Periods, Higgs Fields and Prepotentials in $N=2$ Supersymmetric Yang-Mills Theory

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Abstract

We show how to give the expression for periods, Higgs field and its dual of $N = 2$ supersymmetric Yang-Mills theory around the conformal point. This is achieved by evaluating the integral representation in the weak coupling region explicitly, and by using analytic continuation to the conformal point. We review our previous results to see how effectively the technique to evaluate the integral work, and how the analytic continuation to realize the conformal point. The explicit representation is shown for the $SU(2)$ theory with matter fields and also for pure $SU(N)$ and pure $SO(2N)$ theory around the conformal point where the relation to the beta function of the theory is clarified. Similar calculation can be carried out in the $SU(3)$ theory with matter fields. We also discuss a relation between the fixed points in the $SU(2)$ theories with massless matter fields and the Landau-Ginzburg point of 2-D $N = 2$ SCFT.

1 Introduction

Recent years many progress about four dimensional $N = 2$ supersymmetric Yang-Mills theories have been made. Seiberg and Witten [1] solved the low energy effective theory in the $SU(2)$ theory without matter fields exactly, based on the duality and holomorphy by introducing the elliptic curve. Following this work various generalizations introducing the matter fields or gauge group being higher than $SU(2)$ have been investigated by many people [2, 3, 4, 5, 6, 7, 8, 9, 10]. The exact solution for the prepotential which controls the low energy effective action, can be obtained from the period integrals on the elliptic curve. Associated with singularities of the theories coming from the massless states, these curves for various kinds of $N = 2$ supersymmetric Yang-Mills theories have been studied extensively[4, 5, 6, 9]. Usual approach to obtain the period is to solve the differential equation which the periods obey, so called Pichard-Fuchs equation[10, 12], when these equations turns out to be solved by the well known functions. The $N = 2$ $SU(2)$ supersymmetric Yang-Mills theories with massless hypermultiplets[13] and $SU(3)$ super Yang-Mills theory without matters[12] have been analyzed elegantly by using the method, whereas the method given in Ref.[11] is useful only for the group $SU(2)$. However, the situation changes drastically when it is difficult to find out appropriate variables for solving the Picard-Fuchs equation by any special functions. This situation generally occurs for the theories with massive hypermultiplets or higher rank gauge groups. In these cases, the solution has been studied in the weak coupling region by perturbative treatments to obtain the prepotential [14, 15]. The direct tests have been made by comparison with the instanton method in the case of $SU(2)$ theory with matter fields [16, 17], and even for the higher rank gauge groups [18], which provide the consistent results.

Apart from the analysis in the weak coupling region, the power of the exact results should be used in the analysis in the strong coupling region, where one finds truly non-perturbative results. Among the analysis of the strong coupling region, one of the striking fact is the existence of the conformal points [19, 20, 21] where the prepotentials have no dependence of the dynamical mass scale. These theories are classified by scaling behaviors around the conformal point [21]. It seems interesting to investigate the theories around this points by deriving the explicit form of fields, which seems to provide us of a more concrete behavior of the critical theories.

In above cases, we expect that the explicit evaluation of the integral representation of the solutions is more powerful than finding the solutions of the differential equations. As a matter of fact, the analytic properties of the functions are studied mostly by using the integral representations of the functions rather than the differential equations themselves even in the theory of special functions. In previous works [22, 23] we have evaluated the integral representation about the period, Higgs field and its dual on such situations. The results for $SU(2)$ Yang-Mills theory with matter fields [22] can be analytically continued around the conformal point when the bare masses take the critical values. Another previous work [23] devoted to investigate how to evaluate effectively the integral representation in the weak coupling region. In this article, we combine these approach to investigate the expression around the conformal point. We treat moduli parameters and bare masses as deviations from the conformal point. After evaluating the integral representation explicitely in the region where only one parameter is very large but other parameters are near the confomal point, we perform the analytic continuation of one large parameter to be near the conformal point. By use of the analytic continuation we can get the expression around the conformal point. Usually, analytic continuation to the region where the logarithmic singularity exits must be treated with care because the result may depend on the choice of variables. In other words, some choice of variable are valid only within some branch. However, when we consider the analytic continuation to the critical point where there is no such singularity, the confirmation of such analytic continuation turns out to be easy, as we will see in each cases. As the matter of fact, this approach can be considered as a generalization of the one given to obtain the periods for Calabi-Yau systems [24]. Of course, there are a variety of the class of conformal point [21] so that we cannot exhaust all known cases. In this paper, we will deal with $SU(2)$ Yang-Mills theory with massive hypermultiplets, and $SU(N)$ and also $SO(2N)$ Yang-Mills theory without matters. We also provide expressions of Higgs field and its dual around the conformal point for the pure $SU(N)$ and also $SO(2N)$ Yang-Mills theories.

This article is organized as follows. In section 2, we will review our previous result in terms of the $SU(2)$ theory with massive hypermultiplete, and by analytic continuation a infra-red fixed point of the theory is realized when the mass parameter take critical values. In section 3, we will review our previous investigation about effective evaluation of the integral representation in the theory with more higher rank gauge group to represent the Higgs field and its dual by classical root variables. This method will be generalized in the following section to evaluate explicitely the integral representation to get the expression with respect to the moduli parameters and mass parameters. In section 4, we will obtain the expression of the fields around conformal points in $SU(2)$ theory with matter fields case, and verify that the result recovers our previous result which was obtained by transformations of the hypergeometric functions[22] when the deviation of mass parameters from the conformal point are set to be zero. We will also discuss the relation to 2-D $N = 2$ SCFT through the simple correspondence to the deformation of curve on $W\mathbf{CP^2}$. This relation was pointed out recently in the different context in the ref. [25]. In section 5, we will derive the form of Higgs field and its dual in the pure $SU(N)$ theory around the conformal point, and discuss the relation to the beta function of theory. In section 6, we will study the pure $SO(2N)$ theory around the conformal point and discuss the validity of the expression which contains the unexpected terms. In section 7, we will compare the results obtained by different type of parameterization in pure $SU(3)$ theory and discuss the relation between them. And we will obtain the expression of the field around the conformal point in $SU(3)$ theory with massless and massive matter fields. The last section will be devoted to some discussion.

2 Evaluation of period integrals in the $SU(2)$ theory with massive hypermultipletes

We begin with reviewing some properties of the low-energy effective action of the $N = 2$ supersymmetric $SU(2)$ QCD. In $N = 1$ superfields formulation[1], the theory contains chiral multiplets Φ^a and chiral field strength W^a $(a = 1, 2, 3)$ both in the adjoint representation of $SU(2)$, and chiral superfield Q^i in $\mathbf{2}$ and $\tilde{Q}^i$ $(i = 1, \cdots, N_f)$ in $\bar{\mathbf{2}}$ representation of $SU(2)$. In $N = 2$ formulation Q^i and $\tilde{Q}^i$ are called hypermultiplets. Along the flat direction, the scalar field ϕ of Φ get vacuum expectation values which break $SU(2)$ to $U(1)$, so that the low-energy effective theory contains $U(1)$ vector multiplets (A, W_α), where A^i are $N = 1$ chiral superfields and W_α are $N = 1$ vector superfields. The quantum low-energy effective theory is characterized by effective Lagrangian $\mathcal{L}$ with the holomorphic function $\mathcal{F}(A)$ called prepotential,

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\left(\int d^2\theta d^2\bar{\theta}\frac{\partial\mathcal{F}}{\partial A}\bar{A} + \frac{1}{2}\int d^2\theta\frac{\partial^2\mathcal{F}}{\partial A^2}W_\alpha W^\alpha\right). \tag{2.1}$$

The scalar component of A is denoted by a, and $A_D = \frac{\partial\mathcal{F}}{\partial A}$ which is dual to A by a_D. The pair (a_D, a) is a section of $SL(2, \mathbf{Z})$ and is obtained as the period integrals of the elliptic curve parameterized by u, Λ and m_i $(0 \le i \le N_f)$, where $u = <\mathrm{tr}\phi^2>$ is a gauge invariant moduli parameter, Λ is a dynamical scale and m_i are bare masses of hypermultiplets. Once we know a and a_D as a holomorphic function of u, we can calculate the prepotential $\mathcal{F}(a)$ by using the relation

$$a_D = \frac{\partial\mathcal{F}(a)}{\partial a}. \tag{2.2}$$

General elliptic curves of $SU(2)$ Yang-Mills theories with massive $N_f \le 3$ hypermultiplets are[4]

$$y^2 = C^2(x) - G(x) \tag{2.3}$$

$$\begin{array}{lll}
C(x) = x^2 - u, & G(x) = \Lambda^4, & (N_f = 0) \\
C(x) = x^2 - u, & G(x) = \Lambda^3(x + m_1), & (N_f = 1) \\
C(x) = x^2 - u + \frac{\Lambda^2}{8}, & G(x) = \Lambda^2(x + m_1)(x + m_2), & (N_f = 2) \\
C(x) = x^2 - u + \frac{\Lambda}{4}(x + \frac{m_1+m_2+m_3}{2}), & G(x) = \Lambda(x + m_1)(x + m_2)(x + m_3), & (N_f = 3)
\end{array}$$

These curves are formally denoted by

$$y^2 = C^2(x) - G(x) = (x - e_1)(x - e_2)(x - e_3)(x - e_4), \tag{2.4}$$

where $e_1 = e_4,\ e_2 = e_3$ in the classical limit.

In order to calculate the prepotential, we consider a and a_D as the integrals of the meromorphic differential λ over two independent cycles of these curves,

$$a = \oint_\alpha \lambda, \quad a_D = \oint_\beta \lambda, \tag{2.5}$$

$$\lambda = \frac{x}{2\pi i} \mathrm{d} \ln \left(\frac{C(x) - y}{C(x) + y} \right). \tag{2.6}$$

where α cycle encloses e_2 and e_3, β cycle encloses e_1 and e_3, λ is related to the holomorphic one-form as

$$\frac{\partial \lambda}{\partial u} = \frac{1}{2\pi i} \frac{dx}{y} + d(*). \tag{2.7}$$

Since there are poles coming from mass parameters in the integrant of a and a_D, we instead evaluate the period integrals of holomorphic one-form;

$$\frac{\partial a}{\partial u} = \oint_\alpha \frac{dx}{y}, \quad \frac{\partial a_D}{\partial u} = \oint_\beta \frac{dx}{y}. \tag{2.8}$$

After changing the variable and using the integral representation of hypergeometric function;

$$F(a, b; c; x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \int_0^1 ds s^{b-1} (1-s)^{c-b-1} (1-sx)^{-a} \tag{2.9}$$

where

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{2.10}$$

we obtain $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{pau}$ as follows:

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2\pi} \int_{e_2}^{e_3} \frac{dx}{y} = \frac{\sqrt{2}}{2} (e_2 - e_1)^{-1/2} (e_4 - e_3)^{-1/2} F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right), \tag{2.11}$$

$$\frac{\partial a_D}{\partial u} = \frac{\sqrt{2}}{2\pi} \int_{e_1}^{e_3} \frac{dy}{y} = \frac{\sqrt{2}}{2} \left[(e_1 - e_2)(e_4 - e_3)\right]^{-1/2} F\left(\frac{1}{2}, \frac{1}{2}, 1; 1 - z\right), \tag{2.12}$$

where

$$z = \frac{(e_1 - e_4)(e_3 - e_2)}{(e_2 - e_1)(e_4 - e_3)}, \tag{2.13}$$

and the normalization is fixed so as to be compatible with the asymptotic behavior of a and a_D in the weak coupling region

$$a = \frac{\sqrt{2u}}{2} + \cdots,$$
$$a_D = i\frac{4-N_f}{2\pi} a \ln a + \cdots. \tag{2.14}$$

In this expression a_D is obtained as a hypergeometric function around $z=1$, so we have to do the analytic continuation which gives the logarithmic asymptotic in the weak coupling region.

Since elliptic curves are not factorized in general, it is difficult to obtain their roots in a simple form. Even if we know the form of roots, the variable z in (2.11) and (2.12) is very complicated in terms of u in these representations. So we will transform the variable to the symmetric form with respect to roots, by using the identity of the hypergeometric functions, so that the new variable is given easily from the curve directly without knowing the form of roots. After the analytic continuation to the weak coupling region to (2.12), applying so-called quadratic and cubic transformation of the hypergeometric functions to the expression (2.11) and (2.12) subsequently, we obtain the expression valid in the weak coupling region as

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2}(-D)^{-1/4} F\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3}\right), \tag{2.15}$$

$$\begin{aligned}\frac{\partial a_D}{\partial u} = i\frac{\sqrt{2}}{2}(-D)^{-1/4} \bigg[&\frac{3}{2\pi} \ln 12\ F\left(\frac{1}{12}, \frac{5}{12}, 1, -\frac{27\Delta}{4D^3}\right) \\ &- \frac{1}{2\pi} F^*\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3}\right)\bigg],\end{aligned} \tag{2.16}$$

where

$$z'' = -\frac{27\Delta}{4D^3} = \frac{27z^2(1-z)^2}{4(z^2-z+1)^3}, \quad \Delta = \prod_{i<j}(e_i - e_j)^2, \quad D = \sum_{i<j} \frac{1}{2}(e_i - e_j)^2. \tag{2.17}$$

This variable is completely symmetric with respect to roots e_i, so it is easy to write down z'' by using the coefficient of the curve. Though z is not invariant under the modular transformation of τ, the argument z'' is invariant completely. As a matter of fact, this variable can be written by the absolute invariant form $j(\tau)$ as $z'' = 1/j(\tau)$. Therefore it is quite natural to represent the period in terms of z''. If we consider the case when the curve is factorized to be a simple form such as $N_f = 0$ theory and $N_f = 2$ theory with equal mass parameters, it is not necessary to transform the variable to be a modular invariant form. In this case, we can transform the variable to be simple enough by using the quadratic transformation only.

Next we consider the periods in the strong coupling region. The quadratic and cubic transformation are valid if $|z''| \leq 1$. The region of z-plane which satisfies this condition consists of three parts; one is around $z = 0$, one is around $z = 1$ and the last is around $z = \infty$. The region around $z = 0$ corresponds to the weak coupling region, and the region around $z = 1$ corresponds to the strong coupling region where the monopoles condensate and $z = \infty$ is the dyonic point. So we can construct the formula valid in the strong coupling region by analytic continuation to around $z = 1$ or $z = \infty$ and by using the quadratic and cubic transformation subsequently.

As a example, we consider the theory with a matter hypermultiplet whose curve is given by

$$y^2 = (x^2 - u)^2 - \Lambda^3(x + m), \tag{2.18}$$

from which Δ and D is obtained as

$$\Delta = -\Lambda^6(256u^3 - 256u^2m^2 - 288um\Lambda^3 + 256m^3\Lambda^3 + 27\Lambda^6), \tag{2.19}$$

$$D = -16u^2 + 12m\Lambda^3. \tag{2.20}$$

Substituting these to (2.15) and (2.16), we can obtain a, a_D, by expanding (2.15) and (2.16) at $u = \infty$ and integrating with respect to u. Representing u in terms of a inversely, and substituting u to a_D, and finally integrating a_D with respect to a, we can get the prepotencial in the weak coupling region as

$$\begin{aligned}\mathcal{F}(\tilde{a}) = i\frac{\tilde{a}^2}{\pi}\left[\frac{3}{4}\ln\left(\frac{\tilde{a}^2}{\Lambda^2}\right) + \frac{3}{4}\left(-3 + 4\ln 2 - i\pi\right) - \frac{\sqrt{2}\pi}{2i\tilde{a}}(n'm)\right. \\ \left. - \ln\left(\frac{\tilde{a}}{\Lambda}\right)\frac{m^2}{4\tilde{a}^2} + \sum_{i=2}^{\infty}\mathcal{F}_i\tilde{a}^{-2i}\right].\end{aligned} \tag{2.21}$$

where we introduce $\tilde{a}$ subtracted mass residues from a. These $\mathcal{F}_i$ agree with the perturbative result up to the orders cited in [15]. In principle we can calculate $\mathcal{F}_i$ to arbitrary order in our formalism. Quite similarly, we can obtain the prepotential in the strong coupling region.

The number of the singularity of the theory is the number of the root of the equation $\Delta = 0$ plus one, which is the singularity at $u = \infty$. Since Δ is third order polynomial in terms of u in $N_f = 1$ theory, $\Delta = 0$ must have one double root when the mass takes critical value. This condition is satisfied if $m = 3/4\Lambda$ where the parameters of the periods are given by

$$\Delta = -\Lambda^6(16u + 15\Lambda^2)(4u - 3\Lambda^2)^2, \quad D = -(4u + 3\Lambda)(4u - 3\Lambda), \tag{2.22}$$

$$z'' = -\frac{27}{4}\frac{\Lambda^6(16u + 15\Lambda^2)}{(4u + 3\Lambda^2)^3(4u - 3\Lambda^2)}. \tag{2.23}$$

Such factorization of Δ means that theory has a conformal point $u = 3\Lambda^2/4$ where the curve become [20, 21]

$$y^2 = \left(x + \frac{\Lambda}{2}\right)^3 \left(x - \frac{3\Lambda}{2}\right). \tag{2.24}$$

If we set

$$w = \frac{27\Lambda^2}{16u + 15\Lambda^2}, \tag{2.25}$$

then $z'' = -64w^3/(w-9)^3(w-1)$. In order to obtain a and a_D we need the quartic transformation which makes the variable simple enough. We can prove the following transformation of fourth order;

$$F\left(\frac{1}{12}, \frac{5}{12}, 1, -\frac{64w^3}{(w-9)^3(w-1)}\right) = \left(1 - \frac{w}{9}\right)^{1/4} (1 - w)^{1/12} F\left(\frac{1}{3}, \frac{1}{3}, 1, w\right). \tag{2.26}$$

Using this identity and the identity

$$F(a, b; c; z) = (1 - z)^{-a} F(a, c - b; c; z/(z - 1)), \tag{2.27}$$

we get $\frac{\partial a}{\partial u}$, $\frac{\partial a_D}{\partial u}$

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{8}\left(-27\Lambda^2\right)^{\frac{1}{2}} y^{1/2} F\left(\frac{1}{3}, \frac{2}{3}, 1, y\right) \tag{2.28}$$

$$\frac{\partial a_D}{\partial u} = i\frac{\sqrt{2}}{8}\left(-27\Lambda^2\right)^{\frac{1}{2}} y^{1/2} \left[\frac{(3\ln 3 - i\pi)}{2\pi} F\left(\frac{1}{3}, \frac{2}{3}, 1, y\right) - \frac{3}{2\pi} F^*\left(\frac{1}{3}, \frac{2}{3}, 1, y\right)\right], \tag{2.29}$$

where

$$y = \frac{27\Lambda^2}{-16u + 12\Lambda^2}. \tag{2.30}$$

Performing the analytic continuation to the region $y \sim \infty$, i.e., around the conformal point, and integration with respect to u, we get the expression for the Higgs field a and its dual a_D around the coformal point as

$$\begin{aligned} a = & -\frac{i\sqrt{2}}{8}3\sqrt{3}\Lambda y^{-1/2}\left[\frac{6}{5}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})\Gamma(\frac{2}{3})}(-y)^{-1/3}{}_3F_2\left(\frac{1}{3},\frac{1}{3},\frac{5}{6};\frac{2}{3},\frac{11}{6};\frac{1}{y}\right)\right. \\ & \left. + \frac{6}{7}\frac{\Gamma(-\frac{1}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})}(-y)^{-2/3}{}_3F_2\left(\frac{2}{3},\frac{2}{3},\frac{7}{6};\frac{4}{3},\frac{13}{6};\frac{1}{y}\right)\right], \end{aligned} \tag{2.31}$$

$$\begin{aligned} a_D = & \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{8}3\sqrt{3}\Lambda y^{-1/2}\left[\frac{6}{5}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})\Gamma(\frac{2}{3})}(-y)^{-1/3}{}_3F_2\left(\frac{1}{3},\frac{1}{3},\frac{5}{6};\frac{2}{3},\frac{11}{6};\frac{1}{y}\right)\right. \\ & \left. - \frac{6}{7}\frac{\Gamma(-\frac{1}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})}(-y)^{-2/3}{}_3F_2\left(\frac{2}{3},\frac{2}{3},\frac{7}{6};\frac{4}{3},\frac{13}{6};\frac{1}{y}\right)\right]. \end{aligned} \tag{2.32}$$

where ${}_3F_2$ is the generalized hypergeometric function defined as

$${}_3F_2(a,b,c;1,d;y) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(n!)^2}y^n. \tag{2.33}$$

In this expression, we see that a and a_D contain no logarithmic term, and around the conformal point $u \sim \frac{3\Lambda^2}{4}$ they behave as $a \sim a_D$. This implies that on the conformal point, prepotential does not depend on the dynamical mass scale and the beta function of the theory vanishes on this point. Thus we see that the conformal point is a infra-red fixed point of the theory.

