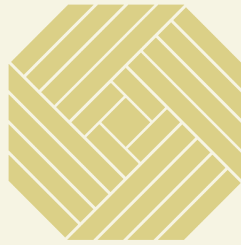




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Irreducible quotients of A -hypergeometric systems

Mutsumi Saito

ABSTRACT

Gel'fand, Kapranov and Zelevinsky proved, using the theory of perverse sheaves, that in the Cohen–Macaulay case an A -hypergeometric system is irreducible if its parameter vector is non-resonant. In this paper we prove, using the theory of the ring of differential operators on an affine toric variety, that in general an A -hypergeometric system is irreducible if and only if its parameter vector is non-resonant. In the course of the proof, we determine the irreducible quotients of an A -hypergeometric system.

1. Introduction

Let K be a field of characteristic 0, and let $A := (a_{ij})$ be a $d \times n$ integer matrix. We assume that $\mathbb{Z}^d$ is generated by the column vectors of A as an abelian group. Given a parameter vector $\beta = (\beta_1, \dots, \beta_d)^T \in K^d$, the A -hypergeometric (or GKZ) system $M_A(\beta)$ with parameter vector β is defined by

$$M_A(\beta) := D(K^n)/D(K^n)I_A(\partial) + D(K^n)\langle A\theta - \beta \rangle, \quad (1)$$

where $D(K^n)$ is the n th Weyl algebra, i.e.

$$D(K^n) = K[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle, \quad (2)$$

$I_A(\partial)$ is the toric ideal of $K[\partial_1, \dots, \partial_n]$ defined by A , and $D(K^n)\langle A\theta - \beta \rangle$ is the left ideal of $D(K^n)$ generated by $\sum_{j=1}^n a_{ij}x_j\partial_j - \beta_i$, $i = 1, \dots, d$.

The irreducibility of $M_A(\beta)$ is one of the most fundamental questions in the theory of A -hypergeometric systems. Gel'fand *et al.* proved, using the theory of perverse sheaves, that when the toric ring is Cohen–Macaulay, $M_A(\beta)$ is irreducible if its parameter vector β is non-resonant; see [GKZ90, Proposition 4.4 and Theorem 4.6]. Schulze and Walther have determined for which parameter vector β the Fourier transform of $M_A(\beta)$ is naturally isomorphic to the direct image of a simple object on the big torus of the affine toric variety defined by A (see [SW09, Corollary 3.7]), which sharpens [GKZ90, Theorem 4.6]. Walther proved in [Wal07, Theorem 3.13] that if $M_A(\beta)$ has irreducible monodromy representation, then so does $M_A(\gamma)$ for any $\gamma \in \beta + \mathbb{Z}^d$, using homological tools developed in [MMW05]. Naturally, an irreducible $D(K^n)$ -module has irreducible monodromy representation; see Proposition 6.8.

In this paper, using the theory of the ring of differential operators on an affine toric variety, we prove that $M_A(\beta)$ is irreducible if and only if β is non-resonant, without assuming that the toric ring is Cohen–Macaulay. Moreover, in the course of the proof, we determine the irreducible quotients of $M_A(\beta)$.

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Let ι be the anti-automorphism of $D(K^n)$ defined by $\iota(x_j) = \partial_j$ and $\iota(\partial_j) = x_j$ for $j = 1, \dots, n$. Then ι gives rise to the equivalence between the category of left $D(K^n)$ -modules and the category of right $D(K^n)$ -modules; the left $D(K^n)$ -module $M_A(\beta)$ corresponds to the right $D(K^n)$ -module $M_{K^n}(\beta)$ (whose definition is given in (8)). Hence the irreducibility of $M_A(\beta)$ is equivalent to that of $M_{K^n}(\beta)$. In this paper, we work with the categories of right D -modules. This has two advantages: one is that the support of $M_{K^n}(\beta)$ is precisely the affine toric variety defined by A ; the other is that we consider direct image functors of D -modules, and for this purpose, right D -modules work more naturally than left D -modules.

In § 2 we introduce the varieties considered in this paper, and in § 3 we briefly recall the rings of differential operators on these varieties and their $\mathbb{Z}^d$ -gradings.

In § 4, for each variety X introduced in § 2 we consider the category $\mathcal{O}_X$, which is analogous to the category $\mathcal{O}$ from the theory of highest-weight modules over semisimple Lie algebras defined in [BGG76] (cf. [MV98, Sai07]). We then recall the simple objects in $\mathcal{O}_X$ for $X = X_A$, the affine toric variety defined by A (see Proposition 4.3), and for $X = T_A$, the big torus of X_A (see Proposition 4.2). Finally, we define Verma-type modules in $\mathcal{O}_X$. The right-module counterpart $M_{K^n}(\beta)$ of the A -hypergeometric system $M_A(\beta)$ is a Verma-type module in $\mathcal{O}_{K^n}$.

In § 5, we explicitly describe the direct image functors of D -modules by inclusions between the varieties under consideration. Using this description, in § 6 we show that the direct image of a simple object in $\mathcal{O}_{T_A}$ by the inclusion of T_A into K^n has a unique irreducible $D(K^n)$ -submodule, and we describe it explicitly (see Theorem 6.4). We then show that each simple object in $\mathcal{O}_{K^n}$ is obtained in a similar way from a possibly smaller torus (Theorem 6.6).

In § 7, we compute the pull-back of each simple object in $\mathcal{O}_{K^n}$ by the inclusion of X_A into K^n (Theorems 7.3 and 7.4). As a consequence, we determine the irreducible quotients of $M_{K^n}(\beta)$ (Corollaries 7.5 and 7.6). In § 8, we prove that $M_{K^n}(\beta)$ is irreducible if and only if β is non-resonant (Theorem 8.3).

2. Varieties

Let $A := \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ be a finite set of column vectors in $\mathbb{Z}^d$. We will sometimes identify A with the matrix $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) = (a_{ij})$. Let $\mathbb{Z}A$ and $\mathbb{R}_{\geq 0}A$ denote, respectively, the abelian group and the cone generated by A . Throughout this paper, we assume that $\mathbb{Z}A = \mathbb{Z}^d$ and that $\mathbb{R}_{\geq 0}A$ is strongly convex.

Let K denote a field of characteristic 0. For a face τ of the cone $\mathbb{R}_{\geq 0}A$, we define the following varieties:

$$\begin{aligned} K^\tau &:= \{\mathbf{x} = (x_1, \dots, x_n) \in K^n : x_j = 0 \text{ when } \mathbf{a}_j \notin \tau\}, \\ (K^\times)^\tau &:= \{\mathbf{x} \in K^\tau : x_j \neq 0 \text{ when } \mathbf{a}_j \in \tau\}, \\ X_\tau &:= \{\mathbf{x} \in K^\tau : x^\mathbf{u} - x^\mathbf{v} = 0 \text{ for } \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ such that } A\mathbf{u} = A\mathbf{v}\}, \\ T_\tau &:= \{\mathbf{x} \in (K^\times)^\tau : x^\mathbf{u} - x^\mathbf{v} = 0 \text{ for } \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ such that } A\mathbf{u} = A\mathbf{v}\}. \end{aligned}$$

Here we have used multi-index notation, where $x^\mathbf{u}$ stands for $x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$, with $\mathbf{u} = (u_1, u_2, \dots, u_n)^\top$. When τ is the whole cone $\mathbb{R}_{\geq 0}A$, we denote the above varieties by K^n , $(K^\times)^n$, X_A and T_A , respectively. Then

$$X_A = \coprod_{\text{faces } \tau \text{ of } \mathbb{R}_{\geq 0}A} T_\tau \quad (3)$$

is the $(K^\times)^d$ -orbit decomposition of the toric variety X_A (see, e.g., [Ful93]). Here $(K^\times)^d$ acts on K^n by

$$(K^\times)^d \times K^n \ni (t, (x_1, \dots, x_n)) \mapsto (t^{\mathbf{a}_1} x_1, \dots, t^{\mathbf{a}_n} x_n) \in K^n,$$

where $t^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\mathbf{a} = (a_1, a_2, \dots, a_d)^T$.

Let $\mathbb{N}A$ denote the monoid generated by A . The semigroup algebra $K[\mathbb{N}A] = \bigoplus_{\mathbf{a} \in \mathbb{N}A} Kt^{\mathbf{a}}$ is the ring of regular functions on the affine toric variety X_A . Then we have $K[\mathbb{N}A] \simeq K[x]/I_A$, where I_A is the ideal of the polynomial ring $K[x] := K[x_1, \dots, x_n]$ generated by all $x^{\mathbf{u}} - x^{\mathbf{v}}$ for $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ with $A\mathbf{u} = A\mathbf{v}$.

3. Rings of differential operators

Let R be a commutative K -algebra, and let M and N be R -modules. We briefly recall the module $D(M, N)$ of differential operators from M to N ; for details, see [SS88]. For $k \in \mathbb{N}$, the subspaces $D^k(M, N)$ of $\text{Hom}_K(M, N)$ are defined inductively by

$$D^0(M, N) := \text{Hom}_R(M, N)$$

and

$$D^{k+1}(M, N) := \{P \in \text{Hom}_K(M, N) : [f, P] \in D^k(M, N) \text{ for all } f \in R\},$$

where $[,]$ denotes the commutator. Set $D(M, N) := \bigcup_{k=0}^{\infty} D^k(M, N)$ and $D(M) := D(M, M)$. Then $D(M)$ is a K -algebra, and $D(M, N)$ is a $(D(M), D(M))$ -bimodule.

The ring $D(K^n) := D(K[x])$ of differential operators on K^n is the n th Weyl algebra (2).

The ring $D((K^\times)^n) := D(K[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ of differential operators on $(K^\times)^n$ is given by

$$\begin{aligned} D((K^\times)^n) &= K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \langle \partial_1, \dots, \partial_n \rangle \\ &= \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} x^{\mathbf{u}} K[\theta_1, \dots, \theta_n], \end{aligned}$$

where $\theta_j = x_j \partial_j$.

The ring $D(T_A) := D(K[t_1^{\pm 1}, \dots, t_d^{\pm 1}])$ of differential operators on T_A is given by

$$\begin{aligned} D(T_A) &= K[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \langle \partial_{t_1}, \dots, \partial_{t_d} \rangle \\ &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} t^{\mathbf{a}} K[s_1, \dots, s_d], \end{aligned}$$

where $s_i = t_i \partial_{t_i}$ and $\partial_{t_i} = \partial / \partial t_i$.

The ring $D(X_A) := D(K[\mathbb{N}A])$ of differential operators on X_A is a subalgebra of $D(T_A)$:

$$D(X_A) = \{P \in D(T_A) : P(K[\mathbb{N}A]) \subseteq K[\mathbb{N}A]\}.$$

Let X be K^n , $(K^\times)^n$, T_A or X_A . For $\mathbf{a} = (a_1, \dots, a_d)^T \in \mathbb{Z}^d$, set

$$D(X)_{\mathbf{a}} := \{P \in D(X) : [s_i, P] = a_i P \text{ for } i = 1, \dots, d\},$$

where $s_i = \sum_{j=1}^n a_{ij} x_j \partial_j$ for $X = K^n$ or $(K^\times)^n$. Then

$$D(X) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} D(X)_{\mathbf{a}}$$

is a $\mathbb{Z}^d$ -graded algebra.

