Two types of local triviality for the structure of a complex analytic singular foliation

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0 Introduction

A singular foliation on a complex manifold M is defined as an integrable coherent subsheaf E of the tangent sheaf of M. In this paper, we study the problem of local (analytical and topological) triviality of the singular foliation along (a subset of) its singular set $S(E)$. In general, $S(E)$ is an analytic variety, so we must stratify it to consider some triviality along the singular set.

For stratified subsets or stratified maps, the local topological triviality has been studied by a number of people and it is generally known that if the stratification satisfies the "Whitney condition" or the "Thom condition", then we have the local topological triviality along each stratum (the Isotopy Lemmas of Thom).

We first review and summarize basic definitions and facts about complex analytic singular foliations in section 1. In the next section, we observe the properties about the singular set of a singular foilation. In particular, we introduce the fundamental "Tangency Lemma" (Theorem (2.5)), which says that every vector field defining the foliation is "tangential" to the singular set $S(E)$. Using this lemma, we prove the existence of the leaf passing through each point of M (even on $S(E)$) in section 3. For the proof we use the method of the "natural Whitney stratification" of the singular set.

In section 4, we mainly explain and prove the local analytical triviality of E along each leaf (Theorem (4.1)). This kind of triviality was studied by P.Baum ([B]) for the point p such that p is a non-singular point of $S(E)$ and $\dim_p S(E) = \dim E(p) = \mathrm{rank} E - 1$. D.Cerveau also took up a similar problem from another viewpoint in [C]

*Research Fellow of the Japan Society for the Promotion of Science. This research is partially supported by The Ministry of Education, Science and Culture, Japan, Grant-in-Aid for Scientific Research.

(for the real case, see [N], [Ss] and [St]). We generalize and arrange their theory, and add new results in this article.

We also give some application and examples of singular foliations in section 4. Under the situation of the last example (4.32), we cannot obtain any information from theorem (4.1) about the structure of singular foliation E near the singular sets, because the dimension of the leaf containing each singular point is zero. For the problem of the triviality along this type of singular set, we consider another kind of triviality of the singular foliation.

We make preparation for the theorem on the second type of triviality in section 5. As another application of the Tangency Lemma, we prove, for a complex analytic singular foliation E, the existence of a Whitney stratification of the singular set $S(E)$ so that E induces a non-singular foliation on each of its strata (Theorem (5.4)).

In the last section, we study the local topological triviality along each stratum of a stratification of $S(E)$ as given in Theorem (5.4). This kind of triviality argument can be applied to the case where a stratum consists of (infinitely) many leaves. A.Kabila studied this problem for the case where the codimension of E is one and $S(E)$ is non-singular ([K]). We give, for a general singular foliation, a regularity condition and prove the local topological triviality under the condition (Theorem (6.10)).

After the preparation of the manuscript, it was informed to me that a result similar to the last one is indicated in an article of D.Trotman and L.Wilson [TW].

The contents of sections 2, 5 and 6 will be published in [Y] and the contents of sections 3 and 4 will appear in [MY]. Some of the results and the idea of the proofs in sections 3 and 4 are originally based on the master's thesis of Y.Mitera which was written in Japanese in 1989. I introduced some new methods and supplemented it with some new results in sections 3 and 4 in this article.

In the process of this work, I received many helpful suggestions and advices, especially from T.Suwa. I would like to thank him for answering my questions and for supporting me in various ways. I also thank J.-P. Brasselet, M.Kwiecinski, T.Ohmoto, A.Saeki and Y.Mitera for helpful conversations and comments.

1 Complex analytic singular foliations

First of all, we recall some generalities about complex analytic singular foliations on complex manifolds. The notation in the following is originally due to T.Suwa. For further details, see [B], [BB] and [Sw].

Let M be a (connected) complex manifold of (complex) dimension n, and let

$\mathcal{O}_M$, Θ_M and Ω_M be, respectively, the sheaf of holomorphic functions on M, the tangent sheaf and the cotangent sheaf of M.

