HOKKAIDO UNIVERSITY

Title	Beating the standard quantum limit : phase super-sensitivity of N-photon interferometers
Author(s)	Okamoto, Ryo; Hofmann, Holger F.; Nagata, Tomohisa et al.
Citation	New Journal of Physics, 10, 073033 https://doi.org/10.1088/1367-2630/10/7/073033
Issue Date	2008-07
Doc URL	https://hdl.handle.net/2115/49847
Rights	© 2008 IOP Publishing Ltd and Deutsche Physikalische Gesellschaft
Type	journal article
File Information	NJP10_073033.pdf

Beating the standard quantum limit: phase super-sensitivity of *N*-photon interferometers

Beating the standard quantum limit: phase super-sensitivity of *N*-photon interferometers

Ryo Okamoto[1,2,5]**, Holger F Hofmann**[3]**, Tomohisa Nagata**[1]**, Jeremy L O'Brien**[4]**, Keiji Sasaki**[1] **and Shigeki Takeuchi**[1,2]

[1] Research Institute for Electronic Science, Hokkaido University, Sapporo 060-0812, Japan
[2] The Institute of Scientific and Industrial Research, Osaka University, Mihogaoka 8-1, Ibaraki, Osaka 567-0047, Japan
[3] Graduate School of Advanced Sciences of Matter, Hiroshima University, Hiroshima 739-8530, Japan
[4] Centre for Quantum Photonics, H H Wills Physics Laboratory and Department of Electrical and Electronic Engineering, University of Bristol, Merchant Venturers Building, Woodland Road, Bristol, BS8 1UB, UK
E-mail: oka@es.hokudai.ac.jp

Abstract. Quantum metrology promises greater sensitivity for optical phase measurements than could ever be achieved classically. Here, we present a theory of the phase sensitivity for the general case where the detection probability is given by an N photon interference fringe. We find that the phase sensitivity has a complicated dependence on both the intrinsic efficiency of detection η and the interference fringe visibility V. Most importantly, the phase that gives maximum phase sensitivity is in general not the same as the phase at which the slope of the interference fringe is a maximum, as has previously been assumed. We determine the parameter range where quantum enhanced sensitivity can be achieved. In order to illustrate these theoretical results, we perform a four-photon experiment with $\eta = 3/4$ and $V = 82 \pm 6\%$ (an extension of our previous work (Nagata *et al* 2007 *Science* **316** 726)) and find a phase sensitivity 1.3 times greater than the standard quantum limit at a phase different to that which gives maximum slope of the interference fringe.

---

[5] Author to whom any correspondence should be addressed.

Contents

1. Introduction

The subwavelength sensitivity offered by optical phase measurements is the reason that they have found applications across all fields of science, from cosmology (gravitational detection) to nanotechnology (phase-contrast microscopy). Given finite resources (energy, number of photons, etc) the phase sensitivity is limited by statistical uncertainty. It has been shown that the use of semi-classical probes, i.e. coherent light fields, limits the sensitivity to the standard quantum limit (SQL): $\delta\phi = 1/\sqrt{N}$, where $\delta\phi$ is the error of the estimated phase and N is the average number of photons used when the probe interacts only once with the phase-changing object [1]. The more fundamental Heisenberg limit is attainable with the use of a quantum probe (e.g. an entangled state of photons): $\delta\phi = 1/N$ [1]–[4]. The possibility to beat the SQL and approach the Heisenberg limit is therefore of great fundamental interest in understanding how quantum effects can be advantageous and may lead to important applications in the precision measurements that are the basis of all quantitative science.

In this context, interference experiments using two- [5]–[10], three- [11], and four-photon states [12]–[14] have been reported. In each case $p \propto \sin(N\phi)$, where p is the detection probability that gives rise to the N-photon interference. Observation of such a 'λ/N' fringe, with a period N times shorter than the single photon fringe of a semi-classical resource, is called *phase super-resolution* [11, 15]. Phase super-resolution has sometimes been associated with beating the SQL: *phase super-sensitivity*. However, Resch *et al* [15] have recently shown that the phase super-resolution can be achieved with purely classical resources. In contrast, the phase super-sensitivity is necessary to gain a quantum advantage in precision, and requires quantum resources.