Similar treatment can be done in the $SU(2)$ theory with $N_f = 2, 3$ matter fields, and the results with critical mass value given by the analytic continuation from the weak coupling region to the region around the conforml point, show that the conformal point of $N_f = 2, 3$ theory is a infra-red fixed point of the theory.

For later use for the comparison to the deformation theory, we list here the expression for $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$ with critical mass value in the weak coupling region of $N_f = 2, 3$ theory as:

$N_f = 2$ theory ($m_1 = m_2 = \frac{\Lambda}{2}$, $y = \frac{8\Lambda^2}{-8u+3\Lambda^2}$)

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2}(-\Lambda^2)^{-1/2}y^{1/2}F\left(\frac{1}{4},\frac{3}{4},1,y\right) \tag{2.34}$$

$$\frac{\partial a_D}{\partial u} = i\frac{\sqrt{2}}{2}(-\Lambda^2)^{-1/2}y^{1/2}\left[\frac{(6\ln 4 - i\pi)}{2\pi}F\left(\frac{1}{4},\frac{3}{4},1,y\right)\right. \quad (2.35)$$

$$\left. -\frac{1}{\pi}F^*\left(\frac{1}{4},\frac{3}{4},1,y\right)\right], \quad (2.36)$$

$N_f = 3$ theory ($m_1 = m_2 = m_3 = \frac{\Lambda}{8}$, $y = \frac{27\Lambda^2}{-256u+8\Lambda^2}$)

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{2}\left(-\frac{27\Lambda^2}{256}\right)^{-\frac{1}{2}} y^{1/2}F\left(\frac{1}{6},\frac{5}{6},1,y\right) \quad (2.37)$$

$$\frac{\partial a_D}{\partial u} = i\frac{\sqrt{2}}{2}\left(-\frac{27\Lambda^2}{256}\right)^{-\frac{1}{2}} y^{1/2}\left[\frac{(3\ln 3 + 2\ln 4 - i\pi)}{2\pi}F\left(\frac{1}{6},\frac{5}{6},1,y\right)\right. \quad (2.38)$$

$$\left. -\frac{1}{2\pi}F^*\left(\frac{1}{6},\frac{5}{6},1,y\right)\right].$$

3 Effective evaluation of the Higgs field in $SU(N_c)$, $SO(2N)$, $Sp(N)$ pure Yang-Mills theories

In this section, we consider $N = 2$ supersymmetric Yang-Mills theory with higher rank gauge group. We will first consider $SU(N_c)$ theories without matter hypermultiplets[12] in detail. The theory has an $N_c - 1$ complex dimensional moduli space of vacua which are parameterized by the expectation value of the Higgs fields as

$$<\phi>=\sum_{i=1}^{N_c} e_i H_i = \mathrm{diag}[a_1, ..., a_{N_c}], \quad (3.1)$$

where H_i are the generators of the Cartan sub-algebra of $U(N_c)$ and

$$\sum_{i=1}^{N_c} a_i = 0. \quad (3.2)$$

The fields a_D^i dual to a_i can be defined as

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a_i}, \quad (3.3)$$

where $\mathcal{F}(a)$ is the prepotential. The curve describing the space of vacua can be identified as

$$y^2 = C^2(x) - \Lambda^{2N_c} = \prod_{k=1}^{N_c}(x - e_k)^2 - \Lambda^{2N_c}, \quad (3.4)$$

where e_i is the value of the classical moduli space with a constraint;

$$\sum_{i=1}^{N_c} e_i = 0. \tag{3.5}$$

It should be noted that the classical root of y, e_i splits into e_i^+ and e_i^- in (3.4).

The meromorphic one form λ can be defined as

$$\lambda = \frac{dx}{2\pi i}\frac{x}{y}[C(x)]'. \tag{3.6}$$

The dual pair of fields a and a_D can be written as periods of a meromorphic one form λ on the curve as

$$a_i = \oint_{\alpha_i} \lambda, \qquad a_D^i = \oint_{\beta_i} \lambda, \tag{3.7}$$

where α^i, β_i form a basis of homology cycles on the curve. Therefore, we can compute the prepotential $\mathcal{F}(a)$ once we can evaluate a_i and a_D^i as functions of e_k and Λ.

We are going to evaluate a^i in the weak coupling region. To begin with, we expand the meromorphic differential with respect to Λ^{2N_c} by using an expansion

$$\frac{1}{y} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n \Lambda^{2N_c n}}{n!(x-e_1)^{2n+1}\cdots(x-e_{N_c})^{2n+1}}, \tag{3.8}$$

where $(a)_n$ is defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{3.9}$$

After the use of a partial integration, we have

$$a_i = \sum_{k=1}^{N_c} \oint_{\alpha^i} \frac{dx}{2\pi i}\frac{x}{x-e_k} + \sum_{n=1}^{\infty} \oint_{\alpha^i} \frac{dx}{2\pi i}\frac{(\frac{1}{2})_n \Lambda^{2N_c n}}{n! 2n (x-e_1)^{2n}\cdots(x-e_{N_c})^{2n}}. \tag{3.10}$$

Originally, the α_i circle was chosen enclosing the roots of y e_i^+, e_i^-. In our expression both of them shrink to the classical value e_i. Therefore, we can take the contour enclosing e_i as α^i cycle to obtain

$$\begin{aligned} a_i &= e_i + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n \Lambda^{2N_c n}}{n! 2n} \sum_{\substack{m_1,\dots m_{N_c}, m_i = 0 \\ m_1 + \cdots m_{N_c} = 2n-1}} \prod_{\substack{k \\ k \neq i}} \frac{\Gamma(2n+m_k)}{\Gamma(2n)\Gamma(m_k+1)} (e_k - e_i)^{-2n-m_k} \\ &= e_i + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n \Lambda^{2N_c n}}{n!(2n)!} \left(\frac{\partial}{\partial e_i}\right)^{2n-1} \prod_{k, k\neq i} (e_k - e_i)^{-2n}. \end{aligned} \tag{3.11}$$

In the case of the dual fields a_D^i, we have to use analytic continuation because of the logarithmic singularity. For such purpose, we write the meromorphic one form λ by Barnes-type representation as

$$\lambda = \frac{dx}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-s)\Gamma(s+1/2)}{\Gamma(1/2)2s} \prod_{k=1}^{N_c} (x - e_k)^{-2s} (-\Lambda^{2N_c})^s, \tag{3.12}$$

where the path of integration is taken around the poles at $s = 0, 1, 2, \ldots$. This expression can be obtained by considering Barnes-type representation for (3.8) and use a partial integration. Note also that we can obtain the strong coupling expansion of λ by taking the poles at $s = -1/2, -3/2, \ldots$

The $\beta_{ij} = \beta_i - \beta_j$ cycle consists of the circle enclosing the root e_j^+ and e_i^+, which can be written as two times the line integral of λ from e_j^+ to e_i^+. When we use the expression (3.12), these roots shrink to the classical value. If we replace the contour integral to the line integral from e_j to e_i, we have to subtract the contribution from the circle around e_i^- and e_j^-, which can be evaluated as the half of the α^i and α^j. We therefore obtain an expression of a_D as follows;

$$a_D^{ij} \equiv a_D^i - a_D^j = 2 \int_{e_j}^{e_i} \lambda - \frac{1}{2}(a_i - a_j). \tag{3.13}$$

The a_D^i can be obtained from the procedure

$$a_D^i = \frac{1}{N_c} \sum_{j=1}^{N_c} a_D^{ij}. \tag{3.14}$$

Although the method of the expansion for a in (3.11) is quite different from those of a_D (3.13), these will provide a consistent result. As a matter of fact, we can write a by using Barnes-type integral representation as

$$\begin{aligned} a_{ij} &\equiv a_i - a_j \\ &= \int_{e_j}^{e_i} \frac{dx}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\sin 2\pi s}{\pi} \frac{\Gamma(-s)\Gamma(s+1/2)}{\Gamma(1/2)2s} \prod_{k=1}^{N_c} (x - e_k)^{-2s} (-\Lambda^{2N_c})^s, \end{aligned} \tag{3.15}$$

which we can derive by considering the integral enclosing e_i and e_j and by replacing it by line integral. The equivalence of (3.11) and (3.15) can also be shown by evaluating the poles of s in (3.15) explicitly.

In the expression for a_D^{ij}, the singularities appear as double poles for the integral with respect to s, which consists of the contribution both from a_i and a_j. Since a_i and a_j have different singularities, it seems not easy to extract these double poles in a concise manner. We therefore consider the path from zero to e_i and define

$$\tilde{a}_D^i \equiv \frac{1}{\pi i}\int_0^{e_i} \lambda - \frac{1}{2}a_i, \tag{3.16}$$

where we have again subtracted the contribution caused by the degeneracy of the roots in the expression of λ in (3.12). In the weak coupling region, a_D^{ij} can be written as

$$a_D^{ij} = \tilde{a}_D^i - \tilde{a}_D^j. \tag{3.17}$$

From the relation (14), we have

$$a_D^i = \tilde{a}_D^i - \frac{1}{N_c}\sum_{k=1}^{N_c} \tilde{a}_D^k. \tag{3.18}$$

Let us evaluate $\tilde{a}_D^i$. By parameterizing $x = e_i(1-t)$, we have

$$\begin{aligned}
\tilde{a}_D^i = & \frac{1}{\pi i \Gamma(1/2)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{\Gamma(-s)\Gamma(s+1/2)}{2s}\sum_{m_k, k\neq i}\int_0^1 t^{-2s+1+\sum m_k}dt \\
& \times \prod_{k,k\neq i}\frac{\Gamma(2s+m_k)}{\Gamma(2s)\Gamma(m_k+1)}(e_i-e_k)^{-2s+m_k}e_i^{-2s+1+\sum m_k}(-\Lambda^{2N_c})^s - \frac{1}{2}a_i, \\
= & \frac{1}{\pi i \Gamma(1/2)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{\Gamma(-s)\Gamma(s+1/2)}{2s}\sum_{m_k, k\neq i}\frac{1}{-2s+\sum m_k+1} \\
& \times \prod_{k,k\neq i}\frac{\Gamma(2s+m_k)}{\Gamma(2s)\Gamma(m_k+1)}(e_i-e_k)^{-2s+m_k}e_i^{-2s+1+\sum m_k}(-\Lambda^{2N_c})^s - \frac{1}{2}a_i.
\end{aligned} \tag{3.19}$$

Let us evaluate the poles separately. At first, we are going to evaluate the pole at $s = 0$. The double pole arises in the case of $m_k = 0$ for all k, and single poles appear when $m_k \neq 0$ for one of k. We next evaluate the contribution from $s = 1$. The double poles appear in the case $\sum m_k = 1$ in (3.19). Other terms have single poles so that we can evaluate them without any analytic continuation by going back to the original expression for λ for $s = 1$.

The prepotential at this order can be obtained as

$$\mathcal{F}(a) = \frac{i}{4\pi}\sum_{i<j}(a_i-a_j)^2\ln(a_i-a_j)^2/\Lambda^2 + \frac{\tau_0}{2N_c}\sum_{i<j}(a_i-a_j)^2 + \mathcal{F}_1(a), \tag{3.20}$$

where τ_0 is the bare coupling;

$$\tau_0 = \frac{i}{2\pi}(2\ln 2 - 3N_c), \tag{3.21}$$

and the one-instanton contribution $\mathcal{F}_1(a)$ is given by

$$\mathcal{F}_1(a) = -\frac{i\Lambda^{2N_c}}{8\pi}\sum_{k=1}^{N_c}\frac{1}{\prod_{l\neq k}(a_k - a_l)^2} \tag{3.22}$$

Therefore, our method for analytic continuation is consistent with the integrability of the prepotential at least up to the leading order of the instanton corrections. It should be noted that the expression (3.22) agrees completely with the known results for $G = SU(2)$ and $SU(3)$[12][1]. Moreover, it coincides with the result obtained by the direct instanton method for $SU(N_c)$[18].

Let us consider the theories with other gauge groups. It is straightforward to apply the method to other groups. We are now going to list the curve and the one instanton contribution of other classical groups.

For $SO(2N+1)$, the curve is identified as[5]

$$y^2 = \prod_{k=1}^{N}(x^2 - e_k^2)^2 - \Lambda^{2(2N-1)}x^2. \tag{3.23}$$

From this curve, we can calculate the one instanton contribution as

$$\mathcal{F}_1(a) = \frac{i\Lambda^{2(2N-1)}}{32\pi}\sum_{k=1}^{N}\frac{1}{\prod_{l\neq k}(a_k^2 - a_l^2)^2}. \tag{3.24}$$

For $SO(2N)$, the curve is given by[6]

$$y^2 = \prod_{k=1}^{N}(x^2 - e_k^2)^2 - \Lambda^{4(N-1)}x^4, \tag{3.25}$$

from which the one instanton contribution is obtained as

$$\mathcal{F}_1(a) = -\frac{i\Lambda^{4(N-1)}}{32\pi}\sum_{k=1}^{N}\frac{a_k^2}{\prod_{l\neq k}(a_k^2 - a_l^2)^2}. \tag{3.26}$$

In the case of $SO(4)$, you can find the decomposition to $SU(2) \times SU(2)$.

[1]The bare coupling agrees when we correct a misprint in ref.[12].

The Weierstrass form of the curve for the groups $SP(2N)$ is given by

$$y^2 = P^2(x) - \Lambda^{2(N+1)}P(x). \tag{3.27}$$

where

$$P(x) = x^2 \prod_{k=1}^{N}(x^2 - e_k^2). \tag{3.28}$$

The equivalent Riemann surface is[9]

$$f = (z + \frac{\Lambda^{2(N+1)}}{z})^2 - 4P(x) = 0, \tag{3.29}$$

whose equivalence can be checked by evaluating the periods. The meromorphic one form of the curve is obtained as

$$\lambda = \frac{dx}{2\pi i}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{\Gamma(-s)\Gamma(s+1/2)}{\Gamma(1/2)s}x^{-2s}\prod_{k=1}^{N}(x^2-e_k^2)^{-s}(-\Lambda^{2(N+1)})^s, \tag{3.30}$$

where the integration over s takes the poles at $s = 0, 1, ...\infty$. It can be shown that the classical part of the prepotential agrees with the general form and we find that the one instanton contribution is given by

$$\mathcal{F}_1(a) = -\frac{i\Lambda^{2(N+1)}}{4\pi}\frac{(-1)^N}{\prod_{k=1}^{N} a_k^2}. \tag{3.31}$$

Note that all these results agree completely with the one obtained by the direct instanton method[18].

We have shown how to calculate the effective action of $N = 2$ supersymmetric Yang-Mills theories without matter hypermultiplets. At this moment, it is not clear whether our method of analytic continuation is consistent with the integrability of the prepotential at all orders. Although we could obtain rather compact expression for a, we have not been able to obtain the general form of the dual fields a_D. Actually the calculation of the next leading order seems very complicated and requires more simplification of our method.

4 $SU(2)$ Yang-Mills theories with matter fields around the conformal point

Until last section, we have mainly studied the behavior of the theory in the weak coupling region. However it seems more interesting to investigate the structure of the theory in the strong coupling region where one finds the truly non-perturbative results. From here to the end of this article, we will analyze the behavior of the theory in the strong coupling region, especially around the conformal point. To this end, we use the method discussed in the last two section, such as analytic continuation and effective evaluation of the integral representation in the weak coupling region.

In this section we treat the $SU(2)$ theory with N_f matter fields ($N_f \leq 3$), to verify that our approach recover the previous results which was obtained by transformations of the hypergeometric functions as in §2[22].