Let E be a coherent subsheaf of Θ_M. Note that, in this case, E is *coherent* if and only if E is locally finitely generated, since Θ_M is locally free. We set

$$S(E) = \{p \in M \mid (\Theta_M/E)_p \text{ is not } (\mathcal{O}_M)_p \text{ -free}\},$$

and call it the *singular set* of E. For each point p of $S(E)$, we also say that p is a *singular point* of E. If we restrict E to a sufficiently small coordinate neighborhood U with coordinates $(z_1, z_2, \ldots, z_n)$, we can express E on U explicitly as follows:

$$(1.1) \qquad E_p = \sum_{i=1}^{m} \mathcal{O}_{M,p}\, v_i, \qquad v_i = \sum_{j=1}^{n} f_{ij}(z) \frac{\partial}{\partial z_j}, \qquad 1 \le i \le m,$$

where $f_{ij}(z)$ are holomorphic functions defined on U, and m is a non-negative integer. Then the singular set $S(E)$ is given on U by

$$S(E) \cap U = \{p \in U \mid \operatorname{rank}(f_{ij}(p)) \text{ is not maximal}\}.$$

A coherent subsheaf E of Θ_M is said to be *integrable* (or *involutive*) if for any point p of M,

$$(1.2) \qquad [E_p, E_p] \subset E_p$$

holds (where [,] denotes the *Lie bracket* of smooth vector fields).

Remark 1.3 This definition is a little different from the one by T.Suwa. According to his definition, E is integrable if and only if (1.2) holds for any point p of $M - S(E)$. If we consider only 'reduced' case (for the definition, see definition (1.5) below), the two definitions are equivalent.

We define the *rank* (we sometimes call it *dimension*) of E to be the rank of locally free sheaf $E|_{M-S(E)}$, and denote it rank$E$. Using the notation in (1.1), we can rewrite it as

$$\operatorname{rank}E = \max_{p \in M} \operatorname{rank}(f_{ij}(p)).$$

Next we give the definition of a singular foliation on M in terms of vector fields. Later, we will introduce it again from another viewpoint.

Definition 1.4 *A* (complex analytic) singular foliation *on M is an integrable coherent subsheaf E of Θ_M.*

It is clear that a singular foliation E induces a non-singular foliation on $M - S(E)$.

Definition 1.5 *Let E be a coherent subsheaf of Θ_M. We say that E is* reduced *if*

$$v \in \Gamma(U, \Theta_M),\ v|_{U-S(E)} \in \Gamma(U - S(E), E) \implies v \in \Gamma(U, E)$$

holds for every open set U in M.

By the preceding two definitions, we can consider *"reduced foliations"* in natural sense, i.e., a reduced foliation on M is a coherent subsheaf of Θ_M which is integrable and reduced.

Remark 1.6 We can check the following facts about reduced foliations:

(i) If a singular foliation E is locally free,

$$E \text{ is reduced} \iff \operatorname{codim} S(E) \geq 2 .$$

(ii) Let E be a reduced coherent subsheaf of Θ_M. Then E is integrable if (1.2) holds for every point $p \in M - S(E)$.

Next, as stated above, let us represent singular foliations in terms of holomorphic 1-forms. However, it is not so difficult to rewrite it from the viewpoint of its "dual".

Definition 1.7 *Let F be a coherent subsheaf of Ω_M. Then we set*

$$S(F) = \{p \in M \mid (\Omega_M/F)_p \text{ is not } (\mathcal{O}_M)_p \text{ -free}\},$$

and call it the singular set *of F. Each point in $S(F)$ is often called a* singular point *of F.*

Definition 1.8 *A coherent subsheaf F of Ω_M is said to be* integrable *(or* involutive*) when*

$$dF_p \subset \Omega_p \wedge F_p$$

holds for every point $p \in M - S(F)$. Moreover, the rank of F is defined to be the rank of the locally free sheaf $F|_{M-S(F)}$, and denoted $\operatorname{rank} F$.

Definition 1.9 *A* (complex analytic) singular foliation *on M is an integrable coherent subsheaf F of Ω_M.*

Definition 1.10 *Let $F(\subset \Omega_M)$ be a coherent subsheaf of Ω_M. We say that F is* reduced *if*

$$\omega \in \Gamma(U, \Omega_M),\ \omega|_{U-S(F)} \in \Gamma(U - S(F), F) \implies \omega \in \Gamma(U, F)$$

holds for every open set U in M.

In the following we discuss the relation between the two definitions, (1.4) and (1.9).