Let τ be a face of the cone $\mathbb{R}_{\geq 0}A$. Let $\mathbb{Z}(A \cap \tau)$ and $\mathbb{N}(A \cap \tau)$ denote, respectively, the abelian group and the monoid generated by $A \cap \tau$. Set

$$\mathbb{Z}^\tau := \{\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^n : u_j = 0 \text{ when } \mathbf{a}_j \notin \tau\}.$$

As in the case where τ is the whole cone $\mathbb{R}_{\geq 0}A$, for K^τ , $(K^\times)^\tau$, T_τ and X_τ we consider the following rings of differential operators:

$$\begin{aligned} D(K^\tau) &= D(K[x_j : \mathbf{a}_j \in \tau]) = K[x_j : \mathbf{a}_j \in \tau] \langle \partial_j : \mathbf{a}_j \in \tau \rangle, \\ D((K^\times)^\tau) &= K[x_j^{\pm 1} : \mathbf{a}_j \in \tau] \langle \partial_j : \mathbf{a}_j \in \tau \rangle = \bigoplus_{\mathbf{u} \in \mathbb{Z}^\tau} x^\mathbf{u} K[\theta_j : \mathbf{a}_j \in \tau], \end{aligned}$$

$$D(T_\tau) = \bigoplus_{\mathbf{a} \in \mathbb{Z}(A \cap \tau)} t^\mathbf{a} K[s_{1|\tau}, \dots, s_{d|\tau}],$$

$$D(X_\tau) = \{P \in D(T_\tau) : P(K[X_\tau]) \subseteq K[X_\tau]\},$$

where $s_{i|\tau}$ is the operator s_i restricted to $K[T_\tau] = K[t^{\pm \mathbf{a}_j} : \mathbf{a}_j \in \tau]$ and $K[X_\tau]$ is the subalgebra of $K[T_\tau]$ defined by

$$K[X_\tau] = K[\mathbb{N}(A \cap \tau)] = K[t^{\mathbf{a}_j} : \mathbf{a}_j \in \tau].$$

These rings of differential operators are graded by $\mathbb{Z}(A \cap \tau)$, and since $\mathbb{Z}(A \cap \tau)$ is a subgroup of $\mathbb{Z}A = \mathbb{Z}^d$, they are also considered to be $\mathbb{Z}^d$ -graded. Note that $s_{i|\tau} = \sum_{\mathbf{a}_j \in \tau} a_{ij} \theta_j$ in x -coordinates.

4. The category $\mathcal{O}_X$

Take X to be K^n , $(K^\times)^n$, T_A or X_A . We shall define a full subcategory $\mathcal{O}_X$ of the category of right $D(X)$ -modules (cf. [MV98]). A right $D(X)$ -module M is an object of $\mathcal{O}_X$ if the support of M is contained in X_A and M has a weight decomposition $M = \bigoplus_{\boldsymbol{\lambda} \in K^d} M_{\boldsymbol{\lambda}}$, where

$$M_{\boldsymbol{\lambda}} = \{x \in M : x.f(s) = f(-\boldsymbol{\lambda})x \text{ for all } f \in K[s]\}$$

with $K[s] = K[s_1, \dots, s_d]$.

PROPOSITION 4.1. *Let M be a simple object in $\mathcal{O}_X$. Then M is an irreducible right $D(X)$ -module.*

Proof. Let N be a right $D(X)$ -submodule of M . Let $x \in N$, and write $x = \sum_{\mathbf{b} \in S} x_{\mathbf{b}}$ for $x_{\mathbf{b}} \in M_{\mathbf{b}}$, where S is a finite subset of K^d . For $\mathbf{b} \in S$, take $f(s) \in K[s]$ such that $f(-\mathbf{b}) \neq 0$ and $f(-\mathbf{c}) = 0$ for all $\mathbf{c} \in S \setminus \{\mathbf{b}\}$. Upon applying $f(s)$ to x , we see that $x_{\mathbf{b}} \in N$. Hence $N \in \mathcal{O}_X$. By the simplicity of M in $\mathcal{O}_X$, we have $N = 0$ or $N = M$. $\square$

In the rest of this section, we define objects $L_{T_A}(\boldsymbol{\lambda})$ and $L_{X_A}(\boldsymbol{\lambda})$ which are simple in the categories $\mathcal{O}_{T_A}$ and $\mathcal{O}_{X_A}$, respectively. Then we define Verma-type modules $M_{X_A}(\boldsymbol{\beta})$, $M_{K^n}(\boldsymbol{\beta})$ and $M_{(K^\times)^n}(\boldsymbol{\beta})$.

Let $\boldsymbol{\lambda} \in K^d$. We define a right $D(T_A)$ -module $L_{T_A}(\boldsymbol{\lambda})$ by

$$L_{T_A}(\boldsymbol{\lambda}) := D(T_A) / \langle s - \boldsymbol{\lambda} \rangle D(T_A) := D(T_A) \Big/ \sum_{i=1}^d (s_i - \lambda_i) D(T_A).$$

Let $K[t^{\pm 1}]$ denote the Laurent polynomial ring $K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. By taking formal adjoint operators, $D(T_A)$ acts on $K[t^{\pm 1}]t^{-\boldsymbol{\lambda}} dT_A$ from the right as follows:

$$(g(t) dT_A).P = P^*(g) dT_A,$$

where

$$P^* = \sum_{\mathbf{a}} f_{\mathbf{a}}(-s)t^{\mathbf{a}}$$

for $P = \sum_{\mathbf{a}} t^{\mathbf{a}} f_{\mathbf{a}}(s) \in \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} t^{\mathbf{a}} K[s] = D(T_A)$ and dT_A is simply a formal symbol. Then $K[t^{\pm 1}]t^{-\boldsymbol{\lambda}} dT_A$ is a realization of $L_{T_A}(\boldsymbol{\lambda})$, and we denote $K[t^{\pm 1}]t^{-\boldsymbol{\lambda}} dT_A$ by $L_{T_A}(\boldsymbol{\lambda})$, so that

$$L_{T_A}(\boldsymbol{\lambda}) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} L_{T_A}(\boldsymbol{\lambda})_{-\boldsymbol{\lambda} + \mathbf{a}} \quad \text{with } L_{T_A}(\boldsymbol{\lambda})_{-\boldsymbol{\lambda} + \mathbf{a}} = Kt^{-\boldsymbol{\lambda} + \mathbf{a}} dT_A. \quad (4)$$

The following proposition is clear.

PROPOSITION 4.2. *Each $L_{T_A}(\boldsymbol{\lambda})$ is a simple object in $\mathcal{O}_{T_A}$. Each simple object in $\mathcal{O}_{T_A}$ is isomorphic to $L_{T_A}(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in K^d$, and $L_{T_A}(\boldsymbol{\lambda}) \simeq L_{T_A}(\boldsymbol{\mu})$ if and only if $\boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathbb{Z}^d$.*

Recall that the ring $D(X_A)$ is described as follows (see [Mus87, Theorem 2.3]):

$$D(X_A)_{\mathbf{a}} = t^{\mathbf{a}} \mathbb{I}(\Omega(\mathbf{a})) \quad \text{for } \mathbf{a} \in \mathbb{Z}^d,$$

where

$$\begin{aligned} \Omega(\mathbf{a}) &:= \Omega_A(\mathbf{a}) := \mathbb{N}A \setminus (-\mathbf{a} + \mathbb{N}A), \\ \mathbb{I}(\Omega(\mathbf{a})) &:= \{f(s) \in K[s] : f(\mathbf{c}) = 0 \text{ for all } \mathbf{c} \in \Omega(\mathbf{a})\}. \end{aligned} \quad (5)$$

Recall also the preorder $\preceq$ defined in [MV98] (see also [ST01]):

$$\text{for } \boldsymbol{\alpha}, \boldsymbol{\beta} \in K^d, \quad \boldsymbol{\alpha} \preceq \boldsymbol{\beta} \iff \mathbb{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\alpha})) \not\subseteq \mathfrak{m}_{\boldsymbol{\alpha}}, \quad (6)$$

where $\mathfrak{m}_{\boldsymbol{\alpha}}$ is the maximal ideal of $K[s]$ at $\boldsymbol{\alpha}$. We define an equivalence relation $\sim$ by setting $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if and only if $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$. We write $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$.

Since the ring $D(X_A)$ is a subalgebra of $D(T_A)$, the right $D(T_A)$ -module

$$L_{T_A}(\boldsymbol{\lambda}) = K[t^{\pm 1}]t^{-\boldsymbol{\lambda}} dT_A = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} Kt^{-\boldsymbol{\lambda} + \mathbf{a}} dT_A$$

is also a right $D(X_A)$ -module. Then the subquotient

$$L_{X_A}(\boldsymbol{\lambda}) := \bigoplus_{\boldsymbol{\mu} \preceq \boldsymbol{\lambda}} Kt^{-\boldsymbol{\mu}} dT_A \Big/ \bigoplus_{\boldsymbol{\mu} \prec \boldsymbol{\lambda}} Kt^{-\boldsymbol{\mu}} dT_A \quad (7)$$

is a right $D(X_A)$ -module (see [ST01, Proposition 4.1.5]). We have the following proposition.

PROPOSITION 4.3. *Each $L_{X_A}(\boldsymbol{\lambda})$ is a simple object in $\mathcal{O}_{X_A}$. Each simple object in $\mathcal{O}_{X_A}$ is isomorphic to $L_{X_A}(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in K^d$. Moreover, $L_{X_A}(\boldsymbol{\lambda}) \simeq L_{X_A}(\boldsymbol{\mu})$ if and only if $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$.*

(See [MV98, Proposition 3.1.7], [ST01, Theorem 4.1.6] or [Sai07, Proposition 3.6(4)].)

For $\boldsymbol{\beta} \in K^d$, we define a right $D(X_A)$ -module $M_{X_A}(\boldsymbol{\beta})$, a right $D(K^n)$ -module $M_{K^n}(\boldsymbol{\beta})$ and a right $D((K^\times)^n)$ -module $M_{(K^\times)^n}(\boldsymbol{\beta})$ by

$$\begin{aligned} M_{X_A}(\boldsymbol{\beta}) &:= D(X_A) / \langle s - \boldsymbol{\beta} \rangle D(X_A), \\ M_{K^n}(\boldsymbol{\beta}) &:= D(K^n) / (I_A D(K^n) + \langle s - \boldsymbol{\beta} \rangle D(K^n)), \\ M_{(K^\times)^n}(\boldsymbol{\beta}) &:= D((K^\times)^n) / (I_A D((K^\times)^n) + \langle s - \boldsymbol{\beta} \rangle D((K^\times)^n)). \end{aligned} \quad (8)$$

Recall that $s_i = t_i \partial_{t_i}$ in t -coordinates and that $s_i = \sum_{j=1}^n a_{ij} \theta_j$ with $\theta_j = x_j \partial_j$ in x -coordinates. Clearly, $M_{X_A}(\boldsymbol{\beta}) \in \mathcal{O}_{X_A}$, $M_{K^n}(\boldsymbol{\beta}) \in \mathcal{O}_{K^n}$ and $M_{(K^\times)^n}(\boldsymbol{\beta}) \in \mathcal{O}_{(K^\times)^n}$.

Let τ be a face of the cone $\mathbb{R}_{\geq 0}A$. Similarly to the case where τ is the whole cone $\mathbb{R}_{\geq 0}A$, for $Y = K^\tau, (K^\times)^\tau, T_\tau$ or X_τ we consider $\mathcal{O}_Y$, replacing $\mathbb{Z}A = \mathbb{Z}^d$, $KA = K^d$ and $f(s) \in K[s]$

by $\mathbb{Z}(A \cap \tau)$, $K(A \cap \tau)$ and $f(s)|_\tau$, respectively, where $f(s)|_\tau$ is the operator $f(s)$ restricted to $K[T_\tau] = K[t^{\pm \mathbf{a}_j} : \mathbf{a}_j \in \tau]$.

5. Direct image functors

In this section, we describe direct image functors explicitly. Using them, we link some of the modules defined in § 4.