First of all we consider $N_f = 1$ case, whose third order curve of $N_f = 1$ theory is given from (2.3) by Möbius transformation:

$$y^2 = x^2(x-u) + \frac{1}{4}m\Lambda^3 x - \frac{\Lambda^6}{64}. \tag{4.1}$$

In order to calculate the periods:

$$\frac{\partial a}{\partial u} = \oint_\alpha \frac{dx}{y}, \quad \frac{\partial a_D}{\partial u} = \oint_\beta \frac{dx}{y}, \tag{4.2}$$

around the conformal point $u = \frac{3}{4}\Lambda^2$, $m = \frac{3}{4}\Lambda$ where the curve becomes degenerate as $y^2 = (x - \frac{\Lambda^2}{4})^3$, we introduce the deviations from this conformal point as $\tilde{u} = u - \frac{3}{4}\Lambda^2$, $\tilde{m} = m - \frac{3}{4}\Lambda$. The strategy is that after calculating the period in the weak coupling region $\tilde{u} \sim \infty$, we analytically continue around the conformal point $\tilde{u} \sim 0$. It should be noted that similar consideration has been used to evaluate periods for Calabi-Yau manifolds [24]. Rescaling the variable as $x = \frac{1}{4}\Lambda^2 z$ the curve becomes

$$y^2 = (\frac{\Lambda^2}{4})^3(z-1)^3 - (\frac{\Lambda^2}{4})^2\tilde{u}z^2 - (\frac{\Lambda^2}{4})^2\Lambda m z, \tag{4.3}$$

and thus we consider $\frac{\tilde{u}}{\Lambda^2}$ and $\frac{\tilde{m}}{\Lambda}$ as perturbations from the conformal point. Expanding the period with respect to $1/\tilde{u}$ we have the expression for the period as follows:

$$\oint \frac{dx}{y} = \tilde{u}^{-1/2} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}+s)\Gamma(s+1)\Gamma(-s)}{\Gamma(\frac{1}{2})\Gamma(s-m+1)m!}$$
$$\times \oint dz\,(1-z)^{3(s-m)} z^{-2s+m-1} \left(\frac{\Lambda^2}{4\tilde{u}}\right)^s \left(\frac{\tilde{m}}{\Lambda}\right)^m, \tag{4.4}$$

where we have introduced Barnes-type integral representation. From this expression we can find that $\frac{\partial a}{\partial u}$ is obtained by picking up poles $z=0$ along α cycle in the weak coupling region. In this way, we find $\frac{\partial a}{\partial u}$ in the weak coupling region is of the form

$$\begin{aligned}\frac{\partial a}{\partial u} &= \frac{\sqrt{3}}{\tilde{u}^{1/2}2\pi}\sum_{n,m=0}^{\infty}\frac{\Gamma(n+m+\frac{1}{2})\Gamma(n+\frac{1}{3})\Gamma(n+\frac{2}{3})}{\Gamma(n+\frac{m}{2}+\frac{1}{2})\Gamma(n+\frac{m}{2}+1)n!m!}\left(-\frac{27\Lambda^2}{16\tilde{u}}\right)^n\left(\frac{\Lambda\tilde{m}}{8\tilde{u}}\right)^m \\ &= \frac{\sqrt{3}}{\tilde{u}^{1/2}2\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{m+1}{2})\Gamma(\frac{m}{2}+1)m!}\left(\frac{\Lambda\tilde{m}}{8\tilde{u}}\right)^m \\ &\qquad \times {}_3F_2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2}+m;\frac{m+1}{2},\frac{m}{2}+1;-\frac{27\Lambda^2}{16\tilde{u}}\right),\end{aligned} \tag{4.5}$$

where ${}_3F_2$ is the generalized hypergeometric function, which is defined as [26]

$${}_pF_{p-1}(a_1,\cdots,a_p;b_1,\cdots,b_{p-1};z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_{p-1})_n}\frac{z^n}{n!},\quad (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}. \tag{4.6}$$

Integrating with respect to $\tilde{u}$ of (2.5) we have Higgs field a up to mass residue in the weak coupling region in the following form

$$\begin{aligned}a &= \frac{\sqrt{3}\tilde{u}^{1/2}}{\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{m+1}{2})\Gamma(\frac{m}{2}+1)m!}\left(\frac{\Lambda\tilde{m}}{8\tilde{u}}\right)^m \\ &\qquad \times {}_3F_2\left(\frac{1}{3},\frac{2}{3},m-\frac{1}{2};\frac{m+1}{2},\frac{m}{2}+1;-\frac{27\Lambda^2}{16\tilde{u}}\right).\end{aligned} \tag{4.7}$$

The analytic continuation from this expression to around the conformal point can be performed to obtain

$$\begin{aligned}a = \sqrt{\pi}\tilde{u}^{1/2}\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}\sum_{m=0}^{\infty}&\left\{\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(m-\frac{5}{6})}{\Gamma(\frac{11}{6})\Gamma(-\frac{1}{6})\Gamma(m+\frac{1}{3})m!}\left(\frac{\Lambda\tilde{m}}{4\tilde{u}}\right)^m\right. \\ &\qquad \times {}_3F_2\left(\frac{1}{3},-\frac{m}{2}+\frac{5}{6},-\frac{m}{2}+\frac{1}{3};\frac{2}{3},-m+\frac{11}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right) \\ -\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}&\frac{\Gamma(\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(m-\frac{7}{6})}{\Gamma(-\frac{7}{6})\Gamma(\frac{13}{6})\Gamma(m-\frac{1}{3})m!}\left(\frac{\Lambda\tilde{m}}{4\tilde{u}}\right)^m \\ &\left.\qquad \times {}_3F_2\left(\frac{2}{3},-\frac{m}{2}+\frac{7}{6},-\frac{m}{2}+\frac{2}{3};\frac{4}{3},-m+\frac{13}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right)\right\}.\end{aligned} \tag{4.8}$$

If we set $\tilde{m}=0$ we can recover the previous result in §2[22] where a is represented by the generalized hypergeometric function ${}_3F_2$ in terms of $\tilde{u}$.

Next we consider a_D. In this case we integrate (2.4) from $z=0$ to $z=1$ and evaluate double poles which give the logarithmic terms as in §3[23]. Quite similarly a_D in the weak coupling region can be written as

$$\frac{\partial a_D}{\partial u} = \frac{-1}{2(-)^{\frac{1}{2}}\tilde{u}^{\frac{1}{2}}\pi}\sum_{n,m}\frac{\Gamma(n+m+\frac{1}{2})\Gamma(3n+1)}{\Gamma(\frac{1}{2})\Gamma(n+1)^2\Gamma(2n+m+1)}\left(\frac{\Lambda^2}{4\tilde{u}}\right)^n\left(\frac{\Lambda\tilde{m}}{4\tilde{u}}\right)^m$$
$$\times\left[\psi(n+m+\frac{1}{2})+3\psi(3n+1)-2\psi(n+1)-2\psi(2n+m+1)+\ln\left(\frac{\Lambda^2}{4\tilde{u}}\right)\right] \quad (4.9)$$

where $\psi(z)$ is defined by $\frac{d\Gamma(z)}{dz}=\psi(z)\Gamma(z)$. Analytic continuation to the region $\tilde{u}\sim 0$ and integration with respect to $\tilde{u}$ give the expression up to mass residue around the conformal point as follows

$$a_D = \frac{-\sqrt{\pi}\tilde{u}^{\frac{1}{2}}}{(-1)^{\frac{1}{2}}}\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}\sum_{m=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(m-\frac{5}{6})}{\Gamma(\frac{11}{6})\Gamma(-\frac{1}{6})\Gamma(m+\frac{1}{3})m!}\left(\frac{\Lambda m}{4\tilde{u}}\right)^m\right.$$
$$\times\,{}_3F_2\left(\frac{1}{3},-\frac{m}{2}+\frac{5}{6},-\frac{m}{2}+\frac{1}{3};\frac{2}{3},-m+\frac{11}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right)$$
$$+\left(\frac{27\Lambda^2}{16\tilde{u}}\right)^{-1/3}\frac{\Gamma(\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(m-\frac{7}{6})}{\Gamma(-\frac{7}{6})\Gamma(\frac{13}{6})\Gamma(m-\frac{1}{3})m!}\left(\frac{\Lambda m}{4\tilde{u}}\right)^m \quad (4.10)$$
$$\left.\times\,{}_3F_2\left(\frac{2}{3},-\frac{m}{2}+\frac{7}{6},-\frac{m}{2}+\frac{2}{3};\frac{4}{3},-m+\frac{13}{6};-\frac{16\tilde{u}}{27\Lambda^2}\right)\right\}.$$

As the parameter approaching the point $\frac{\tilde{m}}{\tilde{u}}\to 0,\ \frac{\tilde{u}}{\Lambda^2}\to 0$, we find $a=a_D$, which implies that the theory is completely free theory. Therefore the conformal point is certainly the fixed point where the beta function of the theory vanishes. Since $a,\ a_D\sim\tilde{u}^{\frac{5}{6}}$ near the conformal point and $a,\ a_D$ are proportional to mass scale of the theory, the conformal dimension of $\tilde{u}$ is $\frac{6}{5}$, which has been observed in [20]. This implies that the conformal point is a non-trivial fixed point where $\tilde{u}$ has a fractional scaling behavior.

In the $N_f=2$ theory, we use the curve of forth order:

$$y^2 = (x^2-u+\frac{\Lambda^2}{8})^2-\Lambda^2(x+m_1)(x+m_2) \quad (4.11)$$
$$= (x^2-u+\frac{\Lambda^2}{8})^2-\Lambda^2(x^2+Mx+N), \quad (4.12)$$

where we introduce symmetrized mass parameters $M=m_1+m_2$ and $N=m_1m_2$. We shift the parameters from the conformal point as $\tilde{u}=u-\frac{3\Lambda^2}{8},\ \ \tilde{M}=M-\Lambda,\ \ \tilde{N}=N-\frac{\Lambda^2}{4}$, and rescale $x+\frac{\Lambda}{2}=\tilde{u}^{1/2}z$, we find that the curve can be written as

$$y^2=\tilde{u}^2(z^2-1)^2-2\Lambda\tilde{u}^{\frac{3}{2}}(z^3-z)-\Lambda^2\tilde{M}\tilde{u}^{\frac{1}{2}}z-\Lambda^2\tilde{N}', \quad (4.13)$$

where $\tilde{N}' = \tilde{N} - \tilde{M}\Lambda/2$. As is the case of $N_f = 1$, we evaluate the period in the weak coupling region by expanding with respect to $1/\tilde{u}$ and mass parameters, and by picking up poles at $z = 1$ along the α cycle to find

$$\begin{aligned}\frac{\partial a}{\partial u} = & \frac{\tilde{u}^{-\frac{1}{2}}}{2\pi}\sum_{m,l=0}^{\infty}\frac{\Gamma(2\alpha_{l,m}+\frac{1}{2})\Gamma(\beta_{l,m}+\frac{1}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+1)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m \\ & \times{}_4F_3\left(\alpha_{l,m}+\frac{1}{4},\alpha_{l,m}+\frac{3}{4},l+\frac{1}{2},\beta_{l,m}+\frac{1}{2};\frac{1}{2},2\alpha_{l,m}+\frac{1}{2},2\alpha_{l,m}+1;-\frac{\Lambda^2}{\tilde{u}}\right) \\ & +\frac{\tilde{u}^{-\frac{1}{2}}}{2\pi}\left(\frac{2\Lambda}{\tilde{u}^{\frac{1}{2}}}\right)\left(\frac{\Lambda^2\tilde{M}}{\tilde{u}^{\frac{3}{2}}}\right)\sum_{m,l=0}^{\infty}\frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(\beta_{l,m}+\frac{5}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+4)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m \\ & \times{}_4F_3\left(\alpha_{l,m}+\frac{5}{4},\alpha_{l,m}+\frac{7}{4},l+\frac{3}{2},\beta_{l,m}+\frac{5}{2};\frac{3}{2},2\alpha_{l,m}+\frac{5}{2},2\alpha_{l,m}+2;-\frac{\Lambda^2}{\tilde{u}}\right),\end{aligned} \tag{4.14}$$

where $\alpha_{l,m} = l + \frac{m}{2}$, $\beta_{l,m} = 2m + 3l$. In the $N_f = 2$ theory, $\frac{\partial a}{\partial u}$ has a additional part which is proportional to $\tilde{M}$ and vanishes when $\tilde{M} = 0$.

Next we consider $\frac{\partial a_D}{\partial u}$. Performing the line integral from $z = 0$ to $z = 1$ and evaluating double poles, we have $\frac{\partial a_D}{\partial u}$ in the weak coupling region as

$$\begin{aligned}\frac{\partial a_D}{\partial u} = & \frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\sum_{l,m,n}\frac{\Gamma(2\alpha_{l,m}+\frac{1}{2})\Gamma(\beta_{l,m}+\frac{1}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+1)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m\left(-\frac{\Lambda^2}{\tilde{u}}\right)^n \\ & \times\frac{(\alpha_{l,m}+\frac{1}{4})_n(\alpha_{l,m}+\frac{3}{4})_n(l+\frac{1}{2})_n(\beta_{l,m}+\frac{1}{2})_n}{(\frac{1}{2})_n(2\alpha_{l,m}+\frac{1}{2})_n(2\alpha_{l,m}+1)_n}\left[\ln\left(-\frac{\Lambda^2}{\tilde{u}}\right)\right. \\ & +\psi_n(\alpha_{l,m}+\frac{1}{4})+\psi_n(\alpha_{l,m}+\frac{3}{4})+\psi_n(\beta_{l,m}+\frac{1}{2}) \\ & \left.+\psi_n(l+\frac{1}{2})-\psi_n(1)-\psi_n(\frac{1}{2})-\psi_n(2\alpha_{l,m}+\frac{1}{2})-\psi_n(2\alpha_{l,m}+1)\right] \\ & +\frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\left(\frac{2\Lambda}{\tilde{u}}\right)\left(\frac{\Lambda^2\tilde{M}}{\tilde{u}^{\frac{3}{2}}}\right)\sum_{l,m,n}\frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(\beta_{l,m}+\frac{5}{2})}{\Gamma(l+1)\Gamma(m+1)\Gamma(4\alpha_{l,m}+4)}\left(-\frac{\Lambda^4\tilde{M}^2}{4\tilde{u}^3}\right)^l\left(\frac{\Lambda^2\tilde{N}'}{\tilde{u}^2}\right)^m \\ & \times\left(-\frac{\Lambda^2}{\tilde{u}}\right)^n\frac{(\alpha_{l,m}+\frac{5}{4})_n(\alpha_{l,m}+\frac{7}{4})_n(l+\frac{3}{2})_n(\beta_{l,m}+\frac{5}{2})_n}{(\frac{3}{2})_n(2\alpha_{l,m}+\frac{5}{2})_n(2\alpha_{l,m}+2)_n}\left[\ln\left(-\frac{\Lambda^2}{\tilde{u}}\right)\right. \\ & +\psi_n(\alpha_{l,m}+\frac{5}{4})+\psi_n(\alpha_{l,m}+\frac{7}{4})+\psi_n(\beta_{l,m}+\frac{5}{2}) \\ & \left.+\psi_n(l+\frac{3}{2})-\psi_n(1)-\psi_n(\frac{3}{2})-\psi_n(2\alpha_{l,m}+2)-\psi_n(2\alpha_{l,m}+\frac{5}{2})\right],\end{aligned} \tag{4.15}$$

where $\psi_n(\alpha) = \psi(n + \alpha)$.

Analytic continuation of $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$ to the region $\tilde{u} \sim 0$ gives four kinds of ${}_4F_3$, and a

and a_D are also represented by ${}_4F_3$ after integration with respect to $\tilde{u}$. By defining Φ as

$$\Phi(\delta,\epsilon;\rho,\sigma,\mu) = {}_4F_3\left(-\alpha_{l,m}+\delta, -\alpha_{l,m}+\delta+\frac{1}{2}, \alpha_{l,m}+\epsilon, \alpha_{l,m}+\epsilon+\frac{1}{2}\right. \qquad (4.16)$$
$$\left. ;\alpha_{l,m}-\beta_{l,m}+\rho, \frac{m}{2}+\sigma, \mu; -\frac{\tilde{u}}{\Lambda^2}\right),$$

and using this function, we find that a around the conformal point can be written in the form:

$$\begin{aligned} a = \frac{\sqrt{\pi}\tilde{u}^{\frac{1}{2}}}{2}\sum_{m,l}\left(-\frac{\Lambda^2\tilde{M}^2}{\tilde{u}^2}\right)^l\left(\frac{\Lambda\tilde{N}'}{2\tilde{u}^{\frac{3}{2}}}\right)^m \frac{1}{\Gamma(\frac{1}{2})\Gamma(m+1)\Gamma(2l+1)\sqrt{2}} \\ \times\left[c_1\left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{1}{4}}\frac{\Gamma(-\frac{m}{2}+\frac{1}{4})\Gamma(\beta_{l,m}-\alpha_{l,m}+\frac{1}{4})}{\Gamma(\alpha_{l,m}+\frac{3}{4})\Gamma(\frac{1}{4}-\alpha_{l,m})}\Phi\left(\frac{1}{4},\frac{1}{4};\frac{7}{4},\frac{3}{4},\frac{1}{2}\right)\right. \\ -c_2\left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{3}{4}}\frac{\Gamma(2\alpha_{l,m}+\frac{1}{2})\Gamma(-\frac{m}{2}-\frac{1}{4})\Gamma(\beta_{l,m}-\alpha_{l,m}-\frac{1}{4})}{\Gamma(\alpha_{l,m}+\frac{1}{4})\Gamma(-\alpha_{l,m}-\frac{1}{4})\Gamma(2\alpha_{l,m}-\frac{1}{2})}\Phi\left(\frac{3}{4},\frac{3}{4};\frac{9}{4},\frac{5}{4},\frac{3}{2}\right) \qquad (4.17) \\ +c_3\left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{3}{4}}\left(\frac{\Lambda^2\tilde{M}}{2\tilde{u}^{\frac{3}{2}}}\right)\frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(\frac{1}{4}-\frac{m}{2})\Gamma(\beta_{l,m}-\alpha_{l,m}+\frac{5}{4})}{\Gamma(\alpha_{l,m}+\frac{3}{4})\Gamma(\frac{1}{4}-\alpha_{l,m})\Gamma(2\alpha_{l,m}+\frac{3}{2})}\Phi\left(-\frac{1}{4},\frac{3}{4};\frac{3}{4},\frac{3}{4},\frac{1}{2}\right) \\ \left.-c_4\left(\frac{\Lambda^2}{\tilde{u}}\right)^{-\frac{5}{4}}\left(\frac{\Lambda^2\tilde{M}}{2\tilde{u}^{\frac{3}{2}}}\right)\frac{\Gamma(2\alpha_{l,m}+\frac{5}{2})\Gamma(-\frac{m}{2}-\frac{1}{4})\Gamma(\beta_{l,m}-\alpha_{l,m}+\frac{3}{4})}{\Gamma(\alpha_{l,m}+\frac{1}{4})\Gamma(-\alpha_{l,m}-\frac{1}{4})\Gamma(2\alpha_{l,m}+\frac{3}{2})}\Phi\left(\frac{1}{4},\frac{5}{4};\frac{5}{4},\frac{5}{4},\frac{3}{2}\right)\right], \end{aligned}$$

where $c_1 = c_2 = c_3 = c_4 = 1$. We find that the expression for a_D is given by changing c_i as $c_1 = c_3 = (-1)^m$, $c_2 = c_4 = -(-1)^m$. If we set $\tilde{M} = \tilde{N}' = 0$ we can recover the previous result in §2[22]. As in the $N_f = 1$ theory, we see that the conformal point is the fixed point of this theory from the relation $a \sim a_D$ on this point. Reading the leading power of the expression (4.17), we see that the conformal dimension of $\tilde{u}$ is $\frac{4}{3}$ [20].