Definition 1.11 *For singular foliations $E \subset \Theta_M$ and $F \subset \Omega_M$, we set*

$$E^a = \{\omega \in \Omega_M \mid \langle v, \omega \rangle = 0 \quad \text{for all} \quad v \in E\},$$

$$F^a = \{v \in \Theta_M \mid \langle v, \omega \rangle = 0 \quad \text{for all} \quad \omega \in F\},$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between a vector field and a 1-form. Then $E^a(\subset \Omega_M)$ and $F^a(\subset \Theta_M)$ define reduced singular foliations on M. We call E^a (resp. F^a) the annihilator *of E (resp. F).*

Remark 1.12 Note that $S(E^a) \subset S(E)$ and $S(F^a) \subset S(F)$ hold.

Definition 1.13 *When $E \subset \Theta_M$ (resp. $F \subset \Omega_M$) is a singular foliation on M, $(E^a)^a$ (resp. $(F^a)^a$) is called the* reduction *of E (resp. F).*

If we use the notations given in (1.11) and (1.13), a singular foliation $E \subset \Theta_M$ (resp. $F \subset \Omega_M$) is reduced if and only if $(E^a)^a = E$ (resp. $(F^a)^a = F$). In this way we can make any singular foliation reduced by taking its reduction. If we consider only reduced foliations, then the two definitions of singular foliation stated above are equivalent, and in this occasion, moreover, there is no difference between the singular set in terms of vecter fields and that in terms of 1-forms.

2 The tangency lemma for singular foliations

In this section, we recall some basic properties of the singular set of a singular foliation, and summarize the "tangency lemma" which have been studied by P.Baum, D.Cerveau, T.Suwa and, for the real case, by T.Nagano, P.Stefan, H.Sussmann, and so on. In the preceding section we defined singular foliations from two different aspects, and observed the relations between the two definitions. We have checked that they produce "almost" the same results, so we often express singular foliations only in terms of vector fields. Hereafter, we assume $E(\subset \Theta_M)$ to be a singular foliation on a complex manifold M and set $r = \operatorname{rank} E$.

Definition 2.1 *For each point p in M, we set*

$$E(p) = \{v(p) \mid v \in E_p\},$$

where $v(p)$ denotes the evaluation of the vector field germ v at p. Note that $E(p)$ is a sub-vector space of the tangent space T_pM.

Definition 2.2 *For an integer k with $0 \leq k \leq r$, we set*

$$L^{(k)} = \{p \in M \mid \dim_{\mathbb{C}} E(p) = k\},$$

$$S^{(k)} = \{p \in M \mid \dim_{\mathbb{C}} E(p) \leq k\},$$

and set $L^{(-1)} = S^{(-1)} = \emptyset$ for convenience. Clearly we have

$$L^{(k)} = S^{(k)} - S^{(k-1)}, \qquad S^{(k)} = \bigcup_{i=0}^{k} L^{(i)}$$

for $k = 0, 1, 2, \ldots, r$.

Proposition 2.3 *$S^{(k)}$ is an analytic set and $L^{(k)}$ is a locally analytic set for every integer k with $0 \leq k \leq r$.*

Proof. If we use the notation in (1.1), $S^{(k)}$ is locally expressed on a small open set U in M as follows:

$$S^{(k)} \cap U = \{z \in U \mid \operatorname{rank}(f_{ij}(z)) \leq k\}.$$

All f_{ij} are holomorphic on U, so $S^{(k)}$ is analytic. And besides, we come to the conclusion that $L^{(k)} (= S^{(k)} - S^{(k-1)})$ is locally analytic because $S^{(k)}$ is analytic and $S^{(k-1)}$ is closed in M.

Q.E.D.

By the proposition stated above, we get the *natural filtration* which consists of analytic sets:

$$(2.4) \qquad \begin{array}{ccccccccccc} S^{(r)} & \supset & S^{(r-1)} & \supset & S^{(r-2)} & \supset \cdots\cdots \supset & S^{(1)} & \supset & S^{(0)} & \supset & S^{(-1)}. \\ \| & & \| & & & & & & & & \| \\ M & & S(E) & & & & & & & & \emptyset \end{array}$$

Theorem 2.5 (Tangency Lemma) *Let k be an integer with $0 \leq k \leq r$ and p a point in $S^{(k)}$. Then we have*

$$E(p) \subset C_p S^{(k)},$$

where $C_p S^{(k)}$ denotes the tangent cone of $S^{(k)}$ at p.

