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AND ITS APPLICATIONS

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0. INTRODUCTION

The divergent diagrams of smooth mappings on smooth manifolds (μ, g) :

$$N \xleftarrow{\mu} M \xrightarrow{g} P$$

appear in several areas in differential topology. For example the envelope theory, the study of singularities of solutions of first order differential equations, the vision theory, the web geometry, and the stability of smooth mappings. Two divergent diagrams $(\mu_j, g_j) : N_j \leftarrow M_j \rightarrow P_j$ ($j = 1, 2$) are said to be *equivalent* if there exist diffeomorphisms $\lambda : N_1 \rightarrow N_2$, $\psi : M_1 \rightarrow M_2$ and $\phi : P_1 \rightarrow P_2$ such that $\lambda \circ \mu_1 = \mu_2 \circ \psi$, $\phi \circ g_1 = g_2 \circ \psi$.

This paper consists of two parts. In Part I, as one of the problems in the vision theory we study a generic semi-local classification for the singularities of orthogonal projections of contour lines of surfaces onto planes (i.e., so-called “topography”). This problem is reduced to the semi-local classification of the divergent diagram $\mathbb{R} \leftarrow M \rightarrow \mathbb{R}^2$.

In Part II, we give a generic classification of the divergent diagrams of map germs $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ corresponding to a certain class of differential equations which are called “Clairaut type”. We also study first order ordinary differential equations with complete integral.

More precisely let us describe how the divergent diagrams $N \xleftarrow{\mu} M \xrightarrow{g} P$ appear in the above several areas.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

(1) **The envelope theory** [1, 7, 14, 15, 18, 36] : The study of this subject can be reduced to the study of divergent diagrams (μ, g) and the envelope is defined as the discriminant set of g , where μ is a submersion (furthermore, Thom[36], Carneiro[7] suppose that g restricted to a fiber of μ is injective immersion). In particular, Arnol'd[1], Dufour[15], Dufour and Jean[18] have classified the diagrams of map germs $(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^n, 0) \xrightarrow{g} (\mathbb{R}^n, 0)$ which are called one-parameter families of hypersurfaces for $n = 2, 3$. We remark that as the result of classifications, some normal forms are parametrized by arbitrary functions in n -variable (i.e., so-called the functional moduli). Hence we need to characterize these functions which appear as functional moduli for normal forms. Dufour[16] has given a characterization of the functional moduli for some normal forms in the case $n = 2$ by investigating the dynamical systems $t \mapsto at + \dots$ ($a \in \mathbb{R} - \{0, 1\}$) associated with the functional moduli, however, we can not justify his result if we adopt his characterization. In Part II we correct the characterization of Dufour and give a generalization for arbitrary n (cf. Theorems D, E).

(2) **The study of singularities for solutions of first order differential equations for one real valued functions** [22, 24-27] : As we shall describe in Part II, a holonomic system of differential equation germs with complete integral corresponds to the diagram of smooth map germs with some condition, which is called a integral diagram: $(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^n, 0) \xrightarrow{g} (\mathbb{R}^n, 0)$, where g describes the discriminant of the equation and the submersion germ μ represents a complete integral. Moreover it is shown that the classification of differential equations of this type under the equivalence relation given by the group of point transformations in the sense of S. Lie is generically reduced to the classification of integral diagrams. In Part II, we give a classification of integral diagrams which correspond to holonomic systems with classical complete integral (briefly, we call a holonomic system of Clairaut type) by using the method of Legendrian singularities [1, 9, 10, 12] (Theorem C). As the result of the classification for such integral diagrams,

we observe that functional moduli appear. We also give a characterization of the functional moduli which is a generalization of the result of Dufour (cf., Theorems D, E). We emphasize that Theorem E gives a complete invariant for the generic classification of holonomic systems of Clairaut type.

By Theorems C, D, E, we have given a complete classification for holonomic systems of Clairaut type, however, the situation is very complicated for general holonomic systems with complete integral. For $n = 1$, that is the case for ordinary differential equations, it has been known a classification under a rather strict equivalence [22]. It also appears functional moduli in some normal forms. We also give a characterization of the functional moduli, however, the normal form is so complicated comparing with the above case, we can not detect the complete invariant.

(3) **The vision theory** [6, 10, 19, 23, 28, 30, 37] : As one of the problems in this area we study the visions for contour lines of surfaces when one looks at it from a distant view in some direction d . The study of such a landscape, which is called a topography with a direction d , is reduced to the study of the divergent diagram which we call a topographic diagram : $\mathbb{R} \xleftarrow{\mu} M \xrightarrow{g} E$, where M is a surface, E is a plane with the normal direction d and the level curves of μ give contour lines on M . In Part I we give a generic semi-local classification of topographic diagrams by using the classification due to Dufour[15] for one-parameter families of plane curves $(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^2, 0) \xrightarrow{g} (\mathbb{R}^2, 0)$.

In [19] Dufour and Tueno studied the local pattern for photographs of lighted surfaces. Their formulation is also summed up by the divergent diagram as above.

(4) **The web geometry** [3, 4, 8, 17, 32] : An r -web in a manifold P is a configuration of r foliations with codimension k in general position of P . Thus the local study of r -web in P is reduced to the study of the following diagram of map

germs :

$$\begin{array}{ccc} (\mathbb{R}^p, 0) & \xrightarrow{f_1} & (\mathbb{R}^k, 0) \\ & \searrow f_r & \\ & & (\mathbb{R}^k, 0) \end{array},$$

where f_i ($i = 1, \dots, r$) is a submersion. The diagram of this type is regarded as a multi germ of a divergent diagram:

$$\begin{array}{ccccc} (\mathbb{R}^k, 0) & \xleftarrow{\mu_1} & (\mathbb{R}^p, 0) & \xrightarrow{g_1} & (\mathbb{R}^p, 0) \\ & \vdots & & \searrow g_r & \\ (\mathbb{R}^k, 0) & \xleftarrow{\mu_r} & (\mathbb{R}^p, 0) & & \end{array},$$

where μ_i is a submersion germ and g_i is a diffeomorphism germ ($i = 1, \dots, r$).

(5) **The stability of diagrams of smooth mappings** [8, 11-15, 18] : The theory of stability of smooth mappings due to R. Thom, J. Mather gave rise to various generalizations and specializations. Particularly it has been known that for divergent diagrams and cyclic diagrams the Thom-Mather theory does not hold ([12]). Dufour has been studied divergent diagrams in detail from the viewpoint of stability theory ([11-15]). One of his attractive results is that the topological classification and the smooth classification of $(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^2, 0) \xrightarrow{g} (\mathbb{R}^2, 0)$ are the same in generic. This result is deduced from the topological rigidity of the web structure formed by the curves $g(\mu^{-1}(c))$, $c \in (\mathbb{R}, 0)$ in $(\mathbb{R}^2, 0)$. Also, by investigating the web structure formed by the curves $g(\mu^{-1}(c))$, in [7] Carneiro has given a characterization of some classes of $(\mu, g) : (\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^{n+1}, 0) \xrightarrow{g} (\mathbb{R}^{n+1}, 0)$ ($n \in \mathbb{N}$) from a different viewpoint. These classes coincide with some classes of integral diagrams of Clairaut type which is described in Part II.

We shall suppose throughout this paper that all mappings, map germs and manifolds are of class C^∞ unless otherwise stated. As usual, we denote by $\mathcal{E}_{x_1, \dots, x_n}$ the ring of all smooth function germs on $\mathbb{R}^n$ at 0 with a coordinate $(x_1, \dots, x_n)$ and denote by $\mathcal{M}_{x_1, \dots, x_n}$ the unique maximal ideal of $\mathcal{E}_{x_1, \dots, x_n}$.

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PART I

SINGULARITIES FOR PROJECTIONS OF CONTOUR LINES OF SURFACES ONTO PLANES

1. FORMULATIONS AND RESULTS

In Part I we give a generic semi-local classification for the singularities of orthogonal projections of contour lines of surfaces onto planes.

Let M be a surface in $\mathbb{R}^3 = \{(x, y, z)\}$ and let E_d be a hyperplane in $\mathbb{R}^3$ with the normal direction d such that $E_d \cap M = \emptyset$. We denote by $Emb(M, \mathbb{R}^3)$ the space of all embeddings $M \rightarrow \mathbb{R}^3$ with the Whitney C^∞ -topology. Let $\pi_d : \mathbb{R}^3 \rightarrow E_d$ be an orthogonal projection along the direction d . Now consider a level set of the height function $z : i(M) \rightarrow \mathbb{R}$, that is $i(M) \cap \{z = \text{constant}\}$ for $i \in Emb(M, \mathbb{R}^3)$. We call the set a *contour line* on $i(M)$. If one looks at a contour line on $i(M)$ from a distant view in some direction d , then one will get $\pi_d(i(M) \cap \{z = c\})$ as the viewing image. Our subject is a semi-local classification of singularities for one parameter families $\{\pi_d(i(M) \cap \{z = c\})\}_{c \in \mathbb{R}}$ which is called a *topography* of $i(M)$ with respect to a direction d .

We now formulate our theorems. Without loss of generality we can suppose that $d = (0, \cos \xi, \sin \xi) \in S^2 \cap \{x = 0\}$ where $0 < \xi \leq \frac{\pi}{2}$. For a direction d , by the transformation of $\frac{\pi}{2} - \xi$ rotation around x -axis in $\mathbb{R}^3$, we choose a new coordinate (x', y', z') . Then the direction d becomes $(0, 0, 1)$ and the height function z is expressed by $-y' \sin(\frac{\pi}{2} - \xi) + z' \cos(\frac{\pi}{2} - \xi)$ in the new coordinate. Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a projection defined by $\pi(u, v, w) = (u, v)$. We call the following divergent diagram of mappings a *topographic diagram* of $i(M)$, which is denoted by $(\mu_i, g) :$

$$\mathbb{R} \xleftarrow{\mu_i} M \xrightarrow{g} \mathbb{R}^2,$$

where $\mu_i = -i_2 \sin \theta + i_3 \cos \theta$ ($0 \leq \theta < \frac{\pi}{2}$), $g = \pi \circ i$.

Since our concern is to describe the discriminant set of $\pi \circ i$ (the outline of $i(M)$) and the bifurcation of $\pi \circ i(\mu_i^{-1}(c))$ along the parameter $c \in \mathbb{R}$ in semi-local situation, we introduce the following definitions. Let $i \in \text{Emb}(M, \mathbb{R}^3)$ and let $\{p_1, \dots, p_r\}$ be a subset of M whose elements are all distinct points in M such that $\pi \circ i(p_1) = \dots = \pi \circ i(p_r)$, where r is a positive integer. Then the multigerms of a topographic diagram at $\{p_1, \dots, p_r\}$ which is denoted by ${}_rT_i$

$$\begin{array}{ccccc} (\mathbb{R}, 0) & \xleftarrow{\mu_1} & (\mathbb{R}^2, 0) & \xrightarrow[g_2]{g_1} & (\mathbb{R}^2, 0) \\ (\mathbb{R}, 0) & \xleftarrow{\mu_2} & (\mathbb{R}^2, 0) & \nearrow & \\ \vdots & & \vdots & & \\ (\mathbb{R}, 0) & \xleftarrow{\mu_r} & (\mathbb{R}^2, 0) & \nearrow & \end{array}$$

where μ_k, g_k are germs of $\mu_i, \pi \circ i$ at p_k respectively ($k = 1, \dots, r$), is called a *topographic multigerms of i* . Let ${}_rT_i$ and ${}_rT_{i'}$ be topographic multigerms. Then ${}_rT_i$ and ${}_rT_{i'}$ are said to be *equivalent* if there exist diffeomorphism germs $\lambda_k : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, $\psi_k : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$, where $k = 1, \dots, r$, and $\phi : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ such that $\lambda_k \circ \mu_k = \mu'_k \circ \psi_k$, $\phi \circ g_k = g'_k \circ \psi_k$.

We shall state a genericity theorem for topographic multigerms.

Theorem A. *There exists a residual subset (hence dense) ${}_r\mathcal{O}$ in $\text{Emb}(M, \mathbb{R}^3)$ such that for any $i \in {}_r\mathcal{O}$ the topographic multigerms ${}_rT_i$ ($1 \leq r \leq 3$) is one of the following types:*

In the case $r = 1$.

(I) μ_1 is a submersion and g_1 is regular.

(II) μ_1 is a Morse type and g_1 is regular.

(III) μ_1 is a submersion, g_1 is a fold, μ_1 restricted to the singular set of g_1 is regular and $(\mu_1, g_1) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ is regular.

(IV) μ_1 is a submersion, g_1 is a fold, μ_1 restricted to the singular set of g_1 is a Morse type and $(\mu_1, g_1) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ is regular.

(V) μ_1 is a submersion, g_1 is a fold, $(\mu_1, g_1) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ is a Whitney's umbrella such that the line of double points of which is transversal at 0 to the

direction $\{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3$.

(VI) μ_1 is a submersion, g_1 is a cusp and $(\mu_1, g_1) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ is regular.

In the case $r = 2$.

$(I, I)_0 :$

$(\mu_1, g_1), (\mu_2, g_2)$ are both of type (I)

and $(\mu_1 \circ g_1^{-1}, \mu_2 \circ g_2^{-1}) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is regular.

$(I, I)_1 :$

$(\mu_1, g_1), (\mu_2, g_2)$ are both of type (I)

and $(\mu_1 \circ g_1^{-1}, \mu_2 \circ g_2^{-1}) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a fold.

$(I, I)_2 :$

$(\mu_1, g_1), (\mu_2, g_2)$ are both of type (I)

and $(\mu_1 \circ g_1^{-1}, \mu_2 \circ g_2^{-1}) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a cusp.

$(II, I) :$

(μ_1, g_1) is of type (II), (μ_2, g_2) is of type (I)

and $(\mu_1 \circ g_1^{-1}, \mu_2 \circ g_2^{-1}) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a fold.

$(III, I)^0 :$

(μ_1, g_1) is of type (III), (μ_2, g_2) is of type (I)

and the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ are transversal.

$(III, I)^1 :$

(μ_1, g_1) is of type (III), (μ_2, g_2) is of type (I)

and the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ have two point contact.

$(IV, I) :$

(μ_1, g_1) is of type (IV), (μ_2, g_2) is of type (I)

and the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ are transversal.

$(V, I) :$

(μ_1, g_1) is of type (V), (μ_2, g_2) is of type (I),

the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ are transversal.

$(VI, I) :$

(μ_1, g_1) is of type (VI) , (μ_2, g_2) is of type (I)

and the tangent cone of the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ are transversal.

$(III, III) :$

$(\mu_1, g_1), (\mu_2, g_2)$ are both of type (III)

and the discriminant sets of g_1, g_2 are transversal.

In the case $r = 3$.

$(I, I, I)_{1,1} : (\mu_j, g_j; \mu_k, g_k)$ is of type $(I, I)_1$ for $1 \leq j < k \leq 3$.

$(III, I, I)_1^{0,0} : (\mu_1, g_1; \mu_2, g_2)$ is of type $(III, I)^0$ and $(\mu_2, g_2; \mu_3, g_3)$ is of type $(I, I)_1$.

$(III, III, I)^{0,0} : (\mu_1, g_1; \mu_2, g_2)$ is of type (III, III) and $(\mu_j, g_j; \mu_3, g_3)$ is of type $(III, I)^0$ for $j = 1, 2$.

Besides the following nine types:

$(I, I, I)_{0,0}, (II, I, I)_0^{0,0}, (III, I, I)_0^{0,0}, (IV, I, I)_0^{0,0}, (V, I, I)_0^{0,0}, (VI, I, I)_0^{0,0} :$
 $(\mu_1, g_1; \mu_k, g_k)$ is of type $(I, I)_0$, $(\nu, I), \nu=II, \dots, VI$ for $k = 2, 3$ and $(\mu_2, g_2; \mu_3, g_3)$ is of type $(I, I)_0$.

$(I, I, I)_{1,0}, (I, I, I)_{2,0}, (III, I, I)_0^{1,0} : (\mu_1, g_1; \mu_2, g_2)$ is of type $(I, I)_1, (I, I)_2, (III, I)^1$ and $(\mu_2, g_2; \mu_3, g_3)$ is of type $(I, I)_0$.

Remark 1.1. In the case $r = 2$, the generic condition of $(I, I)_1$ (resp. $(I, I)_2$) means that $g_1(\mu_1^{-1}(0))$ and $g_2(\mu_2^{-1}(0))$ have second (resp. third) order contact.

Remark 1.2. In the case $r \geq 4$ all of generic types are essentially same as the case $r = 3$. That is, we can add only type (I) to the each type in the case $r = 3$ such that $(\mu_j, g_j; \mu_r, g_r)$ for $j = 1, \dots, r-1$ is not of type $(*, I)_1, (*, I)_2$ in the list of the case $r = 2$. In the same sense for the case $r = 3$, the generic types except three types $(I, I, I)_{1,1}, (III, I, I)_1^{0,0}, (III, III, I)^{0,0}$ are essentially same as in the case $r = 2$.

Next we shall give a normal form for each type stated as above.

Theorem B. *The topographic multigerms of each type are equivalent to one of the following multigerms ${}_rT_i = (\mu_1, g_1; \dots; \mu_r, g_r)$:*

In the case $r = 1$.

(I)

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1).$$

(II)

$$\mu_1 = x_1^2 \pm y_1^2, \quad g_1 = (x_1, y_1).$$

(III)

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2).$$

(IV)

$$\mu_1 = x_1^2 + y_1, \quad g_1 = (x_1, y_1^2).$$

(V)

$$\mu_1 = x_1 + x_1 y_1 + y_1^3, \quad g_1 = (x_1, y_1^2).$$

(VI)

$$\mu_1 = y_1 + \alpha \circ g_1, \quad g_1 = (x_1, y_1^3 + x_1 y_1),$$

where $\alpha \in \mathcal{M}_{u,v}$.

In the case $r = 2$.

$(I, I)_0$

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2, \quad g_2 = (x_2, y_2).$$

$(I, I)_1$

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2^2 + y_2, \quad g_2 = (x_2, y_2).$$

$(I, I)_2$

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2^3 + x_2 y_2 + y_2, \quad g_2 = (x_2, y_2).$$

(II, I)

$$\mu_1 = x_1^2 \pm y_1^2, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2, \quad g_2 = (x_2, y_2).$$

(III, I)⁰

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

where $\theta \in \mathcal{M}_{x_2, y_2}$ with $\frac{\partial \theta}{\partial x_2}(0) = 0$.

(III, I)¹

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = y_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

where $\theta \in \mathcal{M}_{x_2, y_2}^2$ with $\frac{\partial^2 \theta}{\partial x_2^2}(0) \neq 0$.

(IV, I)

$$\mu_1 = x_1^2 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

where $\theta \in \mathcal{M}_{x_2, y_2}$ with $\frac{\partial \theta}{\partial x_2}(0) = 0$.

(V, I)

$$\mu_1 = x_1 + x_1 y_1 + y_1^3, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

where $\theta \in \mathcal{M}_{x_2, y_2}$ with $\frac{\partial \theta}{\partial x_2}(0) = 0$.

(VI, I)

$$\mu_1 = y_1 + \alpha \circ g_1, \quad g_1 = (x_1, y_1^3 + x_1 y_1);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

where $\alpha \in \mathcal{M}_{u, v}$, $\theta \in \mathcal{M}_{x_2, y_2}$ with $\frac{\partial \theta}{\partial x_2}(0) = 0$.

(III, III)

$$\mu_1 = y_1 + \alpha_1 \circ g_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \alpha_2 \circ g_2, \quad g_2 = (x_2^2, y_2),$$

where $\alpha_1, \alpha_2 \in \mathcal{M}_{u,v}$ with $\frac{\partial \alpha_1}{\partial u}(0) \neq 0, \frac{\partial \alpha_2}{\partial v}(0) \neq 0$.

Remark 1.3. The normal forms of type $(I, I, I)_{1,1}, (III, I, I)_1^{0,0}, (III, III, I)^{0,0}$ are the following:

$(I, I, I)_{1,1}$: $\mu_1, g_1; \mu_2, g_2$ have the same form as the normal form of type $(I, I)_1$,

$$\mu_3 = y_3 + ax_3^2 + \theta(x_3, y_3), \quad g_3 = (x_3, y_3)$$

where $a \in \mathbb{R} - \{0, 1\}, \theta \in \mathcal{M}_{x_3, y_3}^2$ with $\frac{\partial^2 \theta}{\partial x_3^2}(0) = 0$.

$(III, I, I)_1^{0,0}$:

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \alpha(x_2, y_2), \quad g_2 = (x_2, y_2);$$

$$\mu_3 = x_3 + \beta(x_3, y_3), \quad g_3 = (x_3, y_3),$$

where $\alpha \in \mathcal{M}_{x_2, y_2}, \beta \in \mathcal{M}_{x_3, y_3}$ with $\frac{\partial \alpha}{\partial x_2}(0) = \frac{\partial \beta}{\partial x_3}(0) = 0, \frac{\partial \alpha}{\partial y_2}(0) = \frac{\partial \beta}{\partial y_3}(0)$ and $\frac{\partial^2 \alpha}{\partial y_2^2}(0) - \frac{\partial \alpha}{\partial y_2}(0) \frac{\partial^2 \alpha}{\partial x_2 y_2}(0) \neq \frac{\partial^2 \beta}{\partial y_3^2}(0) - \frac{\partial \beta}{\partial y_3}(0) \frac{\partial^2 \beta}{\partial x_3 y_3}(0)$.