5.1 From $\mathcal{O}_{T_A}$ to $\mathcal{O}_{(K^\times)^n}$

We shall write $D((K^\times)^n, T_A)$ instead of $D(K[x^{\pm 1}], K[t^{\pm 1}])$, where $K[x^{\pm 1}]$ stands for the Laurent polynomial ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Since T_A is closed in $(K^\times)^n$, the direct image functor

$$\int_{T_A \rightarrow (K^\times)^n}^0 : M \mapsto M \otimes_{D(T_A)} D((K^\times)^n, T_A)$$

gives a category equivalence between $\mathcal{O}_{T_A}$ and $\mathcal{O}_{(K^\times)^n}$, known as Kashiwara's equivalence (see, e.g., [Kas03, Theorem 4.30] or [HTT08, Theorem 1.6.1]). From [SS88, § 1.3, (e) and (f)], we have

$$\begin{aligned} D((K^\times)^n, T_A) &= D((K^\times)^n)/I_A D((K^\times)^n) \\ &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} t^{\mathbf{a}} K[\theta_1, \dots, \theta_n]. \end{aligned} \quad (9)$$

By definition,

$$M_{(K^\times)^n}(\beta) = \int_{T_A \rightarrow (K^\times)^n}^0 L_{T_A}(\beta). \quad (10)$$

Hence, by Kashiwara's equivalence, Proposition 4.2 leads to the following result.

PROPOSITION 5.1. *For each $\beta \in K^d$, $M_{(K^\times)^n}(\beta)$ is a simple object in $\mathcal{O}_{(K^\times)^n}$. Each simple object in $\mathcal{O}_{(K^\times)^n}$ is isomorphic to some $M_{(K^\times)^n}(\beta)$. Moreover, $M_{(K^\times)^n}(\beta) \simeq M_{(K^\times)^n}(\beta')$ if and only if $\beta - \beta' \in \mathbb{Z}^d$.*

5.2 From $\mathcal{O}_{X_A}$ to $\mathcal{O}_{K^n}$

Again from [SS88, § 1.3, (e) and (f)], we have

$$D(K^n, X_A) := D(K[x], K[\mathbb{N}A]) = D(K^n)/I_A D(K^n). \quad (11)$$

Since I_A is $\mathbb{Z}^d$ -homogeneous, $D(K^n, X_A)$ inherits the $\mathbb{Z}^d$ -grading from $D(K^n)$.

The algebra $D(X_A)$ can be identified with

$$\{P \in D(K^n) : PI_A \subseteq I_A D(K^n)\} / I_A D(K^n)$$

(see, e.g., [MR87, Theorem 5.13]). We may therefore consider $D(X_A)$ as being contained in $D(K^n, X_A)$.

Let $\int_{X_A \rightarrow K^n}^0$ denote the functor from $\mathcal{O}_{X_A}$ to $\mathcal{O}_{K^n}$ defined by

$$\int_{X_A \rightarrow K^n}^0 M := M \otimes_{D(X_A)} D(K^n, X_A).$$

Note that, in general, X_A is singular and $\int_{X_A \rightarrow K^n}^0$ does not give a category equivalence. By definition, we have

$$M_{K^n}(\beta) = \int_{X_A \rightarrow K^n}^0 M_{X_A}(\beta). \quad (12)$$

For the following result, see [Sai07, Proposition 4.1 and Corollary 4.2].

PROPOSITION 5.2.

$$D(K^n, X_A) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} D(K^n, X_A)_{\mathbf{a}} \quad \text{with } D(K^n, X_A)_{\mathbf{a}} = t^{\mathbf{a}} \mathbb{I}(\tilde{\Omega}(\mathbf{a})),$$

where

$$\begin{aligned} \tilde{\Omega}(\mathbf{a}) &:= \tilde{\Omega}_A(\mathbf{a}) := \{\mathbf{u} \in \mathbb{N}^n : A\mathbf{u} \notin -\mathbf{a} + \mathbb{N}A\}, \\ \mathbb{I}(\tilde{\Omega}(\mathbf{a})) &= \{f(\theta) \in K[\theta] : f(\mathbf{u}) = 0 \text{ for all } \mathbf{u} \in \tilde{\Omega}(\mathbf{a})\} \end{aligned} \quad (13)$$

and $K[\theta] := K[\theta_1, \dots, \theta_n]$.

5.3 From $\mathcal{O}_{K^\tau}$ to $\mathcal{O}_{K^n}$

Let τ be a face of the cone $\mathbb{R}_{\geq 0}A$. We consider the direct image functor $\int_{K^\tau \rightarrow K^n}^0$ from $\mathcal{O}_{K^\tau}$ to $\mathcal{O}_{K^n}$. Given $M \in \mathcal{O}_{K^\tau}$, we define $\int_{K^\tau \rightarrow K^n}^0 M \in \mathcal{O}_{K^n}$ by

$$\int_{K^\tau \rightarrow K^n}^0 M := M \otimes_{D(K^\tau)} D(K^n, K^\tau),$$

where

$$D(K^n, K^\tau) := D(K[x], K[x_j : \mathbf{a}_j \in \tau]).$$

Put

$$\begin{aligned} K^{\tau^c} &:= \{\mathbf{x} = (x_1, \dots, x_n) \in K^n : x_j = 0 \text{ when } \mathbf{a}_j \in \tau\}, \\ \mathbb{N}^{\tau^c} &:= \{\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n : a_j = 0 \text{ when } \mathbf{a}_j \in \tau\}, \\ \mathbb{Z}^{\tau^c} &:= \{\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n : a_j = 0 \text{ when } \mathbf{a}_j \in \tau\}. \end{aligned}$$

Then

$$\begin{aligned} D(K^n, K^\tau) &= D(K^n) / \langle x_j : \mathbf{a}_j \notin \tau \rangle D(K^n) \\ &= D(K^\tau) \boxtimes D(K^{\tau^c}) / \langle x_j : \mathbf{a}_j \notin \tau \rangle D(K^{\tau^c}). \end{aligned}$$

Since, as right $D(K^{\tau^c})$ -modules,

$$D(K^{\tau^c}) / \langle x_j : \mathbf{a}_j \notin \tau \rangle D(K^{\tau^c}) \simeq \bigoplus_{\mathbf{b} \in \mathbb{Z}^{\tau^c}} Kx^{-\mathbf{b}} d(K^\times)^{\tau^c} \bigg/ \bigoplus_{\mathbf{b} \notin \mathbb{N}^{\tau^c}} Kx^{-\mathbf{b}} d(K^\times)^{\tau^c},$$

we have

$$D(K^n, K^\tau) \simeq D(K^\tau) \boxtimes \bigoplus_{\mathbf{b} \in \mathbb{N}^{\tau^c}} Kx^{-\mathbf{b}} d(K^\times)^{\tau^c}. \quad (14)$$

Hence

$$\int_{K^\tau \rightarrow K^n}^0 M \simeq M \boxtimes \bigoplus_{\mathbf{b} \in \mathbb{N}^{\tau^c}} Kx^{-\mathbf{b}} d(K^\times)^{\tau^c}. \quad (15)$$

6. Simple objects in $\mathcal{O}_{K^n}$

In this section, we describe the simple objects in $\mathcal{O}_{K^n}$ explicitly.

By (9), (10) and the realization (4), we have the following realization of $M_{(K^\times)^n}(\beta)$.

LEMMA 6.1. *Let $\beta \in KA = K^d$. Then*

$$M_{(K^\times)^n}(\beta) = \bigoplus_{\alpha \in \mathbb{Z}^d} K t^{-\beta + \alpha} dT_A \otimes_{K[s]} K[\theta].$$

The $D(K^n)$ -module $\int_{T_A \rightarrow K^n}^0 L_{T_A}(\beta)$ is defined to be the $D((K^\times)^n)$ -module

$$\int_{T_A \rightarrow (K^\times)^n}^0 L_{T_A}(\beta) = M_{(K^\times)^n}(\beta), \quad (16)$$

considered as a $D(K^n)$ -module.

DEFINITION 6.2. Let $\beta \in KA = K^d$. In $\beta + \mathbb{Z}A = \beta + \mathbb{Z}^d$ there exists a unique minimal equivalence class with respect to $\preceq$ (see Remark 6.3), which we denote by β^{empty} . Any fixed element belonging to the class is also denoted by β^{empty} .

Remark 6.3. In [Sai01] we defined, for a face τ and a parameter vector $\alpha \in KA = K^d$, a finite set

$$E_\tau(\alpha) = \{\lambda \in K(A \cap \tau) / \mathbb{Z}(A \cap \tau) : \alpha - \lambda \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\}. \quad (17)$$

The class β^{empty} is given by

$$E_\tau(\beta^{\text{empty}}) = \begin{cases} E_{\mathbb{R}_{\geq 0}A}(\beta) & \text{if } \tau = \mathbb{R}_{\geq 0}A, \\ \emptyset & \text{if } \tau \neq \mathbb{R}_{\geq 0}A. \end{cases} \quad (18)$$

THEOREM 6.4. *Let $\beta \in KA / \mathbb{Z}A = K^d / \mathbb{Z}^d$, and fix an element $e := \beta^{\text{empty}}$. Then*

$$\begin{aligned} L_{K^n}(T_A, \beta) &:= (t^{-e} dT_A \otimes 1) D(K^n) \\ &= \bigoplus_{\alpha \in \mathbb{Z}^d} K t^{-e + \alpha} dT_A \otimes_{K[s]} \mathbb{I}(\tilde{\Omega}(\alpha)) \\ &\simeq D(K^n) / (I_A D(K^n) + D(K^n) \cap \langle s - e \rangle D((K^\times)^n)) \end{aligned}$$

is a unique simple $D(K^n)$ -submodule of $\int_{T_A \rightarrow K^n}^0 L_{T_A}(\beta)$.

Moreover, $L_{K^n}(T_A, \beta) \simeq L_{K^n}(T_A, \beta')$ if and only if $\beta - \beta' \in \mathbb{Z}^d$.

Proof. Recall that $\int_{T_A \rightarrow K^n}^0 L_{T_A}(\beta)$ is the module $M_{(K^\times)^n}(\beta)$ regarded as a $D(K^n)$ -module (16). Hence $L_{K^n}(T_A, \beta)$ is isomorphic to $D(K^n) / (I_A D(K^n) + D(K^n) \cap \langle s - e \rangle D((K^\times)^n))$ by the definition of $M_{(K^\times)^n}(\beta) = M_{(K^\times)^n}(e)$. The first equation is clear from (11) and Proposition 5.2.