In their groundbreaking work, Resch *et al* pointed out that the phase sensitivity depends on both the λ/N fringe visibility V and the efficiency η.[6] Since a lower efficiency means that more photons are necessary to achieve a given measurement precision, they assumed that it is possible to treat the efficiency simply as an increase in the required photon number by a factor of $1/\eta$. However, this assumption requires that the statistical errors of the successful measurements do not depend on the efficiency. As we show in the following, this is not the case in typical experiments, where a particular event is selectively detected by coincident photon counts to observe a λ/N interference fringe, since the detection probability p inseparably combines the effects of η and V. Therefore Resch *et al*'s theory does not apply to previous experiments with $N > 2$ such as the ones reported in [11]–[14], and a more detailed theoretical

---

[6] Note that Resch *et al* used η as a phenomenological parameter that could include various kinds of experimental efficiencies, whereas we consider just the intrinsic efficiency of the experimental scheme.

analysis is required to determine the effects of efficiency η on the phase sensitivity of N-photon interference experiments.

Here, we present a complete theory of the phase sensitivity $\mathcal{S}$ for the general case where the detection probability is given by an N-photon interference fringe. We perform a statistical error analysis for the event-detection probability which is valid for all N-photon interference experiments, including those reported previously [11]–[14]. We find that the phase sensitivity has a complicated dependence on both the intrinsic efficiency of detection η and the interference fringe visibility V. Furthermore, the phase at which $\mathcal{S}$ is a maximum is generally *not* the same as the phase at which the slope of p is a maximum, as has widely been assumed. In order to illustrate these theoretical results, we apply this expression for $\mathcal{S}$ to a new experiment with $N = 4$, $\eta = 3/4$ (improved by a factor of two from our previous experiment [14]) and $V = 82 \pm 6\%$, and find a phase super-sensitivity of 1.3 times greater than the SQL. As anticipated from our theoretical analysis, we find that the maximum phase sensitivity does not occur at the maximum slope of p.

2. Derivation of $\mathcal{S}$

To derive the phase sensitivity in a typical N-photon interference experiment, we start with the probability for the successful detection of the desired N-photon event in a single trial. For a λ/N fringe, the phase dependence of this probability is given as

$$p(\phi) = \frac{\eta}{2} \left[1 + V \sin(N\Phi_0 + N\phi) \right], \tag{1}$$

where Φ_0 is a bias phase, which is the known initial phase of the interferometer, and ϕ is the small phase shift to be measured. The interference fringe is thus characterized by two key parameters: the intrinsic efficiency $0 \leqslant \eta \leqslant 1$ is determined by the experimental scheme used and indicates the probability with which a given input photon contributes to the N-photon interference; and the visibility $0 \leqslant V \leqslant 1$ of the interference fringe observed in the output of the interferometer indicates the quality of the N-photon interference.

Given the dependence of detection probability on phase (equation (1)), the phase sensitivity can be derived using standard methods of metrology [16]. The phase estimate is determined from the average number of times that the selected N-photon event is observed in k trials,

$$C_k = kp(\phi). \tag{2}$$

Small phase shifts can be estimated by the deviation of C_k from its value at Φ_0. If the phase shift is much smaller than $1/N$, an estimate of ϕ can be obtained by dividing the change of C_k by the derivative of C_k at Φ_0. For $k \to \infty$, the difference between the observed number of counts and the expected number at Φ_0 then allows a precise determination of ϕ. In practice, however, k is finite, so that the actual number of N-photon events observed has a finite statistical variance given by

$$\Delta C_k^2 = kp(1 - p). \tag{3}$$

If the change of C_k is smaller than the square root of this variance, the phase shift ϕ cannot be distinguished from the statistical errors. Specifically, the statistical fluctuations in the observed N-photon events cause an error of $\delta\phi$ that limits the accuracy of the estimate of ϕ. If ϕ is

sufficiently smaller than $1/N$, it is possible to use linear error propagation based on the slope of C_k at Φ_0, and the phase error reads

$$\delta\phi^2 = \frac{\Delta C_k^2}{(\mathrm{d}C_k/\mathrm{d}\phi|_{\phi=0})^2}. \tag{4}$$

For direct comparison with the SQL, we can define the phase sensitivity $\mathcal{S}$ as the ratio of the phase error $\delta\phi$ and the phase error of $1/\sqrt{kN}$ at the SQL:

$$\begin{aligned}
\mathcal{S}^2 &\equiv \left(kN\delta\phi^2\right)^{-1} \\
&= N\frac{(\eta/2)V^2\cos(N\Phi_0)^2}{(1+V\sin(N\Phi_0))(1-(\eta/2)(1+V\sin(N\Phi_0)))}.
\end{aligned} \tag{5}$$

A phase sensitivity of $\mathcal{S} > 1$ then beats the SQL, and the Heisenberg limit is reached when $\mathcal{S} = \sqrt{N}$.

