^{(m)}(\mathcal{A})$ has e_i i 's ($i = 0, 1, 2, \dots$).

Proposition 3.5 (cf. Proposition 4.26 in [6]). *Assume that $D^{(m)}(\mathcal{A})$ is free over S , and suppose that*

$$\exp D^{(m)}(\mathcal{A}) = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \dots\}.$$

Then

$$\sum_k e_k = s_m, \quad \sum_k k e_k = rt_m.$$

Proof. Proposition 3.2 is the first statement, and Theorem 3.4 the second. □

4. GENERIC ARRANGEMENTS

In the rest of this paper, we assume that $r \geq n$ and \mathcal{A} is generic. An arrangement \mathcal{A} is said to be *generic*, if every n hyperplanes of \mathcal{A} intersect only at the origin.

For a finite set \mathcal{S} , let $\mathcal{S}^{(k)} \subseteq 2^{\mathcal{S}}$ denote the set of $\mathcal{T} \subseteq \mathcal{S}$ with $\#\mathcal{T} = k$.

Given $\mathcal{H} \in \mathcal{A}^{(n-1)}$, the vector space

$$\{\delta \in \sum_{i=1}^n K \partial_i \mid \delta * p_H = 0 \text{ for all } H \in \mathcal{H}\}$$

is one-dimensional; fix a nonzero element $\delta_{\mathcal{H}}$ of this space. Note that

$$(4.1) \quad \delta_{\mathcal{H}} * p_H = 0 \Leftrightarrow H \in \mathcal{H},$$

since \mathcal{A} is generic.

For $\mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$, put

$$(4.2) \quad P_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} := \prod_{H \notin \bigcap_{i=1}^m \mathcal{H}_i} p_H.$$

Then $P_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} \in D^{(m)}(\mathcal{A})$ by (2.4). In particular, for $\mathcal{H} \in \mathcal{A}^{(n-1)}$,

$$P_{\mathcal{H}} \delta_{\mathcal{H}}^m \in D^{(m)}(\mathcal{A}),$$

where $P_{\mathcal{H}} := P_{\{\mathcal{H}\}}$. Note that

$$(4.3) \quad \deg P_{\mathcal{H}} = r - n + 1.$$

The operator

$$(4.4) \quad \epsilon_m := \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha \partial^\alpha$$

is called *the Euler operator of order m*. Then ϵ_1 is the Euler derivation, and $\epsilon_m = \epsilon_1(\epsilon_1 - 1) \cdots (\epsilon_1 - m + 1)$ [5, Lemma 4.9].

Holm gave a finite set of generators of $D^{(m)}(\mathcal{A})$ as an S -module:

Theorem 4.1 (Theorem 4.22 in [5]).

$$D^{(m)}(\mathcal{A}) = \sum_{\mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}} SP_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} + S\epsilon_m.$$

The following lemma will be used in Sections 7, 8, and 9.

Lemma 4.2. (1) *The set $\{\delta_{\mathcal{H}}^{r-n+1} \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K -basis of $\sum_{|\alpha|=r-n+1} K\partial^\alpha$.*

(2) *The set $\{P_{\mathcal{H}} \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K -basis of $\sum_{|\alpha|=r-n+1} Kx^\alpha = S_{r-n+1}$.*

Proof. The dimensions of $\sum_{|\alpha|=r-n+1} K\partial^\alpha$ and S_{r-n+1} are equal to

$$s_{r-n+1} = \binom{r}{r-n+1} = \binom{r}{n-1} = \#\mathcal{A}^{(n-1)}.$$

Let $\mathcal{H}, \mathcal{H}' \in \mathcal{A}^{(n-1)}$. Then

$$(4.5) \quad \delta_{\mathcal{H}}^{r-n+1} * P_{\mathcal{H}'} = \delta_{\mathcal{H}}^{r-n+1} * \prod_{H \notin \mathcal{H}'} p_H = \begin{cases} (r-n+1)! \prod_{H \notin \mathcal{H}} (\delta_{\mathcal{H}} * p_H) & \text{if } \mathcal{H}' = \mathcal{H} \\ 0 & \text{otherwise.} \end{cases}$$

The assertions follow, since $\delta_{\mathcal{H}} * p_H = 0$ if and only if $H \in \mathcal{H}$. \square

5. THE CASE $n = 2$

In this section, we consider central arrangements with $r \geq 2$ in K^2 , which are always generic. Note that $s_m = m + 1$, and $t_m = m$.

Let $\mathcal{A} = \{H_1, H_2, \dots, H_r\}$. Put $p_i := p_{H_i}$, $P_i := P_{\{H_i\}}$, and $\delta_i := \delta_{\{H_i\}}$ for $i = 1, 2, \dots, r$.

We may assume that there exist distinct $a_2, \dots, a_r \in K$ such that

$$p_1 = x_1, \quad p_i = x_2 - a_i x_1 \quad (i = 2, \dots, r).$$

Then

$$\delta_1 = \partial_2, \quad \delta_i = \partial_1 + a_i \partial_2 \quad (i = 2, \dots, r),$$

and

$$P_i = Q/p_i \quad (i = 1, \dots, r).$$

Proposition 5.1 (Proposition 6.7 III in [4], Proposition 4.14 in [9]). *The S -module $D^{(m)}(\mathcal{A})$ is free with the following basis:*

- (1) $\{\epsilon_m, P_1 \delta_1^m, \dots, P_m \delta_m^m\}$ if $m \leq r - 2$.
- (2) $\{P_1 \delta_1^m, \dots, P_r \delta_r^m\}$ if $m = r - 1$.
- (3) $\{P_1 \delta_1^m, \dots, P_r \delta_r^m, Q \eta_{r+1}, \dots, Q \eta_{m+1}\}$ if $m \geq r$, where $\{\delta_1^m, \dots, \delta_r^m, \eta_{r+1}, \dots, \eta_{m+1}\}$ is a K -basis of $\sum_{i=0}^m K \partial_1^i \partial_2^{m-i}$.

Corollary 5.2.

$$\exp D^{(m)}(\mathcal{A}) = \begin{cases} \{m^1, (r-1)^m\} & (1 \leq m \leq r-2), \\ \{(r-1)^{m+1}\} & (m = r-1), \\ \{(r-1)^r, r^{m-r+1}\} & (m \geq r). \end{cases}$$

6. THE CASE $m = r - n + 1$

In this section, we consider the case $m = r - n + 1$. In this case,

$$(6.1) \quad s_m = \binom{n+m-1}{m} = \binom{r}{m} = \binom{r}{n-1}.$$

Note also that $\deg P_{\mathcal{H}} = r - n + 1 = m$ (4.3).

In Sections 7, 8, and 9, we use Lemma 4.2 in the case $m = r - n + 1$. Lemma 4.2 reads as follows in this case:

Lemma 6.1. (1) *The set $\{\delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K -basis of $\sum_{|\alpha|=m} K \partial^\alpha$.*

(2) *The set $\{P_{\mathcal{H}} \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K -basis of $\sum_{|\alpha|=m} K x^\alpha = S_m$.*

Proposition 6.2 (III Proposition 6.8 in [4]). *The S -module $D^{(m)}(\mathcal{A})$ is free with a basis $\{P_{\mathcal{H}} \delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$.*

Corollary 6.3. *If $m = r - n + 1$, then*

$$\exp D^{(m)}(\mathcal{A}) = \{m \binom{r}{m}\}.$$

7. THE CASE $m \geq r - n + 1$

In this section, we assume that $m \geq r - n + 1$, and we prove Holm's conjecture by giving a basis of $D^{(m)}(\mathcal{A})$.

Set

$$\tilde{r} := n + m - 1,$$

and add $\tilde{r} - r$ hyperplanes to $\mathcal{A} = \{H_1, \dots, H_r\}$ so that

$$(7.1) \quad \tilde{\mathcal{A}} := \mathcal{A} \cup \{H_{r+1}, \dots, H_{\tilde{r}}\}$$

is still generic.

For $\mathcal{H} \in \tilde{\mathcal{A}}^{(n-1)}$, define a homogeneous polynomial $P'_{\mathcal{H}} \in S$ by

$$(7.2) \quad P'_{\mathcal{H}} := \prod_{H \notin \mathcal{H}; H \in \mathcal{A}} p_H.$$

Theorem 7.1. *The S -module $D^{(m)}(\mathcal{A})$ is free with a basis $\{P'_{\mathcal{H}} \delta_{\mathcal{H}}^m \mid \mathcal{H} \in \tilde{\mathcal{A}}^{(n-1)}\}$.*

Proof. By (2.4), $P'_{\mathcal{H}} \delta_{\mathcal{H}}^m \in D^{(m)}(\mathcal{A})$ for each $\mathcal{H} \in \tilde{\mathcal{A}}^{(n-1)}$.

