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計算ホモロジー理論とその応用

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概要

位相幾何学とは、空間をぐにゃぐにゃと連続変形しても不変な性質を研究する数学であり、様々な数学の分野で重要な道具となっている。近年、より現実的な問題に位相幾何学を応用するために、計算機を用いて位相不変量を計算するアルゴリズムの開発が活発になっており、本講演ではその中の一つである計算ホモロジー理論について、その基礎とセンサーネットワークの被覆問題への応用について解説する。

センサーネットワークにおいては、大量のセンサー、例えば人間の存在を感知したり、温度を測定するセンサーがあちこちにばらまかれた状況を考える。このとき、センサーたちの感知領域が、正しく目的の領域を被覆しているか、すなわちどこかに見逃しがないか、という問題が重要になる。GPS などを用いて各センサーの位置情報を求めれば計算幾何を用いることが出来るが、消費電力を小さくおさえるためには、なるべく少ない情報で解決したい。R. Ghrist らの研究により、センサーネットワークから定義される Ris 複体のホモロジーを計算すれば、この被覆問題を少ない情報で綺麗に記述できる事がわかったのだが、今度は「Rips 複体のホモロジーをいかに小さいコストで計算するか」という問題が生じた。

この問題に対し、林和則、平岡裕章両氏との共同研究で得られた、マイヤービートリス系列を用いて分散計算をするアプローチと、Ali Jadbabaie らが進めているネットワーク上のラプラシアンを用いたアプローチを紹介する。

Computational Homology and its Applications

14 May, 2010

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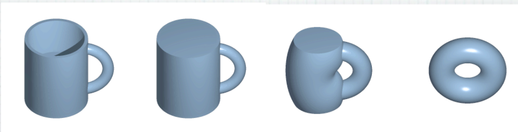
Zin ARAI
CRIS, Hokkaido University/JST PRESTO

Contents

What is Topology?

Topology is the study of the properties of spaces (figures, shapes) that are independent of continuous deformations of the space.



Coffee cups = Doughnuts
(for a topologist)

What is NOT topology?

Topology ≠ Geometry

"length" is not a topological property

"angle" is not a topological property

"area" is not a topological property

Topological Invariants

For a space X

Euler number $\chi(X)$

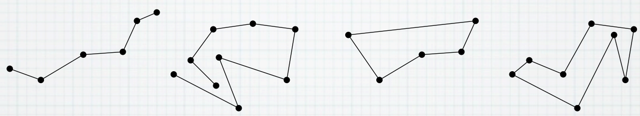
Betti number $\beta_n(X)$

Homology Group $H_n(X)$

are the examples of topological invariants.

(the former can be computed from the latter)

Euler Number (1-dim)



Consider a graph ***G***. Let ***v*** be the number of vertices, ***e*** the number of edges.
Then,

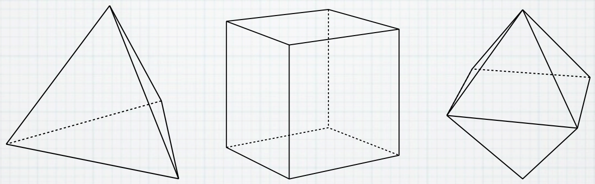
$$v - e = 1$$

when ***G*** is does not have a loop, and

$$v - e = 0$$

when ***G*** has one loop.

Euler Number (2-dim)



These objects can be deformed into a sphere S^2.
Let f be the number of faces. Then the number

$$v - e + f$$

does not depends the combinatorial decomposition of S^2 into a polyhedron. For the tetrahedron $4 - 6 + 4 = 2$, for the cube $8 - 12 + 6 = 2$, and $6 - 12 + 8 = 2$ for the octahedron.