Let $y \in M_{(K^\times)^n}(\beta)_\gamma$ be non-zero. We prove that $yD(K^n) \supseteq L_{K^n}(T_A, \beta)$. By multiplying a suitable x^u from the right, we may assume that

$$y = t^{-\beta'} dT_A \otimes f(\theta) \quad \text{for some } \beta' \sim e. \quad (19)$$

Here $f(\theta) \notin \langle A\theta - \beta' \rangle K[\theta]$ since $y \neq 0$. We shall use the symbols s and $A\theta$ interchangeably. We claim that

$$t^{-\beta''} dT_A \otimes 1 \in yD(K^n) \quad \text{for some } \beta'' \sim e. \quad (20)$$

We take an element of type (19) in $yD(K^n)$ such that the total degree $\deg(f)$ of f is as small as possible, and we call this element y again. If $f(\theta) \in K[s]$, then clearly we have the claim (20). Suppose $f(\theta) \notin K[s]$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ satisfy $A\mathbf{u} = A\mathbf{v}$. Since

$$f(\theta)(x^{\mathbf{u}} - x^{\mathbf{v}}) = (x^{\mathbf{u}} - x^{\mathbf{v}})f(\theta + \mathbf{u}) + x^{\mathbf{v}}(f(\theta + \mathbf{u}) - f(\theta + \mathbf{v})),$$

we have

$$y.(x^{\mathbf{u}} - x^{\mathbf{v}}) = t^{-\beta' + A\mathbf{v}} dT_A \otimes (f(\theta + \mathbf{u}) - f(\theta + \mathbf{v})).$$

By the minimality of $\deg(f)$,

$$f(\theta + \mathbf{u}) - f(\theta + \mathbf{v}) \in \langle A\theta - (\beta' - A\mathbf{v}) \rangle K[\theta].$$

Hence, for all $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ with $A\mathbf{u} = A\mathbf{v}$,

$$f(\theta + \mathbf{u}) - f(\theta + \mathbf{v}) \in \langle A\theta - (\beta' - A\mathbf{v}) \rangle K[\theta].$$

Since $f(\theta) \notin \langle A\theta - \beta' \rangle K[\theta]$, there exists $\mathbf{z} \in K^n$ with $A\mathbf{z} = \beta'$ such that $f(\mathbf{z}) \neq 0$. By Lemma 6.5 below, we have

$$f(\theta) \in f(\mathbf{z}) + \langle A\theta - \beta' \rangle K[\theta].$$

Hence $y = t^{-\beta'} dT_A \otimes f(\mathbf{z})$. We have thus proved claim (20).

Since $\beta'' \sim \mathbf{e}$, there exists $p(s) \in \mathbb{I}(\Omega(\beta'' - \mathbf{e}))$ such that $p(\beta'') \neq 0$. Hence $t^{\beta'' - \mathbf{e}} p(s) \in D(X_A) \subseteq D(K^n)/I_A D(K^n)$, and

$$(t^{-\beta''} dT_A \otimes 1) t^{\beta'' - \mathbf{e}} p(s) = p(\beta'') t^{-\mathbf{e}} dT_A \otimes 1.$$

We have thus proved that $yD(K^n) \supseteq L_{K^n}(T_A, \beta)$ and that $L_{K^n}(T_A, \beta)$ is a unique simple $D(K^n)$ -submodule of $\int_{T_A \rightarrow K^n}^0 L_{T_A}(\beta)$.

Next, we prove the second statement. If $\beta - \beta' \in \mathbb{Z}^d$, then $\beta^{\text{empty}} = \beta'^{\text{empty}}$. Hence $L_{K^n}(T_A, \beta) = L_{K^n}(T_A, \beta')$ by definition. If $\beta - \beta' \notin \mathbb{Z}^d$, then $L_{K^n}(T_A, \beta)$ and $L_{K^n}(T_A, \beta')$ have distinct weight sets and hence are not isomorphic. $\square$

LEMMA 6.5. *Let $f(\theta) \in K[\theta]$ satisfy*

$$f(\theta + \mathbf{l}) - f(\theta) \in \langle A\theta - \mathbf{c} \rangle K[\theta]$$

for all $\mathbf{l}$ with $A\mathbf{l} = 0$. Take $\gamma \in K^n$ such that $A\gamma = \mathbf{c}$. Then

$$f(\theta) \in f(\gamma) + \langle A\theta - \mathbf{c} \rangle K[\theta].$$

Proof.

$$\begin{aligned} f(\theta + \mathbf{l}) - f(\theta) &\in \langle A\theta - \mathbf{c} \rangle K[\theta] \quad \text{for all } \mathbf{l} \text{ such that } A\mathbf{l} = \mathbf{0} \\ \implies f(\mathbf{l} + \gamma) - f(\gamma) &= 0 \quad \text{for all } \mathbf{l} \text{ such that } A\mathbf{l} = \mathbf{0} \\ \iff f(\theta + \gamma) &\in f(\gamma) + \langle A\theta \rangle K[\theta] \\ \iff f(\theta) &\in f(\gamma) + \langle A\theta - \mathbf{c} \rangle K[\theta]. \end{aligned}$$

$\square$

Let τ be a face of $\mathbb{R}_{\geq 0} A$, and let $\lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$. We define a right $D(K^\tau)$ -module $L_{K^\tau}(T_\tau, \lambda)$ in the same way as we defined $L_{K^n}(T_A, \beta)$ in Theorem 6.4. By Theorem 6.4, $L_{K^\tau}(T_\tau, \lambda)$ is a simple $D(K^\tau)$ -module. By Kashiwara's equivalence,

$$L_{K^n}(T_\tau, \lambda) := \int_{K^\tau \rightarrow K^n}^0 L_{K^\tau}(T_\tau, \lambda) \quad (21)$$

is a simple $D(K^n)$ -module.

THEOREM 6.6. *Each simple object in $\mathcal{O}_{K^n}$ is isomorphic to $L_{K^n}(T_\tau, \boldsymbol{\lambda})$ for some face τ and some $\boldsymbol{\lambda} \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$.*

Moreover, $L_{K^n}(T_\tau, \boldsymbol{\lambda}) \simeq L_{K^n}(T_{\tau'}, \boldsymbol{\lambda}')$ if and only if $\tau = \tau'$ and $\boldsymbol{\lambda} - \boldsymbol{\lambda}' \in \mathbb{Z}(A \cap \tau)$.

Proof. Let L be a simple object in $\mathcal{O}_{K^n}$. Suppose that $\text{supp}(L) = \overline{T_A} = X_A$. There exists the following exact sequence in $\mathcal{O}_{K^n}$:

$$0 \rightarrow \Gamma_{K^n \setminus (K^\times)^n}(L) \rightarrow L \rightarrow \Gamma_{(K^\times)^n}(L),$$

where $\Gamma_{K^n \setminus (K^\times)^n}(L) = \{y \in L : \text{supp}(y) \subseteq K^n \setminus (K^\times)^n\}$ and $\Gamma_{(K^\times)^n}(L)$ is the localization of L at the multiplicatively closed set $\{x_j^m : j = 1, \dots, n; m \in \mathbb{N}\}$. By the simplicity of L , $\Gamma_{K^n \setminus (K^\times)^n}(L) = 0$. Hence L is a simple submodule of $\Gamma_{(K^\times)^n}(L)$, and then $\Gamma_{(K^\times)^n}(L)$ is simple in $\mathcal{O}_{(K^\times)^n}$. Indeed, let y be a non-zero element of $\Gamma_{(K^\times)^n}(L)$; then there exists $\mathbf{u} \in \mathbb{N}^n$ such that $y \cdot x^{\mathbf{u}} \in L$. Since L is a simple $D(K^n)$ -module, we have $y \cdot D(K^n) \supseteq L$. Since $\Gamma_{(K^\times)^n}(L)$ is generated by L as a $D((K^\times)^n)$ -module, we obtain $y \cdot D((K^\times)^n) = \Gamma_{(K^\times)^n}(L)$, and hence $\Gamma_{(K^\times)^n}(L)$ is simple in $\mathcal{O}_{(K^\times)^n}$. Then, by Proposition 5.1, $\Gamma_{(K^\times)^n}(L) \simeq M_{(K^\times)^n}(\boldsymbol{\beta})$ for some $\boldsymbol{\beta} \in KA/\mathbb{Z}A$. Since $M_{(K^\times)^n}(\boldsymbol{\beta})$ has the unique simple submodule $L_{K^n}(T_A, \boldsymbol{\beta})$, we conclude that $L \simeq L_{K^n}(T_A, \boldsymbol{\beta})$.

By the simplicity of L , the support of L is the closure of T_τ for some face τ . By the same argument as in the previous paragraph, we obtain $L \simeq L_{K^n}(T_\tau, \boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$.

The second statement is clear from the second statement of Theorem 6.4. $\square$

Example 6.7. Let $A = (1)$. In this case, the cone $\mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}$ has only two faces: $\{0\}$ and $\mathbb{R}_{\geq 0}$. Then

$$L_K(T_{\{0\}}, 0) = \int_{\{0\} \rightarrow K}^0 K \simeq D/xD,$$

where D is the first Weyl algebra.

Let $\beta \in K$. If $\beta \notin \mathbb{Z} = \mathbb{Z}A$, then $\beta = \beta^{\text{empty}}$. If $\beta \in \mathbb{Z}$, then $\beta = \beta^{\text{empty}}$ if and only if $\beta \in \mathbb{Z}_{\leq -1}$. The simple module $L_K(T_A, \beta)$ is the unique simple submodule of $x^{-\beta}K[x, x^{-1}]dT_A$ generated by $x^{-\beta^{\text{empty}}}dT_A$. Hence

$$\begin{aligned} L_K(T_A, \beta) &= x^{-\beta}dT_A \cdot D \simeq D/(x\partial - \beta)D \quad \text{for } \beta \notin \mathbb{Z}, \\ L_K(T_A, \beta) &= L_K(T_A, -1) = xdT_A \cdot D \simeq D/\partial D \quad \text{for } \beta \in \mathbb{Z}. \end{aligned}$$

A left $D(K^n)$ -module M is said to have *irreducible monodromy representation* if $D(K^n)(x) \otimes_{D(K^n)} M$ is an irreducible left $D(K^n)(x)$ -module, where $D(K^n)(x) = K(x) \otimes_{K[x]} D(K^n)$ with $K(x) = K(x_1, \dots, x_n)$ being the field of rational functions (cf. [Wal07]). We naturally have the following proposition.

PROPOSITION 6.8. *Let M be an irreducible left $D(K^n)$ -module. Suppose that $D(K^n)(x) \otimes_{D(K^n)} M \neq 0$. Then M has irreducible monodromy representation.*

Proof. We can write $M = D(K^n)/I$ with I a maximal left ideal of $D(K^n)$. Then

$$D(K^n)(x) \otimes_{D(K^n)} M = D(K^n)(x)/D(K^n)(x)I.$$

Let J be a left ideal of $D(K^n)(x)$ containing $D(K^n)(x)I$. Since $J \cap D(K^n)$ is a left ideal of $D(K^n)$ containing I , we have $J \cap D(K^n) = D(K^n)$ or I . If $J \cap D(K^n) = D(K^n)$, then $1 \in J$ and thus $J = D(K^n)(x)$.

Suppose that $J \cap D(K^n) = I$. Let $P \in J$. Then there exists a non-zero polynomial $f \in K[x]$ such that $fP \in J \cap D(K^n) = I$. Hence $P \in D(K^n)(x)I$, and we have $J = D(K^n)(x)I$. $\square$

7. Pull-back of $L_{K^n}(T_\tau, \lambda)$

Let $i^\sharp$ denote the functor from $\mathcal{O}_{K^n}$ to $\mathcal{O}_{X_A}$ defined by

$$\begin{aligned} i^\sharp(N) &:= \operatorname{Hom}_{D(K^n)}(D(K^n, X_A), N) \\ &= \{x \in N : x.I_A = 0\}. \end{aligned} \quad (22)$$

The following adjointness property holds:

$$\operatorname{Hom}_{D(K^n)}\left(\int_{X_A \rightarrow K^n}^0 M, N\right) \simeq \operatorname{Hom}_{D(X_A)}(M, i^\sharp(N)). \quad (23)$$

In this section, we compute the pull-back of $L_{K^n}(T_\tau, \lambda)$ by $i^\sharp$. As a consequence, we determine the irreducible quotients of $M_{K^n}(\beta)$.

Before considering $i^\sharp(L_{K^n}(T_A, \lambda))$, we present two preparatory lemmas.