By Lemma 6.1 (1), $\{P'_{\mathcal{H}} \delta_{\mathcal{H}}^m \mid \mathcal{H} \in \tilde{\mathcal{A}}^{(n-1)}\}$ is linearly independent over S . Since

$$\deg P'_{\mathcal{H}} = \#\{H \in \mathcal{A} \mid H \notin \mathcal{H}\},$$

the number of $\mathcal{H} \in \tilde{\mathcal{A}}^{(n-1)}$ with $\deg P'_{\mathcal{H}} = j$ is

$$\binom{r}{j} \binom{\tilde{r} - r}{n - 1 - (r - j)} = \binom{r}{j} \binom{m + n - r - 1}{n - r + j - 1}.$$

Then

$$\begin{aligned}
\sum_j j \binom{r}{j} \binom{m+n-r-1}{n-r+j-1} &= r \sum_j \binom{r-1}{j-1} \binom{m+n-r-1}{n-r+j-1} \\
&= r \sum_j \binom{r-1}{j-1} \binom{m+n-r-1}{m-j} \\
&= r \binom{m+n-2}{m-1} = rt_m.
\end{aligned}$$

Hence we have the assertion by Theorem 3.4. \square

Corollary 7.2.

$$\exp D^{(m)}(\mathcal{A}) = \{j \binom{r}{j} \binom{m+n-r-1}{m-j} \mid r-n+1 \leq j \leq \min\{r, m\}\}.$$

8. THE CASE $m < r - n + 1$

Throughout this section, we assume that $m < r - n + 1$.

Recall that $D^{(m)}(\mathcal{A})$ is generated by

$$(8.1) \quad \{P_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} \mid \mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}\} \cup \{\epsilon_m\}$$

over S (Theorem 4.1). In this section, we choose a minimal system of generators from (8.1) (Theorem 8.3), which implies that $D^{(m)}(\mathcal{A})$ is not free (Remark 8.5).

Note that

$$\#\mathcal{A}^{(n-1)} = \binom{r}{n-1} > \binom{n+m-1}{n-1} = s_m.$$

Lemma 8.1. *For any $\mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$, the following hold:*

- (1) $P_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} \in \bigcap_{\cap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} SP_{\mathcal{H}}$.
- (2) $\delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} \in \sum_{\cap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m$.

Proof. (1) If $\cap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}$, then $P_{\mathcal{H}} = \prod_{H \notin \mathcal{H}} p_H$ divides $\prod_{H \notin \cap_{i=1}^m \mathcal{H}_i} p_H = P_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}}$. Hence the assertion is clear.

(2) Let $\bar{r} := n + m - 1$. Take a subarrangement $\mathcal{B} \supset \cap_{i=1}^m \mathcal{H}_i$ of \mathcal{A} with \bar{r} hyperplanes. By Lemma 6.1, there exist $c_{\mathcal{H}} \in K$ ($\mathcal{H} \in \mathcal{B}^{(n-1)}$) such that

$$(8.2) \quad \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}} \delta_{\mathcal{H}}^m.$$

It suffices to show that $c_{\mathcal{H}} = 0$ for all $\mathcal{H} \not\supset \cap_{i=1}^m \mathcal{H}_i$. Fix $\mathcal{H} \in \mathcal{B}^{(n-1)}$ with $\mathcal{H} \not\supset \cap_{i=1}^m \mathcal{H}_i$, and put $\bar{P}_{\mathcal{H}} = \prod_{H \in \mathcal{B} \setminus \mathcal{H}} p_H$. Then $\deg \bar{P}_{\mathcal{H}} = \bar{r} - (n-1) = m$. Since there exists $H_0 \in (\cap_{i=1}^m \mathcal{H}_i) \setminus \mathcal{H}$, we have $\delta_{\mathcal{H}_i} * p_{H_0} = 0$ for all $i = 1, \dots, m$, and hence $\delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} * \bar{P}_{\mathcal{H}} = 0$. Recall from (4.5) that

$$\delta_{\mathcal{H}'}^m * \bar{P}_{\mathcal{H}} = \begin{cases} m! \prod_{H \in \mathcal{B} \setminus \mathcal{H}} (\delta_{\mathcal{H}'} * p_H) \neq 0 & \text{if } \mathcal{H}' = \mathcal{H} \\ 0 & \text{otherwise.} \end{cases}$$

Let the operator (8.2) act on $\bar{P}_{\mathcal{H}}$. Since

$$0 = \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} * \bar{P}_{\mathcal{H}} = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}} \delta_{\mathcal{H}}^m * \bar{P}_{\mathcal{H}} = c_{\mathcal{H}} \cdot m! \prod_{H \in \mathcal{B} \setminus \mathcal{H}} (\delta_{\mathcal{H}} * p_H),$$

we have $c_{\mathcal{H}} = 0$. □

Proposition 8.2. *If $m < r - n + 1$, then*

$$D^{(m)}(\mathcal{A}) = \sum_{\mathcal{H} \in \mathcal{A}^{(n-1)}} SP_{\mathcal{H}} \delta_{\mathcal{H}}^m + S\epsilon_m$$

Proof. Let $\mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$. By Lemma 8.1,

$$\begin{aligned} P_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} \delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} &\in P_{\{\mathcal{H}_1, \dots, \mathcal{H}_m\}} \cdot \sum_{\cap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m \\ &\subset \sum_{\cap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} SP_{\mathcal{H}} \delta_{\mathcal{H}}^m. \end{aligned}$$

Hence we obtain the assertion from (8.1). □

The system of generators for $D^{(m)}(\mathcal{A})$ in Proposition 8.2 is still large. Next, **we fix m hyperplanes H_1, \dots, H_m** , and define an S -submodule $\Xi^{(m)}(\mathcal{A})$ of $D^{(m)}(\mathcal{A})$ by

$$(8.3) \quad \Xi^{(m)}(\mathcal{A}) := \{\theta \in D^{(m)}(\mathcal{A}) \mid \theta * (p_{H_1} \cdots p_{H_m}) = 0\}.$$

For $\mathcal{H} \in \mathcal{A}^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset$, we have $\delta_{\mathcal{H}} * p_{H_i} = 0$ for some i , and hence $P_{\mathcal{H}} \delta_{\mathcal{H}}^m \in \Xi^{(m)}(\mathcal{A})$. Furthermore we have the following.

Theorem 8.3. *If $m < r - n + 1$, then*

$$D^{(m)}(\mathcal{A}) = \Xi^{(m)}(\mathcal{A}) \oplus S\epsilon_m = \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset}} SP_{\mathcal{H}} \delta_{\mathcal{H}}^m \oplus S\epsilon_m$$

Moreover, the set $\{P_{\mathcal{H}} \delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}, \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset\}$ is a minimal system of generators for $\Xi^{(m)}(\mathcal{A})$ over S .

Proof. Let $\theta \in D^{(m)}(\mathcal{A})$. Then $\theta - \frac{1}{m!} \frac{\theta * (p_{H_1} \cdots p_{H_m})}{p_{H_1} \cdots p_{H_m}} \epsilon_m \in \Xi^{(m)}(\mathcal{A})$, since $\theta \in D^{(m)}(\mathcal{A}) \subset D^{(m)}(\langle p_{H_1} \cdots p_{H_m} \rangle)$ by (2.4). So we have $D^{(m)}(\mathcal{A}) = \Xi^{(m)}(\mathcal{A}) + S\epsilon_m$. Moreover, $\epsilon_m * (p_{H_1} \cdots p_{H_m}) = m! p_{H_1} \cdots p_{H_m} \neq 0$ implies that $\Xi^{(m)}(\mathcal{A}) \cap S\epsilon_m = 0$.