Remark 2.6 Theorem (2.5) was proved by P.Baum under the hypotheses that E is reduced, $k = r-1$ and p is a non-singular point of $S^{(k)} \left(= S^{(r-1)} = S(E)\right)$ (see [B]). For the case of real singular foliations, see [N], [Ss] and [St].

This theorem is drawn as a corollary of a theorem by D.Cerveau ([C]). In this paper, let us indicate that we can get a stronger result than (2.5) when E is reduced.

Proposition 2.7 ((Strong) Tangency Lemma) *Suppose $E(\subset \Theta_M)$ is reduced and p is a point of M. Let v be a germ in E_p and let $\{\varphi_t = \exp tv\}$ be the local 1-parameter group of transformations induced by v. For all t sufficiently close to 0, we have*

$$(\varphi_t)_* E_p = E_{\varphi_t(p)},$$

where $(\varphi_t)_*$ denotes the differential map of φ_t.

The following proof of this proposition is due to T. Suwa. We first prepare two lemmas in advance. The first one is a property which is easily drawn from the integrability of E.

Lemma 2.8 *Let v be a germ in E_p and let L_v denote the Lie derivative of v. Then we have*

$$L_v(F_p) \subset F_p,$$

where F is the annihilator of E.

Proof. Take a germ ω in F_p. For any germ u in E_p, we have

$$\langle u, L_v\omega\rangle = v(\langle u, \omega\rangle) - \langle [v, u], \omega\rangle.$$

We have $\langle u, \omega\rangle = 0$ and $\langle [v, u], \omega\rangle = 0$, since $[v, u] \in E_p$. Hence $\langle u, L_v\omega\rangle = 0$ for any $u \in E_p$, and this implies $L_v\omega \in E_p^a = F_p$.

Q.E.D.

Lemma 2.9 *Suppose that E, F, v and $\{\varphi_t\}$ are as above. For any germ u in E_p and any germ in ω in F_p, we have*

$$\frac{\partial}{\partial t} \langle (\varphi_t)_* u, \, \omega(\varphi_t(p)) \rangle = \langle (\varphi_t)_* u, \, L_v\omega(\varphi_t(p)) \rangle.$$

Proof. Choose a coordinate neighborhood U with coordinates $(z_1, z_2, \ldots, z_n)$ about p such that v, u and ω have representatives on U and that E and F have finite numbers of generators on U. Considering only for t sufficiently close to 0, we may assume that $\varphi_t(p)$ stays in U. Now we write explicitly on U as

$$v = \sum_{i=1}^{n} f_i(z)\frac{\partial}{\partial z_i}, \qquad u = \sum_{i=1}^{n} g_i(z)\frac{\partial}{\partial z_i} \quad \text{and} \quad \omega = \sum_{i=1}^{n} h_i(z)dz_i,$$

where f_i, g_i and h_i are holomorphic functions on U. Moreover, we set $\varphi_i(t, z) = z_i \circ \varphi_t(z)$ and $\varphi(t, z) = (\varphi_1(t, z), \ldots, \varphi_n(t, z))$. Then we have

$$\langle (\varphi_t)_* u, \omega(\varphi_t(z)) \rangle = \sum_{i,j=1}^{n} g_i(z) \frac{\partial \varphi_j(t, z)}{\partial z_i} h_j(\varphi(t, z)) \tag{2.10}$$

