Study on lifespan of solutions to quasilinear wave equations

(準線型波動方程式の解の lifespan についての研究)

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STUDY ON LIFESPAN OF SOLUTIONS TO QUASILINEAR WAVE EQUATIONS

AKIRA HOSHIGA

Kitami Institute of Technology
Kitami, 090, Japan

Introduction.

In this paper, we study the behaviour of classical solutions to quasilinear wave equations in two space dimensions with small data, as the following type:

$$(\partial_0^2 - \partial_1^2 - \partial_2^2)u = \sum_{\alpha,\beta=0}^{2} a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u, \qquad (x,t) \in \mathbb{R}^2 \times (0,\infty), \quad (1)$$

$$u(x,0) = \varepsilon f(x), \quad \partial_0 u(x,0) = \varepsilon g(x), \qquad x \in \mathbb{R}^2, \quad (2)$$

where we denote $\partial_0 = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ and $u' = (\partial_0 u, \partial_1 u, \partial_2 u)$. This equation includes the following equation:

$$\partial_0^2 u - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \qquad (x,t) \in \mathbb{R}^2 \times (0,\infty), \quad (3)$$

where $\nabla u = (\partial_1 u, \partial_2 u)$. This equation describes the vertical motion of a vibrating membrane.

We assume that f and g are smooth, have compact support and do not vanish identically. We also assume that any $a_{\alpha\beta}(u')$ is smooth and

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$a_{\alpha\beta}(u') = O(|u'|^2)$ near $u' = 0$. We define *lifespan* T_ε of the Cauchy problem (1) and (2) by the supremum of all τ for which a $C^\infty(\mathbb{R}^2 \times [0, \tau])$-solution exists. Our aim is to obtain some estimates of the lifespan. This paper consists of two parts. In Part I, we obtain a lower bound of the lifespan and sufficient conditions for the global existence of the solution. In Part II, we investigate an upper bound of the lifespan and asymptotic behaviour of the solution near the blowing up point.

M. Kovalyov proved in [6] that

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \geqq C,$$

where C depends on f, g and $a_{\alpha\beta}$. The first aim in Part I is to determine constant C explicitly. To realize that, we use Friedlander's radiation field $\mathcal{F}(\omega, \rho)$ which is defined by f and g, see page 7. Moreover we write the nonlinear term of (1) as

$$a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u = \sum_{\gamma,\delta=0}^{2} Z_{\alpha\beta\gamma\delta}\partial_\delta u \partial_\delta u \partial_\alpha \partial_\beta u + O(|u'|^3|u''|).$$

Then we define an important quantity;

$$H = \max_{\rho \in \mathbb{R}, \omega \in S^1} \{-C(-1, \omega)\partial_\rho \mathcal{F}(\omega, \rho)\partial_\rho^2 \mathcal{F}(\omega, \rho)\},$$

where

$$C(X) = \sum_{\alpha,\beta,\gamma,\delta=0}^{2} Z_{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta, \qquad X = (X_0, X_1, X_2)$$

and $C(-1, \omega)$ is defined by setting $X_0 = -1$ and $(X_1, X_2) = \omega \in S^1$. S^1 stands for the unit circle. When $H > 0$ we prove the following

Theorem A (Theorem 1 in Part I). *If $H > 0$, we have*

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \geqq \frac{1}{H}.$$

This theorem claims that the lifespan goes to infinity as H tends to 0. Thus you may expect that the solution exists globally in time when $H = 0$. In fact, we prove

Theorem B (Theorem 2 in Part *I*). *If $H = 0$, then there exists an $\varepsilon_0 > 0$ such that $T_\varepsilon = +\infty$ for $\varepsilon < \varepsilon_0$.*

Since we assume that f and g do not vanish identically, the condition $H = 0$ is equivalent to the condition $C(-1, \omega) \equiv 0$ in S^1, which is called Klainerman's null-condition. The nonlinear terms satisfying null-condition are represented as linear combinations of the following.

$$\sum_{\alpha,\beta=0}^{2} C_{\alpha\beta}\partial_\alpha\partial_\beta u\{(\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2\} + O(|u'|^3|u''|),$$

$$\sum_{\alpha,\beta=0}^{2} C_{\alpha\beta}\partial_\alpha u\partial_\beta\{(\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2\} + O(|u'|^3|u''|),$$

$$\sum_{\alpha,\beta=0}^{2} C_{\alpha\beta}\partial_\alpha u\{\partial_0^2 u - \partial_1^2 u - \partial_2^2 u\} + O(|u'|^3|u''|),$$

$$\sum_{\alpha=0}^{2} C_\alpha\partial_\alpha u\{\partial_\beta u\partial_\gamma\partial_\delta u - \partial_\gamma u\partial_\beta\partial_\delta u\} + O(|u'|^3|u''|),$$

where $C_{\alpha\beta}$ and C_α are constant. Our method of the proof of Theorem A and Theorem B is based on the one in F. John [3]. He proved some reults similar to ours in three space dimensions.

However, these results mention only the lower bound of the lifespan and thus you may wonder if the estimate in Theorem A is optimal or not. Therefore we have to think about an upper bound of the lifespan. For some technical reason, however, our solution requires to be spatially radially symmetric to show the blowing up result. If the solution to the Cauchy problem (1) and (2) is spatially radially symmetric, it satisfies

$$\partial_0^2 u - c^2(\partial_0 u, \partial_r u)(\partial_r^2 u + \frac{1}{r}\partial_r u) = \frac{1}{r}\partial_r u G(\partial_0 u, \partial_r u), \quad (r,t) \in (0,\infty) \times (0,\infty), \tag{4}$$

$$u(r, 0) = \varepsilon f(r), \quad \partial_0 u(r, 0) = \varepsilon g(r), \quad r \in (0,\infty) \tag{5}$$

where $r = |x|$, $x \in \mathbb{R}^2$ and

$$\begin{aligned} c^2(\partial_0 u, \partial_r u) =& 1 + a_1 \partial_0 u^2 + a_2 \partial_0 u \partial_r u + a_3 \partial_r u^2 + O(|\partial_0 u|^3 + |\partial_r u|^3), \\ G(\partial_0 u, \partial_r u) =& O(|\partial_0 u|^2 + |\partial_r u|^2). \end{aligned}$$

From now on T_ε stands for the lifespan of the Cauchy problem (4) and (5). In Part II we prove

Theorem C (Theorem 2.1 in Part II). *If $a_1 - a_2 + a_3 \neq 0$, we have*

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \leq \frac{1}{H_0}$$

with

$$H_0 = \max_{\rho \in \mathbb{R}} \{-(a_1 - a_2 + a_3)\mathcal{F}'(\rho)\mathcal{F}''(\rho)\},$$

where $\mathcal{F}(\rho)$ is Friedlander's radiation field of $f(r)$ and $g(r)$.

Since the Cauchy problem (4) and (5) is a special case of (1) and (2), the estimate in Theorem A holds for this problem. One can verify that $C(-1, \omega) \equiv a_1 - a_2 + a_3$ and thus $H = H_0$. Therefore, combining Theorem A and Theorem C, we conclude

Theorem D (Corollary in Part II).

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) = \frac{1}{H_0}.$$

The proof of Theorem C goes as follows. At first we construct an ordinary differential equation with respect to $\partial^2 u$ along the characteristic curve $dr/dt = c(\partial_0 u, \partial_r u)$ in (r, t)-plane by using (4). Then, solving the ordinary differential equation we obtain an expression of the solution and find that $\partial^2 u$ goes to infinity as t tends to T_ε. This method was investigated in F. John [2], L. Hörmander [4] and S. Alinhac [1] .

The equation of vibrating membrane (3) is rewritten in the form (4) with $a_1 = a_2 = 0$ and $a_3 = -3/2$, if the solution is radially symmetric. Since $a_1 - a_2 + a_3 \neq 0$, Theorem C is applicable and it implies that the second order derivative of the solution develops singularities pointwisely in finite time. The solution u of (3) stands for the vertical motion of the vibrating membrane. Thus our blowing up result means that the curvature of the membrane breaks at some points ,while the difference of the membrane and the speed of the vibration stay small. When the space dimension is one, (3) stands for the motion of vibrating string. For this equation S. Klainerman and A. Majda [5] proved that the solution always blows up in finite time under the Dirichlet or the Neumann boundary condition.

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PART I

The initial value ploblems
for quasi-linear wave equations
in two space dimensions with small data

ABSTRACT. In Part I, we study the lifespan of solutions to quasilinear wave equations in two space dimensions with small initial data. We shall show a lower bound for the lifespan T_ε;$\liminf_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \geqslant \frac{1}{H}$ $(H \neq 0)$, where H is explicitely represented by initial data and nonlinearities. From the above estimate, one may indicate that $T_\varepsilon = \infty$, if $H = 0$. Indeed, this will be proved in section 3. By virtue of the explicite form of H, We can decide the form of cubic term of the nonlinearity which satisfy $H = 0$. To sum up, Part I says global existence of solutions depends not only on the power of the nonlinearity but on the form of the top term of it.

1. Introduction and Statement of Results.

We study the lifespan of solutions of quasi-linear wave equations in two space dimensions, with small initial data, as following type;

$$\Box u(x,t) = a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u(x,t), \quad (x,t) \in \mathbb{R}^2 \times [0,\infty), \tag{1.1}$$

$$u(x,0) = \varepsilon f(x), \quad \partial_0 u(x,0) = \varepsilon g(x), \quad x \in \mathbb{R}^2. \tag{1.2}$$

Here $a_{\alpha\beta}(u') = a_{\beta\alpha}(u')$ and we denote $\partial_0 = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ $(i = 1,2)$ and $\Box = \partial_0^2 - \partial_1^2 - \partial_2^2$. The gradient of u is denoted by $u' = (\partial_0 u, \partial_1 u, \partial_2 u)$. We use the summation convention with subscripts $\alpha, \beta \cdots$ ranging over 0, 1, 2 and $i, j, \cdots$ over 1,2. Moreover we assume that

$$f, g \in C_0^\infty(\mathbb{R}^2) \quad \text{and} \quad f(x) = g(x) = 0, \quad \text{for} \quad |x| \geqslant M \tag{1.3}$$

$$a_{\alpha\beta}(p) \in C^\infty(\mathbb{R}^3) \tag{1.4a}$$

$$a_{\alpha\beta}(p) = O(|p|^2) \tag{1.4b}$$

$$|a_{\alpha\beta}(p)| < 1/2 \tag{1.4c}$$

for $|p| < \delta$, where δ is a small positive number.

The supremum of all τ for which $C^\infty(\mathbb{R}^2 \times [0,\tau))$-solution of the Cauchy problem (1.1), (1.2) exists is called the "lifespan" T_ε. When $T_\varepsilon = \infty$, we say the Cauchy problem (1.1), (1.2) has a *global solution.*

M. Kovalyov has proved in [12] that

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \geqslant C, \tag{1.5}$$

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where the constant C depends on f, g and $a_{\alpha\beta}$. The first aim is to determine the constant explicitly by Friedlander radiation field. Let $U(x,t)$ be a solution of linear wave equation;

$$\Box U(x,t) = 0, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty), \tag{1.6}$$

$$U(x,0) = f(x), \quad \partial_0 U(x,0) = g(x), \quad x \in \mathbb{R}^2. \tag{1.7}$$

Then we can define the Friedlander radiation field $\mathcal{F}(\omega,\rho)$ by

$$\mathcal{F}(\omega,\rho) = \lim_{r\to\infty} r^{1/2} U(x,t), \quad x = r\omega,\ \omega \in S^1,\ \rho = r - t. \tag{1.8}$$

$\mathcal{F}$ is explicitly expressed by

$$\mathcal{F}(\omega,\rho) = \frac{1}{2\sqrt{2}\pi} \int_\rho^\infty (s-\rho)^{-1/2} \{R_g(\omega,s) - \partial_s R_f(\omega,s)\} ds, \tag{1.9}$$

where R_h is Radon transform of $h \in C_0^\infty(\mathbb{R}^2)$, *i.e.*,

$$R_h(\omega,s) = \int_{\omega\cdot y = s} h(y) dS_y.$$

Note that $\mathcal{F}$ satisfies

$$\mathcal{F}(\omega,\rho) = 0 \quad \text{for} \quad \rho \geqslant M, \tag{1.10}$$

$$|\partial_\rho^\ell \mathcal{F}(\omega,\rho)| \leqslant C(1+|\rho|)^{-1/2-\ell}. \tag{1.11}$$

For the above facts, see Hörmander [3].

We write the non-linear term in (1.1) as

$$a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u = Z_{\alpha\beta\gamma\delta}(\partial_\gamma u)(\partial_\delta u)(\partial_\alpha\partial_\beta u) + O(|u'|^3|u''|), \tag{1.12}$$

where

$$Z_{\alpha\beta\gamma\delta} = \left.\frac{\partial^2 a_{\alpha\beta}(u')}{\partial(\partial_\gamma u)\partial(\partial_\delta u)}\right|_{u'=0}. \tag{1.13}$$

Thus we define an important quantity

$$H = \max_{\rho\in\mathbb{R},\omega\in S^1} \{-C(-1,\omega)\partial_\rho\mathcal{F}(\omega,\rho)\partial_\rho^2\mathcal{F}(\omega,\rho)\}, \tag{1.14}$$

where

$$C(X) = Z_{\alpha\beta\gamma\delta}X_\alpha X_\beta X_\gamma X_\delta, \quad X = (X_0, X_1, X_2), \tag{1.15}$$

and $C(-1,\omega)$ is defined by setting $X_0 = -1$, $(X_1, X_2) = \omega \in S^1$. By (1.10) and (1.11), we find that H is well-defined and non-negative.

Theorem 1.
$$\liminf_{\varepsilon\to 0}\varepsilon^2\log(1+T_\varepsilon)\geqslant\frac{1}{H}.$$

Proof of this theorem is basically owed to the method in F. John [6]. When $a_{\alpha\beta}(u')=O(|u'|)$ in three space dimensions, he proved that

$$\liminf_{\varepsilon\to 0}\varepsilon\log(1+T_\varepsilon)\geqslant\frac{1}{H^*},$$

where
$$H^*=\max_{\rho\in\mathbb{R},\omega\in S^2}\big\{-\frac{1}{2}C(-1,\omega)\partial_\rho^2\mathcal{F}(\omega,\rho)\big\}.$$

We next study the interesting case $H=0$. The condition is equivalent to the condition (i) f and g vanish identically or (ii) $C(-1,\omega)=0$ for any $\omega\in S^1$, see Appendix. Under the condition (i), the Cauchy problem (1.1), (1.2) has a trivial global solution $u\equiv 0$. Under the condition (ii) which is called Klainerman's null-condition, we find from (1.15) that $C(X)$ is devided by $X_0^2-X_1^2-X_2^2$. Hence $a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u$ is represented as a linear combination of the followings:

$$C_{\alpha\beta}(\partial_\alpha\partial_\beta u)\{(\partial_0 u)^2-(\partial_1 u)^2-(\partial_2 u)^2\}+O(|u'|^3|u''|), \tag{1.16a}$$

$$C_{\alpha\beta}(\partial_\alpha u)\partial_\beta\{(\partial_0 u)^2-(\partial_1 u)^2-(\partial_2 u)^2\}+O(|u'|^3|u''|), \tag{1.16b}$$

$$C_{\alpha\beta}(\partial_\alpha u)(\partial_\beta u)\Box u+O(|u'|^3|u''|), \tag{1.16c}$$

$$C_\alpha(\partial_\alpha u)\{(\partial_\beta u)(\partial_\gamma\partial_\delta u)-(\partial_\gamma u)(\partial_\beta\partial_\delta u)\}+O(|u'|^3|u''|),\quad \beta,\gamma,\delta=0,1,2, \tag{1.16d}$$

where $C_{\alpha\beta}$ and C_α are constants.

Theorem 2. *If $H=0$, there exists an $\varepsilon_0>0$ such that $T_\varepsilon=\infty$ for any $0<\varepsilon<\varepsilon_0$.*

S. Klainerman [11], L. Hörmander [3], D. Christodoulou [1] and F. John [6] proved independently that the null-condition implies global existence for small data in three space dimensions. When the non-linear term is cubic form of u' in two space dimensions, P. Godin [2] proved the same results by making use of L^1-L^∞ estimates studied in L. Hörmander [4] and S. Klainerman [9]. Theorem 2 is obtained along the same lines as in P. Godin [2].

We will prove Theorem 1 in section 2 and Theorem 2 in section 3.

2. Proof of Theorem 1.

First we introduce the generalized Sobolev space. Denote by $\Gamma_1,\Gamma_2,\cdots,\Gamma_7$, the vector fields

$$L_0=t\partial_0+x_1\partial_1+x_2\partial_2,\ L_i=x_i\partial_0+t\partial_i\quad(i=1,2),$$
$$\Omega=x_1\partial_2-x_2\partial_1,\quad \partial_0,\partial_1,\partial_2,$$

respectively. These operators satisfy commutation relations

$$[\Gamma_p,\Box]=\Gamma_p\Box-\Box\Gamma_p=2\delta_{1p}\Box,\ p=1,2,\cdots,7,$$
$$[\Gamma,\Gamma]=\overline{\Sigma}\Gamma,\ [\Gamma,\partial]=\overline{\Sigma}\partial. \tag{2.1}$$

$\overline{\Sigma}$ stands for finite linear combination with constant coefficients. For $\sigma \in \mathbb{Z}_+^7$ ($\mathbb{Z}_+$ is the set of non-negative integers), we put $\Gamma^\sigma = \Gamma_1^{\sigma_1}\Gamma_2^{\sigma_2}\cdots\Gamma_7^{\sigma_7}$. We define the norms

$$||v(t)||_k = \sum_{|\sigma|\leqslant k} ||\Gamma^\sigma v(\cdot,t)||_{L_x^2(\mathbb{R}^2)}, \tag{2.2}$$

$$|v(t)|_k = \sum_{|\sigma|\leqslant k} ||\Gamma^\sigma v(\cdot,t)||_{L_x^\infty(\mathbb{R}^2)}. \tag{2.3}$$

For convenience, when $k = 0$, we omit sub-index. Following propositions are very important in proving our theorems.

Propostion 2.1 (Klainerman's inequality [10]). *For smooth function $v(x,t)$ ($x \in \mathbb{R}^n, n \geqslant 2$),*

$$|v(x,t)| \leqslant C_n(1+|x|+t)^{-(n-1)/2}(1+|t-|x||)^{-1/2}||v(t)||_{[\frac{n}{2}]+1}, \tag{2.4}$$

where $[s]$ stands for the largest integer not exceeding s.

Proposition 2.2 (generalized energy estimate). *If the solution u of (1.1), (1.2) exists in $C^\infty(\mathbb{R}^2\times[0,T))$ and satisfies*

$$\sup_{0\leqslant s\leqslant t} |u'(s)|_{[\frac{k+1}{2}]} < 1, \quad \sup_{0\leqslant s\leqslant t} |u'(s)| < \delta \quad \text{for} \quad 0<t<T, \tag{2.5}$$

then

$$||u'(t)||_k \leqslant C_k||u'(0)||_k \exp(C_k \int_0^t |u'(s)|^2_{[\frac{k+1}{2}]} ds), \tag{2.6}$$

where δ is the one in (1.4) and $k \in \mathbb{N}$.

We prove Proposition 2.2. Multiplying Lv by $\partial_0 v$ and integrating with respect to x over $\mathbb{R}^2$, we arrive at the "energy identity" for a scalar v:

$$\frac{d}{dt}\int_{\mathbb{R}^2}\{(\partial_\alpha v)(\partial_\alpha v) - a_{00}(\partial_0 v)^2 + a_{ij}(\partial_i v)(\partial_j v)\}dx = \int_{\mathbb{R}^2} J(t,x)dx,$$

where $L = \Box - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta$ and

$$\begin{aligned} J = 2(\partial_0 v)(Lv) - (\partial_0 a_{00} + 2\partial_i a_{i0})(\partial_0 v)^2 - 2(\partial_j a_{ij})(\partial_0 v)(\partial_i v) \\ + (\partial_0 a_{ij})(\partial_i v)(\partial_j v). \end{aligned} \tag{2.7}$$

Using assumption (1.4c), we get

$$||v'(t)||^2 \leqslant 3||v'(0)||^2 + 2\int_0^t ds \int_{\mathbb{R}^2} |J(s,x)|dx, \tag{2.8}$$

which implies

$$||u'(t)||^2 \leqslant 3||u'(0)||^2 + C\int_0^t |u'(s)|^2_{[\frac{k+1}{2}]} ||u'(s)||_k^2 ds. \tag{2.9}$$

Using (1.1) and (2.1), we verify that for $|\sigma| \leqslant k$

$$L\Gamma^\sigma u = \Sigma\phi(u')(\Gamma^{\xi_1}\partial_{\alpha_1}u)(\Gamma^{\xi_2}\partial_{\alpha_2}u)\cdots(\Gamma^{\xi_q}\partial_{\alpha_q}u), \tag{2.10}$$

where $\phi \in C^\infty$ in u' is formed from the $a_{\alpha\beta}(u')$, and q and the multi-indices ξ_i satisfy

$$3 \leqslant q \leqslant k+2, \quad |\xi_1| + |\xi_2| + \cdots + |\xi_q| \leqslant k+1. \tag{2.11}$$

By (2.9) we can assume that

$$|\xi_p| \leqslant [\frac{k+1}{2}] \quad \text{for} \quad p \geqslant 2. \tag{2.12}$$

Therefore we find from (2.7), (2.10), (2.11), (2.12) and (2.5) that

$$\int_{\mathbb{R}^2} |J(s,x)|dx = O(|u'(s)|^2_{[\frac{k+1}{2}]}||u'(s)||^2_k).$$

Applying (2.8) to $v = \Gamma^\sigma u$ and combining with (2.9), we get

$$||u'(t)||^2_k \leqslant C_k(||u'(0)||^2_k + \int_0^t |u'(s)|^2_{[\frac{k+1}{2}]}||u'(s)||^2_k ds).$$

Therefore Gronwall's inequality yields (2.6).