Next, we show the second equality. By Proposition 8.2, it suffices to show that

$$P_{\mathcal{H}_0} \delta_{\mathcal{H}_0}^m \in \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset}} SP_{\mathcal{H}} \delta_{\mathcal{H}}^m \oplus S\epsilon_m$$

for every $\mathcal{H}_0 \in \mathcal{A}^{(n-1)}$ with $\mathcal{H}_0 \cap \{H_1, \dots, H_m\} = \emptyset$. Put $\mathcal{B} := \mathcal{H}_0 \cup \{H_1, \dots, H_m\}$. By Proposition 6.2,

$$D^{(m)}(\mathcal{B}) = \bigoplus_{\mathcal{H} \in \mathcal{B}^{(n-1)}} S\bar{P}_{\mathcal{H}} \delta_{\mathcal{H}}^m,$$

where $\bar{P}_{\mathcal{H}} = \prod_{H \in \mathcal{B} \setminus \mathcal{H}} p_H$. Since $\epsilon_m \in D^{(m)}(\mathcal{B})$, there exist $c_{\mathcal{H}} \in S$ ($\mathcal{H} \in \mathcal{B}^{(n-1)}$) such that

$$(8.4) \quad \epsilon_m = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}} \bar{P}_{\mathcal{H}} \delta_{\mathcal{H}}^m.$$

(By looking at polynomial degrees, we see $c_{\mathcal{H}} \in K$.) Multiplying $q := \prod_{H \in \mathcal{A} \setminus \mathcal{B}} p_H$ from the left, we have

$$(8.5) \quad q\epsilon_m = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m.$$

Let the operator (8.4) act on $\bar{P}_{\mathcal{H}_0}$. Since

$$0 \neq m! \bar{P}_{\mathcal{H}_0} = m! c_{\mathcal{H}_0} \bar{P}_{\mathcal{H}_0} \cdot \prod_{H \in \mathcal{B} \setminus \mathcal{H}_0} \delta_{\mathcal{H}_0} * p_H = m! c_{\mathcal{H}_0} \bar{P}_{\mathcal{H}_0} \cdot \prod_{i=1}^m (\delta_{\mathcal{H}_0} * p_{H_i}),$$

we have $c_{\mathcal{H}_0} \neq 0$. Hence, we have

$$P_{\mathcal{H}_0} \delta_{\mathcal{H}_0}^m = c_{\mathcal{H}_0}^{-1} \left(q\epsilon_m - \sum_{\substack{\mathcal{H} \in \mathcal{B}^{(n-1)} \\ \mathcal{H} \neq \mathcal{H}_0}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m \right) \in \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset}} SP_{\mathcal{H}} \delta_{\mathcal{H}}^m \oplus S\epsilon_m.$$

Finally, we show the minimality. It suffices to show that the set $\{P_{\mathcal{H}} \delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}, \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset\}$ is linearly independent over K , since all $P_{\mathcal{H}} \delta_{\mathcal{H}}^m$ have the same polynomial degree. Suppose that

$$(8.6) \quad \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m = 0 \quad (c_{\mathcal{H}} \in K).$$

Fix arbitrary hyperplanes $H_{i_1}, \dots, H_{i_m} \in \mathcal{A}$, and put $q' := p_{H_{i_1}} \cdots p_{H_{i_m}}$ and $\mathcal{B}' := \mathcal{A} \setminus \{H_{i_1}, \dots, H_{i_m}\}$. Let the operator (8.6) act on q' . Then we have

$$\sum_{\substack{\mathcal{H} \in \mathcal{B}'^{(n-1)} \\ \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset}} c_{\mathcal{H}} P_{\mathcal{H}} \prod_{\nu=1}^m (\delta_{\mathcal{H}} * p_{H_{i_\nu}}) = 0.$$

By Lemma 4.2, the set $\{P_{\mathcal{H}} \mid \mathcal{H} \in \mathcal{B}'^{(n-1)}\}$ is linearly independent over K . Hence $c_{\mathcal{H}} = 0$ for $\mathcal{H} \in \mathcal{B}'^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset$. For $\mathcal{H} \in \mathcal{A}^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset$, we may take $H_{i_1}, \dots, H_{i_m} \in \mathcal{A}$ so that $\mathcal{H} \in \mathcal{B}'^{(n-1)}$, since $r > m + n - 1$. Hence we have finished the proof. \square

Corollary 8.4 (cf. Conjecture 6.8 in [9]). *The S -module $\Xi^{(m)}(\mathcal{A})$ is minimally generated by $\binom{r}{n-1} - \binom{r-m}{n-1}$ operators of polynomial degree $r - n + 1$.*

Remark 8.5. We can show

$$\binom{r}{n-1} - \binom{r-m}{n-1} + 1 > \binom{n+m-1}{n-1},$$

supposing that $m < r - n + 1$. Then by Proposition 3.2 and Corollary 8.4 we see that, for $n \geq 3$ and $m < r - n + 1$, $D^{(m)}(\mathcal{A})$ is not free over S , which was proved by Holm [4, III Proposition 6.8].

9. GENERALIZATION OF YUZVINSKY'S PAPER [11]

In this section, we assume $m \leq r - n + 1$, and we construct a minimal free resolution of $\Xi^{(m)}(\mathcal{A})$ when $m < r - n + 1$ and $n \geq 3$. We generalize the construction in [11] step by step, and basically we succeed Yuzvinsky's notation.

Let $V := K^n$. Recall that, for $\mathcal{H} \in \mathcal{A}^{(n-1)}$, $\delta_{\mathcal{H}} \in (V^*)^* = V$ is the nonzero derivation with constant coefficients such that $\delta_{\mathcal{H}} * p_H = 0$ for all $H \in \mathcal{H}$. Under the identification $(V^*)^* = V$, $K\delta_{\mathcal{H}}$ corresponds to the linear subspace $[\mathcal{H}] := \bigcap_{H \in \mathcal{H}} H = \bigcap_{H \in \mathcal{H}} (p_H = 0)$ of V . Similarly, $\mathcal{H} \in \mathcal{A}^{(n-j)}$ corresponds to the linear subspace $[\mathcal{H}] = \bigcap_{H \in \mathcal{H}} H \in L_j$, where L_j is the set of elements of dimension j of the intersection lattice of \mathcal{A} .

For $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $1 \leq j \leq n$, set

$$\Delta_{\mathcal{H}} := \sum_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-1)}} K\delta_{\mathcal{H} \cup \mathcal{H}}^m.$$

Note that

$$\Delta_{\mathcal{H}} = K\delta_{\mathcal{H}}^m \quad \text{for } \mathcal{H} \in \mathcal{A}^{(n-1)},$$

and

$$\Delta_{\emptyset} = \sum_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K\delta_{\mathcal{H}}^m.$$

Each $\Delta_{\mathcal{H}}$ is a subspace of Δ_{\emptyset} .

Example 9.1. Let $m = 1$. Then

$$\Delta_{\mathcal{H}} = \{\delta \in (V^*)^* \mid \delta * p_H = 0 \text{ for all } H \in \mathcal{H}\}.$$

Hence, under the identification $(V^*)^* = V$, $\Delta_{\mathcal{H}}$ corresponds to $[\mathcal{H}] = \bigcap_{H \in \mathcal{H}} H = \bigcap_{H \in \mathcal{H}} (p_H = 0)$.

Lemma 9.2. Let $1 \leq j \leq n$, and let $\mathcal{H} \in \mathcal{A}^{(n-j)}$.

Take $\mathcal{A}' := \{H_1, H_2, \dots, H_{\bar{r}}\} \subseteq \mathcal{A}$ with $\bar{r} = m + n - 1$ so that $\mathcal{H} \subseteq \mathcal{A}'$.

Then $\{\delta_{\mathcal{H} \cup \mathcal{H}'}^m \mid \mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}\}$ forms a basis of $\Delta_{\mathcal{H}}$, and $\dim \Delta_{\mathcal{H}} = \binom{\bar{r} - (n-j)}{j-1} = \binom{m+j-1}{j-1}$.

Proof. By Lemma 6.1,

$$(9.1) \quad \sum_{|\alpha|=m} K\partial^{\alpha} = \bigoplus_{\mathcal{H}'' \in (\mathcal{A}')^{(n-1)}} K\delta_{\mathcal{H}''}^m.$$

Hence $\delta_{\mathcal{H} \cup \mathcal{H}'}^m$ ($\mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}$) are linearly independent.

Let $\mathcal{H}''' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)} \setminus (\mathcal{A}')^{(j-1)}$. Then

$$(9.2) \quad \delta_{\mathcal{H} \cup \mathcal{H}'''}^m = \sum_{\mathcal{H}'' \in (\mathcal{A}')^{(n-1)}} \frac{\delta_{\mathcal{H} \cup \mathcal{H}'''}^m * P'_{\mathcal{H}''}}{\delta_{\mathcal{H}''}^m * P'_{\mathcal{H}''}} \delta_{\mathcal{H}''}^m,$$

where

$$(9.3) \quad P'_{\mathcal{H}''} := \prod_{H \in \mathcal{A}' \setminus \mathcal{H}''} p_H.$$

For $\mathcal{H}'' \not\supseteq \mathcal{H}$, there exists $H \in \mathcal{H} \setminus \mathcal{H}''$. Then p_H divides $P'_{\mathcal{H}''}$, and hence $\delta_{\mathcal{H} \cup \mathcal{H}''}^m * P'_{\mathcal{H}''} = 0$. Therefore

$$(9.4) \quad \delta_{\mathcal{H} \cup \mathcal{H}''}^m = \sum_{\mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}} \frac{\delta_{\mathcal{H} \cup \mathcal{H}''}^m * P'_{\mathcal{H} \cup \mathcal{H}'}}{\delta_{\mathcal{H} \cup \mathcal{H}'}^m * P'_{\mathcal{H} \cup \mathcal{H}'}} \delta_{\mathcal{H} \cup \mathcal{H}'}^m.$$

Hence $\{\delta_{\mathcal{H} \cup \mathcal{H}'}^m \mid \mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}\}$ forms a basis of $\Delta_{\mathcal{H}}$, and $\dim \Delta_{\mathcal{H}} = \binom{\bar{r} - (n-j)}{j-1} = \binom{m+j-1}{j-1}$. \square

Let $\mathcal{A} = \{H_1, H_2, \dots, H_r\}$. We write $H_i \prec H_j$ if $i < j$.