LEMMA 7.1. *Let $\mathbf{c} \in \operatorname{ZC}(\Omega(\mathbf{a}))$, where $\Omega(\mathbf{a})$ is as defined in (5) and ZC stands for the Zariski closure in K^d . Then there exist $\mathbf{b} \in \Omega(\mathbf{a})$ and a face τ such that $\mathbf{b} + \mathbb{N}(A \cap \tau) \subseteq \Omega(\mathbf{a})$ and $\mathbf{c} \in \mathbf{b} + K(A \cap \tau)$.*

Proof. This follows from [ST04, Proposition 5.1]. $\square$

LEMMA 7.2. *Suppose that*

$$\mathbb{I}(\Omega(\mathbf{a})) \subseteq \langle s - \mathbf{c} \rangle K[s].$$

Then

$$\{f \in \mathbb{I}(\tilde{\Omega}(\mathbf{a})) : f(\gamma) = f(\gamma') \text{ if } A\gamma = A\gamma' = \mathbf{c}\} \subseteq \langle A\theta - \mathbf{c} \rangle K[\theta], \quad (24)$$

where $\tilde{\Omega}(\mathbf{a})$ is as defined in (13).

Proof. Since $\mathbb{I}(\Omega(\mathbf{a})) \subseteq \langle s - \mathbf{c} \rangle K[s]$, we have $\mathbf{c} \in \operatorname{ZC}(\Omega(\mathbf{a}))$. By Lemma 7.1 there exist $\mathbf{b} \in \Omega(\mathbf{a})$ and a face τ such that $\mathbf{b} + \mathbb{N}(A \cap \tau) \subseteq \Omega(\mathbf{a})$ and $\mathbf{c} \in \mathbf{b} + K(A \cap \tau)$. Take $\mathbf{u} \in \mathbb{N}^n$ such that $A\mathbf{u} = \mathbf{b}$. Then there exists $\gamma' \in \mathbf{u} + K^\tau$ such that $A\gamma' = \mathbf{c}$. Observe that $\gamma' \in \operatorname{ZC}(\tilde{\Omega}(\mathbf{a}))$, since $\mathbf{u} + \mathbb{N}^\tau \subseteq \tilde{\Omega}(\mathbf{a})$.

Let $f(\theta)$ belong to the set on the left-hand side of (24). If $A\gamma = \mathbf{c}$ ($= A\gamma'$), then we have $f(\gamma) = f(\gamma') = 0$ since $\gamma' \in \operatorname{ZC}(\tilde{\Omega}(\mathbf{a}))$. Hence $f \in \langle A\theta - \mathbf{c} \rangle K[\theta]$. $\square$

THEOREM 7.3.

$$i^\sharp(L_{K^n}(T_A, \beta)) = L_{X_A}(\beta^{\text{empty}}).$$

Proof. Fix $\mathbf{e} := \beta^{\text{empty}}$. By Theorem 6.4,

$$\begin{aligned} L_{K^n}(T_A, \beta) &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} t^{-\mathbf{e} + \mathbf{a}} dT_A \otimes_{K[s]} (\mathbb{I}(\tilde{\Omega}(\mathbf{a}))/\mathbb{I}(\tilde{\Omega}(\mathbf{a})) \cap \langle s - \mathbf{e} + \mathbf{a} \rangle K[\theta]) \\ &\subseteq \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} t^{-\mathbf{e} + \mathbf{a}} dT_A \otimes_{K[s]} K[\theta] / \langle s - \mathbf{e} + \mathbf{a} \rangle K[\theta]. \end{aligned}$$

First, we claim that

$$i^\sharp(L_{K^n}(T_A, \beta)) \subseteq \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} K t^{-\mathbf{e} + \mathbf{a}} dT_A. \quad (25)$$

Let $f(\theta) \in K[\theta]$, and fix $\gamma \in K^n$ with $A\gamma = e - a$. Then

$$\begin{aligned}
 t^{-e+a} dT_A \otimes f(\theta).I_A &= 0 \\
 \iff t^{-e+a} dT_A \otimes f(\theta).(x^u - x^v) &= 0 \quad \text{for all } u \text{ and } v \text{ with } Au = Av \\
 \iff t^{-e+a+Au} dT_A \otimes (f(\theta + u) - f(\theta + v)) &= 0 \quad \text{for all } u \text{ and } v \text{ with } Au = Av \\
 \iff f(\theta + u) - f(\theta + v) \in \langle A\theta - e + a + Au \rangle K[\theta] &\quad \text{for all } u \text{ and } v \text{ with } Au = Av \\
 \iff f(\theta + u - v) - f(\theta) \in \langle A\theta - e + a \rangle K[\theta] &\quad \text{for all } u \text{ and } v \text{ with } Au = Av.
 \end{aligned}$$

Hence, by Lemma 6.5, $t^{-e+a} dT_A \otimes f(\theta) \in i^\sharp(L_{K^n}(T_A, \beta))$ implies

$$f(\theta) \in f(\gamma) + \langle A\theta - e + a \rangle K[\theta].$$

Therefore $t^{-e+a} dT_A \otimes f(\theta) = f(\gamma)t^{-e+a} dT_A \otimes 1$ and the claim (25) is proved.

Recall that

$$\begin{aligned}
 e - a \not\sim e &\iff e - a \not\leq e \\
 &\iff \mathbb{I}(\Omega(a)) \subseteq \langle s - e + a \rangle K[s].
 \end{aligned} \tag{26}$$

Suppose $e - a \sim e$. Then there exists $f(s) \in \mathbb{I}(\Omega(a))$ such that $f(s) \notin \langle s - e + a \rangle K[s]$. Hence, for $\gamma \in K^n$ with $A\gamma = e - a$, we have $f(\gamma) = f(A\gamma) \neq 0$. Then

$$i^\sharp(L_{K^n}(T_A, \beta)) \ni t^{-e+a} dT_A \otimes f(A\theta) = f(\gamma)t^{-e+a} dT_A \otimes 1 \neq 0,$$

and thus the weight $-e + a$ appears in $i^\sharp(L_{K^n}(T_A, \beta))$.

Next, suppose $e - a \not\sim e$. Then $\mathbb{I}(\Omega(a)) \subseteq \langle s - e + a \rangle K[s]$. By the proof of (25), if $t^{-e+a} dT_A \otimes f(\theta) \in i^\sharp(L_{K^n}(T_A, \beta))$, then $f(\gamma) = f(\gamma')$ for any $\gamma, \gamma' \in K^n$ with $A\gamma = A\gamma' = e - a$. Hence, by (7), it suffices to prove the inclusion

$$\{f \in \mathbb{I}(\tilde{\Omega}(a)) : f(\gamma) = f(\gamma') \text{ if } A\gamma = A\gamma' = e - a\} \subseteq \langle A\theta - e + a \rangle K[\theta],$$

assuming that $\mathbb{I}(\Omega(a)) \subseteq \langle s - e + a \rangle K[s]$. We finish the proof by invoking Lemma 7.2. $\square$

Given faces τ and τ' of $\mathbb{R}_{\geq 0}A$, $\lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$ and $\lambda' \in K(A \cap \tau')/\mathbb{Z}(A \cap \tau')$, set

$$(\tau', \lambda') \prec (\tau, \lambda) \stackrel{\text{def}}{\iff} \tau' \prec \tau \quad \text{and} \quad \lambda - \lambda' \in \mathbb{Z}(A \cap \tau). \tag{27}$$

THEOREM 7.4. *Let $\lambda \in K(A \cap \tau)/\mathbb{Z}(A \cap \tau)$. Then*

$$\dim_K i^\sharp(L_{K^n}(T_\tau, \lambda))_{-c} = \begin{cases} 1 & \text{if } c \in C_{K^n}(\tau, \lambda), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$C_{K^n}(\tau, \lambda) = \left\{ c \in K^d : \begin{array}{l} E_\tau(c) \ni \lambda \text{ and } E_{\tau'}(c) \not\ni \lambda' \\ \text{whenever } (\tau', \lambda') \prec (\tau, \lambda) \end{array} \right\}. \tag{28}$$

Proof. By (15),

$$L_{K^n}(T_\tau, \lambda) \simeq L_{K^\tau}(T_\tau, \lambda) \boxtimes \left(\bigoplus_{\tilde{\mathbf{b}} \in \mathbb{N}^{\tau^c}} Kx^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c} \right).$$

By the definition of $i^\sharp$,

$$\begin{aligned} i^\sharp(L_{K^n}(T_\tau, \lambda)) &= \{f \in L_{K^n}(T_\tau, \lambda) : f.I_A = 0\} \\ &\subseteq \{f \in L_{K^n}(T_\tau, \lambda) : f.(x^{\mathbf{u}} - x^{\mathbf{v}}) = 0 \text{ for } \mathbf{u}, \mathbf{v} \in \mathbb{N}^\tau \text{ with } A\mathbf{u} = A\mathbf{v}\}. \end{aligned}$$

Hence, by Theorem 7.3,

$$i^\sharp(L_{K^n}(T_\tau, \lambda)) \subseteq \left(\bigoplus_{\mathbf{a} \sim \lambda^{\text{empty}}} Kt^{-\mathbf{a}} dT_\tau \right) \boxtimes \left(\bigoplus_{\tilde{\mathbf{b}} \in \mathbb{N}^{\tau^c}} Kx^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c} \right).$$

Note that for $\mathbf{a} \in K(A \cap \tau)$, $\mathbf{a} \sim \lambda^{\text{empty}}$ if and only if $\mathbf{a} \in C_{K^n}(\tau, \lambda) \cap K(A \cap \tau) =: C_{K^\tau}(\tau, \lambda)$. Let

$$f = \sum_{(\mathbf{a}, \tilde{\mathbf{b}}) \in C} f_{\mathbf{a}, \tilde{\mathbf{b}}} t^{-\mathbf{a}} dT_\tau \otimes x^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c}, \quad (29)$$

where $C = C_{K^\tau}(\tau, \lambda) \times \mathbb{N}^{\tau^c}$. Note that the set of $(\mathbf{a}, \tilde{\mathbf{b}}) \in C$ with a fixed $\mathbf{a} + A\tilde{\mathbf{b}}$ is finite, since $\mathbf{a} \in \lambda + \mathbb{Z}(A \cap \tau)$, $\tilde{\mathbf{b}} \in \mathbb{N}^{\tau^c}$ and $\mathbb{R}_{\geq 0}(A \setminus \tau) \cap \mathbb{R}\tau = \{\mathbf{0}\}$.