In order to prove Theorem 1, it is sufficient to show following lemma.

Lemma 2.1. *For any $k \geqslant 9$ ($k \in \mathbb{N}$), $B > H$ and $m > 0$, there exist $J_k(B) > 0$ and $\varepsilon_k(m, B) > 0$ such that if*

$$\tau < \min\{T_\varepsilon, -1 + \exp(\frac{1}{B\varepsilon^2})\} \tag{2.13}$$

and

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau, \tag{2.14}$$

then for any $0 < \varepsilon < \varepsilon_k(m, B)$

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{J_k(B)\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau, \tag{2.15}$$

where H is given in (1.14).

We shall prove Theorem 1 by assuming Lemma 2.1. Let $U(x,t)$ be the solution of (1.6), (1.7). We find from (1.4b) that

$$\Gamma^\sigma u|_{t=0} = \varepsilon\Gamma^\sigma U|_{t=0} + O(\varepsilon^3) \quad \text{for any} \quad \sigma \in \mathbb{Z}^7_+. \tag{2.16}$$

Therefore by (2.16)

$$||u'(0)||_k \leqslant C_k\varepsilon.$$

Moreover by $k \geqslant 9$ and (2.4)

$$|u'(0)|_{[\frac{k+1}{2}]} \leqslant |u'(0)|_{k-2} \leqslant C_k||u'(0)||_k \leqslant C_k\varepsilon. \tag{2.17}$$

Letting $m > \max\{2J_k(B), C_k\}$, we get for sufficiently small τ

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau. \tag{2.18}$$

If (2.18) holds for any τ, then $T_\varepsilon = \infty$. Hence there exists a τ $(0 < \tau < T_\varepsilon)$ such that (2.18) holds and

$$|u'(\tau)|_{[\frac{k+1}{2}]} = \frac{m\varepsilon}{(1+\tau)^{1/2}}. \tag{2.19}$$

Suppose that $\tau < -1 + \exp(\frac{1}{B\varepsilon^2})$, then we can apply Lemma 2.1 and obtain

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{J_k(B)\varepsilon}{(1+t)^{1/2}} < \frac{m\varepsilon}{2(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau.$$

This contradicts (2.19). Therefore we have

$$T_\varepsilon > \tau \geqslant -1 + \exp(\frac{1}{B\varepsilon^2}) \;\; \text{for} \;\; 0 < \varepsilon < \varepsilon_k(m, B).$$

Since B is arbitrary except for condition "$B > H$", Theorem 1 follows.

Now we prove Lemma 2.1. First we verify that (2.15) holds for $0 \leqslant t \leqslant \varepsilon^{-1}$. By (2.17) we get

$$||u'(0)||_k < C_k\varepsilon.$$

Then for sufficiently small t

$$||u'(t)||_k < 2C_k^2\varepsilon,$$

also by (2.4) and $k \geqslant 9$

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{2C_k^3\varepsilon}{(1+t)^{1/2}} < \delta,$$

for sufficiently small ε. Using Proposition 2.2, we find that these inequality will continue to hold as long as

$$4C_k^7\varepsilon^2 \log(1+t) < \log 2.$$

Therefore if we take ε such that

$$4C_k^7\varepsilon^2 \log(1+\varepsilon^{-1}) < \log 2,$$

(2.15) holds for $0 \leqslant t < \varepsilon^{-1}$.

Moreover we have $u(x,t) = 0$ for $|x| \geqslant t + M$ (see [8], Appendix 1). Therefore we can restrict ourselves in the region

$$\varepsilon^{-1} \leqslant t < \tau, \quad |x| \leqslant t + M. \tag{2.20}$$

In order to show (2.15) in the region (2.20), we introduce "*pseudo characteristic rays*" in (r,t)-plane, which is given by solutions of ordinary differential equations;

$$\frac{dr}{dt} = \kappa(r,t), \tag{2.21}$$

where $\omega \in S^1$ is fixed and

$$\kappa(r,t) = 1 + \frac{1}{2}C(-1,\omega)(\partial_0 u)^2. \tag{2.22}$$

For each point (r,t) with $r \geqslant 0$, $\varepsilon^{-1} \leqslant t < \tau$, there exists such a curve through this point. Continuing this curve backwards, we arrive at a point (r_1, t_1) for which either $r_1 = 0, t_1 > \varepsilon^{-1}$ or $r_1 \geqslant 0, t_1 = \varepsilon^{-1}$. We call S_λ the solution of (2.21) with $t_1 - r_1 = \lambda$.

Along S_λ, we find that

$$\left|\frac{d(t-r-\lambda)}{dt}\right| = |1-\kappa| \leqslant C|u'|^2 \leqslant \frac{Cm^2\varepsilon^2}{1+t}, \tag{2.23}$$

$$|t-r-\lambda| \leqslant Cm^2\varepsilon^2 \log(1+t) \leqslant \frac{Cm^2}{B}, \tag{2.24}$$

where we have used (2.21), (2.22), (2.14) and (2.13). We take $\varepsilon_k(m,B)$ such that $\varepsilon_k(m,B) < \delta m^{-1}$, then Proposition 2.2, (2.14) and (2.13) yield

$$||u'(t)||_k < C_k\varepsilon \exp(C_k \int_0^t \frac{m^2\varepsilon^2}{1+s} ds) < C_k\varepsilon \exp(\frac{C_k m^2}{B}).$$

Therefore by $k \geqslant 9$ and Proposition 2.1

$$|u'(t)|_{[\frac{k+1}{2}]+2} \leqslant |u'(t)|_{k-2} \leqslant \frac{C_k\varepsilon \exp(\frac{C_k m^2}{B})}{(1+t)^{1/2}(1+|t-r|)^{1/2}}. \tag{2.25}$$

We set

$$\lambda_0 = \frac{Cm^2}{B} + \exp(\frac{2C_k m^2}{B}). \tag{2.26}$$

Then by (2.24), (2.25) and (2.26) along S_λ with $\lambda \geqslant \lambda_0$

$$|u'(t)|_{[\frac{k+1}{2}]} \leqslant \frac{C_k\varepsilon \exp(\frac{C_k m^2}{B})}{(1+t)^{1/2}(1+\exp(\frac{2C_k m^2}{B}))^{1/2}} \leqslant \frac{C_k\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau.$$

This implies that (2.15) is valid along S_λ with $\lambda \geqslant \lambda_0$, then it is sufficient to show (2.15) along S_λ with $-M \leqslant \lambda \leqslant \lambda_0$, $\varepsilon^{-1} \leqslant t < \tau$.

For functions $\varphi(x,t;\varepsilon), \psi(x,t;\varepsilon)$ we write $\varphi = O^*(\psi)$, if for any $k \geqslant 9, B > H$ and $m > 0$, there exist $J_k(B) > 0$ and $\varepsilon_k(m,B) > 0$ such that

$$|\varphi(x,t;\varepsilon)| < J_k(B)\psi(x,t;\varepsilon) \quad \text{for} \quad 0 < \varepsilon < \varepsilon_k(m,B),$$

as long as (2.13),(2.14) hold, along S_λ with $-M \leqslant \lambda \leqslant \lambda_0$ and $\varepsilon^{-1} \leqslant t < \tau$. Then our purpose is to prove

$$|u'(t)|_{[\frac{k+1}{2}]} = O^*(\varepsilon(1+t)^{-1/2}). \tag{2.27}$$

If we take $\varepsilon_k(m, B) < \lambda_0^{-1/p}$, then we find

$$\lambda_0 = O^*(\varepsilon^{-p}) \quad \text{for fixed} \quad p > 0. \tag{2.28}$$

For later we shall assume that $p < \dfrac{1}{8}$. We find from (2.28) and (2.24) that

$$t = r + O^*(\varepsilon^{-p}), \quad r^{-1} = t^{-1} + O^*(\varepsilon^{-p}t^{-2}), \quad r^{1/2} = t^{1/2} + O^*(\varepsilon^{-p}t^{-1/2}). \tag{2.29}$$

Then it follows from (2.25), (2.29) and (2.28) that

$$|u'(t)|_{[\frac{k+1}{2}]+2} = O^*(\varepsilon^{1-p}t^{-1/2}). \tag{2.30}$$

Since

$$|\Gamma^\sigma u(x,t)| = \left| -\int_r^{t+M} \omega_i \partial_i \Gamma^\sigma u(s\omega, t) ds \right|$$
$$= O\Big(\int_r^{t+M} |u'(t)|_{[\frac{k+1}{2}]+2} ds \Big) = O^*(\varepsilon^{1-2p}t^{-1/2}),$$

for $|\sigma| \leqslant [\dfrac{k+1}{2}] + 2$, we have

$$|u(t)|_{[\frac{k+1}{2}]+2} = O^*(\varepsilon^{1-2p}t^{-1/2}). \tag{2.31}$$

The operator ∂_α can be written as

$$\partial_i = -\omega_i \partial_0 + \frac{1}{t} L_i + \frac{\omega_i}{t+r} L_0 - \frac{r\omega_i\omega_j}{t(t+r)} L_j, \quad \partial_0 = \frac{1}{t^2 - r^2}(tL_0 - x_i L_i),$$

(see [6], Appendix 2 and [7]) and these representations yield

$$\partial_\alpha v = -\omega_\alpha \partial_0 v + O(t^{-1}|v|_1) = -\omega_\alpha \partial_r v + O(t^{-1}|v|_1), \tag{2.32}$$

$$\partial_\alpha v = O(\frac{1}{|t-r|}|v|_1), \tag{2.33}$$

$$\partial_\alpha \partial_\beta v = \omega_\alpha \omega_\beta \partial_0^2 v + O(t^{-1}|v'|_1) = \omega_\alpha \omega_\beta \partial_r^2 v + O(t^{-1}|v'|_1), \tag{2.34}$$

$$(\partial_0 + \partial_r) v = O(t^{-1}|v|_1), \quad (\partial_0 + \partial_r)^2 v = O(t^{-2}|v|_2), \tag{2.35}$$

where $\partial_r = \omega_i \partial_i$.

The operator L can be written in the form

$$\begin{aligned}Lv = &2t^{-1/2}(\partial_0+\kappa\partial_r)(t^{1/2}\partial_0 v)-(\partial_0+\partial_r)^2v+\frac{t-r}{tr}\partial_0 v\\ &-2(\kappa-1)(\partial_0+\partial_r)\partial_0 v-\frac{\delta_{ij}-\omega_i\omega_j}{t^2}L_iL_jv+\frac{\omega_i}{tr}L_iv\\ &-a_{\alpha\beta}(u')\partial_\alpha\partial_\beta v+C(-1,\omega)\partial_0^2 v.\end{aligned}$$

Then by (2.29), (2.33), (2.34), (2.35) and (2.30)

$$\frac{d}{dt}(t^{1/2}\partial_0 v)=\frac{1}{2}t^{1/2}Lv+O^*(t^{-3/2}|v|_2). \tag{2.36}$$

We apply (2.36) to $v=\Gamma^\sigma u$ with $|\sigma|\leqslant[\frac{k+1}{2}]$ below. When $v=\Gamma^\sigma u$

$$t^{1/2}\partial_0\Gamma^\sigma u(t)|_{t=\varepsilon^{-1}}=O(\varepsilon^{-1/2}|u'(\varepsilon^{-1})|_{[\frac{k+1}{2}]})=O(\varepsilon). \tag{2.37}$$

Now we show (2.27) by induction. We first show

$$|u'(t)|=O^*(\varepsilon t^{-1/2}). \tag{2.38}$$

Let $v=u$ in (2.36). Then it follows from (2.31) and $Lu=0$ that

$$\frac{d}{dt}(t^{1/2}\partial_0 u)=O^*(\varepsilon^{1-2p}t^{-2}). \tag{2.39}$$

Integrating (2.39) from ε^{-1} to t, we find from (2.37) that

$$|t^{1/2}\partial_0 u(t)|\leqslant|s^{1/2}\partial_0 u(s)|_{s=\varepsilon^{-1}}|+O^*\left(\int_{\varepsilon^{-1}}^t\varepsilon^{1-2p}s^{-2}ds\right)=O^*(\varepsilon), \tag{2.40}$$

which implies

$$\partial_0 u(t)=O^*(\varepsilon t^{-1/2}). \tag{2.41}$$

Using (2.32), (2.31) and (2.41),

$$\partial_i u(t)=-\omega_i\partial_0 u+O(t^{-1}|u|)\|=O^*(\varepsilon t^{-1/2})+O^*(\varepsilon^{1-2p}t^{-3/2})=O^*(\varepsilon).$$

Next we shall show

$$|u'(t)|_1=O^*(\varepsilon t^{-1/2}). \tag{2.42}$$

We begin the proof of (2.42) by showing

$$\partial_\alpha\partial_\beta u(t)=O^*(\varepsilon t^{-1/2}). \tag{2.43}$$

Letting $v=\partial_0 u$ in (2.36), it follows from $Lu=0$, (1.15), (2.32), (2.34), (2.30) and (2.31) that

$$\begin{aligned}L(\partial_0 u)&=\Box(\partial_0 u)-a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\partial_0 u\\ &=(\partial_0 a_{\alpha\beta}(u'))\partial_\alpha\partial_\beta u\\ &=Z_{\alpha\beta\gamma\delta}\partial_0\{(\partial_\gamma u)(\partial_\delta u)\}\partial_\alpha\partial_\beta u+O(|u'|^2|u''|^2)\\ &=2C(-1,\omega)(\partial_0^2u)^2\partial_0 u+O(|u'|^2|u''|^2+t^{-1}|u|_1|u'|_1^2)\\ &=2C(-1,\omega)t^{-1}(t^{\frac{1}{2}}\partial_0^2u)^2\partial_0 u+O^*(\varepsilon^{3-4p}t^{-2}).\end{aligned}$$

Then we write the differential equation (2.36) as

$$\frac{d}{dt}W_1(t) = C(-1,\omega)t^{-1/2}W_1(t)^2\partial_0 u(t) + O^*(\varepsilon^{1-p}t^{-3/2}), \tag{2.44}$$

where

$$W_1(t) = t^{1/2}\partial_0^2 u(t). \tag{2.45}$$

Notice that by (2.30),

$$W_1(t) = O^*(\varepsilon^{1-p}). \tag{2.46}$$

The following facts play an important role in our proof.

$$|\partial_0^\ell U(x,t) - (-1)^\ell \partial_\rho^\ell \mathcal{F}(\omega,\rho)| = O(r^{-3/2}), \tag{2.47}$$

$$|\partial_0^\ell u(x,\varepsilon^{-1}) - \varepsilon\partial_0^\ell U(x,\varepsilon^{-1})| = O(\varepsilon^3), \tag{2.48}$$

with $\ell = 1, 2$. (2.47) can be proved by using Lemma 2.1.1 in [3]. We prove (2.48) for $\ell = 1$ because another case can be proved in the similar way. By (2.16), the function $u - \varepsilon U$ satisfies

$$\Box(u - \varepsilon U) = a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u, \tag{2.49}$$

$$||\partial_0 u(0) - \varepsilon\partial_0 U(0)|| = O(\varepsilon^3). \tag{2.50}$$

Applying Proposition 2.1 and classical energy estimate to (2.49), (2.50), we find that

$$\begin{aligned}
&|\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})| \\
\leqslant &C(1+\varepsilon^{-1})^{-1/2}||\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})|| \\
\leqslant &C\varepsilon^{1/2}(||\partial_0 u(0) - \varepsilon\partial_0 U(0)|| + \int_0^{\varepsilon^{-1}} ||a_{\alpha\beta}(u'(s))\partial_\alpha\partial_\beta u(s)||ds).
\end{aligned}$$

Since $0 \leqslant s \leqslant \varepsilon^{-1}$, we find that

$$||a_{\alpha\beta}(u'(s))\partial_\alpha\partial_\beta u(s)|| \leqslant C\left(\int_{\mathbb{R}^2} |u'(s)|^6_{[\frac{k+1}{2}]}ds\right)^{1/2} = O(\varepsilon^3 s^{-1/2}).$$

Therefore we have

$$|\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})| = O(\varepsilon^{7/2} + \varepsilon^{1/2}\int_0^{\varepsilon^{-1}} \varepsilon^3 s^{-1/2}ds) = O(\varepsilon^3).$$

This implies (2.44) with $\ell = 1$.

It follows from (2.40), (2.47) and (2.48) that

$$\begin{aligned}
t^{1/2}\partial_0 u(t) &= (\varepsilon^{-1})^{1/2}\partial_0 u(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) \\
&= (\varepsilon^{-1})^{1/2}\varepsilon\partial_0 U(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) \\
&= (\varepsilon^{-1} - \lambda)^{1/2}\varepsilon\partial_0 U(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) + O(\{(\varepsilon^{-1})^{1/2} - (\varepsilon^{-1} - \lambda)^{1/2}\}\varepsilon) \\
&= -\varepsilon\partial_\rho\mathcal{F}(\omega,-\lambda) + O^*(\varepsilon^{3/2}).
\end{aligned}$$

On the other hand, by (2.47) and (2.48)

$$
\begin{aligned}
W_1(\varepsilon^{-1}) &= (\varepsilon^{-1})^{1/2}\partial_0^2 u(\varepsilon^{-1}) \\
&= (\varepsilon^{-1})^{1/2}\varepsilon\partial_0^2 U(\varepsilon^{-1}) + O(\varepsilon^{5/2}) \\
&= (\varepsilon^{-1}-\lambda)^{1/2}\varepsilon\partial_0^2 U(\varepsilon^{-1}) + O(\varepsilon^{5/2}) + O(\{(\varepsilon^{-1})^{1/2} - (\varepsilon^{-1}-\lambda)^{1/2}\}\varepsilon) \\
&= \varepsilon\partial_\rho^2\mathcal{F}(\omega,-\lambda) + O(\varepsilon^{3/2}).
\end{aligned}
\tag{2.52}
$$

Then by (2.46), (2.51) and (2.52) we rewrite (2.44)

$$
\frac{d}{dt}W_1(t) = -\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)t^{-1}W_1(t)^2 + O^*(\varepsilon^{1-p}t^{-3/2} + \varepsilon^{7/2-2p}t^{-1}), \tag{2.53}
$$

$$
W_1(\varepsilon^{-1}) = \varepsilon\partial_\rho^2\mathcal{F}(\omega,-\lambda) + O(\varepsilon^{3/2}). \tag{2.54}
$$

If the solution $W_1(t)$ of (2.53), (2.54) satisfies

$$
W_1(t) = O^*(\varepsilon), \tag{2.55}
$$

then we obtain (2.43). Indeed, using (2.45), we have

$$
\partial_0^2 u(t) = O^*(\varepsilon t^{-1/2}), \tag{2.56}
$$

then (2.34), (2.30) and (2.56) yield (2.43).

