Tsallis' non-extensive free energy as a subjective value of uncertain reward.

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Summary

Recent studies in neuroeconomics and econophysics revealed the importance of reward expectation in decision under uncertainty. Behavioral neuroeconomic studies have proposed that the unpredictability and the probability of an uncertain reward are distinctly encoded as entropy and a distorted probability weight, respectively, in the brain. However, previous behavioral economic and decision-theoretic models could not quantify reward-seeking and uncertainty-aversion in a theoretically-consistent manner. In this paper, I have proposed that generalized Helmholtz free energy in Tsallis' non-extensive thermostatistics can be utilized to quantify a perceived value of an uncertain reward. Future study directions in neuroeconomics and econophysics by utilizing the Tsallis' free energy model are discussed.

Keywords: uncertainty; entropy; Tsallis' statistics; neuroeconomics; neurofinance

1. **Introduction:**

Humans and non-human animals devalue probabilistic rewards as the receipt becomes more uncertain. The preference for a certain reward over an uncertain reward of an equal expected value is referred to as risk aversion in decision-making under risk/uncertainty. Neuropsychopharmacological studies reported that several types of neurochemical substances such as nicotine and serotonin dramatically modulate human decision under uncertainty [1,2]. In conventional microeconomic theory, risk aversion is defined as a curvature of the utility function [3,4]. Decision-making under uncertainty has been drawing much attention in behavioral neuroeconomics, econophysics and neurofinance, because departures from the prediction of the microeconomic theory (i.e., anomalies) have repeatedly been demonstrated in human choice behavior [4,5,6,7]. To establish quantitatively precise models of actual human decision-making under risk is important for understanding financial markets, risky decision by substance abusers and pathological gamblers [1,8]. Notably, neuroimaging studies have identified neural activities associated with uncertainty in decision under risk [9,10].

Empirical studies in behavioral and neuro- economics on decision-making under risk and uncertainty have reported the following important findings:

(A) People overweight small probabilities and underweight large probabilities [11]

(B) People have aversion to "ignorance" on the outcomes of uncertain rewards (i.e., people prefer predictable over unpredictable gains [4]).

It is important to note that von Neumann-Morgenstern's traditional expected utility theory cannot predict/explain these psychological tendencies observed in humans [3]. In order to explain and formalize anomaly (A), the prospect theory (PT) has been proposed [11]. In PT, it is assumed that a probability of an uncertain reward is non-linearly transformed/distorted into a psychophysical "probability weight" function [5,6,11] which is concave at small probabilities and convex at large probabilities. The anomaly (B) has most dramatically been demonstrated in the Ellsberg paradox experiment in which people prefer uncertain rewards with known over unknown probability distributions [4]. In order to quantify human aversion to ignorance on outcomes with known probabilities, Shannon entropy has been introduced [12]. However, to date, little effort has been spent on unifying PT and Tsallis thermostatistics-based decision theory, and combining the psychophysical and information-theoretic factors in decision under risk. It is to be noted that Antenedo et al (2002) is a pioneering investigation into this direction [13]. Cajueiro's attempt to apply Tsallis' statistics-based deformed algebra to intertemporal choice [14] is also in a similar line to the present study.

This paper is organized in the following manner. In Section 2, I briefly introduce PT and the role of entropy in decision under risk, in Section 3, I explain that a generalized free energy in Tsallis' nonextensive thermostatistics can be utilized as a subjective value of an uncertain reward in decision under risk, and in Section 4, I suggest some conclusions from this study and future study directions by utilizing the present Tsallis entropy model.