We define the complex $C_*(\mathcal{A}) = C_*$ as follows. For $j = 1, 2, \dots, n$, set

$$C_{n-j} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} \Delta_{\mathcal{H}} \mathbf{e}_{\wedge \mathcal{H}},$$

where $\mathbf{e}_{\wedge \mathcal{H}}$ is just a symbol. In particular,

$$C_{n-1} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m \mathbf{e}_{\wedge \mathcal{H}},$$

and

$$C_0 := \Delta_{\emptyset} \mathbf{e}_{\wedge \emptyset}.$$

The differential $\partial_j : C_j \rightarrow C_{j-1}$ is defined by

$$C_j = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \mathbf{e}_{\wedge \mathcal{H}} \ni \xi \mathbf{e}_{\wedge \mathcal{H}} \mapsto \sum_{H \in \mathcal{H}} (-1)^{l_{\mathcal{H}}(H)} \xi \mathbf{e}_{\wedge (\mathcal{H} \setminus \{H\})} \in C_{j-1},$$

where

$$l_{\mathcal{H}}(H) := \#\{H' \in \mathcal{H} \mid H' \prec H\}.$$

Set

$$C_n := \text{Ker } \partial_{n-1}.$$

Lemma 9.3 (cf. Lemma 1.1 in [11]). *The sequence C_* is exact.*

Proof. As in [11, Lemma 1.1], we prove the assertion by induction.

Let $r = m + n - 1$. Then by Lemma 6.1

$$\Delta_{\mathcal{H}} = \bigoplus_{\mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}} K \delta_{\mathcal{H} \cup \mathcal{H}'}^m \quad \text{for } \mathcal{H} \in \mathcal{A}^{(n-j)}.$$

Hence

$$C_{n-j} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}} K \delta_{\mathcal{H} \cup \mathcal{H}'}^m \mathbf{e}_{\wedge \mathcal{H}} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m \otimes \left(\bigoplus_{\mathcal{H}' \in \mathcal{H}^{(n-j)}} K \mathbf{e}_{\wedge \mathcal{H}'} \right).$$

Thus, in this case, with $C_n = 0$,

$$C_* = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m \otimes \tilde{S}(\mathcal{H}),$$

where $\tilde{S}(\mathcal{H})$ is the augmented chain complex of the simplex with vertex set \mathcal{H} . Hence C_* is exact.

For $n = 2$, the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } \partial_1 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \longrightarrow & 0 \\ & & & & \parallel & & \parallel & & \\ & & & & \bigoplus_{H \in \mathcal{A}} K \delta_H^m & \longrightarrow & \sum_{H \in \mathcal{A}} K \delta_H^m & & \end{array}$$

is clearly exact.

Suppose that $n > 2$ and $r > m + n - 1$. Consider the arrangements $\mathcal{A} \setminus \{H_r\}$ and \mathcal{A}^{H_r} . Since $r > m + n - 1$, we have $\Delta_{\mathcal{H}}(\mathcal{A}) = \Delta_{\mathcal{H}}(\mathcal{A} \setminus \{H_r\})$ for $\mathcal{H} \in (\mathcal{A} \setminus \{H_r\})^{(n-j)}$ by Lemma 9.2. Hence

$$0 \rightarrow C_*(\mathcal{A} \setminus \{H_r\}) \rightarrow C_*(\mathcal{A}) \rightarrow C_*(\mathcal{A}^{H_r})(-1) \rightarrow 0$$

is exact. We thus have the assertion by induction. \square

Let $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $j = 1, 2, \dots, n$, and let $C_*^{[\mathcal{H}]} := C_*(\mathcal{A}^{[\mathcal{H}]})$. For $\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-t)}$, we have

$$\Delta_{\mathcal{H}'}(\mathcal{A}^{[\mathcal{H}]}) = \sum_{\mathcal{H}'' \in (\mathcal{A} \setminus \mathcal{H} \cup \mathcal{H}')^{(t-1)}} K(\delta_{\mathcal{H}' \cup \mathcal{H}''}^{[\mathcal{H}]})^m.$$

Since we may identify $\delta_{\mathcal{H}' \cup \mathcal{H}''}^{[\mathcal{H}]}$ with $\delta_{\mathcal{H} \cup \mathcal{H}' \cup \mathcal{H}''}$, we may identify $\Delta_{\mathcal{H}'}(\mathcal{A}^{[\mathcal{H}]})$ with $\Delta_{\mathcal{H} \cup \mathcal{H}'}$. Hence

$$(9.5) \quad C_{j-t}^{[\mathcal{H}]} = \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-t)}} \Delta_{\mathcal{H} \cup \mathcal{H}'} \mathbf{e}_{\mathcal{H}' \cup \mathcal{H}}$$

for $t = 1, 2, \dots, j$, where $\mathbf{e}_{\mathcal{H}}$ is again a symbol.

We put

$$E_{[\mathcal{H}]} := C_j^{[\mathcal{H}]} := \text{Ker}(\partial_{j-1}^{[\mathcal{H}]} : C_{j-1}^{[\mathcal{H}]} \rightarrow C_{j-2}^{[\mathcal{H}]})$$

for $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $j \geq 2$, and

$$E_{[\mathcal{H}]} := K \delta_{\mathcal{H}}^m \mathbf{e}_{\mathcal{H}}$$

for $\mathcal{H} \in \mathcal{A}^{(n-1)}$. Then we put

$$E_j := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} E_{[\mathcal{H}]}.$$

for $j = 1, 2, \dots, n$.

Remark 9.4 (cf. Remark 1.2 in [11]). Let $1 \leq j \leq n$ and $\mathcal{H} \in \mathcal{A}^{(n-j)}$. Then

$$\dim E_{[\mathcal{H}]} = \binom{r - m - n + j - 1}{j - 1}.$$

Proof. By Lemma 9.3,

$$\dim E_{[\mathcal{H}]} = \dim C_j^{[\mathcal{H}]} = \sum_{l=1}^j (-1)^{l-1} \dim C_{j-l}^{[\mathcal{H}]}.$$

Then by Lemma 9.2

$$\dim C_{j-l}^{[\mathcal{H}]} = \binom{r - n + j}{j - l} \binom{m + l - 1}{l - 1}.$$

Hence

$$\begin{aligned}
\dim E_{[\mathcal{H}]} &= \sum_{l=1}^j (-1)^{l-1} \binom{m+l-1}{l-1} \binom{r-n+j}{j-l} \\
&= \sum_{l=1}^j (-1)^{l-1} \binom{m+l-2}{l-1} \binom{r-n+j-1}{j-l} = \dots \\
&= \sum_{l=1}^j (-1)^{l-1} \binom{l-2}{l-1} \binom{r-m-n+j-1}{j-l} = \binom{r-m-n+j-1}{j-1}.
\end{aligned}$$

□

Let

$$\Delta_{ij} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}} \Delta_{\mathcal{H} \cup \mathcal{H}'} e_{\wedge \mathcal{H}'} e_{\mathcal{H}}$$

for $1 \leq i \leq n$, $0 \leq j \leq n-1$ with $i+j \geq n$, and

$$\Delta_{in} := E_i = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} E_{[\mathcal{H}]}.$$

Then

$$\Delta_{ij} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_{i+j-n}^{[\mathcal{H}]}, \quad \text{and hence} \quad \Delta_{i\bullet} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_{\bullet}^{[\mathcal{H}]}(-(n-i)).$$

As differentials of $\Delta_{i\bullet}$, we take $(-1)^i$ times the differentials of $\bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_{\bullet}^{[\mathcal{H}]}(-(n-i))$. We define a linear map $\phi(j)_i : \Delta_{ij} \rightarrow \Delta_{i-1j}$ for $0 \leq j \leq n-1$ by

$$\Delta_{ij} \ni \xi e_{\wedge \mathcal{H}'} e_{\mathcal{H}} \mapsto \sum_{H \in \mathcal{H}'} (-1)^{l_{\mathcal{H}'(H)}} \xi e_{\wedge (\mathcal{H}' \setminus \{H\})} e_{\mathcal{H} \cup \{H\}} \in \Delta_{i-1j}$$

for $\mathcal{H} \in \mathcal{A}^{(n-i)}$, $\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}$, and $\xi \in \Delta_{\mathcal{H} \cup \mathcal{H}'}$. We define $\psi_i : E_i \rightarrow E_{i-1}$ as the restriction of $\phi(n-1)_i$.