We shall show (2.55). Multiplying $\operatorname{sgn} W_1$ to both sides of (2.53),

$$
\frac{d}{dt}|W_1(t)| \leqslant -\varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)t^{-1}|W_1(t)|^2 + |L(t)|, \tag{2.57}
$$

where

$$
L(t) = O^*(\varepsilon^{1-p}t^{-3/2} + \varepsilon^{7/2-2p}t^{-1}).
$$

Note that

$$
\int_{\varepsilon^{-1}}^{t} |L(s)|ds = O^*(\varepsilon^{5/4}). \tag{2.58}
$$

Replacing if necessary W_1 by $-W_1$, we may assume

$$
W_1(\varepsilon^{-1}) \geqslant 0.
$$

We set

$$
\beta(t) = W_1(\varepsilon^{-1}) + J_k(B)\varepsilon^{5/4} - \varepsilon C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda)\int_{\varepsilon^{-1}}^{t} |W_1(s)|^2 s^{-1}ds,
$$

then we find that

$$
0 \leqslant |W_1(t)| \leqslant \beta(t).
$$

If $C(-1,\omega)\partial_\rho\mathcal{F}(\omega,-\lambda) \geqslant 0$, by (2.57)

$$
\frac{d}{dt}|W_1(t)| \leqslant |L(t)|.
$$

Integrating this inequality from ε^{-1} to t, we obtain by (2.37)

$$|W_1(t)| \leqslant W_1(\varepsilon^{-1}) + O^*(\varepsilon^{5/4}) = O^*(\varepsilon).$$

Therefore we assume $C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda) < 0$. In this case,

$$\begin{aligned} \frac{d}{dt}\beta(t) &= -\varepsilon C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda)t^{-1}|W(t)|^2 \\ &\leqslant -\varepsilon C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda)t^{-1}\beta(t)^2. \end{aligned} \tag{2.59}$$

Now we consider a Cauchy problem;

$$\frac{d}{dt}Z(t) = -\varepsilon C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda)t^{-1}Z(t)^2, \tag{2.60}$$

$$Z(\varepsilon^{-1}) = \beta(\varepsilon^{-1}) = W_1(\varepsilon^{-1}) + J_k(B)\varepsilon^{5/4}. \tag{2.61}$$

Then by (2.59) and (2.60)

$$\begin{aligned} &\frac{d}{dt}\{(Z(t)-\beta(t))\exp(\varepsilon C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda)\int_{\varepsilon^{-1}}^{t}(z(s)+\beta(s))s^{-1}ds)\} \\ &=\{\frac{d}{dt}Z(t) - \frac{d}{dt}\beta(t) + \varepsilon C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda)t^{-1}(Z(t)^2-\beta(t)^2)\} \\ &\quad\times \exp(\varepsilon C(-1,\omega)\partial_\rho \mathcal{F}(\omega,-\lambda)\int_{\varepsilon^{-1}}^{t}(Z(s)+\beta(s))s^{-1}ds) \geqslant 0. \end{aligned}$$

Since $Z(\varepsilon^{-1}) = \beta(\varepsilon^{-1})$, we have

$$\beta(t) \leqslant Z(t).$$

Solving (2.60), (2.61) explicitly, we have by (2.13) and (2.45)

$$\begin{aligned} Z(t) &= \frac{Z(\varepsilon^{-1})}{1-\varepsilon Z(\varepsilon^{-1})(-C(-1,\omega))\partial_\rho \mathcal{F}(\omega,-\lambda)\log t} \\ &= \frac{O^*(\varepsilon)}{1-\varepsilon(\varepsilon\partial_\rho^2 \mathcal{F}(\omega,-\lambda)+O^*(\varepsilon^{5/4}))(-C(-1,\omega))\partial_\rho \mathcal{F}(\omega,-\lambda)\log t} \\ &= \frac{O^*(\varepsilon)}{1-\frac{1}{B}\{(-C(-1,\omega))\partial_\rho \mathcal{F}(\omega,-\lambda)\partial_\rho^2 \mathcal{F}(\omega,-\lambda)+O^*(\varepsilon^{1/4})\}} \\ &\leqslant \frac{O^*(\varepsilon)}{\frac{1}{B}(B-H+O^*(\varepsilon^{1/4}))} = O^*(\varepsilon). \end{aligned}$$

Hence we have

$$0 \leqslant |W_1(t)| \leqslant \beta(t) \leqslant Z(t) = O^*(\varepsilon),$$

then (2.55) holds.

Now we prove (2.42). Let $v = \Gamma u$ in (2.36) (Γ is an arbitrary one) and set

$$W(t) = t^{1/2}\partial_0 \Gamma u(t). \tag{2.62}$$

It follows from (2.1), (2.32), (2.38), (2.43) and (2.62) that

$$\begin{aligned}
L(\Gamma u) &= \Box \Gamma u - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta \Gamma u \\
&= \Box \Gamma u - \Gamma \Box u + \Gamma(a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta \Gamma u \\
&= C\Box u + (\Gamma a_{\alpha\beta}(u'))\partial_\alpha\partial_\beta u + a_{\alpha\beta}(u')(\Gamma\partial_\alpha\partial_\beta u - \partial_\alpha\partial_\beta\Gamma u) \\
&= C a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u + (\Gamma a_{\alpha\beta}(u'))\partial_\alpha\partial_\beta u + a_{\alpha\beta}(u')\partial_\gamma\partial_\delta u \\
&= O(|u'|^2|u''| + |\Gamma u'||u'||u''|) \\
&= O(|u'|^2|u''| + |\partial_\alpha \Gamma u||u'||u''|) \\
&= O(|u'|^2|u''| + |\partial_0 \Gamma u||u'||u''| + t^{-1}|u|_2|u'||u''|) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2}|W| + \varepsilon^{3-2p}t^{-5/2}) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2}|W|).
\end{aligned}$$

Therefore by (2.36) and (2.31)

$$\frac{d}{dt}W(t) = O^*(\varepsilon^3 t^{-1} + \varepsilon^2 t^{-1}|W(t)| + \varepsilon^{1-2p}t^{-2}).$$

Integrating this equation from ε^{-1} to t and using (2.13) and (2.37), we have

$$\begin{aligned}
|W(t)| &\leqslant |W(\varepsilon^{-1})| + O^*\left(\int_{\varepsilon^{-1}}^t (\varepsilon^3 s^{-1} + \varepsilon^2|W(s)|s^{-1} + \varepsilon^{1-p}s^{-2})ds\right) \\
&= O^*(\varepsilon) + O^*\left(\varepsilon^2 \int_{\varepsilon^{-1}}^t |W(s)|s^{-1}ds\right).
\end{aligned}$$

Hence Gronwall's inequality and (2.13) lead to

$$|W(t)| = O^*(\varepsilon \exp(\varepsilon^2 \log t)) = O^*(\varepsilon e^{1/B}) = O^*(\varepsilon),$$

i.e.,

$$\partial_0 \Gamma u(t) = O^*(\varepsilon t^{-1/2}). \tag{2.63}$$

We also find from (2.1), (2.32), (2.31), (2.38) and (2.63) that (2.42) holds.

Finally we prove (2.27). It is sufficient to show that

$$|u'(t)|_\ell = O^*(\varepsilon t^{-1/2}), \tag{2.64}$$

under the assumption

$$|u'(t)|_{\ell-1} = O^*(\varepsilon t^{-1/2}), \tag{2.65}$$

where $1 \leqslant \ell \leqslant [\frac{k+1}{2}]$. Let $v = \Gamma^\ell u$ (for $\ell \in \mathbb{Z}_+, \Gamma^\ell$ stands for $\sum_{|\sigma|=\ell} \Gamma^\sigma$) in (2.36) and set

$$W(t) = t^{1/2}\partial_0 \Gamma^\ell u(t). \tag{2.66}$$

Using (2.1), (2.32), (2.38), (2.42) and (2.65), we have

$$
\begin{aligned}
L(\Gamma^\ell u) &= \Box\Gamma^\ell u - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma^\ell u \\
&= \Gamma^\ell\Box u + \sum_{\mu<\ell} C_\mu\Gamma^\mu\Box u - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma^\ell u \\
&= \Gamma^\ell(a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) + \sum_{\mu<\ell} C_\mu\Gamma^\mu(a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma^\ell u \\
&= \sum_{\nu<\ell}\binom{\ell}{\nu}(\Gamma^{\ell-\nu}a_{\alpha\beta}(u'))(\Gamma^\nu\partial_\alpha\partial_\beta u) + a_{\alpha\beta}(u')\Gamma^\ell\partial_\alpha\partial_\beta u \\
&\quad + \sum_{\mu<\ell} C_\mu\Gamma^\mu(a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma^\ell u \\
&= O(|u'|_\eta|u'|_\zeta|u'|_\xi) \quad (\eta,\zeta,\xi \leqslant \ell,\ \eta+\zeta+\xi \leqslant \ell+1) \\
&= O(|u'|^3_{\ell-1} + |u'|_0|u'|_1|u'|_\ell) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-1}|u'|_\ell) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2}V(t)),
\end{aligned}
$$

where

$$V(t) = t^{1/2}|u'(t)|_\ell. \tag{2.67}$$

Therefore by (2.36) and (2.31)

$$\frac{d}{dt}W(t) = O^*(\varepsilon^3 t^{-1} + \varepsilon^2 t^{-1}V(t) + \varepsilon^{1-2p}t^{-2}).$$

Integrating this equation from ε^{-1} to t and using (2.13) and (2.37), we have

$$
\begin{aligned}
|W(t)| &\leqslant |W(\varepsilon^{-1})| + O^*\left(\int_{\varepsilon^{-1}}^t (\varepsilon^3 s^{-1} + \varepsilon^2 V(s)s^{-1} + \varepsilon^{1-2p}s^{-2})ds\right) \\
&= O^*(\varepsilon) + O^*\left(\varepsilon^2\int_{\varepsilon^{-1}}^t V(s)s^{-1}ds\right).
\end{aligned}
$$

Then by (2.66), (2.1), (2.32), (2.37), (2.65) and (2.31)

$$V(t) = O^*(\varepsilon) + O^*(\varepsilon^2\int_{\varepsilon^{-1}}^t V(s)s^{-1}ds).$$

Gronwall's inequality and (2.13) yield

$$V(t) = O^*(\varepsilon\exp(\varepsilon^2\log t)) = O^*(\varepsilon e^{1/B}) = O^*(\varepsilon). \tag{2.68}$$

We find from (2.67) and (2.68) that (2.64) holds and this complete the proof of Theorem 1.

3. Proof of Theorem 2.

For a function $h \in C^\infty(\mathbb{R}^2 \times [0,T))$, we denote by $E_T(h)$ the solution of the Cauchy problem;

$$\Box E_T(h)(x,t) = h(x,t), \quad (x,t) \in \mathbb{R}^2 \times [0,T),$$
$$E_T(h)(x,0) = 0, \quad \partial_0 E_T(h)(x,0) = 0, \quad x \in \mathbb{R}^2.$$

Using an L^1-L^∞ estimate in Corollary 6.2 in [4] (also see [2], Lemma 4.1), we can prove;

Proposition 3.1. *Suppose that $b \in \mathbb{R}$, and $h_1, h_2 \in C^\infty(\mathbb{R}^2 \times [0,T))$ have a compact support in x for fixed t, then there exists $C > 0$ such that*

$$(1+t)|E_T(h_1h_2)(x,t)|^2 \leqslant C\Big(\sum_{|\theta|\leqslant 1}\int_0^t \frac{||\Gamma^\theta h_1(s)||^2}{(1+s)^b}ds\Big)\Big(\sum_{|\theta|\leqslant 1}\int_0^t \frac{||\Gamma^\theta h_2(s)||^2}{(1+s)^{1-b}}ds\Big), \tag{3.1}$$

for $0 \leqslant t < T$.

We also need following proposition which is proved in [13].

Proposition 3.2. *If functions u and v are smooth and*

$$u(x,t) = v(x,t) = 0 \quad \text{for} \quad |x| \geqslant t + M,$$

then we have

$$||u(t)v'(t)|| \leqslant C_M \sum_{|\theta|=1} |\Gamma^\theta v(t)|\,||u'(t)||. \tag{3.2}$$

Let k be fixed and $k \geqslant 9$. For functions $\varphi(x,t;\varepsilon) and \psi(x,t;\varepsilon)$, we denote $\varphi = O^*(\psi)$ if for any $m > 0$, there exist $J > 0$ and $\varepsilon(m) > 0$ such that for $0 < \varepsilon < \varepsilon(m)$,

$$|\varphi(x,t;\varepsilon)| < J\psi(x,t;\varepsilon) \quad \text{for} \quad 0 \leqslant t < \tau,$$

holds as long as $\tau < T_\varepsilon$ and

$$|u(t)|_{[\frac{k+1}{2}]+1} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau. \tag{3.3}$$

By the similar argument in Section 2, it is sufficient to prove;

Lemma 3.1.

$$|u(t)|_{[\frac{k+1}{2}]} = O^*(\varepsilon(1+t)^{-1/2}).$$

We conclude this section by proving Lemma 3.1. Denote by F the right-hand side of (1.1). By (2.1) we have

$$\Box\Gamma^\sigma u = \sum_{\lambda\leqslant\sigma} C_\lambda \Gamma^\lambda F \quad (C_\sigma = 1).$$

Then we can write

$$\Gamma^\sigma u = W^\sigma + \sum_{\lambda \leqslant \sigma} C_\lambda E_\tau(\Gamma^\lambda F),$$

where W^σ is a solution of linear wave equation;

$$\Box W^\sigma(x,t) = 0, \quad (x,t) \in \mathbb{R}^2 \times [0,\tau),$$
$$W^\sigma(x,0) = \Gamma^\sigma u(x,0), \quad \partial_0 W^\sigma(x,0) = \partial_0 \Gamma^\sigma u(x,0), \quad x \in \mathbb{R}^2.$$

Since, as well known,

$$|W^\sigma(x,t)| \leqslant \frac{C_\sigma \varepsilon}{(1+t)^{1/2}} \quad \text{for} \quad 0 \leqslant t < \tau,$$

it is sufficient to show

$$E_\tau(\Gamma^\lambda F) = O^*(\varepsilon(1+t)^{-1/2}) \quad \text{for} \quad |\lambda| \leqslant [\frac{k+1}{2}] + 1. \tag{3.4}$$

As stated in introduction, we may assume that F has a form in (1.16a-d). We verify (3.4) for each case.

Case (1.16a). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha \partial_\beta u Q(u'))) = O^*(\varepsilon(1+t)^{-1/2}), \tag{3.5}$$

$$E_\tau(\Gamma^\lambda(|u'|^3 |u''|)) = O^*(\varepsilon(1+t)^{-1/2}), \tag{3.6}$$

where $Q(u') = (\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2$. It follows from Proposition 2.2 and (3.3) that

$$\begin{aligned} ||u'(t)||_k &\leqslant C\varepsilon \exp(C \int_0^t |u'(s)|^2_{[\frac{k+1}{2}]} ds) \\ &\leqslant C\varepsilon \exp(Cm^2\varepsilon^2 \log(1+t)) \\ &\leqslant C\varepsilon(1+t)^p \quad \text{for} \quad 0 \leqslant t < \tau, \end{aligned}$$

if $Cm^2\varepsilon^2 < p$. For later we shall assume that $p < \frac{1}{4}$. Hence we have

$$||u'(t)||_k = O^*(\varepsilon(1+t)^p). \tag{3.7}$$

Now we shall prove (3.5). Using Proposition 3.1 with $b = -1$, we get

$$(1+t)^{1/2} |E_\tau(\Gamma^\lambda(\partial_\alpha \partial_\beta u Q(u')))| \leqslant C \sum_{\mu+\nu=\lambda} I_\mu(t) J_\nu(t), \tag{3.8}$$

where

$$I_\mu(t) = \left(\sum_{|\theta| \leqslant 1} \int_0^t ||\Gamma^\theta \Gamma^\mu \partial_\alpha \partial_\beta u(s)||^2 (1+s)^{-2} ds \right)^{1/2},$$

$$J_\nu(t) = \left(\sum_{|\theta| \leqslant 1} \int_0^t ||\Gamma^\theta \Gamma^\nu Q(u'(s))||^2 (1+s) ds \right)^{1/2}.$$

Since $k \geqslant 9, |\theta| + |\mu| + 1 \leqslant [\frac{k+1}{2}] + 3 \leqslant k$. Therefore, by (3.7) we get

$$I_\mu(t) = O^*\Big(\Big(\int_0^t \varepsilon^2(1+s)^{2p-2}ds\Big)^{1/2}\Big) = O^*(\varepsilon). \tag{3.9}$$

On the other hand, since

$$Q(u') = t^{-1}(L_0 u \partial_0 u - L_i u \partial_i u),$$

$\Gamma^\theta \Gamma^\nu Q(u')$ is represented as the sum of

$$\Gamma^\rho(t^{-1})\Gamma^\eta(L_\alpha u)\Gamma^\xi(\partial_\alpha u), \quad \rho + \eta + \xi = \theta + \nu, \ \alpha = 0, 1, 2.$$

We can verify in the support of u

$$\Gamma^\rho(t^{-1}) \leqslant Ct^{-1}(1 + t^{-|\rho|}). \tag{3.10}$$

Moreover we find that

$$||\Gamma^\eta(L_\alpha u(t))\Gamma^\xi(\partial_\alpha u(t))|| \leqslant C|u(t)|_{[\frac{\ell+1}{2}]}||u'(t)||_\ell, \tag{3.11}$$

where $\ell = [\frac{k+1}{2}] + 3$. Indeed, if we set $\zeta = \eta + \chi$ ($|\chi| = 1$, $\Gamma^\chi = L_\alpha$), we obtain $|\zeta + \xi| \leqslant [\frac{k+1}{2}] + 3 = \ell$. If $|\xi| \geqslant [\frac{\ell+1}{2}]$, then $|\zeta| \leqslant [\frac{\ell+1}{2}]$ and (3.11) holds. If $|\xi| < [\frac{\ell+1}{2}]$, *i.e.*, $|\xi| \leqslant [\frac{\ell+1}{2}] - 1$, we have by using Proposition 3.2

$$\begin{aligned} ||\Gamma^\zeta u(t)\Gamma^\xi \partial_\alpha u(t)|| &\leqslant C \sum_{|\iota|=1} |\Gamma^{\xi+\iota} u(t)|\, ||\Gamma^\zeta u'(t)|| \\ &\leqslant C|u(t)|_{[\frac{\ell+1}{2}]}||u'(t)||_\ell, \end{aligned}$$

which implies (3.11). Since $k \geqslant 9$, we get $\ell < k$ and $[\frac{\ell+1}{2}] \leqslant [\frac{k+1}{2}]$. Then, by (3.3) and (3.7)

$$\begin{aligned} ||\Gamma^\zeta u(t)\Gamma^\xi \partial_\alpha u(t)|| &\leqslant C|u(t)|_{[\frac{k+1}{2}]}||u'(t)||_k \\ &= O^*(\varepsilon(1+t)^{p-1/2}). \end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12), we get

$$||\Gamma^\theta \Gamma^\nu Q(u'(t))|| = O^*(\varepsilon t^{-1}(1 + t^{-|\theta+\nu|})(1+t)^{p-1/2}). \tag{3.13}$$

On the other hand, as shown in section 2,

$$||\Gamma^\theta \Gamma^\nu Q(u'(t))|| \leqslant C\varepsilon \quad \text{for} \quad 0 \leqslant t \leqslant 1. \tag{3.14}$$

Hence by (3.13) and (3.14)

$$
\begin{aligned}
&J_\nu(t)\\
=&\Big(\sum_{|\theta|\leqslant 1}\int_0^1 \|\Gamma^\theta\Gamma^\nu Q(u'(s))\|^2(1+s)ds+\sum_{|\theta|\leqslant 1}\int_1^t \|\Gamma^\theta\Gamma^\nu Q(u'(s))\|^2(1+s)ds\Big)^{1/2}\\
\leqslant&\Big(C\varepsilon+O^*\Big(\int_1^t \varepsilon^2(1+s)^{2p-2}ds\Big)\Big)^{1/2}=O^*(\varepsilon).
\end{aligned}
\tag{3.15}
$$

Therefore (3.5) follows from (3.9) and (3.15).

Next we shall prove (3.6). Using Proposition 3.1 with $b=\frac{1}{2}$, we get

$$
(1+t)^{1/2}|E_\tau(\Gamma^\lambda(|u'|^3|u''|))|\leqslant C\sum_{\mu+\nu=\lambda} I_\mu(t)J_\nu(t), \tag{3.16}
$$

where

$$
I_\mu(t)=\Big(\sum_{|\theta|\leqslant 1}\int_0^t \|\Gamma^\theta\Gamma^\mu(|u'(s)|^2)\|^2(1+s)^{-1/2}ds\Big)^{1/2},
$$

$$
J_\nu(t)=\Big(\sum_{|\theta|\leqslant 1}\int_0^t \|\Gamma^\theta\Gamma^\mu(|u'(s)|\,|u''(s)|)\|^2(1+s)^{-1/2}ds\Big)^{1/2}.
$$

Since $|\theta+\mu|\leqslant[\frac{k+1}{2}]+2\leqslant k$, we can verify

$$
\|\Gamma^\theta\Gamma^\mu(|u'(s)|^2)\|\leqslant C|u(s)|_{[\frac{k+1}{2}]+1}\|u'(s)\|_k.
$$

Using (3.3) and (3.7), we get

$$
\|\Gamma^\theta\Gamma^\mu(|u'(s)|^2)\|\leqslant Cm\varepsilon^2(1+s)^{p-1/2}=O^*(\varepsilon(1+s)^{p-1/2}).
$$

Then we have

$$
I_\mu(t)=O^*\Big(\Big(\int_0^t \varepsilon^2(1+s)^{2p-3/2}ds\Big)^{1/2}\Big)=O^*(\varepsilon). \tag{3.17}
$$

Similarly we also find that

$$
J_\nu(t)=O^*(\varepsilon). \tag{3.18}
$$

Therefore (3.16), (3.17) and (3.18) imply (3.6).

Case (1.16b). We have to show

$$
E_\tau(\Gamma^\lambda(\partial_\alpha u\partial_\beta Q(u')))=O^*(\varepsilon(1+t)^{-1/2}).
$$

This can be obtained similarly to (3.5), by using (3.10), (3.11) and Proposition 3.1, 3.2 but $\ell=[\frac{k+1}{2}]+4$ in (3.11).

Case (1.16c). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u\partial_\beta u\Box u)) = O^*(\varepsilon(1+t)^{-1/2}).$$

Using (1.1), we get

$$\begin{aligned}\Gamma^\lambda(\partial_\alpha u\partial_\beta u\Box u) &= \Gamma^\lambda(\partial_\alpha u\partial_\beta u(\partial_\gamma u\partial_\delta u\Box u + O(|u'|^3|u''|))) \\ &= O(\Gamma^\lambda(|u'|^4|u''|)).\end{aligned}$$

Therefore this case can be reduced to (3.6).

