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# A probabilistic choice model based on Tsallis' statistics.

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**Abstract**

Decision under risk and uncertainty (probabilistic choice) has been attracting attention in econophysics and neuroeconomics. This paper proposes a probabilistic choice model based on a mathematical equivalence of delay and uncertainty in decision-making, and the deformed algebra developed in the Tsallis' non-extensive thermodynamics. Furthermore, it is shown that this model can be utilized to quantify the degree of consistency in probabilistic choice in humans and animals. Future directions in the application of the model to studies in econophysics, neurofinance, neuroeconomics, and social physics are discussed.

Keywords: Neuroeconomics; Econophysics; Neurofinance; risk; Tsallis's statistics

## **1. Introduction:**

Decision under risk and probabilistic uncertainty (probabilistic choice) has been a major topic in microeconomics (e.g., the expected utility theory [1]), behavioral neuroeconomics (e.g. the prospect theory [2]), and econophysics [3]. Studies in behavioral and neuro- economics have revealed that humans and non-human animals discount the value of probabilistic rewards as the receipt becomes more uncertain ("probability discounting") [4]. In order to develop decision theory on probabilistic choice, recent efforts in neuroeconomics have started to combine probabilistic choice processes with intertemporal choice process [4]. Delay discounting in intertemporal choice refers to the devaluation of a delayed reward compared to the value of a sooner reward [5-9]. In this line of investigations into the unification of delay and probability discounting, recent studies in behavioral psychology and neuroeconomics have demonstrated a mathematical equivalence of delay and probabilistic uncertainty (risk) [4,9,10] in reward-seeking behavior and optimal foraging, in several manners. It is noteworthy here that neurofinancial studies reported substance misusers and patients with brain lesions in the orbitofrontal cortex have low degree of risk aversion in investment behavior under risk [11,12].

I introduce, in this paper, a novel framework combining a probabilistic model based on the equivalence of delay and uncertainty, with an intertemporal choice model utilizing the  $q$ -exponential function [13,14], of which usefulness has been examined in our previous behavioral study [15]. Notably, the  $q$ -exponential function is a well-studied function in a deformed algebra inspired by and developed in Tsallis' non-extensive thermodynamics [16].

This paper is organized in the following manner. In Section 2, I briefly introduce the mathematical equivalence of delay and probabilistic uncertainty, in Section 3, I explain the intertemporal choice model based on Tsallis' statistics and how the model is extended into probabilistic choice, and in Section 4, some conclusions from this study and future study directions by utilizing the present probabilistic choice models are discussed.

## **2. A mathematical equivalence of delay and uncertainty in decision making**

As noted above, it has been, in several studies, demonstrated that delay until receipt of gains in intertemporal choice and uncertainty of winning of gains in probabilistic choice may be equivalent. The most well-known account in behavioral ecology and psychopharmacology was proposed by Rachlin et al [4]. In unifying delay

and probability discounting, Rachlin et al hypothesized that a decrease in a probability of winning an uncertain reward corresponds to an increase in a delay until winning the reward. Specifically, an average waiting time until winning an uncertain reward is proportional to  $(1/p)-1$  ("odds against"), where  $p$  is a probability of winning the uncertain reward. Therefore, according to Rachlin et al's hypothesis, decision-making models in intertemporal choice (delay discounting) can straightforwardly be extended into probabilistic choice, after replacing a parameter of delay in intertemporal choice models with the odds against parameter. A recent behavioral study on delay and probability discounting by human subjects has supported the hypothesis [17]. Furthermore, I have also proposed the mathematical equivalence of delay and uncertainty under a Markovian competitive social foraging condition [9], in which a delay is also proportional to  $(1/p)-1$ , which may explain anti-social behavior by subjects with strong delay discounting. Taken together, it appears to be a promising direction to establish probabilistic choice models based on the equivalence of delay and odds against.

### 3. An intertemporal choice model based on Tsallis' statistics

Behavioral neuroeconomic studies have been focusing on impulsivity and dynamic consistency in intertemporal choice [7,8,9,15,17]. If agent's intertemporal choice is dynamically consistent, the agent's choice can be considered as rational, i.e., utility-maximizing, behavior, even when the agents dependent on addictive drugs (e.g. a theory of rational addiction) [18]. It can easily be shown that dynamically consistent intertemporal choice behavior is described with the following exponential discount function:

$$V(D) = V(0)\exp(-kD) \quad (\text{Equation 1})$$

where  $V(D)$  is the subjective value of a reward at delay  $D$  until receipt. The free parameter  $k$  is an index of impulsivity (preference of sooner but small rewards), i.e., larger  $k$  values correspond to steeper delay discounting. In the exponential discount function, a discount rate defined as  $-(dV/dD)/V=k$ , independent of delay, confirming the absence of preference reversal.