Then we have the double complex $\Delta_{\bullet\bullet}$:

$$\begin{array}{ccccccccccc}
& & 0 & & 0 & & 0 & & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_n & \longrightarrow & \Delta_{n,n-1} & \longrightarrow & \Delta_{n,n-2} & \longrightarrow & \cdots & \longrightarrow & \Delta_{n,1} & \longrightarrow & \Delta_{n,0} & \longrightarrow & 0 \\
& & \psi_n \downarrow & & \downarrow & & \downarrow & & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_{n-1} & \longrightarrow & \Delta_{n-1,n-1} & \longrightarrow & \Delta_{n-1,n-2} & \longrightarrow & \cdots & \longrightarrow & \Delta_{n-1,1} & \longrightarrow & 0 \\
& & \psi_{n-1} \downarrow & & \downarrow & & \downarrow & & & \downarrow & & 0 \\
& & \vdots & & \vdots & & \vdots & & & & & & & & \\
& & \psi_3 \downarrow & & \downarrow & & \downarrow & & & & & & & & \\
0 & \longrightarrow & E_2 & \longrightarrow & \Delta_{2,n-1} & \longrightarrow & \Delta_{2,n-2} & \longrightarrow & 0 \\
& & \psi_2 \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_1 & \longrightarrow & \Delta_{1,n-1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \\
& & 0 & & 0.
\end{array}$$

We add

$$\psi_1 : E_1 = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m e_{\mathcal{H}} \ni \delta_{\mathcal{H}}^m e_{\mathcal{H}} \mapsto \delta_{\mathcal{H}}^m \in E_0 := \Delta_{\emptyset} = \sum_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K \delta_{\mathcal{H}}^m.$$

Lemma 9.5 (cf. Lemma 1.3 in [11]). *The sequence*

$$E_* : 0 \rightarrow E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow 0$$

is exact.

Proof. All rows of $\Delta_{\bullet\bullet}$ are exact by Lemma 9.3 and the argument in the paragraph just after the proof of Lemma 9.3.

For $1 \leq j < n$, since we have

$$\begin{aligned}
\Delta_{ij} &= \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}} \Delta_{\mathcal{H} \cup \mathcal{H}'} e_{\mathcal{H} \setminus \mathcal{H}'} e_{\mathcal{H}} \\
&= \bigoplus_{\mathcal{H} \in \mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \otimes_K \left(\bigoplus_{\mathcal{H}' \in \mathcal{H}^{(i+j-n)}} K e_{\mathcal{H}'} e_{\mathcal{H} \setminus \mathcal{H}'} \right),
\end{aligned}$$

the j -th column $\Delta_{\bullet j}$ is the same as $\bigoplus_{\mathcal{H} \in \mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \otimes_K \tilde{S}(\mathcal{H})$, where $\tilde{S}(\mathcal{H})$ is the augmented chain complex of the simplex with vertex set \mathcal{H} :

$$0 \rightarrow K e_{\mathcal{H}} \rightarrow \bigoplus_{B \in \mathcal{H}^{(j-1)}} K e_{\mathcal{A} \setminus B} \rightarrow \bigoplus_{B \in \mathcal{H}^{(j-2)}} K e_{\mathcal{A} \setminus B} \rightarrow \cdots \rightarrow \bigoplus_{H \in \mathcal{H}} K e_H \rightarrow K e_{\emptyset} \rightarrow 0.$$

Thus the j -th columns ($1 \leq j \leq n-1$) are exact. The 0-th column has the unique nonzero term $\Delta_{\emptyset} e_{\emptyset} (= E_0)$ at $i = n$. Hence by the spectral sequence argument we see that E_* is exact. \square

Let $\sigma \subseteq \{1, 2, \dots, r\}$ and $\sigma \neq \emptyset$. Put

$$\mathcal{L}_j[\sigma] := \{ \mathcal{H} \in \mathcal{A}^{(n-j)} \mid \mathcal{H} \cap \{H_i \mid i \in \sigma\} \neq \emptyset \}.$$

For $1 \leq j \leq n$,

$$E_j[\sigma] := \bigoplus_{\mathcal{H} \in \mathcal{L}_j[\sigma]} E_{[\mathcal{H}]}.$$

Then

$$E_n[\sigma] = 0, \quad E_1[\sigma] = \bigoplus_{\mathcal{H} \in \mathcal{L}_1[\sigma]} K \delta_{\mathcal{H}}^m \mathbf{e}_{\mathcal{H}}.$$

We put

$$E_0[\sigma] := \sum_{\mathcal{H} \in \mathcal{L}_1[\sigma]} K \delta_{\mathcal{H}}^m.$$

We also put

$$E_j[\emptyset] := 0.$$

for all j . Then $\{(E_*[\sigma], \psi_*[\sigma])\}$ is a subcomplex of $\{(E_*, \psi_*)\}$.

Lemma 9.6 (cf. Lemma 1.4 in [11]). *For every σ with $|\sigma| \leq n + m - 1$, $E_*[\sigma]$ is exact.*

Proof. We prove the assertion by induction on $|\sigma|$. If $|\sigma| = 0$, then the assertion is trivial.

When $n = 2$, we have

$$0 \rightarrow E_1[\sigma] = \bigoplus_{H \in \mathcal{L}_1[\sigma]} K \delta_H^m \mathbf{e}_H = \bigoplus_{H \in \sigma} K \delta_H^m \mathbf{e}_H \rightarrow E_0[\sigma] = \sum_{H \in \mathcal{L}_1[\sigma]} K \delta_H^m = \sum_{H \in \sigma} K \delta_H^m \rightarrow 0.$$

This is an isomorphism, since $|\sigma| \leq 2 + m - 1$ (see Lemma 6.1).

Now assume that $|\sigma| \geq 1$ and $n \geq 3$. Fix $j \in \sigma$ and put $\tau := \sigma \setminus \{j\}$. Then $E_*[\tau]$ and $E_*[\{j\}] = E_*(\mathcal{A}^{H_j})$ (by (9.5)) are subcomplexes of $E_*[\sigma]$, which are exact by the induction hypothesis and Lemma 9.5. Moreover there exists an exact sequence of complexes:

$$0 \rightarrow E_*[\tau] \cap E_*[\{j\}] \rightarrow E_*[\tau] \oplus E_*[\{j\}] \rightarrow E_*[\sigma] \rightarrow 0.$$

Since $E_*[\tau] \cap E_*[\{j\}] = E_*[\tau](\mathcal{A}^{H_j})$ and $|\tau| \leq (n - 1) + m - 1$, we are done. \square

Put

$$\sigma_0 := \{1, 2, \dots, m\},$$

and

$$\bar{E}_* := E_*[\sigma_0].$$

We use notation

$$\psi_j : \bar{E}_j \rightarrow \bar{E}_{j-1} \quad (j = 1, 2, \dots, n - 1).$$

Put

$$F_j := S \otimes \bar{E}_j \quad (j = 0, 1, \dots, n - 1).$$

Note that F_i is a submodule of

$$S \Delta_{i, n-1}[\sigma_0] = \bigoplus_{\mathcal{H} \in \mathcal{L}_i[\sigma_0]} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i-1)}} S \delta_{\mathcal{H} \cup \mathcal{H}'}^m \mathbf{e}_{\mathcal{H} \cup \mathcal{H}'}$$

For $i \geq 2$, the morphism $d_i : F_i \rightarrow F_{i-1}$ is defined by

$$\mathbf{e}_{\mathcal{H} \cup \mathcal{H}'} \mapsto \sum_{H' \in \mathcal{H}'} (-1)^{l_{\mathcal{H}'}(H')} p_{H'} \mathbf{e}_{\mathcal{H} \cup (\mathcal{H}' \setminus \{H'\})} \mathbf{e}_{\mathcal{H} \cup \{H'\}}.$$

Note that

$$F_0 = S \otimes_K \sum_{\mathcal{H} \in \mathcal{L}_1[\sigma_0]} K \delta_{\mathcal{H}}^m,$$

and

$$F_1 = \bigoplus_{\mathcal{H} \in \mathcal{L}_1[\sigma_0]} S \delta_{\mathcal{H}}^m e_{\mathcal{H}}.$$

We define a morphism $d_1 : F_1 \rightarrow F_0$ by

$$\delta_{\mathcal{H}}^m e_{\mathcal{H}} \mapsto P_{\mathcal{H}} \delta_{\mathcal{H}}^m.$$

Lemma 9.7. *The sequence*

$$0 \rightarrow F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow 0$$

is a complex.

Proof. By the definition of d_i , clearly $d_i \circ d_{i+1} = 0$ for $i \geq 2$. We prove $d_1 \circ d_2 = 0$.

