Title	Equivariant multiplicities of Coxeter arrangements and invariant bases
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Citation	Advances in Mathematics, 230(4-6), 2364-2377 https://doi.org/10.1016/j.aim.2012.04.015
Issue Date	2012-07
Doc URL	https://hdl.handle.net/2115/49577
Type	journal article
File Information	AiM230-4-6_2364-2377.pdf

Equivariant multiplicities of Coxeter arrangements and invariant bases

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November 22, 2011

Abstract

Let $\mathcal{A}$ be an irreducible Coxeter arrangement and W be its Coxeter group. Then W naturally acts on $\mathcal{A}$. A multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is said to be equivariant when $\mathbf{m}$ is constant on each W-orbit of $\mathcal{A}$. In this article, we prove that the multi-derivation module $D(\mathcal{A}, \mathbf{m})$ is a free module whenever $\mathbf{m}$ is equivariant by explicitly constructing a basis, which generalizes the main theorem of [T2002]. The main tool is a primitive derivation and its covariant derivative. Moreover, we show that the W-invariant part $D(\mathcal{A}, \mathbf{m})^W$ for any multiplicity $\mathbf{m}$ is a free module over the W-invariant subring.

Keywords: arrangement of hyperplanes; Coxeter arrangements; equivariant multiplicities; invariant bases.

1 Introduction

Let V be an ℓ-dimensional Euclidean space with an inner product $I : V \times V \to \mathbb{R}$. Let S denote the symmetric algebra of the dual space V^* and F be its quotient field. Let Der_S be the S-module of $\mathbb{R}$-linear derivations from S to itself. Let Ω^1_S be the S-module of regular 1-forms. Similarly define Der_F and

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Ω^1_F over F. The dual inner product $I^* : V^* \times V^* \to \mathbb{R}$ naturally induces an F-bilinear form $I^* : \Omega^1_F \times \Omega^1_F \to F$. Then one has an F-linear bijection

$$I^* : \Omega^1_F \to \mathrm{Der}_F$$

defined by $[I^*(\omega)]\,(f) := I^*(\omega, df)$ for $f \in F$.

Let $\mathcal{A}$ be an irreducible Coxeter arrangement with its Coxeter group W. For each $H \in \mathcal{A}$, choose $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. Let $Q = \prod_{H\in\mathcal{A}} \alpha_H \in S$. Recall the S-module of logarithmic forms

$$\begin{aligned}\Omega^1(\mathcal{A}, \infty) = \{\omega \in \Omega^1_F \quad | \quad & Q^N\omega \text{ and } (Q/\alpha_H)^N\omega \wedge d\alpha_H \text{ are both regular} \\ & \text{for any } H \in \mathcal{A} \text{ and } N \gg 0\}\end{aligned}$$

and the S-module of logarithmic derivations

$$D(\mathcal{A}, -\infty) \quad = \quad I^*(\Omega^1(\mathcal{A}, \infty))$$

from [AT2010]. A map $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is called a multiplicity. For an arbitrary multiplicity, let

$$\begin{aligned} D(\mathcal{A}, \mathbf{m}) &= \{\theta \in D(\mathcal{A}, -\infty) \mid \theta(\alpha_H) \in \alpha_H^{\mathbf{m}(H)}\ S_{(\alpha_H)} \text{ for all } H \in \mathcal{A}\}, \\ \Omega^1(\mathcal{A}, \mathbf{m}) &= (I^*)^{-1} D(\mathcal{A}, -\mathbf{m}), \end{aligned}$$

where $S_{(\alpha_H)}$ is the localization of S at the prime ideal (α_H). These two modules were introduced in [Sa1980] (when $\mathbf{m}$ is constantly equal to one), in [Z1989] (when $\mathrm{im}(\mathbf{m}) \subset \mathbb{Z}_{>0}$), and in [A2008, AT2010, AT2011A] (when $\mathbf{m}$ is arbitrary). A derivation $0 \neq \theta \in \mathrm{Der}_F$ is said to be **homogeneous of degree** d, or $\deg\theta = d$, if $\theta(\alpha) \in F$ is homogeneous of degree d whenever $\theta(\alpha) \neq 0$ ($\alpha \in V^*$). A multiarrangement $(\mathcal{A}, \mathbf{m})$ is called to be **free** with **exponents** $\exp(\mathcal{A}, \mathbf{m}) = (d_1, \ldots, d_\ell)$ if $D(\mathcal{A}, \mathbf{m}) = \oplus_{i=1}^{\ell} S \cdot \theta_i$ with a homogeneous basis θ_i such that $\deg(\theta_i) = d_i$ ($i = 1, \ldots, \ell$). A multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is said to be **equivariant** when $\mathbf{m}(H) = \mathbf{m}(wH)$ for any $H \in \mathcal{A}$ and any $w \in W$, i.e., $\mathbf{m}$ is constant on each orbit. The main result of this article is

Theorem 1.1 *For any irreducible Coxeter arrangement $\mathcal{A}$ and any equivariant multiplicity $\mathbf{m}$, the multiarrangement $(\mathcal{A}, \mathbf{m})$ is free.*

For a fixed arrangement $\mathcal{A}$, we say that a multiplicity $\mathbf{m}$ is **free** if $(\mathcal{A}, \mathbf{m})$ is free. Although we have a limited knowledge about the set of all free multiplicities for a fixed irreducible Coxeter arrangement $\mathcal{A}$, it is known that there exist infinitely many non-free multiplicities unless $\mathcal{A}$ is either one- or two-dimensional [ATY2009]. Theorem 1.1 claims that any equivariant multiplicity is free for any irreducible Coxeter arrangement.

