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Criticality governed by the stable renormalization fixed point of the Ising model in the hierarchical small-world network

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We study the Ising model in a hierarchical small-world network by renormalization group analysis and find a phase transition between an ordered phase and a critical phase, which is driven by the coupling strength of the shortcut edges. Unlike ordinary phase transitions, which are related to unstable renormalization fixed points (FPs), the singularity in the ordered phase of the present model is governed by the FP that coincides with the stable FP of the ordered phase. The weak stability of the FP yields peculiar criticalities, including logarithmic behavior. On the other hand, the critical phase is related to a nontrivial FP, which depends on the coupling strength and is continuously connected to the ordered FP at the transition point. We show that this continuity indicates the existence of a finite correlation-length-like quantity inside the critical phase, which diverges upon approaching the transition point.

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Recently, various physical phenomena in non-Euclidean graphs have been studied, especially in the context of complex networks, and their properties have been found to be beyond the scope of the conventional theory for Euclidean graphs [1]. Of particular interest are systems regarded as infinite-dimensional in the sense that the equidistant surface S_r of radius r grows exponentially as $S_r \propto e^{y_d r}$ with a positive constant y_d, which is faster than any power function r^{d-1} as in d-dimensional Euclidean graphs. Typical examples are trees and hyperbolic lattices [2]. Remarkably, such infinite-dimensional systems often exhibit the critical phases in which the (nonlinear) susceptibility diverges [3–5]. Although the critical phase is also observed in some Euclidean systems, e.g., the quasi-long-range ordered phase in the two-dimensional XY model [6,7], the critical phases in infinite-dimensional systems are considered to be due to rather geometrical effects. Indeed, the exponential growth of a graph admits the divergence of the susceptibility χ, which is calculated by the integral of the two-point correlation function [8], even with finite correlation length $\hat{\xi}$ [9]; for $\hat{\xi} > 1/y_d$,

$$\chi \propto \int_0^{\ln N/y_d} dr S_r e^{-r/\hat{\xi}} \propto N^{\psi}, \quad \psi \equiv 1 - 1/y_d\hat{\xi}, \qquad (1)$$

diverges in the thermodynamic limit $N \to \infty$. Here we set the upper bound $\ln N/y_d$ to ensure $\chi \propto N$ for $\hat{\xi} = \infty$. A critical phase, if it exists, lies between a disordered phase with $\hat{\xi} < 1/y_d$ and an ordered phase with $\hat{\xi} = \infty$. Such a phase with a fractal exponent $0 < \psi < 1$ is actually observed in the percolation transitions in enhanced trees [5], hyperbolic lattices [10], hierarchical graphs [11,12], and growing random networks [13].

The property of the phase transitions between a critical phase and an ordered phase is an interesting issue. Quite recently, some models in the simple hierarchical network shown in Fig. 1(a) were investigated to examine these transitions. This network is very useful because rigorous real-space renormalization is possible for various models in the simplest way. Furthermore various types of phase transition are observed depending on the model used, e.g., a discontinuous transition of the bond percolation model [12], equivalent to the one-state Potts model [14,15], and continuous transitions with a power-law singularity (PLS) or an essential singularity (ES) for the q-state Potts model with $q \geqslant 3$ [16]. These are observed in other graphs [3–5,17,18]. Thus this hierarchical network is a good stage to investigate what determines the type of phase transitions in a systematic way. In particular the two-state Potts model, equivalent to the Ising model, stands between $q = 1$ and $q = 3$ and has special importance to understand how the transition class changes.

In this Rapid Communication, we study the phase transition of the two-state Potts model in the network mentioned above by renormalization group (RG) analysis, which reveals a new class of phase transition governed by the stable fixed point. Furthermore the singularity for $q = 2$ is found to be special due to the marginal bifurcation of the RG fixed point (FP) between pitchfork type for $q < 2$ and saddle-node type for $q > 2$.