In the $N_f = 3$ theory, the curve is given by

$$\begin{aligned} y^2 &= (x^2 - u + \frac{\Lambda}{4}x + \frac{(m_1+m_2+m_3)\Lambda}{8})^2 - \Lambda(x+m_1)(x+m_2)(x+m_3) \\ &= (x^2 - u + \frac{\Lambda}{4}x + \frac{\Lambda L}{8})^2 - \Lambda(x^3 + Lx^2 + Mx + N), \end{aligned} \qquad (4.18)$$

where $L = m_1 + m_2 + m_3$, $M = m_1m_2 + m_2m_3 + m_3m_1$, $N = m_1m_2m_3$. We shift the parameter from the conformal point as $u' = u - \frac{\Lambda^2}{32}$, $\tilde{L} = L - \frac{3\Lambda}{8}$, $\tilde{M} = M - \frac{3\Lambda^2}{64}$, $\tilde{N} = N - \frac{\Lambda^3}{512}$, the curve becomes

$$y^2 = (x+\frac{\Lambda}{8})^3(x-\frac{7\Lambda}{8}) - 2(u'-\frac{\Lambda\tilde{L}}{8})(x+\frac{\Lambda}{8})^2 - \Lambda(\tilde{L}x^2 + \tilde{M}x + \tilde{N}) + (u'-\frac{\Lambda\tilde{L}}{8})^2 \qquad (4.19)$$

Setting $\tilde{u} = u' - \frac{\Lambda\tilde{L}}{8}$ and rescaling $x + \frac{\Lambda}{8} = \tilde{u}^{1/2}z$, we find that the curve can be written as

$$y^2 = \tilde{u}^2(z^2-1)^2 - \tilde{u}^{\frac{3}{2}}\Lambda z^3 - \tilde{L}\Lambda\tilde{u}z^2 - \tilde{u}^{\frac{1}{2}}Az + B, \tag{4.20}$$

where $A = \Lambda\tilde{M} - \frac{\Lambda^2\tilde{L}}{4}$, $B = \frac{\Lambda^2\tilde{L}^2}{64} + \frac{\Lambda^2\tilde{M}}{8} - \Lambda\tilde{N}$. Evaluation of the integral for the period and analytic continuation from $\tilde{u} \sim \infty$ to $\tilde{u} \sim 0$ are same as $N_f = 2$ case. In this way, we can obtain the period in the weak coupling region in the form:

$$\begin{aligned}\frac{\partial a}{\partial u} = &\frac{\tilde{u}^{-\frac{1}{2}}}{2\sqrt{\pi}}\sum_{l,m,p,q}^{\infty}\frac{\Gamma(3\eta_{l,p}+\frac{1}{2})\Gamma(\omega_{l,p,q}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2\chi_{l,p,q}+1)l!m!(2p)!}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}^3}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q\\
&\times {}_4F_3\Big(\eta_{l,p}+\frac{1}{6},\eta_{l,p}+\frac{1}{2},\eta_{l,p}+\frac{5}{6},\omega_{l,p,q}+\frac{1}{2}\\
&\qquad ;\frac{1}{2},\chi_{l,p,q}+\frac{1}{2},\chi_{l,p,q}+1;-\frac{27\Lambda^2}{256\tilde{u}}\Big).\end{aligned} \tag{4.21}$$

$$\begin{aligned}&-\frac{\tilde{u}^{-\frac{1}{2}}}{2\sqrt{\pi}}\left(\frac{\Lambda A}{\tilde{u}^2}\right)\sum_{l,m,p,q}^{\infty}\frac{\Gamma(3\eta_{l,p}+\frac{5}{2})\Gamma(\omega_{l,p,q}+\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(2\chi_{l,p,q}+3)l!m!(2p+1)!}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}^3}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q\\
&\times {}_4F_3\Big(\eta_{l,p}+\frac{5}{6},\eta_{l,p}+\frac{7}{6},\eta_{l,p}+\frac{9}{6},\omega_{l,p,q}+\frac{5}{2}\\
&\qquad ;\frac{3}{2},\chi_{l,p,q}+\frac{3}{2},\chi_{l,p,q}+2;-\frac{27\Lambda^2}{256\tilde{u}}\Big),\end{aligned} \tag{4.22}$$

$$\begin{aligned}\frac{\partial a_D}{\partial u} = &\frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\sum_{l,m,p,q}^{\infty}\frac{\Gamma(3\eta_{l,p}+\frac{1}{2})\Gamma(\omega_{l,p,q}+\frac{1}{2})}{\Gamma(2\chi_{l,p,q}+1)l!m!(2p)!}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q\\
&\times\sum_{n=0}^{\infty}\frac{(\eta_{l,p}+\frac{1}{6})_n(\eta_{l,p}+\frac{1}{2})_n(\eta_{l,p}+\frac{5}{6})_n(\omega_{l,p,q}+\frac{1}{2})_n}{(\frac{1}{2})_n(\chi_{l,p,q}+\frac{1}{2})_n(\chi_{l,p,q}+1)_n n!}\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)^n\\
&\times\Big[\ln\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)+\psi_n(\eta_{l,p}+\frac{1}{6})+\psi_n(\eta_{l,p}+\frac{1}{2})+\psi_n(\eta_{l,p}+\frac{5}{6})\\
&\quad+\psi_n(\omega_{l,p,q}+\frac{1}{2})-\psi_n(\frac{1}{2})-\psi_n(\chi_{l,p,q}+\frac{1}{2})-\psi_n(\chi_{l,p,q}+1)-\psi_n(1)\Big]\\
&-\frac{\tilde{u}^{-\frac{1}{2}}}{4\pi^2 i}\left(\frac{\Lambda A}{\tilde{u}^2}\right)\sum_{l,m,p,q}^{\infty}\frac{\Gamma(3\eta_{l,p}+\frac{5}{2})\Gamma(\omega_{l,p,q}+\frac{5}{2})}{\Gamma(2\chi_{l,p,q}+3)l!m!(2p+1)!}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^l\left(\frac{A^2}{\tilde{u}}\right)^p\left(\frac{B}{\tilde{u}^2}\right)^q\\
&\times\sum_{n=0}^{\infty}\frac{(\eta_{l,p}+\frac{5}{6})_n(\eta_{l,p}+\frac{7}{6})_n(\eta_{l,p}+\frac{9}{6})_n(\omega_{l,p,q}+\frac{5}{2})_n}{(\frac{3}{2})_n(\chi_{l,p,q}+\frac{3}{2})_n(\chi_{l,p,q}+2)_n n!}\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)^n\\
&\times\Big[\ln\left(-\frac{27\Lambda^2}{256\tilde{u}}\right)+\psi_n(\eta_{l,p}+\frac{5}{6})+\psi_n(\eta_{l,p}+\frac{7}{6})+\psi_n(\eta_{l,p}+\frac{9}{6})\\
&\quad+\psi_n(\omega_{l,p,q}+\frac{5}{2})-\psi_n(\frac{3}{2})-\psi_n(\chi_{l,p,q}+\frac{3}{2})-\psi_n(\chi_{l,p,q}+2)-\psi_n(1)\Big],\end{aligned} \tag{4.23}$$

where

$$\eta_{l,p}=\frac{l}{3}+\frac{p}{3},\quad \omega_{l,p,q}=l+3p+2q,\quad \chi_{l,p,q}=\frac{l}{2}+p+\frac{q}{2}. \tag{4.24}$$

By analytic continuation and by integration with respect to $\tilde{u}$, we obtain a around the conformal point in the form:

$$\begin{aligned}
a=&-2u^{\frac{1}{2}}\sum_{l,p,q}\frac{2^{\chi_{l,p,q}+1}}{l!(2p)!q!3^{\eta_{l,p}}}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^{l}\left(\frac{A^2}{\tilde{u}^3}\right)^{p}\left(\frac{B}{\tilde{u}^2}\right)^{q}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{-\eta_{l,p}}\\
&\times\left\{\frac{c_1\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+\frac{1}{3})\Gamma(\eta_{l,p}+\frac{1}{6})}{\Gamma(\frac{1}{3}-\eta_{l,p})\Gamma(\frac{1}{3}+\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{5}{6}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{1}{6}}\Psi\left(\frac{1}{6},\frac{1}{6};\frac{1}{3},\frac{2}{3},\frac{5}{3}\right)\right.\\
&+\frac{c_2\Gamma(-\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p})\Gamma(\eta_{l,p}+\frac{1}{2})}{\Gamma(-\eta_{l,p})\Gamma(\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{1}{2}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{1}{2}}\Psi\left(\frac{1}{2},\frac{1}{2};\frac{2}{3},\frac{4}{3},2\right)\\
&\left.+\frac{c_3\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}-\frac{1}{3})\Gamma(\eta_{l,p}+\frac{5}{6})}{\Gamma(-\eta_{l,p}-\frac{1}{3})\Gamma(\chi_{l,p,q}-\eta_{l,p}-\frac{1}{3})\Gamma(\chi_{l,p,q}-\eta_{l,p}+\frac{1}{6})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{5}{6}}\Psi\left(\frac{5}{6},\frac{5}{6};\frac{4}{3},\frac{5}{3},\frac{7}{3}\right)\right\}\\
&+\frac{2\pi\sqrt{\pi}\Lambda A}{\tilde{u}^{\frac{1}{2}}}\sum_{l,p,q}\frac{2^{\chi_{l,p,q}+2}}{l!(2p)!q!3^{\eta_{l,p}+2}}\left(\frac{\Lambda\tilde{L}}{\tilde{u}}\right)^{l}\left(\frac{A^2}{\tilde{u}^3}\right)^{p}\left(\frac{B}{\tilde{u}^2}\right)^{q}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{-\eta_{l,p}}\\
&\times\left\{\frac{c_4\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+\frac{5}{3})\Gamma(\eta_{l,p}+\frac{5}{6})}{\Gamma(\frac{2}{3}-\eta_{l,p})\Gamma(\frac{2}{3}+\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{7}{6}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{5}{6}}\Psi\left(\frac{1}{3},-\frac{1}{6};\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)\right.\\
&+\frac{c_5\Gamma(-\frac{1}{3})\Gamma(\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+\frac{4}{3})\Gamma(\eta_{l,p}+\frac{7}{6})}{\Gamma(\frac{1}{3}-\eta_{l,p})\Gamma(\frac{1}{3}+\chi_{l,p,q}-\eta_{l,p})\Gamma(\frac{5}{6}+\chi_{l,p,q}-\eta_{l,p})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{7}{6}}\Psi\left(\frac{2}{3},\frac{1}{6};\frac{2}{3},\frac{4}{3},\frac{2}{3}\right)\\
&\left.+\frac{c_6\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{3})\Gamma(\omega_{l,p,q}-\eta_{l,p}+1)\Gamma(\eta_{l,p}+\frac{3}{2})}{\Gamma(-\eta_{l,p})\Gamma(\chi_{l,p,q}-\eta_{l,p})\Gamma(\chi_{l,p,q}-\eta_{l,p}+\frac{1}{2})}\left(-\frac{256\tilde{u}}{27\Lambda^2}\right)^{\frac{3}{2}}\Psi\left(1,1;\frac{4}{3},\frac{5}{3},1\right)\right\},
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
\Psi(a,b;c,d,e)=&{}_4F_3\left(\eta_{l,p}+a,\eta_{l,p}+a+\frac{1}{2},\eta_{l,p}-\chi_{l,p,q}+b,\eta_{l,p}-\chi_{l,p,q}+b+\frac{1}{2}\right.\\
&\left.;c,d,\eta_{l,p}-\omega_{l,p,q}+e;-\frac{256\tilde{u}}{27\Lambda^2}\right),
\end{aligned} \tag{4.26}$$

and $c_1=\cdots=c_6=1$. The expression for a_D is given by changing c_i as $c_1=\cot(\eta_{l,p}+\frac{1}{6})\pi,\ c_2=\cot(\eta_{l,p}+\frac{1}{2})\pi,\ c_3=c_4=\cot(\eta_{l,p}+\frac{5}{6})\pi,\ c_5=\cot(\eta_{l,p}+\frac{7}{6})\pi,\ c_6=\cot(\eta_{l,p}+\frac{3}{2})\pi$. If we set $\tilde{L}=\tilde{M}=\tilde{N}=0$, i.e., $A=B=0$ and $\tilde{u}=u'=u-\frac{\Lambda^2}{32}$, we can recover the previous result in §2 [22]. As was the case of $N_f=1,2$, the relation $a\sim a_D$ hold on the conformal point, therefore we can recognize the conformal point is the fixed point of this theory.

Let us compare our expressions to the ones obtained by the expansion around the different point from the conformal point. If we consider the massive theory as the generalization from the massless theory, we would treat the bare mass parameter as the deviation from the massless theory. In order to see the behavior of the field a and a_D in the weak coupling region in this case, we expand the meromorphic differential λ with respect to Λ and mass parameter, and evaluate the integral representation along the corresponding cycle. For example in the case of $N_f = 1$, λ, a and a_D are given by

$$\begin{aligned} \lambda &= \frac{x\,dx}{2\pi i\,y}\left(\frac{(x^2-u)}{2(x+m)} - (x^2-u)'\right), \\ a &= \oint_\alpha \lambda, \qquad a_D = \oint_\beta \lambda, \end{aligned} \tag{4.27}$$

where we use the curve of forth order. The result of the calculation for the field a can be written as

$$\begin{aligned} a &= \frac{\sqrt{u}}{12\sqrt{\pi}} \sum_{n,l=0} \frac{\Gamma(n-\frac{1}{6})\Gamma(n+\frac{1}{6})\Gamma(l-n)\Gamma(l+3n-\frac{1}{2})}{\Gamma(n+1)\Gamma(-n)\Gamma(3n-\frac{1}{2})\Gamma(l+\frac{1}{2})n!l!}\left(\frac{m^2}{u}\right)^l\left(\frac{-27\Lambda^6}{256u^3}\right)^n \\ &+ \frac{3\sqrt{u}}{32\sqrt{\pi}}\left(\frac{\Lambda^3 m}{u^2}\right)\sum_{n,l=0}\frac{\Gamma(n+\frac{7}{6})\Gamma(n+\frac{5}{6})\Gamma(l-n)\Gamma(l+3n+\frac{3}{2})}{(2n+1)\Gamma(n+1)\Gamma(-n)\Gamma(3n+\frac{3}{2})\Gamma(l+\frac{3}{2})n!l!}\left(\frac{m^2}{u}\right)^l\left(\frac{-27\Lambda^6}{256u^3}\right)^n. \end{aligned} \tag{4.28}$$

In the massless limit, this expression reduces to the previous result obtained by solving the Picard-Fucks equation [13], which is represented by using the Gauss' hypergeometric function. The expression (2.29) can be verified by expanding the following expression which is represented by using the modular invariant form given in §2 [22]:

$$\begin{aligned} \frac{\partial a}{\partial u} &= (-D)^{-\frac{1}{4}} F\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{27\Delta}{4D^3}\right), \\ \Delta &= -\Lambda^6(256u^3 - 256u^2m^2 - 288um\Lambda^3 + 256m^3\Lambda^3 + 27\Lambda^6), \\ D &= -16u^2 + 12m\Lambda^3. \end{aligned} \tag{4.29}$$

in the weak coupling region, and by comparing two expressions order by order after u integration. In the $N_f = 2, 3$ case, instead of integrating λ to obtain fields a and a_D, we can evaluate $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$ by expanding around the massless point in a similar manner. The results are expressed in terms of the following arguments:

$$\frac{1}{64}\left(\frac{\Lambda^2}{u}\right)^2 \ (N_f = 2), \qquad \frac{1}{256}\left(\frac{\Lambda^2}{u}\right) \ (N_f = 3), \tag{4.30}$$

and appropriate combinations of mass parameters. These are identical to the argument of the hypergeometric function describing the massless theories [13]. These powers of Λ are equivalent to the powers of the instanton term of the curve, and vary as the number of matters we have introduced. On the contrary, the argument of the expression we have derived in this section is simple compared to (4.29), which is the argument based of the the deviation from the conformal point, and the form of these deviations does not depend on the number of the matters as we have seen in this section. Thus if we use the parameterization from the conformal point, the theory can be described by using the simple deviation from the conformal point even in such case that we discuss the weak coupling behavior. Furthermore the expression around the massless point in the $N_f = 1, 2$ case can be obtained from our expression for the $N_f = 3$ case by taking suitable double scaling limit to decouple the irrelevant mass parameters. These are obvious advantages to observe the behavior of the theory by using the expression around the conformal point.

Before closing this section, we discuss the relation between 4-D $SU(2)$ $N = 2$ supersymmetric QCD and 2-D $N = 2$ SCFT, which has been partially analyzed in our previous paper [22]. Let us review the Landau-Ginzburg description of 2-D $N = 2$ superconformal minimal models with $c = 3$ which describe the torus. Since the theory with central charge $c = 3k/k + 2$ corresponds to the Landau-Ginzburg potential x^{k+2}, we have three type of description; $(k = 1)^3$, $(k = 2)^2$ and $(k = 1)(k = 4)$, as

$$\begin{aligned} f_1 &= x^3 + y^3 + z^3, \\ f_2 &= x^4 + y^4 + z^2, \\ f_3 &= x^6 + y^3 + z^2. \end{aligned} \tag{4.31}$$

These are known as the algebraic curve on the (weighted) complex projective space $(W)\mathbf{CP^2}$ with homogeneous coordinates $[x, y, z]$ describing singular torus, and their typical deformation in one parameter ψ are following

$$\begin{aligned} \tilde{E}_6 &: f = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - \psi_6 xyz = 0, \\ \tilde{E}_7 &: f = \frac{x^4}{4} + \frac{y^4}{4} + \frac{z^2}{2} - \psi_7 xyz = 0, \end{aligned} \tag{4.32}$$

$$\tilde{E}_8 \ : f = \frac{x^6}{6} + \frac{y^3}{3} + \frac{z^2}{2} - \psi_8 xyz = 0,$$

where we have used appropriate normalization. We can evaluate the period $\mathcal{W}$:

$$\mathcal{W} = \psi \int_\Gamma \frac{dxdydz}{f}, \tag{4.33}$$

on each curve in the region $\psi \sim \infty$ by picking up poles of the integrant expanded by $1/\psi$. Alternative approach to obtain the period is solving the Picard-Fuchs equation corresponding to these curves

$$(1-y)y\frac{d^2\mathcal{W}}{dy^2} + (1-2y)\frac{d\mathcal{W}}{dy} - \frac{1}{\alpha}(1-\frac{1}{\alpha})\mathcal{W} = 0, \tag{4.34}$$

where $y = \psi^{-\alpha}$ and $\alpha = 3$ $(\tilde{E}_6)$, 4 $(\tilde{E}_7)$, 6 $(\tilde{E}_8)$. As a result, periods are expressed as linear combinations of $F(\frac{1}{\alpha}, 1-\frac{1}{\alpha}, 1; y)$ and $F^*(\frac{1}{\alpha}, 1-\frac{1}{\alpha}, 1; y)$ around $y = 0$ where F is Gauss' hypergeometric function ${}_2F_1$, and F^* is another independent solution corresponding to F. Comparing these results to the expression obtained by setting mass deviations zero in the results we have derived in this section, or the more obvious expression in the last part of §2[22], we find that periods of $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ curves are identical to the periods $\frac{\partial a}{\partial u}$, $\frac{\partial a_D}{\partial u}$ of 4-D $N=2$ supersymmetric $SU(2)$ QCD with $N_f = 1$, 2 and 3 matter fields respectively in the weak coupling region $\tilde{u} \sim \infty$ when the theory has the conformal point. In this way, we can find a simple identification between the moduli parameter of each theory, which is

$$\psi^\alpha \longleftrightarrow \tilde{u}, \tag{4.35}$$

up to irrelevant constant factors, and Landau-Ginzburg point $\psi = 0$ of torus corresponds to the fixed point $\tilde{u} = 0$ of N=2 supersymmetric SU(2) QCD. This is another confirmation of our expression around the critical points. It is also interesting that another toric description of torus:

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 - \psi zw = 0, \quad \frac{1}{2}z^2 + \frac{1}{2}w^2 - xy = 0, \tag{4.36}$$

can be regarded as the curve which corresponds to $N_f = 0$ curve whose parameter ψ^2 can be identified by deviation from the dyon point.