$(III, III, I)^{0,0}$: $\mu_1, g_1; \mu_2, g_2$ have the same form as the normal form of type (III, III, I) ,

$$\mu_3 = x_3 + y_3 + \theta(x_3, y_3), \quad g_3 = (x_3, y_3) \text{ where } \theta \in \mathcal{M}_{x_3, y_3}^2.$$

Remark 1.4. The divergent diagram $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ have been studied by Arnol'd [1], Carneiro [7], Dufour [15] from the viewpoint of envelope, stability theory. Also, the normal forms of each type stated in Theorem A for $r = 1$ has been obtained by Arnol'd [1], Dufour [15]. (The classification of Arnol'd is not C^∞ case but formal case.) For $r = 2$, essentially types $(I, I)_k$ ($k = 0, 1, 2$), (II, I) have been studied by Dufour [11, 12, 13] from the viewpoint of "bi-stability" and its normal

forms have been obtained. On the other hand, singularities for certain visual images have been studied by several authors [6, 10, 19, 28, 37]. In particular, Dufour and Tueno have investigated in [19] the pattern of illuminance due to a point source of light which coincide with our Theorem A in the case $r = 1$.

2. GENERICITY

2.1. Preliminary of proof of Theorem A.

In order to describe the genericity we shall prepare a transversality lemma. Let $M^{(r)} = \{(p_1, \dots, p_r) \in M^r \mid p_j \neq p_k \text{ for } 1 \leq j < k \leq r\}$, where M^r is the r -fold product of M and let ${}_r J^l(M, \mathbb{R}^3)$ denote the multi jet space from M to $\mathbb{R}^3$ of multiplicity r and order l . For a map $f : M \rightarrow \mathbb{R}^3$, denote by ${}_r j^l f : M^{(r)} \rightarrow {}_r J^l(M, \mathbb{R}^3)$ the multi l -extension of f . Denote by ${}_r J_{emb}^l(M, \mathbb{R}^3)$ the set of all multi jets of $i \in Emb(M, \mathbb{R}^3)$. Note that ${}_r J_{emb}^l(M, \mathbb{R}^3)$ is open in ${}_r J^l(M, \mathbb{R}^3)$. We suppose that l is a positive integer with $l \geq 3$.

Consider the map ${}_r \Phi : {}_r J_{emb}^l(M, \mathbb{R}^3) \rightarrow {}_r J_{emb}^l(M, \mathbb{R}^3)$ defined by

$${}_r \Phi(j^l i(p_1), \dots, j^l i(p_r)) = (j^l T_i(p_1), \dots, j^l T_i(p_r)),$$

where $T_i = (\mu_i, \pi \circ i) \in Emb(M, \mathbb{R}^3)$ defined in section 1. We can easily see that the map ${}_r \Phi$ is a diffeomorphism.

The following is our transversality lemma. Let S be a smooth submanifold of ${}_r J_{emb}^l(M, \mathbb{R}^3)$. Then by the multi jet transversality theorem we can easily have

Lemma 2.1. *If $\text{codim} S > 2r$*

$$\mathcal{O}_S := \{i \in Emb(M, \mathbb{R}^3) \mid {}_r j^l T_i(M^{(r)}) \cap S = \emptyset\}$$

is residual in $Emb(M, \mathbb{R}^3)$.

Remark 2.2. Any algebraic set $A \subset \mathbb{R}^N$ can be decomposed into a finite disjoint union of analytic manifolds with dimensions less than or equal to $\dim A$ (see [5]). Hence our transversality lemma is valid even if S is an algebraic set in ${}_r J^l(\mathbb{R}^2, \mathbb{R}^3)$.

More precisely, let $S \subset {}_rJ^l(\mathbb{R}^2, \mathbb{R}^3)$ be an algebraic set and let $S = \cup_j \tilde{S}_j$ be its decomposition as above. If $\text{codim} S > 2r$, applying Lemma 2.1 to each $\tilde{S}_j$ we define $\mathcal{O}_S := \cup_j \mathcal{O}_{\tilde{S}_j}$. Then this $\mathcal{O}_S$ also has the same property in Lemma 2.1.

According to Dufour[15], we now define subvariety of $J^l(2, 3)$ as follows.

$$\Sigma_a = \{j^3(\mu, g)(0) \mid g \text{ is a fold map at } 0 \text{ and } \mu|_{S_g} \text{ is degenerate at } 0\},$$

$$\Sigma_b = \{j^3(\mu, g)(0) \mid g \text{ is a fold map at } 0 \text{ and } (\mu, g) \text{ is Whitney's umbrella}$$

such that the line of double points of which is non-transversal at 0 to the direction $\{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3\}$,

$$\Sigma_c = \{j^3(\mu, g)(0) \mid g \text{ is a cusp at } 0 \text{ and } (\mu, g) \text{ is singular at } 0\},$$

$$\Sigma_d = \{j^3(\mu, g)(0) \mid \mu \text{ is singular and degenerate at } 0\},$$

where $\mu : \mathbb{R}^2, 0 \rightarrow \mathbb{R}, 0, g : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^2, 0$.

For Σ_d , by its definition, Σ_d is a subvariety of codimension 3 in $J^l(2, 3)$.

Lemma 2.3(Dufour[15]). $\Sigma_a, \Sigma_b, \Sigma_c$ are subvarieties of codimension 3 in $J^l(2, 3)$.

2.2. Proof of Theorem A in the case $r = 1$.

In the case $r = 1$, as was shown in [15], by Lemma 2.1 and Lemma 2.3 we obtain Theorem A.

2.3. Proof of Theorem A in the case $r = 2$.

Denote by ${}_r\Delta$ the diagonal set in r -fold product of $\mathbb{R}^2$. Then ${}_r\Delta$ is codimension $2r - 2$ submanifold of $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ (r -times). Set

$$S_* = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \Sigma_* \times J^l(2, 3),$$

which is algebraic sets in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$, where $*$ = a, b, c, d and Σ_* is as above. By Lemma 2.3, $\text{codim} S_* > 4$. Hence, by Lemma 2.1 a residual subset $\mathcal{O}_0 = \mathcal{O}_{S_a} \cup \mathcal{O}_{S_b} \cup \mathcal{O}_{S_c} \cup \mathcal{O}_{S_d}$ in $\text{Emb}(M, \mathbb{R}^3)$ has the following property: for any $i \in \mathcal{O}_0$, (μ_1, g_1) is one of the types $(I), \dots, (VI)$, (μ_2, g_2) is of type (I) .

Moreover we need the following to see relations between type $(I), \dots, (VI)$ and type (I) .

Proposition 2.4. *There exist residual subsets $\mathcal{O}_{(\nu, I)}$ in $\text{Emb}(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_{(\nu, I)}$, ${}_2T_i$ is of type (ν, I) , where $\nu = I, \dots, VI$ and of type (I, I) means that of type $(I, I)_0, (I, I)_1, (I, I)_2$ and similarly of type (III, I) means that of type $(III, I)^0, (III, I)^1$.*

Proof. Define $\Sigma_{fold}, \Sigma_{cusp} \subset J^l(2, 2)$ by

$$\Sigma_{fold} = \{j^l g(0) \mid g \text{ is a fold map at } 0\},$$

$$\Sigma_{cusp} = \{j^l g(0) \mid g \text{ is a cusp map at } 0\}.$$

It is a well-known fact that Σ_{fold} is a codimension 1 subvariety and Σ_{cusp} is a codimension 2 subvariety in $J^l(2, 2)$.

Proof of the case $\nu = III, IV, V$. Let $g : M \rightarrow \mathbb{R}^2$ and let $p_1, p_2 \in M$ with $g(p_1) = g(p_2)$. Suppose that p_1 is a fold point of g . We take a local coordinate (x_1, y_1) of M around p_1 and (u, v) of the target space of g such that $g(x_1, y_1) = (x_1, y_1^2)$. Then, to describe the contact of the discriminant set $g_1(S_{g_1})$ and the curve $g_2(\mu_2^{-1}(0))$, we define a function germ $K : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ by

$$K(u) = \mu_2 \circ g_2^{-1}(u, 0).$$

By direct calculations, we see that $\{\frac{dK}{du}(0) = 0\}, \{\frac{d^2K}{du^2}(0) = 0\}$ are algebraic sets in $J^l(2, 3)$.

Define algebraic sets $\mathcal{S}_{(\nu, I)}$ in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$\mathcal{S}_{(III, I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times J^l(2, 1) \times \Sigma_{fold} \times \left\{ \frac{dK}{du}(0) = \frac{d^2K}{du^2}(0) = 0 \right\},$$

$$\mathcal{S}_{(IV, I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \left\{ \frac{\partial \mu}{\partial x_1}(0) = 0 \right\} \cap \left\{ \frac{\partial^2 \mu}{\partial x_1^2}(0) \neq 0 \right\} \times \Sigma_{fold} \times \left\{ \frac{dK}{du}(0) = 0 \right\},$$

$$\mathcal{S}_{(V, I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta$$

$\times \{(\mu, g) \text{ is Whitney's umbrella such that}$

the line of double points and $\{0\} \times \mathbb{R}^2$ are transversal at $0\}$

$$\times \Sigma_{fold} \times \left\{ \frac{dK}{du}(0) = 0 \right\}.$$

Since $\text{codim} S_{(\nu, I)} = 5$, by Lemma 2.1 ${}_2T_i$ is of type (ν, I) for any element i of the residual set $\mathcal{O}_{S_{(\nu, I)}}$. $\square$

Proof of the case $\nu = VI$. Suppose that g_1 is a cusp map. We take a coordinate (x_1, y_1) such that $g_1 = (x_1, y_1^3 + x_1 y_1)$. Then the function defined as above $K(u) = \mu_2 \circ g_2^{-1}(u, 0)$ describe the contact of the tangent cone of the discriminant set $g_1(S_{g_1})$ and the curve $g_2(\mu_2^{-1}(0))$. Define an algebraic set in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$S_{(VI, I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times J^l(2, 1) \times \Sigma_{\text{cusp}} \times \left\{ \frac{dK}{du}(0) = 0 \right\}.$$

Since $\text{codim} S_{(VI, I)} = 5$, by Lemma 2.1 the residual set $\mathcal{O}_{S_{(VI, I)}}$ has the required properties. $\square$

Proof of the case $\nu = I$. Suppose that $(\mu_1, g_1), (\mu_2, g_2)$ are both of type (I) . Let (u, v) be a coordinate in the target space of g_1 and g_2 . Since $\mu_1 \circ g_1^{-1}$ is a submersion, we may suppose that $\frac{\partial(\mu_1 \circ g_1^{-1})}{\partial u}(0) \neq 0$. So that consider the coordinate transformation $(u, v) \mapsto (u', v') = (\mu_1 \circ g_1^{-1}, v)$. Then in the new coordinate (u', v') , the map germ $(\mu_1 \circ g_1^{-1}, \mu_2 \circ g_2^{-1}) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ becomes the following form:

$$(u', G(u', v')) \text{ where } G(u', v') = \mu_2 \circ g_2^{-1}(u(u', v'), v').$$

Define algebraic sets V_1, V_2 in $J^l(2, 3) \times J^l(2, 3)$ by

$$V_1 = \left\{ \frac{\partial G}{\partial v'}(0) = \frac{\partial^2 G}{\partial v'^2}(0) = \frac{\partial^3 G}{\partial v'^3}(0) = 0 \right\},$$

$$V_2 = \left\{ \frac{\partial G}{\partial v'}(0) = \frac{\partial^2 G}{\partial v'^2}(0) = \frac{\partial^2 G}{\partial u' \partial v'}(0) = 0 \right\}.$$

By direct calculations, we see that the codimensions of V_1, V_2 in $J^l(2, 3) \times J^l(2, 3)$ are both three. As is well-known $(j^l \mu_1(0), j^l g_1(0), j^l \mu_2(0), j^l g_2(0)) \in J^l(2, 3) \times J^l(2, 3) - (V_1 \cup V_2)$ if and only if the map germ $(\mu_1 \circ g_1^{-1}, \mu_2 \circ g_2^{-1})$ is regular or a fold or a cusp.

We now define an algebraic set $S_{(I, I)}$ by

$$S_{(I, I)} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times V_1 \cup V_2.$$

Since $\text{codim} S_{(I,I)} = 5$, by Lemma 2.1 the residual set $\mathcal{O}_{S_{(I,I)}}$ has the required properties. $\square$

Proof of the case $\nu = II$. Let (μ_1, g_1) is of type (I), and let (μ_2, g_2) is of type (II). By the same argument as in the case (I, I), we see that $(\mu_1 \circ g_1^{-1}, \mu_2 \circ g_2^{-1}) = (u', G(u', v'))$ for some coordinate (u', v') . Note that by the direct calculation we have

$$\left\{ \frac{\partial \mu}{\partial x}(0) = \frac{\partial \mu}{\partial y}(0) = 0 \right\} \subset \left\{ \frac{\partial G}{\partial v'}(0) = 0 \right\}.$$

Define an algebraic set $S_{(II,I)}$ in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$\begin{aligned} S_{(II,I)} &= \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \\ &\times \left\{ \frac{\partial \mu}{\partial x}(0) = \frac{\partial \mu}{\partial y}(0) = 0 \right\} \cap \left\{ \frac{\partial G}{\partial v'}(0) = 0 \right\} \cap \left\{ \frac{\partial^2 G}{\partial v'^2}(0) \neq 0 \right\} \\ &\times \left\{ \frac{\partial^2 \mu}{\partial x^2}(0) \frac{\partial^2 \mu}{\partial y^2}(0) - 2 \frac{\partial^2 \mu}{\partial x \partial y}(0) = 0 \right\}. \end{aligned}$$

Since $\text{codim} S_{(II,I)} = 5$, by Lemma 2.1 the residual set $\mathcal{O}_{S_{(II,I)}}$ has the required properties. $\square$

Thus this completes the proof of Proposition 2.4. $\square$

Next we shall prove the genericity of type (III, III). The following lemma is obtained by a theorem of Whitney and Lemma 2.1.

Lemma 2.5. *There exists a residual set $\mathcal{O}_1$ in $\text{Emb}(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_1$, the topographic multigerms ${}_2T_i$ is the following type: g_1 and g_2 are both fold map germs such that the discriminant sets of g_1, g_2 are transversal at 0 each other.*

Proposition 2.6. *There exists a residual set $\mathcal{O}_{(III,III)}$ in $\text{Emb}(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_{(III,III)}$, ${}_2T_i$ is of type (III, III).*

Proof. Suppose that g_1 is a fold map germ. We can take a coordinate (x, y) such that $g = (x, y^2)$. In this coordinate, $\mu|_{S_g}$ is singular at 0, which means that $\frac{\partial \mu}{\partial x}(0) = 0$ and $(\mu, g) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ is singular at 0, which means that $\frac{\partial \mu}{\partial y}(0) = 0$. So that we shall define algebraic sets S_1, S_2 in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ as follows.

$$S_1 := \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \left\{ \frac{\partial \mu}{\partial x}(0) = 0 \right\} \times \Sigma_{\text{fold}} \times J^l(2, 1) \times \Sigma_{\text{fold}},$$

$$S_2 := \mathbb{R}^{2^{(2)}} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \left\{ \frac{\partial \mu}{\partial y}(0) = 0 \right\} \times \Sigma_{fold} \times J^l(2, 1) \times \Sigma_{fold}.$$

Since the codimensions of S_1, S_2 are both five, by Lemma 2.1 $\mathcal{O}_{(III, III)} := \mathcal{O}_{S_1} \cap \mathcal{O}_{S_2} \cap \mathcal{O}_1$ has required properties. $\square$

Proof of Theorem A in the case $r = 2$. Set $\mathcal{O} = \bigcup_{\nu=I}^{VI} (\mathcal{O}_{(\nu, I)} \cap \mathcal{O}_0) \cup \mathcal{O}_{(III, III)}$. Then by Proposition 2.4 and Proposition 2.6, $\mathcal{O}$ has required properties in Theorem A. This completes the proof in the case $r = 2$. $\square$

2.4. Proof of Theorem A in the case $r = 3$.

We shall give the proof which is likewise in the case $r = 2$ by using Lemma 2.1. So that we admit only $\text{codim} S \cap (\{0\} \times (J^l(2, 3))^3) \leq 2$ in $(J^l(2, 3))^3$, where $S \subset \mathbb{R}^{2^{(3)}} \times \mathbb{R}^3 \times {}_3\Delta \times (J^l(2, 3))^3$, because $\text{codim}_3 \Delta = 4$ in ${}_3J^l(\mathbb{R}^2, \mathbb{R}^3)$.

Proposition 2.7. *There exist residual sets $\mathcal{O}_n$ in $\text{Emb}(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_n$, the topographic multigerm ${}_3T_i$ is one of the following types:*

- (1) $(I, I, I)_{0,0}, (I, I, I)_{1,0}, (I, I, I)_{1,1}, (I, I, I)_{2,0}$ if $n = 1$;
- (2) $(III, I, I)_{0,0}^{0,0}, (III, I, I)_{0,0}^{1,0}, (III, I, I)_{1,1}^{0,0}$ if $n = 2$;
- (3) $(III, III, I)_{0,0}^{0,0}$ if $n = 3$.

Proof of the case $n = 1$. Let (μ_k, g_k) be of type (I) and let $F_k = \mu_k \circ g_k^{-1}$. Suppose that the map germ $(F_j, F_k) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is regular or a fold or a cusp, where $1 \leq j < k \leq 3$. We may take a coordinate (u, v) of the source space of (F_j, F_k) such that $\frac{\partial F_j}{\partial u}(0) \neq 0$. Hence it follows that $(F_j, F_k) = (u', G_{j,k}(u', v'))$ for some coordinate (u', v') , where $G_{j,k}(u', v') = F_k(u(u', v'), v')$. Then define an algebraic set $W_{j,k}$ in $(J^l(2, 3))^3$ by

$$W_{j,k} = \left\{ \frac{\partial G_{j,k}}{\partial v'}(0) = 0 \right\} \cap \left\{ \frac{\partial^2 G_{j,k}}{\partial v'^2}(0) \neq 0 \right\}.$$

By the direct calculation, we see that

$$W_{1,2} \cap W_{1,3} \subset \left\{ \frac{\partial G_{2,3}}{\partial v'}(0) = 0 \right\}.$$

Hence $\text{codim} W_{1,2} \cap W_{1,3} \cap W_{2,3} = 2$. Therefore we see that combinations of types (F_j, F_k) are voided of the following two cases in generic: for any two (F_j, F_k) , both types are cusp; one is a cusp and the other is a fold. This implies the assertion.

Proof of the case $n = 2$. We use the same notations in the proof of Proposition 2.3 and also in the proof of genericity of type (I, I). Suppose that g_1 is a fold map germ. Let us take coordinates $(x_1, y_1), (u, v)$ such that $g_1(x_1, y_1) = (x_1, y_1^2)$. Then define an algebraic set $S_{(III, I, I)}$ in ${}_3J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$\begin{aligned} S_{(III, I, I)} = & \mathbb{R}^{2(3)} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times {}_3\Delta \\ & \times J^l(2, 1) \times \Sigma_{fold} \times \left\{ \frac{dK}{du}(0) = 0 \right\} \cap \left\{ \frac{d^2K}{du^2}(0) \neq 0 \right\} \\ & \cap \left\{ \frac{\partial G}{\partial v'}(0) = 0 \right\} \cap \left\{ \frac{\partial^2 G}{\partial v'^2}(0) \neq 0 \right\}. \end{aligned}$$

Since $\text{codim} S_{(III, I, I)} = 7$, by Lemma 2.1 it follows the assertion.

Proof of the case $n = 3$. It is clear that the condition for type (III, III, I) is generic.

This implies the assertion. $\square$

Clearly the other types in the case $r = 3$ are also generic. Thus these fact implies the Theorem A in the case $r = 3$.

3. NORMAL FORMS

3.1. Proof of Theorem B in the case $r = 1$.

For the case $r = 1$, the normal forms have been obtained by Dufour [15] except type (II). For type (II), by Morse's lemma we immediately obtain the normal form.

3.2. Proof of Theorem B in the case $r = 2$.

In the case $r = 2$, we shall detect normal forms as follows.