2. **Psychophysics and information theory of decision under uncertainty**

Suppose that the outcome (lottery) $L(x_1,p_1; \ldots; x_n, p_n)$, where $x_i>0$ (gain) occurs with probability p_i (an integer i satisfies $0<i<n+1$). PT assumes that a subjective value ("prospect") $V(x,p) = V(x_1,p_1; \ldots; x_n, p_n)$ of an uncertain reward is equal to $\Sigma_i\, v(x_i)w(p_i)$, where $v(x_i)$ is a subjective value of a certain reward x_i when probability of receipt $p_i=1$ and $w(p_i)$ is a probability weight function which reflects distortion in the perception of probability values [5,6,11]. Note that $w(p)$ is an increasing function of probability p_i. By assuming that $d^2w(p)/dp^2<0$ for small probabilities (typically $p_i < 1/e$, according to Prelec's proposal [5]) and $d^2w(p)/dp^2>0$ for large probabilities (an inverted S-shape probability weight [5,6,7,12]), PT can capture one of the human biases in decision under uncertainty; i.e., overweighting of small probabilities and underweighting of large probabilities. Behavioral economic studies have proposed several functional forms of the probability weight function. For instance, Gonzalez and Wu proposed $w(p)=\delta p^\gamma/(\delta p^\gamma+(1-p)^\gamma)$ [6] and Prelec proposed $w(p) =\exp(-\delta(-\log(p))^\gamma)$ [5], where δ and γ are positive free parameters. Irrespective of the functional forms of the probability weight, the nonlinear distortion of probability p into $w(p)$ may be the result of psychophysics of probability perception [6,7,12]. In Antenedo et al's study and Takahashi's entropy model [7,12,13], the probability weight has been assumed to be q-probability (i.e. an escort probability in Tsallis non-extensive thermostatistics) [15,16,17]. It is to be noted here that although PT can describe the aforementioned anomaly (A) in Section 1, PT has several limitations. For instance, PT does not predict that humans have aversion to unpredictability/ignorance on uncertain outcomes. A recent neuroimaging study on human decision-making under risk reported psychological processes of a reward expectation and aversion to unpredictability on uncertain rewards are distinctly represented in the brain [10]. Because PT can only describe subject's reward expectation ("prospect"), this finding implies the necessity of modifying traditional PT.

In order to quantify the aversion to ignorance/unpredictability on outcomes, Takahashi introduced Shannon's information-theoretic entropy as a parameter of ignorance on probabilistic outcomes [12]. In the entropy model, a subjective value of the uncertain reward is $V(x,p)=p^a-TS_{shannon}$, where free parameter a indicates the psychophysical effect on small probability perception, $S_{shannon}= -\Sigma_i p_i \log p_i$ is a conventional Shannon entropy in information theory, and free parameter T indicates a subject's degree of unpredictability aversion [12]. It is to be noted that the parameter of ignorance/unpredictability $S_{shannon}$ is maximal at p=0.5 and minimal at p=0 or 1. Our recent behavioral economic study demonstrated that the entropy model fit human subjects' aggregated probabilistic choice behavior better than simple hyperbolic model which has been utilized in neuropsychopharmacology [1,7]. Also, a recent neuroeconomic study utilizing a psychologically similar model reported that unpredictability in probabilistic choice was associated with the activation of brain regions such as the insula (a neural circuit for disgust) [10]. It can be seen that, as in Antenedo et al's model, the entropy model assumes that the probability weight function is q-probability in the Tsallis statistics-based framework. However, no study to date applies Tsallis' thermostatistics-based generalized free energy to decision under risk in a theoretically consistent manner. Antenedo et al's study also had a limitation that the model was not intended to describe agents' probabilistic choice behavior at small probabilities [13].