5 $SU(N)$ pure Yang-Mills theories around the conformal point

In this section we study pure $SU(N)$ theory around the conformal point. In this case the curve is given by

$$y^2 = (x^N - \sum_{i=2}^{N} s_i x^{N-i})^2 - \Lambda^{2N}, \tag{5.1}$$

where s_i $(2 \leq i \leq N)$ are gauge invariant moduli parameter. We treat meromorphic differential λ directly, and calculate the period of meromorphic differential λ, i.e. Higgs field and its dual, which are defined by

$$\lambda = \frac{x}{2\pi i}(x^N - \sum_{i=2}^{N} s_i x^{N-i})' \frac{dx}{y} \tag{5.2}$$

$$a_i = \oint_{\alpha_i} \lambda, \quad a_D^i = \oint_{\beta_i} \lambda. \tag{5.3}$$

We consider so called Z_N critical point $s_2 = \cdots = s_{N-1} = 0$, $s_N = \pm\Lambda^N$ where the curve becomes [19, 20, 21]

$$y^2 = x^N(x^N \pm 2\Lambda^N). \tag{5.4}$$

First we evaluate the integral in the region $s_i \sim 0$ $(2 \leq i \leq N-1)$, $s_N \sim \infty$. To this end, we expand the meromorphic differential λ with respect to Λ^{2N} in the form

$$\lambda = \frac{dx}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})(\Lambda^{2N})^n}{\Gamma(\frac{1}{2})\ n!\ 2n}(x^N - \sum_{i=2}^{N} s_i x^{N-i})^{-2n}. \tag{5.5}$$

Rescaling $x = s_N^{1/N} z$ and $\alpha_i = s_i s_N^{-i/N}$, and expanding with respect to $1/s_N$ and α_i, λ becomes

$$\lambda = \frac{s_N^{1/N} dz}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2}) n! 2n} \left(\frac{\Lambda^{2N}}{s_N^2}\right)^n (z^N - 1)^{-2n}$$
$$\times \sum_{\{m\}}^{\infty} \frac{\Gamma(a_{\{m\}} + 2n)}{\Gamma(2n)} \prod_{i=2}^{N-1} \frac{1}{m_i!} \left(\frac{\alpha_i z^{N-i}}{z^N - 1}\right)^{m_i}, \tag{5.6}$$

where $\{m\} = \{m_2, \cdots, m_{N-1}\}$ and $a_{\{m\}} = \sum_{i=2}^{N-1} m_i$. In order to calculate a_i, we pick up the poles at $e^{\frac{2\pi i k}{N}}$ in meromorphic differential along α_i cycle. By introducing Barnes-type integral representation [23] and multiplying $\sin 2s\pi/\pi$ as in §3, we integrate from $z=0$ to $z = e^{\frac{2\pi i k}{N}}$ to pick up the poles as

$$a_k = s_N^{\frac{1}{N}} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \sum_{\{m\}}^{\infty} \int_0^{e^{\frac{2\pi i k}{N}}} dz \frac{\Gamma(s+\frac{1}{2})(-1)^s \Gamma(-s)\Gamma(a_{\{m\}}+2s)}{\Gamma(\frac{1}{2})2s\Gamma(2s)} \frac{\sin 2s\pi}{\pi}$$
$$\times (z^N-1)^{-2s-a_{\{m\}}} z^{Na_{\{m\}}-b_{\{m\}}} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \left(\frac{\Lambda^{2N}}{s_N^2}\right)^s, \tag{5.7}$$

where $b_{\{m\}} = \sum_{i=2}^{N-1} i\, m_i$. Therefore we find that a_k in the region where $s_N \sim \infty$ is given by

$$a_k = \frac{s_N^{\frac{1}{N}}}{N} \sum_{n,\{m_i\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(n+\frac{1}{2})\Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(\frac{1}{2})\Gamma(2n+1)n!\Gamma(-2n-b'_{\{m\}}+1)} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \left(\frac{\Lambda^{2N}}{s_N^2}\right)^n, \tag{5.8}$$

where $b'_{\{m\}} = (b_{\{m\}}-1)/N$. Note that the phase factor guarantees the constraint $\sum_{i=1}^N a_i = 0$. In order to continue analytically to the region $s_N \sim \Lambda^N$ and to use various identities, we re-express (5.8) by using the hypergeometric function as

$$a_k = \frac{s_N^{\frac{1}{N}}}{N} \sum_{n,\{m_i\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1-b'_{\{m\}})} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} F\left(\frac{b'_{\{m\}}}{2}, \frac{b'_{\{m\}}+1}{2}; 1; \frac{\Lambda^{2N}}{s_N^2}\right). \tag{5.9}$$

Quite generally the expression of a_k differs by the choice of the branch, therefore we cannot perform analytic continuation of the expression beyond the convergence domain. In the case of $SU(2)$, this process can be justified by comparison to the elliptic singular curve made for torus. For general hyper-elliptic curve, there is no such guarantee for the process. However in our expression, $\frac{\Lambda^{2N}}{s_N^2} = 1$ is the critical point which is just on the boundary of the convergence domain, therefore we can obtain expression around $\frac{\Lambda^{2N}}{s_N^2} = 1$. Performing analytic continuation to $\frac{\Lambda^{2N}}{s_N^2} \sim 1$, and using the identity for the hypergeometric function

$$F(a,b,c,w) = (1-w)^{-a} F(a, c-b, c, \frac{w}{w-1}), \tag{5.10}$$

and the quadratic transformation [26]

$$F(2a, 2b, a+b+\frac{1}{2}, z) = F(a, b, a+b+\frac{1}{2}, 4z(1-z)), \tag{5.11}$$

and also using another identity

$$F(a,b,c,z) = (1-z)^{c-a-b} F(c-a, c-b, c, z), \tag{5.12}$$

where $z = \frac{1}{2} - \frac{s_N}{2\Lambda^N}$, we can put a_k around the conformal point in the form:

$$a_k = \frac{s_N^{\frac{1}{N}}}{N} \sum_{\{m\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \left(\frac{s_N + \Lambda^N}{2\Lambda^N} \right)^{\frac{1}{2}}$$
$$\times \left\{ \frac{\Gamma(\frac{1}{2} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \left(\frac{\Lambda^N + s_N}{s_N} \right)^{-b'_{\{m\}}} F\left(\frac{1}{2}, \frac{1}{2}; b'_{\{m\}} + \frac{1}{2}; z\right) \right. \tag{5.13}$$
$$\left. + \frac{\Gamma(b'_{\{m\}} - \frac{1}{2})}{\Gamma(b'_{\{m\}})} \left(\frac{\Lambda^N + s_N}{s_N} \right)^{-\frac{1}{2}} \left(\frac{s_N - \Lambda^N}{s_N} \right)^{\frac{1}{2} - b'_{\{m\}}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} - b'_{\{m\}}; z\right) \right\}.$$

Next we consider a_D^i. In this case we integrate from $z = 0$ to $z = e^{\frac{2\pi i k}{N}}$ without multiplying $\sin 2s\pi$ as

$$a_D^k = \frac{s_N^{\frac{1}{N}}}{\pi i N} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \sum_{\{m\}} e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}}) (-1)^{-2s - a_{\{m\}}} \tag{5.14}$$
$$\times \frac{\Gamma(s + \frac{1}{2}) \Gamma(-s) \Gamma(a_{\{m\}} + 2s) \Gamma(-2s - a_{\{m\}} + 1)}{\Gamma(\frac{1}{2}) \Gamma(2s + 1) \Gamma(-2s - b'_{\{m\}} + 1)} \left(\prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \right) \left(-\frac{\Lambda^{2N}}{s_N^2} \right)^s,$$

which is defined modulo a_k in the weak coupling region. We evaluate double poles of this integral and also subtract the contribution from $z = 0$ as discussed in §3[23] to obtain a_D^k

$$a_D^k = \frac{s_N^{\frac{1}{N}}}{\pi i N} \sum_{\{m\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \left(\prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \right)$$
$$\times \sum_{n=0}^{\infty} \frac{(\frac{b'_{\{m\}}}{2})_n (\frac{b'_{\{m\}}}{2} + \frac{1}{2})_n}{\Gamma(n+1) n!} \left(\frac{\Lambda^{2N}}{s_N^2} \right)^n \tag{5.15}$$
$$\times \left\{ \ln\left(\frac{\Lambda^{2N}}{s_N^2} \right) + \psi(n + \frac{b'_{\{m\}}}{2}) \right.$$
$$\left. + \psi(n + \frac{b'_{\{m\}}}{2} + \frac{1}{2}) - 2\psi(n+1) + \pi \cot(b'_{\{m\}} \pi) \right\}$$

Using the analytic continuation to the region $s_N \sim \Lambda$, we have

$$a_D^k = \frac{s_N^{\frac{1}{N}}}{\pi i N} \sum_{\{m\}} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \left(\prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \right) \left(\frac{s_N + \Lambda^N}{2\Lambda^N} \right)^{\frac{1}{2}}$$
$$\times \left\{ \frac{\pi \cot(b'_{\{m\}} \pi) \Gamma(b'_{\{m\}} - \frac{1}{2})}{\Gamma(b'_{\{m\}})} \right. \tag{5.16}$$
$$\left. \times \left(\frac{s_N + \Lambda^N}{s_N} \right)^{-\frac{1}{2}} \left(\frac{s_N - \Lambda^N}{s_N} \right)^{\frac{1}{2} - b'_{\{m\}}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} - b'_{\{m\}}; z\right) \right\}$$

Around the critical point, the original roots of the curve e_k^+, e_k^- which both reduce to e_k for $\Lambda = 0$, become $e_k^+ \simeq \Lambda e^{\frac{2\pi i k}{N}}$, $e_k^- \simeq 0$. The expression (5.13) and (5.16) show that a_k consists of the contribution from both poles whereas a_D^k consists of the contribution from e_k^- which vanishes at the critical point. Of course, we can find an expression for a_D^k which reduces to a_k at the conformal point, i.e., $a_D^{'k} = a_D^k + a_k$, because a_D^k was defined modulo a_k, which cannot be determined by analytic continuation but by the consistency. Therefore, around the conformal point a_k and a_D^k behave as

$$a_k \sim a_D^k \sim (s_N - \Lambda^N)^{\frac{N+2}{2N}} + const. \tag{5.17}$$

From this result, we recognize that the conformal point is certainly the non-trivial fixed point of the theory, and the conformal dimension of s_N is $\frac{2N}{N+2}$ [21].

We have used the ordinary type of the analytic continuation but the presence of the factor $\Gamma(-b'_{\{m\}} + \frac{1}{2})$ shows that this factor has poles and the expression (5.13) and (5.16) contain the logarithmic terms. To see this, we decompose $b_{\{m\}}$ mod N as $b_{\{m\}} = Nl + \lambda$ where $l = 0, 1, 2, \cdots$ and $0 \leq \lambda \leq N - 1$. Noticing that $b'_{\{m\}} = (b_{\{m\}} - 1)/N$, when N is even and $\lambda = 1 + \frac{N}{2}$, $b'_{\{m\}} - \frac{1}{2}$ becomes integer, thus we find that $\Gamma(-b'_{\{m\}} + \frac{1}{2})$ has poles. That is, around the conformal point of the moduli space of pure $SU(2n)$ theory, there are unstable directions that a_i and a_D^i have logarithmic terms. However except these directions a_i and a_D^i contain no logarithmic term, and since just on the conformal point $\Gamma(-b'_{\{m\}} + \frac{1}{2}) = \Gamma(-\frac{1}{N} + \frac{1}{2})$ there is no logarithmic singularity except $N = 2$, the conformal point is still the fixed point of the theory. When we set $N = 2$, i.e., gauge group $G = SU(2)$, the point we considered is a dyon point. Therefore it is natural that a and a_D have such logarithmic contribution.

As a check of our result and an example, we consider the gauge group $G = SU(3)$. We set $u = s_2$, $v = s_3$, $\alpha_2 = u/v^{\frac{2}{3}}$ and $a_{\{m\}} = m$, $b'_{\{m\}} = (2m - 1)/3$. In the weak coupling region $v \sim \infty$, our expression reduces to Appell's F_4 system [26] with argument $\frac{4u^3}{27v^2}$, $\frac{\Lambda^6}{v^2}$. Analytic continuation to the region $u \sim \infty$ recovers the result in ref.[12] up to the choice of branch for the logarithmic term of a_D^i, which is again represented by Appell's F_4 system. By analytic continuation to around the conformal point, we can find that our expression becomes Horn's H_7 system [26]. To see this, we set $m = 3l + \lambda$ $(l = 0, 1, 2, \cdots,\ \lambda = 0, 1, 2)$

so that a_k and a_D^k are decomposed to $a_k = \sum_{\lambda=0}^{2} a_k^{\lambda}$, $a_D^k = \sum_{\lambda=0}^{2} a_D^{k\lambda}$. For example, a_k^{λ} can be written as

$$a_k^{\lambda} = \frac{v^{\frac{1}{3}} e^{-\frac{2\pi i k}{3}(2\lambda-1)} \sin(\frac{2\lambda-1}{3})\pi \; 2^{\frac{2\lambda-1}{3}}}{i6\pi\Gamma(\frac{1}{2})^3} \left(\frac{u^3}{v^2}\right)^{\frac{\lambda}{3}} \left(\frac{v}{\Lambda^3}\right)^{\frac{1}{2}} \sum_{n,l} \frac{\Gamma(l+\frac{\lambda+1}{3})}{\Gamma(3l+\lambda+1)} \tag{5.18}$$
$$\times \left\{ \left(\frac{\Lambda^3}{v}\right)^{-\frac{2\lambda}{3}+\frac{5}{6}} \frac{\Gamma(2l+n+\frac{2\lambda}{3}-\frac{1}{3})^2 \sin(\frac{2\lambda}{3}-\frac{1}{3})\pi \; z^n}{\Gamma(2l+n+\frac{2\lambda}{3}+\frac{1}{6})\sin(\frac{2\lambda}{3}+\frac{1}{6})\pi \; n!} \left(\frac{u^3}{4\Lambda^6}\right)^l \right.$$
$$\left. + \left(\frac{2(v-\Lambda^3)}{v}\right)^{-\frac{2\lambda}{3}+\frac{5}{6}} \Gamma(n+\frac{1}{2})^2 \Gamma(2l-n+\frac{2\lambda}{3}-\frac{5}{6}) \frac{(-z)^n}{n!} \left(\frac{u^3}{(v-\Lambda^3)^2}\right)^l \right\},$$

where $z = \frac{1}{2}(1-\frac{v}{\Lambda^3})$. Because of a factor $\sin(\frac{2\lambda-1}{3})\pi$ in (5.18), the component for $\lambda = 2$ disappears, i.e., $a_k^2 = 0$. For $\lambda = 0, 1$, the second term can be expressed by Horn's H_7 function as

$$H_7\left(-\frac{5-4\lambda}{6}, \frac{1}{2}, \frac{1}{2}, \frac{2+2\lambda}{3}, \frac{u^3}{27(v-\Lambda^3)^2}, -\frac{1}{2}(1-\frac{v}{\Lambda^3})\right), \tag{5.19}$$

where $H_7(a,b,c,d,x,y)$ is given by [26]

$$H_7(a,b,c,d,x,y) = \sum_{n,m} \frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m m! n!} x^m y^n. \tag{5.20}$$

This means that if we choose the variable $x = \frac{u^3}{27(v-\Lambda^3)^2}$ and $y = -\frac{1}{2}(1-\frac{v}{\Lambda^3})$, Picard-Fuchs equations of the theory should reduce to differential equations of $H_7(a,b,c,d,x,y)$ system which is given by [26]

$$\left\{-y(1+y)\partial_y^2 + 2x\partial_x\partial_y + (a-1-(b+c+1)y)\partial_y - bc\right\} H_7 = 0, \tag{5.21}$$
$$\left\{x(1-4x)\partial_x^2 + 4xy\partial_x\partial_y - y^2\partial_y^2 + (d-(4d+6)x)\partial_x + 2ay\partial_y - a(a+1)\right\} H_7 = 0,$$

where we have corrected a misprint in ref.[26]. Furthermore, noticing that four independent solutions of this system can be written as

$$H_7(a,b,c,d,x,y)$$
$$x^{1-d} H_7(a-2d+2, b, c, 2-d, x, y),$$
$$y^a \sum_{m,n=0}^{\infty} \frac{(b+a)_{2m+n}(c+a)_{2m+n}}{(d)_m (1+a)_{2m+n} m! n!} (xy^2)^m (-y)^n, \tag{5.22}$$
$$y^{a-2d+2} \sum_{m,n}^{\infty} \frac{(b+a-2d+2)_{2m+n}(c+a-2d+2)_{2m+n}}{(2-d)_m (a-2d+3)_{2m+n}} (xy^2)^m (-y)^n,$$

first and second term of (5.18) with $\lambda = 0, 1$ correspond to above solutions of this system. Let us check this point. We start with the Picard-Fuchs equation in this theory for $\Pi = \oint \lambda$ [12]:

$$\mathcal{L}_1 \Pi = \left\{ (27\Lambda^6 - 4u^3 - 27v^2)\partial_u^2 - 12u^2 v \partial_u \partial_v - 3uv\partial_v - u \right\} \Pi = 0,$$
$$\mathcal{L}_2 \Pi = \left\{ (27\Lambda^6 - 4u^3 - 27v^2)\partial_v^2 - 36uv \partial_u \partial_v - 9v\partial_v - 3 \right\} \Pi = 0. \tag{5.23}$$

By direct change of variables $x = \frac{u^3}{27(v-\Lambda^3)^2}$ and $y = -\frac{1}{2}(1-\frac{v}{\Lambda^3})$, and some linear combinations of these equations, we can check that the Picard-Fuchs equation (5.23) can be written as

$$x(1-4x)\partial_x^2 \Pi_0 - y^2 \partial_y^2 \Pi_0 + 4xy\partial_x\partial_y \Pi_0 + \frac{2}{3}(1-4x)\partial_x \Pi_0 - \frac{5}{3} y \partial_y \Pi_0 + \frac{5}{36}\Pi_0 = 0,$$
$$y(1+y)\partial_y^2 \Pi_0 - 2x\partial_x \partial_y \Pi_0 + \frac{11+12y}{6}\partial_y \Pi_0 + \frac{1}{4}\Pi_0 = 0. \tag{5.24}$$

where $\Pi_0 = y^{-\frac{5}{6}}\Pi$. Comparing this to (5.21), we see that this system is identical to (5.21) with $a = -\frac{5}{6},\ b = c = \frac{1}{2},\ d = \frac{2}{3}$. Substituting these to (5.22), we can find directly that four solutions of the Picard-Fuchs equation of this theory are identical to four functions of the expression (5.18) with $\lambda = 0, 1$, although the first term of (5.18) are not within the Horn's list.

6 $SO(2N)$ pure Yang-Mills theories around the conformal point

In this section we discuss pure $SO(2N)$ theory whose singular points in the strong coupling region are known in arbitrary N [21].