Case of type $(III, I)^0, (III, I)^1$. We can suppose that

$$\mu_1 = x_1 + y_1, g_1 = (x_1, y_1^2); (\mu_2, g_2) \text{ is of type (I).}$$

By the coordinate change g_2^{-1} in the source space of g_2 and μ_2 , $(\mu_1, g_1; \mu_2, g_2)$ is equivalent to $(\mu_1, g_1; \mu'_2, g'_2)$, where

$$\mu'_2(x'_2, y'_2) = \mu_2 \circ g_2^{-1}(x'_2, y'_2), \quad g'_2(x'_2, y'_2) = (x'_2, y'_2).$$

Then in the case $(III, I)^0$, the generic condition about transversality means that $\frac{\partial \mu'_2}{\partial x'_2}(0) \neq 0$. By the coordinate change $(\frac{\partial \mu'_2}{\partial x'_2}(0))^{-1}$ which is a scalar multiple in the target space of μ'_2 , $(\mu_1, g_1; \mu'_2, g'_2)$ is equivalent to $(\mu_1, g_1; \mu''_2, g'_2)$, where

$$\mu'_2(x''_2, y'_2) = x'_2 + \theta(x'_2, y'_2), \quad \theta \in \mathcal{M}_{x'_2, y'_2} \text{ with } \frac{\partial \theta}{\partial x'_2}(0) = 0.$$

On the other hand, in the case $(III, I)^1$, the non-transversality condition means that $\frac{\partial \mu'_2}{\partial x'_2}(0) = 0$. Since $\mu'_2 = \mu_2 \circ g_2^{-1}$ is a submersion, we have $\frac{\partial \mu'_2}{\partial y'_2}(0) \neq 0$. Moreover, from the generic condition of the second order contact, we have $\frac{\partial^2 \mu'_2}{\partial x'^2_2}(0) \neq 0$. Hence, by the coordinate change $(\frac{\partial \mu'_2}{\partial y'_2}(0))^{-1}$ which is a scalar multiple in the target space of μ'_2 , $(\mu_1, g_1; \mu'_2, g'_2)$ is equivalent to $(\mu_1, g_1; \mu''_2, g'_2)$, where

$$\mu''_2(x'_2, y'_2) = y'_2 + \theta(x'_2, y'_2),$$

$$\theta \in \mathcal{M}_{x'_2, y'_2} \text{ with } \frac{\partial \theta}{\partial x'_2}(0) = \frac{\partial \theta}{\partial y'_2}(0) = 0 \text{ and } \frac{\partial^2 \theta}{\partial x'^2_2}(0) \neq 0.$$

Thus we obtained normal forms of type $(III, I)^0, (III, I)^1$.

Case of type $(IV, I), (V, I), (VI, I)$. By the same argument in the case of type $(III, I)^0$, we obtain the normal forms in Theorem B.

Case of type (III, III) . To detect the normal form of type (III, III) , we need the following lemmas.

Lemma 3.1. *Let $(\mathbb{R}^2, 0) \xrightarrow{g_1} (\mathbb{R}^2, 0) \xleftarrow{g_2} (\mathbb{R}^2, 0)$ be a bi-germ such that g_1, g_2 are both fold map germs and that the discriminant sets of g_1, g_2 are transversal at 0 each other. Then we can express the bi-germ for some coordinates as follows:*

$$g_1(x_1, y_1) = (x_1, y_1^2), \quad g_2(x_2, y_2) = (x_2^2, y_2).$$

Proof. We can choose coordinates (x_1, y_1) in the source space of g_1 , (u, v) in the target space of g_1 and g_2 such that $g_1(x_1, y_1) = (x_1, y_1^2)$. Let (x_2, y_2) be a coordinate in the source space of g_2 . By direct calculations, the transversality condition for the discriminant sets of g_1, g_2 means that $\frac{\partial(v \circ g_2)}{\partial x_2}(0) \neq 0$ or $\frac{\partial(v \circ g_2)}{\partial y_2}(0) \neq 0$. If $\frac{\partial(v \circ g_2)}{\partial x_2}(0) \neq 0$ (resp. $\frac{\partial(v \circ g_2)}{\partial y_2}(0) \neq 0$), then by using the coordinate transformation in the source space of g_2

$$(x_2, y_2) \mapsto (y_2, v \circ g_2) \text{ (resp. } (x_2, y_2) \mapsto (x_2, v \circ g_2)),$$

we can suppose that

$$g_2(x_2, y_2) = (u \circ g_2, y_2).$$

Since g_2 is a fold map germ, from the Malgrange preparation theorem, $\mathcal{E}_{x_2, y_2}$ is generated by $1, x_2$ as $\mathcal{E}_{u, v}$ -module via g_2^* . Thus there exist $A, B \in \mathcal{E}_{u, v}$ such that

$$\begin{aligned} x_2^2 &= A \circ g_2 + 2(B \circ g_2)x_2 \\ (*) \quad &= A(u \circ g_2, y_2) + 2B(u \circ g_2, y_2)x_2. \end{aligned}$$

By simple calculations, we see that $A(0) = B(0) = 0$, $\frac{\partial A}{\partial u}(0) \neq 0$. Now we can define coordinate transformations ϕ_1, ψ, ϕ_2 by

$$\phi_1(x_1, y_1) = (A \circ g_1(x_1, y_1) + B^2 \circ g_1(x_1, y_1), y_1),$$

$$\psi(u, v) = (A(u, v) + B(u, v)^2, v),$$

$$\phi_2(x_2, y_2) = (x_2 - B(u \circ g_2(x_2, y_2), y_2), y_2).$$

By the definitions of ψ, ϕ_1 , it is clear that

$$\psi \circ g_1 = g_1 \circ \phi_1.$$

Set $\phi_2(x_2, y_2) = (x'_2, y'_2)$, $\psi(u, v) = (u', v')$. Then by using (*) we see that

$$\begin{aligned} (x'_2)^2 &= (x_2 - B(u \circ g_2, y_2))^2 \\ &= A(u \circ g_2, y_2) + B(u \circ g_2, y_2)^2 \\ &= u' \circ g_2. \end{aligned}$$

Therefore, in these new coordinates preserving g_1 , we have

$$(u' \circ g_2)(x_2', y_2') = (x_2')^2, (v' \circ g_2)(x_2', y_2') = y_2'.$$

This completes the proof of Lemma 3.1 $\square$

Lemma 3.2. *Let $g = (x, y^2) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a fold map germ and let $A, B \in \mathcal{E}_{u,v}$. If $A \circ g + yB \circ g = 0$ in $\mathcal{E}_{x,y}$, then $A \circ g = B \circ g = 0$ in $\mathcal{E}_{x,y}$. That is, $\mathcal{E}_{x,y}$ is a free $\mathcal{E}_{u,v}$ -module via g^* .*

Proof. Suppose that $A \circ g + yB \circ g = 0$. Then, since g is even with respect to y -variable, we have $A \circ g - yB \circ g = 0$. This means that $A \circ g = yB \circ g = 0$. Hence we have $B \circ g = 0$. $\square$

Lemma 3.3. *Let $g = (x, y^2) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a fold map germ and let $H : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$, $K : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be diffeomorphism germs. Then $g \circ H = K \circ g$ if and only if H, K have the following forms:*

$$H = (\alpha \circ g, y\beta \circ g), K = (\alpha, v\beta^2),$$

where $\alpha, \beta \in \mathcal{E}_{u,v}$ with $\alpha(0) = 0$, $\frac{\partial \alpha}{\partial u}(0) \neq 0$, $\beta(0) \neq 0$.

Proof. The sufficiency is clear, by direct calculations. We shall prove the necessity. Suppose that $g \circ H = K \circ g$. Set $H = (h_1, h_2)$, $K = (k_1, k_2)$. Then by the assumption, $h_1 = k_1 \circ g$ and $h_2^2 = k_2 \circ g$. On the other hand, using the Malgrange preparation theorem we have

$$h_2(x, y) = A \circ g(x, y) + yB \circ g(x, y)$$

for some $A, B \in \mathcal{E}_{u,v}$. Thus

$$(A^2 + vB^2 - k_2) \circ g + y(2AB) \circ g = 0.$$

By Lemma 3.2, we have $AB \circ g = 0$. Note that $B(0) \neq 0$ and $\frac{\partial k_1}{\partial u}(0) \neq 0$, because $H = (k_1 \circ g, A \circ g + yB \circ g)$ is a diffeomorphism germ. Thus $B \circ g$ does not vanish on

$(\mathbb{R}^2, 0)$. Hence $A \circ g = 0$, and $h_2 = yB \circ g$. Consequently a form of H is determined to

$$H = (k_1 \circ g, yB \circ g).$$

Then by the form of H and the assumption $g \circ H = K \circ g$, the form of K is

$$K(u, v) = (k_1(u, v), vB(u, v)^2).$$

These forms of H, K are the required forms. $\square$

Lemma 3.4. *Let $(\mathbb{R}^2, 0) \xrightarrow{g_1} (\mathbb{R}^2, 0) \xleftarrow{g_2} (\mathbb{R}^2, 0)$ be a bi-germ such that $g_1(x_1, y_1) = (x_1, y_1^2)$, $g_2(x_2, y_2) = (x_2^2, y_2)$. Let $H_1, H_2, K : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be diffeomorphism germs. Then $g_1 \circ H_1 = K \circ g_1$ and $g_2 \circ H_2 = K \circ g_2$ if and only if H_1, H_2, K have the following forms:*

$$H_1(x_1, y_1) = (x_1 \alpha^2 \circ g_1, y_1 \beta \circ g_1),$$

$$H_2(x_2, y_2) = (x_2 \alpha \circ g_2, y_2 \beta^2 \circ g_2),$$

$$K(u, v) = (u \alpha^2, v \beta^2),$$

where $\alpha, \beta \in \mathcal{E}_{u,v}$ with $\alpha(0) \neq 0, \beta(0) \neq 0$.

Proof. By the same argument as in Lemma 3.3, the following fact holds: $g_2 \circ H_2 = K \circ g_2$ if and only if

$$H_2(x_2, y_2) = (x_2 \alpha_2 \circ g_2, \beta_2 \circ g_2), \quad K(u, v) = (u \alpha_2^2, \beta_2),$$

where $\alpha_2, \beta_2 \in \mathcal{E}_{u,v}$ with $\alpha_2(0) \neq 0, \beta_2(0) = 0, \frac{\partial \beta_2}{\partial v}(0) \neq 0$.

Combining this fact and Lemma 3.3, we can deduce Lemma 3.4. $\square$

We are now ready to detect a normal form of type (III, III) .

Let $(\mu_1, g_1; \mu_2, g_2)$ be of type (III, III) . By Lemma 3.1, we can suppose that

$$g_1(x_1, y_1) = (x_1, y_1^2),$$

$$g_2(x_2, y_2) = (x_2^2, y_2).$$

Applying the Malgrange preparation theorem, μ_1, μ_2 have the following forms:

$$\mu_1 = A_1 \circ g_1 + y_1 B_1 \circ g_1,$$

$$\mu_2 = A_2 \circ g_2 + x_2 B_2 \circ g_2,$$

where $A_k, B_k \in \mathcal{E}_{u,v}$ ($k = 1, 2$). Note that $B_k(0) \neq 0$ ($k = 1, 2$) because the map germ (μ_k, g_k) is non-singular. Thus we can define coordinate transformations H_1, H_2, K by

$$H_1(x_1, y_1) = (x_1 B_2^2 \circ g_1, y_1 B_1 \circ g_1),$$

$$H_2(x_2, y_2) = (x_2 B_2 \circ g_2, y_2 B_1^2 \circ g_2),$$

$$K(u, v) = (u B_2^2, v B_1^2).$$

In the new coordinates by H_1, H_2, K , it follows from Lemma 3.4 that $(\mu_1, g_1; \mu_2, g_2)$ is equivalent to $(\mu_1', g_1; \mu_2', g_2)$, where

$$(**) \quad \mu_1'(x_1', y_1') = \alpha_1 \circ g_1(x_1', y_1') + y_1',$$

$$\mu_2'(x_2', y_2') = \alpha_2 \circ g_2(x_2', y_2') + x_2',$$

where $\alpha_k = A_k \circ K^{-1} \in \mathcal{M}_{u,v}$ ($k = 1, 2$).

Since $\mu_1'|_{S_{g_1}}$ is non-singular, we have $\frac{\partial \alpha_1}{\partial u}(0) \neq 0$. Similarly $\frac{\partial \alpha_2}{\partial v}(0) \neq 0$. Conversely, for any $\alpha_1, \alpha_2 \in \mathcal{M}_{u,v}$ with $\frac{\partial \alpha_1}{\partial u}(0) \neq 0, \frac{\partial \alpha_2}{\partial v}(0) \neq 0$, a germ whose form is given by $(**)$ is of type (III, III) . Therefore we obtain the normal form of type (III, III) .

Case of $(I, I)_0, (I, I)_1, (I, I)_2$. The normal forms of this case have been detected essentially by Dufour [11, 12, 13], however, we shall give a process of detecting the normal forms.

Let $(\mu_1, g_1; \mu_2, g_2)$ be a topographic multigerms. If g_k ($k = 1, 2$) is a diffeomorphism, then by the coordinate transformation g_k^{-1} in the source space of μ_k and g_k , $(\mu_1, g_1; \mu_2, g_2)$ is equivalent to $(F, g_1'; G, g_2')$, where $F = \mu_1 \circ g_1^{-1}, G =$

$\mu_2 \circ g_2^{-1}, g'_k = id_{\mathbb{R}^2} (k = 1, 2)$. From now on, we only consider the coordinate transformations preserving both $g'_1 = id$ and $g'_2 = id$. Hence it is sufficient to detect normal forms of the following divergent diagrams up to coordinate transformations:

$$(\mathbb{R}, 0) \xleftarrow{F} (\mathbb{R}^2, 0) \xrightarrow{G} (\mathbb{R}, 0).$$

Let (x, y) be a coordinate of the source space of F, G . Then we detect the normal form of each type by using the deformation theory of functions ([2]) as follows.

In the case of type $(I, I)_0$: From the generic condition, $(F, G) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a diffeomorphism germ. Hence, clearly we obtain the normal form $F = x, G = y$.

In the case of type $(I, I)_1$: From the generic condition, G is a submersion. Thus we may suppose that $G(x, y) = y$. Then the generic condition such that the map germ $(F, G) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a fold and besides F is a submersion means that

$$\frac{\partial F}{\partial x}(0) = 0, \quad \frac{\partial^2 F}{\partial x^2}(0) \neq 0; \quad \frac{\partial F}{\partial y}(0) \neq 0.$$

Namely $F(x, 0)$ has A_1 -singularity and F is a deformation of $F(x, 0)$. So that let $\psi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be the diffeomorphism germ such that $F(\psi(x), 0) = \pm x^2$. Then trivially the diagram (F, y) is equivalent to (F', y') , where $F'(x', y') = \pm F(\psi(x'), y')$. We can easily verify that F' is an R -infinitesimally versal (hence R -versal) deformation of x'^2 . On the other hand, $x^2 + y$ is also an R -versal deformation of x^2 . Therefore by the uniqueness of the R -versal deformation, the following holds: there exist a diffeomorphism germ $\lambda \in \mathcal{M}_y$ and $h \in \mathcal{M}_{x,y}$ with $h(x, 0) = x$ such that the two diagrams $(F', y'), (x^2 + y, y)$ are equivalent transforming the coordinates in the target space of y' by λ and in the source space of both F' and y' by $(x, y) \mapsto (h(x, y), \lambda(y))$. Thus we obtain the normal form of type $(I, I)_1$.

In the case of type $(I, I)_2$. We may suppose that $G(x, y) = y$. The generic condition on this type means that

$$\frac{\partial F}{\partial x}(0) = \frac{\partial^2 F}{\partial x^2}(0) = 0, \quad \frac{\partial^3 F}{\partial x^3}(0) \neq 0, \quad \frac{\partial^2 F}{\partial x \partial y}(0) \neq 0; \quad \frac{\partial F}{\partial y}(0) \neq 0.$$

Hence $F(x, 0)$ has A_2 -singularity. So that let $\psi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be the diffeomorphism germ such that $F(\psi(x), 0) = \pm x^3$. Then trivially the diagram (F, y) is equivalent to (F', y') , where $F'(x', y') = \pm F(\psi(x'), y')$. Due to the condition $\frac{\partial^2 F}{\partial x \partial y}(0) \neq 0$, we can easily verify that F' is a R^+ -infinitesimally versal (hence R^+ -versal) deformation of x'^3 . On the other hand, $x^3 + xy$ is also an R^+ -versal deformation of x^3 . Therefore by the uniqueness of versal deformation, the diagram (F', y') is equivalent to

$$(x^3 + xy + \alpha(y), y)$$

for some $\alpha \in \mathcal{M}_y$. Here we need the following lemma which was shown by Dufour in [11].

Lemma 3.5. *For any $\alpha \in \mathcal{M}_y$, the diagram $(x^3 + xy + \alpha(y), y)$ is equivalent to $(x^3 + xy, y)$.*

That is, we obtain the normal form.

In the case of type (II, I): We may suppose that $G(x, y) = y$. By the generic condition, since the map germ $(F, G) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is a fold and F is a Morse germ, we have

$$\frac{\partial F}{\partial x}(0) = 0, \quad \frac{\partial^2 F}{\partial x^2}(0) \neq 0; \quad \frac{\partial F}{\partial y}(0) = 0.$$

Thus F is an R^+ -infinitesimally versal (hence R^+ -versal) deformation of $F(x, 0)$ which has A_1 -singularity. So that we see the diagram (F, y) is equivalent to (F', y') , where F' is an R^+ -versal deformation of x'^2 . On the other hand the trivial deformation of x^2 is also R^+ -versal. Therefore the diagram (F, y) is equivalent to $(x^2 + \alpha(y), y)$, where $\alpha \in \mathcal{M}_y$. Moreover by the generic condition, $x^2 + \alpha(y)$ must be a Morse germ. That is

$$\frac{\partial \alpha}{\partial y}(0) = 0, \quad \frac{\partial^2 \alpha}{\partial y^2}(0) \neq 0.$$

Hence $\alpha(y) = \pm \psi(y)^2$ for some coordinate transformation ψ in y -variable. Consequently by the coordinate transformation $(x, y) \mapsto (x, \psi(y))$ we obtain the normal form $(x^2 \pm y^2, y)$.

This completes the proof of Theorem B in the case $r = 2$.

3.3. Normal form in the case $r = 3$.

We shall detect the normal forms of types $(I, I, I)_{1,1}$, $(III, I, I)_1^{0,0}$, $(III, III, I)^{0,0}$ stated in Remark 1.3.

Case of type $(I, I, I)_{1,1}$. Let $(\mu_1, g_1; \mu_2, g_2; \mu_3, g_3)$ be of type $(I, I, I)_{1,1}$. Set $F_k = \mu_k \circ g_k^{-1}$ ($k = 1, 2, 3$). From the generic condition such that the map germ (F_1, F_2) is a fold, we may suppose that $F_1 = y, F_2 = y + x^2$. Moreover since (F_1, F_3) is a fold, F_3 can be written

$$F_3 = y + ax^2 + \theta(x, y),$$

where $a \neq 0$, $\theta \in \mathcal{M}_{x,y}^2$ with $\frac{\partial^2 \theta}{\partial x^2}(0) = 0$. Then to see the condition such that (F_3, F_2) is a fold, we consider the coordinate transformation $(x, y) \mapsto (x, F_2)$. In the new coordinate the map germ (F_3, F_2) has the following form: $(G(x, y), y)$, where $G(x, y) = y - x^2 + ax^2 + \theta(x, y)$. Thus the condition, (F_3, F_2) is a fold, means that $\frac{\partial G}{\partial x}(0) = 0$, $\frac{\partial^2 G}{\partial x^2}(0) \neq 0$. That is, $\frac{\partial \theta}{\partial x}(0) = 0$, $a \neq 1$ since $\frac{\partial^2 \theta}{\partial x^2}(0) = 0$. Therefore by adding this condition to the above F_3 , we obtain the normal form.

Case of type $(III, I, I)_1^{0,0}$. Let $(\mu_1, g_1; \mu_2, g_2; \mu_3, g_3)$ be of type $(III, I, I)_1^{0,0}$. We may suppose that (μ_1, g_1) is the normal form of type (III) and that $(\mu_2, g_2; \mu_3, g_3) = (F_2, id; F_3, id)$, where $F_k = x_k + \theta_k(x_k, y_k)$ ($k = 1, 2$), $\theta_k \in \mathcal{M}_{x_k, y_k}$ with $\frac{\partial \theta_k}{\partial x_k}(0) = 0$. We remark that the generic condition such that (F_2, F_3) is a fold is equivalent to the condition such that $F_2^{-1}(0)$ and $F_3^{-1}(0)$ have second order contact at 0. Hence by the direct calculation we get the condition θ_2, θ_3 so that we obtain the normal form stated in Remark 1.3.