We consider the two-state Potts model in the hierarchical small-world network shown in Fig. 1(a). This consists of one-dimensional backbone edges (BBEs) and nested shortcut edges (SCEs). The spin variable σ, taking a value 0 or 1, is put on every node of the network. The dimensionless energy function is written as

$$-\frac{E}{k_B T} = \sum_{\langle i,j\rangle \in \mathrm{BBE}} K\delta_{\sigma_i\sigma_j} + \sum_{\langle i,j\rangle \in \mathrm{SCE}} J\delta_{\sigma_i\sigma_j} + \sum_i H\delta_{\sigma_i 0}, \qquad (2)$$

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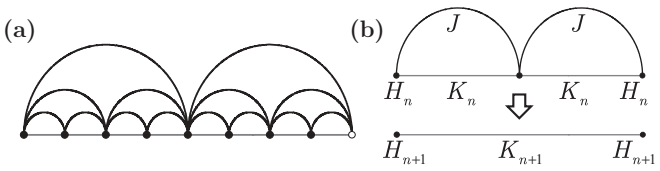

FIG. 1. (a) Hierarchical small-world network with eight nodes. The periodic boundary condition is imposed on the horizontal direction. (b) One-step decimation of the central spin in the partial sum of partition function. This is an inverse procedure to grow the graph in (a).

where K and J are the coupling constants on BBEs and SSEs, respectively, and H (>0) denotes the external magnetic field coupled with state 0.

Let us consider the partial sum of the partition function $Z = \prod_i \sum_{\sigma_i} e^{-E/k_BT}$ over the states of spins in the youngest generation having two neighbors [see Fig. 1(b)]. This is rigorously equivalent to replacing the parameters as

$$C_{n+1}e^{K_{n+1}\delta_{\sigma_1\sigma_2}+(H_{n+1}/2)(\delta_{\sigma_1 0}+\delta_{\sigma_2 0})} = C_n^2 e^{(H_n/2)(\delta_{\sigma_1 0}+\delta_{\sigma_2 0})}\sum_{\sigma} e^{(K_n+J)(\delta_{\sigma\sigma_1}+\delta_{\sigma\sigma_2})+H_n\delta_{\sigma 0}}, \quad (3)$$

with the recursion relations

$$g_{n+1} = g_n + 2^{-(n+1)}\left[K_n + J - A_n^{(1)}/2 + A_n^{(2)}/2 + A_n^{(3)}\right],$$
$$K_{n+1} = -(K_n + J) + A_n^{(1)}/2 + A_n^{(2)}/2 - A_n^{(3)}, \quad (4)$$

$$H_{n+1} = H_n + A_n^{(1)} - A_n^{(2)}, \quad (5)$$

where $g_n = 2^{-n}\ln C_n$, $e^{A_n^{(1)}} = e^{2(K_n+J)+H_n} + 1$, $e^{A_n^{(2)}} = e^{2(K_n+J)} + e^{H_n}$, and $e^{A_n^{(3)}} = e^{H_n} + 1$.

For $H = 0$, the recursion relation is rewritten as

$$k_{n+1} - k_n = -\frac{k_n\left[j^2k_n^2 - (2j-1)\right]}{1 + j^2k_n^2}, \quad (6)$$

where we put $k_n = e^{-K_n}$ and $j = e^{-J}$. The FP is given by $k_{n+1} = k_n = k^*$, which is solved as

$$k^* = 0, \pm\sqrt{2t/(1+t)}, \qquad t \equiv 2j - 1. \quad (7)$$

There is a pitchfork bifurcation point (PF BP) at $(j,k) = (1/2,0)$; the stable FP is $k^* = 0$ for $j \leqslant 1/2$ and $k^* = \sqrt{2t/(1+t)}$ for $j > 1/2$ (see Fig. 2). The region where the flow goes to the nontrivial fixed *line* is regarded as a critical phase because each FP on the line represents a self-similar structure like an ordinary critical point. It is noteworthy that in this renormalization procedure the couplings on the SCEs remain the same, while the ones on the BBEs are updated. The SCE couplings $0 < J \ll 1$ prevent K_n from converging to zero, and therefore the critical phase appears instead of a disordered phase. A similar mechanism is observed in the RG analysis of the Ising models exhibiting the critical phases, e.g., in graphs: the decorated (2,2)-flower [3] and the Hanoi network with average degree 5 (HN5) [18].