In pure $SO(2N)$ theory the curve and meromorphic differential are given by

$$y^2 = P(x)^2 - \Lambda^{4(N-1)}x^4 = \left(x^{2N} - \sum_{i=1}^{N} x^{2(N-i)} s_i \right)^2 - \Lambda^{4(N-1)}x^4, \tag{6.1}$$
$$\lambda = (2P(x) - xP'(x))\frac{dx}{y}. \tag{6.2}$$

Since the difference from $SU(N)$ theory is only powers of Λ in the instanton correction term, the calculation is almost same as $SU(N)$ theory. What we need is the expression around the point $s_i = 0\,(i \neq N-1)$, $s_{N-1} = \pm\Lambda^{2N-2}$ where the curve is degenerate as [21]

$$y^2 = x^{2N+2}(x^{2N-2} \pm 2\Lambda^{2N-2}). \tag{6.3}$$

To this end, it is convenient to evaluate integral in the region $s_i \sim 0\,(i \neq N-1)$, $s_{N-1} \sim \infty$. Expanding λ with respect to $\Lambda^{4(N-1)}$ and integrating by part, we can rewrite λ in the following form:

$$\lambda = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\frac{dx}{2\pi i}\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)}{\Gamma(\frac{1}{2})2s}\left(-\Lambda^{4(N-1)}x^4\right)^s P(x)^{-2s}, \tag{6.4}$$

where we have introduced Barnes-type integral representation as before. Rescaling the variable as

$$x = s_{N-1}^{1/(2N-2)} z = uz, \quad s_i = u^{2i}\alpha_i\,(i \neq N-1). \tag{6.5}$$

and expanding with respect to α_i and Λ^{4N-4}/u^{4N-4}, we have λ in the form:

$$\begin{aligned}\lambda = u\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)\Gamma(2s+a_{\{m\}})}{\Gamma(\frac{1}{2})\Gamma(2s+1)}\left(-\frac{\Lambda^{4N-4}}{u^{4N-4}}\right)^s\sum_{\{m\}}\prod_{i\neq N-1}^{N}\left(\frac{\alpha_i}{m_i!}\right)^{m_i} \\ \times\int\frac{dz}{2\pi i}z^{2(N-1)a_{\{m\}}-2b_{\{m\}}}\left(z^{2N-2}-1\right)^{-2s-a_{\{m\}}},\end{aligned} \tag{6.6}$$

where $\{m\} = \{m_1, \cdots, m_{N-2}, m_N\}$ and $a_{\{m\}} = \sum_{i=1,i\neq N-1}^{N} m_i$, $b_{\{m\}} = \sum_{i=1,i\neq N-1}^{N} i m_i$. In order to obtain a_k, we pick up poles at $z = e^{\frac{2\pi ik}{2N-2}}$ $(0 \leq k \leq N-1)$ along α_k cycle and $z = 0$ along α_N cycle. First we calculate a_k $(0 \leq k \leq N-1)$. To pick up poles at $z = e^{\frac{2\pi ik}{2N-2}}$ we integrate from $z = 0$ to $z = e^{\frac{2\pi ik}{2N-2}}$ multiplying $\sin 2s\pi/\pi$ to find that a_k can be expressed in the form:

$$\begin{aligned}a_k &= \frac{u}{2N-2}\sum_{n,\{m\}}^{\infty}\frac{e^{-2\pi ikb'_{\{m\}}}\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\Gamma(2n+1)n!}\left(\frac{\Lambda^{4N-4}}{u^{4N-4}}\right)^n\prod_{i\neq N-1}\left(\frac{\alpha_i}{m_i!}\right)^{m_i}\frac{\Gamma(a_{\{m\}}-b'_{\{m\}})}{\Gamma(-2n-b'_{\{m\}}+1)} \\ &= \frac{2u}{2N-2}\sum_{n,\{m\}}^{\infty}\frac{e^{-2\pi ikb'_{\{m\}}}\Gamma(a_{\{m\}}-b'_{\{m\}})}{\Gamma(1-b'_{\{m\}})} \\ &\quad\times\prod_{i\neq N-1}\left(\frac{\alpha_i}{m_i!}\right)^{m_i}F\left(\frac{b'_{\{m\}}}{2},\frac{b'_{\{m\}}+1}{2};1;\frac{\Lambda^{4N-4}}{u^{4N-4}}\right),\end{aligned} \tag{6.7}$$

where $b'_{\{m\}} = \frac{b_{\{m\}}}{(N-1)} - \frac{1}{(2N-2)}$. The analytic continuation to the region $u^{2N-2} = s_{N-1} \sim \Lambda^{2N-2}$ and the quadratic transformation show that the result is

$$
\begin{aligned}
a_k = \frac{2u}{2N-2} \sum_{n,\{m\}}^{\infty} \frac{e^{-2\pi i b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \prod_{i \neq N-1} \left(\frac{\alpha_i}{m_i!} \right)^{m_i} \\
\times \left[\frac{\Gamma(\frac{1}{2} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \left(\frac{\Lambda^{2N-2}}{u^{2N-2}} \right)^{-b'_{\{m\}}} \left(\frac{\Lambda^{2N-2} + u^{2N-2}}{\Lambda^{2N-2}} \right)^{\frac{1}{2} - b'_{\{m\}}} F\left(\frac{1}{2}, \frac{1}{2}; b'_{\{m\}} + \frac{1}{2}; z \right) \right. \\
+ \frac{\Gamma(b'_{\{m\}} - \frac{1}{2})}{\Gamma(b'_{\{m\}})} (1 - \frac{\Lambda^{4N-4}}{u^{4N-4}})^{\frac{1}{2} - b'_{\{m\}}} \left(\frac{\Lambda^{2N-2}}{u^{2N-2}} \right)^{b'_{\{m\}} - 1} \\
\left. \times \left(\frac{\Lambda^{2N-2} + u^{2N-2}}{\Lambda^{2N-2}} \right)^{b'_{\{m\}} - \frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} - b'_{\{m\}}; z \right) \right],
\end{aligned}
\tag{6.8}
$$

where $z = \frac{1}{2}(1 - \frac{u^{2N-2}}{\Lambda^{2N-2}})$.

Next we consider a_D^k $(1 \leq k \leq N-1)$. In this case we integrate meromorphic differential λ from $z = -e^{\frac{2\pi i k}{2N-2}}$ to $z = e^{\frac{2\pi i k}{2N-2}}$ and evaluate double pole of the integrant without multiplied by $\sin 2s\pi$, and subtract $\frac{1}{2} a_k$ [23]. We have a_D^k in the form:

$$
\begin{aligned}
a_D^k = \frac{u}{2\pi^2 i} \sum_{n,\{m\}} \frac{e^{-2\pi i k (b'_{\{m\}})} \Gamma(a_{\{m\}} - b'_{\{m\}}) \sin(b'_{\{m\}}\pi) 2^{b'_{\{m\}}}}{(2N-2)\Gamma(\frac{1}{2})\Gamma(n+1)^2} \prod_{i \neq N-1}^{N} \left(\frac{\alpha_i}{m_i!} \right)^{m_i} \\
\times \Gamma(n + \frac{b'_{\{m\}}}{2}) \Gamma(n + \frac{b'_{\{m\}}}{2} + \frac{1}{2}) \left(\frac{\Lambda^{4N-4}}{u^{4N-4}} \right)^n \\
\times \left[\psi(n + \frac{b'_{\{m\}}}{2}) + \psi(n + \frac{b'_{\{m\}}}{2} + \frac{1}{2}) - 2\psi(n+1) + \ln \left(\frac{\Lambda^{4N-4}}{u^{4N-4}} \right) + 2\pi \cot(b'_{\{m\}}\pi) \right].
\end{aligned}
\tag{6.9}
$$

We make use of the analytic continuation of a_D^k around the conformal point to get

$$
\begin{aligned}
a_D^k = \frac{2u}{(2N-2)i} \sum_{n,\{m\}}^{\infty} \frac{e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b_{\{m\}})} \prod_{i \neq N-1} \left(\frac{\alpha_i}{m_i!} \right)^{m_i} \\
\times \cot(b'_{\{m\}}\pi) \frac{\Gamma(b'_{\{m\}} - \frac{1}{2})}{\Gamma(b'_{\{m\}})} (1 - \frac{\Lambda^{4N-4}}{u^{4N-4}})^{\frac{1}{2} - b'_{\{m\}}} \left(\frac{\Lambda^{2N-2}}{u^{2N-2}} \right)^{b'_{\{m\}} - 1} \\
\times \left(\frac{\Lambda^{2N-2} + u^{2N-2}}{\Lambda^{2N-2}} \right)^{b'_{\{m\}} - \frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} - b'_{\{m\}}; z \right).
\end{aligned}
$$

As in the pure $SU(N)$ theory, we can claim that $a_k \sim a_D^k$ at the critical point. The behavior of a and a_D near $s_{N-1} = \Lambda^{2N-2},\ s_i = 0\ (i \neq N-1)$ is

$$
a_k \sim a_D^k \sim (s_{N-1} - \Lambda^{2N-2})^{\frac{1}{2} - \frac{1}{2N-2}} + \mathit{const.} \tag{6.10}
$$

Therefore, we see that the confomal dimension of s_{N-1} is $\frac{2N-2}{N}$ [21].

As was the case of $SU(2n)$, a_i and a_D^i contain the logarithmic terms coming from the factor $\Gamma(\frac{1}{2}-b'_{\{m\}})$ when N of $SO(2N)$ is even, which vanish at the conformal point.

Next we consider a_N and a_D^N. Until now the calculation is same as $SU(N)$ case. However in order to calculate a_N and a_D^N, we have to pick up the pole $x\sim 0$. To this end we rescale the variable of the curve as

$$x^2=-\frac{s_N}{s_{N-1}}z^2,\quad \beta_i=\frac{s_i}{s_N}\left(-\frac{s_N}{s_{N-1}}\right)^{N-i}, \tag{6.11}$$

where $s_0=-1$, and λ becomes as

$$\begin{aligned}\lambda=&\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\sum_m\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)\Gamma(2s+c_{\{m\}})}{\Gamma(\frac{1}{2})2s\Gamma(2s)}\left(-\frac{\Lambda^{4N-4}}{s_{N-1}^2}\right)^s\\&\times\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)z^{4s+2Nc_{\{m\}}-2d_{\{m\}}}(z^2-1)^{-2s-c_{\{m\}}},\end{aligned} \tag{6.12}$$

where $\{m\}=\{m_0,m_1,\cdots,m_{N-2}\}$ and $c_{\{m\}}=\sum_{i=0}^{N-2}m_i$, $d_{\{m\}}=\sum_{i=0}^{N-2}(N-i)m_i$. By evaluating the line integral from $z=0$ to $z=1$ and by multiplying $\sin 2s\pi/\pi$ to pick up the pole at $z=1$, we get a_N in the region $\frac{s_{N-1}^2}{\Lambda^{4N-4}}>>1$ in the form:

$$\begin{aligned}a_N=&\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\sum_{n,\{m\}}\frac{\Gamma(n+\frac{1}{2})\Gamma(2n+d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2n+1)\Gamma((N-1)c_{\{m\}}-d_{\{m\}}+\frac{3}{2})}\left(\frac{\Lambda^{4N-4}}{s_{N-1}^2}\right)^n\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)\\=&\,2\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\sum_{\{m\}}\frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})}\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)\\&\times F\left(\frac{d_{\{m\}}}{2}+\frac{1}{4},\frac{d_{\{m\}}}{2}+\frac{3}{4};1;\frac{\Lambda^{4N-4}}{s_{N-1}^2}\right).\end{aligned} \tag{6.13}$$

Notice that this hypergeometic function gives logarithmic term by analytic continuation to the region $\frac{\Lambda^{4N-4}}{s_{N-1}^2}\sim 1$. To see this, we set the variable as

$$y=\frac{\Lambda^{4N-4}}{s_{N-1}^2},\quad z=\frac{\Lambda^{2N-2}-s_{N-1}}{2\Lambda^{2N-2}}, \tag{6.14}$$

and perform the analytic continuation to the region $\frac{s_{N-1}}{\Lambda^{4N-4}}\sim 1$ as

$$a_N=\left(-\frac{s_N}{s_{N-1}}\right)^{\frac{1}{2}}\sum_{\{m\}}\frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})}\prod_{i=0}^{N-2}\left(\frac{\beta_i^{m_i}}{m_i!}\right)$$

$$\times \Bigg\{ (1-y)^{-d_{\{m\}}-\frac{1}{2}} y^{\frac{d_{\{m\}}}{2}-\frac{1}{4}} \frac{\Gamma(d_{\{m\}})}{\Gamma(\frac{d_{\{m\}}}{2}+\frac{1}{4})^2} \sum_{n=0}^{d_{\{m\}}-1} \frac{(\frac{1}{4}-\frac{d_{\{m\}}}{2})_n (\frac{1}{4}-\frac{d_{\{m\}}}{2})_n}{n!(-d_{\{m\}}+1)_n} (1-y)^n$$

$$+ \frac{y^{\frac{d_{\{m\}}}{2}-\frac{1}{4}} (1-z)^{-d_{\{m\}}}}{\Gamma(\frac{3}{4}-\frac{d_{\{m\}}}{2})\Gamma(\frac{1}{4}-\frac{d_{\{m\}}}{2})\Gamma(d_{\{m\}}+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{n!(d_{\{m\}}+1)_n} z^n \tag{6.15}$$

$$\times \Big[\psi(n+1) + \psi(n+d_{\{m\}}+1) - 2\psi(n+\frac{1}{2}) - \pi - \log(-z) \Big] \Bigg\}.$$

Next we calculate a_D^N. In the region $s_{N-1} \sim \infty$, a_D^N is given by integrating meromorphic differential λ from $z=-1$ to $z=1$ without multiplying $\sin 2s\pi$ and subtracting $\frac{1}{2}a_N$, and by evaluating double poles as

$$a_D^N = \frac{i s_N^{\frac{1}{2}}}{s_{N-1}^{\frac{1}{2}} 2\pi i} \sum_{n,\{m\}} \frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})} \prod_{i=0}^{N-2} \left(\frac{\beta_i^{m_i}}{m_i!} \right) \frac{(\frac{d_{\{m\}}}{2}+\frac{1}{4})_n (\frac{d_{\{m\}}}{2}+\frac{3}{4})_n}{(n!)^2} y^n$$

$$\times \left[\psi(n+\frac{d_{\{m\}}}{2}+\frac{1}{4}) + \psi(n+\frac{d_{\{m\}}}{2}+\frac{3}{4}) - 2\psi(n+1) + \ln y \right], \tag{6.16}$$

where $y = \frac{\Lambda^{4N-4}}{s_{N-1}^2}$. Although this logarithmic term disappears by the analytic continuation to the region $s_{N-1} \sim \Lambda^{2N-2}$, another logarithmic term appears

$$a_D^N = \frac{s_N^{\frac{1}{2}}}{2 s_{N-1}^{\frac{1}{2}}} \sum_{n,\{m\}} \frac{\Gamma(d_{\{m\}}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(-c_{\{m\}}+d_{\{m\}}+\frac{3}{2})} \prod_{i=0}^{N-2} \left(\frac{\beta_i^{m_i}}{m_i!} \right)$$

$$\times \Bigg\{ (1-y)^{-d_{\{m\}}-\frac{1}{2}} y^{\frac{d_{\{m\}}}{2}-\frac{1}{4}} \frac{i\pi\Gamma(d_{\{m\}})}{\Gamma(\frac{d_{\{m\}}}{2}+\frac{1}{4})^2} \sum_{n=0}^{d_{\{m\}}-1} \frac{(\frac{1}{4}-\frac{d_{\{m\}}}{2})_n (\frac{1}{4}-\frac{d_{\{m\}}}{2})_n}{n!(-d_{\{m\}}+1)_n} (1-y)^n$$

$$+ \frac{y^{-\frac{d_{\{m\}}}{2}-\frac{1}{4}} (1-z)^{-d_{\{m\}}}}{\Gamma(\frac{3}{4}-\frac{d_{\{m\}}}{2})\Gamma(\frac{1}{4}-\frac{d_{\{m\}}}{2})\Gamma(d_{\{m\}}+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n z^n}{n!(d_{\{m\}}+1)_n} \tag{6.17}$$

$$\times \Big[\psi(n+1) + \psi(n+d_{\{m\}}+1) - 2\psi(n+\frac{1}{2}) - \log(-z) - \pi \Big] \Bigg\}.$$

Thus in $SO(2N)$ theory a_N and a_D^N have the logarighmic terms around this point though the curve become degenerate multiple. Let us consider what is happening. Near $x \sim 0$, α_N cycle and β_N cycle form a small torus, and the curve looks like the curve of pure $SU(2)$ theory. In this case due to our choice of approaching to the point $s_{N-1} = \Lambda^{2N-2}$, $s_N = 0$, this point corresponds to the dyon point for a_N and a_D^N and these have certainly the logarithmic terms. These logarithmic terms are simply caused by the fact that we consider a branch where two of the singularity approach to zero before the theory is going to be at the critical point. This point has been understood in the framework of the $SU(3)$ theory near $u = 0,\ v = \Lambda^2$ [19]. From the expression (4.16) and (4.18), we see that $a_N \sim a_D^N$ on the conformal point. Therefore the existence of logarithmic terms in the expression (4.16) and (4.18) is not harmful.

7 Various expansion in $SU(3)$ theory

7.1 Comparison to other expansion in pure $SU(3)$ theory

Since the analytic continuation of the logarithmic function depends on the choice of the branch, we have to retain the parameterization used in §4 and §5 in the theory with the gauge group higher than $SU(2)$, to get the expression around the conformal point by using the analytic continuation from the weak coupling region. However we need the parameterization which is based on the deviation from the conformal point, to obtain the expression around the conformal point of the theory with massive matter multipletes where mass parameters take non-zero critical values. Unfortunately we have not found the relation between Yang-Mills theories whose rank of the gauge group being more than 2, and the deformation theory from the singular curve describing the Riemann surface whose genus being more than 1. So if we intend to express the behavior of the theory by using the deviation from the conformal point, we don't know how to confirm the validity of the result obtained by the analytic continuation. In this subsection we will discuss the difference between these two kind of parameterization, by comparing one expression represented with the reliable parameterization retained in §4,5, to another expression based on the deviation from the conformal point convenient to describe the theory with massive matter fields.