Case of type $(III, III, I)^{0,0}$. Let $(\mu_1, g_1; \mu_2, g_2; \mu_3, g_3)$ be of type $(III, III, I)^{0,0}$. We may suppose that the form of $(\mu_1, g_1; \mu_2, g_2)$ is the normal form of type (III, III) and that $(\mu_3, g_3) = (F_3, id)$, where $F_3 = \mu_3 \circ g_3^{-1}$. Then by the generic condition, F_3 can be written $F_3 = ax_3 + by_3 + \theta(x, y)$, where $a \neq 0$, $b \neq 0$, $\theta \in \mathcal{M}_{x,y}^2$. Consider the following six coordinate transformations preserving g_1, g_2 and $g_3 = id$:

$(x_1, y_1) \mapsto (ax_1, \sqrt{b}y_1)$ in the source space of g_1 ,

$(x_2, y_2) \mapsto (\sqrt{a}x_2, by_2)$ in the source space of g_2 ,

$(x_3, y_3) \mapsto (ax_3, by_3)$ in the source space of g_3 ,

$(u, v) \mapsto (au, bv)$ in the common target space of g_1, g_2 and g_3 ,

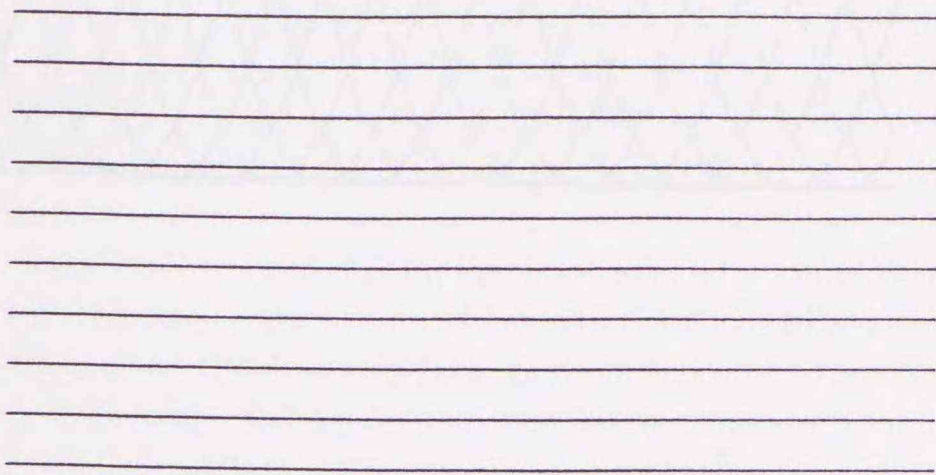
$\sqrt{b}, \sqrt{a}$ which are scalar multiples in the target spaces of μ_1, μ_2 respectively.

Then in these new coordinates we obtain the normal form.

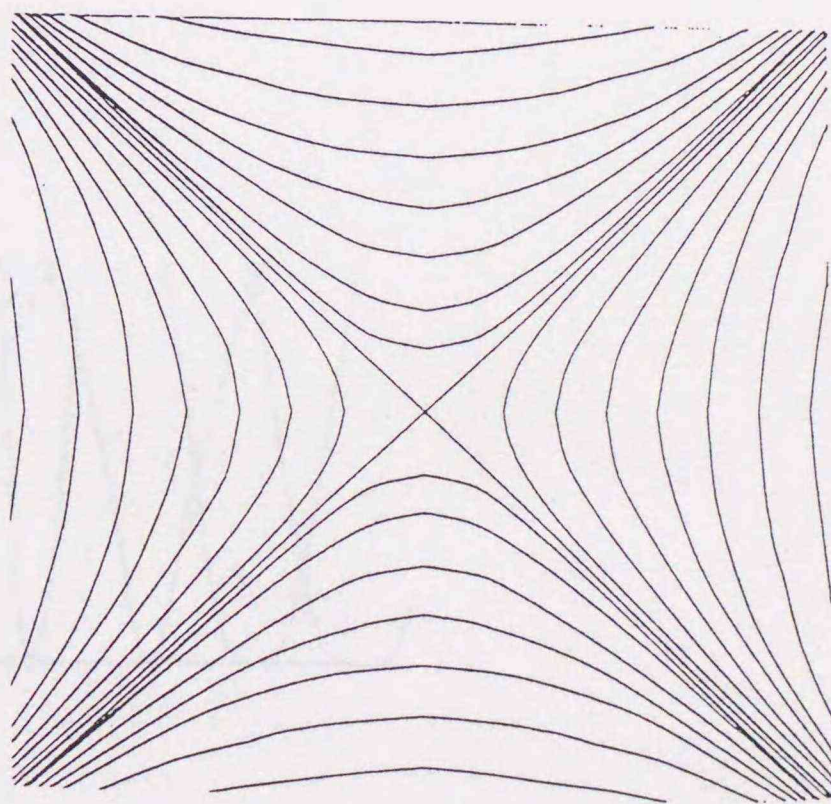
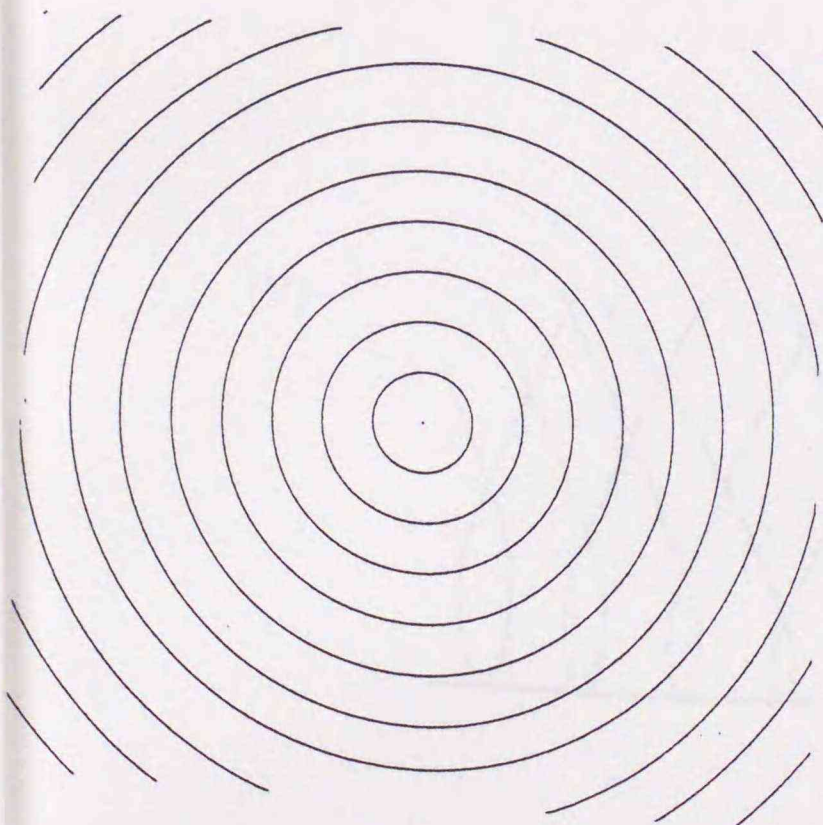
4. GEOMETRIC DESCRIPTION OF THE NORMAL FORMS

In order to understand our classification geometrically, let us describe the level curves $\{g_k(\mu_k^{-1}(c))\}$ and the discriminant set of g_k , $1 \leq k \leq r$.

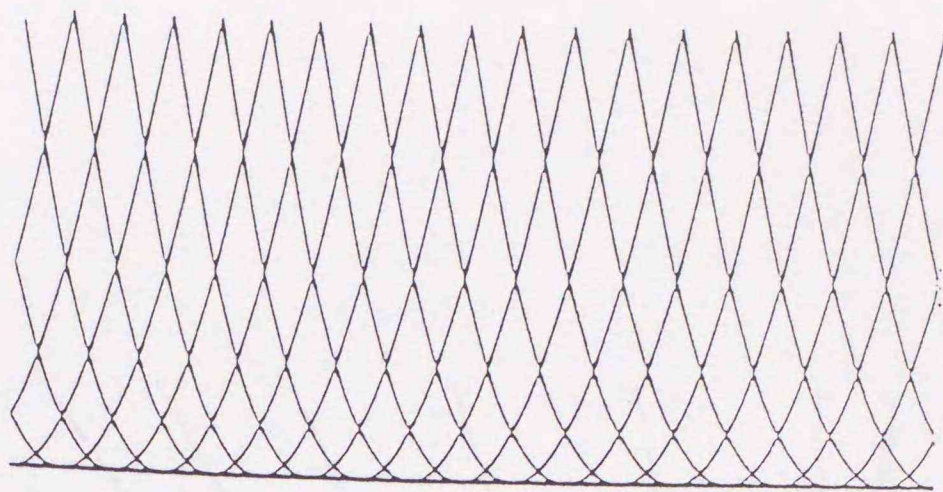
(I)



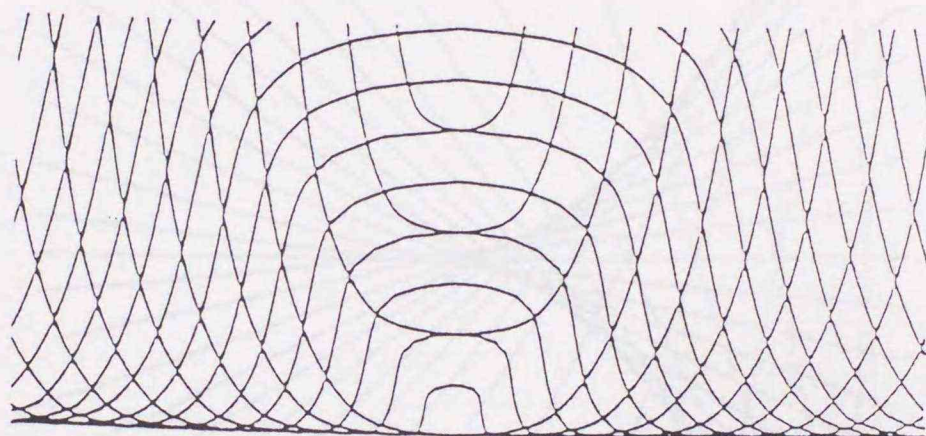
(II)



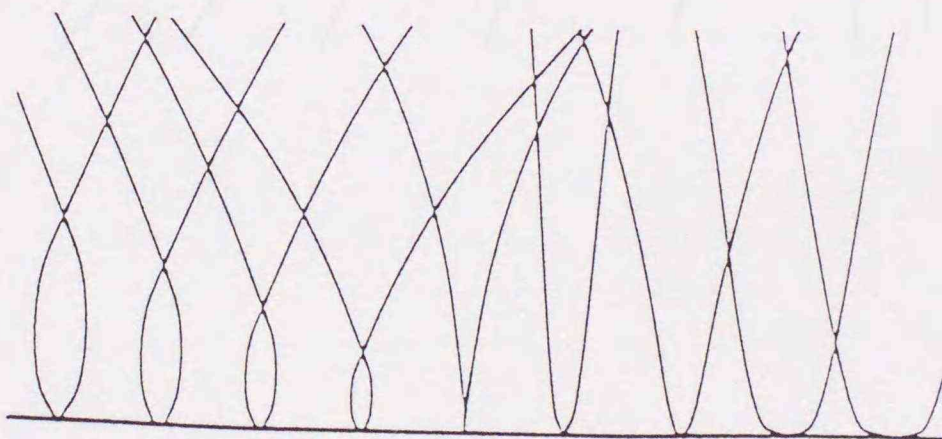
(III)



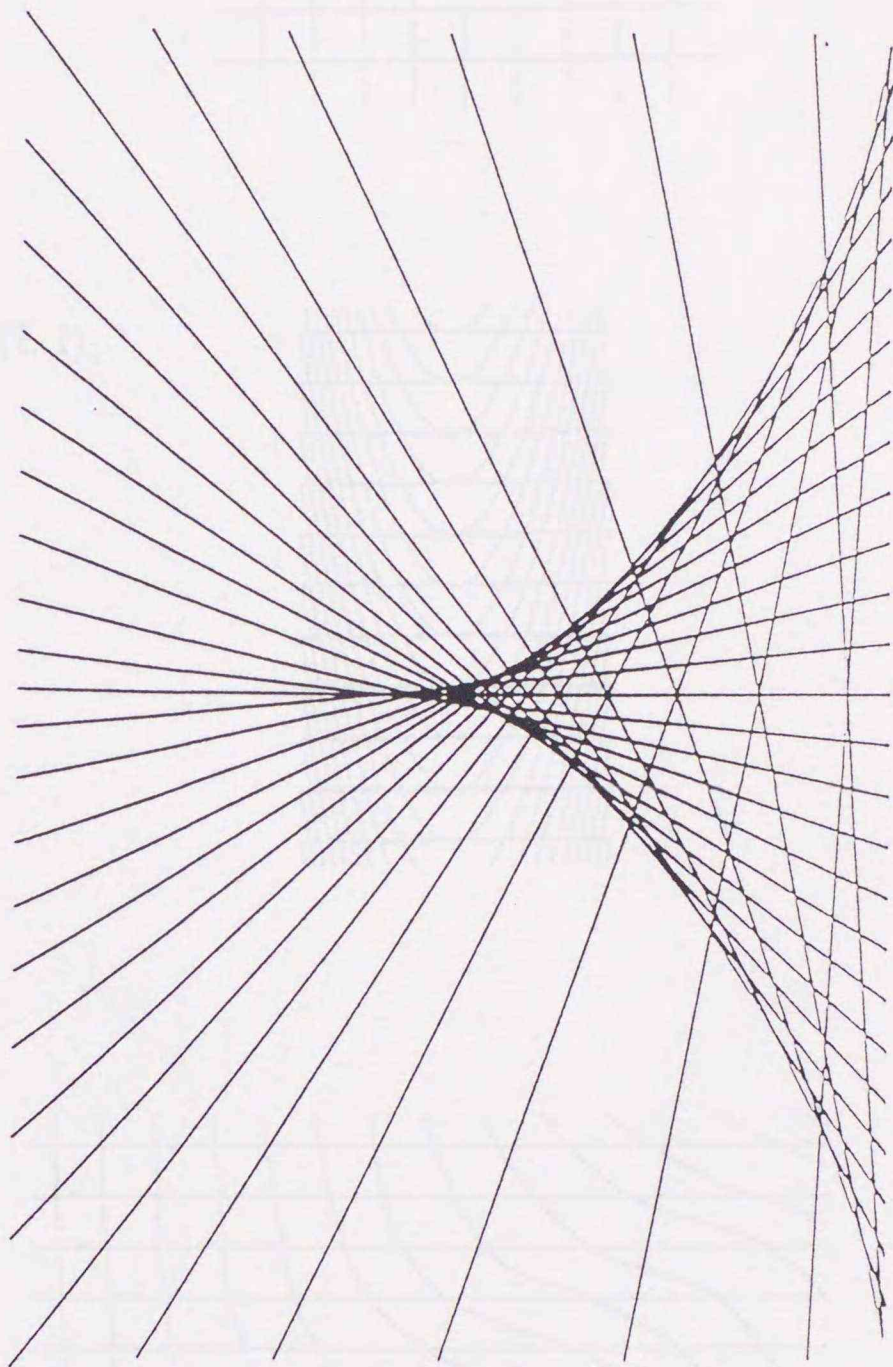
(IV)



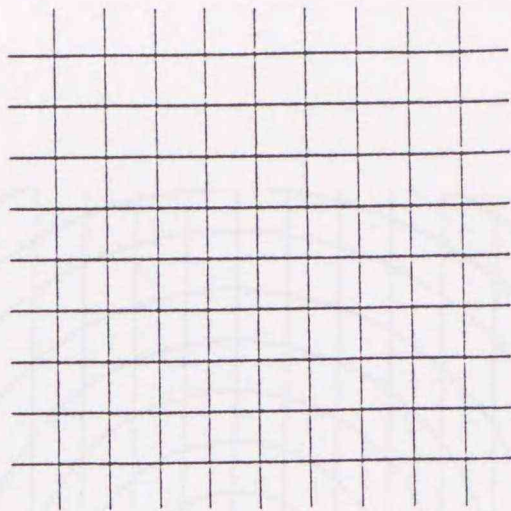
(V)



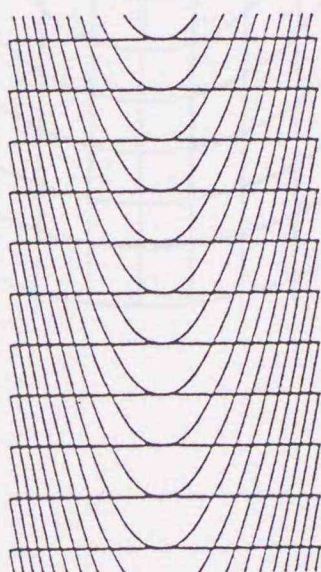
(VI)



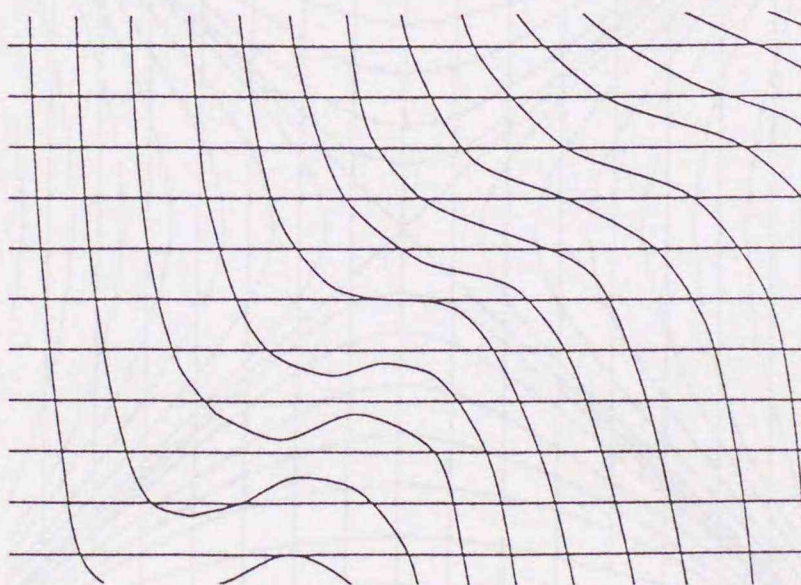
$(I, I)_0$



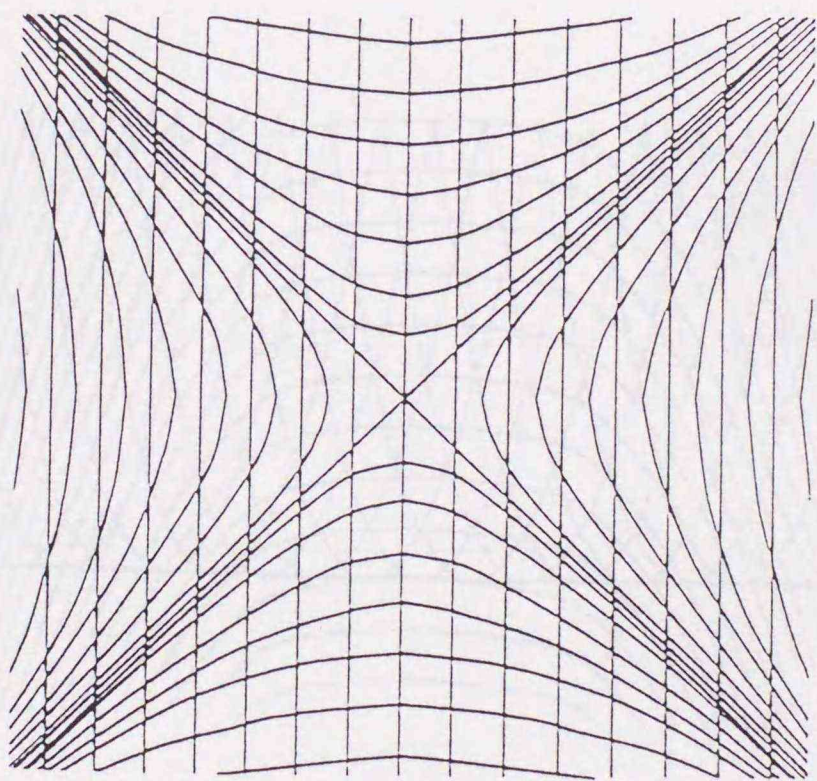
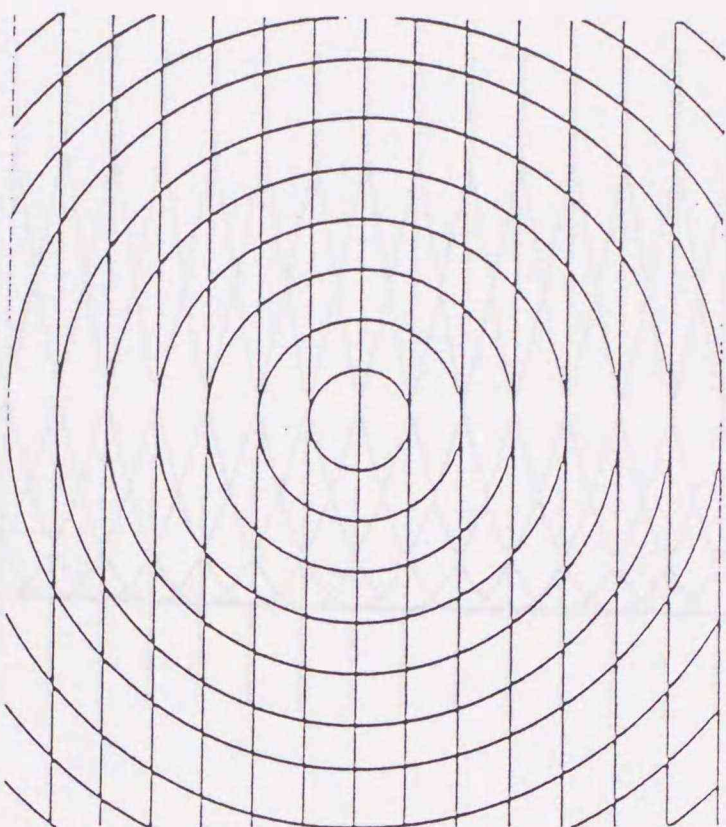
$(I, I)_1$