When j increases with k fixed, a transition from the ordered phase to the critical phase occurs at $j = j_c = 1/2$, irrespective of the value of k. Remarkably, the FP at j_c corresponds to $K = \infty (k = 0)$ as well as in the ordered phase $j < j_c$ and is stable, so that the critical behavior is quite unlike the conventional

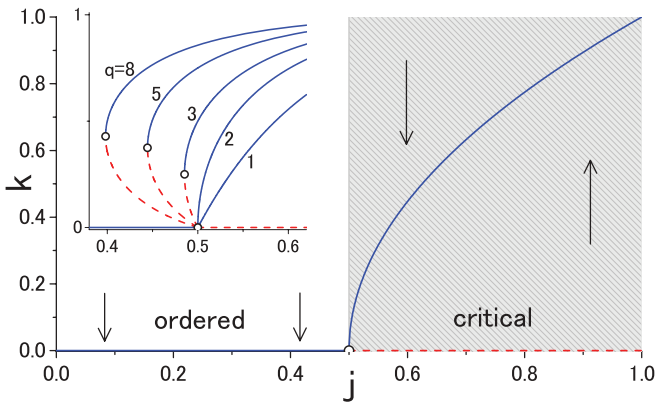

FIG. 2. (Color online) Phase diagram in the j vs k plain. The shaded region indicates the critical phase. The blue solid lines and red dashed lines denote stable and unstable fixed lines, respectively. The arrows indicate the RG flow of k. The inset shows the fixed lines for $q = 1, 2, 3, 5$, and 8.

one governed by unstable FP (US FP). This yields a curious singularity, as shown below.

Equation (6) is approximated by a differential equation:

$$\frac{ds_n}{dn} = -\frac{2s_n(s_n + |t|)}{1 + s_n}, \qquad s_n \equiv k_n^2 j^2. \quad (8)$$

This is solved as $e^{-2n|t|} = \frac{s_n}{s_0}\left(\frac{s_0+|t|}{s_n+|t|}\right)^{1-|t|}$, leading to

$$s_n \approx \frac{|t|}{e^{2n|t|} - 1} = \begin{cases} 1/2n & \text{for } n|t| \ll 1 \\ |t|e^{-2n|t|} & \text{for } n|t| \gg 1 \end{cases} \quad (9)$$

for $|t| \ll s_0$. Since s_n goes to zero for $n \to \infty$, it is (and thus K is) an irrelevant parameter.

In the lowest order of H_n, Eq. (9) is still valid as a solution of Eq. (4). On the other hand, Eq. (5) is rewritten as

$$H_{n+1} = H_n + \ln\frac{e^{2(K_n+J)+H_n} + 1}{e^{2(K_n+J)} + e^{H_n}} = \frac{2H_n}{1 + s_n} + O\left(H_n^2\right). \quad (10)$$

Note that we leave s_n in the first term; the cross term of s_n and H_n remains and thus this is beyond the linear stability analysis. By putting $H_n = 2^n e^{-x_n}$, we obtain

$$x_n \approx \sum_{\ell=n_0}^{n-1} s_\ell \approx \int_{n_0}^{n} \frac{|t|d\ell}{e^{2\ell|t|} - 1} = \frac{1}{2}\ln\frac{1 - e^{-2n|t|}}{1 - e^{-2n_0|t|}}, \quad (11)$$

where we substitute Eq. (9) into the summation and approximate it by the integral. Here we give the lower bound of summation n_0 to avoid the divergence of s_ℓ at $\ell = 0$. Finally, we obtain

$$H_n = He^{ny_d}u_n,$$
$$u_n \approx \sqrt{\frac{|t|}{1 - e^{-2n|t|}}} \propto \begin{cases} n^{-1/2} & \text{for } n|t| \ll 1 \\ |t|^{1/2} & \text{for } n|t| \gg 1 \end{cases}, \quad (12)$$

where we use $y_d = \ln 2$. Note that the factor u_n does not appear in the linear-order treatment for s_n and H_n.

The RG evolution [Eq. (12)] leads to a scale-invariant formula of the free energy as

$$g(H, N^{-1}) = e^{-ny_d}g(Hu_ne^{ny_d}, N^{-1}e^{ny_d}) \quad (13)$$

for $n \gg 1$. Here t is not included explicitly in the arguments of g because it is (and thus J is) not a scaling field but just an external parameter. The irrelevant scaling field s_n is also omitted. However, the effect of s_n is indirectly included in H_n as in Eq. (10) and gives the t dependence as in Eq. (12). The first and second order derivatives of g with H are written as

$$g_H(H,N^{-1}) = u_n g_H(Hu_n e^{ny_d}, N^{-1}e^{ny_d}), \tag{14}$$

$$g_{H^2}(H,N^{-1}) = u_n^2 e^{ny_d} g_{H^2}(Hu_n e^{ny_d}, N^{-1}e^{ny_d}), \tag{15}$$

respectively. In the following, we show the behaviors of the magnetization $m = g_H$ and the susceptibility $\chi = g_{H^2}$ in three asymptotic regimes: (i) $t < 0$, $H = 0$, and $N = \infty$, (ii) $t = 0$, $H > 0$, and $N = \infty$, and (iii) $t = 0$, $H = 0$, and $N < \infty$.