and

$$\frac{\partial \varphi_i(t, z)}{\partial t} = f_i(\varphi(t, z)) \tag{2.11}$$

for all z in a small neighborhood around p. Differentialing (2.10) with respect to t and using (2.11), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} \langle (\varphi_t)_* u, \omega(\varphi_t(z)) \rangle \\
&= \sum_{i,j=1}^{n} g_i(z) \frac{\partial^2 \varphi_j(t, z)}{\partial t \partial z_i} h_j(\varphi(t, z)) + \sum_{i,j,k=1}^{n} g_i(z) \frac{\partial \varphi_j(t, z)}{\partial z_i} \frac{\partial h_j(\varphi(t, z))}{\partial z_k} \frac{\partial \varphi_k(t, z)}{\partial t} \\
&= \sum_{i,j,k=1}^{n} g_i(z) \frac{\partial \varphi_j(t, z)}{\partial z_i} \left\{ \frac{\partial f_k}{\partial z_j}(\varphi(t, z)) h_k(\varphi(t, z)) + f_k(\varphi(t, z)) \frac{\partial h_j}{\partial z_k}(\varphi(t, z)) \right\} \\
&= \langle (\varphi_t)_* u, L_v \omega(\varphi_t(z)) \rangle .
\end{aligned}$$

Q.E.D.

Proof of (2.7). We take a coordinate neighborhood U with coordinates $(z_1, z_2, \ldots, z_n)$ about p such that v has a representative on U and that E and F have finite numbers of generators on U. In order to prove this proposition, it suffices to show that

$$(\varphi_t)_* E_p \subset E_{\varphi_t(p)} \tag{2.12}$$

hold for all t sufficiently close to 0. Once we have (2.12), then $(\varphi_t^{-1})_* E_{\varphi_t(p)} = (\varphi_{-t})_* E_{\varphi_t(p)} \subset E_p$ and thus this proposition.

Now we take two sections $u \in \Gamma(U, E)$ and $\omega \in \Gamma(U, F)$ arbitrarily. Using (2.9) repeatedly, we have

$$\frac{\partial^m}{\partial t^m} \langle (\varphi_t)_* u, \omega(\varphi_t(p)) \rangle \Big|_{t=0} = \langle u, \underbrace{L_v \cdots L_v}_{m\text{-times}} \omega \rangle$$

for all non-negative integer m, and the right-hand side of this equation is equal to zero by (2.8). So we have

$$\langle (\varphi_t)_* u, \omega(\varphi_t(p)) \rangle = 0$$

for all t sufficiently close to 0. This implies

$$(\varphi_t)_* u \in F^a_{\varphi_t(p)} = E_{\varphi_t(p)},$$

hence we have (2.12).

Q.E.D.

Now let us look back at the tangency lemma (2.5). Take a germ $v \in E_p$ and set $\varphi_t = \exp tv$. Suppose $\varphi_t(p) \notin S^{(k)}$ for some t. Then we have

$$\dim E(p) \leq k < \dim E(\varphi_t(p)),$$

which contradicts proposition (2.7). So we have $\varphi_t(p) \in S^{(k)}$ for all t sufficiently close to 0. Hence

$$v(p) = \lim_{t \to 0} \frac{\varphi_t(p) - p}{t}$$

is in the tangent cone $C_p S^{(k)}$ of $S^{(k)}$ at p.

Thus, in the case that E is reduced, theorem (2.5) is easily proved as a corollary of (2.7).

3 Existence of the integral submanifolds

Let E be a singular foliation of rank r on M. In §1, we recalled that E induces a non-singular foliation on $M - S(E)$. Thus if a point $p \in M$ does not belong to $S(E)$, it is clear that there exists an integral submanifold (of dimension r) passing through p. The main purpose in this section is to prove that there also exist integral submanifolds on the singular set $S(E)$, whose dimensions are lower than r.

Since the singular set $S(E)$ is not a smooth submanifold of M in general, we have to take a stratification of $S(E)$. However we must be careful in the choice of the stratification, because if we take a stratification too much fine, then the space $E(p)$

is not always contained in the tangent space of the stratum at p. We adopt here the famous method of *natural Whitney stratification* which is due to H.Whitney. We introduce just the essence below. (For details, see [W].)

Let A be an analytic set. We denote by $\mathrm{Sing}(A)$ the singular set of A and denote by $\mathrm{Reg}(A)$ the set of regular (i.e., non-singular) points of A. Moreover we set

$$\Sigma(A) = \mathrm{Sing}(A) \cup \{p \in \mathrm{Reg}(A) \mid \dim_p A < \dim A\},$$

where $\dim_p A$ denotes the dimension of A at each point $p \in \mathrm{Reg}(A)$. For two manifolds X and Y, we define a subset $B(X,Y)$ of X by

$$B(X,Y) = \{p \in X \mid Y \text{ is not Whitney regular over } X \text{ at } p\}.$$

Also for two analytic sets A and A' we set

$$W(A, A') = \Sigma(A) \cup B(A - \Sigma(A),\ A' - \Sigma(A')). \tag{3.1}$$

$W(A,A')$ is a analytic subset of A whose dimension is lower than $\dim A$.