Let $\mathbf{u} = \mathbf{u}_\tau + \mathbf{u}_{\tau^c}$ and $\mathbf{v} = \mathbf{v}_\tau + \mathbf{v}_{\tau^c}$, with $\mathbf{u}_\tau, \mathbf{v}_\tau \in \mathbb{N}^\tau$ and $\mathbf{u}_{\tau^c}, \mathbf{v}_{\tau^c} \in \mathbb{N}^{\tau^c}$, satisfy $A\mathbf{u} = A\mathbf{v}$. We claim that for f as in (29),

$$f \in i^\sharp(L_{K^n}(T_\tau, \lambda)) \iff \begin{cases} \text{(i)} & f_{\mathbf{a}+A\mathbf{u}_\tau, \tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}} = f_{\mathbf{a}+A\mathbf{v}_\tau, \tilde{\mathbf{b}}+\mathbf{v}_{\tau^c}} \\ & \text{for } (\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a}+A\mathbf{u}_\tau, \tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}), (\mathbf{a}+A\mathbf{v}_\tau, \tilde{\mathbf{b}}+\mathbf{v}_{\tau^c}) \in C, \\ \text{(ii)} & f_{\mathbf{a}+A\mathbf{u}_\tau, \tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}} = 0 \\ & \text{for } (\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a}+A\mathbf{u}_\tau, \tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}) \in C, (\mathbf{a}+A\mathbf{v}_\tau, \tilde{\mathbf{b}}+\mathbf{v}_{\tau^c}) \notin C. \end{cases} \quad (30)$$

We have

$$\begin{aligned} f.(x^{\mathbf{u}} - x^{\mathbf{v}}) &= \sum_{(\mathbf{a}, \tilde{\mathbf{b}}) \in C} f_{\mathbf{a}, \tilde{\mathbf{b}}} t^{-\mathbf{a}+A\mathbf{u}_\tau} dT_\tau \otimes x^{-\tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}} d(K^\times)^{\tau^c} \\ &\quad - \sum_{(\mathbf{a}, \tilde{\mathbf{b}}) \in C} f_{\mathbf{a}, \tilde{\mathbf{b}}} t^{-\mathbf{a}+A\mathbf{v}_\tau} dT_\tau \otimes x^{-\tilde{\mathbf{b}}+\mathbf{v}_{\tau^c}} d(K^\times)^{\tau^c} \\ &= \sum_{(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a}-A\mathbf{u}_\tau, \tilde{\mathbf{b}}-\mathbf{u}_{\tau^c}) \in C} f_{\mathbf{a}, \tilde{\mathbf{b}}} t^{-\mathbf{a}+A\mathbf{u}_\tau} dT_\tau \otimes x^{-\tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}} d(K^\times)^{\tau^c} \\ &\quad - \sum_{(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a}-A\mathbf{v}_\tau, \tilde{\mathbf{b}}-\mathbf{v}_{\tau^c}) \in C} f_{\mathbf{a}, \tilde{\mathbf{b}}} t^{-\mathbf{a}+A\mathbf{v}_\tau} dT_\tau \otimes x^{-\tilde{\mathbf{b}}+\mathbf{v}_{\tau^c}} d(K^\times)^{\tau^c} \\ &= \sum_{(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a}+A\mathbf{u}_\tau, \tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}) \in C} f_{\mathbf{a}+A\mathbf{u}_\tau, \tilde{\mathbf{b}}+\mathbf{u}_{\tau^c}} t^{-\mathbf{a}} dT_\tau \otimes x^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c} \\ &\quad - \sum_{(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a}+A\mathbf{v}_\tau, \tilde{\mathbf{b}}+\mathbf{v}_{\tau^c}) \in C} f_{\mathbf{a}+A\mathbf{v}_\tau, \tilde{\mathbf{b}}+\mathbf{v}_{\tau^c}} t^{-\mathbf{a}} dT_\tau \otimes x^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a} + A\mathbf{u}_\tau, \tilde{\mathbf{b}} + \mathbf{u}_{\tau^c}) \in C \\ (\mathbf{a} + A\mathbf{v}_\tau, \tilde{\mathbf{b}} + \mathbf{v}_{\tau^c}) \in C}} (f_{\mathbf{a} + A\mathbf{u}_\tau, \tilde{\mathbf{b}} + \mathbf{u}_{\tau^c}} - f_{\mathbf{a} + A\mathbf{v}_\tau, \tilde{\mathbf{b}} + \mathbf{v}_{\tau^c}}) t^{-\mathbf{a}} dT_\tau \otimes x^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c} \\
 &+ \sum_{\substack{(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a} + A\mathbf{u}_\tau, \tilde{\mathbf{b}} + \mathbf{u}_{\tau^c}) \in C \\ (\mathbf{a} + A\mathbf{v}_\tau, \tilde{\mathbf{b}} + \mathbf{v}_{\tau^c}) \notin C}} f_{\mathbf{a} + A\mathbf{u}_\tau, \tilde{\mathbf{b}} + \mathbf{u}_{\tau^c}} t^{-\mathbf{a}} dT_\tau \otimes x^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c} \\
 &- \sum_{\substack{(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a} + A\mathbf{v}_\tau, \tilde{\mathbf{b}} + \mathbf{v}_{\tau^c}) \in C \\ (\mathbf{a} + A\mathbf{u}_\tau, \tilde{\mathbf{b}} + \mathbf{u}_{\tau^c}) \notin C}} f_{\mathbf{a} + A\mathbf{v}_\tau, \tilde{\mathbf{b}} + \mathbf{v}_{\tau^c}} t^{-\mathbf{a}} dT_\tau \otimes x^{-\tilde{\mathbf{b}}} d(K^\times)^{\tau^c},
 \end{aligned}$$

so (30) is established.

Let us keep $f \in i^{\sharp}(L_{K^n}(T_\tau, \boldsymbol{\lambda}))$ as in (29) and take $(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a}', \tilde{\mathbf{b}}') \in C$ with $\mathbf{a} + A\tilde{\mathbf{b}} = \mathbf{a}' + A\tilde{\mathbf{b}}'$. We claim that then

$$f_{\mathbf{a}, \tilde{\mathbf{b}}} = f_{\mathbf{a}', \tilde{\mathbf{b}}'}. \quad (31)$$

Indeed, let $\mathbf{w} \in K^\tau$ and $\tilde{\mathbf{a}}, \tilde{\mathbf{a}}' \in \mathbb{Z}^\tau$ satisfy $\boldsymbol{\lambda} = A\mathbf{w}$, $\mathbf{a} = A(\mathbf{w} + \tilde{\mathbf{a}})$ and $\mathbf{a}' = A(\mathbf{w} + \tilde{\mathbf{a}}')$. Put $\mathbf{u}_\tau := (\tilde{\mathbf{a}} - \tilde{\mathbf{a}}')_+ \in \mathbb{N}^\tau$, $\mathbf{v}_\tau := (\tilde{\mathbf{a}} - \tilde{\mathbf{a}}')_- \in \mathbb{N}^\tau$, $\mathbf{u}_{\tau^c} := (\tilde{\mathbf{b}} - \tilde{\mathbf{b}}')_+ \in \mathbb{N}^{\tau^c}$ and $\mathbf{v}_{\tau^c} := (\tilde{\mathbf{b}} - \tilde{\mathbf{b}}')_- \in \mathbb{N}^{\tau^c}$. Here, $(\tilde{\mathbf{a}} - \tilde{\mathbf{a}}')_+$ is the non-negative part of $\tilde{\mathbf{a}} - \tilde{\mathbf{a}}'$, and $(\tilde{\mathbf{a}} - \tilde{\mathbf{a}}')_-$ is the negative of the non-positive part of $\tilde{\mathbf{a}} - \tilde{\mathbf{a}}'$. Then $A(\mathbf{u}_\tau + \mathbf{u}_{\tau^c}) = A(\mathbf{v}_\tau + \mathbf{v}_{\tau^c})$ and $\tilde{\mathbf{b}} - \mathbf{u}_{\tau^c} = \tilde{\mathbf{b}}' - \mathbf{v}_{\tau^c} \in \mathbb{N}^{\tau^c}$. Furthermore, $\mathbf{a} - A\mathbf{u}_\tau = \mathbf{a}' - A\mathbf{v}_\tau \in C_{K^\tau}(\tau, \boldsymbol{\lambda})$, since $\mathbf{a} \sim \mathbf{a}' \sim \boldsymbol{\lambda}^{\text{empty}}$ is the minimal class (see [Sai01, Proposition 2.2(5)]). Hence, from (30)(i) we obtain (31).

We can rewrite (30)(ii) as

$$f_{\mathbf{a}, \tilde{\mathbf{b}}} = 0 \quad (32)$$

for $(\mathbf{a}, \tilde{\mathbf{b}}), (\mathbf{a} - A\mathbf{u}_\tau, \tilde{\mathbf{b}} - \mathbf{u}_{\tau^c}) \in C$ and $(\mathbf{a} - A\mathbf{u}_\tau + A\mathbf{v}_\tau, \tilde{\mathbf{b}} - \mathbf{u}_{\tau^c} + \mathbf{v}_{\tau^c}) \notin C$.

We prove next that (32) is equivalent to the following condition:

$$\begin{aligned}
 &\text{if there exists } (\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda}) \text{ such that } E_{\tau'}(\mathbf{a} + A\tilde{\mathbf{b}}) \ni \boldsymbol{\lambda}', \\
 &\text{then } f_{\mathbf{a}, \tilde{\mathbf{b}}} = 0.
 \end{aligned} \quad (33)$$

For this purpose, when $(\mathbf{a}, \tilde{\mathbf{b}}) \in C$ we prove the equivalence

$$\text{there exists } (\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda}) \text{ such that } E_{\tau'}(\mathbf{a} + A\tilde{\mathbf{b}}) \ni \boldsymbol{\lambda}' \quad (34)$$

$$\iff \text{there exist } \mathbf{u}_\tau, \mathbf{v}_\tau \in \mathbb{N}^\tau \text{ and } \mathbf{u}_{\tau^c}, \mathbf{v}_{\tau^c} \in \mathbb{N}^{\tau^c} \text{ such that} \quad (35)$$

$$A(\mathbf{u}_\tau + \mathbf{u}_{\tau^c}) = A(\mathbf{v}_\tau + \mathbf{v}_{\tau^c}), (\mathbf{a} - A\mathbf{u}_\tau, \tilde{\mathbf{b}} - \mathbf{u}_{\tau^c}) \in C$$

$$\text{and } (\mathbf{a} - A\mathbf{u}_\tau + A\mathbf{v}_\tau, \tilde{\mathbf{b}} - \mathbf{u}_{\tau^c} + \mathbf{v}_{\tau^c}) \notin C.$$

First, suppose that (35) holds. Then $\tilde{\mathbf{b}} - \mathbf{u}_{\tau^c} \in \mathbb{N}^{\tau^c}$, and there exists $(\tau', \boldsymbol{\lambda}') \prec (\tau, \boldsymbol{\lambda})$ such that $E_{\tau'}(\mathbf{a} - A\mathbf{u}_\tau + A\mathbf{v}_\tau) \ni \boldsymbol{\lambda}'$. It follows from $\tilde{\mathbf{b}} - \mathbf{u}_{\tau^c} \in \mathbb{N}^{\tau^c}$ and $A(\mathbf{u}_\tau + \mathbf{u}_{\tau^c}) = A(\mathbf{v}_\tau + \mathbf{v}_{\tau^c})$ that $A\mathbf{v}_\tau - A\mathbf{u}_\tau \in A(\tilde{\mathbf{b}} - \mathbb{N}^{\tau^c})$. Hence $E_{\tau'}(\mathbf{a} + A\tilde{\mathbf{b}}) \ni \boldsymbol{\lambda}'$ (cf. [Sai01, Proposition 2.2(5)]).