Since kN is the total photon number used in k trials and $1/\delta\phi^2$ is the Fisher information [17] of the phase estimate, the squared phase sensitivity $\mathcal{S}^2$ can also be interpreted as the Fisher information per photon. The SQL therefore corresponds to a Fisher information of one per photon, and beating it indicates a quantum mechanical increase of the Fisher information density beyond this limit, up to a maximal Fisher information of N per photon at the Heisenberg limit.

It may also be worth noting that the sensitivity determined here can be applied to a Bayesian phase estimation of a completely unknown phase [18], even though this means that the phase Φ_0 is initially unknown. In that case, Φ_0 is the result of the Bayesian estimate and $\delta\phi^2$ is the variance of the Gaussian probability distribution around Φ_0 for sufficiently large numbers of trials k. Effectively, the assumption of small ϕ corresponds to the 'fine tuning' of a measurement result near Φ_0. The condition for the linearization of C_k around Φ_0 is then that the number of trials k should be large enough to achieve an error $\delta\phi$ much smaller than $1/N$.

As equation (5) shows, the phase sensitivity depends on the bias phase Φ_0 in a rather nontrivial manner. Moreover, the dependences of the phase sensitivity on the efficiency η and the visibility V are quite different from each other. Figure 1 illustrates these different dependences for $N = 4$. Figure 1(a) shows the interference fringe $p(\phi)$, as given by equation (1). Figure 1(b) shows the phase sensitivity $\mathcal{S}$ for various visibilities V at $\eta = 1$. As might be expected, the maximum sensitivities $\mathcal{S}_{\mathrm{M}} = \sqrt{N}V$ are obtained where the slope $|\mathrm{d}p/\mathrm{d}\Phi_0|$ of the fringes is maximum. In the case of $V = 0.4$, $\mathcal{S}$ is always smaller than 1, indicating that the SQL cannot be beaten with V lower than a threshold value of $1/\sqrt{N} = 0.5$. Figure 1(c) shows $\mathcal{S}$ for various efficiencies η at $V = 1$. In contrast to figure 1(b), maximum sensitivity is now obtained at the minima of the interference fringes $p(\Phi_0)$, even though the slope at these points is equal to zero. The key to understanding this curious phenomena is that, since $V = 1$, p is exactly zero at these points. Thus the variance of C_k given by equation (3) is also zero, and equation (5) defines a finite phase sensitivity depending on the asymptotic ratio of the squared slope and the variance. The maximum sensitivity thus obtained is $\mathcal{S}_{\mathrm{M}} = \sqrt{N\eta}$, indicating that the SQL cannot be beaten with η lower than a threshold value of $1/N = 0.25$. Finally, figure 1(d) shows the phase sensitivity for $\eta = 3/4$ and $V = 0.82$, which correspond to the data obtained in the experiment described below. Interestingly, the combined effects of $V < 1$ and $\eta < 1$ result in zero sensitivity at the minima of $p(\Phi_0)$, but bend the maxima away from the maximum slope of the fringes and towards the positions of the minima. As a result, the phase bias which gives the
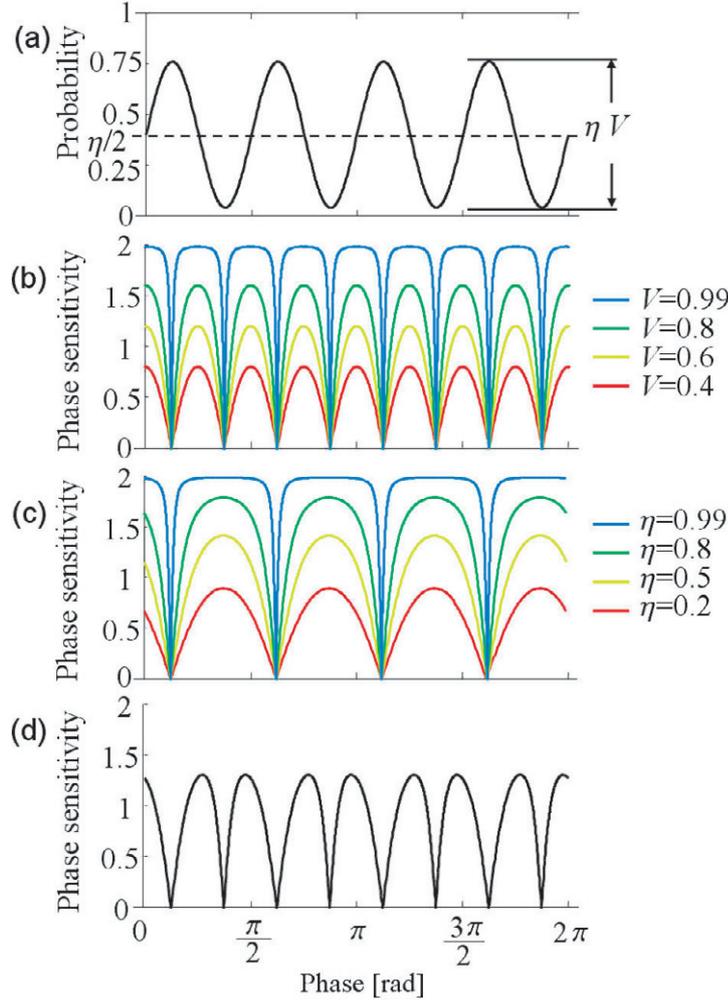
Figure 1. Phase sensitivity for N-photon interference. (a) Detection probability fringe with $N = 4$, where $\eta/2$ and ηV correspond to the average and the amplitude of the fringe, respectively. (b) Phase sensitivities $\mathcal{S}$ for $\eta = 1$ and various visibilities V. (c) Phase sensitivities $\mathcal{S}$ with $V = 1$ and various efficiencies η. (d) Phase sensitivity $\mathcal{S}$ for $\eta = 3/4$ and $V = 0.82$, corresponding to our experimental results.