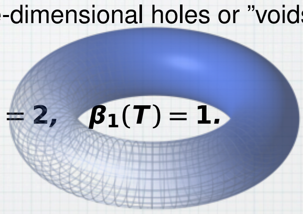
Betti Number

Informally, the betti number $\beta_i(X)$ of a space X counts the number of independent i-dimensional objects in X.
For example,

- $\beta_0(X)$ is the number of connected components
- $\beta_1(X)$ is the number of two-dimensional or "circular" holes
- $\beta_2(X)$ is the number of three-dimensional holes or "voids"

In the case of the torus T,

$$\beta_0(T) = 1, \quad \beta_1(T) = 2, \quad \beta_1(T) = 1.$$



The betti number is the rank of the homology group.
That is, $\beta_i(X) = \operatorname{rank} H_i(X)$

Why topology matters?

Why topology matters?

We can see the structure of high dimensional spaces.

We can use topological properties to prove the existence of fixed points, periodic orbits, etc, in dynamical systems.

We can quantify the difference of spaces.

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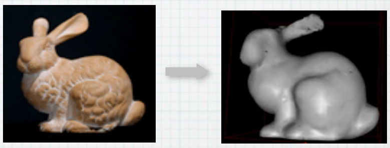
Image Processing

N (bitmap img)

$$H_0(N) = \mathbb{Z}^{10}$$
$$H_1(N) = \mathbb{Z}^{36}$$

Stanford Bunny
(Stanford 3D Scanning Repository)

http:graphics.stanford.edu/data/3Dscanrep/



Stanford Bunny

Polygonal Approximation

$$H_0 = \mathbb{Z}$$
$$H_1 = \mathbb{Z}^7$$

This indicates there exist 7 holes in the approximation.

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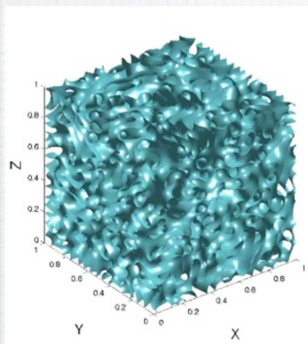
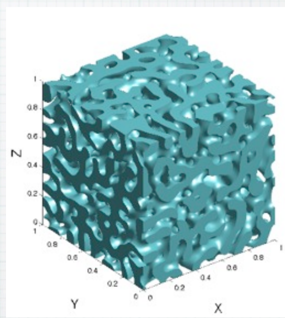
Quantifying the difference



$$H_i = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}^{1705}, & i = 1 \\ 0, & i \neq 0, 1 \end{cases}$$



$$H_i = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}^{847}, & i = 1 \\ 0, & i \neq 0, 1 \end{cases}$$

The coverage problem of sensor networks

Ubiquitous Sensor

Sensor network consists of spatially distributed sensors to monitor physical or environmental conditions.

Sensors today: Single, powerful sensor

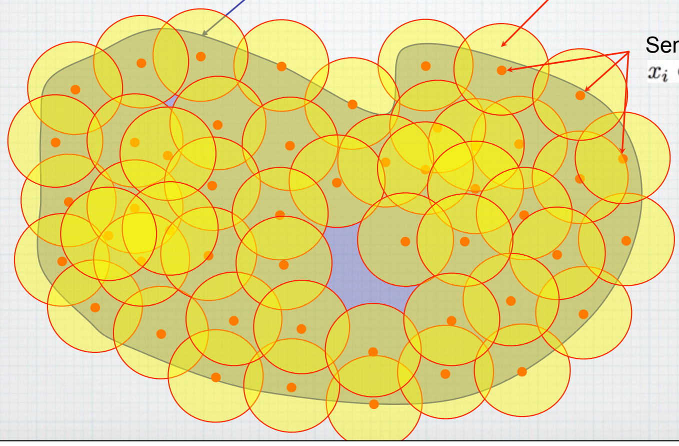
Future sensors: many, cheap sensors working cooperatively

The coverage problem

D : the region we want to monitor

Each sensor can monitor the disc of radius r_c

Sensors $x_i \in D$



Traditional Approaches

- Computational Geometry

  Need to know the absolute position of sensors. We can use GPS, etc, to know the position, but this is battery consuming.