Let $X = \sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H}, H} \delta_{\mathcal{H} \cup \{H\}}^m e_{\mathcal{H}} \in F_2$. Then

$$\sum_{H \notin \mathcal{H}} f_{\mathcal{H}, H} \delta_{\mathcal{H} \cup \{H\}}^m = 0 \quad \text{for all } \mathcal{H} \in \mathcal{L}_2[\sigma_0].$$

We have

$$\begin{aligned} d_1 \circ d_2(X) &= d_1 \left(\sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H}, H} P_H \delta_{\mathcal{H} \cup \{H\}}^m e_{\mathcal{H} \cup \{H\}} \right) \\ &= \sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H}, H} P_H P_{\mathcal{H} \cup \{H\}} \delta_{\mathcal{H} \cup \{H\}}^m \\ &= \sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H}, H} P_{\mathcal{H}} \delta_{\mathcal{H} \cup \{H\}}^m \quad (\text{Here } P_{\mathcal{H}} := \prod_{H \notin \mathcal{H}} P_H.) \\ &= \sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} P_{\mathcal{H}} \sum_{H \notin \mathcal{H}} f_{\mathcal{H}, H} \delta_{\mathcal{H} \cup \{H\}}^m = 0. \end{aligned}$$

□

The following is Theorem 8.3.

Lemma 9.8 (cf. Lemma 2.1 in [11]). *Assume that $m < r - n + 1$. Then the image of d_1 coincides with $\Xi^{(m)}(\mathcal{A})$.*

By Remark 9.4, we have the following.

Remark 9.9 (cf. Remark 2.2 in [11]).

$$\text{rank}_S(F_j) = \binom{r - m - n + j - 1}{j - 1} \left(\binom{r}{n - j} - \binom{r - m}{n - j} \right) =: w_j^{(m)}.$$

Under the above preparations, we can prove the following theorem. Since the proof is almost the same as that of [11, Theorem 2.3], we omit it.

Theorem 9.10 (cf. Theorem 2.3 in [11]). *Assume that $n \geq 3$ and $m < r - n + 1$. Then the complex*

$$F_* : 0 \rightarrow F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} \Xi^{(m)}(\mathcal{A}) \rightarrow 0$$

is a minimal free resolution of $\Xi^{(m)}(\mathcal{A})$. In particular, the projective dimensions of S -modules $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to $n - 2$.

By Theorem 8.3, Remark 9.9, and the construction of the complex F_* in Theorem 9.10, we have the following corollary:

Corollary 9.11 (cf. Corollary 4.4.3 in [7]). *Assume that $n \geq 3$ and $m < r - n + 1$. Then there exist exact sequences*

$$\begin{aligned} 0 &\rightarrow S(m+1-r)w_{n-1}^{(m)} \rightarrow \cdots \rightarrow S(m+n-j-r)w_j^{(m)} \rightarrow \cdots \\ &\rightarrow S(m+n-2-r)w_2^{(m)} \rightarrow S(m+n-1-r)w_1^{(m)} \rightarrow \Xi^{(m)}(\mathcal{A}) \rightarrow 0, \\ 0 &\rightarrow S(m+1-r)w_{n-1}^{(m)} \rightarrow \cdots \rightarrow S(m+n-j-r)w_j^{(m)} \rightarrow \cdots \\ &\rightarrow S(m+n-2-r)w_2^{(m)} \rightarrow S(m+n-1-r)w_1^{(m)} \bigoplus S \rightarrow D^{(m)}(\mathcal{A}) \rightarrow 0, \end{aligned}$$

where $w_j^{(m)}$ were defined in Remark 9.9, and all maps are homogeneous of degree 0.

In particular, the Castelnuovo-Mumford regularities of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to $r - m - n + 1$.

Remark 9.12. If we use the polynomial degrees in $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ as the degrees of graded S -modules, then the degrees are shifted by m . Then the Castelnuovo-Mumford regularities of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to $r - n + 1$ as stated for $D^{(1)}(\mathcal{A})$ in [1, Section 5.2], and the Poincaré-Betti series of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ coincide with the ones conjectured by Snellman [9, Conjecture 6.8].

10. MINIMAL FREE RESOLUTION OF $J_m(\mathcal{A})$

In this section, we generalize the minimal free resolution of S/J given in [7], where J is the Jacobian ideal of Q . We retain the assumptions $n \geq 3$ and $n + m - 1 < r$.

Let $J_m(\mathcal{A})$ denote the S -submodule of $S^{\binom{n+m-1}{m-1}} = \bigoplus_{|\beta| \leq m-1} S e_\beta$ generated by all

$$(10.1) \quad \frac{1}{\alpha!} \partial^\alpha \bullet Q := \left(\frac{1}{(\alpha - \beta)!} \partial^{\alpha - \beta} * Q : |\beta| \leq m - 1 \right) = \sum_{|\beta| \leq m-1} \frac{1}{(\alpha - \beta)!} \partial^{\alpha - \beta} * Q e_\beta$$

with $1 \leq |\alpha| \leq m$. Here we agree $\partial^{\alpha - \beta} = 0$ for $\beta \not\leq \alpha$.

Example 10.1. Let $m = 1$. Then $J_1(\mathcal{A})$ is the S -submodule of S generated by $\partial_j * Q$ ($j = 1, \dots, n$), i.e., $J_1(\mathcal{A})$ is nothing but the Jacobian ideal J of Q .

Lemma 10.2. *For all $\alpha, \beta \in \mathbb{N}^n$,*

$$\frac{1}{(\alpha - \beta)!} \partial^{\alpha - \beta} = (-1)^{|\beta|} \frac{(\text{adx})^\beta}{\alpha!} (\partial^\alpha).$$

Here we denote by adx_i the endomorphism of $D(S)$: $D(S) \ni P \mapsto \text{adx}_i(P) = [x_i, P] \in D(S)$. For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, we set $(\text{adx})^\beta = (\text{adx}_1)^{\beta_1} \circ \cdots \circ (\text{adx}_n)^{\beta_n}$.

Proof. We prove the assertion by induction on $|\beta|$. For all $\alpha, \beta \in \mathbb{N}^n$,

$$\begin{aligned} \text{adx}_i((-1)^{|\beta|} \frac{(\text{adx})^\beta}{\alpha!} (\partial^\alpha)) &= \frac{1}{(\alpha - \beta)!} \text{adx}_i(\partial^{\alpha - \beta}) \\ &= -\frac{1}{(\alpha - \beta)!} (\alpha_i - \beta_i) \partial^{\alpha - \beta - \mathbf{1}_i} \\ &= -\frac{1}{(\alpha - \beta - \mathbf{1}_i)!} \partial^{\alpha - \beta - \mathbf{1}_i}. \end{aligned}$$

□

By Lemma 10.2,

$$\frac{1}{\alpha!} \partial^\alpha \bullet Q = ((-1)^{|\beta|} (\text{adx})^\beta (\frac{1}{\alpha!} \partial^\alpha) * Q : |\beta| \leq m - 1).$$

We define an S -module morphism

$$\delta_0 : F_0^{[1, m]} := D^{[1, m]}(S) := \bigoplus_{k=1}^m D^{(k)}(S) \rightarrow S^{\binom{n+m-1}{m-1}} = \bigoplus_{|\beta| \leq m-1} S e_\beta$$

by

$$(10.2) \quad \delta_0(\theta) := \theta \bullet Q := ((-1)^{|\beta|} (\text{adx})^\beta (\theta) * Q : |\beta| \leq m - 1).$$

By definition,

$$(10.3) \quad \text{Im } \delta_0 = J_m(\mathcal{A}).$$

Lemma 10.3. *Let $\theta \in D(S)$. Then*

$$\theta x^\beta = \sum_{\gamma \leq \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} x^{\beta - \gamma} (\text{adx})^\gamma (\theta),$$

where $\binom{\beta}{\gamma} = \prod_{i=1}^n \binom{\beta_i}{\gamma_i}$.

Proof. We prove the assertion by induction on $|\beta|$. We have

$$\begin{aligned} \theta x_i x^\beta &= -(\text{adx}_i(\theta)) x^\beta + x_i \theta x^\beta \\ &= -\sum_{\gamma \leq \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} x^{\beta - \gamma} (\text{adx})^\gamma (\text{adx}_i(\theta)) + x_i \sum_{\gamma \leq \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} x^{\beta - \gamma} (\text{adx})^\gamma (\theta) \\ &= \sum_{\gamma \leq \beta} (-1)^{|\gamma+1_i|} \binom{\beta}{\gamma} x^{\beta+1_i - \gamma - 1_i} (\text{adx})^{\gamma+1_i} (\theta) + \sum_{\gamma \leq \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} x^{\beta+1_i - \gamma} (\text{adx})^\gamma (\theta) \\ &= \sum_{\gamma - 1_i \leq \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma - 1_i} x^{\beta+1_i - \gamma} (\text{adx})^\gamma (\theta) + \sum_{\gamma \leq \beta} (-1)^{|\gamma|} \binom{\beta}{\gamma} x^{\beta+1_i - \gamma} (\text{adx})^\gamma (\theta) \\ &= \sum_{\gamma \leq \beta+1_i} (-1)^{|\gamma|} \binom{\beta+1_i}{\gamma} x^{\beta+1_i - \gamma} (\text{adx})^\gamma (\theta). \end{aligned}$$

□

Let $\bar{\delta}_0$ denote the composite of δ_0 with the canonical projections of Se_β onto $(S/SQ)e_\beta$ for $\beta \neq \mathbf{0}$:

$$(10.4) \quad \bar{\delta}_0 : D^{[1,m]}(S) \xrightarrow{\delta_0} \bigoplus_{|\beta| \leq m-1} Se_\beta \rightarrow Se_{\mathbf{0}} \bigoplus \bigoplus_{0 \neq |\beta| \leq m-1} (S/SQ)e_\beta.$$

Here note that $\bar{\delta}_0$ is a graded S -module homomorphism homogeneous of degree 0 if we put $\deg(e_\beta) = -r - |\beta|$.