maximum phase sensitivity is not the same as the phase with the maximum slope of the event probability $p(\Phi_0)$.

In order to find the phase bias at which optimal phase sensitivity is obtained, it is convenient to express the phase error in terms of the Heisenberg limit plus excess noise,

$$\delta\phi^2 = \frac{1}{kN^2}\left(1 + \frac{(2 - \eta(1 + V^2)) + 2(1 - \eta)V\sin(N\Phi_0)}{(\eta V^2(1 - \sin(N\Phi_0)^2))}\right). \quad (6)$$

By minimizing this function of $\sin(N\Phi_0)$, we can find the following relation for the optimal phase bias Φ_{opt}:

$$\sin(N\Phi_{\text{opt}}) = \frac{(\eta/2)(1 + V^2) - 1 + \sqrt{(1 - V^2)((1 - \eta/2)^2 - (\eta V/2)^2)}}{(1 - \eta)V}. \quad (7)$$
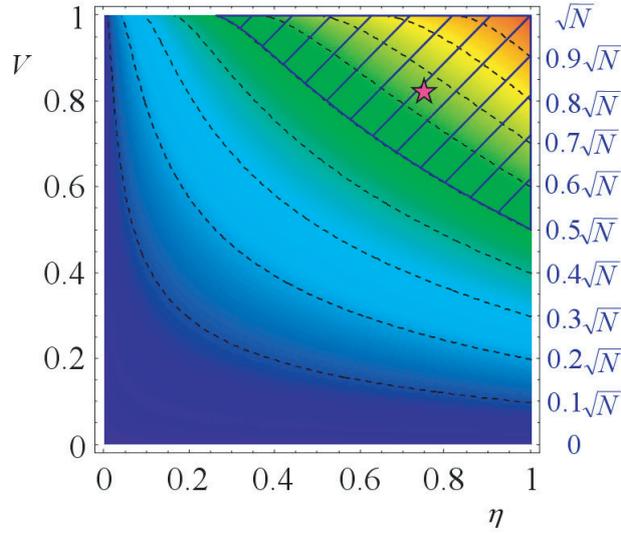
Figure 2. Contour plot of the maximum phase sensitivity $\mathcal{S}_{\mathrm{M}}$. The values on the right end are the sensitivities along the contours. The phase sensitivities in the area covered by diagonal lines beat the SQL for $N = 4$. The star indicates the performance of our four-photon interferometer for the data in figure 3.