To this end, we consider pure $SU(3)$ theory by introducing the parameterization from the conformal point $u = 0,\ v = \Lambda^3$, such as $\tilde{u} = u,\ \tilde{v} = v - \Lambda^3$. In this case the curve of the theory can be written as:

$$y^2 = (x^3 - ux - v)^2 - \Lambda^6 \tag{7.1}$$
$$= (x^3 - \tilde{u}x - \tilde{v})^2 - 2\Lambda^3(x^3 - \tilde{u}x - \tilde{v}).$$

Here after we regard $\tilde{u},\ \tilde{v}$ as the physical parameters to obtain the expression of Higgs field and its dual. Setting $\tilde{C}(x) = x^3 - \tilde{u}x - \tilde{v}$, and expanding $1/y$ in the weak coupling region $\Lambda \sim 0$ as

$$\frac{\partial\lambda}{\partial\tilde{v}} = \frac{dx}{2\pi i\ y} = \frac{dx}{2\pi i\ \tilde{C}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})n!} \left(\frac{2\Lambda^3}{\tilde{C}}\right)^n, \tag{7.2}$$

we find that the meromorphic differential λ can be written as

$$\lambda = \frac{dx}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})n!n} \left(\frac{2\Lambda^3}{\tilde{C}}\right)^n . \tag{7.3}$$

As in $SU(2)$ theory with massive matter fields, first we evaluate the integral in the region $\tilde{u} \sim 0,\ \tilde{v} \sim \infty$, and after that analytically continuation the expression to the conformal point. Setting the variable as $x = \tilde{v}^{\frac{1}{3}}z$, and expanding with respect to $1/\tilde{v}$, and introducing Barnes-type integral representation we obtain λ in this region in the following form:

$$\lambda = \frac{\tilde{v}^{\frac{1}{3}}dz}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(s+\frac{1}{2})\Gamma(-s)}{\Gamma(\frac{1}{2})s} \left(-\frac{2\Lambda^3}{\tilde{v}}\right)^s \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{\Gamma(s)m!} \left(\frac{\tilde{u}}{\tilde{v}^{\frac{2}{3}}}\right)^m z^m(z^3-1)^{-s-m}. \tag{7.4}$$

First of all, we evaluate a_k. In the weak coupling region, the way of the evaluation along the α_k cycle is same as in §3. We find the result can be written as

$$a_k^\lambda = \frac{2\sqrt{3}\tilde{v}^{\frac{1}{3}}}{9} \sin(\frac{2\lambda}{3} - \frac{1}{3})\pi \left(\frac{\tilde{u}}{3\tilde{v}^{\frac{2}{3}}}\right)^\lambda \tag{7.5}$$
$$\times \sum_{l,n=0}^{\infty} \frac{e^{\frac{2\pi i k}{3}(\lambda+1)}\Gamma(n+\frac{1}{2})\Gamma(n+2l+\frac{2\lambda}{3}-\frac{1}{3})}{\Gamma(\frac{1}{2})\Gamma(n+1)n!\Gamma(l+\frac{\lambda+2}{3})\Gamma(l+\frac{\lambda+3}{3})} \left(-\frac{2\Lambda^3}{\tilde{v}}\right)^n \left(\frac{\tilde{u}^3}{27\tilde{v}^2}\right)^l,$$

where as before we decompose a_k as $a_k = \sum_{\lambda=0}^{2} a_k^\lambda$ with mod 3. Quite similarly we can evaluate β_k cycle and decompose a_D^k to $a_D^{k\lambda}$ $(k = 0, 1, 2)$

$$a_D^{k,\lambda} = -\frac{\sqrt{3}\tilde{v}^{\frac{1}{3}}}{9\pi i} \sin(\frac{2\lambda}{3} - \frac{1}{3})\pi \left(\frac{\tilde{u}}{3\tilde{v}^{\frac{2}{3}}}\right)^\lambda$$
$$\times \sum_{l,n=0}^{\infty} \frac{e^{\frac{2\pi i k}{3}(\lambda+1)}\Gamma(n+\frac{1}{2})\Gamma(n+2l+\frac{2\lambda}{3}-\frac{1}{3})}{\Gamma(\frac{1}{2})\Gamma(n+1)n!\Gamma(l+\frac{\lambda+2}{3})\Gamma(l+\frac{\lambda+3}{3})} \left(-\frac{2\Lambda^3}{\tilde{v}}\right)^n \left(\frac{\tilde{u}^3}{27\tilde{v}^2}\right)^l$$

$$\times\left[\log\left(\frac{2\Lambda^3}{\tilde{v}}\right)+\psi(n+\frac{1}{2})+\psi(n+2l+\frac{2\lambda-1}{3})\right. \tag{7.6}$$
$$\left.-2\psi(n+1)+\pi\cot(\frac{2\lambda-1}{3})\pi\right].$$

In above expressions, a_k^λ and $a_D^{k\lambda}$ can be represented by using Horn's H_4 function [26]. Recalling that Higgs field and its dual form Appell's F_4 system with respect to the original moduli parameters u, v as mentioned in §5, we see that this result corresponds to the change of the variables in the weak coupling region by using following identity between F_4 and H_4:

$$H_4(a,b,c,2b;x,y)=\left(1-\frac{1}{2}y\right)^{-a}F_4\left(\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},c,b+\frac{1}{2};\frac{16x}{(2-y)^2},\frac{y^2}{(2-y)^2}\right), \tag{7.7}$$

where F_4 and H_4 are two functions of 2-parameter generalization of the hypergeometric function defined as

$$F_4(a,b,c,d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{n+m}(b)_{n+m}}{(c)_m(d)_n m!n!}x^m y^n, \tag{7.8}$$

$$H_4(a,b,c,d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n m!n!}x^m y^n. \tag{7.9}$$

Performing the analytic continuation to the region $\tilde{v}\sim 0$, we can obtain a_k, a_D^k around the conformal point

$$a_k^\lambda=\frac{2\sqrt{3}\tilde{v}^{\frac{1}{3}}}{9}\left(\frac{\tilde{u}}{3\tilde{v}^{\frac{2}{3}}}\right)^\lambda e^{\frac{2\pi ik}{3}(\lambda+1)}\sin(\frac{2\lambda}{3}-\frac{1}{3})\pi$$
$$\times\sum_{l,n=0}^{\infty}\left\{\left(\frac{\tilde{v}}{2\Lambda^3}\right)^{\frac{1}{2}}\frac{\Gamma(2l-n+\frac{2\lambda}{3}-\frac{5}{6})\Gamma(n+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})n!\Gamma(l+\frac{\lambda+2}{3})\Gamma(l+\frac{\lambda+3}{3})}\left(-\frac{\tilde{v}}{\Lambda^3}\right)^n\left(\frac{\tilde{u}^3}{27\tilde{v}^2}\right)^l\right. \tag{7.10}$$
$$\left.+\left(\frac{2\tilde{v}}{\Lambda^3}\right)^{\frac{2\lambda}{3}-\frac{1}{3}}\frac{\Gamma(-n-2l-\frac{2\lambda}{3}+\frac{5}{6})\Gamma(n+2l+\frac{2\lambda}{3}-\frac{1}{3})}{\Gamma(\frac{1}{2})\Gamma(-n-2l-\frac{2\lambda}{3}+\frac{4}{3})n!\Gamma(l+\frac{\lambda+2}{3})\Gamma(l+\frac{\lambda+3}{3})}\left(-\frac{\tilde{v}}{\Lambda^3}\right)^n\left(\frac{\tilde{u}^3}{2\Lambda^3}\right)^l\right\}.$$

$$a_D^{k\lambda}=\frac{\sqrt{3}\tilde{v}^{\frac{1}{3}}}{9\pi i}\left(\frac{\tilde{u}}{3\tilde{v}^{\frac{2}{3}}}\right)^\lambda e^{\frac{2\pi ik}{3}(\lambda+1)}\sin(\frac{2\lambda}{3}-\frac{1}{3})\pi\pi\cot(\frac{2\lambda}{3}-\frac{1}{3})\pi$$
$$\times\sum_{l,n=0}^{\infty}\left(\frac{\tilde{v}}{2\Lambda^3}\right)^{\frac{1}{2}}\frac{\Gamma(2l-n+\frac{2\lambda}{3}-\frac{5}{6})\Gamma(n+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})n!\Gamma(l+\frac{\lambda+2}{3})\Gamma(l+\frac{\lambda+3}{3})}\left(-\frac{\tilde{v}}{\Lambda^3}\right)^n\left(\frac{\tilde{u}^3}{27\tilde{v}^2}\right)^l. \tag{7.11}$$

Comparing above expressions to the ones obtained in §5, we find that they are identical expressions around the conformal point. This indicate that in pure $SU(3)$ theory, the description based on the deviation from the conformal point can be reliable except the ambiguity adding a_k to a_D^k in the weak coupling region. Note also that in this expression a_D^k contains no contribution from $e_k^+ \sim \Lambda e^{\frac{2\pi i k}{3}}$ again.

7.2 Direct expansion around the coformal point

Until last subsection we have derived the expression around the conformal point by using analytic continuation of the one evaluated in the weak coupling region. On this course we choose the base of homology cycles of the hyperelliptic curve corresponding to a_k, a_D^k convenient to deal with in the weak coupling region. However this homology base is not always useful in the expression around the conformal point through the analytic continuation; as illustrated in §4 and §5, a_k contains the contributions both from $e_k^+ \sim \Lambda$ and $e_k^- \sim 0$. In this section we take the holomogy base around the conformal point in a way following the ref.[19], to get the expression directly valid in this region. The hyperelliptic curve which describes pure $SU(3)$ theory is realized as the Riemann surface of genus 2. Since on the conformal point the curve become degenerate as $y^2 = x^3(x^3 - 2\Lambda^3)$, the Riemann surface looks like decoupled to two independent tori near the conformal point. One is a large torus consisting of $e_k^+ \sim \Lambda e^{\frac{2\pi i k}{3}}$ $(k = 1, 2, 3)$, and another one is a small torus consisting of $e_k^- \sim 0$ $(k = 1, 2, 3)$. In this case we take α_l, β_l as independent cycles enclosing two of e_k^+ and a_l, a_D^l as corresponding Higgs field and its dual respectively. Similarly we define α_s, β_s concerning to e_k^-, and a_s, a_D^s as corresponding Higgs field and its dual.

In order to obtain a_s, a_D^s around the conformal point directly, we evaluate near $x \sim \tilde{v} \sim 0$. In this case Λ is large enough comparing to $\tilde{u}$, $\tilde{v}$ where we certainly use the deviation parameter from the coformal point. Expanding the meromorphic differential λ with respect to $1/\Lambda$

$$\lambda = \frac{dx}{\sqrt{2}\Lambda^{\frac{3}{2}} 2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2}) n! (n+\frac{1}{2})} \left(-\frac{1}{2\Lambda^3}\right)^n (x^3 - \tilde{u}x - \tilde{v})^{n+\frac{1}{2}}, \tag{7.12}$$

and setting the variable as $x = \tilde{v}^{\frac{1}{3}} z$, and expanding $\tilde{v}$ where $\tilde{v} \sim 0$, we find that the curve can be written as

$$\lambda = \frac{\tilde{v}^{\frac{1}{3}} dz}{\sqrt{2}\Lambda^{\frac{3}{2}} 2\pi i} \sum_{n,m=0}^{\infty} \frac{\Gamma^2(\frac{1}{2}+n)}{\Gamma(\frac{1}{2}) n! \Gamma(n+\frac{3}{2}-m) m!} \left(-\frac{\tilde{v}}{2\Lambda^3}\right)^n \left(-\frac{\tilde{u}}{\tilde{v}^{\frac{2}{3}}}\right)^m z^m (z^3-1)^{n-m+\frac{1}{2}} \tag{7.13}$$

We define a_s, a_D^s in the following form

$$a_s = \int_1^{e^{\frac{2\pi i}{3}}} \lambda, \quad a_D^s = \int_{e^{\frac{2\pi i}{3}}}^{e^{\frac{4\pi i}{3}}} \lambda. \tag{7.14}$$

We can evaluate this line integral easily to get a_s in the following form

$$\begin{aligned} a_s^\lambda = & \frac{\tilde{v}^{\frac{1}{3}} 2\sqrt{3} e^{\frac{2\pi i}{3}(\lambda+1)} \sin 2(\frac{\lambda+1}{3})\pi}{\sqrt{2} 3 \Lambda^{\frac{3}{2}}} \left(\frac{\tilde{u}}{3\tilde{v}^{\frac{2}{3}}}\right)^\lambda \\ & \times \sum_{l,n=0}^{\infty} \frac{\Gamma^2(n+\frac{1}{2})\Gamma(2l-n+\frac{2\lambda}{3}-\frac{5}{6})\sin(\frac{2\lambda}{3}-\frac{5}{6})\pi}{\Gamma(\frac{1}{2})n!\Gamma(l+\frac{\lambda+2}{3})\Gamma(l+\frac{\lambda+3}{3})} \left(-\frac{\tilde{v}}{2\Lambda^3}\right)^n \left(\frac{\tilde{u}^3}{27\tilde{v}^2}\right)^l, \end{aligned} \tag{7.15}$$

where we again decompose a_s to $\sum_{\lambda=0}^{2} a_s^\lambda$. The expression for $a_D^{s\lambda}$ can be given by $e^{\frac{2\pi i}{3}(\lambda+1)} a_s^\lambda$. This expression corresponds to one of the functions obtained by analytic continuation, which is represented by using H_7 function

Next we consider a_l, a_D^l. In this case we use formally the expansion valid in the case $\Lambda \sim 0$, and evaluate near $x \sim \Lambda$ by changing the variable as $x = \Lambda z$. In this way the meromorphic differential λ is given by

$$\lambda = \frac{dx}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Lambda^{6n}}{\Gamma(\frac{1}{2})n!2n} (x^3 - \tilde{u}x - \tilde{v} - \Lambda^3)^{-2n}. \tag{7.16}$$

Setting $x = \Lambda z$ we evaluate around $z^3 \sim 1$. Expanding λ with respect to $\tilde{v}$, we obtain λ of the form:

$$\lambda = \frac{\Lambda dz}{2\pi i} \sum_{n,m,l}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(2n+m+l)}{\Gamma(\frac{1}{2})n!\Gamma(2n+1)l!m!} \left(\frac{\tilde{u}}{\Lambda^2}\right)^l \left(\frac{\tilde{v}}{\Lambda^3}\right)^m z^l (z^3-1)^{-2n-l-m}. \tag{7.17}$$

We define a_l, a_D^l in a same manner as a_s, a_D^s. We have already known that a_l and a_D^l contain no logarithmic term in the suitable limit $\tilde{u} \to 0$, $\tilde{v} \to 0$, therefore we regard the power function obtained in this evaluation as the expression for a_l, a_D^l. Using the special value of the hypergeometric function

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{7.18}$$

we find that a_l can be written as

$$\begin{aligned} a_l^\lambda = & \frac{\Lambda e^{\frac{2\pi i k}{3}(\lambda+1)} 2\sqrt{3}}{3} \left(\frac{\tilde{u}}{\Lambda^2}\right)^\lambda \frac{\sin^2(\frac{2\lambda}{3}-\frac{1}{3})\pi}{\sin(\frac{2\lambda}{3}+\frac{1}{6})\pi} \\ & \times \sum_{n,m=0}^{\infty} \frac{\Gamma^2(2n+m+\lambda-\frac{1}{3})\pi}{\Gamma(2n+m+\Lambda+\frac{1}{6})l!\Gamma(n+\frac{\lambda+2}{3})\Gamma(n+\frac{\lambda+3}{3})} \left(\frac{\tilde{u}^3}{27\cdot 4\Lambda^6}\right)^n \left(-\frac{\tilde{v}}{\Lambda^3}\right)^m. \end{aligned} \tag{7.19}$$

The expression for a_D^l is obtained by $a_D^{l\lambda} = e^{\frac{2\pi i}{3}(\lambda+1)} a_l^\lambda$. We can see that this expression is identical to the other function than H_7 obtained by using the analytic continuation. In this way it is shown that four independent functions of the expression around the conformal point obtained by analytic continuation, consist of some linear combinations of the contribution from the large and the small torus.

Although it is possible to write down the expression around the conformal point directly as we have derived, the way of evaluation in terms of large torus is not so reliable that we prefer to use the analytic continuation from the weak coupling region. In the weak coupling region, we have to add a_k to a_D^k to be consistent after analytic continuation around the conformal point. After the change of the homology base with $Sp(4;\mathbf{Z})$ transformation it is natural to arise the picture that on the conformal point mutually non-local charged 3 dyons are massless, which correspond to the massless state $a_s,\ a_D^s,\ a_s + a_D^s$ [19].

On the contrary, concerning with the weak coupling behavior the convenient base around the conformal point is certainly not so good. As a matter of fact, if we analytically continue $a_l,\ a_D^l,\ a_s,\ a_D^s$ derived above, to the weak coupling region, there appear the logarithmic terms in every expressions. Thus we have to combine these expression properly to be a_k to show the correct asymptotic behavior in the weak coupling region.