3. Generalized free energy as subjective value of uncertain reward

It has been expected that Tsallis' thermostatistics explain human perception and decision-making [15]. Let us briefly review the characteristics of Tsallis' non-extensive thermodynamics. The Boltzmann constant k_B is set to be 1 throughout because the present application is not physical. Based on above considerations, it is supposable that a subjective value of an uncertain reward may be expressed as the generalized Helmholtz free energy, because the non-extensive free energy can describe both subject's probability weight (q-probability) and aversion to ignorance about uncertain outcomes. The generalized Helmholtz free energy F_q is [16,17]:

$$F_q:=U_q-T_qS_q \qquad \text{(Equation 1)}$$

where U_q is the internal energy (reward expectation), T_q is a parameter of subject's unpredictability aversion in decision under uncertainty, and S_q is the Tsallis entropy. The functional form of U_q and S_q will be presented below for a special case for the applications in behavioral neuroeconomic studies. Note that parameter q is real number and this expression of the generalized free energy should recover the usual Boltzmann-Gibbs' Helmholtz free energy in the limit $q \rightarrow 1$. The explicit expressions of the quantities in Equation 1 in terms of escort probabilities depend on the constraint [16,17].

Let us now consider the Bernoulli-type lottery $L(x, p; 0, 1-p)$. In this case, the reward expectation (corresponding to the internal energy/probability weight) is $U_q^{(1)}=xp^q/\Sigma_i p^q=xp^q/(p^q+(1-p)^q)$ (or $U_q^{(2)}=p^q x$, depending on the constraint), and the unpredictability aversion (Tsallis entropy) is $S_q=(1-\Sigma_i p_i^q)/(1-q)=(1-p^q-(1-p)^q)/(1-q)$. Therefore, the explicit expression of a subjective value is:

$$F_q^{(1)}(x,p)=x\, p^q/(p^q+(1-p)^q)-T_q(1-p^q-(1-p)^q)/(1-q) \quad \text{(Equation 2)}$$

or

$$F_q^{(2)}(x,p)=xp^q-T_q(1-p^q-(1-p)^q)/(1-q). \quad \text{(Equation 3)}$$

In this way, the Tsallis generalized free energy may describe human subject's choice in the simple lottery. The important point here is that the expression is capable of capturing both of the two distinct neuropsychological tendencies (A) and (B) introduced in Section 1. It is important to note that $U_q^{(1)}=p^q/(p^q+(1-p)^q)$ exactly matches one type of probability weight functions proposed by Gonzalez et al. in behavioral economic studies [6]. Because traditional PT cannot describe the biased tendencies (A) and (B) simultaneously, it is expected that present framework based on the non-extensive thermostatistics has an advantage in the application in neuroeconomics and econophysics. Furthermore, previous Takahashi's entropy model [7,12] has a limitation that the model does not treat the free energy-like function in a consistent manner; namely, it includes a *q*-probability-like probability weight but the entropy in the model is extensive.

4. **Conclusions and implications for behavioral and neuroeconomics**

The present study shows that Tsallis entropy may be able to express subject's unpredictability aversion. Future studies should examine whether the present models (Equation 2 and 3) fit human probabilistic choice data better than the previous probabistic choice models [5,6,7]. Which model of Equation 2 and 3 better fits human choice behavior under uncertainty is a problem of future empirical examinations. The answer for this question may depend on psychophysics of human perception of probability values. Specifically, if human subjects' probability perception is additive (i.e., the sum of the perceived probabilities of all outcomes is 1), Equation 3 may better fit. It is to be noted that in decision under uncertainty with an unknown probability distribution ("ambiguity" or Knightian uncertainty), subjective probability is known to be non-additive [4,18]. Therefore, human decision under ambiguity may be described with Equation 2 (in this case the probability weight should be replaced with the subjective probability). Also, future studies in neuroeconomics should examine the relations between subjects' T_q parameter values and neural activities encoding unpredictability aversion [10]. Regarding neurofinance and econophysics, the present model may be quite helpful because we have shown that aggregated human probabilistic choice may be described with entropy-based models better than other well-known models such as Prelec's probability weight function (often adopted in behavioral economics) and the one-parameter hyperbolic probability discount function (often adopted in psychopharmacology) [1,7]. Moreover, future neuropsychopharmacological studies should examine how anti-depressants such as selective serotonin reuptake inhibitors modify human subjects' parameters in the Tsallis' generalized free energy, in order to establish more effective medical treatments for mood disorders such as depression and anxiety.

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