When the W-action on $\mathcal{A}$ is transitive, an equivariant multiplicity is constant and a basis was constructed in [SoT1998, T2002, AY2007, AT2010]. So we may assume, in order to prove Theorem 1.1, that the W-action on $\mathcal{A}$ is not transitive. In other words, we may only study the cases when $\mathcal{A}$ is of the type either B_ℓ, F_4, G_2 or $I_2(2n)$ $(n \geq 4)$. In these cases, $\mathcal{A}$ has exactly two W-orbits: $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. The orbit decompositions are explicitly given by $B_\ell = A_1^\ell \cup D_\ell$, $F_4 = D_4 \cup D_4$, $G_2 = A_2 \cup A_2$ or $I_2(2n) = I_2(n) \cup I_2(n)$ $(n \geq 4)$ as we will see in Section 4. Note that A_1^ℓ is not irreducible. Denote the reflection group of $\mathcal{A}_i$ by W_i $(i = 1, 2)$.

We will prove Theorem 1.1 by giving an explicit construction (Theorem 2.2) of a basis for $D(\mathcal{A}, \mathbf{m})$ when $\mathbf{m}$ is equivariant. Our recipe for the construction needs the following three ingredients:

(1) **primitive derivations** D_i and D with respect to W_i and W respectively,

(2) the **Levi-Civita connection** ∇ with respect to the inner product I, and

(3) **basic invariants** of W_i as well as of W.

In order to define the D_1 and D_2 in (1), we need the theory of primitive derivations in the case of non-irreducible Coxeter arrangements introduced in [AT2011A].

The organization of this article is as follows. In Section 2, we introduce the above-mentioned ingredients (1)-(3). Then we explicitly describe our basis construction in Theorem 2.2 and prove it under a certain restriction. In Section 3 we prove the **primitive decomposition** of the W-invariant part $D(\mathcal{A}, \mathbf{m})^W$. It is a key to complete the proof of Theorem 2.2 at the end of Section 3. In Section 4 we verify that the primitive derivation D_1 can be chosen to be W-invariant when $\mathcal{A}$ is a Coxeter arrangement of either the type B_ℓ or F_4. We also review the cases of G_2 and $I_2(2n)$ $(n \geq 4)$ and find that the primitive derivation D_1 is W_2-antiinvariant. In Section 5, combining Theorem 2.2 with earlier results in [T2002, AT2010, W2011], we finally prove Theorem 1.1. Moreover, we show that the W-invariant part $D(\mathcal{A}, \mathbf{m})^W$ for any multiplicity $\mathbf{m}$ is a free module over the W-invariant subring $R := S^W$ in Theorem 5.3.

2 An explicit construction of basis

Let

$$\begin{aligned} \nabla : \mathrm{Der}_F \times \mathrm{Der}_F &\longrightarrow \mathrm{Der}_F \\ (\theta, \delta) \quad &\mapsto \quad \nabla_\theta\, \delta \end{aligned}$$

be the **Levi-Civita connection** with respect to the inner product I on V. It is characterized by the following four properties (e.g., [Sa1993]):

i) $\nabla_{f\theta}\delta = f\nabla_\theta\delta$ ($f \in F$ and $\theta, \delta \in \mathrm{Der}_F$),

ii) $\nabla_\theta(f\delta) = \theta(f)\delta + f\nabla_\theta\delta$ ($f \in F$ and $\theta, \delta \in \mathrm{Der}_F$),

iii) $\theta(I(\delta, \delta')) = I(\nabla_\theta\delta, \delta') + I(\delta, \nabla_\theta\delta')$ ($\theta, \delta, \delta' \in \mathrm{Der}_F$),

iv) $\nabla_\delta\delta' - \nabla_{\delta'}\delta = [\delta, \delta']$ ($\delta, \delta' \in \mathrm{Der}_F$) (torsion-freeness).

Then $\nabla_\delta\nabla_{\delta'} - \nabla_{\delta'}\nabla_\delta = \nabla_{[\delta,\delta']}$ ($\delta, \delta' \in \mathrm{Der}_F$) (integrability) (e.g., [Sa1993]). Also note that the equality $(\nabla_\theta\delta)(\alpha) = \theta(\delta(\alpha))$ holds true for any $\alpha \in V^*$.

When $\mathcal{A}$ is irreducible, the **primitive derivations** play the central role to define the Hodge filtration introduced by K. Saito. (See [Sa2003] for example.) For the invariant subring $R = S^W$, let D be an element of the lowest degree in Der_R, which is called a primitive derivation corresponding to $\mathcal{A}$. Then D is unique up to a nonzero constant multiple. A theory of primitive derivations in the case of non-irreducible Coxeter arrangements was introduced in [AT2011A]. Thus we may consider a primitive derivation D_i corresponding with the W-orbit $\mathcal{A}_i$ ($1 \le i \le 2$). We only use D_1 because of symmetricity. Note that D_1 is not unique up to a nonzero multiple when $\mathcal{A}_1 = A_1^\ell$ (non-irreducible).

We need the following theorem for our proof of Theorem 1.1:

Theorem 2.1 *([AT2010, AT2011A]) Let $D(\mathcal{A}, -\infty)^W$ be the W-invariant part of $D(\mathcal{A}, -\infty)$. Then*

$$\nabla_D : D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W$$

is a T-linear automorphism where $T := \{f \in R \mid Df = 0\}$. When $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition,

$$\nabla_{D_1} : D(\mathcal{A}_1, -\infty)^{W_1} \xrightarrow{\sim} D(\mathcal{A}_1, -\infty)^{W_1}$$

is a T_1-linear automorphism where

$$R_1 := S^{W_1},\ T_1 := \{f \in R_1 \mid D_1 f = 0\}.$$

Suppose that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition and that the primitive derivation D_1 is W-invariant. Define

$$E^{(p,q)} := \nabla_D^{-q}\nabla_{D_1}^{q-p}E$$

for $p, q \in \mathbb{Z}$ with the Euler derivation E. Here, thanks to Theorem 2.1, we may interpret $\nabla_D^m = (\nabla_D^{-1})^{-m}$ and $\nabla_{D_1}^m = (\nabla_{D_1}^{-1})^{-m}$ when m is negative.