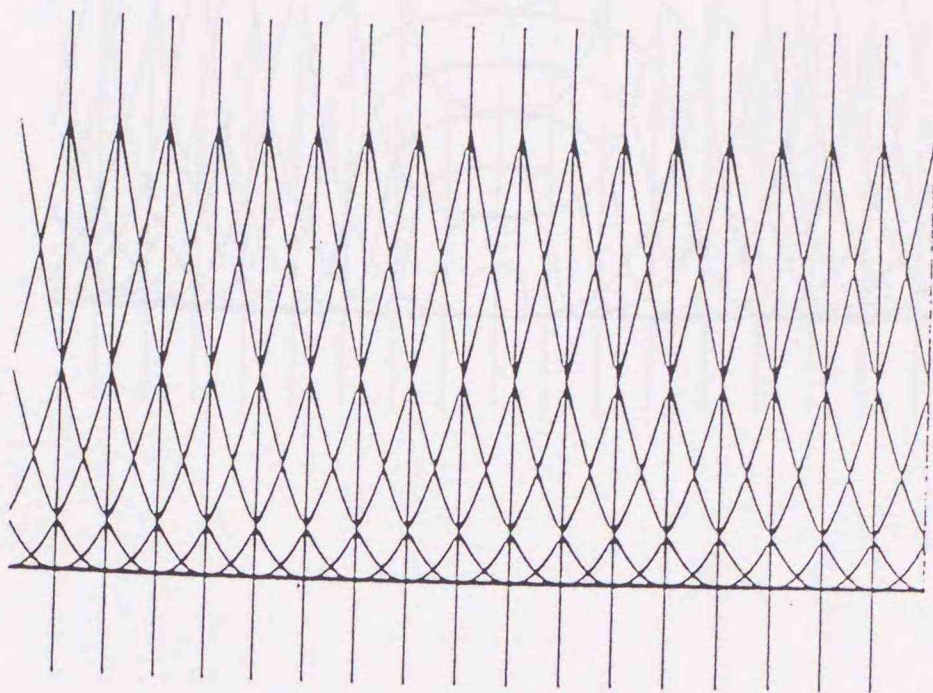
$(I, I)_2$



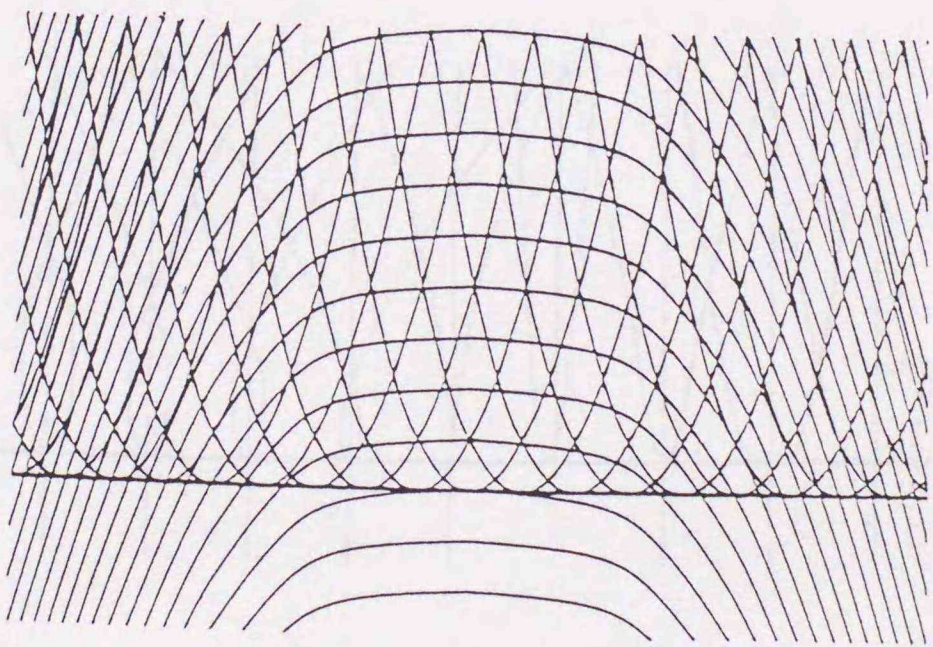
(II, I)



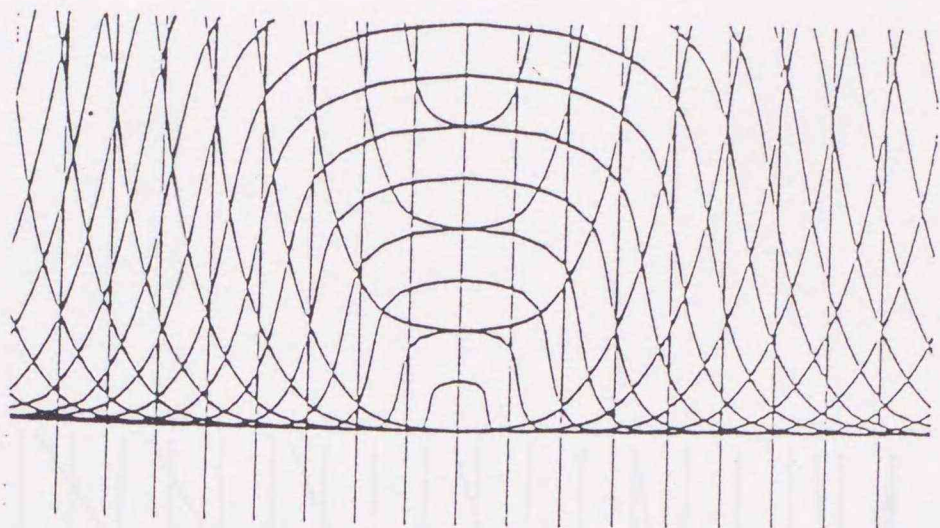
(III, I)⁰



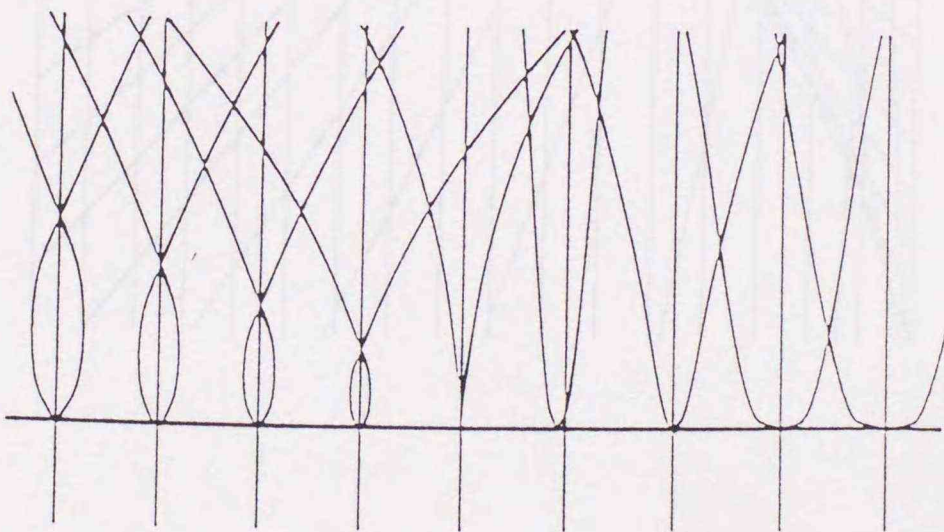
(III, I)¹



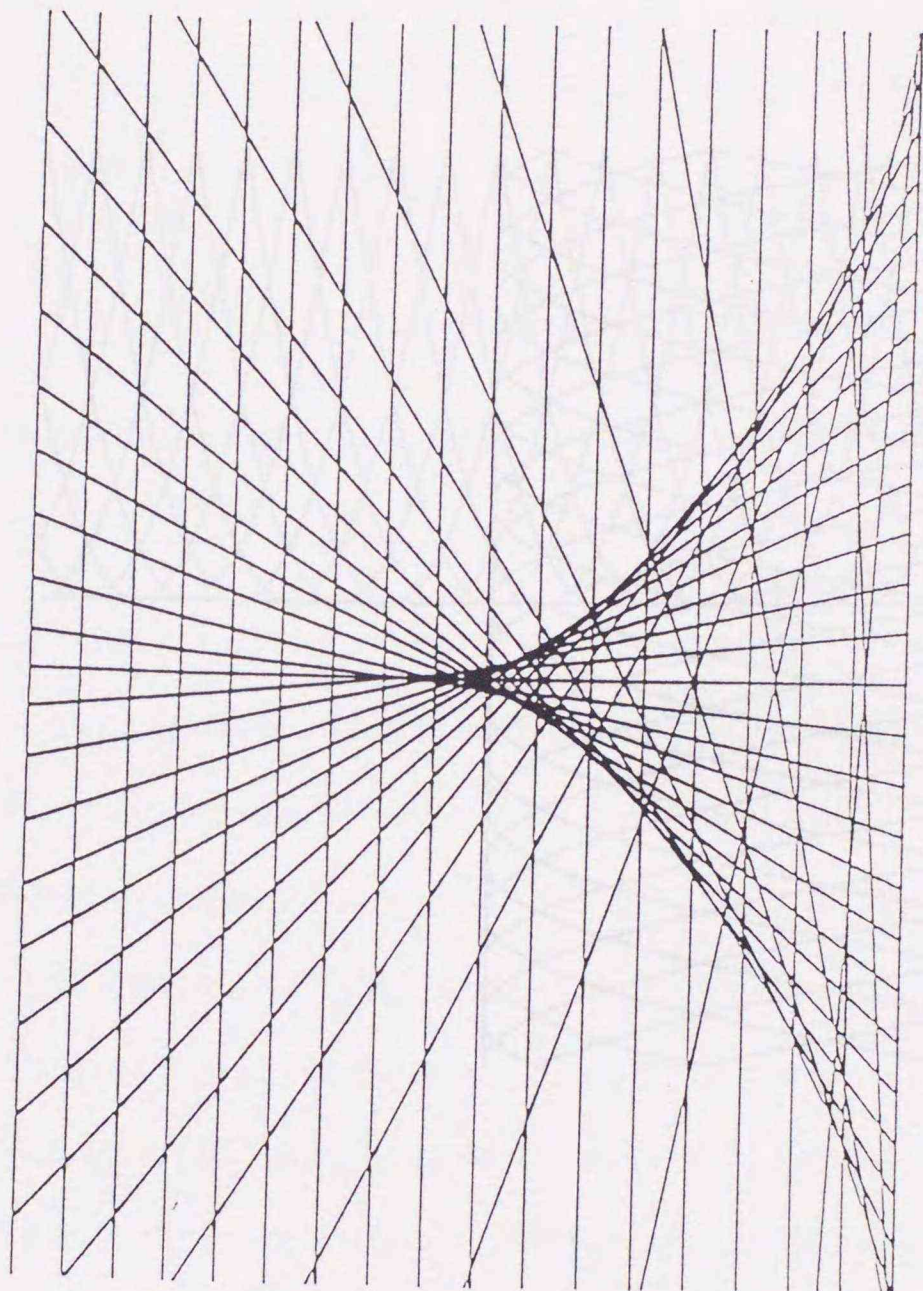
(IV. I)



(V. I)



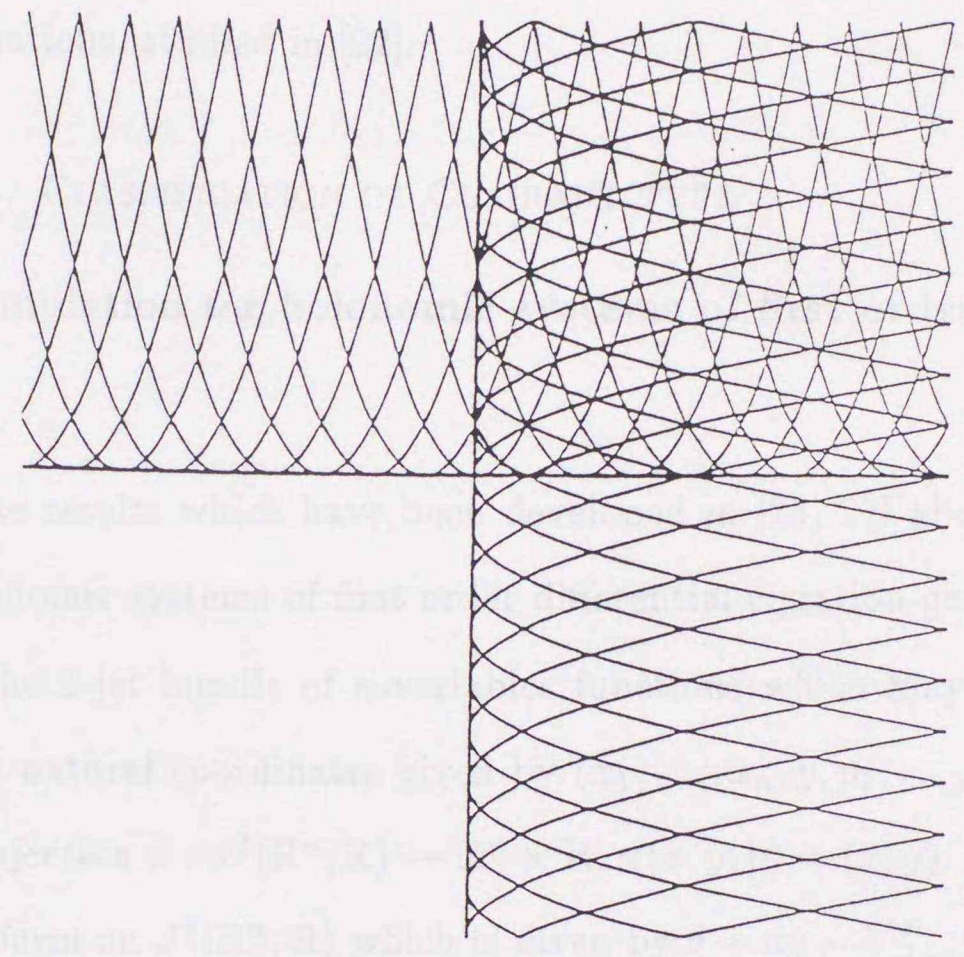
(VI, I)



HOLONOMIC SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

Chapter II contains two main results. Firstly we give a classification of holonomic systems of first order differential equations with constant coefficients, which is called the *normal form*. Secondly we show that any holonomic system of first order differential equations can be brought to normal form by a suitable change of variables.

(III, III)



PART II

HOLONOMIC SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

Part II contains two main results. Firstly we give a classification of holonomic systems of first order differential equations with classical complete solutions, which are called Clairaut type, studied in [26, 27]. Secondly we characterize functional moduli which appear in the normal forms for completely integrable first order ordinary differential equations, studied in [22].

5. CLASSIFICATION OF CLAIRAUT TYPE

5.1. Geometric formulation for holonomic systems of first order differential equations.

We shall recall some results which have been developed in [25, 27] about completely integrable holonomic systems of first order differential equation germs.

Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of n -variables functions which may be considered as $\mathbb{R}^{2n+1}$ with natural coordinates given by $(x_1, \dots, x_n, y, p_1, \dots, p_n)$. We have the canonical projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}^n \times \mathbb{R}$, $\pi(x, y, p) = (x, y)$. Let θ be the canonical contact form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$. A *holonomic system of first order differential equation germ* (or, briefly, a *holonomic system*) is defined to be an immersion germ $f : (\mathbb{R}^{n+1}, 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$. We also say that f is *completely integrable* if there exists a submersion germ $\mu : (\mathbb{R}^{n+1}, 0) \rightarrow \mathbb{R}$ such that $\langle d\mu \rangle_{\mathcal{E}_u} \supset \langle f^*\theta \rangle_{\mathcal{E}_u}$, where $\mathcal{E}_u$ is the ring of smooth function germs on $(\mathbb{R}^{n+1}, 0)$. Here, $u = (u_1, \dots, u_{n+1})$ are canonical coordinates of $(\mathbb{R}^{n+1}, 0)$. Then we call the pair $(\mu, f) : (\mathbb{R}^{n+1}, 0) \rightarrow \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$ a *holonomic system with complete integral*. We observe that $\pi \circ f(\mu^{-1}(t))$ is the graph of the solution in $\mathbb{R}^n \times \mathbb{R}$. If $\pi \circ f(\mu^{-1}(t))$ are non-singular map germs for each $t \in (\mathbb{R}, \mu(0))$, then $\{\pi \circ f(\mu^{-1}(t))\}_{t \in \mathbb{R}}$ is the family of graphs of a (classical) complete solution. We call

such a system a *holonomic system of Clairaut type*. Let (μ, g) be a pair of a map germ $g : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ and a submersion germ $\mu : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}, 0)$. Then the diagram

$$(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^{n+1}, 0) \xrightarrow{g} (\mathbb{R}^n \times \mathbb{R}, 0)$$

or briefly (μ, g) , is called a *(holonomic) integral diagram* if there exists a holonomic system $f : (\mathbb{R}^{n+1}, 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$ such that (μ, f) is an equation germ with complete integral and $\pi \circ f = g$, and we say that the integral diagram (μ, g) is *induced by f* . If f is a system of Clairaut type, then $(\mu, \pi \circ f)$ is called *of Clairaut type*. Furthermore we introduce an equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are *equivalent* (respectively, *strictly equivalent*) if the diagram

$$\begin{array}{ccccc} (\mathbb{R}, 0) & \xleftarrow{\mu} & (\mathbb{R}^{n+1}, 0) & \xrightarrow{g} & (\mathbb{R}^n \times \mathbb{R}, 0) \\ \lambda \downarrow & & \psi \downarrow & & \downarrow \phi \\ (\mathbb{R}, 0) & \xleftarrow{\mu'} & (\mathbb{R}^{n+1}, 0) & \xrightarrow{g'} & (\mathbb{R}^n \times \mathbb{R}, 0) \end{array}$$

commutes for some diffeomorphism germs λ , ψ and ϕ (respectively, $\lambda = id_{\mathbb{R}}$).

In [25] it has been defined an equivalence relation among systems under the group of point transformations in the sense of S. Lie and shown that two completely integrable holonomic systems f and f' are equivalent if and only if induced integral diagrams $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent for generic (μ, f) and (μ', f') .

Under the above formulation it has been given generic classifications of holonomic integral diagrams by the strictly equivalence in the case $n = 1$ [22], $1 \leq n \leq 3$ [25]. Particularly we shall give a generic classification of holonomic systems of Clairaut type by the equivalence (not the strictly equivalence) in the following subsection 5.2.

5.2. Statement of results.

Theorem C. *For a generic holonomic system of Clairaut type*

$$(\mu, f) : (\mathbb{R}^{n+1}, 0) \rightarrow \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}),$$

the integral diagram $(\mu, \pi \circ f)$ is strictly equivalent to one of germs in the following list :

DA_1 ;

$$\mu = u_{n+1},$$

$$g = (u_1, \dots, u_{n+1}).$$

DA_2 ;

$$\mu = u_{n+1} - \frac{1}{2}u_1,$$

$$g = (u_1, \dots, u_n, u_{n+1}^2).$$

DA_ℓ ($3 \leq \ell \leq n+1$) ;

$$\mu = u_{n+1},$$

$$g = (u_1, \dots, u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-1} u_i u_{n+1}^i).$$

$\widetilde{DA}_{n+2}$;

$$\mu_\alpha = u_{n+1} + \alpha \circ g \quad \text{for } \alpha \in \mathcal{M}_{x,y},$$

$$g = (u_1, \dots, u_n, u_{n+1}^{n+2} + \sum_{i=1}^n u_i u_{n+1}^i).$$

This theorem gives a generic classification of integral diagrams of Clairaut type under the strict equivalence, however, our purpose is to classify these subjects under the equivalence. We remark that two germs of types DA_ℓ ($1 \leq \ell \leq n+1$) and $\widetilde{DA}_{n+2}$ are not equivalent each other if two types are different. Thus the problem is reduced to classify germs which are contained in the family $\widetilde{DA}_{n+2}$ under the equivalence. This family is parametrized by function germs α which are called *functional moduli*. We shall characterize functional moduli relative to the equivalence.