Conversely, suppose that (34) holds. Then $\mathbf{a} + A\tilde{\mathbf{b}} - \boldsymbol{\lambda}' \in \mathbb{N}A + \mathbb{Z}(A \cap \tau')$. Let $\mathbf{w}' \in K^{\tau'}$, $\tilde{\mathbf{a}} \in \mathbb{Z}^\tau$, $\tilde{\mathbf{b}}' \in \mathbb{N}^{\tau^c}$ and $\tilde{\mathbf{a}}' \in \mathbb{N}^{\tau \setminus \tau'} \times \mathbb{Z}^{\tau'}$ satisfy $\boldsymbol{\lambda}' = A\mathbf{w}'$, $\mathbf{a} = A(\mathbf{w}' + \tilde{\mathbf{a}})$ and $\mathbf{a} + A\tilde{\mathbf{b}} - \boldsymbol{\lambda}' = A\tilde{\mathbf{b}}' + A\tilde{\mathbf{a}}'$. As before, put $\mathbf{u}_\tau := (\tilde{\mathbf{a}} - \tilde{\mathbf{a}}')_+ \in \mathbb{N}^\tau$, $\mathbf{v}_\tau := (\tilde{\mathbf{a}} - \tilde{\mathbf{a}}')_- \in \mathbb{N}^\tau$, $\mathbf{u}_{\tau^c} := (\tilde{\mathbf{b}} - \tilde{\mathbf{b}}')_+ \in \mathbb{N}^{\tau^c}$ and $\mathbf{v}_{\tau^c} := (\tilde{\mathbf{b}} - \tilde{\mathbf{b}}')_- \in \mathbb{N}^{\tau^c}$. Then $(\mathbf{a} - A\mathbf{u}_\tau, \tilde{\mathbf{b}} - \mathbf{u}_{\tau^c}) \in C$. Furthermore, $\mathbf{a} - A\mathbf{u}_\tau + A\mathbf{v}_\tau = \mathbf{a} - A(\tilde{\mathbf{a}} - \tilde{\mathbf{a}}') = \boldsymbol{\lambda}' + A\tilde{\mathbf{a}}' \in \boldsymbol{\lambda}' + \mathbb{N}A + \mathbb{Z}(A \cap \tau')$. Hence $\boldsymbol{\lambda}' \in E_{\tau'}(\mathbf{a} - A\mathbf{u}_\tau + A\mathbf{v}_\tau)$, and thus $(\mathbf{a} - A\mathbf{u}_\tau + A\mathbf{v}_\tau, \tilde{\mathbf{b}} - \mathbf{u}_{\tau^c} + \mathbf{v}_{\tau^c}) \notin C$. Finally, $A(\mathbf{u}_\tau + \mathbf{u}_{\tau^c}) - A(\mathbf{v}_\tau + \mathbf{v}_{\tau^c}) = A(\tilde{\mathbf{a}} - \tilde{\mathbf{a}}') + A(\tilde{\mathbf{b}} - \tilde{\mathbf{b}}') = \mathbf{a} - \boldsymbol{\lambda}' - A\tilde{\mathbf{a}}' + A(\tilde{\mathbf{b}} - \tilde{\mathbf{b}}') = \mathbf{0}$. Therefore we have established the equivalence between (34) and (35) and hence the equivalence between (32) and (33).

In summary, we have shown that

$$i^{\natural}(L_{K^n}(T_{\tau}, \boldsymbol{\lambda})) = \bigoplus_{\mathbf{c} \in C_{K^n}(\tau, \boldsymbol{\lambda})} K \sum_{(\mathbf{a}, \tilde{\mathbf{b}}), \mathbf{c} = \mathbf{a} + A\tilde{\mathbf{b}}} t^{-\mathbf{a}} dT_{\tau} \otimes x^{-\tilde{\mathbf{b}}} d(K^{\times})^{\tau^c}, \quad (36)$$

so the proof of Theorem 7.4 is complete. $\square$

COROLLARY 7.5.

$$\dim_K \operatorname{Hom}_{D(R)}(M_{K^n}(\boldsymbol{\beta}), L_{K^n}(T_{\tau}, \boldsymbol{\lambda})) = \begin{cases} 1 & \text{if } \boldsymbol{\beta} \in C_{K^n}(\tau, \boldsymbol{\lambda}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} & \dim_K \operatorname{Hom}_{D(K^n)}(M_{K^n}(\boldsymbol{\beta}), L_{K^n}(T_{\tau}, \boldsymbol{\lambda})) \\ &= \dim_K \operatorname{Hom}_{D(K^n)}\left(\int_{X_A \rightarrow K^n}^0 M_{X_A}(\boldsymbol{\beta}), L_{K^n}(T_{\tau}, \boldsymbol{\lambda})\right) \\ &= \dim_K \operatorname{Hom}_{D(X_A)}(M_{X_A}(\boldsymbol{\beta}), i^{\natural}(L_{K^n}(T_{\tau}, \boldsymbol{\lambda}))) \\ &= \dim_K (i^{\natural}(L_{K^n}(T_{\tau}, \boldsymbol{\lambda})))_{-\boldsymbol{\beta}}. \end{aligned}$$

The first equality comes from (12) and the second from the adjointness (23). The third follows from [MV98, Proposition 3.1.7] (see also [Sai07, Proposition 3.6]). Theorem 7.4 then finishes the proof of this corollary. $\square$

For $\boldsymbol{\beta} \in K^d$, set

$$E(\boldsymbol{\beta}) := \{(\tau, \boldsymbol{\lambda}) : \tau \text{ a face of } \mathbb{R}_{\geq 0}A, \boldsymbol{\lambda} \in E_{\tau}(\boldsymbol{\beta})\}. \quad (37)$$

Then Corollary 7.5 can be rephrased as follows.

COROLLARY 7.6.

$$\dim_K \operatorname{Hom}_{D(R)}(M_{K^n}(\boldsymbol{\beta}), L_{K^n}(T_{\tau}, \boldsymbol{\lambda})) = \begin{cases} 1 & \text{if } (\tau, \boldsymbol{\lambda}) \text{ is minimal in } E(\boldsymbol{\beta}), \\ 0 & \text{otherwise.} \end{cases}$$

Here the minimality is with respect to (27).

Example 7.7. Let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3].$$

Then the cone $\mathbb{R}_{\geq 0}A$ has exactly four faces: $\mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}^2$, $\sigma_1 := \mathbb{R}_{\geq 0}\mathbf{a}_1$, $\sigma_3 := \mathbb{R}_{\geq 0}\mathbf{a}_3$ and $\{\mathbf{0}\}$. The semigroup $\mathbb{N}A$ is shown in Figure 1.

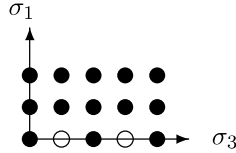


FIGURE 1. The semigroup $\mathbb{N}A$.

Let τ be a face of $\mathbb{R}_{\geq 0}A$. Then

$$|\mathbb{Z}^2 \cap K(A \cap \tau) / \mathbb{Z}(A \cap \tau)| = \begin{cases} 1 & \text{if } \tau \neq \sigma_3, \\ 2 & \text{if } \tau = \sigma_3. \end{cases}$$

Hence the category $\mathcal{O}_{K^3}$ has exactly five simple objects with weights in $\mathbb{Z}^2$, namely $L_{K^3}(T_A, \mathbf{0})$, $L_{K^3}(T_{\sigma_1}, \mathbf{0})$, $L_{K^3}(T_{\sigma_3}, \mathbf{0})$, $L_{K^3}(T_{\sigma_3}, (1, 0)^T)$ and $L_{K^3}(T_{\{0\}}, \mathbf{0})$. For each of these, we write down the weight set $(C_{K^n}(\tau, \lambda)$ in Theorem 7.4) of the pull-back by $i^\sharp$.

- (i) $i^\sharp(L_{K^3}(T_A, \mathbf{0}))$: the weights in $C_{K^3}(\mathbb{R}_{\geq 0}A, \mathbf{0})$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_1}(\beta) = \emptyset$ and $E_{\sigma_3}(\beta) = \emptyset$, shown in Figure 2.

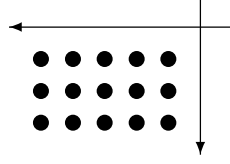


FIGURE 2. The weight space of $i^\sharp(L_{K^3}(T_A, \mathbf{0}))$.

- (ii) $i^\sharp(L_{K^3}(T_{\sigma_1}, \mathbf{0}))$: the weights in $C_{K^3}(\sigma_1, \mathbf{0})$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_1}(\beta) = \{0\}$ and $E_{\{0\}}(\beta) = \emptyset$, shown in Figure 3.

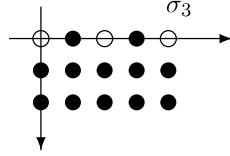


FIGURE 3. The weight space of $i^\sharp(L_{K^3}(T_{\sigma_1}, \mathbf{0}))$.

- (iii) $i^\sharp(L_{K^3}(T_{\sigma_3}, \mathbf{0}))$: the weights in $C_{K^3}(\sigma_3, \mathbf{0})$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_3}(\beta) \ni \mathbf{0}$ and $E_{\{0\}}(\beta) = \emptyset$, shown in Figure 4.

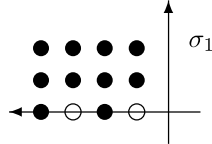


FIGURE 4. The weight space of $i^\sharp(L_{K^3}(T_{\sigma_3}, \mathbf{0}))$.

- (iv) $i^\sharp(L_{K^3}(T_{\sigma_3}, (1, 0)^T))$: the weights in $C_{K^3}(\sigma_3, (1, 0)^T)$ are $\beta \in \mathbb{Z}^2$ with $E_{\sigma_3}(\beta) \ni (1, 0)^T$, shown in Figure 5.

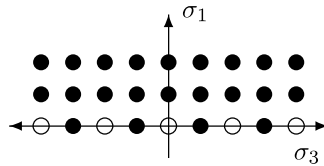


FIGURE 5. The weight space of $i^\sharp(L_{K^3}(T_{\sigma_3}, (1, 0)^T))$.

- (v) $i^\sharp(L_{K^3}(T_{\{0\}}, \mathbf{0}))$: the weights in $C_{K^3}(\{0\}, \mathbf{0})$ are $\beta \in \mathbb{Z}^2$ with $E_{\{0\}}(\beta) = \{0\}$; hence the weight set is $\mathbb{N}A$, shown in Figure 1.

Let $\beta \in \mathbb{Z}^2$. By Corollary 7.5, the irreducible quotients of $M_{K^3}(\beta)$ are precisely the above $L_{K^3}(T_\tau, \lambda)$ such that β appears in the weight set of $i^\sharp(L_{K^3}(T_\tau, \lambda))$.

Recall that $M_{K^3}(\beta) \simeq M_{K^3}(\beta')$ if and only if $\beta \sim \beta'$ (see [Sai01, Theorem 2.1]). There are eight equivalence classes in $\{M_{K^3}(\beta) : \beta \in \mathbb{Z}^2\}$. The following table lists the irreducible quotients for each equivalence class.

$M_{K^3}(\beta)$	Irreducible quotients
$M_{K^3}((0, 1)^T)$	$L_{K^3}(T_{\{0\}}, \mathbf{0}), L_{K^3}(T_{\sigma_3}, (1, 0)^T)$
$M_{K^3}((-1, 1)^T)$	$L_{K^3}(T_{\sigma_3}, \mathbf{0}), L_{K^3}(T_{\sigma_3}, (1, 0)^T)$
$M_{K^3}((0, 0)^T)$	$L_{K^3}(T_{\{0\}}, \mathbf{0})$
$M_{K^3}((1, 0)^T)$	$L_{K^3}(T_{\sigma_1}, \mathbf{0}), L_{K^3}(T_{\sigma_3}, (1, 0)^T)$
$M_{K^3}((-1, 0)^T)$	$L_{K^3}(T_{\sigma_3}, (1, 0)^T)$
$M_{K^3}((-2, 0)^T)$	$L_{K^3}(T_{\sigma_3}, \mathbf{0})$
$M_{K^3}((0, -1)^T)$	$L_{K^3}(T_{\sigma_1}, \mathbf{0})$
$M_{K^3}((-1, -1)^T)$	$L_{K^3}(T_A, \mathbf{0})$

8. The irreducibility of $M_{K^n}(\beta)$

If $\beta = \beta^{\text{empty}}$, then, by Corollary 7.6, there exists a surjective homomorphism

$$M_{K^n}(\beta) \rightarrow L_{K^n}(T_A, \beta). \quad (38)$$

In this section, we analyze the kernel of (38) and prove that $M_{K^n}(\beta)$ is irreducible if and only if β is non-resonant.

Given a facet (maximal proper face) σ of $\mathbb{R}_{\geq 0}A$, we denote by F_σ the *primitive integral support function* of σ ; that is, F_σ is the uniquely determined linear form on $\mathbb{R}^d$ satisfying:

- (i) $F_\sigma(\mathbb{R}_{\geq 0}A) \geq 0$;
- (ii) $F_\sigma(\sigma) = 0$;
- (iii) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Then, by [Sai01, Proposition 2.2] and Remark 6.3, we know that $\beta = \beta^{\text{empty}}$ if and only if $F_\sigma(\beta) \notin F_\sigma(\mathbb{N}A)$ for all facets σ of $\mathbb{R}_{\geq 0}A$.