Let $x_1, \dots, x_\ell$ be a basis for V^* and $P_1, \dots, P_\ell$ be a set of **basic invariants** with respect to W: $R = \mathbb{R}[P_1, \dots, P_\ell]$. Let $P_1^{(i)}, \dots, P_\ell^{(i)}$ be a set of basic invariants with respect to W_i: $R_i = \mathbb{R}[P_1^{(i)}, \dots, P_\ell^{(i)}]$ $(i = 1, 2)$. Define

$$d_j := \deg P_j, \ d_j^{(i)} := \deg P_j^{(i)} \ (i = 1, 2, 1 \le j \le \ell).$$

We assume

$$d_1 \le d_2 \le \cdots \le d_\ell, \ d_1^{(i)} \le d_2^{(i)} \le \cdots \le d_\ell^{(i)} \ (i = 1, 2).$$

Then $h := d_\ell$ is the **Coxeter number** of W. We call $h_i := \deg P_\ell^{(i)}$ the Coxeter number of W_i $(i = 1, 2)$. Using the notation

$$\partial_{x_j} := \partial / \partial x_j, \ \partial_{P_j} := \partial / \partial P_j, \ \partial_{P_j^{(i)}} := \partial / \partial P_j^{(i)} \ \ (1 \le j \le \ell, 1 \le i \le 2),$$

we have the following explicit construction of a basis:

Theorem 2.2 *Let $\mathcal{A}$ be an irreducible Coxeter arrangement. Suppose that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition and that the primitive derivation D_1 is W-invariant. Denote the equivariant multiplicity $\mathbf{m}$ by (m_1, m_2) when $\mathbf{m}(H) = m_1$ $(H \in \mathcal{A}_1)$ and $\mathbf{m}(H) = m_2$ $(H \in \mathcal{A}_2)$. Then*

(1) the S-module $D(\mathcal{A}, (2p-1, 2q-1))$ is free with W-invariant basis

$$\nabla_{\partial_{P_1}} E^{(p,q)}, \dots, \nabla_{\partial_{P_\ell}} E^{(p,q)}$$

with $\deg \nabla_{\partial_{P_i}} E^{(p,q)} = ph_1 + qh_2 - d_i + 1$ for $i = 1, \dots, \ell$,

(2) the S-module $D(\mathcal{A}, (2p-1, 2q))$ is free with basis

$$\nabla_{\partial_{P_1^{(1)}}} E^{(p,q)}, \dots, \nabla_{\partial_{P_\ell^{(1)}}} E^{(p,q)}$$

with $\deg \nabla_{\partial_{P_i^{(1)}}} E^{(p,q)} = ph_1 + qh_2 - d_i^{(1)} + 1$ for $i = 1, \dots, \ell$,

(3) the S-module $D(\mathcal{A}, (2p, 2q-1))$ is free with basis

$$\nabla_{\partial_{P_1^{(2)}}} E^{(p,q)}, \dots, \nabla_{\partial_{P_\ell^{(2)}}} E^{(p,q)}$$

with $\deg \nabla_{\partial_{P_i^{(2)}}} E^{(p,q)} = ph_1 + qh_2 - d_i^{(2)} + 1$ for $i = 1, \dots, \ell$,

(4) the S-module $D(\mathcal{A}, (2p, 2q))$ is free with basis

$$\nabla_{\partial_{x_1}} E^{(p,q)}, \dots, \nabla_{\partial_{x_\ell}} E^{(p,q)}$$

with $\deg \nabla_{\partial_{x_i}} E^{(p,q)} = ph_1 + qh_2$ for $i = 1, \dots, \ell$.

Remark. Theorems 2.2 and 1.1 are used to prove the freeness of the Shi-Catalan arrangements associated with any Weyl arrangements in [AT2011E].

Definition 2.3 *([Y2002, W2011]) Let $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ and $\zeta \in D(\mathcal{A}, -\infty)^W$. We say that ζ is* **m-universal** *when ζ is homogeneous and the S-linear map*

$$\begin{aligned} \Psi_\zeta : \mathrm{Der}_S &\longrightarrow D(\mathcal{A}, 2\mathbf{m}) \\ \theta &\longmapsto \nabla_\theta\, \zeta \end{aligned}$$

is bijective.

Example 2.4 *The Euler derivation E is $\mathbf{0}$-universal because $\Psi_E(\delta) = \nabla_\delta\, E = \delta$ and $D(\mathcal{A}, \mathbf{0}) = \mathrm{Der}_S$.*

Recall the T-automorphisms

$$\nabla_D^k \colon D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W \ (k \in \mathbb{Z})$$

from Theorem 2.1. Recall the following two results concerning the $\mathbf{m}$-universality:

Theorem 2.5 *([W2011, Theorem 2.8]) If ζ is $\mathbf{m}$-universal, then $\nabla_D^{-1}\zeta$ is $(\mathbf{m} + \mathbf{1})$-universal.*

Proposition 2.6 *([W2011, Proposition 2.7]) Suppose that ζ is $\mathbf{m}$-universal. Let $\mathbf{k} \colon \mathcal{A} \to \{+1, 0, -1\}$. Then an S-homomorphism*

$$\Phi_\zeta : D(\mathcal{A}, \mathbf{k}) \to D(\mathcal{A}, \mathbf{k} + 2\mathbf{m})$$

defined by

$$\Phi_\zeta(\theta) := \nabla_\theta\, \zeta$$

gives an S-module isomorphism.

Proposition 2.7 *Suppose $q \geq 0$. The derivation $E^{(p,q)} = \nabla_D^{-q} \nabla_{D_1}^{q-p} E$ is (p, q)-universal.*

Proof. When $\mathcal{A}_1$ is irreducible, [AY2007] and [AT2010] imply that $\nabla_{D_1}^{q-p} E$ is $(p - q, 0)$-universal. When $\mathcal{A}_1$ is not irreducible, $\nabla_{D_1}^{q-p} E$ is $(p - q, 0)$-universal because of [AT2011A]. Thus $E^{(p,q)} = \nabla_D^{-q} \nabla_{D_1}^{q-p} E$ is (p, q)-universal by Theorem 2.5. $\square$

Since $E^{(p,q)}$ is (p, q)-universal, Proposition 2.6 yields the following:

Proposition 2.8 *Let $q \geq 0$ and $\mathbf{m}\colon \mathcal{A} \to \{+1, 0, -1\}$. Then an S-homomorphism*

$$\Phi_{p,q} : D(\mathcal{A}, \mathbf{m}) \to D(\mathcal{A}, (2p, 2q) + \mathbf{m})$$

defined by

$$\Phi_{p,q}(\theta) := \nabla_\theta E^{(p,q)}$$

gives an S-module isomorphism.