Using the notation stated above, for an analytic subset A we define a family of analytic subsets $\{\Pi^i A\}_{i=0,1,2,\cdots}$ (inductively) as follows:

$$\begin{cases} \Pi^0 A = A \\ \Pi^1 A = \Sigma(A) \\ \text{For each integer } i \text{ with } i \geq 2, \\ \Pi^i A = \mathrm{Cl}_A\left(\bigcup_{j=0}^{i-2} W(\Pi^{i-1}A,\ \Pi^j A - \Pi^{j+1}A)\right), \end{cases} \tag{3.2}$$

where $\mathrm{Cl}_A(\quad)$ denotes the closure in A. Note that $\Pi^i A = \emptyset$ for sufficiently large i. Thus we have a sequence of analytic subsets of A:

$$\begin{array}{l} \Pi^0 A \supset \Pi^1 A \supset \Pi^2 A \supset \cdots\cdots \supset \Pi^l A \supset \Pi^{l+1} A \supset \cdots\cdots \supset \emptyset. \\ \ \ \ || \qquad\ \ || \\ \ \ \ A \qquad \Sigma(A) \end{array} \tag{3.3}$$

Then we set

$$\mathcal{A} = \{\Pi^i A - \Pi^{i+1} A (\neq \emptyset) \mid i = 0, 1, 2, \cdots\}. \tag{3.4}$$

By the construction of $\Pi^i A$, $\mathcal{A}$ is a Whitney stratification of A. This stratification is called the natural Whitney stratification of A. Note that each stratum of $\mathcal{A}$ is not

always connected, but if X and Y are connected components of a stratum $\Pi^i A - \Pi^{i+1} A$ then $\dim X = \dim Y$.

Now let us prepare a lemma which plays an important role in the proof of the existence of integral submanifolds.

Lemma 3.5 *Let $E(\subset \Theta_M)$ be a singular foliation on a complex manifold M and S be an analytic subset of M. Suppose that $E(p) \subset C_p S$ holds for every point $p \in S$ ($C_p S$ denotes the tangent cone of S at p, same notation as in §2). Let $\mathcal{S}$ be the natural Whitney stratification of S. Then we have $E(p) \subset T_p X$ for every point $p \in S$ where $X (\in \mathcal{S})$ is the stratum containing p.*

Proof. It is sufficient to show that

$$E(p) \subset C_p \Pi^i S \qquad \left(\text{for } \forall p \in \Pi^i S\right) \tag{3.6}$$

holds for every non-negative integer i. Suppose we have already showed (3.6). For any point $p \in S$, take the stratum $X \in \mathcal{S}$ passing through p. By the definition of the natural Whitney stratification, X can be expressed as

$$X = \Pi^i S - \Pi^{i+1} S$$

for some integer i. p belongs to $\Pi^i S - \Pi^{i+1} S$ and the singular points of $\Pi^i S$ are contained in $\Pi^{i+1} S$, hence p is a non-singular point of $\Pi^i S$. Then, by (3.6), we have

$$E(p) \subset C_p \Pi^i S = T_p \Pi^i S = T_p X,$$

which completes the proof.

In the following let us show (3.6) by the induction for i under the assumption of lemma (3.5). In the case of $i = 0$, (3.6) is nothing but the assumption of Lemma (3.5). Suppose (3.6) holds for every integer i with $0 \leq i \leq l$ and take a point $p \in \Pi^{l+1} S$ arbitrarily. Our purpose is to show that

$$E(p) \subset C_p \Pi^{l+1} S.$$

In order to do this, it is enough to prove that

$$v(p) \in C_p \Pi^{l+1} S \tag{3.7}$$

holds for any vector field germ v in the stalk of E at p. If $v(p) = 0$ then (3.7) is clearly fulfilled, so let us consider the case of $v(p) \neq 0$. Take a coordinate neighborhood U of

p with coordinates $(z_1, z_2, \ldots, z_n)$ on U such that $p = (0, 0, \ldots, 0)$. Since $v(p) \neq 0$, we may assume that the expression of v using the local coordinates $(z_1, z_2, \ldots, z_n)$ is given by

$$v = \frac{\partial}{\partial z_1}. \qquad (3.8)$$

Next, for each point $q \in U$ we set

$$L(q) = \{q' \in U \mid z_k(q') = z_k(q) \ \ (\text{for } k = 2, 3, \ldots, n)\}.$$