(i) For $t < 0$, $H = 0$, and $N = e^{ny_d} \to \infty$, we obtain

$$m = |t|^{1/2} g_H(0,1) \propto |t|^{1/2}, \tag{16}$$

$$\chi = N|t| g_{H^2}(0,1) \propto N|t|, \tag{17}$$

with $u_n \to |t|^{1/2}$. The magnetization shows the PLS as if it were an ordinary second order transition. Within the linear analysis $u_n = 1$, m does not vanish at $t = 0$, and the transition would look discontinuous. Surprisingly, χ diverges in the whole ordered phase, meaning the coexistence of a divergent fluctuation and a nonzero order parameter. Such coexistence may be realized by spatial segregation into the fluctuating region and the ordered region around the root node, the leftmost node in Fig. 1(a).

(ii) For $t \to 0$ and $N = e^{ny_d}$, $u_n \to n^{-1/2}$, Eqs. (14) and (15) become

$$g_H(H,N^{-1}) = n^{-1/2} g_H(Hn^{-1/2}e^{ny_d}, 1), \tag{18}$$

$$g_{H^2}(H,N^{-1}) = n^{-1}e^{ny_d} g_{H^2}(Hn^{-1/2}e^{ny_d}, 1). \tag{19}$$

We set n so as to satisfy $Hn^{-1/2}e^{ny_d} = 1$, which is approximately solved as $e^{ny_d} \approx \ln(H^{-1})/y_d H$. By substituting this into Eqs. (18) and (19), we obtain

$$m \approx \sqrt{\frac{y_d}{\ln(H^{-1})}}\, g_H(1,1) \propto [\ln(H^{-1})]^{-1/2}, \tag{20}$$

$$\chi \approx \frac{y_d}{\ln H^{-1}} \frac{\ln H^{-1}}{y_d H} g_{H^2}(1,1) \propto \frac{1}{H}. \tag{21}$$

(iii) For $t = 0$ and $H = 0$, Eqs. (18) and (19) become

$$m = \sqrt{\frac{y_d}{\ln N}}\, g_H(0,1) \propto (\ln N)^{-1/2}, \tag{22}$$

$$\chi = \frac{N}{\ln N} g_{H^2}(0,1) \propto \frac{N}{\ln N}. \tag{23}$$

On the crossover between the two limits, Eqs. (16) and (20), we expect a one-parameter scaling formula for $N = \infty$, such as

$$m = t^{1/2}\tilde{m}(t \ln H^{-1}) = [\ln(H^{-1})]^{-1/2}\tilde{\tilde{m}}(t \ln H^{-1}), \tag{24}$$

which is confirmed in Fig. 3. We numerically calculate m by integrating the recursion equations of g_n, K_n, H_n, and their derivatives.

Next, we investigate the property of the critical phase ($t > 0$). By considering the linear stability of Eqs. (4) and (5) at the points on the fixed line, we obtain

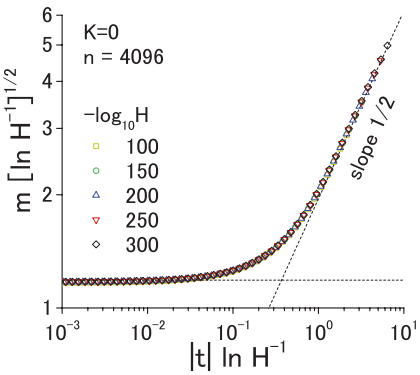

FIG. 3. (Color online) Scaling plot of the magnetization in the ordered phase. Data with $m < 0.20$ are used.