Proof of Theorem 2.2 ($q \geq 0$). We may apply Proposition 2.8 because

(1) $\partial_{P_1}, \dots, \partial_{P_\ell}$ form a basis for $D(\mathcal{A}, (-1, -1))$,
(2) $\partial_{P_1^{(1)}}, \dots, \partial_{P_\ell^{(1)}}$ form a basis for $D(\mathcal{A}, (-1, 0))$,
(3) $\partial_{P_1^{(2)}}, \dots, \partial_{P_\ell^{(2)}}$ form a basis for $D(\mathcal{A}, (0, -1))$, and
(4) $\partial_{x_1}, \dots, \partial_{x_\ell}$ form a basis for $D(\mathcal{A}, (0, 0))$. $\square$

3 Primitive decompositions

Before we complete the proof of Theorem 2.2 at the end of this section, we need to prove the following theorem which asserts the existence of the **primitive decomposition** of $D(\mathcal{A}, (2p-1, 2q-1))^W$:

Theorem 3.1 *Under the same assumption of Theorem 2.2 define*

$$\theta_i^{(p,q)} : = \nabla_{\partial_{P_i}} E^{(p,q)} = \nabla_{\partial_{P_i}} \nabla_D^{-q} \nabla_{D_1}^{q-p} E \quad (1 \leq i \leq \ell)$$

for $p, q \in \mathbb{Z}$. Then the set

$$\{\theta_i^{(p+k,q+k)} \mid k \geq 0, 1 \leq i \leq \ell\}$$

is a T-basis for $D(\mathcal{A}, (2p-1, 2q-1))^W$. Put

$$\mathcal{G}^{(p,q)} := \bigoplus_{i=1}^{\ell} T \cdot \theta_i^{(p,q)}.$$

Then we have a T-module decomposition (called the primitive decomposition)

$$D(\mathcal{A}, (2p-1, 2q-1))^W = \bigoplus_{k \geq 0} \mathcal{G}^{(p+k,q+k)}.$$

For $\mathbf{P} := [P_1, \dots, P_\ell]$, the Jacobian matrix $J(\mathbf{P})$ is defined as the matrix whose (i,j)-entry is $\dfrac{\partial P_j}{\partial x_i}$. Define $A := [I^*(dx_i, dx_j)]_{1 \leq i,j \leq \ell}$ and $G := [I^*(dP_i, dP_j)]_{1 \leq i,j \leq \ell} = J(\mathbf{P})^T A J(\mathbf{P})$.

Proposition 3.2 *Let ζ be* $\mathbf{m}$*-universal. Then*

(1) the set $\{\nabla_{\partial_{P_j}} \nabla_D^{-k} \zeta \mid 1 \le j \le \ell, k \ge 0\}$ *is linearly independent over* T.

(2) Define $\mathcal{G}^{(k)}$ *to be the free* T*-module with basis* $\{\nabla_{\partial_{P_j}} \nabla_D^{-k} \zeta \mid 1 \le j \le \ell\}$ *for* $k \ge 0$*. Then the Poincaré series* $\mathrm{Poin}(\bigoplus_{k\ge 0} \mathcal{G}^{(k)}, t)$ *satisfies:*

$$\mathrm{Poin}(\bigoplus_{k\ge 0} \mathcal{G}^{(k)}, t) \quad = \quad \left(\prod_{i=1}^{\ell} \frac{1}{1-t^{d_i}} \right) (\sum_{j=1}^{\ell} t^{p-d_j}),$$

where $p = \deg \zeta$ *and* $d_j = \deg P_j$ $(1 \le j \le \ell)$*.*

(3)

$$D(\mathcal{A}, 2\mathbf{m}-1)^W = \bigoplus_{k\ge 0} \mathcal{G}^{(k)}.$$

Proof. Let $k \in \mathbb{Z}_{\ge 0}$. By Theorem 2.5, $\zeta^{(k)} := \nabla_D^{-k}\zeta$ is $(\mathbf{m}+k)$-universal, where the "k" in the $(\mathbf{m}+k)$ stands for the constant multiplicity k by abuse of notation. Thus by Proposition 2.6 we have the following two bases:

$$\nabla_{\partial_{P_1}} \zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}} \zeta^{(k)},$$

for the S-module $D(\mathcal{A}, 2\mathbf{m}+2k-1)$ and

$$\nabla_{\partial_{I^*(dP_1)}} \zeta^{(k)}, \dots, \nabla_{\partial_{I^*(dP_\ell)}} \zeta^{(k)},$$

for the S-module $D(\mathcal{A}, 2\mathbf{m}+2k+1)$. Note that the two bases are also R-bases for $D(\mathcal{A}, 2\mathbf{m}+2k-1)^W$ and $D(\mathcal{A}, 2\mathbf{m}+2k+1)^W$ respectively. Since the T-automorphism

$$\nabla_D \colon D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W$$

in Theorem 2.1 induces a T-linear bijection

$$\nabla_D \colon D(\mathcal{A}, 2\mathbf{m}+2k+1)^W \xrightarrow{\sim} D(\mathcal{A}, 2\mathbf{m}+2k-1)^W$$

as in [AT2011A, Theorem 4.4], we may find an $\ell \times \ell$-matrix $B^{(k)}$ with entries in R such that

$$\begin{aligned} \nabla_D \left(\left[\nabla_{\partial_{P_1}} \zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}} \zeta^{(k)} \right] G \right) &= \nabla_D \left[\nabla_{\partial_{I^*(dP_1)}} \zeta^{(k)}, \dots, \nabla_{\partial_{I^*(dP_\ell)}} \zeta^{(k)} \right] \\ &= \left[\nabla_{\partial_{P_1}} \zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}} \zeta^{(k)} \right] B^{(k)}. \end{aligned}$$