In the following two lemmas, we describe the cokernel and the kernel of $\bar{\delta}_0$.

Lemma 10.4.

$$\text{Coker } \bar{\delta}_0 = S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}}).$$

Proof. By (10.3), we only need to show $Qe_{\mathbf{0}} \in \text{Im } \bar{\delta}_0$. We have $\epsilon_1 * Q = rQ$. Since $\epsilon_1 \in D(\mathcal{A})$, we see $\delta_0(\epsilon_1) \in \bigoplus_{|\beta| \leq m-1} SQe_\beta$ by the definition of δ_0 (10.2). Hence

$$Qe_{\mathbf{0}} = \bar{\delta}_0\left(\frac{1}{r}\epsilon_1\right) \in \text{Im } \bar{\delta}_0.$$

□

Lemma 10.5.

$$\text{Ker } \bar{\delta}_0 = \bigoplus_{k=1}^m D^{(k)}(\mathcal{A})' =: D^{[1,m]}(\mathcal{A})',$$

where $D^{(k)}(\mathcal{A})' := \{\theta \in D^{(k)}(\mathcal{A}) : \theta * Q = 0\}$.

Proof. If $\theta \in D^{[1,m]}(\mathcal{A})'$, then $(\text{ad } x)^\beta(\theta) \in D(\mathcal{A})$ for all β , and $\theta * Q = 0$. Hence $\theta \in \text{Ker } \bar{\delta}_0$ by the definitions of $D(\mathcal{A})$ and $\bar{\delta}_0$.

Next we suppose that $\theta \in \text{Ker } \bar{\delta}_0$. Then by Lemma 10.3

$$(10.5) \quad \theta * x^\beta Q \in \langle Q \rangle = QS \text{ for all } \beta \text{ with } |\beta| \leq m-1.$$

By [5, Proposition 2.3], we conclude that $\theta \in D^{[1,m]}(\mathcal{A})'$.

□

Lemma 10.6. *Let $k \leq r$. As S -modules,*

$$\Xi^{(k)}(\mathcal{A}) \simeq D^{(k)}(\mathcal{A})'.$$

Proof. It is easy to see that

$$\gamma_k : \Xi^{(k)}(\mathcal{A}) \ni \theta \mapsto \theta - \frac{\theta * Q}{Q} \frac{\epsilon_k}{r(r-1)\cdots(r-k+1)} \in D^{(k)}(\mathcal{A})'$$

and

$$D^{(k)}(\mathcal{A})' \ni \theta \mapsto \theta - \frac{\theta * (p_1 \cdots p_k)}{p_1 \cdots p_k} \frac{\epsilon_k}{k!} \in \Xi^{(k)}(\mathcal{A})$$

are inverse to each other.

□

For $1 \leq k \leq m$, let $F_*^{(k)}$ denote the minimal free resolution of $\Xi^{(k)}$ in Theorem 9.10.

We consider the following complex:

$$(10.6) \quad 0 \rightarrow \tilde{F}_{n-1} \xrightarrow{\tilde{\delta}_{n-1}} \cdots \xrightarrow{\tilde{\delta}_2} \tilde{F}_1 \xrightarrow{\tilde{\delta}_1} \tilde{F}_0 \xrightarrow{\tilde{\delta}_0} \tilde{F}_{-1} \rightarrow \text{Coker}(\tilde{\delta}_0) \rightarrow 0,$$

where

$$\begin{aligned}\tilde{F}_{-1} &= \bigoplus_{|\beta| \leq m-1} S e_\beta, \\ \tilde{F}_0 &= D^{[1,m]}(S) \bigoplus_{0 \neq |\beta| \leq m-1} S e_\beta, \\ \tilde{F}_j &= \bigoplus_{k=1}^m F_j^{(k)} \quad (j = 1, \dots, n-1),\end{aligned}$$

and

$$\begin{aligned}\tilde{\delta}_0(\theta, \sum_{\beta \neq \mathbf{0}} f_\beta e_\beta) &= \delta_0(\theta) + \sum_{\beta \neq \mathbf{0}} f_\beta Q e_\beta = \theta * Q e_{\mathbf{0}} + \sum_{\beta \neq \mathbf{0}} ((-1)^{|\beta|} (\text{ad} x)^\beta(\theta) * Q + f_\beta Q) e_\beta, \\ \tilde{\delta}_1(\delta_{\mathcal{H}}^k e_{\mathcal{H}}^{(k)}) &= (\gamma_k(P_{\mathcal{H}} \delta_{\mathcal{H}}^k), -\frac{1}{Q} \sum_{\beta \neq \mathbf{0}} (-1)^{|\beta|} (\text{ad} x)^\beta(\gamma_k(P_{\mathcal{H}} \delta_{\mathcal{H}}^k)) * Q e_\beta), \\ \tilde{\delta}_j &= \bigoplus_{k=1}^m d_j^{(k)} \quad (j \geq 2).\end{aligned}$$

Recall that $D^{(k)}(S) = F_0^{(k)}$, and $d_1(\delta_{\mathcal{H}}^k e_{\mathcal{H}}^{(k)}) = P_{\mathcal{H}} \delta_{\mathcal{H}}^k$ for $1 \leq k \leq m$.

Theorem 10.7 (cf. Theorem 4.5.3 in [7]). *The complex (10.6) is a minimal free resolution of $\text{Coker}(\tilde{\delta}_0) = S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$.*

Proof. The complex (10.6) is exact by Theorem 8.3, Theorem 9.10, Lemma 10.4, Lemma 10.5, and Lemma 10.6. The operator $P_{\mathcal{H}} \delta_{\mathcal{H}}^k$ is of order k and homogeneous of polynomial degree $\deg(P_{\mathcal{H}}) = r - (n-1)$. Then each term of $\gamma_k(P_{\mathcal{H}} \delta_{\mathcal{H}}^k)$ is of order k and of polynomial degree greater than or equal to k . Hence each term of the operator $(\text{ad} x)^\beta(\gamma_k(P_{\mathcal{H}} \delta_{\mathcal{H}}^k))$ is of order $k - |\beta|$ and of polynomial degree greater than or equal to k . Therefore each term of the polynomial

$$\frac{1}{Q} (-1)^{|\beta|} (\text{ad} x)^\beta(\gamma_k(P_{\mathcal{H}} \delta_{\mathcal{H}}^k)) * Q$$

is of degree greater than or equal to

$$r - (k - |\beta|) + k - r = |\beta| > 0.$$

Thus the free resolution (10.6) of $\text{Coker}(\tilde{\delta}_0)$ is minimal. Clearly by (10.4)

$$\text{Coker}(\tilde{\delta}_0) = S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}}) = \text{Coker}(\bar{\delta}_0).$$

□

The following corollary is clear from Theorem 10.7 and the Auslander-Buchsbaum formula.

Corollary 10.8 (cf. Corollary 4.5.5 [7]). *The projective dimension of the S -module $S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ is n , and the depth is 0.*

In the complex (10.6), the degrees of elements of bases are as follows:

$$\begin{aligned} \deg(\mathbf{e}_\beta) &= -r - |\beta| && \text{in } \tilde{F}_{-1}, \\ \deg(\partial^\alpha) &= -|\alpha| && \text{in } \tilde{F}_0, \\ \deg(\mathbf{e}_\beta) &= -|\beta| && \text{in } \tilde{F}_0, \\ \deg(\delta_{\mathcal{H}}^k \mathbf{e}_{\mathcal{H}}) &= -k + r - (n-1) = r - n - k + 1 && \text{in } \tilde{F}_1. \end{aligned}$$

Hence we have the following corollary:

Corollary 10.9 (cf. Corollary 4.5.4 in [7]). *Assume that $n \geq 3$ and $m < r - n + 1$. Then there exists an exact sequence*

$$\begin{aligned} 0 &\rightarrow \bigoplus_{k=1}^m S(k+1-r)w_{n-1}^{(k)} \rightarrow \cdots \rightarrow \bigoplus_{k=1}^m S(k+n-j-r)w_j^{(k)} \rightarrow \cdots \\ &\rightarrow \bigoplus_{k=1}^m S(k+n-1-r)w_1^{(k)} \rightarrow \bigoplus_{k=1}^m S(k)^{s_k} \bigoplus_{k=1}^{m-1} \bigoplus_{k=1}^m S(k)^{s_k} \rightarrow \\ &\quad \bigoplus_{k=0}^{m-1} S(r+k)^{s_k} \rightarrow \text{Coker}(\bar{\delta}_0) \rightarrow 0, \end{aligned}$$

where $w_j^{(k)}$ were defined in Remark 9.9, $s_k = \binom{n+k-1}{k}$, and all maps are homogeneous of degree 0.