- Probability theoretical approach

  Compute the probability of the happening of a hole.

  Need to assume that sensors are distributed uniformly, which is too strong in practice.

The setting

A1 – A4:

$D \subset \mathbb{R}^2$ is compact, connected.

The boundary ∂D is connected.

$\mathcal{X} := \{x_i \in D \mid i = 1, \cdots, N\}$ sensors

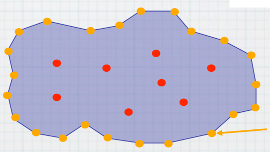
$\mathcal{X}_f$: sensors on the boundary ∂D, we call them "fence nodes"

r_b : a sensor can communicate with another sensor within the distance r_b

r_c: the radius of monitoring region. We assume that $r_c \geq \dfrac{r_b}{\sqrt{3}}$

$\mathcal{U}$: the union of all monitoring region.

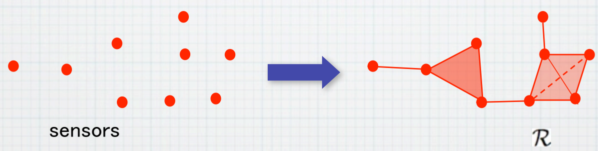
Problem: $D \subset \mathcal{U}$?



Rips Complex

Rips complex $\mathcal{R}$

- 0-simplex ⟷ each sensor $x_i,\ i = 1, \cdots, N$
- k-simplex $|x_0 \cdots x_k|$ ⟷ sensors $\{x_0, \cdots, x_k\}$ are mutually within the distance r_b



$\mathcal{F}$: the Rips complex of the "fence" nodes, which is on the boundary of D.

Robert Ghrist's theorem

Theorem [Ghrist]:

If there exists $[\sigma] \in H_2(\mathcal{R}, \mathcal{F})$ such that

$\delta_2[\sigma] \neq 0 \ (\in H_1(\mathcal{F}))$ then $D \subset \mathcal{U}$.

We can simplify this condition as follows:

Theorem [ZA-Hayashi-Hiraoka]:

If $H_1(\mathcal{R}, \mathcal{F}) = 0$ then $D \subset \mathcal{U}$.

Distributed Computation?

Problems

Who compute the homology?

We do not want to set a central node in the network.

Homology computation is slow

Homology computation is $O(n^3)$, therefore requires a lot of CPU power when the network is large.

We want to distribute the homology computation!

Idea 1: Mayer-Vietoris sequence (ZA-Hayashi-Hiraoka)

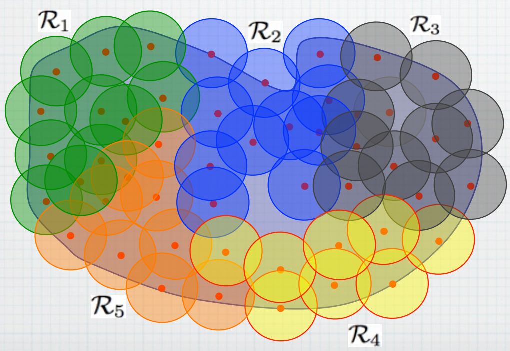
Idea 2: Discrete Laplacian and the gossip algorithm

Mayer-Vietoris Sequence

The Mayer-Vietoris long exact sequence

$$\to H_k(\mathcal{R}_1 \cap \mathcal{R}_2) \to H_k(\mathcal{R}_1) \oplus H_k(\mathcal{R}_2) \to H_k(\mathcal{R}_1 \cup \mathcal{R}_2) \to H_{k-1}(\mathcal{R}_1 \cap \mathcal{R}_2) \to$$