Case (1.16d). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u(\partial_\beta u\partial_\gamma\partial_\delta u - \partial_\gamma u\partial_\beta\partial_\delta u))) = O^*(\varepsilon(1+t)^{-1/2}).$$

Using (2.32) and (2.34), we have

$$\begin{aligned}\partial_\alpha u(\partial_\beta u\partial_\gamma\partial_\delta u - \partial_\gamma u\partial_\beta\partial_\delta u) &= O(t^{-1}(|u'|^2|u'|_1 + |u'||u''||u|_1)) \\ &= O(t^{-1}|u'||u|_1|u'|_1).\end{aligned}$$

Therefore this case can be verified similarly to (3.5) as $Q = t^{-1}|u'||u|_1$, then the proof of Theorem 2 is complete.

Appendix.

If $H = 0$, we find from the definition of H (1.14) that

$$C(-1,\omega)\partial_\rho\mathcal{F}(\omega,\rho)\partial_\rho^2\mathcal{F}(\omega,\rho) = \frac{1}{2}C(-1,\omega)\partial_\rho((\partial_\rho\mathcal{F}(\omega,\rho))^2) \geqslant 0,$$

for any $\omega \in S^1$ and $\rho \in \mathbb{R}$. For fixed ω, if $C(-1,\omega) \neq 0$, then $\partial_\rho((\partial_\rho\mathcal{F}(\omega,\rho))^2)$ is of constant sign in ρ. Using (1.10) and (1.11), we find that $\partial_\rho\mathcal{F}(\omega,\rho) = 0$ for any $\rho \in \mathbb{R}, i.e., \mathcal{F}(\omega,\rho) \equiv$ const in ρ. Therefore (1.10) implies $\mathcal{F}(\omega,\rho) = 0$ for any $\rho \in \mathbb{R}$.

We set

$$\Omega = S^1\backslash\{\omega \in S^1 | C(-1,\omega) = 0 \text{ and } \mathcal{F}(\omega,\rho) = 0 \quad \text{for any } \rho \in \mathbb{R}\}.$$

We claim that Ω is either the set of ω such that $C(-1,\omega) = 0$, or the set of ω such that $\mathcal{F}(\omega,\rho) = 0$ for any $\rho \in \mathbb{R}$. Set

$$F = \{\omega \in \Omega | \ C(-1,\omega) \neq 0\}.$$

Clearly F is open in Ω. On the other hand, by the above argument we have

$$F = \{\omega \in \Omega | \ \mathcal{F}(\omega,\rho) = 0 \text{ for any } \rho \in \mathbb{R}\}.$$

Then we find that F is also closed in Ω. Therefore F is equal to either ϕ or Ω. When $F=\phi, \Omega$ is the set of ω such that $C(-1,\omega)=0$. When $F=\Omega$, Ω is the set of ω such that $\mathcal{F}(\omega,\rho)=0$ for any $\rho \in \mathbb{R}$.

Since

$$S^1=\Omega \cup \{\omega \in S^1 \big| C(-1,\omega)=0 \text{ and } \mathcal{F}(\omega,\rho)=0 \text{ for any } \rho \in \mathbb{R}\},$$

either "$C(-1,\omega)=0$ for any $\omega \in S^1$" or "$\mathcal{F}(\omega,\rho)=0$ for any $\omega \in S^1$ and $\rho \in \mathbb{R}$", when $H=0$. Moreover if $\mathcal{F}(\omega,\rho)=0$ for any $\omega \in S^1$ and $\rho \in \mathbb{R}$, then, by (1.9),

$$\mathcal{F}(\omega,\rho)+\mathcal{F}(-\omega,-\rho)=\frac{1}{\sqrt{2\pi}}\int_\rho^\infty (s-\rho)^{-1/2}R_g(\omega,s)ds=0,$$

$$\mathcal{F}(\omega,\rho)-\mathcal{F}(-\omega,-\rho)=-\frac{1}{\sqrt{2\pi}}\int_\rho^\infty (s-\rho)^{-1/2}\partial_s R_f(\omega,s)ds=0.$$

Using Tichmarsh's Theorem (see [14], p.166), Radon problem (see [5], p.162) and integrating by parts, we find that f and g vanish identically. Therefore the condition "$H=0$" implies either "$C(-1,\omega)=0$ for any $\omega \in S^1$" or "f and g vanish identically".

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PART II

The asymptotic behaviour of the radially symmetric solutions to quasilinear wave equations in two space dimensions

ABSTRACT. In Part II, we study the behaviour of solutions to quasilinear wave equations in two space dimensions. We obtain blow-up results near the wave front. More precisely, any radially symmetric solution with small initial data is shown to develop singularities in the second order derivatives in finite time, while the first order derivatives and itself remain small. Moreover, we succeed to represent the solution explicitely near the blowing up point.

1. Introduction.

Part II deals with the initial value problem:

$$u_{tt} - c^2(u_t, u_r)(u_{rr} + \frac{1}{r}u_r) = \frac{1}{r}u_r G(u_t, u_r), \quad (r,t) \in (0,\infty) \times (0, T_\varepsilon), \tag{1.1}$$

$$u(r,0) = \varepsilon f(r), \qquad u_t(r,0) = \varepsilon g(r), \qquad r \in (0,\infty) \tag{1.2}$$

where

$$\begin{aligned} c^2(u_t, u_r) =& 1 + a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2 + O(|u_t|^3 + |u_r|^3), \\ G(u_t, u_r) =& O(|u_t|^2 + |u_r|^2) \end{aligned}$$

near $u_t = u_r = 0$ and the initial data are smooth and have compact support. The equation (1.1) is the radially symmetric form of quasi-linear wave equations in two space dimensions. In [2], we have shown that the smooth solution to the initial value problem (1.1) and (1.2) exists almost globally, that is

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \geqq \frac{1}{H_0}$$

where

$$H_0 = \max_{\rho \in \mathbb{R}}(-(a_1 - a_2 + a_3)\mathcal{F}'(\rho)\mathcal{F}''(\rho)).$$

The quantity T_ε stands for the lifespan of the smooth solution to (1.1) and (1.2) and the function $\mathcal{F}(\rho)$ is the Friedlander radiation field with respect to f and g defined in below. It has been proved that the smooth solution to (1.1) and (1.2) exists globally provided $c^2(u_t, u_r) - 1 = O(|u_t|^3 + |u_r|^3)$ by M. Kovalyov [9]. Even in the

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present case, by provided $H_0 = 0$ [2] has proved the smooth solution exists globally. In other words, *null condition* guarantees the existence of global solutions.

Our main purpose in this paper is to show that the smooth solution to (1.1) and (1.2) blows up in finite time if H_0 does not vanish. We have two points of view to study. One is to determine the blowing up time of the smooth solution to (1.1) and (1.2) exactly. We will prove

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \leqq \frac{1}{H_0}$$

in below. The constant H_0 is the same as the earlier one, thus we conclude

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) = \frac{1}{H_0}.$$

When the coefficient c^2 of the Laplacian has the form

$$c^2(u_t) = 1 + au_t + O(|u_t|^2), \qquad a \neq 0$$

and $G(u_t, u_r) \equiv 0$, F. John [3]-[5] and L. Hörmander [1] have obtained in three space dimensions,

$$\lim_{\varepsilon\to 0} \varepsilon \log(1+T_\varepsilon) = \frac{1}{H_1} \tag{1.3}$$

where

$$H_1 = \max_{\rho\in\mathbb{R}}\left(\frac{a}{2}\mathcal{F}''(\rho)\right)$$

and L. Hörmander [1] has also shown in two space dimensions,

$$\lim_{\varepsilon\to 0} \varepsilon\sqrt{T_\varepsilon} = \frac{1}{H_2} \tag{1.4}$$

where

$$H_2 = \max_{\rho\in\mathbb{R}}(a\mathcal{F}''(\rho)).$$

Secondly, we turn our interest to the behaviour of the solution near the blowing up point. To make our purpose clear, it is worth noting how the blowing up of the smooth solution occurs. Let $w_1(r,t)$ be a directional derivative of $u_r(r,t)$, whose direction intersects orthogonally the pseudo-characteristic curve to the equation (1.1) in (r,t)-plane. If we denote the value of $w_1(r,t)$ along the pseudo-characteristic curve by $w_1(t)$, then $w_1(t)$ diverges as t tends to T_ε for sufficiently small initial data, while u and the first order derivatives of u are still small. Thus it is natural for you to wonder about the action of $w_1(t)$. We will represent $w_1(t)$ explicitely near the blowing up point as a limit when ε tends to 0. It has not been studied yet for the cases of F. John and L. Hörmander.

For an application of our results we consider the equation of vibrating membrane:

$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \qquad (x,t) \in \mathbb{R}^2 \times (0, T_\varepsilon).$$

The radially symmetric form of this equation is written in the form of (1.1) with $a_1 = a_2 = 0$ and $a_3 = -3/2$. The solution u stands for the vertical motion of the vibrating membrane, thus our blowing up results imply that the curvature of the membrane brakes at some points while the difference of the membrane and the speed of the vibration become small. Further consideration for the vibrationg membrane will be developed in section 7.

2. Statement of results.

To state our results for the initial value problem (1.1) and (1.2) we set the assumption more clearly. We assume $c, G \in C^\infty(\mathbb{R}^2)$,

$$\begin{aligned} c^2(u_t, u_r) =& 1 + a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2 + O(|u_t|^3 + |u_r|^3), \\ G(u_t, u_r) =& O(|u_t|^2 + |u_r|^2), \end{aligned}$$

near $u_t = u_r = 0$ and assume $f(|x|),\ g(|x|) \in C_0^\infty(\mathbb{R}^2)$, $|f| + |g| \not\equiv 0$ and $\mathrm{supp} f,\ \mathrm{supp} g \subset [-M, M]$. We also need $a_1 - a_2 + a_3 \neq 0$ so that H_0 does not vanish. We define the Friedlander radiation field $\mathcal{F}(\rho)$ by

$$\mathcal{F}(\rho) = r^{\frac{1}{2}} u^0(r, t) \qquad \text{along} \qquad \rho = r - t,$$

where $u^0(r, t)$ is the solution of linear wave equation:

$$u_{tt}^0 - u_{rr}^0 - \frac{1}{r} u_r^0 = 0, \qquad (r, t) \in (0, \infty) \times (0, \infty), \tag{2.1}$$

$$u^0(r, 0) = f(r), \quad u_t^0(r, 0) = g(r) \qquad r \in (0, \infty) \tag{2.2}$$

$\mathcal{F}(\rho)$ is strictly expressed as

$$\mathcal{F}(\rho) = \frac{1}{\sqrt{2\pi}} \int_\rho^\infty (s - \rho)^{-\frac{1}{2}} (R_g(s) - R_f'(s)) ds,$$

where $R_h(s)$ is the Radon transform of $h(|x|) \in C_0^\infty(\mathbb{R}^2)$, *i.e.*,

$$R_h(s) = \int_s^\infty \frac{\xi h(\xi)}{\sqrt{\xi^2 - s^2}} d\xi.$$

Moreover, $\mathcal{F}(\rho)$ has the properties:

$$\begin{aligned} \left| \frac{d^k}{d\rho^k} \mathcal{F}(\rho) \right| \leqq& C_k (1 + |\rho|)^{-\frac{1}{2} - k} \quad \text{for} \quad \rho \in \mathbb{R}, \\ \mathcal{F}(\rho) =& 0 \quad \text{for} \quad \rho \geqq M. \end{aligned}$$

(e.g. L. Hörmander [1]). Thus the quantity

$$H_0 = \max_{\rho \in \mathbb{R}} (-a\mathcal{F}'(\rho)\mathcal{F}''(\rho)) = -a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0)$$

is well-defined for some ρ_0 and non negative. Our assumption $|f| + |g| \not\equiv 0$ and $a \equiv a_1 - a_2 + a_3 \neq 0$ guarantee that $H_0 > 0$, which is shown in [2]. The lifespan T_ε of the solution u to (1.1) and (1.2) means the supremum of τ such that the solution exists in $C^\infty((0, \infty) \times (0, \tau))$.

At first, we will prove the following

Theorem 1.

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \leqq \frac{1}{H_0}.$$

Combining the result

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \geqq \frac{1}{H_0}$$

obtained in [2] with Theorem 1, we have

Corollary.

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) = \frac{1}{H_0}.$$

This blowing up result will be obtained as

$$w_1(t) \to \infty \qquad \text{as} \qquad t \to T_\varepsilon$$

for some function $w_1(t)$ constructed by the second order derivatives of u. With regard to $w_1(t)$, we will prove

Theorem 2.

$$\lim_{\varepsilon\to 0,\ \varepsilon^2\log(1+t)\to\frac{1}{H_0}} \left(\frac{1}{H_0} - \varepsilon^2 \log(1+t)\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_0}\mathcal{F}''(\rho_0).$$

We define the function $w_1(t)$ describing the outline of the proof of Theorem 1 and Theorem 2. First we fix a constant $B > H_0$. Set $\rho = r - t$, $s = \varepsilon^2 \log(1+t)$ and consider the Burgers' equation:

$$\begin{aligned} U_s(\rho, s) + \frac{a}{6}(U_\rho(\rho, s))^3 =&0, \qquad (\rho, s) \in \mathbb{R} \times [0, \frac{1}{B}] \\ U(\rho, 0) = \mathcal{F}(\rho),& \qquad \rho \in \mathbb{R}. \end{aligned}$$

For the solutions U of the above Burgers' equation and u of the initial value problem (1.1) and (1.2), we will find that

$$|\partial_r^l \partial_t^m u(r, t_{\frac{1}{B}}) - \varepsilon r^{-\frac{1}{2}}(-1)^m \partial_\rho^{l+m} U(r - t_{\frac{1}{B}}, \frac{1}{B})| \leqq C\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}}$$

$$\text{for} \quad r - t_{\frac{1}{B}} \geqq -\frac{1}{3\varepsilon} \quad \text{and} \quad l, m \in \mathbb{N} \cup \{0\} \quad (l + m \neq 0),$$

where $t_{1/B} = \exp(1/\varepsilon^2 B) - 1$, *i.e.*, $\varepsilon^2 \log(1 + t_{1/B}) = 1/B$. Moreover, on characteristic curves Λ in (ρ, s)-plane, we approximate U by the Friedlander radiation field $\mathcal{F}$ for $0 \leqq s \leqq 1/B$. These give u approximation by $\mathcal{F}$ at $t = t_{1/B}$. These will be proved in section 3. Next we investigate the behaviour of u after $t = t_{1/B}$ in section 4. If we set $v(r,t) = r^{1/2} u(r,t)$ and

$$\begin{aligned} w_1(r,t) =& \frac{cv_{rr} - v_{rt}}{2c}, \\ w_2(r,t) =& \frac{cv_{rr} + v_{rt}}{2c}, \end{aligned}$$

the following *a priori* estimates hold:

$$|v(r,t)| < C\varepsilon^{\frac{1}{2}}, \qquad |v_t(r,t)|,\ |v_r(r,t)| < C\varepsilon,$$
$$|w_2(r,t)| < C\varepsilon^3$$

as long as u exists. On the other hand, we define a pseudo-characteristic curve Z^1 in (r,t)-plane as a solution of

$$\frac{dr}{dt} = c,$$

connected with some Λ at $t = t_{1/B}$. We denote $(r(t),t) \in Z^1$ and set $w_1(t) = w_1(r(t),t)$, this is the definition of $w_1(t)$. Using above *a priori* estimates, we construct an ordinary differential equation with respect to $w_1(t)$. Solving the ordinary differential equation, we will find that Theorem 1 and Theorem 2 hold in section 5 and 6.

In the end of this section, we mention the case of F. John and L. Hörmander (1.3) and (1.4). We also expect

$$\lim_{\varepsilon\to 0,\ \varepsilon\log(1+t)\to\frac{1}{H_1}} \left(\frac{1}{H_1} - \varepsilon\log(1+t)\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_1}\mathcal{F}''(\rho_0),$$

$$\lim_{\varepsilon\to 0,\ \varepsilon\sqrt{t}\to\frac{1}{H_2}} \left(\frac{1}{H_2} - \varepsilon\sqrt{t}\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_2}\mathcal{F}''(\rho_0)$$

respectively. These would be proved in parallel.

3. Approximation for u by the solution of Burgers' equation.

It can be easily seen that the following lemma leads Theorem 1.

Main Lemma. *For any $A > H_0$, there exists an $\varepsilon_A > 0$ such that for $0 < \varepsilon < \varepsilon_A$,*

$$\varepsilon^2\log(1+T_\varepsilon) \leqq \frac{1}{H_0}$$

holds.

To prove Main Lemma we consider the following Bergers' equation:

$$U_s + \frac{a}{6}(U_\rho)^3 = 0, \qquad (\rho,s) \in \mathbb{R}\times[0,\frac{1}{B}], \tag{3.1a}$$

$$U(\rho,0) = \mathcal{F}(\rho), \qquad \rho \in \mathbb{R}, \tag{3.2a}$$

or

$$U_{\rho s} + \frac{a}{2}(U_\rho)^2 U_{\rho\rho} = 0, \quad (\rho,s) \in \mathbb{R}\times[0,\frac{1}{B}], \tag{3.1b}$$

$$U_\rho(\rho,0) = \mathcal{F}'(\rho), \qquad \rho \in \mathbb{R}, \tag{3.2b}$$

where $a = a_1 - a_2 + a_3$, $B > H_0$, $\rho = r - t$ and $s = \varepsilon^2 \log(1+t)$. We find that the Cauchy problem (3.1a) and (3.2a) is equivalent to (3.1b) and (3.2b) because there exists a smooth solution U_ρ to (3.1b) and (3.2b) and integral of U_ρ satisfies (3.1a) and (3.2a). For the solutions U of (3.1a) and (3.2a) and u of (1.1) and (1.2), we will prove

$$
\begin{aligned}
&|\partial_r^l \partial_t^m u(r, t_{\frac{1}{B}}) - \varepsilon r^{-\frac{1}{2}}(-1)^m \partial_\rho^{l+m} U(r - t_{\frac{1}{B}}, \frac{1}{B})| \leqq C_{l,m,B}\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}} \\
&\qquad \text{for} \quad r - t_{\frac{1}{B}} > -\frac{1}{3\varepsilon} \quad \text{and} \quad l + m \neq 0,
\end{aligned} \tag{3.3}
$$

where we denote $t_{1/B} = \exp(1/B\varepsilon^2) - 1$.

The main task in this section is to prove (3.3). To do this, we introduce the vector fields used in S. Klainerman [6] and state some results used through this paper.

$$
\begin{gathered}
L_0 = t\partial_t + x_1 \partial_{x_1} + x_2 \partial_{x_2}, \quad L_i = x_i \partial_t + t\partial_{x_i}, \quad \text{for} \quad i = 1, 2, \\
\partial_{x_1}, \quad \partial_{x_2}, \quad \partial_t,
\end{gathered}
$$

named $\Gamma_1, \Gamma_2, \cdots, \Gamma_6$ respectively. These operators satisfy commutation relations:

$$
\begin{aligned}
[\Gamma_p, \Box] &= \Gamma_p \Box - \Box \Gamma_p = 2\delta_{1p}\Box \quad \text{for} \quad p = 1, 2, \cdots, 6, \\
[\Gamma, \Gamma] &= \overline{\Sigma}\Gamma, \quad [\Gamma, \partial] = \overline{\Sigma}\partial,
\end{aligned} \tag{3.4}
$$

where $\Box = \partial_t^2 - \triangle$ and $\overline{\Sigma}$ stands for a finite linear combination with constant coefficients. For $\alpha \in \mathbb{Z}_+^6 (\mathbb{Z}_+ = \mathbb{N} \cup \{0\})$ we write $\Gamma^\alpha = \Gamma_1^{\alpha_1} \Gamma_2^{\alpha_2} \cdots \Gamma_6^{\alpha_6}$ and define the norms

$$
||v(t)||_k = \sum_{|\alpha| \leqq k} ||\Gamma^\alpha v(t)||_{L_x^2(\mathbb{R}^2)},
$$

$$
|v(t)|_k = \sum_{|\alpha| \leqq k} ||\Gamma^\alpha v(t)||_{L_x^\infty(\mathbb{R}^2)}.
$$

In [2], we proved that

$$
|\Gamma^\alpha \partial_r u|, |\Gamma^\alpha \partial_t u| \leqq C_{\alpha,B}\varepsilon(1+t)^{-\frac{1}{2}} \quad \text{for} \quad 0 \leqq t \leqq t_{\frac{1}{B}}, \quad \alpha \in \mathbb{Z}_+^6. \tag{3.5}
$$

For the solution u^0 to (2.1), (2.2), we set $F(1/r, \rho) = r^{1/2} u^0(r, t)$. Then L. Hörmander showed in [1] that

$$
|\partial_z^l \partial_\rho^m F(z, \rho)| \leqq C_{l,m}(1 + |\rho|)^{-\frac{1}{2}+l-m} \quad \text{for} \quad 0 < z \leqq \frac{1}{2M} \tag{3.6}
$$

and

$$
\begin{aligned}
&\left| \Gamma^\alpha (\partial_\rho^k F(z, \rho) - \frac{d^k}{d\rho^k}\mathcal{F}(\rho)) \right| \leqq C_{\alpha,k,L}(1 + |\rho|)^{\frac{1}{2}-k}(1+t)^{-1} \\
&\qquad \text{for} \quad r \geqq Lt \quad \text{and} \quad t \geqq 1.
\end{aligned} \tag{3.7}
$$

Here $M > 0$ is the radius of support of initial data and $L > 0$. Furthermore, $U(\rho, s)$ satisfies

$$|\partial_\rho^l \partial_s^m U(\rho, s)| \leqq C_{l,m,B}(1+|\rho|)^{-\frac{1}{2}-l-4m} \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.8}$$

$$U(\rho, s) = 0 \quad \text{for} \quad \rho \geqq M,\ 0 \leqq s \leqq \frac{1}{B}, \tag{3.9}$$

which will be proved in Appendix 1.