The degree of (i,j)-th entry of $B^{(k)}$ is $m_i + m_j - h \le h-2 < h$. In particular, the degree of $B^{(k)}_{i,\ell+1-i}$ is 0 and $B^{(k)}_{i,j} = 0$ if $i+j < \ell+1$. Hence each entry of $B^{(k)}$ lies in T and $\det B^{(k)} \in \mathbb{R}$. Since D is a derivation of the minimum degree in Der_R, one gets $[D, \partial_{P_i}] = 0$. Thus $\nabla_D \nabla_{\partial_{P_i}} = \nabla_{\partial_{P_i}} \nabla_D$. Operate ∇_D^{-1} on the both sides of the equality above, and get

$$\left[\nabla_{\partial_{P_1}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k)}\right] G = \left[\nabla_{\partial_{P_1}} \zeta^{(k+1)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k+1)}\right] B^{(k)}.$$

This implies that $\det B^{(k)} \in \mathbb{R}^\times$ because $\nabla_{\partial_{P_1}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k)}$ are linearly independent over S. Inductively we have

$$\begin{aligned}
&\left[\nabla_{\partial_{P_1}} \zeta^{(k+1)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k+1)}\right] = \left[\nabla_{\partial_{P_1}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k)}\right] G(B^{(k)})^{-1} \\
&= \left[\nabla_{\partial_{P_1}} \zeta, \ldots, \nabla_{\partial_{P_\ell}} \zeta\right] G(B^{(0)})^{-1} G(B^{(1)})^{-1} \cdots G(B^{(k)})^{-1} \\
&= \left[\nabla_{\partial_{P_1}} \zeta, \ldots, \nabla_{\partial_{P_\ell}} \zeta\right] G_{k+1},
\end{aligned}$$

where $G_i = G(B^{(0)})^{-1} G(B^{(1)})^{-1} \cdots G(B^{(i-1)})^{-1}$ $(i \geq 0)$. Note that G appears i times in the definition of G_i. For $M = (m_{ij}) \in M_\ell(F)$, define $D[M] = (D(m_{ij})) \in M_\ell(F)$. Then $D^j[G_i] = O$ when $j > i$ and $\det D^i[G_i] \neq 0$ because $\det D[G] \neq 0$ and $D^2[G] = O$ (e.g., see [Sa1993, AT2011A]).

(1) Suppose that $\{\nabla_{\partial_{P_j}} \zeta^{(k)} \mid 1 \leq j \leq \ell, k \geq 0\}$ is linearly dependent over T. Then there exist ℓ-dimensional column vectors $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_q \in T^\ell (q \geq 0)$ with $\mathbf{g}_q \neq \mathbf{0}$ such that

$$\mathbf{0} = \sum_{i=0}^{q} \left[\nabla_{\partial_{P_1}} \zeta^{(i)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(i)}\right] \mathbf{g}_i = \left[\nabla_{\partial_{P_1}} \zeta, \ldots, \nabla_{\partial_{P_\ell}} \zeta\right] \left(\sum_{i=0}^{q} G_i \mathbf{g}_i\right).$$

Since $\nabla_{\partial_{P_1}} \zeta, \ldots, \nabla_{\partial_{P_\ell}} \zeta$ are linearly independent over R, one has

$$\mathbf{0} = \sum_{i=0}^{q} G_i \mathbf{g}_i.$$

Applying the operator D on the both sides q times, we get $D^q[G_q]\mathbf{g}_q = \mathbf{0}$. Thus $\mathbf{g}_q = \mathbf{0}$ which is a contradiction. This proves (1).

(2) Compute

$$\begin{aligned}
\mathrm{Poin}(\bigoplus_{k \geq 0} \mathcal{G}^{(k)}, t) &= \sum_{k \geq 0} \left(\prod_{i=1}^{\ell-1} \frac{1}{1 - t^{d_i}}\right) (\sum_{j=1}^{\ell} t^{p - d_j + k d_\ell}) \\
&= \left(\prod_{i=1}^{\ell-1} \frac{1}{1 - t^{d_i}}\right) (\sum_{k \geq 0} t^{k d_\ell})(\sum_{j=1}^{\ell} t^{p - d_j}) \\
&= \left(\prod_{i=1}^{\ell} \frac{1}{1 - t^{d_i}}\right) (\sum_{j=1}^{\ell} t^{p - d_j}).
\end{aligned}$$

(3) We have

$$D(\mathcal{A}, 2\mathbf{m}-1)^W \supseteq \bigoplus_{k\geq 0} \mathcal{G}^{(k)}$$

by (1). So it suffices to prove

$$\mathrm{Poin}(D(\mathcal{A}, 2\mathbf{m}-1)^W, t) = \mathrm{Poin}(\bigoplus_{k\geq 0} \mathcal{G}^{(k)}, t).$$

Since $D(\mathcal{A}, 2\mathbf{m}-1)^W$ is a free R-module with a basis

$$\nabla_{\partial_{P_1}}\zeta, \dots, \nabla_{\partial_{P_\ell}}\zeta,$$

we obtain

$$\mathrm{Poin}(D(\mathcal{A}, 2\mathbf{m}-1)^W, t) \quad = \quad \left(\prod_{i=1}^{\ell} \frac{1}{1-t^{d_i}}\right)(\sum_{i=1}^{\ell} t^{p-d_j}) = \mathrm{Poin}(\bigoplus_{k\geq 0} \mathcal{G}^{(k)}, t),$$

which completes the proof. □

We require that the assumption of Theorem 2.2 is satisfied in the rest of this section.