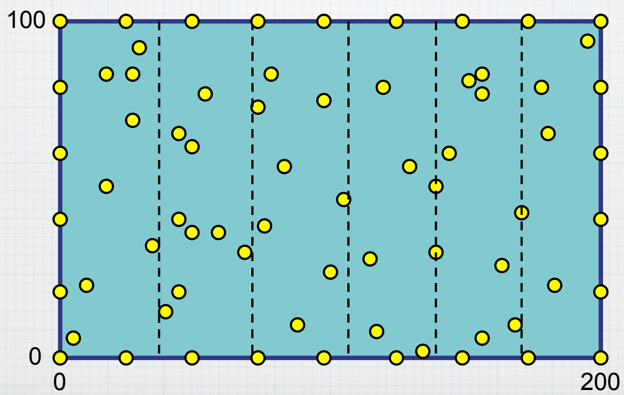
Local information ⇒ Global Information



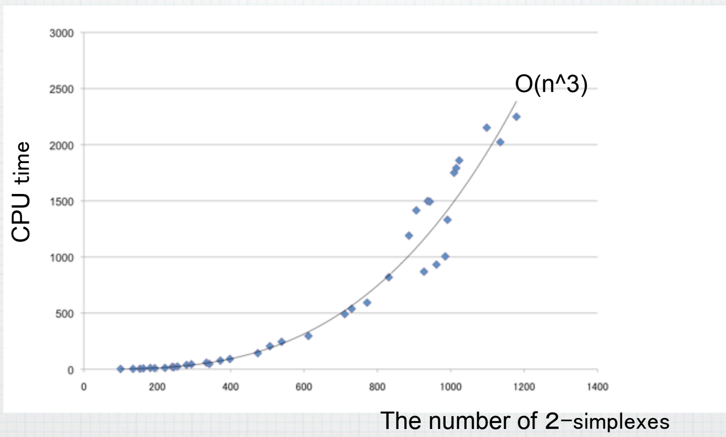
- 1. Decompose $\mathcal{R}$
  $$\mathcal{R} = \bigcup_{i=1}^{K} \mathcal{R}_i$$
- 2. Compute $H_*(\mathcal{R}_i), H_*(\mathcal{R}_i \cap \mathcal{R}_j)$ in parallel.
- Use Mayer-Vietoris to compute $H_*(\mathcal{R})$

Computational Experiment

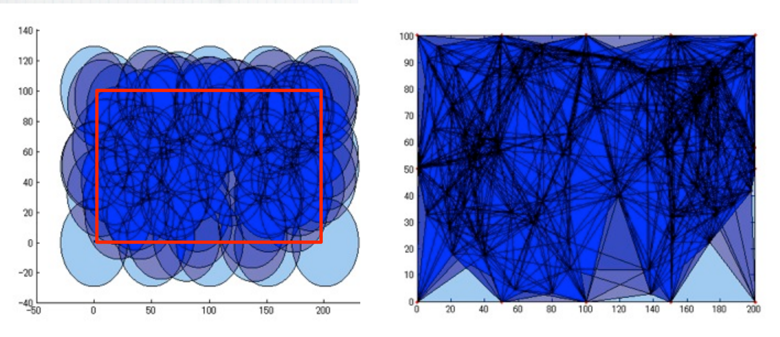
Decompose D into 1, 2, or 5 vertical rectangles overlapping each other