We choose a cut-off function $\chi \in C^\infty(\mathbb{R})$ equal to 1 in $(-\infty, 1)$ and 0 in $(2, \infty)$, and define a function $w(r,t)$ by

$$w(r,t) = \varepsilon\chi(\varepsilon t)u^0(r,t) - \varepsilon(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)r^{-\frac{1}{2}}U(\rho, s).$$

Using (3.5), (3.6), (3.7) and (3.8), we will prove

$$|\Gamma^\alpha w(r,t)| \leqq C_{\alpha,B}\varepsilon(1+t)^{-\frac{1}{2}}(1+|\rho|)^{-\frac{1}{2}} \quad \text{for} \quad 0 \leqq t \leqq t_{\frac{1}{B}}, \tag{3.10}$$

$$||\Gamma^\alpha J(t)||_0 \leqq C_{\alpha,B}(\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{4}{5}} + \varepsilon^4(1+t)^{-1}) \quad \text{for} \quad 0 \leqq t \leqq t_{\frac{1}{B}}, \tag{3.11}$$

where

$$J(r,t) = \Box w - (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\triangle w.$$

First we prove (3.10). Since the following decay estimate for u^0 (showed in L. Hörmander [1]) holds

$$|\Gamma^\alpha u^0(r,t)| \leqq C_\alpha(1+t)^{-\frac{1}{2}}(1+|\rho|)^{-\frac{1}{2}}, \tag{3.12}$$

we find that the first term of w satisfies (3.10). On the other hand, we get

$$t+1 \leqq 6r \leqq 6(t+M),$$

in the support of $(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)U(\rho, s)$. The second term of w satisfies (3.10) if we prove

$$|\Gamma^\beta(r^{-\frac{1}{2}})| \leqq C_\beta(1+t)^{-\frac{1}{2}}, \tag{3.13a}$$

$$|\Gamma^\beta(1-\chi(\varepsilon t))| \leqq C_\beta, \tag{3.13b}$$

$$|\Gamma^\beta(\chi(-3\varepsilon\rho))| \leqq C_\beta, \tag{3.13c}$$

$$|\Gamma^\beta(U(\rho, s))| \leqq C_\beta(1+|\rho|)^{-\frac{1}{2}}. \tag{3.13d}$$

Indeed, (3.13b) follows in principle from the inequalities

$$\begin{aligned} |L_i^k(1-\chi(\varepsilon t))| &\leqq C_k \sum_{j=0}^{k}\sum_{l=0}^{j} \varepsilon^j |x_i|^l t^{j-l} |\chi^{(j)}(\varepsilon t)| \\ &\leqq C_k \sum_{j=0}^{k} \varepsilon^j t^j |\chi^{(j)}(\varepsilon t)| \\ &\leqq C_k \qquad \text{for} \quad i = 1, 2, \end{aligned}$$

where the last inequality holds since $\varepsilon t \leqq 2$, in the support of $\chi^{(j)}(\varepsilon t)$. (3.13c) follows from the inequalities

$$
\begin{aligned}
|L_i^k(\chi(-3\varepsilon\rho))| &\leqq C_k \sum_{j=0}^{k} \sum_{l=0}^{j} \varepsilon^j \frac{|x_i|^l t^{j-l}}{r^j} |\rho|^j |\chi^{(j)}(-3\varepsilon\rho)| \\
&\leqq C_k \sum_{j=0}^{k} \varepsilon^j |\rho|^j |\chi^{(j)}(-3\varepsilon\rho)| \\
&\leqq C_k \qquad \text{for} \quad i = 1, 2,
\end{aligned}
$$

where the last inequality holds since $\varepsilon|\rho| \leqq 2/3$ in the support of $\chi^{(j)}(-3\varepsilon\rho)$. (3.13d) follows from (3.8) and a similar calculation as above.

Next we show (3.11) by dividing the proof into three cases.

Case 1. $0 \leqq \varepsilon t \leqq 1$. Since

$$
w(r,t) = \varepsilon u^0(r,t),
$$

we find

$$
J(r,t) = -\varepsilon^3 (a_1 {u_t^0}^2 + a_2 u_t^0 u_r^0 + a_3 {u_r^0}^2) \triangle u^0.
$$

It follows from (3.12) that

$$
|\Gamma^\alpha J(r,t)| \leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{3}{2}}.
$$

Since

$$
\int_{\mathbb{R}^2} |\Gamma^\alpha J(r,t)|^2 dx = 2\pi \int_0^{t+M} |\Gamma^\alpha J(r,t)|^2 r dr,
$$

we get

$$
\begin{aligned}
||\Gamma^\alpha J(r,t)||_0 &\leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{3}{2}} (t+M) \\
&\leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{1}{2}} \\
&\leqq C_\alpha \varepsilon^{\frac{5}{4}} (1+t)^{-\frac{5}{4}},
\end{aligned}
$$

where the last inequality follows from the fact

$$
\varepsilon(1+t) \leqq \varepsilon + 1 \leqq 2.
$$

This is what we wanted.

Case 2. $1 \leqq \varepsilon t \leqq 2$. Since the same estimate holds for nonlinear term $-(a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\triangle w$, we have only to examine

$$
\begin{aligned}
\Box w =& \varepsilon \Box [(1-\chi(\varepsilon t))\{\chi(-3\varepsilon\rho) r^{-\frac{1}{2}} U(\rho,s) - u^0(r,t)\}] \\
=& \varepsilon \Box \{(1-\chi(\varepsilon t))(\chi(-3\varepsilon\rho) - 1) u^0\} \\
&+ \varepsilon \Box \{(1-\chi(\varepsilon t)) \chi(-3\varepsilon\rho) r^{-\frac{1}{2}} (U(\rho,s) - F(\frac{1}{r},\rho))\} \\
=& J_1 + J_2,
\end{aligned}
$$

where $F(1/r, \rho)$ is the one in (3.6) and the last equality is the definition of J_1 and J_2. In the support of $1 - \chi(-3\varepsilon\rho)$, we have $6r \leqq 5t$. Hence we find

$$|\Gamma^\alpha \partial_r^l \partial_t^m u^0(r,t)| \leqq C_{\alpha,l,m}(1+t)^{-1-l-m} \quad \text{for} \quad t \geqq 1. \tag{3.14}$$

Since

$$|\partial_r^l \partial_t^m \{(1-\chi(\varepsilon t))(\chi(-3\varepsilon\rho)-1)\}| \leqq C_{l,m}\varepsilon^{l+m}$$

and the support of u^0 is the same as that of U, it follows from (3.13b), (3.13c) and (3.14) that

$$\begin{aligned}
|\Gamma^\alpha J_1(r,t)| &\leqq C_\alpha(\varepsilon^3(1+t)^{-1} + \varepsilon^2(1+t)^{-2}) \\
&\leqq C_\alpha \varepsilon^2 (1+t)^{-2}, \\
\|\Gamma^\alpha J_1(t)\|_0 &\leqq C_\alpha \varepsilon^2 (1+t)^{-1} \\
&\leqq C_\alpha \varepsilon^{\frac{5}{4}} (1+t)^{-\frac{5}{4}}.
\end{aligned}$$

On the other hand, in the support of J_2, we have $1 + t \leqq 6r \leqq 6(t+M)$ and then obtain (3.13). Moreover we prove that

$$|\Gamma^\alpha(\partial_\rho^l U(\rho,s) - \partial_\rho^l F(\frac{1}{r},\rho))| \leqq C_l(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1}, \tag{3.15}$$

for $0 \leqq s \leqq 1/B$, $r \geqq 1/(2M)$. Indeed,

$$\begin{aligned}
\partial_\rho^l U(\rho,s) &= \partial_\rho^l U(\rho,0) + \int_0^1 \frac{d}{d\lambda}\partial_\rho^l U(\rho,\lambda s)d\lambda \\
&= \frac{d^l}{d\rho^l}\mathcal{F}(\rho) + \varepsilon^2\log(1+t)\int_0^1 \partial_\rho^l \partial_s U(\rho,\lambda s)d\lambda.
\end{aligned}$$

By (3.8), (3.13d), $\varepsilon t \leqq 2$ and the fact

$$|\Gamma^\beta(\varepsilon\log(1+t))| \leqq C_\beta,$$

we find that

$$\begin{aligned}
|\Gamma^\alpha(\varepsilon^2\log(1+t)\int_0^1 \partial_\rho^l \partial_s U(\rho,s\lambda)d\lambda)| &\leqq C_\alpha(\varepsilon^2\log(1+t)(1+|\rho|)^{-\frac{1}{2}-l-4}) \\
&\leqq C_\alpha(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1}.
\end{aligned}$$

Thus it follows from (3.7) that

$$|\Gamma^\alpha(\partial_\rho^l U(\rho,s) - \partial_\rho^l F(\frac{1}{r},\rho))| \leqq C_\alpha(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1},$$

which implies (3.15). Now, since

$$\Box v = r^{-\frac{1}{2}}(\partial_t^2 - \partial_r^2 - \frac{1}{4r^2})(r^{\frac{1}{2}}v),$$

we get

$$\begin{aligned}J_2 =&\varepsilon r^{-\frac{1}{2}}(\partial_t-\partial_r)(\partial_t+\partial_r)\{(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)(U-F)\}\\&+\varepsilon r^{-\frac{5}{2}}(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)(U-F)\\=&J_2'+J_2'',\end{aligned}$$

where the last equality is the definition of J_2' and J_2''. By (3.6), we have

$$|\Gamma^\beta F(\frac{1}{r},\rho)| \leqq C_\beta(1+|\rho|)^{-\frac{1}{2}}. \tag{3.16}$$

Then it follows from (3.13), (3.16) and $1+t \leqq 6r$ that

$$|\Gamma^\alpha J_2''| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}.$$

Since $(\partial_t+\partial_r)\rho = 0$, we obtain

$$\begin{aligned}|\Gamma^\alpha J_2'| \leqq& C_\alpha(\varepsilon^2 r^{-\frac{1}{2}}|\Gamma^\alpha\{(\partial_t-\partial_r)(\chi'(\varepsilon t)\chi(-3\varepsilon\rho)(U-F))\}|\\&+\varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}),\end{aligned}$$

where we have used (3.13), (3.16), $1+t \leqq 6r \leqq 6(t+M)$ and $\varepsilon t \leqq 2$. Moreover using (3.13) and (3.15), we find that

$$\begin{aligned}&|\Gamma^\alpha\{(\partial_t-\partial_r)(\chi'(\varepsilon t)\chi(-3\varepsilon\rho)(U-F))\}|\\\leqq& C_\alpha(\varepsilon(1+t)^{-1}(1+|\rho|)^{\frac{1}{2}}+(1+t)^{-1}(1+|\rho|)^{-\frac{1}{2}})\\\leqq& C_\alpha(1+t)^{-1}(1+|\rho|)^{-\frac{1}{2}}.\end{aligned}$$

Thus we get

$$|\Gamma^\alpha J_2'| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}$$

and then we have

$$\begin{aligned}|\Gamma^\alpha J_2| &\leqq C_\alpha\varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}},\\||\Gamma^\alpha J_2||_0 &\leqq C_\alpha\varepsilon(1+t)^{-2}(\log(t+M))^{\frac{1}{2}}\\&\leqq C_\alpha\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}},\end{aligned}$$

which implies (3.11) for $1 \leqq \varepsilon t \leqq 2$.

Case 3. $2 \leqq \varepsilon t \leqq \varepsilon t_{\frac{1}{B}}$. In this case, we have

$$w(r,t) = \varepsilon r^{-\frac{1}{2}}\chi(-3\varepsilon\rho)U(\rho,s) = \varepsilon r^{-\frac{1}{2}}\hat{U}(\rho,s).$$

We divide J into three parts:

$$\Gamma^\alpha J = Q_1 + Q_2 + Q_3,$$

where

$$
\begin{aligned}
Q_1 =&\Gamma^\alpha(\Box w + 2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{\rho s}),\\
Q_2 =&\Gamma^\alpha(-2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{\rho s} - (a_1 - a_2 + a_3)(\hat{U}_\rho)^2\hat{U}_{\rho\rho}),\\
Q_3 =&\Gamma^\alpha((a_1 - a_2 + a_3)\varepsilon^3 r^{-\frac{3}{2}}(\hat{U}_\rho)^2\hat{U}_{\rho\rho} - (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\triangle w).
\end{aligned}
$$

Thus our purpose is converted to

$$
||Q_i||_0 = O(\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}} + \varepsilon^4(1+t)^{-1}) \quad \text{for} \quad i = 1,2,3.
$$

In the support of Q_i, we have $1 + t \leqq 3r \leqq 3(t + M)$ and then we have (3.13). First we consider Q_1. We get

$$
\begin{aligned}
\Gamma^\alpha \Box w(r,t) =&\Gamma^\alpha(\varepsilon r^{-\frac{1}{2}}(\partial_t - \partial_r)(\partial_t + \partial_r)\hat{U}(\rho, s) + \frac{1}{4}\varepsilon r^{-\frac{5}{2}}\hat{U}(\rho, s))\\
=&R_1 + R_2,
\end{aligned}
$$

where the last equality is the definition of R_1 and R_2. Using (3.13) and $1 + t \leqq 3r$, we get

$$
|R_2| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}
$$

and then

$$
\begin{aligned}
||R_2||_0 \leqq& C_\alpha \varepsilon (1+t)^{-2}(\log(t+M))^{\frac{1}{2}}\\
\leqq& C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Since $(\partial_t + \partial_r)\rho = 0$, we have

$$
R_1 = \Gamma^\alpha(\varepsilon^3 r^{-\frac{1}{2}}(\partial_t - \partial_r)\{(1+t)^{-1}\hat{U}_s(\rho, s)\}).
$$

Moreover, by (3.8) and (3.13), we get

$$
\begin{aligned}
|R_1 + 2\varepsilon^3\Gamma^\alpha(r^{-\frac{1}{2}}(1+t)^{-2}\hat{U}_{s\rho}(\rho, s))| \leqq& C_\alpha \varepsilon^3 r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{9}{2}},\\
|R_1 + 2\varepsilon^3\Gamma^\alpha(r^{-\frac{3}{2}}\hat{U}_{s\rho}(\rho, s))| \leqq& C_\alpha \varepsilon^3 r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{7}{2}},
\end{aligned}
$$

where we have used the fact

$$
|\Gamma^\alpha(\rho\hat{U}_{s\rho}(\rho, s))| \leqq C_\alpha(1+|\rho|)^{-\frac{7}{2}}.
$$

Thus we obtain

$$
\begin{aligned}
||Q_1||_0 \leqq& ||R_1 + \Gamma^\alpha(2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{s\rho}(\rho, s))||_0 + ||R_2||_0\\
\leqq& C_\alpha(\varepsilon^3(1+t)^{-2} + \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}) \qquad (3.17)\\
\leqq& C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Next we consider Q_3. Using (3.13), we get

$$
|\Gamma^\alpha(\partial_t^l\partial_r^m w(r,t) - \varepsilon r^{-\frac{1}{2}}(-1)^l\partial_\rho^{l+m}\hat{U}(\rho, s))| \leqq C_{\alpha,l,m}\varepsilon r^{-\frac{1}{2}}(1+t)^{-1}.
$$

This estimate yields

$$|Q_3| \leqq C_\alpha \varepsilon^3 r^{-\frac{3}{2}}(1+t)^{-1}$$

and then we get

$$\begin{aligned} ||Q_3||_0 &\leqq C_\alpha \varepsilon^3 (1+t)^{-2}(t+M) \\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}. \end{aligned} \tag{3.18}$$

Finally we estimate Q_2. When $\chi(-3\varepsilon\rho)$ is equal to 1 or 0, we find $Q_2 = 0$ by (3.1b). Thus we can assume $1 \leqq -3\varepsilon\rho \leqq 2$, *i.e.*, $(1+|\rho|)^{-1} \leqq 3\varepsilon$. Using (3.8), (3.12) and $1+t \leqq 3r$, we have

$$\begin{aligned} |Q_2| &\leqq C_\alpha \varepsilon^4 r^{-\frac{1}{2}}(1+t)^{-1}(1+|\rho|)^{-\frac{3}{2}}, \\ ||Q_2||_0 &\leqq C_\alpha \varepsilon^4 (1+t)^{-1}. \end{aligned} \tag{3.19}$$

Combining (3.17), (3.18) and (3.19), we find that (3.11) is valid for $2 \leqq \varepsilon t \leqq \varepsilon t_{1/B}$ and then that is valid for $0 \leqq t \leqq t_{1/B}$.

To finish the poof of (3.3), we need the following propositions.

Proposition 3.1. *Let $v \in C^2$ satisfy a wave equation:*

$$\Box v(x,t) = \sum_{\alpha,\beta=0}^{2} \gamma_{\alpha\beta}(x,t)\partial_\alpha \partial_\beta v(x,t) + h(x,t), \qquad (x,t) \in \mathbb{R}^2 \times [0,\infty),$$

where $\partial_0 = \partial_t$ and

$$|\gamma(t)|_0 = \sum_{\alpha,\beta=0}^{2} |\gamma_{\alpha\beta}(t)|_0 < \frac{1}{2} \quad \text{for} \quad 0 \leqq t < T.$$

Assume that for any fixed t, v vanishes for large $|x|$. Then we have for $0 \leqq t < T$

$$||Dv(t)||_0 \leqq 3(||Dv(0)||_0 + \int_0^t ||h(\tau)||_0 d\tau) \exp(\int_0^t |D\gamma(\tau)|_0 d\tau),$$

where

$$Dv = (\partial_0 v, \partial_1 v, \partial_2 v) \quad \text{and} \quad |D\gamma(\tau)|_0 = \sum_{\alpha,\beta,\delta=0}^{2} |\partial_\delta \gamma_{\alpha\beta}(\tau)|_0.$$

Proposition 3.2. *For a smooth function $v(x,t)$ radially symmetric with respect to x,*

$$|v(x,t)| \leqq C_n (1+|x|+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}} ||v(t)||_{[\frac{n}{2}]+1}$$

holds where $[s]$ stands for the largest integer not exceeding s.

Proposition 3.1 is obtained by integration by parts and Gronwall's inequality. Proposition 3.2 is so-called Klainerman's inequality which has proved in S. Klainerman [7] and F. John [5].