$\tilde{K}_{n+1} = e^{ny_K(t)}\tilde{K}_n$ and $H_{n+1} = e^{ny_H(t)}H_n$, where $\tilde{K}_n \equiv K_n - K^*(t)$, $y_K(t) = \ln\frac{1-t}{1+t}$, and $y_H(t) = \ln\frac{2}{1+t}$. These lead to $\tilde{K}_n \propto e^{ny_K}$ and $H_n \propto e^{ny_H}$. While $y_K(t)$ is negative and therefore $\tilde{K}$ is irrelevant in the critical phase, $y_H(t)$ is positive and approaches $y_d = \ln 2$ from below as $t \to 0_+$.

The free energy in the critical phase satisfies

$$g(H,N^{-1}) = e^{-ny_d} g(He^{ny_H(t)}, N^{-1}e^{ny_d}).$$

Again we omit the irrelevant parameters $\tilde{K}$ and t from the arguments, but t dependence is included in y_H. The ℓth order derivative of g is written as

$$g_{H^\ell}(H,N^{-1}) = e^{n[\ell y_H(t) - y_d]} g_{H^\ell}(He^{ny_H(t)}, N^{-1}e^{ny_d}).$$

Since $y_H(t) < y_d$, the first order derivative, i.e., the magnetization, is zero for $H = 0$ and $N = e^{ny_d} \to \infty$ in the critical phase. The susceptibility for $H = 0$ and $N = e^{ny_d}$ is

$$\chi = N^{\psi(t)} g_{H^2}(0,1) \propto N^{\psi(t)}, \tag{25}$$

$$\psi(t) = 2\frac{y_H(t)}{y_d} - 1 = 1 - \frac{2}{\ln 2}\ln(1+t). \tag{26}$$

For $1/2 < j < 1/\sqrt{2}$, ψ varies from 1 to 0, as shown in Fig. 4. In this region, χ diverges in the thermodynamic limit. For $j > 1/\sqrt{2}$, χ is finite, but higher order derivatives, i.e., nonlinear susceptibilities, diverge; the ℓth order derivative of the free energy with H diverges in the region where $\ell y_H(t) > y_d$.

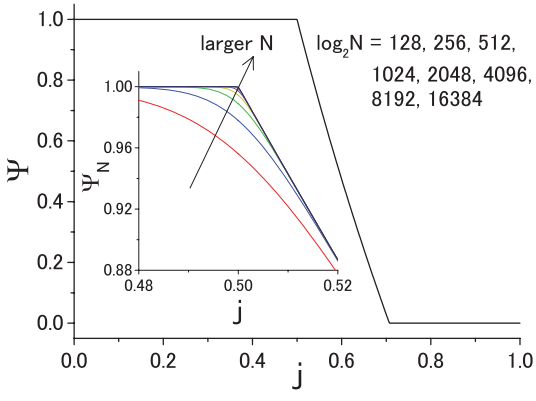

FIG. 4. (Color online) The fractal exponent is plotted with respect to the BBE coupling constant. The inset enlarges the finite size behavior ψ_N around $j = j_c = 1/2$.

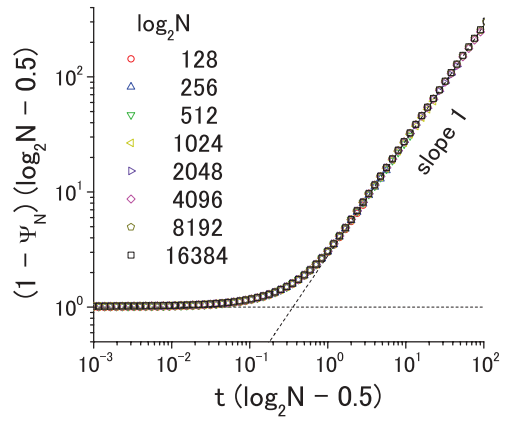

FIG. 5. (Color online) Scaling plot of the fractal exponent ψ_N in the critical phase. Data with $\psi_N > 0.60$ are used.