However, behavioral economic studies observed inconsistency in human intertemporal choice. For instance, most people prefer "two cups of coffee available one year and a week later" over "a cup of coffee available one year later", but prefer "a cup of coffee available now" over "two cups of coffee available one week later". This

intertemporal choice behavior is inconsistent, since time-intervals between sooner and delayed rewards are identical (i.e., 7 days) in the two intertemporal choice examples. The traditional account for the observed preference reversal over time is that the following hyperbolic discount equation significantly fits their behavioral data better than Equation 1:

$$V(D) = V(0)/(1+jD). \quad (\text{Equation 2})$$

where  $j$  is a free parameter ( $V$  and  $D$  have the same definition as Equation 1). Large  $j$  values again correspond to rapid discounting. In the hyperbolic discount function, the discounting rate  $=j/(1+jD)$  (a hyperbolic discount rate) is a decreasing function of delay ("increasing patience"), resulting in preference reversal as time passes [5,6,7,8].

Recently, the following time-discount function (q-exponential discount function) has been proposed by Cajueiro [14] and empirically examined in our previous study [15], in order to continuously parameterize agent's consistency in intertemporal choice:

$$V(D) = V(0)/\exp_q(kD) = V(0)/[1+(1-q)kD]^{1/(1-q)} \quad (\text{Equation 3})$$

where  $\exp_q()$  is the q-exponential function in the deformed algebra inspired by Tsallis' non-extensive thermodynamics,  $0 \leq q < 1$  is a consistency parameter, and  $k$  is an impulsivity parameter ( $V$  and  $D$  have the same definitions as Equation 1 and 2). It is to be noted that large  $q$  values corresponds to more consistent intertemporal choice; namely,  $q \rightarrow 1$  corresponds to exponential discounting (complete consistency), while  $q=0$ , hyperbolic discounting (complete inconsistency). By utilizing the q-exponential discount function, we have empirically shown that parameter  $q$  can continuously parameterize human agents' consistency in intertemporal choice, and most people's intertemporal choice has no consistency [15]. Considering that it is well established, as introduced above, that delay and risk/probabilistic uncertainty may be equivalent in decision-making processes, it is a logical next step to combine the q-exponential discount model with the equivalence of delay and uncertainty.

Let us now put the odds against introduced in Equation 3 (Section 2), in stead of delay  $D$ . We obtain:

$$\begin{aligned} V(p) &= V(0)/\exp_q(k_p(1-p)/p) \\ &= V(0)/[1+k_p(1-q)(1-p)/p]^{1/(1-q)} \end{aligned} \quad (\text{Equation 4})$$

where  $p$  is the probability of winning an uncertain reward and  $k_p$  is a parameter of delay aversion in repeated probabilistic choices; i.e., larger  $k_p$  values indicate stronger risk (i.e., delay until winning) aversion (other parameters have the same definitions in Equation 3). This q-exponential probability discounting model, which is a natural extension of the q-exponential discount function combined with the reported equivalence of delay and uncertainty in decision processes, may allow us to continuously parameterize agents' consistency in probabilistic choice in gambling, financial decisions, and lottery. Notably, the equivalence of delay and odds-against in decision-making indicates that the agent spontaneously assumes the system is ergodic, in that a time-averaged frequency of winning is equal to a probability of winning in each trial. As far as I know, this study is the first to apply the concept of ergodicity to social physics.

#### **4. Conclusions and implications for neuroeconomics and econophysics**

As introduced in Section 3,  $q$  in the q-exponential probability discounting model may be capable of expressing each subject's inconsistency in probabilistic choice in a continuous manner (with smaller  $q$  values indicating more inconsistent probabilistic choices). As stated earlier, drug addicts, orbitofrontal lesion patients, and pathological gamblers have impaired decision-making processes in probabilistic choice. Therefore, future studies should examine whether these subjects are more inconsistent in probabilistic choice, in addition to conventional risk attitude defined in microeconomics, by utilizing the q-exponential probability discount function (equation 4). Since the mathematical characteristics of q-algebra have extensively been examined and generalized [19], the present formulation may readily be utilized in future econophysical studies. Also, because a recent neuroeconomic study reported that neural activity in the orbitofrontal cortex in the brain was correlated with consistency in decision under risk across gain and loss domains [20], future neuroimaging studies in neurofinance and neuroeconomics may be able to identify brain regions responsible for consistency in probabilistic choice defined as parameter  $q$ . Moreover, investigating the relationship between the parameter  $q$  in the present probability discount model and the risk attitude parameter also based on Tsallis' statistics [2] is important in future studies in social physics and econophysics.

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