If we show that

$$||\Gamma^\alpha D(u(r,t)-w(r,t))||_0 \leqq C_{\alpha,B}\varepsilon^{\frac{5}{4}} \quad \text{for any} \quad \alpha \in \mathbb{Z}_+^6, \tag{3.20}$$

we find that (3.3) is valid. Indeed, it follows from (3.20) and Proposition 3.2 that

$$|\partial_r^l \partial_t^m (u(r,t)-w(r,t))| \leqq C_{l,m,B}\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}} \quad 0 \leqq t \leqq t_{\frac{1}{B}}$$

for any l and m. Moreover when $t \geqq 2/\varepsilon$ and $r-t \geqq -1/3\varepsilon$, $w(r,t) = \varepsilon r^{-1/2}U(\rho,s)$. Then

$$\partial_r^l \partial_t^m w(r,t) = \varepsilon r^{-\frac{1}{2}}(-1)^m \partial_\rho^{l+m} U(\rho,s) + O(\varepsilon r^{-\frac{3}{2}})$$

holds. By combining above inequality and equality, the desired estimate is obtained. Thus we have only to prove (3.20). If we set $v(r,t) = u(r,t) - w(r,t)$, by (2.1) v satisfies

$$\begin{aligned}\Box v =&(a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2 + O(|Du|^4))\triangle u + \frac{1}{r}u_r G(u_t, u_r) - J(r,t)\\ &- (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\triangle w\\ =&(a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2)\triangle v + O(|Du|^4|\triangle u|) + \frac{1}{r}u_r G(u_r, u_t)\\ &+ \{a_1(u_t + w_t)v_t + a_2(u_t v_r + w_r v_t) + a_3(u_r + w_r)v_r\}\triangle w - J(r,t).\end{aligned} \tag{3.21}$$

By (3.5), we have for sufficiently small $\varepsilon > 0$

$$|a_1 u_t^2(t) + a_2 u_t(t) u_r(t) + a_3 u_r^2(t)|_0 \leqq \frac{1}{4}.$$

Thus we can apply Proposition 3.1 to (3.21). Since $v(r,0) \equiv 0$, we obtain for $0 \leqq t \leqq t_{\frac{1}{B}}$

$$\begin{aligned}||Dv(t)||_0 \leqq& C\int_0^t || \, (|Du(\tau)| + |Dw(\tau)|)|Dv(\tau)| \cdot |\triangle w(\tau)| + |J(\tau)|\\ &+ |Du|^4|D^2 u| + |u_{rr}G(\tau)| \, ||_0 d\tau\\ &\times \exp(C\int_0^t |Du(\tau)| \cdot |D^2 u(\tau)| d\tau).\end{aligned}$$

It follows from (3.5), (3.10), (3.11) and $\varepsilon^2 \log(1 + t_{1/B}) = 1/B$ that

$$\begin{aligned}||Dv(t)||_0 \leqq& Ce^{\frac{C}{B}} \int_0^t \{\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}} + \varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-1}||Dv(\tau)||_0\} d\tau\\ \leqq& C\varepsilon^{\frac{5}{4}} + C\int_0^t \varepsilon^2(1+\tau)^{-1}||Dv(\tau)||_0 d\tau.\end{aligned}$$

Gronwall's inequality yields

$$||Dv(t)||_0 \leqq C\varepsilon^{\frac{5}{4}} \exp(C\int_0^t \varepsilon^2(1+t)^{-1} d\tau) \leqq C\varepsilon^{\frac{5}{4}}.$$

This implies that (3.20) is valid for $\alpha = 0$. To prove (3.20) by induction, we assume that (3.20) holds for $|\alpha| = s-1$. For any α with $|\alpha| = s$, (3.3) admits

$$\begin{aligned}
\Box\Gamma^\alpha v = &\sum_{|\beta|<|\alpha|} \Gamma^\beta(\Box v) + \Gamma^\alpha\{(a_1u_t^2 + a_2u_tu_r + a_3u_r^2)\triangle v\} \\
&+ \Gamma^\alpha\{(a_1(u_t+w_t)v_t + a_2(u_tv_r + w_rv_t) + a_3(u_r+w_r)v_r)\triangle w\} \\
&+ O(|Du|^4|\triangle u|) + \frac{1}{r}u_rG(u_r,u_t) - \Gamma^\alpha J \\
=&(a_1u_t^2 + a_2u_tu_r + a_3u_r^2)\triangle\Gamma^\alpha v + O((|Du| + |Dw|)|\triangle w| \cdot |D\Gamma^\alpha v| \\
&+ |\Gamma^{s-1}Dv|(|\Gamma^s Du|^2 + |\Gamma^{s+1}w|^2) + |\Gamma^s J| + |\Gamma^s Du|^4|\Gamma^s\triangle u| \\
&+ |\Gamma^s(\frac{1}{r}u_rG(u_r,u_t))|),
\end{aligned}$$

where $Df = (\partial_t f, \partial_r f)$ and $\Gamma^s = \sum_{|\lambda|=s}\Gamma^\lambda$. By Proposition 3.1, we get for $0 \leqq t \leqq t_{1/B}$

$$\begin{aligned}
||D\Gamma^\alpha v(t)||_0 \leqq &C\int_0^t ||\ (|Du(\tau)| + |Dw(\tau)|)|\triangle w(\tau)| \cdot |D\Gamma^\alpha v(\tau)| \\
&+ |\Gamma^{s-1}Dv(\tau)|(|\Gamma^s Du(\tau)|^2 + |\Gamma^{s+1}Dw(\tau)|^2) + |\Gamma^s J(\tau)| \\
&|\Gamma^s Du(\tau)|^4|\Gamma^s\triangle u(\tau)| + |\Gamma^s(\frac{1}{r}u_rG(\tau))|\ ||_0 d\tau \\
&\times \exp(C\int_0^t |Du(\tau)| \cdot |D^2u(\tau)| d\tau).
\end{aligned}$$

Proceeding as above, by (3.5), (3.10), (3.11), $\varepsilon^2\log(1+t_{1/B}) = 1/B$ and the assumption, we have

$$\begin{aligned}
||D\Gamma^\alpha v(t)||_0 \leqq &C\int_0^t \{\varepsilon^{\frac{5}{4}}(1+\tau)^{-\frac{5}{4}} + \varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-1}||D\Gamma^\alpha v(\tau)||_0\}d\tau \\
\leqq &C\varepsilon^{\frac{5}{4}} + C\int_0^t \varepsilon^2(1+\tau)^{-1}||D\Gamma^\alpha v(\tau)||_0 d\tau.
\end{aligned}$$

Gronwall's inequality yields

$$||D\Gamma^\alpha v(t)||_0 \leqq C\varepsilon^{\frac{5}{4}}\exp(C\int_0^t \varepsilon^2(1+\tau)^{-1}d\tau) \leqq C\varepsilon^{\frac{5}{4}}.$$

Again using (3.4) and the assumption, we obtain

$$||\Gamma^\alpha Dv(t)||_0 \leqq C\varepsilon^{\frac{5}{4}},$$

for any α with $|\alpha| = s$. This completes the proof of (3.20).

At the end of this section, we investigate the value of the solution $U = U(\rho, s)$ at $s = 1/B$ *i.e.*, $t = t_{1/B}$. We assume that the maximum in the definition of H_0 is attained at $\rho = \rho_0$, *i.e.*,

$$H_0 = -a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0). \tag{3.22}$$

In (ρ, s)-plane, we consider a characteristic curve $\Lambda_q (q \in \mathbb{R})$ which is defined by the solution of the following differential equation:

$$\frac{d\rho}{ds} = \frac{a}{2}(U_\rho(\rho, s))^2 \quad \text{for} \quad s \geqq 0, \qquad \rho = q \quad \text{for} \quad s = 0.$$

If we denote a point on Λ_{ρ_0} by $(\rho(s), s)$, then we find

$$U_\rho(\rho(\frac{1}{B}), \frac{1}{B}) = \mathcal{F}'(\rho_0) \tag{3.23}$$

$$\frac{1}{-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B})} = \frac{1}{H_0} - \frac{1}{B}. \tag{3.24}$$

Indeed, by (3.1b) and (3.2b), we have along Λ_{ρ_0}

$$\frac{d}{ds}U_\rho(\rho(s), s) = U_{\rho s} + \frac{a}{2}(U_\rho)^2 U_{\rho\rho} = 0 \quad 0 \leqq s \leqq \frac{1}{B}.$$

Hence we have

$$U_\rho(\rho(s), s) = U_\rho(\rho_0, 0) = \mathcal{F}'(\rho_0) \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.25}$$

which implies (3.23). Similarly, it follows from (3.1b) and (3.25) that

$$\begin{aligned}\frac{d}{ds}U_{\rho\rho}(\rho(s), s) =& U_{\rho\rho s}(\rho(s), s) + \frac{a}{2}(U_\rho(\rho(s), s))^2 U_{\rho\rho\rho}(\rho(s), s)\\ =& -aU_\rho(\rho(s), s)(U_{\rho\rho}(\rho(s), s))^2\\ =& -a\mathcal{F}'(\rho_0)(U_{\rho\rho}(\rho(s), s))^2.\end{aligned}$$

Solving this equation , we obtain by (3.2b) and (3.22)

$$U_{\rho\rho}(\rho(s), s) = \frac{U_{\rho\rho}(\rho_0, 0)}{1 + a\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho_0, 0)s},$$

i.e.,

$$\frac{1}{U_{\rho\rho}(\rho(s), s)} = \frac{1}{\mathcal{F}''(\rho_0)} + a\mathcal{F}'(\rho_0)s,$$

i.e.,

$$\frac{1}{-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho(s), s)} = \frac{1}{H_0} - s,$$

for $0 \leqq s \leqq 1/B$. Thus (3.24) follows from this equality.

4. *A priori* estimates.

From now on, we investigate the behaviour of u after $t = t_{1/B}$. If we set $v(r, t) = r^{\frac{1}{2}}u(r, t)$, the equation (1.1) can be written as

$$v_{tt} - c^2(u_t, u_r)(v_{rr} + \frac{1}{4}r^{-2}v_r) = r^{-\frac{1}{2}}u_r G(u_t, u_r). \tag{4.1}$$

Moreover we define functions $w_1(r,t), w_2(r,t)$ by

$$\begin{aligned} w_1(r,t) =& \frac{cv_{rr} - v_{rt}}{2c} = -\frac{\mathcal{L}_2 v_r}{2c}, \\ w_2(r,t) =& \frac{cv_{rr} + v_{rt}}{2c} = \frac{\mathcal{L}_1 v_r}{2c}, \end{aligned}$$

where $\mathcal{L}_1 = \partial_t + c\partial_r, \mathcal{L}_2 = \partial_t - c\partial_r$. We find that w_1 and w_2 satisfy

$$w_1 + w_2 = v_{rr}, \quad c(w_2 - w_1) = v_{rt},$$

and these imply

$$\begin{aligned} u_r =& r^{-\frac{1}{2}} v_r - \frac{1}{2} r^{-\frac{3}{2}} v, \\ u_{rr} =& r^{-\frac{1}{2}}(w_1 + w_2) - r^{-\frac{3}{2}} v_r + \frac{3}{4} r^{-\frac{5}{2}} v, \\ u_{rt} =& c r^{-\frac{1}{2}}(w_2 - w_1) - \frac{1}{2} r^{-\frac{3}{2}} v_t. \end{aligned} \tag{4.2}$$

Then using (4.2), we obtain the equalities:

$$\begin{aligned} \mathcal{L}_1 w_1 =& \{c(a_1 u_t + \frac{a_2}{2} u_r) - \frac{a_2}{2} u_t - a_3 u_r + O(|Du|^3)\} r^{-\frac{1}{2}} w_1^2 \\ &+ O(\{r^{-\frac{1}{2}}|Du| \cdot |w_2| + r^{-\frac{3}{2}}|Dv| \cdot |Du| + r^{-\frac{5}{2}}|Du| \cdot |v|\}|w_1| \\ &+ r^{-\frac{5}{2}}|w_2| \cdot |Du| \cdot |v| + r^{-\frac{3}{2}}|w_2| \cdot |Du| \cdot |Dv| + r^{-2}|Dv| + r^{-3}|v|) \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathcal{L}_2 w_2 =& O(\{r^{-\frac{1}{2}}|w_2| \cdot |Du| + r^{-\frac{3}{2}}|Du| \cdot |Dv| + r^{-\frac{5}{2}}|Du| \cdot |v|\}|w_1| \\ &+ r^{-\frac{1}{2}}|Du| \cdot |w_2|^2 + r^{-\frac{3}{2}}|Du| \cdot |Dv| \cdot |w_2| + r^{-\frac{5}{2}}|Dw_2| \cdot |Du| \cdot |v| \\ &+ r^{-2}|Dv| + r^{-3}|v|). \end{aligned} \tag{4.4}$$

In what follows, we assume that there exists a T $(t_{1/B} < T < t_{1/A})$ such that the Cauchy problem (1.1), (1.2) has a solution $u(r,t)$ for $0 \leqq t \leqq T$. In (r,t)-plane, we consider pseudo-characteristic curves Z^1_λ and Z^2_μ which are given by solutions of differential equations:

$$Z^1_\lambda : \frac{dr}{dt} = c(u_t, u_r) \quad \text{for} \quad t \geqq t_{\frac{1}{B}}, \qquad r = \lambda + t_{\frac{1}{B}} \quad \text{for} \quad t = t_{\frac{1}{B}},$$

$$Z^2_\mu : \frac{dr}{dt} = -c(u_t, u_r) \quad \text{for} \quad t \geqq t_{\frac{1}{B}}, \qquad r = \mu - t_{frac1B} \quad \text{for} \quad t = t_{\frac{1}{B}}.$$

We set

$$\begin{aligned} D =& \{(r,t) \mid t_{\frac{1}{B}} \leqq t \leqq T, (r,t) \in Z^1_\lambda, \ -N \leqq \lambda \leqq M\}, \\ D_{t_*} =& D \cap \{(r,t) \mid t_{\frac{1}{B}} \leqq t \leqq t_*\}, \end{aligned}$$

where the constant N is sufficiently greater than $|\rho_0|$. Moreover we define the functions

$$\begin{aligned} I(t) =& \max_{t_{\frac{1}{B}} \leqq \tau \leqq t} \int_{r_1(\tau)}^{r_2(\tau)} |w_1(r,\tau)| dr, \\ V(t) =& \max_{(r,\tau) \in D_t} |v(r,\tau)|, \\ \dot{V}(t) =& \max_{(r,\tau) \in D_t} (|v_r(r,\tau)| + |v_t(r,\tau)|), \\ W_2(t) =& \max_{(r,\tau) \in D_t} |w_2(r,\tau)|, \end{aligned}$$

where $(r_1(\tau),\tau)\in Z^1_{-N}$ and $(r_2(\tau),\tau)\in Z^1_M$. Then the purpose of this section is following.

There exists a constant $\hat{C}>0$ independent of A and an $\varepsilon_A>0$ such that

$$
\begin{aligned}
I(t)<\hat{C}\varepsilon,\quad V(t)<\hat{C}\varepsilon^{\frac{1}{2}},\\
\dot{V}(t)<\hat{C}\varepsilon,\quad W_2(t)<\hat{C}\varepsilon^3,\quad r>\frac{1+t}{2},
\end{aligned}
\tag{4.5}
$$

for $(r,t)\in D$ and $0<\varepsilon<\varepsilon_A$.

To obtain (4.5) we just have to show:
(1) (4.5) holds at $t=t_{1/B}$,
(2) If (4.5) holds for $t_{1/B}\leqq t<t_1$, (4.5) also holds at $t=t_1$.
At first we prove (1). If $(r,t_{1/B})\in Z^1_\lambda\cap D$, it follows that

$$
r=t_{\frac{1}{B}}+\lambda,\qquad -N\leqq\lambda\leqq M,
\tag{4.6}
$$

then we find that

$$
t_{\frac{1}{B}}-N\leqq r\leqq t_{\frac{1}{B}}+M.
$$

If we take ε sufficiently small as

$$
t_{\frac{1}{B}}=\exp(\frac{1}{B\varepsilon^2})-1>\max(M-2,2N+1),
$$

then we obtain

$$
\frac{1+t_{\frac{1}{B}}}{2}<r(t_{\frac{1}{B}})<2(1+t_{\frac{1}{B}}).
\tag{4.7}
$$

For $(r,t_{\frac{1}{B}})\in Z^1_\lambda$, it follows from (3.5), (4.6) and (4.7) that

$$
\begin{aligned}
|u(r,t_{\frac{1}{B}})|=&\left|-\int_r^{t_{\frac{1}{B}}+M}\frac{\partial}{\partial\lambda}(u(\lambda,t_{\frac{1}{B}}))d\lambda\right|\\
\leqq&|t_{\frac{1}{B}}+M-r|\cdot|u_r(t_{\frac{1}{B}})|_0\\
\leqq&C\varepsilon(1+t_{\frac{1}{B}})^{-\frac{1}{2}}|\lambda+M|\\
\leqq&C(M+N)\varepsilon(1+t_{\frac{1}{B}})^{-\frac{1}{2}}\\
<&\sqrt{2}C(M+N)\varepsilon r^{-\frac{1}{2}}=C_0\varepsilon r^{-\frac{1}{2}},
\end{aligned}
$$

which implies $V(t_{1/B})<C_0\varepsilon^{1/2}$. It follows from (3.4), (4.7) and $V(t_{1/B})<C_0\varepsilon^{1/2}$ that for $(r,t_{1/B})\in D$,

$$
\begin{aligned}
|v_r(r,t_{\frac{1}{B}})|=&|r^{\frac{1}{2}}u_r(r,t_{\frac{1}{B}})+\frac{1}{2}r^{-1}v(r,t_{\frac{1}{B}})|\\
\leqq&Cr^{\frac{1}{2}}\varepsilon(1+t_{\frac{1}{B}})^{-\frac{1}{2}}+\frac{1}{2}C_0r^{-1}\varepsilon^{\frac{1}{2}}\\
<&\sqrt{2}C\varepsilon+\frac{1}{\sqrt{2}}C_0\varepsilon^{\frac{1}{2}}(1+t_{\frac{1}{B}})^{-1}.
\end{aligned}
$$

If we take ε sufficiently small, we obtain

$$(1+t_{\frac{1}{B}})^{-1} = (\exp(\frac{1}{B\varepsilon^2}))^{-1} < \varepsilon^3. \tag{4.8}$$

Thus we find

$$|v_r(r, t_{\frac{1}{B}})| < \frac{C_1}{2}\varepsilon.$$

Similarly we have

$$|v_t(r, t_{\frac{1}{B}})| < \frac{C_1}{2}\varepsilon.$$

Therefore we obtain $\dot{V}(t_{1/B}) < C_1\varepsilon$. Using (3.5), (4.8), $V(t_{1/B}) < C_0\varepsilon^{1/2}$, $\dot{V}(t_{1/B}) < C_1\varepsilon$ and an equality

$$\partial_t + \partial_r = \frac{1}{t+r}(L_0 + \frac{x_1}{r}L_1 + \frac{x_2}{r}L_2),$$

we have for $(r, t_{1/B}) \in D$,

$$\begin{aligned}
|w_2(r, t_{\frac{1}{B}})| = & \left|\frac{v_{tr} + cv_{rr}}{2c}\right| \\
= & \frac{|v_{rt} + v_{rr}|}{2} + O((|v_{rt}| + |v_{rr}|)|Du|^2) \\
= & O((t_{\frac{1}{B}} + r)^{-1}|v_r(t_{\frac{1}{B}})|_1 + (|v_{rt}(t_{\frac{1}{B}})|_0 + |v_{rr}(t_{\frac{1}{B}})|_0)|Du(t_{\frac{1}{B}})|_0^2) \\
= & O(\varepsilon(1+t_{\frac{1}{B}})^{-1} + \varepsilon^3(1+t_{\frac{1}{B}})^{-1}) \\
= & O(\varepsilon^4).
\end{aligned}$$

This implies $W_2(t_{1/B}) < C_2\varepsilon^3$. Finally we consider $I(t_{1/B})$. It follows from (3.5), (4.8) $V(t_{1/B}) < C_0\varepsilon^{1/2}$ and $\dot{V}(t_{1/B}) < C_1\varepsilon$ that for $(r, t_{1/B}) \in D$,

$$\begin{aligned}
|w_1(r, t_{\frac{1}{B}})| = & \left|\frac{v_{rt} - cv_{rr}}{2}\right| \\
\leqq & \frac{|v_{rt}| + |v_{rr}|}{2} + O((|v_{rt}| + |v_{rr}|)|Du|^2) \\
\leqq & C''(\varepsilon + \varepsilon^3(1+t_{\frac{1}{B}})^{-1}) \\
\leqq & C'\varepsilon.
\end{aligned}$$

On the other hand, it follows from $(r_1(t_{1/B}), t_{1/B}) \in Z^1_{-N}, (r_2(t_{1/B}), t_{1/B}) \in Z^1_M$ and (4.6) that

$$|r_2(t_{\frac{1}{B}}) - r_1(t_{\frac{1}{B}})| = |t_{\frac{1}{B}} + M - t_{\frac{1}{B}} + N| = M + N.$$

Then we have

$$I(t_{\frac{1}{B}}) = \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1(r, t_{\frac{1}{B}})|dr \leqq C'(M+N)\varepsilon < C_3\varepsilon.$$

If we take $\hat{C} > 0$

$$\hat{C} > \max\{C_0, C_1, C_2, C_3\},$$

(4.5) is valid at $t = t_{1/B}$ for sufficiently small ε. Thus we have proved (1).

To prove (2) we assume that for fixed t_1, (4.5) holds for $t_{1/B} \leqq t < t_1$. The smoothness of the solution u guarantees that the inequalities which are altered $<$ by $\leqq$ in (4.5) hold at $t = t_1$. First we show $r > (1 + t_1)/2$ if $\varepsilon < \varepsilon_A$. By (4.5) and the assumption $\varepsilon^2 \log(1 + T) < 1/A$, we obtain for $(r(t), t) \in Z^1_\lambda$, $t_{1/B} \leqq t \leqq t_1$

$$\frac{d(r-t)}{dt} = c - 1 = O(|Du|^2) = O(\varepsilon^2(1+t)^{-1}),$$

$$\begin{aligned} |r(t) - t - \lambda| \leqq & C \int_{t_{\frac{1}{B}}}^{t} \varepsilon^2 (1+\tau)^{-1} d\tau \\ \leqq & C\varepsilon^2 \log(1+t) \\ \leqq & \frac{C}{A}. \end{aligned} \tag{4.9}$$

This leads to

$$r(t_1) \geqq t_1 + \lambda - \frac{C}{A} \geqq t_1 - M - \frac{C}{A} > \frac{1+t_1}{2},$$

provided $t_{1/B} > 2M + 2C/A + 1$, which is attained for $0 < \varepsilon < \varepsilon_A$ if ε_A is sufficiently small. Next we estimate $v(r, t_1)$. By (4.5) and (4.9), we obtain for $0 < \varepsilon < \varepsilon_A$,

$$\begin{aligned} |v(r, t_1)| = & \left| - \int_r^{t_1 + M} v_r(\lambda, t_1) d\lambda \right| \\ \leqq & \hat{C}\varepsilon |t_1 + M - r| \\ \leqq & \hat{C}(\frac{C}{A} + M + N)\varepsilon \\ < & \hat{C}\varepsilon^{\frac{1}{2}}, \end{aligned}$$

if $\varepsilon_A < (C/A + M + N)^{-2}$. Thus $V(t_1) < \hat{C}\varepsilon^{\frac{1}{2}}$ holds.