The order of the computation



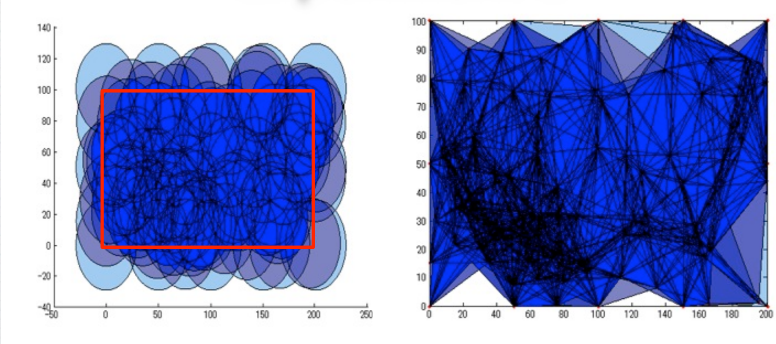
Experimental Result



#{2−simplex}=519, #{1−simplexes}=184, #{0−simplexes}=30

# of partition elements	1	2	5
computational time	81.0	49.2	1.0

Experiment 2

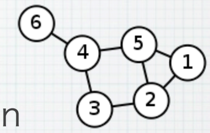


#{2−simplexes}=426, #{1−simplexes}=295, #{0−simplexes}=68

# of partition elements	1	2	5
computational time	589.3	138.8	6.2

Discrete Laplacian and Homology Computation

Graph Laplacian



For a graph G, the matrix

$L = D - A$ is called its Laplacian

where

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

D : degree matrix (diagonal)

A : adjacency matrix (off-diagonal)

Using the directed Incidence matrix B ,

we can write $L = BB^T$

Laplacian for simplicial cpx

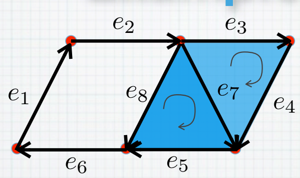
Recall that the incidence matrix of a graph is just the boundary operator of the graph considerd as a simplicial complex.

For a simplicial complex X, define its k-th Laplacian by

$$L_k : C_k(X) \to C_k(X)$$
$$L_k := \partial_k^* \circ \partial_k + \partial_{k+1} \circ \partial_{k+1}^*$$

(L_0 is the usual graph Laplacian)

Example



$$L_1 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 3 \end{pmatrix}$$

Laplacian and Homology

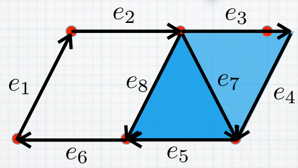
Theorem :

$$C_k(X) = \mathrm{im}\,\partial_{k+1} \oplus \ker L_k \oplus \mathrm{im}\,\partial_k^*$$

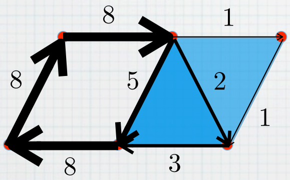
Therefore, $H_k(X) \cong \ker\ L_k$

This is a discrete analogue of the Hodge decomposition of a manifold!

Example



$\ker\ L_1 = \langle (8,8,1,1,3,8,2,5) \rangle \cong \mathbb{R}$



Recovers $H_1(X,\mathbb{R}) = \mathbb{R}$

At each vertex, in = out.

Weights on edges represents the corresponding loop

Application to the Coverage Problem of Sensor Networks

If $\ker\ L_1 = \{0\}$ then $D \subset \mathcal{U}$

That is, if the kernel of L_1 is trivial, there is no hole in the coverage.

How can we check $\ker L_1 = \{0\}$?

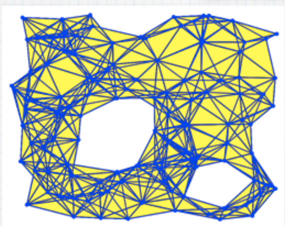
Use ODE to compute

Consider ODE $\dfrac{dx}{dt} = -L_1 x$

0-solution is globally stable $\Longleftrightarrow$ $\ker\ L_1 = \{0\}$

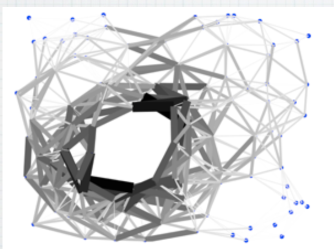
Therefore, if we compute the orbit of this ODE from several initial points, and the solution always converges to 0, then we can conclude $D \subset \mathcal{U}$.

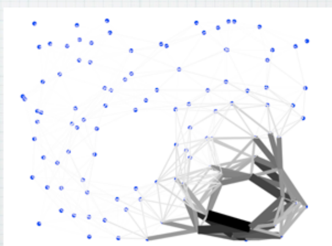
Example



$H_1(\mathcal{R},\mathbb{R}) = \mathbb{R}^2$

There should be two independent solutions





References for sensor network problem and the Laplacian method

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Thank you.