Proof of Theorem 3.1. Suppose $q \geq 0$ to begin with. Then, by Proposition 3.5, $E^{(p,q)}$ is (p,q)-universal. Apply Proposition 3.2 for $\zeta = E^{(p,q)}$ and $\mathbf{m} = (p,q)$, and we have Theorem 3.1:

$$D(\mathcal{A}, (2p-1, 2q-1))^W = \bigoplus_{k\geq 0} \mathcal{G}^{(p+k,q+k)}$$

when $q \geq 0$. Apply ∇_D to the both sides, and we get

$$D(\mathcal{A}, (2p-3, 2q-3))^W = \bigoplus_{k\geq 0} \mathcal{G}^{(p+k-1,q+k-1)}$$

because $\nabla_D \left(D(\mathcal{A}, (2p-1, 2q-1))^W\right) = D(\mathcal{A}, (2p-3, 2q-3))^W$ as in [AT2011A, Theorem 4.4] and $\nabla_D(\theta_i^{(p,q)}) = \theta_i^{(p-1,q-1)}$. Apply ∇_D repeatedly to complete the proof for all $q \in \mathbb{Z}$. □

Note that we do not assume $p \geq 0$ in the following proposition:

Proposition 3.3 *For $p, q \in \mathbb{Z}$, the S-module $D(\mathcal{A}, (2p-1, 2q-1))$ has a W-invariant basis.*

Proof. Recall that

$$\nabla_{\partial_{P_1}} E^{(p,q)}, \nabla_{\partial_{P_2}} E^{(p,q)}, \ldots \nabla_{\partial_{P_\ell}} E^{(p,q)},$$

which are W-invariant, form an S-basis for $D(\mathcal{A},(2p-1,2q-1))$ when $q \geq 0$ by Theorem 2.2 (1). It is then easy to see that they are also an R-basis for $D(\mathcal{A},(2p-1,2q-1))^W$ for $q \geq 0$. By [A2008] [AT2010], there exists a W-equivariant nondegenerate S-bilinear pairing

$$(\ ,\ ) : D(\mathcal{A},(2p-1,2q-1)) \times D(\mathcal{A},(-2p+1,-2q+1)) \longrightarrow S,$$

characterized by

$$(I^*(\omega),\theta) = \langle \omega, \theta \rangle$$

where $\omega \in \Omega^1(\mathcal{A},(-2p+1,-2q+1))$ and $\theta \in D(\mathcal{A},(-2p+1,-2q+1))$. Let $\theta_1,\ldots,\theta_\ell$ denote the dual basis for $D(\mathcal{A},(-2p+1,-2q+1))$ satisfying

$$\left(\nabla_{\partial_{P_i}} E^{(p,q)}, \theta_j\right) = \delta_{ij}$$

for $1 \leq i,j \leq \ell$. Then $\theta_1,\ldots,\theta_\ell$ are W-invariant because the pairing $(\ ,\ )$ is W-equivariant. $\square$

Although the following lemma is standard and easy, we give a proof for completeness.

Lemma 3.4 *Let M be an S-submodule of Der_F. The following two conditions are equivalent:*

(1) M has a W-invariant basis Θ over S.

(2) The W-invariant part M^W is a free R-module with a basis Θ and there exists a natural S-linear isomorphism

$$M^W \otimes_R S \simeq M.$$

Proof. It suffices to prove that (1) implies (2) because the other implication is obvious. Suppose that $\Theta = \{\theta_\lambda\}_{\lambda\in\Lambda}$ is a W-invariant basis for M over S. Since it is linearly independent over S, so is over R. Let $\theta \in M^W$. Express

$$\theta = \sum_{i=1}^{n} f_i \theta_i$$

with $f_i \in S$ and $\theta_i \in \Theta$ $(i = 1,\ldots,n)$. Let $w \in W$ act on the both sides. Then we get

$$\theta = \sum_{i=1}^{n} w(f_i)\theta_i.$$

This implies $f_i = w(f_i)$ for every $w \in W$. Hence $f_i \in R$ for each i. Therefore Θ is a basis for M^W over R. This is (2). $\square$

Proposition 3.5 *For any $p, q \in \mathbb{Z}$, $E^{(p,q)}$ is (p,q)-universal.*

Proof. By Theorem 3.1 we have the decomposition:

$$D(\mathcal{A}, (2p-1, 2q-1))^W = \bigoplus_{k \geq 0} \mathcal{G}^{(p+k, q+k)}$$

for $p, q \in \mathbb{Z}$. As we saw in Proposition 3.2 (2), we have

$$\text{(3.1)} \quad \begin{aligned} \mathrm{Poin}(D(\mathcal{A}, (2p-1, 2q-1))^W, t) &= \mathrm{Poin}(\bigoplus_{k \geq 0} \mathcal{G}^{(p+k, q+k)}, t) \\ &= \left(\prod_{i=1}^{\ell} \frac{1}{1 - t^{d_i}} \right) (\sum_{i=1}^{\ell} t^{m - d_j}), \end{aligned}$$

where $m := \deg E^{(p,q)}$. Recall that the S-module $D(\mathcal{A}, (2p-1, 2q-1))$ has a W-invariant basis $\theta_1, \ldots, \theta_\ell$ by Proposition 3.3. By Lemma 3.4, we know that $\theta_1, \ldots, \theta_\ell$ form a basis for the R-module $D(\mathcal{A}, (2p-1, 2q-1))^W$. Thanks to (3.1) we may assume that $\deg \theta_j = m - d_j = \deg \nabla_{\partial_{P_j}} E^{(p,q)}$. Therefore there exists $M \in M_\ell(R)$ such that

$$[\theta_1, \ldots, \theta_\ell] M = [\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)}]$$

with $\det M \in \mathbb{R}$. Since

$$\max_{1 \leq i,j \leq \ell} \left| \deg \theta_i - \deg \nabla_{\partial_{P_j}} E^{(p,q)} \right| = d_\ell - d_1 < \deg P_\ell,$$

we get $M \in M_\ell(T)$. Since $\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)}$ are linearly independent over T by Proposition 3.2 (1), we have $\det M \in \mathbb{R}^\times$. Thus

$$\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)}$$

form an S-basis for $D(\mathcal{A}, (2p-1, 2q-1))$. Since

$$\left[\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)} \right] J(\mathbf{P})^T = \left[\nabla_{\partial_{x_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{x_\ell}} E^{(p,q)} \right],$$

we may apply the multi-arrangement version of Saito's criterion [Sa1980, Z1989, A2008] to prove that $\nabla_{\partial_{x_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{x_\ell}} E^{(p,q)}$ form an S-basis for $D(\mathcal{A}, (2p, 2q))$ for any $p, q \in \mathbb{Z}$. This shows that $E^{(p,q)}$ is (p,q)-universal for any $p, q \in \mathbb{Z}$. $\square$