The maximum phase sensitivity obtained at this phase bias is given by

$$\mathcal{S}_{\mathrm{M}}^2 = N \left(1 + \frac{(1 - \eta)^2}{\eta(1 - (\eta/2)(1 + V^2) - \sqrt{(1 - V^2)((1 - \eta/2)^2 - (\eta V/2)^2))}} \right)^{-1}. \tag{8}$$

Equation (8) defines the phase sensitivity of N-photon interference in terms of the experimental parameters V and η. It thus provides the basis for evaluating the actual phase sensitivities achieved in a specific experiment. A contour plot of equation (8) is shown in figure 2. The values on the right-hand side give the sensitivities corresponding to each contour. Note that the sensitivities increase with $\sqrt{N}$, indicating that the SQL can be beaten at lower values of V and η as the photon number N increases. The area shaded with diagonal lines indicates the region where the sensitivity $\mathcal{S}_{\mathrm{M}}$ is greater than one for $N = 4$, which is the condition for beating the SQL with a four-photon interferometer. Note that our theory does not consider the quantum efficiencies of the single photon detectors or the optical losses. The result therefore represents the phase sensitivity that could be achieved by the N-photon state in the interferometer if no further photons are lost in the detection process.

3. Experiment

We now apply the evaluation of phase sensitivity derived above to an improved version of the four-photon interference experiment reported in [14], and proposed in [19]. A product state $|22\rangle_{ab}$ of two two-photon Fock states is generated by parametric down-conversion and injected into a Mach–Zehnder (MZ) interferometer (figure 3(a)). The state after the first beam splitter of the interferometer is

$$|\psi_{\mathrm{path}}\rangle = \sqrt{\tfrac{3}{8}}|40\rangle_{cd} + \sqrt{\tfrac{1}{4}}|22\rangle_{cd} + \sqrt{\tfrac{3}{8}}|04\rangle_{cd}, \tag{9}$$
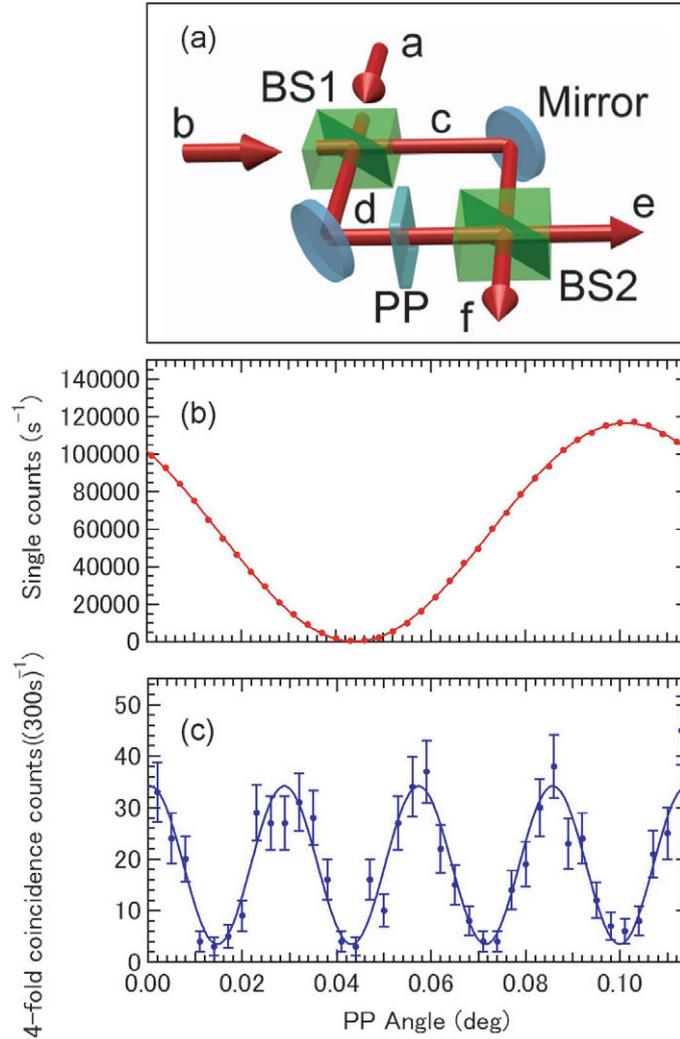
Figure 3. Experimental $\lambda/4$ fringe with $\eta = 3/4$. (a) A schematic of a MZ interferometer consisting of two $50:50$ beamsplitters (BS1 and BS2). (b) Single photon counts as a function of PP angle for a single photon input. (c) Four-fold coincidence counts of three photons in mode e (or f) and one photon in mode f (or e).