To prove $I(t_1) < \hat{C}\varepsilon$, we consider exterior derivatives of differential forms $w_1 dr - cw_1 dt$ and $w_2 dr + cw_2 dt$:

$$d(w_1(dr - cdt)) = -(\mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1) dr \wedge dt, \tag{4.10}$$

$$d(w_2(dr + cdt)) = -(\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2) dr \wedge dt. \tag{4.11}$$

We set

$$\mathcal{K} = \{(r, t_1) \in D_{t_1} | w_1(r, t_1) > 0\},$$

$$\mathcal{K}' = \{(r, t_1) \in D_{t_1} | w_1(r, t_1) < 0\}.$$

Since these are open sets in $\mathbb{R}$, $\mathcal{K}$ and $\mathcal{K}'$ are the unions of at most denumerable families $\{K_i\}$ and $\{K'_i\}$ of open intervals, no two of which have common points.

Assume that $\mathcal{K} = \{(r,t_1)|r_1(t_1) \leqq r \leqq r_2(t_1)\}$. Then, integrating (4.10) over D_{t_1} and using Green's formula, we obtain

$$-\iint_{D_{t_1}} (\mathcal{L}_1 w_1 + \frac{\partial c}{\partial t} w_1) dr\, dt$$
$$= \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} w_1 dr + \int_{Z_M^1} w_1(dr - cdt) - \int_{\mathcal{K}} w_1 dr - \int_{Z_{-N}^1} w_1(dr - cdt).$$

Since

$$\int_{Z_\lambda^1} w_1(dr - cdt) = 0 \quad \text{for any} \quad \lambda,$$

we have

$$\int_{\mathcal{K}} w_1 dr \leqq \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| dr dt.$$

Furthermore, assume that $\mathcal{K}' = \{(r,t_1)|r_1(t_1) \leqq r \leqq r_2(t_1)\}$. Then, the same argument gives

$$-\int_{\mathcal{K}'} w_1 dr \leqq \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| dr dt.$$

Summing up such inequalities corresponding to K_i and K_i', we obtain

$$\begin{aligned} \int_{r_1(t_1)}^{r_2(t_1)} |w_1| dr \leqq & \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| dr dt \\ = & I(t_{\frac{1}{B}}) + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| dr dt. \end{aligned} \tag{4.12}$$

It follows from (4.2), (4.3) and (4.5) that

$$\begin{aligned} \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 = & O(\{r^{-\frac{1}{2}}|Du| \cdot |w_2| + r^{-\frac{3}{2}}|Dv| \cdot |Du| + r^{-\frac{5}{2}}|Du| \cdot |v|\}|w_1| \\ & + r^{-\frac{5}{2}}|w_2| \cdot |Du| \cdot |v| + r^{-\frac{3}{2}}|w_2| \cdot |Du| \cdot |Dv| + r^{-2}|Dv| + r^{-3}|v|) \\ = & O((\varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-2})|w_1| + \varepsilon(1+t)^{-2}). \end{aligned}$$

Note that, from (4.9), we have

$$\begin{aligned} |r_1(t) - r_2(t)| \leqq & |r_1(t) - t + N| + |t - r_2(t) + M| + M + N \\ \leqq & \frac{2C}{A} + M + N. \end{aligned}$$

Then it follows from (4.5), (4.8) and the assumption $\varepsilon^2 \log(1+T) < 1/A$ that

$$\begin{aligned} & \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| dr dt \\ = & O(\int_{t_{\frac{1}{B}}}^{t_1} (\varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-2}) dt \int_{r_1(t)}^{r_2(t)} |w_1| dr + \int_{t_{\frac{1}{B}}}^{t_1} \varepsilon(1+t)^{-2} dt \int_{r_1(t)}^{r_2(t)} dr) \\ = & O(\varepsilon^5 \log(1+t_1) + (\frac{2C}{A} + M + N)\varepsilon(1+t_{\frac{1}{B}})^{-1}) \\ = & O(\frac{\varepsilon^3}{A} + (\frac{2C}{A} + M + N)\varepsilon^4). \end{aligned}$$

Thus we obtain

$$I(t_1) < C_3\varepsilon + O(\varepsilon^2) < \hat{C}\varepsilon,$$

for $\varepsilon < \varepsilon_A$ if ε_A is sufficiently small.

Next we estimate v_r. We fix a point $(r, t_1) \in D_{t_1}$, then there exist λ_0 and μ_0 such that $(r, t_1) \in Z^1_{\lambda_0} \cap Z^2_{\mu_0}$. Integrating the following equality

$$\mathcal{L}_1 v_r = v_{rt} + c v_{rr} = 2cw_2,$$

along $Z^1_{\lambda_0}$ from $t_{1/B}$ to t_1, we find

$$\begin{aligned} v_r(r, t_1) - v_r(\lambda_0 + t_{\frac{1}{B}}, t_{\frac{1}{B}}) &= \int_{t_0}^{t_1} \frac{d}{dt}(v_r(r(t), t))dt \\ &= \int_{t_{\frac{1}{B}}}^{t_1} \mathcal{L}_1 v_r(r(t), t)dt \\ &= 2\int_{t_{\frac{1}{B}}}^{t_1} cw_2(r(t), t)dt \\ &= O(\int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t), t)|dt), \end{aligned}$$

where $(r(t), t) \in Z^1_{\lambda_0}$. To estimate the last integral in the above equality, we set

$$E = \{(r, t) \in D_{t_1} | (r, t) \in Z^1_\lambda \cap Z^2_\mu, \quad \lambda_0 \leqq \lambda \quad \text{and} \quad \mu \leqq \mu_0\}.$$

By the same argument to obtain (4.12), we get from (4.11)

$$\int_{Z^1_\lambda} |w_2|(dr + cdt) \leqq \int_{E\cap\{t=t_{\frac{1}{B}}\}} |w_2|dr + \iint_E \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt.$$

It follows from (4.5) and (4.6) that

$$\int_{E\cap\{t=t_{\frac{1}{B}}\}} |w_2|dr \leqq \int_{D_{t_{\frac{1}{B}}}} |w_2|dr \leqq W_2(t_{\frac{1}{B}})|r_1(t_{\frac{1}{B}}) - r_2(t_{\frac{1}{B}})| = O(\varepsilon^3).$$

The same argument to estimate the integral of $|\mathcal{L}_1 w_1 + (\partial c/\partial r)w_1|$ over D_{t_1} gives

$$\iint_E \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt \leqq \iint_{D_{t_1}} \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt = O(\varepsilon^2).$$

On the other hand, we find

$$\begin{aligned} \int_{Z^1_\lambda} |w_2|(dr + cdt) &= \int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t), t)|(\frac{dr}{dt} + c)dt \\ &= 2\int_{t_{\frac{1}{B}}}^{t_1} c|w_2(r(t), t)|dt \\ &\geqq \int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t), t)|dt, \end{aligned}$$

for sufficiently small ε. These imply

$$\int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t),t)|dt = O(\varepsilon^2). \tag{4.13}$$

Thus we obtain

$$v_r(r,t) = v_r(\lambda_0 + t_{\frac{1}{B}}, t_{\frac{1}{B}}) + O(\varepsilon^2), \tag{4.14}$$

and

$$|v_r(r,t_1)| \leqq \frac{C_1}{2}\varepsilon + O(\varepsilon^2) < \frac{\hat{C}}{2}\varepsilon.$$

Similarly we have

$$v_t(r,t_1) = v_t(\lambda_0 + t_{\frac{1}{B}}, t_{\frac{1}{B}}) + O(\varepsilon^2), \tag{4.15}$$

and

$$|v_t(r,t_1)| < \frac{\hat{C}}{2}\varepsilon.$$

Thus $\dot{V}(t_1) < \hat{C}\varepsilon$ holds. More precisely, we have for $(r(t),t) \in Z^1_{\rho(1/B)}$,

$$\partial_r^l \partial_t^m u(r(t),t) = (-1)^m \varepsilon r^{-\frac{1}{2}} \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}}) \quad \text{for} \quad l+m=1, \tag{4.16}$$

where $\rho(1/B)$ is the one in (3.23) or (3.24). Indeed, if we write $r_{1/B} = \rho(1/B)+t_{1/B}$, (3.3) and (3.23) imply

$$\begin{aligned} r_{\frac{1}{B}}^{\frac{1}{2}} \partial_r^l \partial_t^m u(r_{\frac{1}{B}}, t_{\frac{1}{B}}) =& \varepsilon(-1)^m U_\rho(\rho(\frac{1}{B}), \frac{1}{B}) + O(\varepsilon^{\frac{5}{4}}) \\ =& (-1)^m \varepsilon \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}). \end{aligned} \tag{4.17}$$

When $m=1$ and $l=0$, using (4.15) with $\lambda_0 = \rho(1/B)$ and (4.17) we obtain for $(r(t),t) \in Z^1_{\rho(1/B)}$

$$\begin{aligned} r^{\frac{1}{2}} u_t(r(t),t) =& r_{\frac{1}{B}}^{\frac{1}{2}} u_t(r_{\frac{1}{B}}, t_{\frac{1}{B}}) + O(\varepsilon^2) \\ =& -\varepsilon \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}). \end{aligned}$$

The other case shall be obtained by using (4.14) and (4.17).

Finally we estimate $w_2(r,t_1)$. We fix a point $(r,t_1) \in D_{t_1}$ and take a constant μ such that $(r,t_1) \in Z^2_\mu$. Then, it follows from (4.4), (4.8) and the assumption $\varepsilon^2 \log(1+T) < 1/A$ that for $(r(t),t) \in Z^2_\mu$,

$$\begin{aligned} w_2(r,t_1) - w_2(\mu - t_{\frac{1}{B}}, t_{\frac{1}{B}}) =& \int_{t_{\frac{1}{B}}}^{t_1} \frac{d}{dt} w_2(r(t),t)dt \\ =& \int_{t_{\frac{1}{B}}}^{t_1} \mathcal{L}_2 w_2(r(t),t)dt \\ =& O(\int_{t_{\frac{1}{B}}}^{t_1} \{\varepsilon^7(1+t)^{-1} + \varepsilon(1+t)^{-2} + \varepsilon^7|w_1(r(t),t)|\}dt) \\ =& O(\varepsilon^4 + \varepsilon^7 \int_{t_{\frac{1}{B}}}^{t_1} |w_1(r(t),t)|dt). \end{aligned}$$

By the same argument to obtain (4.13), we have

$$\int_{t_{\frac{1}{B}}}^{t_1} |w_1(r(t),t)|dt = O(\varepsilon) \qquad \text{for} \qquad (r(t),t) \in Z_\mu^2.$$

This implies

$$\begin{aligned}|w_2(r,t_1)| =& |w_2(\mu - t_{\frac{1}{B}}, t_{\frac{1}{B}})| + O(\varepsilon^4)\\ \leqq& C_3\varepsilon^3 + O(\varepsilon^4)\\ <& C\varepsilon^3,\end{aligned}$$

for $\varepsilon < \varepsilon_A$ if ε_A is sufficiently small. Thus we have finished proving (2) and then (4.5).

5. Proof of the Main Lemma.

The following lemma play an important role in the proof of Main Lemma. It will be proved in Appendix 2.

Lemma. *Let w be a solution in $[t_0, T]$ of the ordinary differential equation:*

$$\frac{dw}{dt} = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t),$$

where α_j are continuous and $\alpha_0 \geqq 0$. Let

$$K = \int_{t_0}^{T} |\alpha_2(t)|dt \exp(\int_{t_0}^{T} |\alpha_1(t)|dt).$$

If $w(t_0) > K$, $w(t)$ must satisfy

$$w(t)\exp(-\int_{t_0}^{t} \alpha_0(\tau)d\tau) \geqq \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)\exp(\int_{t_0}^{\tau} \alpha_1(\xi)d\xi)d\tau} \tag{5.1}$$

and

$$w(t)\exp(-\int_{t_0}^{t} \alpha_0(\tau)d\tau) \leqq \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)\exp(\int_{t_0}^{\tau} \alpha_1(\xi)d\xi)d\xi} \tag{5.2}$$

for $t_0 \leqq t \leqq T$.

By (4.3) and (4.5), we find that $w_1(t) = w_1(r(t),t)$ satisfies

$$\frac{d}{dt}w_1(t) = \alpha_0(t)w_1(t)^2 + \alpha_1(t)w_1(t) + \alpha_2(t) \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq T, \tag{5.3}$$

along $Z^1_{\rho(1/B)}$, where

$$\begin{aligned}\alpha_0(t) =&\{c(a_1 u_t + \frac{a_2}{2} u_r) - \frac{a_2}{2} u_t - a_3 u_r\} r^{-\frac{1}{2}},\\ \alpha_1(t) =&O(\varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-2}),\\ \alpha_2(t) =&O(\varepsilon(1+t)^{-2}).\end{aligned}$$

It follows from (4.5), (4.9) and (4.16) that

$$\begin{aligned}\alpha_0(t) =& - a\varepsilon \mathcal{F}'(\rho_0) r^{-1} + O(\varepsilon^{\frac{5}{4}} r^{-1})\\ =& - a\varepsilon \mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}}(1+t)^{-1} + (\frac{1}{r} - \frac{1}{1+t})\varepsilon)\\ =& - a\varepsilon \mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}}(1+t)^{-1}).\end{aligned}$$

Since $H_0 = -a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0) > 0$, we can assume without loss of generality that $-a\mathcal{F}'(\rho_0) > 0$ and $\mathcal{F}''(\rho_0) > 0$. This assumption guarantees $\alpha_0(t) > 0$ for sufficiently small ε. Moreover we find that

$$\begin{aligned}\exp(\pm \int_{t_{\frac{1}{B}}}^t \alpha_1(\tau) d\tau =& \exp(O(\int_{t_{\frac{1}{B}}}^t \varepsilon^4 (1+\tau)^{-1} d\tau)))\\ =& \exp(O(\varepsilon^4 \log(1+t)) + O(\varepsilon^4 \log(1+t_{\frac{1}{B}})))\\ =& \exp(O(\varepsilon))\\ =& 1 + O(\varepsilon) \qquad \text{for} \qquad t_{\frac{1}{B}} \leqq t \leqq T,\end{aligned}$$

$$\begin{aligned}K = & \int_{t_{\frac{1}{B}}}^T |\alpha_2(t)| \exp(-\int_{t_{\frac{1}{B}}}^t \alpha_1(\tau) d\tau) dt\\ =&O((1+\varepsilon^2)\varepsilon \int_{t_{\frac{1}{B}}}^T (1+t)^{-2} dt)\\ =&O(\varepsilon(1+t_{\frac{1}{B}})^{-1}) + O(\varepsilon(1+T)^{-1})\\ =&O(\varepsilon^3),\end{aligned}$$

$$\begin{aligned}&\int_{t_{\frac{1}{B}}}^t \alpha_0(\tau) \exp(\int_{t_{\frac{1}{B}}}^\tau \alpha_1(\xi) d\xi) d\tau\\ =&(1+O(\varepsilon))(-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}})) \int_{t_{\frac{1}{B}}}^t (1+\tau)^{-1} d\tau\\ =&(-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}))(\log(1+t) - \log(1+t_{\frac{1}{B}}))\\ &\text{for } t_{\frac{1}{B}} \leqq t \leqq T,\end{aligned}$$

if $\varepsilon < \varepsilon_A (< A)$. On the other hand, by (3.5) and (4.5)

$$\begin{aligned}
w_1(t_{\frac{1}{B}}) =& \frac{cv_{rr}(t_{\frac{1}{B}}) - v_{rt}(t_{\frac{1}{B}})}{2c} \\
=& \frac{1}{2} v_{rr}(t_{\frac{1}{B}}) - v_{rt}(t_{\frac{1}{B}}) + O(|D^2 v||Du|^2) \\
=& \frac{1}{2} r_{\frac{1}{B}}^{\frac{1}{2}} u_{rr}(t_{\frac{1}{B}}) - \frac{1}{2} r_{\frac{1}{B}}^{\frac{1}{2}} u_{rt}(t_{\frac{1}{B}}) + \frac{1}{2} r_{\frac{1}{B}}^{-\frac{1}{2}} u_r(t_{\frac{1}{B}}) - \frac{1}{4} r_{\frac{1}{B}}^{-\frac{1}{2}} u_t(t_{\frac{1}{B}}) \\
&+ \frac{1}{8} r_{\frac{1}{B}}^{-\frac{3}{2}} u + O(\varepsilon^6) \\
=& \frac{1}{2} r_0^{\frac{1}{2}} u_{rr}(t_{\frac{1}{B}}) - \frac{1}{2} r_{\frac{1}{B}}^{\frac{1}{2}} u_{rt}(t_{\frac{1}{B}}) + O(\varepsilon^4),
\end{aligned}$$

where $r_{1/B} = t_{1/B} + \rho(1/B)$. Using (3.2b), we obtain

$$w_1(t_{\frac{1}{B}}) = \varepsilon U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B}) + O(\varepsilon^{\frac{5}{4}}).$$

By (3.24), we have $U_{\rho\rho}(\rho(1/B), 1/B) > 0$ and therefore $w_1(t_{1/B}) > K$. Thus, applying (5.1) to $w = w_1$ with $t_0 = t_{1/B}$, we find that w_1 must satisfy

$$\begin{aligned}
&(1 + C\varepsilon) w_1(t) \\
&\geqq \frac{w_1(t_{\frac{1}{B}}) - C\varepsilon^3}{1 - (w_1(t) - C\varepsilon^3)(-a\varepsilon \mathcal{F}'(\rho_0) U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}})(\log(1+t) - \log(1 + t_{\frac{1}{B}}))} \\
&= \frac{\varepsilon U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}}}{1 - (-a\mathcal{F}'(\rho_0) U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}})(\varepsilon^2 \log(1+t) - \frac{1}{B})} \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq T,
\end{aligned}$$

where $U_{\rho\rho}(1/B) = U_{\rho\rho}(\rho(1/B), 1/B)$ and C is a constant depending only on B, f, g, ρ_0, a and M and it varies from line to line. By (3.24), we get

$$\begin{aligned}
\frac{w_1(t)}{\varepsilon} &\geqq (1 - C\varepsilon) \frac{U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}}}{1 - (-a\mathcal{F}'(\rho_0) U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}})(\varepsilon^2 \log(1+t) - \frac{1}{B})} \\
&= \frac{U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}}}{1 - \frac{\varepsilon^2 \log(1+t) - \frac{1}{B}}{\frac{1}{H_0} - \frac{1}{B}} + C(\frac{1}{A} - \frac{1}{B})\varepsilon^{\frac{1}{4}}} \qquad (5.4) \\
&= \frac{\frac{1}{H_0} \mathcal{F}''(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}}}{\frac{1}{H_0} - \varepsilon^2 \log(1+t) + C\varepsilon^{\frac{1}{8}}} \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq T,
\end{aligned}$$

if $\varepsilon < \varepsilon_A (< (1/A - 1/B)^{-8})$. Since the right term of (5.3) is positive, the follwing must hold

$$\varepsilon^2 \log(1+T) < \frac{1}{H_0} + C\varepsilon^{\frac{1}{8}}.$$

If we take ε_A such that $1/H_0 + C\varepsilon^{1/8} < 1/A$ for $\varepsilon < \varepsilon_A$, we have

$$\varepsilon^2 \log(1+T) < \frac{1}{A} \quad \text{for} \quad \varepsilon < \varepsilon_A.$$

This completes the proof of the Main Lemma.

6. The asymptotic behaviour of the solution near the blow up point

As we stated in section 2, we study the behavior of $w_1(t)$. Note that since w_1 is defined by the solution u, w_1 does not always exist.

Theorem 2*. *For any $\delta > 0$ there exists an $\varepsilon_\delta > 0$ such that $w_1(t)$ is well-defined in $t_{1/B} \leqq t \leqq t_{1/H_0-\delta}$, if $\varepsilon < \varepsilon_\delta$ and at the point $t = t_{1/H_0-\delta}$*

$$\lim_{\varepsilon\to 0}\left(\frac{1}{H_0} - \varepsilon^2\log(1+t)\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_0}\mathcal{F}''(\rho_0)$$

holds. Here we use the notation $\varepsilon^2\log(1+t_{1/H_0-\delta}) = 1/H_0 - \delta$.