Proof of Theorem 2.2 ($q \in \mathbb{Z}$). Theorem 2.5 and Proposition 3.5 complete the proof by the same argument as that in Section 2 for $q \geq 0$. $\square$

4 The cases of B_ℓ, F_4, G_2 and $I_2(2n)$

• **The case of B_ℓ**

The roots of the type B_ℓ are:

$$\pm x_i, \ \pm x_i \pm x_j \quad (1 \le i < j \le \ell)$$

in terms of an orthonormal basis $x_1, \dots, x_\ell$ for V^*. Altogether there are $2\ell^2$ of them. Define

$$Q_1 := \prod_{i=1}^{\ell} x_i, \ \ Q_2 := \prod_{1 \le i < j \le \ell} (x_i \pm x_j), \ \ Q = Q_1 Q_2.$$

Then the arrangement $\mathcal{A}_1$ defined by Q_1 is of the type $A_1 \times \cdots \times A_1 = A_1^\ell$. The arrangement $\mathcal{A}_2$ defined by Q_2 is of the type D_ℓ. The arrangement $\mathcal{A}$ defined by Q is of the type B_ℓ and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition. Note that A_1^ℓ is not irreducible. Define

$$D_1 := \sum_{i=1}^{\ell} \frac{1}{x_i} \partial_{x_i}$$

which is a primitive derivation in the sense of [AT2011A]. Obviously D_1 is W-invariant. Let $P_j = \sum_{i=1}^{\ell} x_i^{2j}$ $(j \ge 1)$. Then $P_1, \dots, P_\ell$ form a set of basic invariants under W while $Q_1, P_1, \dots, P_{\ell-1}$ form a set of basic invariants under W_2. Define a primitive derivation D_2 with respect to $\mathcal{A}_2$ so that

$$D_2(Q_1) = D_2(P_j) = 0 \ (j = 1, \dots, \ell - 2), \ \ D_2(P_{\ell-1}) = 1.$$

Thus

$$(wD_2)(P_{\ell-1}) = D_2(w^{-1} P_{\ell-1}) = D_2(P_{\ell-1}) = 1 \ \ (w \in W).$$

This implies that D_2 is W-invariant.

• **The case of F_4**

The roots of the type F_4 are:

$$\pm x_i, \ (\pm x_1 \pm x_2 \pm x_3 \pm x_4)/2, \ \pm x_i \pm x_j \quad (1 \le i < j \le 4)$$

in terms of an orthonormal basis x_1, x_2, x_3, x_4 for V^*. Altogether there are 48 of them. Define

$$Q_1 := \prod_{1 \le i < j \le 4} (x_i \pm x_j), \ Q_2 := \prod_{i=1}^{4} x_i \prod (x_1 \pm x_2 \pm x_3 \pm x_4), \ Q = Q_1 Q_2.$$

The arrangement $\mathcal{A}_i$ defined by Q_i is of the type D_4 $(i = 1, 2)$. Then the arrangement $\mathcal{A}$ defined by Q is of the type F_4 and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition. Define

$$P_1^{(1)} = \sum_{i=1}^{4} x_i^2,\ P_2^{(1)} = \sum_{i=1}^{4} x_i^4,\ P_3^{(1)} = x_1x_2x_3x_4,\ P_4^{(1)} = \sum_{i=1}^{4} x_i^6 + 5\sum_{i\neq j} x_i^2 x_j^4.$$

Compute

$$P_4^{(1)} = -4\sum_{i=1}^{4} x_i^6 + 5P_1^{(1)}P_2^{(1)}.$$

Thus $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}, P_4^{(1)}$ are a set of basic invariants under W_1. The reflection τ with respect to $x_1 + x_2 + x_3 + x_4 = 0$ is given by

$$\tau(x_i) = \frac{2x_i - \sum_{j=1}^{4} x_j}{2}\ (i = 1, 2, 3, 4).$$

A calculation shows that $P_4^{(1)}$ is τ-invariant. Let s_i denote the reflection with respect to $x_i = 0$ $(1 \le i \le 4)$. Since the Coxeter group W_2 is generated by τ and s_i $(1 \le i \le 4)$, we know that $P_4^{(1)}$ is W_2-invariant thus W-invariant. Define a primitive derivation D_1 with respect to $\mathcal{A}_1$ so that

$$D_1(P_j^{(1)}) = 0\ (j = 1, 2, 3),\ \ D_1(P_4^{(1)}) = 1.$$

Thus

$$(wD_1)(P_4^{(1)}) = D_1(w^{-1}P_4^{(1)}) = D_1(P_4^{(1)}) = 1\ \ (w \in W).$$

This implies that D_1 is W-invariant. We conclude that D_2 is also W-invariant because an orthonormal coordinate change

$$x_1 = \frac{y_1 - y_2}{\sqrt{2}},\ x_2 = \frac{y_1 + y_2}{\sqrt{2}},\ x_3 = \frac{y_3 - y_4}{\sqrt{2}},\ x_4 = \frac{y_3 + y_4}{\sqrt{2}}$$

switches $\mathcal{A}_1$ and $\mathcal{A}_2$.

- **The cases of G_2 and $I_2(2n)$ $(n \ge 4)$**

The arrangement $\mathcal{A}$ of the type G_2 has exactly two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$, each of which is of the type A_2. Let $n \ge 4$. Then the arrangement $\mathcal{A}$ of the type $I_2(2n)$ has exactly two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$, each of which is of the type $I_2(n)$. In both cases, by [W2011], one may choose

$$D_1 = Q_2D,\ \ D_2 = Q_1D.$$

Since Q_2 is W_2-antiinvariant and D is W-invariant, D_1 is W_2-antiinvariant. Similarly D_2 is W_1-antiinvariant.