For the purpose, we now adopt coordinates $(x_1, \dots, x_n, y)$ of $\mathbb{R}^n \times \mathbb{R}$ and define $D^n = \{(x_1, \dots, x_n, y) \in \mathbb{R}^n \times \mathbb{R} \mid F = \frac{\partial F}{\partial t} = \dots = \frac{\partial^n F}{\partial t^n} = 0 \text{ for some } t\}$, where $F(t, x_1, \dots, x_n, y) = t^{n+2} + x_1 t + \dots + x_n t^n - y$. Then we have the following characterization theorem.

Theorem D. Let (μ_α, g) be an integral diagram of $\widetilde{DA}_{n+2}$. Then for any α , there exists a function germ $\alpha' : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}, 0)$ such that

(1) (μ_α, g) is equivalent to $(\mu_{\alpha'}, g)$.

(2) $\alpha'|D^n = 0$.

This theorem is a generalization of Dufour's result in [16]. He has also shown that the uniqueness of functional moduli relative to the equivalence, so that we now consider a generalization of his uniqueness result. Define

$\Delta = \{(x_1, \dots, x_n, y) \in \mathbb{R}^n \times \mathbb{R} \mid \text{The } (n+2)\text{-degree algebraic equation}$

$$F_{(x_1, \dots, x_n, y)}(t) = 0 \text{ has } (n+2)\text{-real roots with multiplicity}\},$$

where $F_{(x_1, \dots, x_n, y)}(t) = F(t, x_1, \dots, x_n, y)$.

We say that α and α' are *equivalent as moduli* if there exists $a \in \mathbb{R} - \{0\}$ such that $a\alpha(x_1, \dots, x_n, y) = \alpha'(a^{n+1}x_1, a^n x_2, \dots, a^2 x_n, a^{n+2}y)$ for any $(x_1, \dots, x_n, y) \in \Delta$.

We remark that this definition of the equivalence among functional moduli is slightly different from Dufour's definition of it in [16]. If we adopt his definition, we cannot assert the necessity of the condition that functional moduli are equivalent. Then we correct the definition as the above.

Theorem E. Let (μ_α, g) and $(\mu_{\alpha'}, g)$ be integral diagrams of $\widetilde{DA}_{n+2}$ such that $\alpha|D^n = \alpha'|D^n = 0$. Then (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent if and only if α and α' are equivalent as moduli.

We emphasize that this theorem asserts that the equivalence classes of functional moduli α with $\alpha|D^n = 0$ are the complete invariant for generic classifications of holonomic systems of Clairaut type under the equivalence relation given by the group of point transformations.

We define $\mathcal{M}(D^n) = \{\alpha \in \mathcal{M}_{x,y} \mid \alpha|D^n = 0\}$ and $\mathfrak{M}(\widetilde{DA}_{n+2}) = \mathcal{M}(D^n)/\sim$, where $\sim$ denotes the equivalence relation as moduli. The above theorem asserts that the moduli space for generic holonomic systems of Clairaut type is $\mathfrak{M}(\widetilde{DA}_{n+2})$.

In §6 we shall prepare basic tools to prove the theorems including quick reviews of ([24, 25]). We shall give a proof of Theorem C in §7. Theorems D and E will be proved in §8 by using an analogous method of Dufour[16].

6. PREPARATIONS (INCLUDING QUICK REVIEWS OF [24, 25])

In this section we review some results on completely integrable systems and the theory of Legendrian unfoldings [24, 25] which will be used in later sections.

Firstly we introduce an equivalence relation among systems under the group of point transformations of $\mathbb{R}^n \times \mathbb{R}$. A *point transformation* ϕ on $\mathbb{R}^n \times \mathbb{R}$ is, by definition, a diffeomorphism of $\mathbb{R}^n \times \mathbb{R}$ onto itself.

To define a lift of ϕ , we give a contact manifold which is a fiberwise compactification of $J^1(\mathbb{R}^n, \mathbb{R})$. Let $\tilde{\pi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \rightarrow \mathbb{R}^n \times \mathbb{R}$ be a projective cotangent bundle over $\mathbb{R}^n \times \mathbb{R}$ which contains $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}^n \times \mathbb{R}$ as an affine part. Then we have a canonical contact lift $\hat{\phi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \rightarrow PT^*(\mathbb{R}^n \times \mathbb{R})$ of ϕ . Let $f, g : (\mathbb{R}^{n+1}, 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$ be equation germs. Following Lie, we say that f and g are *equivalent as equations* if there exist a diffeomorphism germ $\psi : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ and a point transformation $\phi : (\mathbb{R}^n \times \mathbb{R}, \pi(z_0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}, \pi(z_1))$ such that the lift $\hat{\phi}$ of ϕ satisfies that $\hat{\phi}(z_0) = z_1$ and $\hat{\phi} \circ f = g \circ \psi$, where $z_0 = f(0)$ and $z_1 = g(0)$. In [25] it has been shown the following theorem.

Theorem 6.1. *Let (μ, f) and $(\mu', f') : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}), 0 \times v)$ be holonomic system with complete integral such that the set of critical points of $\pi \circ f$ and $\pi \circ f'$ are closed sets without interior points. Then the followings are equivalent:*

- (1) *f and f' are equivalent as equations.*
- (2) *$(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent as integral diagrams.*

Secondly, we briefly review the theory of one-parameter Legendrian unfoldings in [24, 25]. We now consider the 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on the space. Let $(t, x_1, \dots, x_n)$ be the canonical coordinate on $\mathbb{R} \times \mathbb{R}^n$ and $(t, x_1, \dots, x_n, y, q, p_1, \dots, p_n)$ be the corresponding coordinate on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.

Then the canonical 1-form is given by $\Theta = dy - \sum_{i=1}^n p_i dx_i - q dt = \theta - q dt$. We also have the natural projection

$$\Pi : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \rightarrow (\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}$$

defined by $\Pi(t, x, y, q, p) = (t, x, y)$. We call the above 1-jet bundle an *unfolded 1-jet bundle*. Let (μ, f) be a holonomic system with complete integral. Then there exists a unique element $h \in \mathcal{E}_u$ such that $\ell^* \theta = h \cdot d\mu$. Define a map germ

$$\ell_{(\mu, f)} : (\mathbb{R}^{n+1}, 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\ell_{(\mu, f)}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u)).$$

Then we can easily show that $\ell_{(\mu, f)}$ is a Legendrian immersion germ. We call $\ell_{(\mu, f)}$ a *complete Legendrian unfolding associated with (μ, f)* . By the aid of the notion of Legendrian unfoldings, holonomic systems of Clairaut type are characterized as follows :

Proposition 6.2. [24]. *Let $(\mu, f) : (\mathbb{R}^{n+1}, 0) \rightarrow \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$ be a holonomic system with complete integral. Then (μ, f) is a holonomic system of Clairaut type if and only if $\ell_{(\mu, f)}$ is Legendrian non-singular.*

A complete Legendrian unfolding $\ell_{(\mu, f)}$ associated with (μ, f) is called a *Legendrian unfolding of Clairaut type* if $\ell_{(\mu, f)}$ is a holonomic system of Clairaut type.

Returning to the study of equations with complete integral, we now establish the notion of the genericity.

Let $U \subset \mathbb{R}^{n+1}$ be an open set. We denote by $\text{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$ the set of systems with complete integral $(\mu, f) : U \rightarrow \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$. We also define $L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ to be the set of complete Legendrian unfoldings $\ell_{(\mu, f)} : U \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.

These sets are topological spaces equipped with the Whitney C^∞ -topology. A subset of either spaces is said to be *generic* if it is an open dense subset in the space.

The genericity of a property of germs are defined as follows. Let P be a property of equation germs with complete integral $(\mu, f) : (\mathbb{R}^{n+1}, 0) \rightarrow \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$ (respectively, Legendrian unfoldings $\ell_{(\mu, f)} : (\mathbb{R}^{n+1}, 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$). For an open set $U \subset \mathbb{R}^r$, we define $\mathcal{P}(U)$ to be the set of $(\mu, f) \in \text{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$ (respectively, $\ell_{(\mu, f)} \in L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$) such that the germ at x whose representative is given by (μ, f) (respectively, $\ell_{(\mu, f)}$) has property P for any $x \in U$.

The property P is said to be *generic* if for some neighbourhood U of 0 in $\mathbb{R}^r$, the set $\mathcal{P}(U)$ is a generic subset in $\text{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$ (respectively, $L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_* : L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \rightarrow \text{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

defined by $(\Pi_1)_*(\ell_{(\mu, f)}) = \Pi_1 \circ \ell_{(\mu, f)} = (\mu, f)$, where $\Pi_1 : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$ is the canonical projection. Then it has been shown that the following fundamental theorem.

Theorem 6.3. [24, 25]. *The continuous map*

$$(\Pi_1)_* : L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \rightarrow \text{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

is a homeomorphism.

On the other hand, since $\ell_{(\mu, f)}$ is a Legendrian immersion germ, then there exists a generating family of $\mathcal{L}$ by the Arnol'd-Zakalyukin's theory ([2, 38]). In this case the generating family is naturally constructed by an one-parameter family of generating families associated with (μ, ℓ) . Let $F : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ such that $d_2 F|_{0 \times \mathbb{R}^n \times \mathbb{R}^k}$ is non-singular, where $d_2 F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \dots, \frac{\partial F}{\partial q_k}(t, x, q))$. We call F a *generalized phase family*. Then $C(F) = d_2 F^{-1}(0)$ is a smooth $(n+1)$ -manifold germ and $\pi_F : (C(F), 0) \rightarrow \mathbb{R}$ is a submersion germ, where $\pi_F(t, x, q) = t$. We call the submanifold $C(F)$ a *catastrophe set* of F . Define

$$\tilde{\Phi}_F : (C(F), 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and

$$\Phi_F : (C(F), 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t, x, q) = (t, x, F(t, x, q), \frac{\partial F}{\partial t}(t, x, q), \frac{\partial F}{\partial x}(t, x, q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that $(\tilde{\Phi}_F)^*\theta = \frac{\partial F}{\partial t}|_{C(F)} \cdot dt|_{C(F)} = 0$.

By the definition, Φ_F is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. By the same method of the theory of Arnol'd-Zakalyukin ([2, 38]), we can show the following proposition.

Proposition 6.4. *All Legendrian unfolding germs are constructed by the above method.*

Let (μ, f) be a holonomic system of Clairaut type. By Proposition 6.2, $\ell_{(\mu, f)}$ is Legendrian non-singular. Then we can choose a family of function germ

$$F : (\mathbb{R} \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$$

such that $\text{Image } j^1 F_t = f(\mu^{-1}(t))$ for any $t \in (\mathbb{R})$ and

$$j_1^1 F : (\mathbb{R} \times \mathbb{R}^n, 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$$

is an immersion germ, where $F_t(x) = F(t, x)$ and $j_1^1 F(t, x) = j^1 F_t(x)$. The fact that $j_1^1 F$ is an immersion leads us to the following equality :

$$\text{rank} \left(\frac{\partial F}{\partial t} \frac{\partial^2 F}{\partial t \partial x_j} \right) = 1.$$

In this case, we have $(C(F), 0) = (\mathbb{R} \times \mathbb{R}^n, 0)$ and

$$\Phi_F = j^1 F : (\mathbb{R} \times \mathbb{R}^n, 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}),$$

so that it is a complete Legendrian unfolding associated with $(\pi_{r-n}, j_1^1 F)$. Thus the generating family of a Legendrian unfolding of Clairaut type is given by the above germ.

7. PROOF OF THEOREM C

The main idea of the proof is to define an equivalence relation which can ignore functional moduli and to do everything in terms of generating families for Legendrian unfoldings like as those of in [25]. Let (μ, g) and (μ', g') be holonomic integral diagrams. Then (μ, g) and (μ', g') are $\mathcal{R}^+$ -equivalent if there exist a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Psi(t, x, y) = (t + \alpha(x, y), \psi(x, y))$ and a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ such that $\Psi \circ (\mu, g) = (\mu', g') \circ \Phi$. We remark that if (μ, g) and (μ', g') are $\mathcal{R}^+$ -equivalent by the above diffeomorphisms, then we have $\mu(u) + \alpha \circ g(u) = \mu' \circ \Phi(u)$ and $\psi \circ g(u) = g' \circ \Phi(u)$ for any $u \in (\mathbb{R}^{n+1}, 0)$. Thus the diagram $(\mu + \alpha \circ g, g)$ is strictly equivalent to (μ', g') .

We now define the corresponding equivalence relation among Legendrian unfoldings. Let $\ell_{(\mu, f)}, \ell_{(\mu', f')} : (\mathbb{R}^{n+1}, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0)$ be Legendrian unfoldings. We say that $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are *S.P⁺-Legendrian equivalent* (respectively, *S.P-Legendrian equivalent*) if there exist a contact diffeomorphism germ $K : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z'_0)$, a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ and a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0)) \rightarrow (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z'_0))$ of the form $\Psi(t, x, y) = (t + \alpha(x, y), \psi(x, y))$ (respectively, $\Psi(t, x, y) = (t, \psi(x, y))$) such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \mathcal{L} = \mathcal{L}' \circ \Phi$. It is clear that if $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are *S.P⁺-Legendrian equivalent* (respectively, *S.P-Legendrian equivalent*), then $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are $\mathcal{R}^+$ -equivalent (respectively, strictly equivalent). By Theorem 6.1 in [38], the converse is also true for generic (μ, f) and (μ', f') . The notion of the stability of Legendrian unfoldings with respect to *S.P⁺-Legendrian equivalence* (respectively, *S.P-Legendrian equivalence*) is analogous to the usual notion of the stability of Legendrian immersion germs with respect to Legendrian equivalence (cf. Part III in [2]).

On the other hand, we can interpret the above equivalence relation in terms of generating families. For the purpose, we use some notations and results in [2,

9, 20, 25, 38]. Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type. We say that $\tilde{F}$ and $\tilde{G}$ are $P-C^+$ -equivalent (respectively, $P-C$ -equivalent) if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t, x, y) = (t + \alpha(x, y), \phi_1(x, y), \phi_2(x, y))$ (respectively, $\Phi(t, x, y) = (t, \phi_1(x, y), \phi_2(x, y))$) such that $\langle F \circ \Phi \rangle_{\mathcal{E}_{(t, x, y)}} = \langle G \rangle_{\mathcal{E}_{(t, x, y)}}$ where $\langle G \rangle_{\mathcal{E}_{(t, x, y)}}$ is the ideal generated by G in $\mathcal{E}_{(t, x, y)}$. We also say that $\tilde{F}(t, x, y)$ is C^+ (respectively, C)-versal deformation of $f = F|_{\mathbb{R} \times 0}$ if

$$\mathcal{E}_t = \left\langle \frac{df}{dt} \right\rangle_{\mathbb{R}} + \langle f \rangle_{\mathcal{E}_t} + \left\langle \frac{\partial F}{\partial x_1} \Big|_{\mathbb{R} \times 0}, \dots, \frac{\partial F}{\partial x_n} \Big|_{\mathbb{R} \times 0}, 1 \right\rangle_{\mathbb{R}}$$

(respectively,

$$\mathcal{E}_t = \langle f \rangle_{\mathcal{E}_t} + \left\langle \frac{\partial F}{\partial x_1} \Big|_{\mathbb{R} \times 0}, \dots, \frac{\partial F}{\partial x_n} \Big|_{\mathbb{R} \times 0}, 1 \right\rangle_{\mathbb{R}}).$$

By the similar arguments like as those of Theorems 20.8 and 21.4 in [2], we can show the following :

Theorem 7.1. *Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type Φ_F, Φ_G respectively. Then*

- (1) Φ_F and Φ_G are $S.P^+$ (respectively, $S.P$)-Legendrian equivalent if and only if $\tilde{F}$ and $\tilde{G}$ are $P-C^+$ (respectively, C)-equivalent.
- (2) Φ_F is $S.P^+$ (respectively, $S.P$)-Legendrian stable if and only if $\tilde{F}$ is a $P-C^+$ (respectively, C)-versal deformation of $f = F|_{\mathbb{R} \times 0}$.

The following theorem is a corollary of Damon's general versality theorem in [9].

Theorem 7.2. *Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type such that Φ_F, Φ_G are $S.P^+$ (respectively, $S.P$)-Legendrian stable. Then Φ_F, Φ_G are $S.P^+$ (respectively, $S.P$)-Legendrian equivalent if and only if $f = F|_{\mathbb{R} \times 0}$, $g = G|_{\mathbb{R} \times 0}$ are C -equivalent (i.e. $\langle f \rangle_{\mathcal{E}_t} = \langle g \rangle_{\mathcal{E}_t}$).*

Then the classification theory of function germs by the C -equivalence is quite

useful for our purpose. For each function germ $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, we set

$$\begin{aligned} \mathcal{C}\text{-cod}(f) &= \dim_{\mathbb{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t}, \\ \mathcal{C}^+\text{-cod}(f) &= \dim_{\mathbb{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t} + \left\langle \frac{df}{dt} \right\rangle_{\mathbb{R}}, \\ \mathcal{K}\text{-cod}(f) &= \dim_{\mathbb{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t} + \left\langle \frac{df}{dt} \right\rangle_{\mathcal{E}_t}. \end{aligned}$$

Then we have the following well-known classification (cf. [20]).

Lemma 7.3. *Let $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ with $\mathcal{K}\text{-cod}(f) < \infty$. Then f is $\mathcal{C}$ -equivalent to the map germ $t^{\ell+1}$ for some $\ell \in \mathbb{N}$.*

By the direct calculation, we have

$$\begin{aligned} \mathcal{C}\text{-cod}(t^{\ell+1}) &= \ell + 1, \\ \mathcal{C}^+\text{-cod}(t^{\ell+1}) &= \ell. \end{aligned}$$

Thus we can easily determine $\mathcal{C}$ (respectively, $\mathcal{C}^+$)-versal deformations of the above germs by the usual method as follows :

The $\mathcal{C}$ -versal deformation :

$$t^{\ell+1} + \sum_{i=0}^{\ell} u_{i+1} t^i.$$

The $\mathcal{C}^+$ -versal deformation :

$$t^{\ell+1} + \sum_{i=0}^{\ell-1} u_{i+1} t^i.$$

We are now ready to prove Theorem C.

Proof of Theorem C. Let (μ, f) be a holonomic system of Clairaut type such that the corresponding Legendrian unfolding $\ell_{(\mu, f)}$ is S.P⁺-Legendrian stable. By the assumption, the generating family $\tilde{F}(t, x, y)$ of $\ell_{(\mu, f)}$ is $\mathcal{C}^+$ -versal deformation of $f = F|_{\mathbb{R} \times 0}$. By Theorem 7.2 and Lemma 7.3, $\tilde{F}(t, x, y)$ is P- $\mathcal{C}^+$ -equivalent to one of germs in the following list :

$$DA_{\ell} \ (1 \leq \ell \leq n+1) ; \ t^{\ell} + \sum_{i=1}^{\ell-1} x_i t^i + \sum_{j=\ell}^n x_j - y,$$

$$\widetilde{DA}_{n+2} ; t^{n+2} + \sum_{i=1}^n x_i t^i - y,$$

We now detect the corresponding normal forms of integral diagrams as follows :

DA_ℓ ($1 \leq \ell \leq n+1$) ; We can choose

$$F(t, x,) = t^\ell + \sum_{i=1}^{\ell-1} x_i t^i + \sum_{j=\ell}^n x_j$$

as a generalized phase family, so that

$$\begin{aligned} \Phi_F = (t, x_1, \dots, x_n, \pm t^\ell + \sum_{i=1}^{\ell-1} x_i t^i + \sum_{j=\ell}^n x_j, \\ \pm \ell t^{\ell-1} + \sum_{i=1}^{\ell-1} i x_i t^{i-1}, t, \dots, t^{\ell-1}, 1, \dots, 1). \end{aligned}$$

Then we can easily calculate that the corresponding integral diagram is strictly equivalent to

$$\mu = u_{n+1},$$

$$g = (u_1, \dots, u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-1} u_i u_{n+1}^i).$$

This is the normal form in the case of $3 \leq \ell \leq n+1$. If $\ell = 2$, we have

$$\mu = u_{n+1},$$

$$g = (u_1, \dots, u_n, u_{n+1}^2 + u_1 u_{n+1}).$$

We now define a transformation by

$$U_i = u_i \quad (i = 1, \dots, n), \quad U_{n+1} = u_{n+1} + \frac{1}{2} u_1,$$

then (μ, g) is strictly equivalent to

$$\mu = u_{n+1} - \frac{1}{2} u_1,$$

$$g = (u_1, \dots, u_n, (u_{n+1}^2 - \frac{1}{4} u_1^2)).$$

We also apply a transformation which is defined by

$$X_i = x_i \quad (i = 1, \dots, n), \quad Y = y + \frac{1}{4} x_1^2,$$

then we have the normal form.