Let $\beta = \beta^{\text{empty}}$, and let

$$\mathbf{v}_{-\beta} := t^{-\beta} dT_A \otimes 1 \in L_{K^n}(T_A, \beta)_{-\beta}.$$

Then, by Theorem 6.4,

$$\text{Ann}_{D(K^n)}(\mathbf{v}_{-\beta}) = I_A D(K^n) + D(K^n) \cap \langle A\theta - \beta \rangle D((K^\times)^n).$$

Let

$$N := \text{Ann}_{D(K^n)}(\mathbf{v}_{-\beta}) / (I_A D(K^n) + \langle A\theta - \beta \rangle D(K^n)). \quad (39)$$

Then N is the kernel of (38). By (11) and Proposition 5.2, for $\mathbf{a} \in \mathbb{Z}^d$ we have

$$N_{-\beta-\mathbf{a}} = t^{-\mathbf{a}}(\mathbb{I}(\tilde{\Omega}(-\mathbf{a})) \cap \langle A\theta - \beta - \mathbf{a} \rangle) / t^{-\mathbf{a}}(\mathbb{I}(\tilde{\Omega}(-\mathbf{a})) \langle A\theta - \beta - \mathbf{a} \rangle). \quad (40)$$

Since $\{\mathbf{u} \in \mathbb{N}^n : A\mathbf{u} \in \mathbf{a} + \mathbb{N}A\}$ is $\mathbb{N}^n$ -stable, there exists a finite set $\{(\mathbf{u}^{(j)}, I_j) : j \in J\}$ of pairs made up of a $\mathbf{u}^{(j)} \in \mathbb{N}^n$ and a subset I_j of $\{1, \dots, n\}$ (the set of so-called *standard pairs* of $\{\mathbf{u} \in \mathbb{N}^n : A\mathbf{u} \in \mathbf{a} + \mathbb{N}A\}$; see, e.g., [SST00, § 3.2]) such that:

- the i th coordinate of $\mathbf{u}^{(j)}$ is 0 for each $i \in I_j$;
- for all $i \notin I_j$, $(\mathbf{u}^{(j)} + \mathbb{N}^{I_j \cup \{i\}}) \cap \{\mathbf{u} \in \mathbb{N}^n : A\mathbf{u} \in \mathbf{a} + \mathbb{N}A\} \neq \emptyset$;
- $\tilde{\Omega}(-\mathbf{a}) = \mathbb{N}^n \setminus \{\mathbf{u} \in \mathbb{N}^n : A\mathbf{u} \in \mathbf{a} + \mathbb{N}A\} = \bigcup_{j \in J} (\mathbf{u}^{(j)} + \mathbb{N}^{I_j})$.

LEMMA 8.1. Let $\mathbf{a} \in \mathbb{Z}^d$, and let $\{(\mathbf{u}^{(j)}, I_j) : j \in J\}$ be the set of standard pairs of $\{\mathbf{u} \in \mathbb{N}^n : A\mathbf{u} \in \mathbf{a} + \mathbb{N}A\}$. Then for each $j \in J$ there exists a face $\tau^{(j)}$ of $\mathbb{R}_{\geq 0}A$ such that $I_j = \{k \in \{1, \dots, n\} : \mathbf{a}_k \in \tau^{(j)}\}$, and either $\tau^{(j)}$ is a facet with $F_{\tau^{(j)}}(A\mathbf{u}^{(j)}) \notin F_{\tau^{(j)}}(\mathbf{a} + \mathbb{N}A)$ or $F_{\sigma}(A\mathbf{u}^{(j)}) \in F_{\sigma}(\mathbf{a} + \mathbb{N}A)$ for all facets $\sigma \succeq \tau^{(j)}$.

Proof. Put $S_c = \{\mathbf{d} \in \mathbb{Z}^d : F_{\sigma}(\mathbf{d}) \in F_{\sigma}(\mathbb{N}A) \text{ for all facets } \sigma\}$. Then there exist finitely many pairs $(\mathbf{b}_i, \tau_i)$ of $\mathbf{b}_i \in S_c$ and a face τ_i such that

$$S_c \setminus \mathbb{N}A = \bigcup_i (\mathbf{b}_i + \mathbb{Z}(A \cap \tau_i)) \cap S_c$$

(see [ST04, proof of Proposition 5.1]). Then

$$\begin{aligned} \Omega(-\mathbf{a}) = & \left(\bigcup_{\text{facets } \sigma} \bigcup_{m \in F_{\sigma}(\mathbb{N}A) \setminus F_{\sigma}(\mathbf{a} + \mathbb{N}A)} F_{\sigma}^{-1}(m) \cap \mathbb{N}A \right) \\ & \cup \bigcup_{\mathbf{b}_i + \mathbf{a} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)} (\mathbf{b}_i + \mathbf{a} + \mathbb{Z}(A \cap \tau_i)) \cap \mathbb{N}A. \end{aligned}$$

Since $\tilde{\Omega}(-\mathbf{a}) = \{\mathbf{u} \in \mathbb{N}^n : A\mathbf{u} \in \Omega(-\mathbf{a})\}$ by definition, the assertion follows. $\square$

LEMMA 8.2. Let $\beta = \beta^{\text{empty}}$ and $\mathbf{a} \in \mathbb{Z}^d$.

- If $\beta + \mathbf{a} \sim \beta$, then $N_{-\beta-\mathbf{a}} = \{0\}$.
- Suppose that there exists a facet σ such that $F_{\sigma}(\beta + \mathbf{a}) \in F_{\sigma}(\mathbb{N}A)$ and $F_{\sigma'}(\beta + \mathbf{a}) \notin F_{\sigma'}(\mathbb{N}A)$ for every facet $\sigma' \neq \sigma$. Then $N_{-\beta-\mathbf{a}} \neq \{0\}$.

Proof. (i) Suppose that $\beta + \mathbf{a} \sim \beta$. Then $\mathbb{I}(\Omega(-\mathbf{a})) \not\subseteq \mathfrak{m}_{\beta+\mathbf{a}}$ or $\mathbb{I}(\Omega(-\mathbf{a})) + \mathfrak{m}_{\beta+\mathbf{a}} = K[s]$. Hence $\mathbb{I}(\tilde{\Omega}(-\mathbf{a})) + \langle A\theta - \beta - \mathbf{a} \rangle K[\theta] = K[\theta]$. Therefore $\mathbb{I}(\tilde{\Omega}(-\mathbf{a})) \cap \langle A\theta - \beta - \mathbf{a} \rangle K[\theta] = \langle A\theta - \beta - \mathbf{a} \rangle \mathbb{I}(\tilde{\Omega}(-\mathbf{a}))$, or $N_{-\beta-\mathbf{a}} = \{0\}$ by (40).

(ii) Since $F_{\sigma}(\beta + \mathbf{a}) \in \mathbb{N}A$, there exist $\mathbf{u} \in \mathbb{N}^n$ and $\gamma \in K^{\sigma}$ such that $\beta + \mathbf{a} = A(\mathbf{u} + \gamma)$. Then, for any $\mathbf{v} \in \mathbb{N}^{\sigma}$, $A(\mathbf{u} + \mathbf{v}) \in \mathbb{N}A \setminus (\mathbf{a} + \mathbb{N}A) = \Omega(-\mathbf{a})$ since $F_{\sigma}(A(\mathbf{u} + \mathbf{v})) = F_{\sigma}(\beta + \mathbf{a} - A\gamma + A\mathbf{v}) = F_{\sigma}(\beta + \mathbf{a}) \notin F_{\sigma}(\mathbf{a} + \mathbb{N}A)$. Hence $\mathbf{u} + \mathbb{N}^{\sigma} \subseteq \tilde{\Omega}(-\mathbf{a})$. Put $\xi := \mathbf{u} + \gamma$. Then $A\xi = \beta + \mathbf{a}$ and $\xi + K^{\sigma} = \mathbf{u} + K^{\sigma} \subseteq \text{ZC}(\tilde{\Omega}(-\mathbf{a}))$. By Lemma 8.1 we have

$$\text{ZC}(\tilde{\Omega}(-\mathbf{a})) = \bigcup_{j \in J} (\mathbf{u}^{(j)} + K^{\tau^{(j)}}),$$

and we see that, by the assumption, $\xi + K^{\sigma}$ is the unique irreducible component of $\text{ZC}(\tilde{\Omega}(-\mathbf{a}))$ containing ξ . Hence, by localizing at ξ , to prove the assertion it is enough to show that $\mathbb{I}(\xi + K^{\sigma}) \cap \langle A\theta - (\beta + \mathbf{a}) \rangle \neq \mathbb{I}(\xi + K^{\sigma}) \cdot \langle A\theta - (\beta + \mathbf{a}) \rangle$ (see (40)) or, upon translating by ξ , that $\mathbb{I}(K^{\sigma}) \cap \langle A\theta \rangle \neq \mathbb{I}(K^{\sigma}) \cdot \langle A\theta \rangle$. Since it is clearly true that

$$F_{\sigma}(A\theta) = \sum_{j=1}^n F_{\sigma}(\mathbf{a}_j)\theta_j \in \mathbb{I}(K^{\sigma}) \cap \langle A\theta \rangle \setminus \mathbb{I}(K^{\sigma}) \cdot \langle A\theta \rangle,$$

we have finished the proof. $\square$

THEOREM 8.3. $M_{K^n}(\beta)$ is irreducible if and only if β is non-resonant, i.e. $F_\sigma(\beta) \notin \mathbb{Z}$ for all facets σ of $\mathbb{R}_{\geq 0}A$.

Proof. Suppose that β is non-resonant. Then $\beta + \mathbf{a} \sim \beta$ for all $\mathbf{a} \in \mathbb{Z}^d$. Hence, by Lemma 8.2(i), $M_{K^n}(\beta) \simeq L_{K^n}(T_A, \beta)$.

Suppose that β is resonant and that $F_\sigma(\beta) \in \mathbb{Z}$. If $\beta = \beta^{\text{empty}}$, then, by Corollary 7.6, there exists a surjective homomorphism

$$M_{K^n}(\beta) \rightarrow L_{K^n}(T_A, \beta). \quad (41)$$

Since σ is a facet of $\mathbb{R}_{\geq 0}A$, there exists $\mathbf{b} \in \mathbb{Z}^d$ such that $F_\sigma(\mathbf{b}) < 0$ while $F_{\sigma'}(\mathbf{b}) > 0$ for every facet $\sigma' \neq \sigma$. Hence, for a sufficiently large $n \in \mathbb{N}$, $F_\sigma(\beta - n\mathbf{b}) \in F_\sigma(\mathbb{N}A)$ and $F_{\sigma'}(\beta - n\mathbf{b}) \notin F_{\sigma'}(\mathbb{N}A)$ for every facet $\sigma' \neq \sigma$. Thus the homomorphism (41) has a non-trivial kernel by Lemma 8.2(ii).

Let $\beta \neq \beta^{\text{empty}}$. Then there exists a minimal $(\tau, \lambda) \in E(\beta)$ (see (37)) with $\tau \neq \mathbb{R}_{\geq 0}A$. Hence, by Corollary 7.6, $L_{K^n}(T_\tau, \lambda)$ is a quotient of $M_{K^n}(\beta)$. Since the support of $L_{K^n}(T_\tau, \lambda)$ is strictly contained in the support of $M_{K^n}(\beta)$, the kernel of the homomorphism $M_{K^n}(\beta) \rightarrow L_{K^n}(T_\tau, \lambda)$ is non-trivial. $\square$

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