where c and d are the two paths inside the interferometer. A phase shift of ϕ is then applied to mode d. The state after the second beam splitter of the interferometer is

$$|\psi_{\text{out}}\rangle = \frac{\sqrt{6}}{16}(1 - 2e^{i2\phi} + e^{i4\phi})(|40\rangle_{ef} + |04\rangle_{ef}) + \frac{1}{8}(3 + 2e^{i2\phi} + 3e^{i4\phi})|22\rangle_{ef}$$
$$+ \frac{\sqrt{6}}{8}(1 - e^{i4\phi})(|31\rangle_{ef} + |13\rangle_{ef}), \tag{10}$$

where e and f are the output modes of the MZ interferometer. Significantly, the amplitudes of the $|31\rangle_{ef}$ and $|13\rangle_{ef}$ components do not include the phase oscillation of $\exp(i2\phi)$ associated with the $|22\rangle_{cd}$ component inside the interferometer. It is therefore possible to observe pure four-photon interference fringes in the detection probabilities P_{3ef} of $|31\rangle_{ef}$ and P_{3fe} of $|13\rangle_{ef}$. In our previous experiment [14], we counted only the detection of three photons in mode e and

one in mode f, for an efficiency of $\eta = 3/8$. However, we can also observe a four-photon fringe by detecting one photon in mode e and three in mode f. By using both detection events, we can improve η to 3/4. Ideally, the total detection probability then reads $P_{3ef+3fe} = \frac{3}{8}(1 - \cos 4\phi)$.

A frequency doubled 780 nm fs pulsed laser (repetition interval 13 ns, power 200 mW) pumps a type-I phase-matched beta barium borate (BBO) crystal (2 mm thickness) to generate the state $|22\rangle_{ab}$ via spontaneous parametric down-conversion. The down-converted photons pass through interference filters with 4 nm bandwidth and are then guided via polarization maintaining fibres (PMFs) to a displaced-Sagnac interferometer, which is essentially equivalent to a MZ interferometer [14]. A variable phase shift in mode d is realized by changing the angle of a phase plate (PP) in the interferometer. Photons are collected in single mode fibres (SMFs) at the output modes and detected using three cascade single photon counting modules (SPCM, detection efficiency 60% at 780 nm) in modes e and f (a total of six detectors). To test the performance of the four-photon interferometer, we used a relatively low efficiency source and modest efficiency detectors which means that many more photons pass through the interferometer than lead to a four-photon detection event. For applications (such as biological sensing) where photon flux is important, high efficiency number resolving visible light photon counters [20, 21] would dramatically improve detection efficiency. Using a home-made coincidence counter, we counted either of the following four-fold coincidence events: (i) detection signals from one of the three counters in mode e and all of the three counters in mode f, and (ii) from one of the three counters in mode f and all of the three counters in mode e.

Figure 3(b) shows a single-photon interference fringe with $V = 99.2 \pm 0.3\%$, obtained by inputting single photons in mode a and detecting the rate of single photons in mode e. The result of the four-photon interference ($P_{3ef+3fe}$) is shown in figure 3(c). As expected, the fringe period is 1/4 that of figure 3(b), demonstrating phase super-resolution. The visibility V of the fitted curve is $82 \pm 6\%$. Using equation (8), we can now determine the phase sensitivity for the experimental parameters $\eta = 3/4$ and $V = 0.82$. The maximum phase sensitivity achieved by the four-photon interferometry is then found to be $\mathcal{S}_M = 1.30$, i.e. the phase sensitivity of our interferometer is 1.3 times greater than the SQL. Note that $S_M = 1.30$ is the sensitivity value that could be reached if the experiment is performed with unit quantum efficiency detectors and without losses. This experimental result is indicated by the star mark in figure 2, which illustrates the relation between the experimental parameters and the conditions for beating the SQL. Note that $\mathcal{S}_M = 1.30$ is achievable not at the maximum slope of p but at the points shifted to the valleys of p as shown in figure 1(d).