$\widetilde{DA}_{n+2}$; In this case the generalized phase family is given by

$$F(t, x, q) = t^{n+2} + \sum_{i=1}^n x_i t^i.$$

By the same calculations as those of the case when $\ell \leq n+1$, we can show that the corresponding integral diagram is $\mathcal{R}^+$ -equivalent to

$$\mu = u_{n+1},$$

$$g = (u_1, \dots, u_n, \pm u_{n+1}^{n+2} + \sum_{i=1}^n u_i u_{n+1}^i).$$

Since the generalized phase family is $\mathcal{C}^+$ -versal and not $\mathcal{C}$ -versal, then the integral diagram is strictly equivalent to the normal form. This completes the proof of Theorem C.

8. PROOF OF THEOREMS D AND E

For the proof of Theorem D, we now introduce another equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be holonomic integral diagrams. Then (μ, g) and (μ', g') are *weak equivalent* if there exist a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Psi(t, x, y) = (\psi_1(t, x, y), \psi_2(x, y))$ and a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ such that $\Psi \circ (\mu, g) = (\mu', g') \circ \Phi$. By the definition, if either (μ, g) and (μ, g') are equivalent or $\mathcal{R}^+$ -equivalent, then these are weak equivalent.

There also exists the corresponding equivalence relation among Legendrian unfoldings. Let $\ell_{(\mu, f)}, \ell_{(\mu', f')} : (\mathbb{R}^{n+1}, 0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0)$ be Legendrian unfoldings. We say that $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are *P-Legendrian equivalent* if there exist a contact diffeomorphism germ $K : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z'_0)$, a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ and a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0)) \rightarrow (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z'_0))$ of the form $\Psi(t, x, y) = (\psi_1(t, x, y), \psi_2(x, y))$ such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \mathcal{L} = \mathcal{L}' \circ \Phi$. It is clear that

if $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are P -Legendrian equivalent, then $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are weak equivalent. By Theorem 1.1 in [38], the converse is also true for generic (μ, f) and (μ', f') . We can also define the notion of the stability of Legendrian unfoldings with respect to P -Legendrian equivalence exactly the same way as in the previous section. The corresponding equivalence relation among generating families are also given as follows: Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type. We say that $\tilde{F}$ and $\tilde{G}$ are P - $\mathcal{K}$ -equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t, x, y) = (\phi_1(t, x, y), \phi_2(x, y), \phi_3(x, y))$ such that $\langle F \circ \Phi \rangle_{\mathcal{E}_{(t, x, y)}} = \langle G \rangle_{\mathcal{E}_{(t, x, y)}}$. We also say that $\tilde{F}(t, x, y)$ is $\mathcal{K}$ -versal deformation of $f = F|_{\mathbb{R} \times 0}$ if

$$\mathcal{E}_t = \left\langle \frac{df}{dt}, f \right\rangle_{\mathcal{E}_t} + \left\langle \frac{\partial F}{\partial x_1} |_{\mathbb{R} \times 0}, \dots, \frac{\partial F}{\partial x_n} |_{\mathbb{R} \times 0}, 1 \right\rangle_{\mathbb{R}}.$$

By the similar arguments like as those of Theorems 7.1 and 7.2, we can show the following:

Theorem 8.1. *Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \rightarrow (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type Φ_F, Φ_G respectively. Then*

- (1) Φ_F and Φ_G are P -Legendrian equivalent if and only if $\tilde{F}$ and $\tilde{G}$ are P - $\mathcal{K}$ -equivalent.
- (2) Φ_F is P -Legendrian stable if and only if $\tilde{F}$ is a P - $\mathcal{K}$ -versal deformation of $f = F|_{\mathbb{R} \times 0}$.
- (3) Suppose that $\tilde{F}, \tilde{G}$ be generating families of P -Legendrian stable Legendrian unfoldings Φ_F, Φ_G of Clairaut type. Then Φ_F, Φ_G are P -Legendrian equivalent if and only if $f = F|_{\mathbb{R} \times 0}$, $g = G|_{\mathbb{R} \times 0}$ are $\mathcal{K}$ -equivalent.

Since $f = F|_{\mathbb{R} \times 0}$ is a function germ of one-variable, we have $\langle \frac{df}{dt} \rangle_{\mathbb{R}} + \langle f \rangle_{\mathcal{E}_t} = \langle \frac{df}{dt}, f \rangle_{\mathcal{E}_t}$, so that $\tilde{F}$ is a C^+ -versal deformation of f if and only if it is a $\mathcal{K}$ -versal deformation. By Theorems 7.1 and 8.1, a Legendrian unfolding of Clairaut type is $S.P^+$ -Legendrian stable if and only if it is P -Legendrian stable. Then the following lemma is a corollary of Theorem C.

Lemma 8.2. *Let (μ', g') be an integral diagram of Clairaut type which is equivalent to (μ_α, g) for some $\alpha \in \mathfrak{M}_{(x,y)}$. Then there exists $\alpha' \in \mathfrak{M}_{(x,y)}$ such that (μ', g') and $(\mu_{\alpha'}, g)$ are strictly equivalent.*

The main idea of the proof is a generalization of methods studied by Dufour [16]. Let (μ, g) be an integral diagram which is equivalent to (μ_α, g) for some $\alpha \in \mathcal{M}_{x,y}$. Define

$$S_\mu = \{(c_1, \dots, c_{n+2}) \mid \text{All of } c_1, \dots, c_{n+2} \text{ are different and } \bigcap_{j=1}^{n+2} g(\mu^{-1}(c_j)) \neq \emptyset\}.$$

We call it the *Carneiro hypersurface* of (μ, g) . We can detect a defining equation of the closure of the Carneiro hypersurface of (μ_α, g) .

Proposition 8.3. *The closure $\overline{S_{\mu_\alpha}}$ of the Carneiro surface of (μ_α, g) in $\mathbb{R}^{n+2}$ is defined by*

$$s_1 = (n+2)\alpha(x_1, \dots, x_n, y),$$

where

$$s_j = \sum_{\{i_1, \dots, i_j\} \subset \{1, \dots, n+2\}} c_{i_1} \cdots c_{i_j}$$

for $j = 1, \dots, n+2$ and

$$\begin{aligned} x_{n+2-k} &= \sum_{i=0}^{k-1} b_i s_{k-i} s_1^i \quad (k = 2, \dots, n+1) \\ y &= \sum_{i=0}^{n+1} b_i s_{n+2-i} s_1^i \end{aligned}$$

for some real numbers b_i which depend on n, k .

Proof. Let $t_1, \dots, t_{n+2}$ with $t_1 \leq \dots \leq t_{n+2}$ be the $(n+2)$ -real roots of $F_{(x,y)}(t) = 0$ for each $(x, y) \in \Delta$. We define map germs $c_j: \Delta \rightarrow (\mathbb{R}, 0)$ by $c_j(x, y) = t_j + \alpha(x, y)$, where $j = 1, \dots, n+2$. Then we set $\mathcal{C} = (c_1, \dots, c_{n+2}): \Delta \rightarrow (\mathbb{R}^{n+2}, 0)$. Moreover let $s = (s_1, \dots, s_{n+2}): (\mathbb{R}^{n+2}, 0) \rightarrow (\mathbb{R}^{n+2}, 0)$ be a map germ defined by s_j = elementary symmetric polynomial of degree j . By the relation between roots and coefficients

we have

$$\begin{aligned} t_1 + \cdots + t_{n+2} &= 0 \\ \sum_{\{i_1, \dots, i_k\} \subset \{1, \dots, n+2\}} t_{i_1} \cdots t_{i_k} &= (-1)^k x_{n+2-k} \quad (k = 2, \dots, n+1) \\ t_1 \cdots t_{n+2} &= (-1)^{n+2} (-y). \end{aligned}$$

Putting $s \circ \mathcal{C}(x, y) = (s_1, \dots, s_{n+2})$, we have

$$\begin{aligned} s_1 &= (n+2)\alpha(x, y) \\ s_k &= \sum_{i=2}^k \binom{n+2-i}{k-i} (-1)^i x_{n+2-i} \alpha(x, y)^{k-i} + \binom{n+2}{k} \alpha(x, y)^k \\ s_{n+2} &= (-1)^{n+2} (-y) + \sum_{i=2}^{n+1} (-1)^i x_{n+2-i} \alpha(x, y)^{n+2-i} + \alpha(x, y)^{n+2}, \end{aligned}$$

where $k = 2, \dots, n+1$, $\binom{m}{r}$ denotes the binomial coefficient and we presumably use $\binom{m}{0} = 1$.

From these formulae and by direct calculations we can inductively show that x_{n+2-k}, y are uniquely expressed as in the statement of this proposition. Furthermore we can show that $\overline{S_{\mu_\alpha}} = \cup_{\sigma \in \mathfrak{S}_{n+2}} \text{Image}(c_{\sigma(1)}, \dots, c_{\sigma(n+2)})$, where $\mathfrak{S}_{n+2}$ is the symmetric group of degree $n+2$. It follows that $\overline{S_{\mu_\alpha}}$ is included by the hypersurface which is defined by the equation $s_1 = (n+2)\alpha(x, y)$, where (x, y) is the preimage of $(s_1, \dots, s_{n+2})$ by the map $\text{ger } s \circ \mathcal{C}$ and s_j is the elementary symmetric polynomial of degree j with respect to $c_1, \dots, c_{n+2}$. Therefore we obtain a defining equation of $\overline{S_{\mu_\alpha}}$. This completes the proof. $\square$

By Proposition 8.3 we can easily have

Lemma 8.4. *Under the same hypothesis as in Proposition 8.3, $\alpha|D^n = 0$ if and only if*

$$\left. \begin{aligned} (c_1, \dots, c_{n+2}) &\in \overline{S_{\mu_\alpha}} \\ c_2 &= \cdots = c_{n+2} \end{aligned} \right\} \Rightarrow c_1 = -(n+1)c_{n+2}.$$

We are now ready to prove Theorem D.

Proof of Theorem D. By Proposition 8.3 and the implicit function theorem, there exists a function germ $h: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ whose form is given by $h(c) = -(n+1)c + o(|c|)$ such that

$$\begin{aligned} \overline{S_{\mu_\alpha}} \cap \{(c_1, c_2, \dots, c_{n+2}) | c_2 = \dots = c_{n+2}\} \\ = \{(h(c_{n+2}), c_2, \dots, c_{n+2}) | c_2 = \dots = c_{n+2}\}. \end{aligned}$$

Applying a linearization theorem of Sternberg [34], there exists a diffeomorphism germ $\lambda: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $\lambda \circ h \circ \lambda^{-1}(c) = -(n+1)c$. Thus we have $\lambda(c_1) + \lambda(c) + \dots + \lambda(c) = 0$ for any $(c_1, c, \dots, c) \in \overline{S_{\mu_\alpha}}$. By this relation and the fact that the $(n+2)$ -fold product of λ sends $\overline{S_{\mu_\alpha}}$ onto $\overline{S_{\lambda \circ \mu_\alpha}}$, we can show that if $(c_1, \dots, c_{n+2}) \in \overline{S_{\lambda \circ \mu_\alpha}}$ with $c_2 = \dots = c_{n+2}$ then $c_1 = -(n+1)c_{n+2}$. Furthermore, by Lemma 8.2 there exists $\alpha' \in \mathcal{M}_{x,y}$ such that the following diagram commutes

$$\begin{array}{ccccc} (\mathbb{R}, 0) & \xleftarrow{\lambda \circ \mu_\alpha} & (\mathbb{R}^{n+1}, 0) & \xrightarrow{g} & (\mathbb{R}^n \times \mathbb{R}, 0) \\ \parallel & & \downarrow \psi & & \downarrow \phi \\ (\mathbb{R}, 0) & \xleftarrow{\mu_{\alpha'}} & (\mathbb{R}^{n+1}, 0) & \xrightarrow{g} & (\mathbb{R}^n \times \mathbb{R}, 0). \end{array}$$

The commutativity of the diagram implies that $\overline{S_{\lambda \circ \mu_\alpha}} = \overline{S_{\mu_{\alpha'}}}$. Thus by Lemma 8.4 we have $\alpha'|D^n = 0$. This completes the proof of Theorem D. $\square$

Proof of Theorem E. Firstly we prove the necessity of the condition that functional moduli are equivalent. Assume that (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent, where $\alpha|D^n = \alpha'|D^n = 0$. Let us assume that $\lambda \circ \mu_\alpha = \mu_{\alpha'} \circ \psi$ and $\phi \circ g = g \circ \psi$ for some diffeomorphism germs $\lambda: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$, $\psi: (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ and $\phi: (\mathbb{R}^n \times \mathbb{R}, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$.

We determine the form of λ . Since $(\lambda \times \dots \times \lambda)(\overline{S_{\mu_\alpha}}) = \overline{S_{\mu_{\alpha'}}}$ and by Lemma 8.4 we have $\lambda(-(n+1)c) = -(n+1)\lambda(c)$, so that we inductively have

$$\lambda(c) = (-(n+1))^p \lambda\left(\frac{c}{(-(n+1))^p}\right)$$

for all integers p . By the same arguments as those of Dufour ([15] p.466, [16] p.231, [17] p.274) we can show that the form of λ is given by $\lambda(c) = ac$ ($a \neq 0$). Also we determine the form of $\phi|_{\Delta}$. Let $\mathcal{C}$ be the map germ as in Proposition 8.3. By the assumption we have $(\lambda \times \cdots \times \lambda) \circ \mathcal{C} = (c_{\sigma(1)}, \dots, c_{\sigma(n+2)}) \circ \phi|_{\Delta}$ for some $\sigma \in \mathfrak{S}_{n+2}$. We recall the fact that $s \circ \mathcal{C}$ is invertible and the inverse map germ is given as in Proposition 8.3. Thus we have $\phi(x, y) = (a^{n+1}x_1, a^nx_2, \dots, a^2x_n, a^{n+2}y)$ on Δ , where $a \in \mathbb{R} - \{0\}$ such that $\lambda(c) = ac$. Now for each $(x, y) \in \Delta$ write $\mathcal{C}(x, y) = (c_1, \dots, c_{n+2})$. Then by Proposition 8.3 we have $c_1 + \cdots + c_{n+2} = (n+2)\alpha(x, y)$ and $ac_1 + \cdots + ac_{n+2} = (n+2)\alpha'(\phi(x, y))$. Hence $a\alpha(x, y) = \alpha'(\phi(x, y))$ on Δ . Therefore we obtain the necessity.

Conversely suppose that $a\alpha(x_1, \dots, x_n, y) = \alpha'(a^{n+1}x_1, a^nx_2, \dots, a^2x_n, a^{n+2}y)$ on Δ for some $a \in \mathbb{R} - \{0\}$. Then we have the following commutative diagram

$$\begin{array}{ccccc} (\mathbb{R}, 0) & \xleftarrow{\mu_{\alpha}} & g^{-1}(\Delta) & \xrightarrow{g} & \Delta \\ \lambda \downarrow & & \downarrow \psi & & \downarrow \phi \\ (\mathbb{R}, 0) & \xleftarrow{\mu_{\alpha'}} & g^{-1}(\Delta) & \xrightarrow{g} & \Delta, \end{array}$$

where

$$\lambda(c) = ac,$$

$$\psi(u_1, \dots, u_{n+1}) = (a^{n+1}u_1, a^nu_2, \dots, a^2u_n, au_{n+1}),$$

$$\phi(x_1, \dots, x_n, y) = (a^{n+1}x_1, a^nx_2, \dots, a^2x_n, a^{n+2}y).$$

The proof is based on the following result of Carneiro [7].

Theorem 8.5. *Let (μ_{α}, g) and $(\mu_{\alpha'}, g)$ be two integral diagrams of $\widetilde{DA}_{n+2}$. Then (μ_{α}, g) and $(\mu_{\alpha'}, g)$ are equivalent if and only if there exists a diffeomorphism germ $\lambda: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $(\lambda \times \cdots \times \lambda)(\overline{S_{\mu_{\alpha}}}) = \overline{S_{\mu_{\alpha'}.}$*

We remark that the commutativity of the above diagram implies that the $(n+2)$ -fold product of λ sends $\overline{S_{\mu_{\alpha}}}$ onto $\overline{S_{\mu_{\alpha'}.}$ Thus it follows from Theorem 8.5 that (μ_{α}, g) and $(\mu_{\alpha'}, g)$ are equivalent. This completes the proof of Theorem E.

9. FUNCTIONAL MODULI FOR FIRST ORDER

ORDINARY DIFFERENTIAL EQUATIONS

Under the same formulation in subsection 5.1, in [22] it has been studied a generic classification of completely integrable holonomic systems in the case $n = 1$, that is the case of ordinary differential equations. They showed that any generic integral diagram in the case $n = 1$, (μ, g) is strictly equivalent to one of the following germs:

- (1) $\mu = v, g = (u, v)$.
- (2) $\mu = v - \frac{1}{3}u^3, g = (u^2, v)$.
- (3) $\mu = v - \frac{1}{2}u, g = (u, v^2)$.
- (4) $\mu_\alpha = \frac{3}{4}u^4 + \frac{1}{2}u^2v + \alpha \circ g, g = (u^3 + uv, v)$ for $\alpha \in \mathcal{M}_{x,y}$ with $\frac{\partial \alpha}{\partial y}(0) = \pm 1$.
- (5) $\mu_\alpha = v + \alpha \circ g, g = (u, v^3 + uv)$ for $\alpha \in \mathcal{M}_{x,y}$.
- (6) $\mu_\alpha = \frac{1}{2}v^2 + \alpha \circ g, g = (u, v^3 + uv^2)$ for $\alpha \in \mathcal{M}_{x,y}$ with $\frac{\partial \alpha}{\partial x}(0) = \pm 1$.

A function germ α which appear in the normal forms (4), (5), (6) is called a *functional modulus* of each normal form.

Our concern is to classify generic integral diagrams by the equivalence (not the strict equivalence). We remark that the germs (1) to (6) are not equivalent each other. Hence we shall classify the families of the germs (4), (5), (6) by the equivalence. On the other hand, the normal form (5) is one of the normal forms for generic divergent diagrams studied by Dufour [15] and also one of the normal forms for Clairaut type in the case $n = 1$ as was described in §5. Thus we shall classify the normal forms (4), (6) by the equivalence, which is an analogy of Dufour's result for (5).

Denote by $\mathcal{A}_y$ (resp. $\mathcal{A}_x$) as the set of functional moduli of (4) (resp. (6)).

We now state our results.

Theorem F. *Let (μ_α, g) be an integral diagram of (4). Then, for any $\alpha \in \mathcal{A}_y$ there exists an $\alpha' \in \mathcal{A}_y$ such that*

- (i) (μ_α, g) is equivalent to $(\mu_{\alpha'}, g)$,

(ii) $\alpha'(0, y) = \frac{\partial \alpha}{\partial y}(0, 0)y + \frac{1}{2}\chi_\alpha y^2$ for all $y \leq 0$,
 where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0, 0)$.

Theorem G_j. ($j=1,2$) Let (μ_α, g) be an integral diagram of (6). Then, for any $\alpha \in \mathcal{A}_x$ there exists an $\alpha' \in \mathcal{A}_x$ such that

(i) (μ_α, g) is equivalent to $(\mu_{\alpha'}, g)$,

(ii) $\alpha' = \delta x + \frac{1}{2}\chi_\alpha x^2$ on D_j ,

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0, 0)$, $\delta = \pm 1$, $D_1 = \{(x, y) | y = 0\}$ and $D_2 = \{(x, y) | 27y = 4x^3\}$.

Dufour also has shown uniqueness of functional modulus. However the germs (4), (6) are so complicated that we can not obtain the uniqueness result in this paper.

10. PROOF OF THEOREM F

The proof is based on the following two propositions.

Proposition 10.1.(Takens[35]). Let $\psi: \mathbb{R}, 0 \rightarrow \mathbb{R}, 0$ be a diffeomorphism such that ψ^2 has the form $\psi^2(x) = x + x^k F(x)$ with $F(0) \neq 0$ and $k \geq 2$. Then there is an orientation preserving diffeomorphism $\lambda: \mathbb{R}, 0 \rightarrow \mathbb{R}, 0$ such that, in some neighbourhood of $0 \in \mathbb{R}$,

$$\lambda \circ \psi \circ \lambda^{-1}(x) = \pm x + \delta x^k + \chi x^{2k-1}$$

where $\delta = \pm 1$ and $\chi \in \mathbb{R}$.

The following is implicitly proved in [22].

Proposition 10.2. Let (μ', g') be an integral diagram which is equivalent to (μ_α, g) of (4) (resp. (6)) for some $\alpha \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$). Then (μ', g') is strictly equivalent to $(\mu_{\alpha'}, g)$ of (4) (resp. (6)) for some $\alpha' \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$).