As a corollary to this theorem we obtain Theorem 2 in section 2. In [2], we have proved that there exists an $\varepsilon_1(\delta) > 0$ such that for $\varepsilon < \varepsilon_1(\delta)$ the Cauchy problem (1.1), (1.2) has a smooth solution in $t_{1/B} \leqq t \leqq t_{1/H_0-\delta}$ and therefore $w_1(t)$ is well-defined in the same interval. Thus we have only to prove that for any $\eta > 0$ there exists an $\varepsilon_0(\delta,\eta) > 0$ such that for $\varepsilon < \varepsilon_0(\delta,\eta)$

$$\left|\frac{1}{H_0} - \varepsilon^2\log(1+t)\frac{w_1(t)}{\varepsilon} - \frac{1}{H_0}\mathcal{F}''(\rho_0)\right| < \eta$$

holds at $t = t_{1/H_0-\delta}$. As we stated in above, for $\varepsilon < \varepsilon_1(\delta)$ since $w_1(t)$ is well-defined, the ordinary differential equation (5.3) make sense in $t_{1/B} \leqq t \leqq t_{1/H_0-\delta}$. Thus we obtain (5.4) with $T = t_{1/H_0-\delta}$. Notice that we are able to change $\varepsilon^{1/8}$ with $\varepsilon^{1/4}$ in (5.4) because of $1/H_0 - \delta < 1/H_0$. If we take $t = t_{1/H_0-\delta}$ in (5.4),

$$\begin{aligned}
\left(\frac{1}{H_0} - \varepsilon^2\log(1+t)\right)\frac{w_1(t)}{\varepsilon} &\geqq \left(\frac{1}{H_0}\mathcal{F}''(\rho_0) - C\varepsilon^{\frac14}\right)\frac{\frac{1}{H_0} - \varepsilon^2\log(1+t)}{\frac{1}{H_0} - \varepsilon^2\log(1+t) + C\varepsilon^{\frac14}} \\
&= \frac{1}{H_0}\mathcal{F}''(\rho_0)\frac{\delta}{\delta + C\varepsilon^{\frac14}} - \frac{C\delta\varepsilon^{\frac14}}{\delta + C\varepsilon^{\frac14}} \\
&= \frac{1}{H_0}\mathcal{F}''(\rho_0) - \frac{C\varepsilon^{\frac14}}{\delta + C\varepsilon^{\frac14}} - \frac{C\delta\varepsilon^{\frac14}}{\delta + C\varepsilon^{\frac14}}
\end{aligned}$$

holds. There exists an $\varepsilon_2(\delta,\eta) > 0$ such that for $\varepsilon < \varepsilon_2(\delta,\eta)$

$$\frac{C\varepsilon^{\frac14}}{\delta + C\varepsilon^{\frac14}} + \frac{C\delta\varepsilon^{\frac14}}{\delta + C\varepsilon^{\frac14}} < \eta,$$

i.e.,

$$\left(\frac{1}{H_0} - \varepsilon^2\log(1+t)\right)\frac{w_1(t)}{\varepsilon} - \frac{1}{H_0}\mathcal{F}''(\rho_0) > -\eta$$

holds. Similarly, using (5.2) we find that there exists an $\varepsilon_3(\delta, \eta) > 0$ such that for $\varepsilon < \varepsilon_3(\delta, \eta)$

$$(\frac{1}{H_0} - \varepsilon^2 \log(1+t))\frac{w_1(t)}{\varepsilon} - \frac{1}{H_0}\mathcal{F}''(\rho_0) < \eta$$

holds. Thus if we take $\varepsilon_0(\delta, \eta) = \min(\varepsilon_1(\delta), \varepsilon_2(\delta, \eta), \varepsilon_3(\delta, \eta))$, we get for $\varepsilon < \varepsilon_0(\delta, \eta)$

$$|(\frac{1}{H_0} - \varepsilon^2 \log(1+t))\frac{w_1(t)}{\varepsilon} - \frac{1}{H_0}\mathcal{F}''(\rho_0)| < \eta,$$

which implies that Theorem 2* holds.

7. Application.

The vertical motion of nonlinear vibrating membrane is governed by the equation:

$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \qquad (x,t) \in \Omega \times (0,T). \tag{6.1}$$

The total energy $E(t)$ at time t has a form

$$E(t) = \int_\Omega (u_t^2 + \sqrt{1+|\nabla u|^2})dx,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Let a solution u to (6.1) satisfy initial condition and Diriclet or Nuemann boundary condition,

$$u(x,0) = \varepsilon f(x), \quad u_t(x) = \varepsilon g(x), \qquad x \in \Omega, \tag{6.2}$$

$$u = 0 \quad \text{ or } \quad n \cdot \nabla u = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{6.3}$$

where n stands for the outer unit normal vector to $\partial\Omega$. Then the conservation law of the energy holds:

$$E(t) = E(0).$$

For the equation of nonlinear vibrating string corresponding to one space dimension, S. Klainerman and A. Majda [8] have proved that smooth solutions with small initial data and with Diriclet or Neumann boundary condition always develop singularities in the second order derivatives in finite time.

For our problem when Ω is a ball in $\mathbb{R}^2$ with radius R, radially symmetric solutions to the initial-boundary value problem (6.1), (6.2) and (6.3) blow up in finite time, though we can not determine the radius R in advance. In fact, if we write $r = |x|$ the equation (6.1) is rewritten as

$$u_{tt} - c^2(u_r)(u_{rr} + \frac{1}{r}u_r) = \frac{u_r}{r}G(u_r)$$

with

$$c^2(u_r) = 1 - \frac{3}{2}u_r^2 + O(|u_r|^3), \quad G(u_r) = O(|u_r|^2) \qquad \text{near} \quad u_r = 0.$$

Thus applying Theorem 1 to the initial value problem (6.1) and (6.2), we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) = \frac{1}{H_*},$$

where

$$H_* = \max_{\rho\in\mathbb{R}}(\frac{3}{2}\mathcal{F}'(\rho)\mathcal{F}''(\rho)).$$

This fact implies that if we take $T > 1/H_*$, then for sufficiently small ε_0 we have

$$T_{\varepsilon_0} < \exp(\frac{T}{\varepsilon_0^2}) - 1.$$

If we take the radius R greater than $\exp(T/\varepsilon_0^2) - 1 + M$, the solution s blow up before the distervances reach the boundary. Thus the solution to (6.1) and (6.2) is also the solution to (6.1), (6.2) and (6.3) which blows up in finite time.

Appendix 1.

It remains to prove

$$|\partial_\rho^l \partial_s^m U(\rho,s)| \leqq C_{l,m,B}(1+|\rho|)^{-\frac{1}{2}-l-4m} \qquad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.8}$$

$$U(\rho,s) = 0 \qquad \text{for} \qquad \rho \geqq M, \tag{3.9}$$

for the solution $U(\rho,s)$ of the initial value problem (3.1a), (3.2a). Along the same argument to obtain (3.23) we get for $(\rho(s),s) \in \Lambda_q$

$$U_\rho(\rho(s),s) = U_\rho(q,0) = \mathcal{F}'(q) \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}. \tag{A.1}$$

Hence, by the definition of characteristic curves Λ_q, $\rho(s)$ can be written as

$$\rho(s) = q + \frac{a}{2}(\mathcal{F}'(q))^2 s \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}. \tag{A.2}$$

On the other hand, it has been known that $\mathcal{F}$ satisfies

$$\left|\frac{d^k}{d\rho^k}\mathcal{F}(\rho)\right| \leqq \widetilde{C}_k(1+|\rho|)^{-\frac{1}{2}-k} \qquad \text{for} \qquad \rho \in \mathbb{R}, \tag{A.3}$$

$$\mathcal{F}(\rho) = 0 \qquad \text{for} \qquad \rho \geqq M \tag{A.4}$$

e.g. L. Hörmander [1]. Then we have

$$\left|\frac{a}{2}(\mathcal{F}'(q))^2 s\right| \leqq \frac{|a|\widetilde{C}_1^2}{2B} \equiv C_1',$$

where the last inequality is the definition of C_1'. At first we prove (3.8) for $l=1$ and $m=0$. When $|\rho(s)| \leqq 2C_1'$, we find that for $(\rho(s), s) \in \Lambda_q$

$$\begin{aligned}|U_\rho(\rho(s), s)| =&|\mathcal{F}_\rho(q)| \leqq \widetilde{C}_1 \leqq \widetilde{C}_1(1+2C_1')^{\frac{3}{2}}(1+2C_1')^{-\frac{3}{2}} \\ \leqq&\widetilde{C}_1(1+2C_1')^{\frac{3}{2}}(1+|\rho|)^{-\frac{3}{2}}.\end{aligned}$$

When $|\rho(s)| \geqq 2C_1'$, it follows from (A.2) that

$$|q| = \left|\rho(s) - \frac{a}{2}(\mathcal{F}'(q))^2 s\right| \geqq |\rho| - C_1' \geqq \frac{1}{2}|\rho|.$$

Thus we obtain

$$|U_\rho(\rho(s), s)| = |\mathcal{F}'(q)| \leqq \widetilde{C}_1(1+|q|)^{-\frac{3}{2}} \leqq 2\sqrt{2}\widetilde{C}_1(1+|\rho|)^{-\frac{3}{2}}.$$

Therefore if we take $C_{1,0,B} = \widetilde{C}_1(1+2C_1')^{3/2} + 2\sqrt{2}\widetilde{C}_1$, we find that (3.8) is valid for $l=1$ and $m=0$. When $l=0$ and $m=0$, (3.1a) and (3.2a) imply that for any $(\rho, s) \in \mathbb{R} \times [0, 1/B]$

$$\begin{aligned}U(\rho, s) =&U(\rho, 0) + \int_0^s \frac{\partial}{\partial s} U(\rho, s) ds \\ =&\mathcal{F}(\rho) - \frac{a}{6}\int_0^s (U_\rho(\rho, s))^3 ds.\end{aligned}$$

Thus we obtain

$$\begin{aligned}|U(\rho, s)| \leqq&\widetilde{C}_0(1+|\rho|)^{-\frac{1}{2}} + \frac{|a|}{6B}C_{1,0,B}^3(1+|\rho|)^{-\frac{9}{2}} \\ \leqq&(\widetilde{C}_0 + \frac{|a|}{6B}C_{1,0,B}^3)(1+|\rho|)^{-\frac{1}{2}}.\end{aligned}$$

This implies that (3.8) is valid for $l=0$ and $m=0$ if we take

$$C_{0,0,B} = \widetilde{C}_0 + \frac{|a|}{6B}C_{1,0,B}^3.$$

Next we prove (3.8) for general $l \geqq 2$ and $m=0$. Let s $(0 \leqq s \leqq 1/B)$ be fixed arbitrarily. Then for any point (ρ, s), there exist a smooth curve $q = q_s(\rho)$ such that $(\rho, s) \in \Lambda_q$. Differentiating (A.1) with respect to ρ, we find that for $l \geqq 2$

$$\partial_\rho^l U(\rho, s) = \sum_{j=1}^{l-1} \mathcal{F}^{(j+1)}(q) \sum_{m(j) \in X} C_{m(j)}(\frac{\partial q}{\partial \rho})^{m_1(j)}(\frac{\partial^2 q}{\partial \rho^2})^{m_2(j)} \cdots (\frac{\partial^{l-1} q}{\partial \rho^{l-1}})^{m_{l-1}(j)}, \tag{A.5}$$

where

$$\begin{aligned}X = \{m(j) \in \mathbb{Z}_+^{l-1} |\ & m_1(j) + m_2(j) + \cdots + m_{l-1}(j) = j, \\ & m_1(j) + 2m_2(j) + \cdots + (l-1)m_{l-1}(j) = l-1\}.\end{aligned}$$

On the other hand, differentiating $\partial q/\partial\rho = (\partial\rho/\partial q)^{-1}$ with respect to ρ, we find that for $k \geqq 2$

$$\frac{\partial^k q}{\partial\rho^k} = \sum_{j=2}^{k} \frac{\partial^j\rho}{\partial q^j} \sum_{N(j)\in Y} C_{N(j)} \left(\frac{\partial q}{\partial\rho}\right)^{N_1(j)} \left(\frac{\partial^2 q}{\partial\rho^2}\right)^{N_2(j)} \cdots \left(\frac{\partial^{k-1} q}{\partial\rho^{k-1}}\right)^{N_{k-1}(j)}, \tag{A.6}$$

where

$$Y = \{N(j) \in \mathbb{Z}_+^{k-1} \mid N_1(j) + N_2(j) + \cdots + N_{k-1}(j) = j+1,$$
$$N_1(j) + 2N_2(j) + \cdots + (k-1)N_{k-1}(j) = k+1\}.$$

Moreover by (A.2), (A.3) and the same argument in the case $l=1$ and $m=0$, we obtain

$$\begin{aligned} \left|\frac{\partial\rho}{\partial q}\right| &\leqq \hat{C}_1, \\ \left|\frac{\partial^k\rho}{\partial q^k}\right| &\leqq \hat{C}_k(1+|\rho|)^{-3-k} \quad \text{for} \quad k \geqq 2. \end{aligned} \tag{A.7}$$

Using (A.7), we get

$$\begin{aligned} \left|\frac{\partial q}{\partial\rho}\right| &\leqq \overline{C}_1, \\ \left|\frac{\partial^k q}{\partial\rho^k}\right| &\leqq \overline{C}_k(1+|\rho|)^{-3-k} \quad \text{for} \quad k \geqq 2. \end{aligned} \tag{A.8}$$

Thus it follows from (A.5) and (A.8) that

$$\begin{aligned} |\partial_\rho^l U(\rho,s)| &\leqq C_{l,B} \sum_{j=2}^{l-1} (1+|\rho|)^{-\frac{1}{2}-j-1} \prod_{k=2}^{l-1} (1+|\rho|)^{(-k-3)m_k(j)} \\ &\leqq C_{l,B} \sum_{j=1}^{l-1} (1+|\rho|)^{-\frac{1}{2}-j-1} (1+|\rho|)^{-\sum_{k=2}^{l-1}(k-1)m_k(j) - 4\sum_{k=2}^{l-1} m_k(j)}. \end{aligned}$$

Since

$$\begin{aligned} m_2(j) + 2m_3(j) + \cdots + (l-2)m_{l-1}(j) &= l-j-1, \\ m_2(j) + m_3(j) + \cdots + m_{l-1}(j) &\geqq 0, \end{aligned}$$

we have

$$\begin{aligned} |\partial_\rho^l U(\rho,s)| &\leqq C_{l,0,B}(1+|\rho|)^{-\frac{1}{2}-j-1-l+1+j} \\ &\leqq C_{l,0,B}(1+|\rho|)^{-\frac{1}{2}-l}. \end{aligned}$$

Next we assume that (3.8) holds for any l and $0 \leqq m \leqq k-1$. Differentiating the equation (3.1a), we have

$$\partial_\rho^l \partial_s^k U(\rho,s) = \sum C \partial_\rho^{\alpha_1} \partial_s^{1+\beta_1} U(\rho,s) \partial_\rho^{\alpha_2} \partial_s^{1+\beta_2} U(\rho,s) \partial_\rho^{\alpha_3} \partial_s^{1+\beta_3} U(\rho,s),$$

where

$$\alpha_1+\alpha_2+\alpha_3=l \quad \text{and} \quad \beta_1+\beta_2+\beta_3=k-1.$$

Thus we have

$$\begin{aligned}|\partial_\rho^l\partial_s^k U(\rho,s)| &\leqq C_{l,k,B}(1+|\rho|)^{-\frac{3}{2}-4(k-1)-3-l}\\ &\leqq C_{l,k,B}(1+|\rho|)^{-\frac{1}{2}-4k-l}.\end{aligned}$$

This completes the proof of (3.8).

Finally we prove (3.9). If $\rho \geqq M$ and $(\rho,s)\in\Lambda_q$, we find $q\geqq M$ because of the uniqueness of Λ_q. It follows from (A.2) and (A.4) that

$$U_\rho(\rho,s)=\mathcal{F}'(q)=0 \quad \text{for} \quad \rho\geqq M,\ 0\leqq s\leqq \frac{1}{B}.$$

Thus we have

$$U(\rho,s)=0 \qquad \text{for} \qquad \rho\geqq M,\ 0\leqq s\leqq \frac{1}{B},$$

which implies (3.9).

Appendix 2.

Here we prove Lemma in section 5. At first we consider the case $\alpha_1(t)\equiv 0$. Let $W_1(t)$ be a solution of

$$W_1'(t)=\alpha_0(t)(W_1(t)-K)^2, \tag{A.9}$$

$$W_1(t_0)=w(t_0) \tag{A.10}$$

and set

$$W_2(t)=\int_{t_0}^t |\alpha_2(\tau)|d\tau.$$

Since $\alpha_0(t)\geqq 0$, we find that

$$W_1(t)\geqq w(t_0)>K=W_2(T)>W_2(t)$$

and that

$$\begin{aligned}(W_1(t)-W_2(t))' &=\alpha_0(t)(W_1(t)-K)^2-|\alpha_2(t)|\\ &\leqq\alpha_0(t)(W_1(t)-W_2(t))^2+\alpha_2(t),\\ W_1(t_0)-W_2(t_0) &=w(_0).\end{aligned}$$

Thus the usual comparison theorem leads to

$$W_1(t)-W_2(t)\leqq w(t). \tag{A.11}$$

By solving (A.9) and (A.10), $W_1(t)$ is represented by

$$W_1(t)=K+\frac{w(t_0)-K}{1-(w(t_0)-K)\int_{t_0}^t \alpha_0(\tau)d\tau}.$$

Substituting this equality in (A.11), we have

$$w(t) \geqq K + \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)d\tau} - W_2(t)$$
$$\geqq \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)d\tau}.$$

This implies (5.1) for $\alpha_1(t) \equiv 0$. On the other hand, if we let $W_3(t)$ be a solution of

$$W_3'(t) = \alpha_0(t)(W_3(t) + K)^2,$$
$$W_3(t_0) = w(t_0),$$

then we find

$$\begin{aligned}(W_3(t) + W_2(t))' =&\alpha_0(t)(W_3(t) + K)^2 + |\alpha_2(t)| \\ \geqq&\alpha_0(t)(W_3(t) + W_2(t))^2 + |\alpha_2(t)|, \\ W_3(t_0) + W_2(t_0) =&w(t_0).\end{aligned}$$

Since $W_3(t)$ is represented by

$$W_3(t) = -K + \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau},$$

we obtain

$$w(t) \leqq -K + \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau} + W_2(t)$$
$$\leqq \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)d\tau}.$$

this implies (5.2) for $\alpha_1(t) \equiv 0$. For the general case, setting

$$W(t) = w(t)\exp(-\int_{t_0}^{t} \alpha_1(\tau)d\tau)$$

and applying the results just proved to $W(t)$, we would obtain the inequalities which we wanted.

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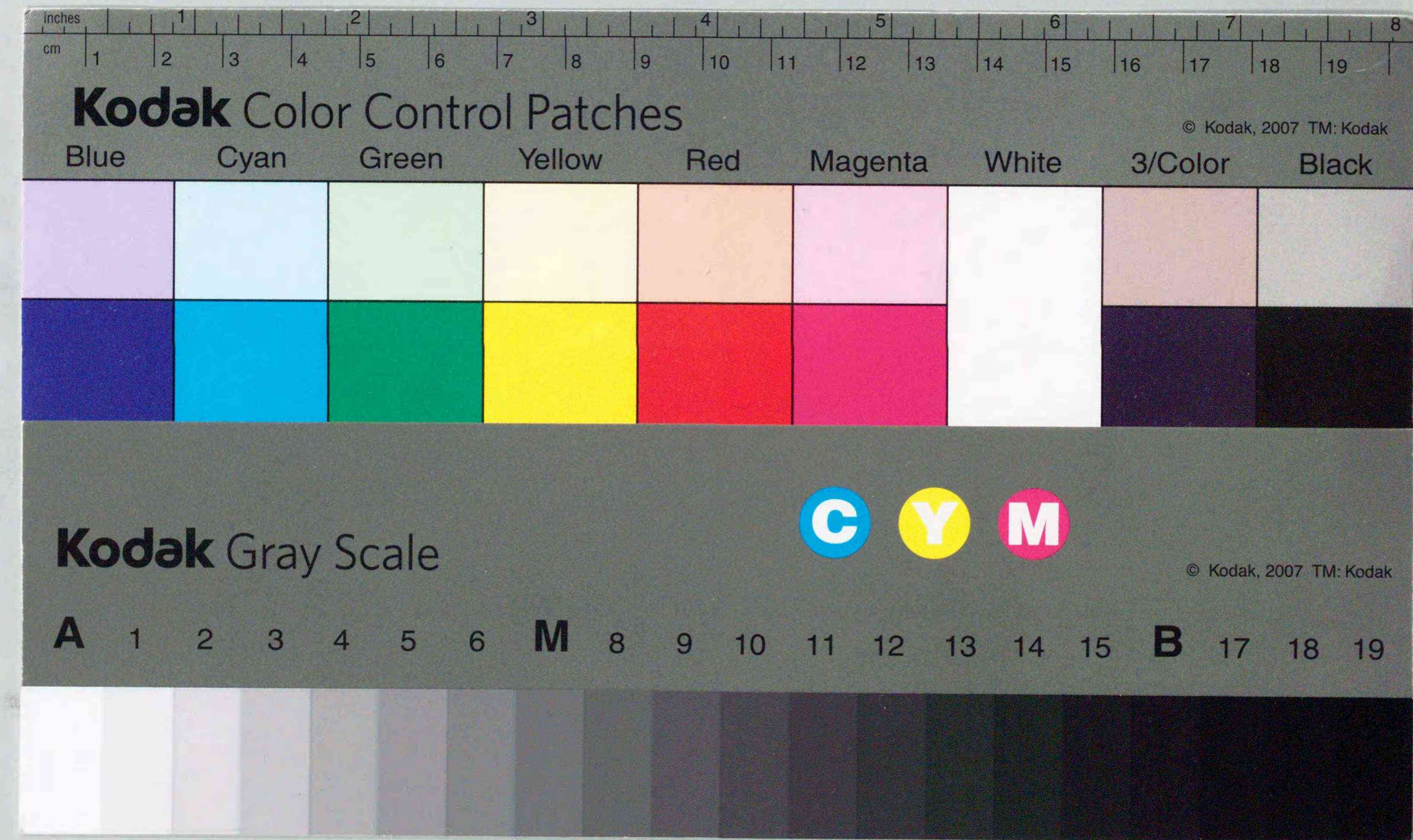
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