As noted in Eq. (1), ψ is related to the correlation length $\hat{\xi}$, and the fact that ψ approaches 1 from below means the divergence of $\hat{\xi}$. Here we consider the finite size effect of this singularity. The inset of the Fig. 4 shows the fractal exponent for size N, defined as $\psi_N \equiv \log_2[\chi(N)/\chi(N/2)]$. If we assume a power-law divergence of $\hat{\xi}$, a finite size scaling formula,

$$\hat{\xi}^{-1} \equiv 1 - \psi_N = t^{\nu} F(t^{\nu}[\ln N]^{\mu}) = \begin{cases} t^{\nu} & \text{for } t^{\nu}[\ln N]^{\mu} \gg 1 \\ [\ln N]^{-\mu} & \text{for } t^{\nu}[\ln N]^{\mu} \ll 1 \end{cases}, \quad (27)$$

is expected for $H = 0$. If $\ln N$ is a proper cutoff length, μ should be unity. Equations (26) and (23) lead to $1 - \psi_N \propto t$ and $1 - \psi_N \propto [\ln N]^{-1}$, respectively. Thus $\nu = 1$ and $\mu = 1$. We confirm the scaling behavior [Eq. (27)] in Fig. 5.

We have investigated the two-state Potts (Ising) model in the simple hierarchical network with the real-space RG method. The phase transition between the ordered and critical phases is governed by the PF BP at $K = \infty$ independently of the bare value of K. The singular behavior of the present model is summarized in Table I, where we also show the Potts model with $q = 1$ [12] and $q \geqslant 3$ [16]. See also the inset of Fig. 2 for the fixed lines for various q systems.

TABLE I. Summary of singularities. We show the singular formula of m and χ in the ordered phase $t < 0$ and of $\hat{\xi}$ in the critical phase $t > 0$. A discontinuous change is noted as t^0. The fact that $\hat{\xi} \propto [\ln N]^2$ for $q = 1$ is our preliminary result of a Monte Carlo simulation.

q	FP	m	χ	$\hat{\xi} \equiv 1/(1-\psi)$
1	PF BP		$N\lvert t\rvert^0$	t^{-2}, $[\ln N]^2$
2	PF BP	$\sqrt{\lvert t\rvert}$, $[\ln H^{-1}]^{-1/2}$	$N\lvert t\rvert$, H^{-1}	t^{-1}, $\ln N$
$\geqslant 3$	SN BP	$e^{-\lvert t\rvert^{-1/2}}$, $H^{1/\delta}$	$e^{\lvert t\rvert^{-1/2}}$, $H^{1/\delta-1}$	t^0
$\geqslant 3$	US FP	$\lvert t\rvert^{\beta}$, $H^{1/\delta}$	$\lvert t\rvert^{-\gamma}$, $H^{1/\delta-1}$	t^0

Boettcher *et al.* studied the bond percolation model, i.e., the one-state Potts model, in the same graph without backbone ($K = 0$) [12] and found a PF bifurcation of the RG FP similar to the present model ($q = 2$). They also calculated the maximum cluster size $\langle s_{\max}\rangle$, which is a quantity corresponding to unconnected susceptibility $\tilde{\chi} \equiv \chi + Nm^2$ because both quantities are defined as a summation of the two-point correlation. For $q = 1$, however, discontinuity of $\langle s_{\max}\rangle/N$ is observed at the transition point. This is in contrast to the fact that $\tilde{\chi}/N \propto t$ for $q = 2$. By taking the analytic continuation of the recursion relation, corresponding to Eq. (6), for $q \geqslant 3$ in Ref. [16], we obtain the PF BP $(j,k) = (1/2,0)$ for $q \leqslant 2$ and the recursion relation at $j = 1/2$: $k_{n+1} - k_n = -\frac{2-q}{4}k^2 - \frac{q-1}{4}k^3 + O(k^4)$. Thus the FP of $q = 2$ is marginal and has weaker stability. This is presumably the origin of the continuity of the transition.

For $q \geqslant 3$, the saddle-node (SN) bifurcation of the FP is observed [16]. Consequently, two kinds of singularity appear depending on $k = e^{-K}$: ES corresponding to SN BP for $k \geqslant k_{\rm SN}$ and PLS corresponding to USFP for $k < k_{\rm SN}$. The SN BP $(j_{\rm SN},k_{\rm SN})$ approaches the PF BP $(1/2,0)$ as $q \to 2$. While the PLS and ES for $q \geqslant 3$ are governed by USFP and marginally unstable SN BP, respectively, the phase transition for $q \leqslant 2$ is governed by the stable FP. Thus the generalized scaling theory in Ref. [16] assuming instability of a FP cannot be applied to the latter. We emphasize that the stability or instability of the FP of the transition point is the most fundamental criterion of phase transitions.

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