For each case F, G₁, G₂ we will define a map germ $\gamma_\alpha: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$, as follows. In the case F : Put $\Delta = \{(x, y) | 27x^2 + 4y^3 < 0\}$. Note that $\Delta = \{(x, y) | \#(g^{-1}(x, y)) = 3\}$. Let (u_1, y) , (u_2, y) and (u_3, y) be the preimages of (x, y) by

g for each $(x, y) \in \Delta$ near $(0, 0)$, where $u_j = u_j(x, y)$ ($j = 1, 2, 3$) are three real roots of the equation $U^3 + yU - x = 0$ and ordered by $u_1 < u_2 < u_3$. For each $\alpha \in \mathcal{A}_y$ set $c_j(x, y) = \mu_\alpha(u_j, y)$. We see clearly $c_1(0, y) = c_3(0, y) = \frac{1}{4}y^2 + \alpha(0, y)$, $c_2(0, y) = \alpha(0, y)$ for any $(0, y) \in \Delta$. We set $\gamma_\alpha(y) = (\alpha(0, y), \frac{1}{4}y^2 + \alpha(0, y))$ for each $\alpha \in \mathcal{A}_y$.

In the case G_1 (resp. G_2): Note that $\sharp(g^{-1}(x, y)) = 2$ for any $(x, y) \in D_1$ (resp. D_2). Let $(x, v_1), (x, v_2)$ be the preimages by g for each $(x, y) \in D_1$ (resp. D_2), where $v_j = v_j(x, y)$ ($j = 1, 2$) are three real roots of $V^3 + xV^2 - y = 0$ (v_1 is the multiple root). For any $(x, y) \in D_1$ (resp. D_2), set $c_j(x, y) = \mu_\alpha(x, v_j)$. We see clearly $c_1(x, y) = \alpha(x, y)$, $c_2(x, y) = \frac{1}{2}x^2 + \alpha(x, y)$ (resp. $c_1(x, y) = \frac{2}{9}x^2 + \alpha(x, y)$, $c_2(x, y) = \frac{1}{18}x^2 + \alpha(x, y)$). We set $\gamma_\alpha(x) = (c_1(x, 0), c_2(x, 0))$ (resp. $\gamma_\alpha(x) = (c_1(x, \frac{4}{27}x^3), c_2(x, \frac{4}{27}x^3))$) for each $\alpha \in \mathcal{A}_x$.

Lemma 10.3. *Let $\theta : (\mathbb{R}, 0) \rightarrow \mathbb{R}$ be a function germ such that*

$$(10.1) \quad \theta(0) = \theta'(0) = \theta''(0) = \theta'''(0) = 0.$$

In the case E for any $\alpha \in \mathcal{A}_y$ there exists an $\alpha' \in \mathcal{A}_y$ such that

- (i) (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent,
- (ii) $-y^2 + \alpha'(0, y)^2 - \chi_\alpha \alpha'(0, y)^3 + \theta(\alpha'(0, y)) = 0$ for all $y \leq 0$,

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0, 0)$.

In the case F_1 (resp. F_2) for any $\alpha \in \mathcal{A}_x$ there exists an $\alpha' \in \mathcal{A}_x$ such that

- (i) (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent,
- (ii) $-x^2 + \alpha'(x, 0)^2 - \chi_\alpha \alpha'(x, 0)^3 + \theta(\alpha'(x, 0)) = 0$ on D_1
(resp. $-x^2 + c^2 - (\frac{1}{9} + \chi_\alpha)c^3 + \theta(c) = 0$ on D_2)

where $c = \frac{1}{18}x^2 + \alpha'(x, \frac{4}{27}x^3)$, $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0, 0)$.

Proof. Since any $\alpha \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$) verifies the condition $\frac{\partial \alpha}{\partial y}(0, 0) = \pm 1$ (resp. $\frac{\partial \alpha}{\partial x}(0, 0) = \pm 1$), by the implicit function theorem, there exists the function-germ $\psi_\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $Im \gamma_\alpha = graph \psi_\alpha$ in each case. By direct calculations, we see respectively in the case F, G_1, G_2

$$\psi_\alpha(c) = c + \frac{1}{4}c^2 - \frac{1}{4}\chi_\alpha c^3 + o(|c|^3),$$

$$\psi_\alpha(c) = c + \frac{1}{2}c^2 - \frac{1}{2}\chi_\alpha c^3 + o(|c|^3), \quad \psi_\alpha(c) = c + \frac{1}{6}c^2 - \frac{1}{6}\left(\frac{1}{9} + \chi_\alpha\right)c^3 + o(|c|^3).$$

In the case F, G_1 , G_2 respectively, define the function germ $\overline{\psi}_\alpha : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ by

$$\overline{\psi}_\alpha(c) = c + \frac{1}{4}c^2 - \frac{1}{4}\chi_\alpha c^3 + \frac{1}{4}\theta(c),$$

$$\overline{\psi}_\alpha(c) = c + \frac{1}{2}c^2 - \frac{1}{2}\chi_\alpha c^3 + \frac{1}{2}\theta(c), \quad \overline{\psi}_\alpha(c) = c + \frac{1}{6}c^2 - \frac{1}{6}\left(\frac{1}{9} + \chi_\alpha\right)c^3 + \frac{1}{6}\theta(c).$$

Then by Proposition 10.1 there exists an orientation preserving diffeomorphism-germ $\lambda : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $\overline{\psi}_\alpha = \lambda \circ \psi_\alpha \circ \lambda^{-1}$ in each case. Since $(\lambda \circ \mu_\alpha, g)$ is equivalent to (μ_α, g) , by Proposition 10.2 there exists a functional modulus $\alpha' \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$) in the case F (resp. G_1, G_2) such that the following diagram commute, hence (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent:

$$\begin{array}{ccccc} (\mathbb{R}, 0) & \xleftarrow{\mu_\alpha} & (\mathbb{R}^2, 0) & \xrightarrow{g} & (\mathbb{R}^2, 0) \\ \lambda \downarrow & & \parallel & & \parallel \\ (\mathbb{R}, 0) & \xleftarrow{\lambda \circ \mu_\alpha} & (\mathbb{R}^2, 0) & \xrightarrow{g} & (\mathbb{R}^2, 0) \\ \parallel & & \downarrow k & & \downarrow h \\ (\mathbb{R}, 0) & \xleftarrow{\mu_{\alpha'}} & (\mathbb{R}^2, 0) & \xrightarrow{g} & (\mathbb{R}^2, 0) \end{array}$$

In the case F since the set $\{(0, y) \mid y \leq 0\}$ is preserved by h and since λ is orientation preserving, the above commutative diagram implies

$$(10.2) \quad \lambda \times \lambda(Im \gamma_\alpha|_{y \leq 0}) = graph \overline{\psi}_\alpha \cap \{\delta c \leq 0\} = Im(\gamma_{\alpha'}|_{y \leq 0})$$

where $\delta = \frac{\partial \alpha}{\partial y}(0, 0)$. In the case G_1 (resp. G_2) since a discriminant set D_1 (resp. D_2) is preserved by h , from the commutativity of the above diagram we see that $\lambda \times \lambda(Im \gamma_\alpha) = graph \overline{\psi}_\alpha = Im \gamma_{\alpha'}$. Therefore by the definitions of $\overline{\psi}_\alpha$ and γ_α we have the equation in Lemma 10.3.(ii) in each case. This completes the proof. $\square$

Remark 10.4. In the case F, by (10.2) if $\delta = 1$, then $c = \alpha'(0, y) \leq 0$, thus $\frac{\partial \alpha'}{\partial y}(0, 0) = 1$. Similarly, if $\delta = -1$ then $\frac{\partial \alpha'}{\partial y}(0, 0) = -1$. That is $\frac{\partial \alpha}{\partial y}(0, 0) = \frac{\partial \alpha'}{\partial y}(0, 0)$.

The functional moduli α' in Lemma 10.3 depends on θ and α . Now, by means of special choice of θ , we normalize α' such that $\alpha'(0, y)$ (resp. $\alpha'(x, 0)$, $\alpha'(x, \frac{4}{27}x^3)$) is the polynomial as low degree as possible in the case F (resp. G_1 , G_2). Note that the degree of the normalization is not zero for the condition of the functional moduli $\frac{\partial \alpha}{\partial y}(0, 0) = \pm 1$ (resp. $\frac{\partial \alpha}{\partial x}(0, 0) = \pm 1$). In the case of $\chi_\alpha = 0$, we have the normalization to degree one. In the case of $\chi_\alpha \neq 0$, we can not have the normalization to degree one because of the condition (10.1) for θ , thus we consider the normalization to degree two. In fact, it is possible as follows: For any $\chi \in \mathbb{R}$ we define the function germ $\theta_\chi: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ by

$$\theta_\chi = \begin{cases} 0 & \text{if } \chi = 0 \\ a \circ \xi & \text{if } \chi \neq 0 \end{cases}$$

where $\xi(t) = \frac{-1 + \sqrt{1 + 2\chi t}}{\chi}$, $a(t) = \frac{5}{4}\chi^2 t^4 + \frac{3}{4}\chi^3 t^5 + \frac{1}{8}\chi^4 t^6$.

Then the θ_χ satisfy the condition (10.1) in Lemma 10.3. Moreover we define the function germ $h_\chi: (\mathbb{R} \times \mathbb{R}, (0, 0)) \rightarrow (\mathbb{R}, 0)$ by $h_\chi(t, c) = -t^2 + c^2 - \chi c^3 + \theta_\chi(c)$ for any $\chi \in \mathbb{R}$. By the definition of θ_χ it can be directly shown that $h_\chi(\pm t, t + \frac{\chi}{2}t^2) = 0$ for all $t \in (\mathbb{R}, 0)$. Hence we can easily have

Lemma 10.5. *If $(t, c) \in h_\chi^{-1}(0)$, then $c = \pm t + \frac{\chi}{2}t^2$ for any $\chi \in \mathbb{R}$.*

For any $\alpha \in \mathcal{A}_y$ (resp. $\mathcal{A}_x$), set $\chi = \frac{\partial^2 \alpha}{\partial y^2}(0, 0)$ (resp. $\chi = \frac{\partial^2 \alpha}{\partial x^2}(0, 0)$, $\frac{1}{9} + \frac{\partial^2 \alpha}{\partial x^2}(0, 0)$). Then, by Lemma 10.3, Lemma 10.5 (and Remark 10.4 in the case F), we obtain Theorem F, G_1 , G_2 .

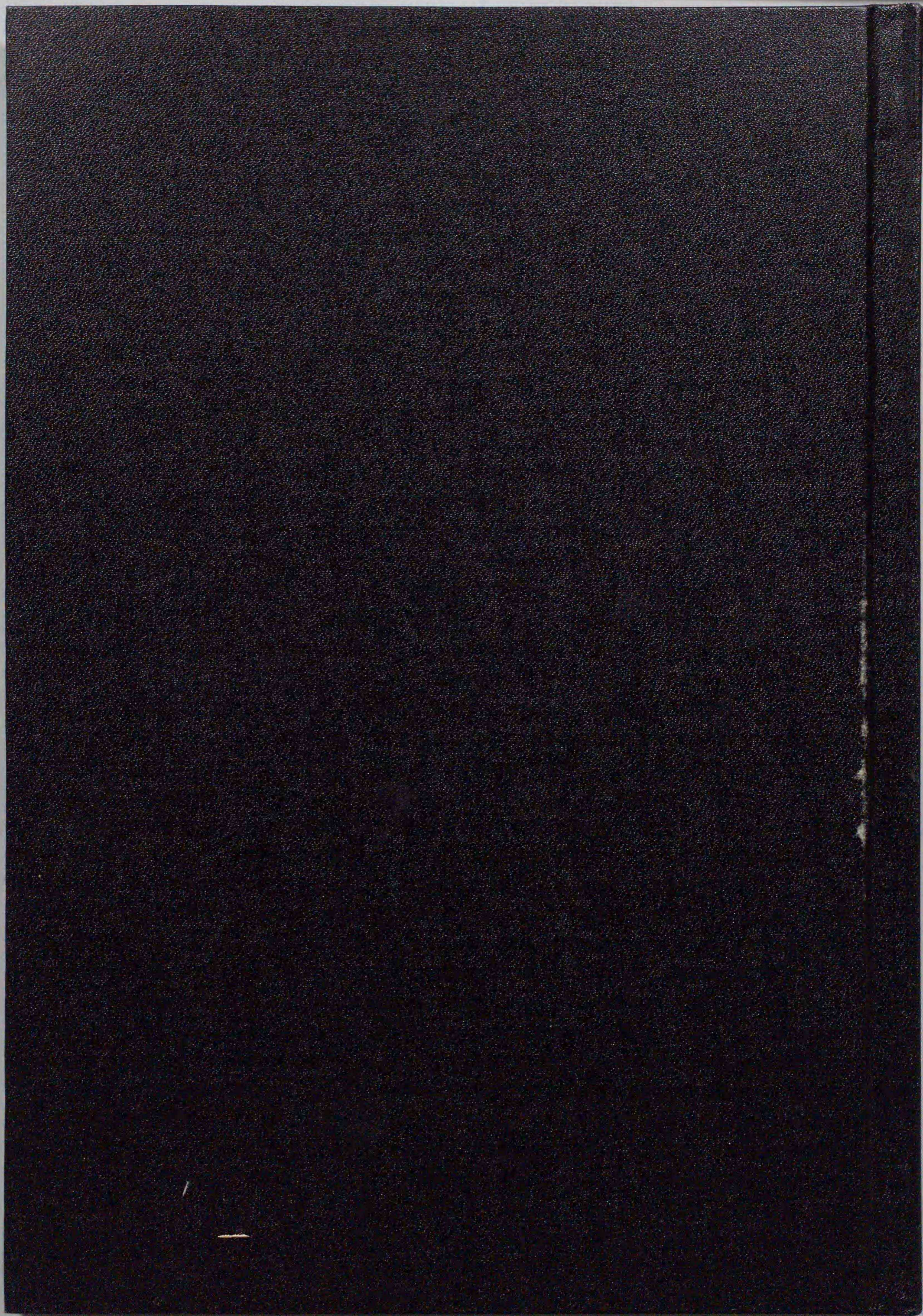
Remark 10.6. χ_α is an invariant of (4), (6) relative to the equivalence because we can see that by the direct calculation the third coefficient of the Taylor expansion at the origin for ψ_α is invariant under the conjugate $\psi_\alpha = \lambda^{-1} \circ \psi_{\alpha'} \circ \lambda$ for a diffeomorphism germ λ .

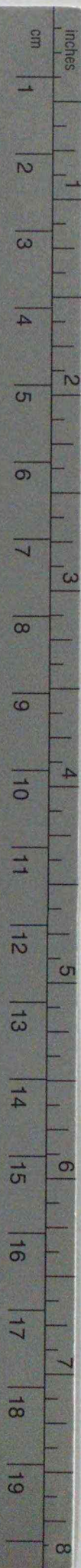
REFERENCES

1. V. I. Arnol'd, *Wave front evolution and equivariant Morse lemma*, Comm. Pure Appl. Math. **29** (1976), 557-582.
2. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps I*, Birkhäuser, 1985.
3. W. Blaschke, *Einführung in die Geometrie der Waben*, Birkhäuser, 1955.
4. W. Blaschke and G. Bol, *Geometrie der Gewebe*, Springer, 1938.
5. T. H. Bröcker, *Differentiable germs and Catastrophes*, Cambridge University Press, 1975.
6. J. W. Bruce, *Seeing-the Mathematical Viewpoint*, Math. Intelligencer **6** (1984), 18-25.
7. M. J. Dias Carneiro, *Singularities of envelopes of families of submanifolds in $\mathbb{R}^N$* , Ann. Scient. Éc. Norm. Sup. **16** (1983), 173-192.
8. S. S. Chern, *Web geometry*, Bull. Am. Math. Soc **6** (1982), 1-8.
9. J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$* , Memoirs of AMS. **50** (1984).
10. J. M. S. David, *Projection-generic curves*, J. London. Math. Soc. **27** (1983), 552-562.
11. J. P. Dufour, *Bi-stabilité des fronces*, C. R. Acad. Sci. Paris. **285** (1977), 445-448.
12. J. P. Dufour, *Sur la stabilité des diagrammes d'applications différentiables*, Ann. scient. Éc. Norm. Sup. **10** (1977), 153-174.
13. J. P. Dufour, *Stabilité simultanée de deux fonctions différentiables*, Ann. Inst. Fourier **29** (1979), 263-282.
14. J. P. Dufour, *Triplets de fonctions et stabilité des enveloppes*, C. R. Acad. Sci. Paris **293** (1981), 509-512.
15. J. P. Dufour, *Families de courbes planes différentiables*, Topology **22** (1983), 449-474.
16. J. P. Dufour, *Modules pour les familles de courbes planes*, Ann. Inst. Fourier, Grenoble. **39-1** (1989), 225-238.
17. J. P. Dufour and P. Jean, *Rigidity of webs and families of hypersurfaces*, Singularities and Dynamical Systems (ed S. N. Pnevmatikos), North-Holland (1985), 271-283.
18. J. P. Dufour and P. Jean, *Families de surfaces différentiables*, J. London Math. Soc. **(2)42** (1990), 175-192.
19. J. P. Dufour and J. P. Tueno, *Singularitiés et photographies de surfaces*, J. Geometry and Physics **9** (1992), 173-182.

20. C. G. Gibson, *Singular Points of Smooth Mappings*, Pitman, London, 1979.
21. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer, Graduate Texts in Math. 14 (1973).
22. A. Hayakawa, G. Ishikawa, S. Izumiya and K. Yamaguchi, *Classification of generic integral diagram and first order ordinary differential equations*, International J. Math. 5 (1994), 447-489.
23. J. -P. Henry and M. Merle, *Shade, Shadow and Shape*, Computational algebraic geometry 109, Birkhäuser, Progress in Math, 105-128.
24. S. Izumiya, *The theory of Legendrian unfoldings and first order differential equations*, Proceeding of the Royal Society of Edinburgh Sect. A 123 (1993), 517-532.
25. S. Izumiya, *Completely integrable holonomic systems of first order differential equations*, Proceeding of the Royal Society of Edinburgh Sect. A 125 (1995), 567-586.
26. S. Izumiya and Y. Kurokawa, *On systems of Clairaut type*, Kodai Math. J. 17-3 (1994), 636-643.
27. S. Izumiya and Y. Kurokawa, *Holonomic systems of Clairaut type*, Differential Geometry and its Applications 5 (1995), 219-235.
28. J. J. Koenderink, *Solid Shape*, The MIT Press (1990).
29. Y. Kurokawa, *On functional moduli for first order ordinary differential equations*, C. R. Acad. Sci. Paris. 317 (1993), 233-238.
30. Y. Kurokawa, *Singularities for projections of contour lines of surfaces onto planes*, Preprint.
31. P. L. Lions, E. Rouy and A. Tourin, *Shape-from-Shading, viscosity solutions and edges*, Numer. Math. 64 (1993), 323-353.
32. I. Nakai, *Topology of complex webs of codimension one and geometry of projective space curves*, Topology 26-4 (1987), 475-504.
33. H. Noguchi and T. Fukuda, *Elementary catastrophes*, Kyouritsuzensho 208, 1976 (in Japanese).
34. S. Sternberg, *Local C^n transformations of the real line*, Duke Math. J. 24 (1957), 97-112.
35. F. Takens, *Normal forms for certain singular vector fields*, Ann. Inst. Fourier, Grenoble. 23-2 (1973), 163-195.
36. R. Thom, *Sur la theorie des enveloppes*, J. Math. pure et appl. t. XLI, fasc. 2 (1962), 177-192.

37. C. T. C. Wall, *Geometric properties of generic differentiable manifolds*, Springer, Lecture Notes in Math. **597** (1976), 707-774.
38. V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, Journal of Soviet Math. **27** (1983), 2713-2735.

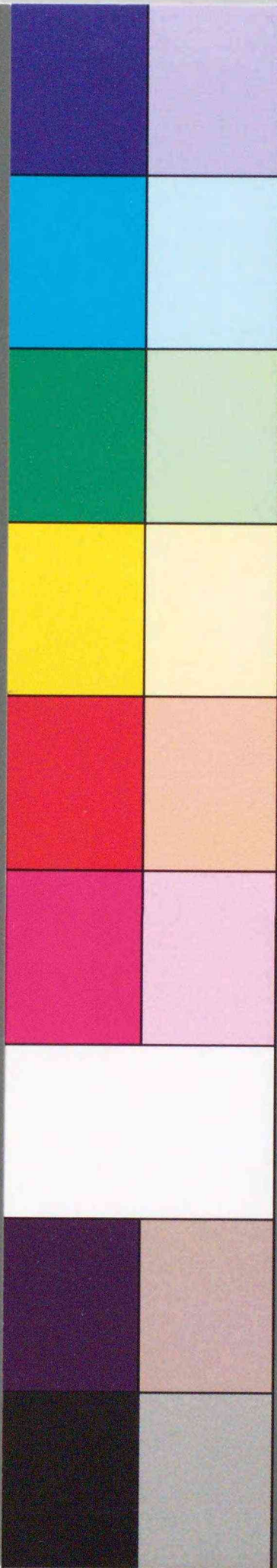




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