5 Proof of Theorem 1.1 and invariant bases

Assume that $\mathcal{A}$ is an irreducible Coxeter arrangement in the rest of the article.

Proof of Theorem 1.1. If $\mathcal{A}$ has the single orbit, then the result in [T2002, AY2007, AT2010] completes the proof. If not, then $\mathcal{A}$ has exactly two orbits. If $\mathcal{A}$ is of the type either G_2 or $I_2(2n)$ with $n \geq 4$, then $D(\mathcal{A}, \mathbf{m})$ is a free S-module because $\mathcal{A}$ lies in a two-dimensional vector space. For the remaining cases of the type B_ℓ and F_4, Section 4 allows us to apply Theorem 2.2 to complete the proof. □

A multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is said to be **odd** if its image lies in $1 + 2\mathbb{Z}$.

Proposition 5.1 *If* $\mathbf{m}$ *is equivariant and odd, then* $D(\mathcal{A}, \mathbf{m})$ *has a* W*-invariant basis over* S*.*

Proof. When $\mathcal{A}$ has the single orbit, $\mathbf{m}$ is constant. In this case Proposition was proved in [T2002, AY2007, AT2010]. If $\mathcal{A}$ is of the type either G_2 or $I_2(2n)$ $(n \geq 4)$, then Proposition was verified in [W2011]. For the remaining cases of B_ℓ and F_4, Proposition 3.3 completes the proof. □

Recall the W-action on $\mathcal{A}$:

$$W \times \mathcal{A} \longrightarrow \mathcal{A}$$

by mapping (w, H) to wH $(w \in W,\ H \in \mathcal{A})$. For any multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$, define a new multiplicity $\mathbf{m}^*$ by

$$\mathbf{m}^*(H) := \max_{w \in W} \left(2 \cdot \lfloor \mathbf{m}(wH)/2 \rfloor + 1\right),$$

where $\lfloor a \rfloor$ stands for the greatest integer not exceeding a. Then $\mathbf{m}^*$ is obviously equivariant and odd.

Proposition 5.2 *For any irreducible Coxeter arrangement* $\mathcal{A}$ *and any multiplicity* $\mathbf{m}$*,*

$$D(\mathcal{A}, \mathbf{m})^W = D(\mathcal{A}, \mathbf{m}^*)^W.$$

Proof. Since $\mathbf{m}(H) \leq \mathbf{m}^*(H)$ for any $H \in \mathcal{A}$, we have

$$D(\mathcal{A}, \mathbf{m})^W \supseteq D(\mathcal{A}, \mathbf{m}^*)^W.$$

We will show the other inclusion. Let $H \in \mathcal{A}$ and $\theta \in D(\mathcal{A}, \mathbf{m})^W$. It suffices to verify the following two statements:

(A) $\theta(\alpha_H) \in \alpha_H^{\mathbf{m}(wH)} S_{(\alpha_H)}$ for any $w \in W$,

(B) $\theta(\alpha_H) \in \alpha_H^{2m} S_{(\alpha_H)}$ implies $\theta(\alpha_H) \in \alpha_H^{2m+1} S_{(\alpha_H)}$ for any $m \in \mathbb{Z}$.

For $w \in W$ let w^{-1} act on the both sides of

$$\theta(\alpha_{wH}) \in \alpha_{wH}^{\mathbf{m}(wH)} S_{(\alpha_{wH})}$$

to get

$$\theta(\alpha_H) \in \alpha_H^{\mathbf{m}(wH)} S_{(\alpha_H)}.$$

This verifies (A).

Fix $H \in \mathcal{A}$. Let s be the orthogonal reflection through H. Then $s(\alpha_H) = -\alpha_H$. Suppose that $\theta(\alpha_H) = \alpha_H^{2m} p$ with $p \in S_{(\alpha_H)}$. Let s act on the both sides and we have $\theta(-\alpha_H) = (-\alpha_H)^{2m} s(p)$. This implies $-p = s(p)$. Since $s(p) = p$ on H, one has $p = 0$ on H, which implies $p \in \alpha_H S_{(\alpha_H)}$. This verifies (B). $\square$

Theorem 5.3 *For any irreducible Coxeter arrangement $\mathcal{A}$ and any multiplicity $\mathbf{m}$, the R-module $D(\mathcal{A}, \mathbf{m})^W$ is free.*

Proof. Thanks to Proposition 5.2 we may assume that $\mathbf{m}$ is equivariant and odd. Apply Proposition 5.1 and Lemma 3.4. $\square$

Corollary 5.4

$$D(\mathcal{A}, \mathbf{m})^W \otimes_R S \simeq D(\mathcal{A}, \mathbf{m}^*).$$

Proof. Apply Proposition 5.1 and Lemma 3.4 to get

$$D(\mathcal{A}, \mathbf{m}^*)^W \otimes_R S \simeq D(\mathcal{A}, \mathbf{m}^*).$$

Then Proposition 5.2 completes the proof. $\square$

The following corollary shows that the converse of Proposition 5.1 is true.

Corollary 5.5 *The S-module $D(\mathcal{A}, \mathbf{m})$ has a W-invariant basis if and only if $\mathbf{m}$ is odd and equivariant.*

Proof. Assume that $D(\mathcal{A}, \mathbf{m})$ has a W-invariant basis over S. Then, by Lemma 3.4, we get

$$D(\mathcal{A}, \mathbf{m})^W \otimes_R S \simeq D(\mathcal{A}, \mathbf{m}).$$

Compare this with Corollary 5.4 and we know that there exsits a common S-basis for both $D(\mathcal{A}, \mathbf{m})$ and $D(\mathcal{A}, \mathbf{m}^*)$. By the multi-arrangement version of Saito's criterion [Sa1980, Z1989, A2008], we have $\mathbf{m} = \mathbf{m}^*$. $\square$

Acknowledgment. We would like to thank the referee whose comments were very useful for us to revise this article.

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