In particular, the Castelnuovo-Mumford regularity of $\text{Coker}(\bar{\delta}_0)$ is equal to $r - n - 2$.

Remark 10.10. In Corollary 10.9, to make the degrees of all the minimal generators of $\text{Coker}(\bar{\delta}_0)$ nonnegative, we can shift the degrees by $r + (m-1)$ as in [7, Corollary 4.5.5]. Then the Castelnuovo-Mumford regularity of $\text{Coker}(\bar{\delta}_0)$ is equal to $2r + m - n - 3$.

11. JET MODULES

In this section, we prove that $\text{Coker}(\bar{\delta}_0) = S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ in Section 10 is the transpose of the m -jet module $\Omega^{[1,m]}(S/SQ)$. For the basics of jet modules, see [2], [3], and [10].

Let $I := \langle f_1, \dots, f_k \rangle$ be an ideal of S . Let $R := S/I$. Define jet modules

$$(11.1) \quad \begin{aligned} \Omega^{[1,m]}(S) &:= J_S/J_S^{m+1}, & \Omega^{\leq m}(S) &:= S \otimes_K S/J_S^{m+1}, \\ \Omega^{[1,m]}(R) &:= J_R/J_R^{m+1}, & \Omega^{\leq m}(R) &:= R \otimes_K R/J_R^{m+1}, \end{aligned}$$

where

$$\begin{aligned} J_S &:= \langle 1 \otimes a - a \otimes 1 \mid a \in S \rangle \subseteq S \otimes_K S, \\ J_R &:= \langle 1 \otimes a - a \otimes 1 \mid a \in R \rangle \subseteq R \otimes_K R. \end{aligned}$$

Then $\Omega^{\leq m}(R)$ is the representative object of the functor $M \rightarrow D_R^m(R, M)$, i.e., there exists a natural isomorphism of R -modules:

$$D_R^m(R, M) \simeq \text{Hom}_R(\Omega^{\leq m}(R), M),$$

where M is an R -module, and $D_R^m(R, M)$ is the module of differential operators of order $\leq m$ from R to M .

As S -modules,

$$\Omega^{\leq m}(S) = \Omega^{[1,m]}(S) \bigoplus S \otimes 1, \quad \Omega^{\leq m}(R) = \Omega^{[1,m]}(R) \bigoplus R \otimes 1.$$

Here note that S acts as $S \otimes 1$. We have

$$\{P \in D_R^m(R, M) \mid P * 1 = 0\} \simeq \text{Hom}_R(\Omega^{[1,m]}(R), M)$$

for an R -module M .

For $a \in S$ (or R), we denote $1 \otimes a - a \otimes 1 \bmod J_S^{m+1}$ (or J_R^{m+1} , respectively) by da . Then, for $f, g \in R$, we have

$$(11.2) \quad d(fg) = f dg + g df + (df)(dg).$$

As an S -module

$$\Omega^{[1,m]}(S) = \bigoplus_{1 \leq |\alpha| \leq m} S(dx)^\alpha.$$

For $f \in S$, we have

$$(11.3) \quad df = \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} (\partial^\alpha * f)(dx)^\alpha.$$

We have a surjective $S \otimes S$ -module homomorphism

$$\varphi : \Omega^{[1,m]}(S) \ni (dx)^\alpha \mapsto (d\bar{x})^\alpha \in \Omega^{[1,m]}(R).$$

Lemma 11.1. *As an S -module,*

$$\text{Ker } \varphi = \sum_{i; 1 \leq |\alpha| \leq m} S f_i (dx)^\alpha + \sum_{i; 0 \leq |\alpha| \leq m-1} S (df_i)(dx)^\alpha.$$

Proof. The inclusion ‘ \supset ’ is clear. We prove the other inclusion.

First we prove that

$$(11.4) \quad \text{Ker } \varphi = I dS + S dI.$$

Clearly the kernel of the $S \otimes S$ -module homomorphism :

$$\Omega^{\leq m}(S) \ni f \otimes g \mapsto \bar{f} \otimes \bar{g} \in \Omega^{\leq m}(R)$$

equals $(S \otimes I + I \otimes S)/J_S^{m+1}$ or $(S dI + I \otimes S)/J_S^{m+1}$. Hence, to prove (11.4), it is enough to show that

$$(11.5) \quad (I \otimes S) \cap J_S = I dS.$$

Let $\sum_k i_k \otimes g_k \in J_S$ with $i_k \in I, g_k \in S$. Then $\sum_k i_k g_k = 0$. We have

$$\sum_k i_k \otimes g_k = \sum_k (i_k \otimes g_k - i_k g_k \otimes 1) + \sum_k i_k g_k \otimes 1 = \sum_k i_k dg_k + 0 \in I dS.$$

Hence we have proved (11.5) and in turn (11.4). Thus as an S -module

$$\text{Ker } \varphi = \sum_{1 \leq |\alpha| \leq m} I(dx)^\alpha + \sum_{0 \leq |\alpha| < m} S dI(dx)^\alpha.$$

To finish the proof, we only need to show that $d(f_i x^\alpha)$ belongs to the right hand of the assertion for any α . This is done by (11.2):

$$d(f_i x^\alpha) = f_i d(x^\alpha) + x^\alpha df_i + (df_i)(d(x^\alpha)).$$

□

Hence we have an S -free presentation of $\Omega^{[1,m]}(R)$:

$$(11.6) \quad \left(\bigoplus_{i; 1 \leq |\alpha| \leq m} S f_i(dx)^\alpha \right) \oplus \left(\bigoplus_{i; 0 \leq |\beta| \leq m-1} S(df_i)(dx)^\beta \right) \rightarrow \Omega^{[1,m]}(S) \rightarrow \Omega^{[1,m]}(R) \rightarrow 0.$$

Now we consider the case $I = SQ$:

$$(11.7) \quad \left(\bigoplus_{1 \leq |\alpha| \leq m} SQ(dx)^\alpha \right) \oplus \left(\bigoplus_{0 \leq |\beta| \leq m-1} S(dQ)(dx)^\beta \right) \rightarrow \Omega^{[1,m]}(S) \rightarrow \Omega^{[1,m]}(S/SQ) \rightarrow 0.$$

Hence, as an S/SQ -module, $\Omega^{[1,m]}(S/SQ)$ has a presentation:

$$(11.8) \quad \bigoplus_{0 \leq |\beta| \leq m-1} (S/SQ)(dQ)(dx)^\beta \rightarrow \bigoplus_{1 \leq |\alpha| \leq m} (S/SQ)(dx)^\alpha \rightarrow \Omega^{[1,m]}(S/SQ) \rightarrow 0.$$

Note that by (11.3)

$$\begin{aligned} (dQ)(dx)^\beta &= \sum_{|\alpha+\beta| \leq m, \alpha \neq 0} \frac{1}{\alpha!} (\partial^\alpha * Q)(dx)^{\alpha+\beta} \\ &= \sum_{|\gamma| \leq m, \gamma \neq \beta} \frac{1}{(\gamma - \beta)!} (\partial^{\gamma-\beta} * Q)(dx)^\gamma. \end{aligned}$$

Hence the (β, γ) -component of the matrix of (11.8) equals $\frac{1}{(\gamma - \beta)!} (\partial^{\gamma-\beta} * Q)$.

By Lemma 10.4, the S/SQ -module $S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ has a presentation:

$$(11.9) \quad \bigoplus_{1 \leq |\gamma| \leq m} (S/SQ) \frac{1}{\gamma!} \partial^\gamma \xrightarrow{\bullet} \bigoplus_{0 \leq |\beta| \leq m-1} (S/SQ) e_\beta \rightarrow S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}}) \rightarrow 0,$$

and the (γ, β) -component of the matrix of the map \bullet in (11.9) (recall (10.1)) equals $\frac{1}{(\gamma - \beta)!} (\partial^{\gamma-\beta} * Q)$.

Thus we have proved the following theorem.

Theorem 11.2. *The S/SQ -module $S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ is the transpose of $\Omega^{[1,m]}(S/SQ)$.*

Corollary 11.3. *The S/SQ -modules $S^{\binom{n+m-1}{m-1}} / (J_m(\mathcal{A}) + QS^{\binom{n+m-1}{m-1}})$ and $\Omega^{[1,m]}(S/SQ)$ share the same Fitting ideals.*

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