4. Summary

We have derived the phase sensitivity when the detection probability of an output event is given by an N-photon interference fringe. We find that the phase sensitivity shows quite different dependences on the efficiency η and the visibility V. As a result, the phase bias that gives the maximum phase sensitivity is in general not the same as the phase with the maximum slope of the detection probability. We have determined the optimal phase bias and the corresponding maximum phase sensitivity as a function of efficiency η and visibility V obtained for a specific experiment. With this result, we can determine the quantitative enhancement of the phase sensitivity achieved in any N-photon interference experiment. In order to illustrate the theoretical results, we have applied this evaluation method to an improved four-photon

IOP Institute of Physics   Φ DEUTSCHE PHYSIKALISCHE GESELLSCHAFT

interference experiment with an efficiency of $\eta = 3/4$ and obtained a maximum phase sensitivity of 1.3 above the SQL given the experimentally observed visibility of 0.82. Finally, it may be possible to extend the analysis presented here to include schemes which allow multiple passes of single photons [22] and those using trapped ions [23, 24].

Acknowledgments

References

[1] Giovannetti V, Lloyd S and Maccone L 2006 *Phys. Rev. Lett.* **96** 010401
[2] Caves C M 1981 *Phys. Rev.* D **23** 1693
[3] Yurke B, McCall S L and Klauder J R 1986 *Phys. Rev.* A **33** 4033
[4] Giovannetti V, Lloyd S and Maccone L 2004 *Science* **306** 1330
[5] Ou Z Y, Zou X Y, Wang L J and Mandel L 1990 *Phys. Rev.* A **42** 2957
[6] Rarity J G, Tapster P R, Jakeman E, Larchuk T, Campos R A, Teich M C and Saleh B E A 1990 *Phys. Rev. Lett.* **65** 1348
[7] Kuzmich A and Mandel L 1998 *Quantum Semiclass. Opt.* **10** 493
[8] Fonseca E J S, Monken C H and Pádua S 1999 *Phys. Rev. Lett.* **82** 2868
[9] Edamatsu K, Shimizu R and Itoh T 2002 *Phys. Rev. Lett.* **89** 213601
[10] Eisenberg H S, Hodelin J F, Khoury G and Bouwmeester D 2005 *Phys. Rev. Lett.* **94** 090502
[11] Mitchell M W, Lundeen J S and Steinberg A M 2004 *Nature* **429** 161
[12] Walther P, Pan J W, Aspelmeyer M, Ursin R, Gasparoni S and Zeilinger A 2004 *Nature* **429** 158
[13] Sun F W, Liu B H, Huang Y F, Ou Z Y and Guo G C 2006 *Phys. Rev.* A **74** 033812
[14] Nagata T, Okamoto R, O'Brien J L, Sasaki K and Takeuchi S 2007 *Science* **316** 726
[15] Resch K J, Pregnell K L, Prevedel R, Gilchrist A, Pryde G J, O'Brien J L and White A G 2007 *Phys. Rev. Lett.* **98** 223601
[16] Barford N C 1985 *Experimental Measurements: Precision, Error, and Truth* 2nd edn (New York: Wiley)
[17] Cramér H 1946 *Mathematical Methods of Statistics* (Princeton, NJ: Princeton University Press)
[18] Pezze L, Smerzi A, Khoury G, Hodelin J F and Bouwmeester D 2007 *Phys. Rev. Lett.* **99** 223602
[19] Steuernagel O 2002 *Phys. Rev.* A **65** 033820
[20] Takeuchi S, Kim J, Yamamoto Y and Hogue H H 1999 *Appl. Phys. Lett.* **74** 1063
[21] Kim J, Takeuchi S, Yamamoto Y and Hogue H H 1999 *Appl. Phys. Lett.* **74** 902
[22] Higgins B L, Berry D W, Bartlett S D, Wiseman H M and Pryde G J 2007 *Nature* **450** 393
[23] Bollinger J J, Itano W M, Wineland D J and Heinzen D J 1996 *Phys. Rev.* A **54** R4649
[24] Leibfried D *et al* 2005 *Nature* **438** 639