7.3 $SU(3)$ theory with $N_f = 2$ matter fields

In this subsection we consider $SU(3)$ theory with $N_f = 2$ hypermultipletes whose curve is given by

$$y^2 = C(x)^2 - G(x) = (x^3 - ux - v)^2 - \Lambda^2(x+m_1)(x+m_2). \tag{7.20}$$

When the mass parameters take the equal value $m_1 = m_2 = m_0$, the conformal point of the theory seems to be located at

$$u = \pm\Lambda^2,\ v = \pm\Lambda^2 m_0, \tag{7.21}$$

and on this point the curve become degenerate in the following form:

$$y^2 = x^3(x^3 \mp 2\Lambda^2 x \mp 2\Lambda^2 m_0). \tag{7.22}$$

Thus we expect that the infra-red fixed point is realized with the arbitrary value of m_0, and the theory has a conformal line parameterized with m_0. Furthermore if we set $m_0 = 0$, the degeneracy of the curve becomes forth order:

$$y^2 = x^4(x^2 \pm 2\Lambda^2). \tag{7.23}$$

If we consider the case with $m_1 \neq m_2$, there exists a conformal point of fifth order certainly. However the expansion at hand and analytic continuation reliable in terms of the logarithmic function are only ones for the massless case. So we treat the conformal point realized at $m_0 = 0$. This situation can happen generally in $SU(N_c)$ theory with $N_f = 2$ hypermultiplete, and $SO(2N+1)$ theory with $N_f = 1$ hypermultiplete whose curves are defined as

$$SU(N_c) \ : y^2 = (x_c^N - \sum_{i=2}^{N_c} s_i x^{N_c - i})^2 - \Lambda^{2N_c - 2}(x+m)(x+m), \tag{7.24}$$

$$SO(2N+1) \ : y^2 = (x^{2N} - \sum_{i=1}^{N} s_{2i}^{2N-2i})^2 - \Lambda^{2(2N-2)}(x^2 - m^2). \tag{7.25}$$

Let us turn to the $SU(3)$ theory with $N_f = 2$ matter fields. The meromorphic differential λ of this theory is given by

$$\lambda = \frac{xdx}{2\pi i y}\left(\frac{C(x)G'(x)}{2G(x)} - C'(x)\right). \tag{7.26}$$

What we need is the expression at weak coupling region $u \sim \infty$, $\Lambda \sim 0$ as before. To this end, we expand λ around $u \sim \infty$ and integrate by parts to give the curve of the form as:

$$\begin{aligned} \lambda &= -\frac{u^{\frac{1}{2}}dz}{2\pi i} \sum_{n,m=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(2n+m)}{\Gamma(\frac{1}{2})n!\Gamma(2n+1)m!}\left(\frac{\Lambda^4}{u^2}\right)^n \left(\frac{v}{u^{\frac{3}{2}}}\right)^m z^{-m}(z^2-1)^{-2n-m} \qquad (7.27) \\ &= -\frac{u^{\frac{1}{2}}dz}{2\pi i}\int \frac{ds}{2\pi i} \sum_{m=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})\Gamma(-s)(-1)^s\Gamma(2s+m)}{\Gamma(\frac{1}{2})\Gamma(2s+1)m!}\left(\frac{\Lambda^4}{u^2}\right)^s \left(\frac{v}{u^{\frac{3}{2}}}\right)^m z^{-m}(z^2-1)^{-2s-m}, \end{aligned}$$

where we set $x = u^{1/2}z$. In order to obtain a_k $(k = 1, 2)$, we pick up poles at $z = 1, -1$ along α_1, α_2 cycle and $z = 0$ along α_3 cycle. First we calculate a_1 by picking up the poles at $z = 1$ in a same way as before. But in order to get rid of the logarithmic singularity at $z = 0$, we integrate from $z = -1$ to $z = 1$ multiplying the phase factor $\sin 2\pi s/\pi$. Secondly we evaluate a_3 near $z \sim 0$ by suitably re-expanding the expression for λ, and after that from the consistency

$$a_2 = -a_1 - a_3, \tag{7.28}$$

we can determine the rest component of Higgs field a_2. First of all, we can evaluate a_1 in the following form:

$$\begin{aligned} a_1 &= -u^{\frac{1}{2}} \int \frac{ds}{2\pi i} \frac{\sin 2\pi s}{\pi} \sum_{m=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})\Gamma(-s)(-1)^s \Gamma(2s+m)}{\Gamma(\frac{1}{2})\Gamma(2s+1)m!} \left(\frac{\Lambda^4}{u^2}\right)^s \left(\frac{v}{u^{\frac{3}{2}}}\right)^m \\ &\qquad \times \frac{\Gamma(-\frac{m}{2}+\frac{1}{2})\Gamma(-2s-m+1)}{\Gamma(-2s-\frac{3m}{2}+\frac{3}{2})} \left(1+(-1)^{-m}\right) \\ &= u^{\frac{1}{2}} \sum_{n,l=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(2n+3l-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2n+1)n!\Gamma(2l+1)\Gamma(l+\frac{1}{2})} \left(\frac{\Lambda^4}{u^2}\right)^n \left(\frac{v^2}{u^3}\right)^l . \end{aligned} \tag{7.29}$$

Next we calculate a_D^1. We replace $\sin 2\pi s/\pi$ in (7.28) to $\frac{1}{\pi i}$, and evaluate double poles coming from the line integral to get a_D^1 in the following form

$$\begin{aligned} a_D^1 &= -\frac{u^{\frac{1}{2}}}{2\pi i} \int \frac{ds}{2\pi i} \sum_{m=0}^{\infty} \frac{\Gamma(s+\frac{1}{2})\Gamma(-s)(-1)^s \Gamma(2s+m)}{\Gamma(\frac{1}{2})\Gamma(2s+1)m!} \left(\frac{\Lambda^4}{u^2}\right)^s \left(\frac{v}{u^{\frac{3}{2}}}\right)^m \\ &\qquad \times \frac{\Gamma(-\frac{m}{2}+\frac{1}{2})\Gamma(-2s-m+1)}{\Gamma(-2s-\frac{3m}{2}+\frac{3}{2})} \left(1+(-1)^{-m}\right) \\ &= \frac{u^{\frac{1}{2}}}{2\pi i} \sum_{n,l=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(2n+3l-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2n+1)n!\Gamma(2l+1)\Gamma(l+\frac{1}{2})} \left(\frac{\Lambda^4}{u^2}\right)^n \left(\frac{v^2}{u^3}\right)^l \\ &\qquad \times \left[\psi(n+\frac{3l}{2}-\frac{1}{4}) + \psi(n+\frac{3l}{2}+\frac{1}{4}) - 2\psi(n+1) + \log\left(-\frac{\Lambda^4}{u^2}\right)\right]. \end{aligned} \tag{7.30}$$

Applying the usual analytic continuation to the expression for a_1, a_D^1 to get the expression around $u \sim \Lambda^2$, we find that a_1, a_D^1 both contain the logarithmic terms. However using the degree of freedom to add a_1 to a_D^1 in the weak coupling region, we can claim that after analytic continuation the relation $a_1 \sim a_D^1$ holds near the critical point as follows:

$$\begin{aligned} a_1 &= u^{\frac{1}{2}} \sum_{l=0}^{\infty} \frac{\Gamma(3l-\frac{1}{2})}{\Gamma(2l+1)\Gamma(l+\frac{1}{2})} \left(\frac{v^2}{u^3}\right)^l \\ &\quad \times \left\{ \frac{\Gamma(3l-1)(1-y)^{-3l+1}}{\Gamma(\frac{3l}{2}-\frac{1}{4})\Gamma(\frac{3l}{2}+\frac{1}{4})} \sum_{n=0}^{3l-2} \frac{(-\frac{3l}{2}+\frac{3}{4})_n(-\frac{3l}{2}+\frac{5}{4})_n}{(2-3l)_n\ n!}(1-y)^n \right. \\ &\quad \left. -\frac{(1-y)^{3l-1}y^{-\frac{3l}{2}+\frac{1}{4}}(1-z)^{-3l+1}}{\Gamma(-\frac{3l}{2}+\frac{3}{4})\Gamma(-\frac{3l}{2}+\frac{5}{4})\Gamma(3l)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n z^n}{(3l)_n\ n!} \right. \end{aligned} \tag{7.31}$$

$$\times\left[\log z-\psi_n(1)-\psi_n(3l)+2\psi_n(\frac{1}{2})+\log 4 \right.$$
$$\left.\left. +\psi(\frac{3l}{2}+\frac{1}{4})+\psi(\frac{3l}{2}-\frac{1}{4})+i\pi-2\psi(\frac{1}{2})\right]\right\},$$

$$a_D^1 = u^{\frac{1}{2}}\sum_{l=0}^{\infty}\frac{\Gamma(3l-\frac{1}{2})}{\Gamma(2l+1)\Gamma(l+\frac{1}{2})}\left(\frac{v^2}{u^3}\right)^l$$
$$\times\left\{\frac{\Gamma(3l-1)(1-y)^{-3l+1}}{\Gamma(\frac{3l}{2}-\frac{1}{4})\Gamma(\frac{3l}{2}+\frac{1}{4})}\sum_{n=0}^{3l-2}\frac{(-\frac{3l}{2}+\frac{3}{4})_n(-\frac{3l}{2}+\frac{5}{4})_n}{(2-3l)_n\ n!}(1-y)^n\right.$$
$$-\frac{(1-y)^{3l-1}y^{-\frac{3l}{2}+\frac{1}{4}}(1-z)^{-3l+1}}{\Gamma(-\frac{3l}{2}+\frac{3}{4})\Gamma(-\frac{3l}{2}+\frac{5}{4})\Gamma(3l)}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\frac{1}{2})_n z^n}{(3l)_n\ n!} \tag{7.32}$$
$$\times\left[\log z-\psi_n(1)-\psi_n(3l)+2\psi_n(\frac{1}{2})+\log 4\right.$$
$$\left.\left. +\psi(\frac{3l}{2}+\frac{1}{4})+\psi(\frac{3l}{2}-\frac{1}{4})-2\psi(\frac{1}{2})\right]\right\},$$

where $y=\frac{\Lambda^4}{u^2}$ and $z=\frac{\Lambda^2-u}{2\Lambda^2}$. This situation is same as $SO(2N)$ theory.

Next we want to obtain a_3, a_D^3 by evaluating around $z\sim 0$. In this case, we instead take the variable as $x=-\frac{v}{u}z$, and expand λ around $u\sim\infty$ to give the following form:

$$\lambda = -\frac{vdz}{u2\pi i}\int\frac{ds}{2\pi i}\sum_{m=0}^{\infty}\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)(-1)^s\Gamma(2s+m)}{\Gamma(\frac{1}{2})\Gamma(2s+1)m!}\left(\frac{\Lambda^4}{u^2}\right)^s\left(\frac{v}{u^{\frac{3}{2}}}\right)^m \tag{7.33}$$

Using the effective evaluation of the cycle α_3 by integrating from $z=0$ to $z=1$ multiplying the factor $\sin 2s\pi/\pi$, we can calculate a_3 quite similarly

$$a_3 = -\frac{v}{u}\int\frac{ds}{2\pi i}\frac{\sin 2s\pi}{\pi}\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)(-1)^s\Gamma(2s+m)}{\Gamma(\frac{1}{2})\Gamma(2s+1)m!}\left(\frac{\Lambda^4}{u^2}\right)^s\left(\frac{v}{u^{\frac{3}{2}}}\right)^m$$
$$\times\frac{\Gamma(2s+3m+1)\Gamma(-2s-m+1)}{\Gamma(2m+2)(-1)^{2s+m}} \tag{7.34}$$
$$= -\frac{v}{u}\sum_{n,m=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\Gamma(2n+3m+1)}{\Gamma(\frac{1}{2})\Gamma(2n+1)n!\Gamma(2m+2)}\left(\frac{\Lambda^4}{u^2}\right)^n\left(\frac{v^2}{u^3}\right)^m.$$

In order to obtain the expression for a_D^3, we replace $\sin 2\pi s/\pi$ in (1.13) to $\frac{1}{2\pi i}$ and evaluate double poles. The result can be written as:

$$a_D^3 = -\frac{v}{u2\pi i}\int\frac{ds}{2\pi i}\frac{\Gamma(s+\frac{1}{2})\Gamma(-s)(-1)^s\Gamma(2s+m)}{\Gamma(\frac{1}{2})\Gamma(2s+1)m!}\left(\frac{\Lambda^4}{u^2}\right)^s\left(\frac{v}{u^{\frac{3}{2}}}\right)^m$$
$$\times\frac{\Gamma(2s+3m+1)\Gamma(-2s-m+1)}{\Gamma(2m+2)(-1)^{2s+m}} \tag{7.35}$$

$$= -\frac{v}{u2\pi i}\sum_{n,m=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\Gamma(2n+3m+1)}{\Gamma(\frac{1}{2})\Gamma(2n+1)n!\Gamma(2m+2)}\left(\frac{\Lambda^4}{u^2}\right)^n\left(\frac{v^2}{u^3}\right)^m$$
$$\times\left[\psi(n+\frac{3m}{2}+\frac{1}{2})+\psi(n+\frac{3m}{2}+1)-2\psi(n+1)+\log\left(-\frac{\Lambda^4}{u^2}\right)\right].$$

Performing the analytic continuation to the region $u \sim \Lambda^2$, we find that these expressions for a_3, a_D^3 contain no logarithmic term

$$a_3 = \frac{v}{u}\sum_{m=0}^{\infty}\frac{\Gamma(3m+1)}{\Gamma(2m+2)m!}\left(\frac{v^2}{u^3}\right)^m$$
$$\times\left\{\frac{\Gamma(-3m-\frac{1}{2})y^{-\frac{3m}{2}-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\frac{3m}{2})\Gamma(-\frac{3m}{2})}(1-z)^{-3m-\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2};3m+\frac{3}{2};z\right)\right. \tag{7.36}$$
$$\left.+\frac{\Gamma(3m+\frac{1}{2})y^{\frac{3m}{2}}}{\Gamma(\frac{3m}{2}+\frac{1}{2})\Gamma(\frac{3m}{2}+1)}(1-y)^{\frac{1}{2}-3m}(1-z)^{3m+\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2};\frac{1}{2}-3m;z\right)\right\}$$

$$a_D^3 = \frac{v}{u\pi i}\sum_{m=0}^{\infty}\frac{\Gamma(3m+1)}{\Gamma(2m+2)m!}\left(\frac{v^2}{u^3}\right)^m$$
$$\times\frac{\Gamma(3m+\frac{1}{2})\Gamma(\frac{3m}{2}+1)}{\Gamma(3m+\frac{3}{2})}y^{-\frac{3m}{2}-\frac{1}{2}}(1-z)^{-3m-\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2};3m+\frac{3}{2};z\right), \tag{7.37}$$

and we can claim that the relation $a_3 \sim a_D'^3$ holds near the conformal point by setting $a_D'^3 = a_3 + a_D^3$. Therefore on the massless conformal point of $SU(3)$ theory with two hypermultipletes, the behavior of Higgs fields and dual fields obtained by the analytic continuation from the weak coupling region is similar to pure $SO(2N)$ theory. If we retain the parameterization $\tilde{u}$, $\tilde{v}$ based on the deviation from the conformal point, and evaluate the integral in the region $\tilde{u} \sim \infty$, $\tilde{v} \sim 0$ and analytically continue to the region $\tilde{u} \sim 0$, we can obtain the same result around the conformal point as was the case of pure $SU(3)$ theory. Notice that above treatments in this subsection for the massless conformal point can be easily generalized to the cases of $SU(N_c)$ theory with two hypermultiplets and $SO(2N+1)$ theory with one hypermultiplet.

8 Discussion

We have derived the expression for the periods and Higgs fields and its dual around the conformal point of $SU(2)$ Yang-Mills theory with matter fields, pure $SU(N)$ and pure $SO(2N)$ Yang-Mills theory. In the $SU(2)$ theory with matter fields and the pure $SU(N)$ theory, we have directly recognized the structure of the theories near the conformal points. We find a simple correspondence between the fixed point of 4-D $N = 2$ $SU(2)$ Yang-Mills theory with matter fields and Landau-Ginzburg description of 2-D $N = 2$ SCFT with $c = 3$. For $SU(N)$ and $SO(2N)$ theories we could show a verification of the analytic continuation due to the well known formula of the hypergeometric functions. Especially for $SU(3)$ theory we have compared various expressions with respect to the different parameterization. For $SU(3)$ theory with two massless matter fields, we could show a verification even in the case where the theory contain the matter field.

It seems interesting that we could obtain the explicit expression of fields around the conformal point even for the theories with higher rank gauge groups. But the examples we treated in this paper is elementary compared to more complicated varieties of critical points as was shown in [21]. For a non-trivial example there is a conformal point in the $SU(N_c)$ with $(2N_c - 2)$ hypermultipletes. Indeed looking at the curve of this theory

$$y^2 = \left(x^{N_c} - \sum_{i=2}^{N_c} s_i x^{N_c - i} + f_{N_f}(x, m_i)\right)^2 - \Lambda^{2N_c - N_f}(x+m)^{N_f},$$

$$f_{N_f}(x, m_i) = \frac{\Lambda^{2N_c - N_f}}{4} \sum_{k=0}^{N_f - N_c} \frac{N_f! m^k}{k!(N_f - k)!} x^{N_f - N_c - 1},$$

and comparing the coefficient of the curve when $N_f = 2N_c - 2$, we can find the conformal point

$$m = \frac{\Lambda}{N_c},$$

$$s_{k+2} = \left(\frac{\Lambda}{N_c}\right)^{k+2} N_c \left(\frac{N_c(2N_c - 2)!}{4(2N_c - 2 - k)!k!} + \frac{(N_c - 1)!(k+1)}{(k+2)!(N_c - 2 - k)!}\right).$$

where the curve become degenerate in the form as

$$y^2 = \left(x + \frac{\Lambda}{N_c}\right)^{2N_c - 1} \left(x - \frac{2N_c - 1}{N_c}\Lambda\right).$$

Although it is interesting to obtain the explicit form of the fields in this case, an important question is the verification of the validity of the analytic continuation for these cases, which require further investigation.

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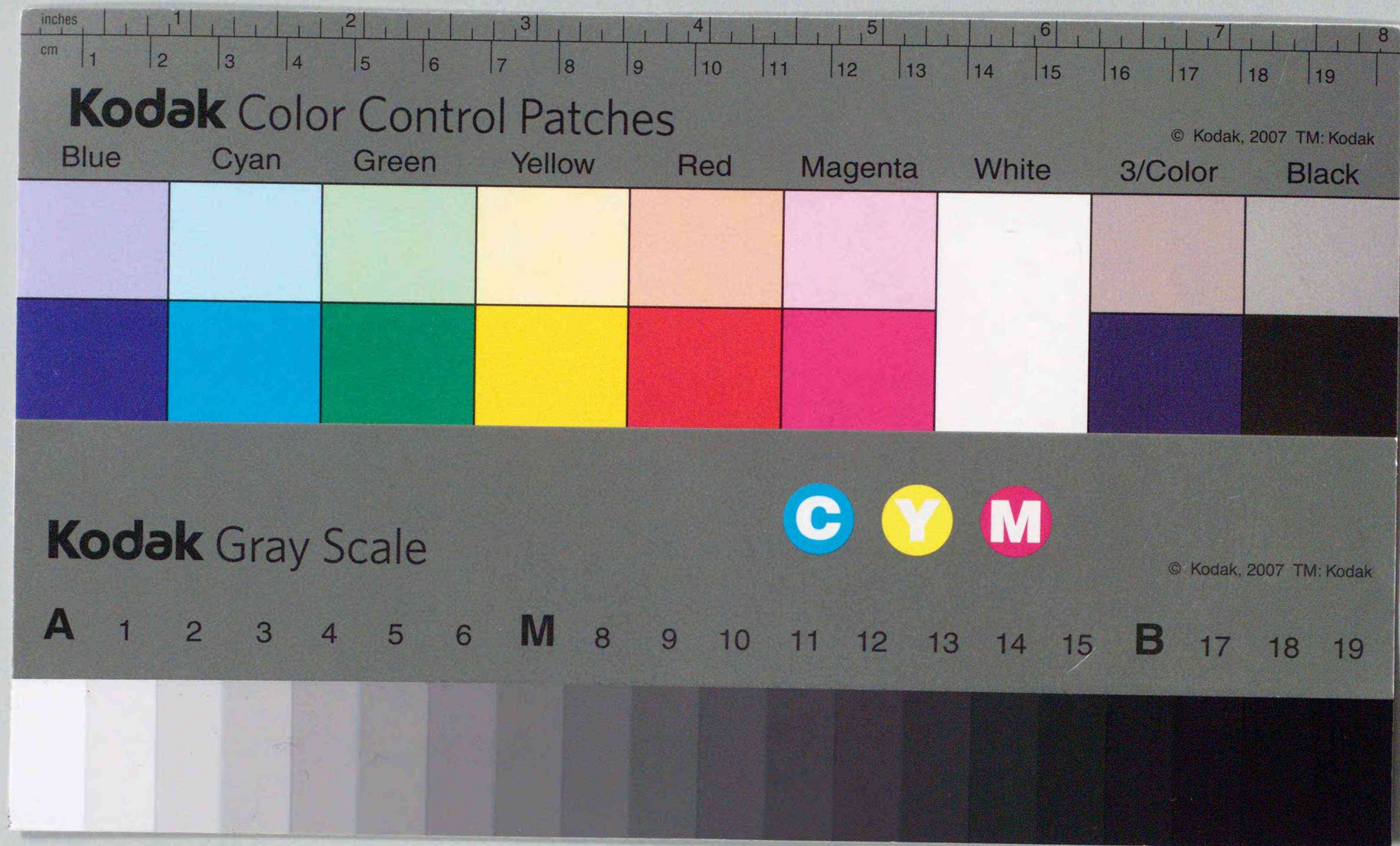

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