It may be assumed that U has been chosen such that all $L(q)$ are connected. Note that $L(q)$ is the integral curve of $v = \frac{\partial}{\partial z_1}$ passing through q. Furthermore, we set

$$D = \{z \in U \mid z_1 = 0\},$$

and let $\pi : U \longrightarrow D$ be the natural projection from U onto D (i.e. $\pi(z_1, z_2, \ldots, z_n) = (0, z_2, \ldots, z_n)$).

Our purpose was to prove (3.7) under the inductive assumptions. However, in fact, it suffices to show the following claim:

$$L(q) \cap \Pi^i S \neq \emptyset \ \Rightarrow \ L(q) \subset \Pi^i S \quad \text{holds for } i = 0, 1, \ldots, l. \qquad (3.9)$$

If we assume that (3.9) is true, then for any point $y \in \Pi^i S$ we have $y \in L(y) \cap \Pi^i S$, so (3.9) assures $L(y) \subset \Pi^i S$. This implies that the structures of $\Pi^i S \cap U$ are trivial along the z_1-axis. To be more precise, there exist analytic subsets A^i of D such that $\pi^{-1}(A^i) = \Pi^i S \cap U$ (in fact A^i coincide with $\Pi^i S \cap D$). On the other hand, the way of construction of $\Pi^i S$ given in (3.2) tells us that the local structure of $\Pi^{l+1} S$ is determined using only the local structures of $\Pi^i S$ for $i = 0, 1, \ldots, l$. Therefore the structure of $\Pi^{l+1} S \cap U$ is also trivial along the z_1-axis, i.e., there exists an analytic subset A^{l+1} of D such that $\pi^{-1}(A^{l+1}) = \Pi^{l+1} S \cap U$. Taking (3.8) into consideration, we obtain (3.7), thus the induction is completed.

From the preceding argument, all we have to do is to show (3.9) under the inductive assumptions. We set

$$L^+ = L(q) \cap \Pi^i S, \qquad L^- = L(q) - \Pi^i S.$$

Note that $L(q)$ is the disjoint union L^+ and L^-. The inductive assumption implies that the vector field v is logarithmic for $(\Pi^i S,\ q)$, so the flow generated by v preserves $(\Pi^i S,\ q)$ (see, for example, [BR] §1). This fact tells us that L^+ is open in $L(q)$. On the other hand, L^- is also open in $L(q)$ since $\Pi^i S$ is a closed set of M. Then either

L^+ or L^- must be empty by the connectedness of $L(q)$. In other words if L^+ is not empty then L^- is empty, and this is clearly equivalent to (3.9).

Q.E.D.

The following corollary is an immediate consequence from (2.5) and (3.5).

Corollary 3.10 *Let $E(\subset \Theta_M)$ be a singular foliation of rank r on a complex manifold M. Let k be an integer with $0 \leq k \leq r$ and $\mathcal{S}^{(k)}$ the natural Whitney stratification of $S^{(k)}$. Then for any stratum $X \in \mathcal{S}^{(k)}$ and each point $p \in X$ we have $E(p) \subset T_pX$.*

Now let us prove the main theorem in this section.

Theorem 3.11 (Existence of Integral Submanifolds) *There exist integral submanifolds (whose dimensions are lower than r) also on $S(E)$. To be more precise, there is a family $\mathcal{L}$ of submanifolds of M such that $M = \bigcup_{L \in \mathcal{L}} L$ is a disjoint union and that any $L \in \mathcal{L}$ and $p \in L$, we have $E(p) = T_pL$.*

Proof. For each point $p \in M$, take the unique integer k such that $p \in L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$. Let $\mathcal{S}^{(k)}$ be the natural Whitney stratification of $S^{(k)}$ and $X \in \mathcal{S}^{(k)}$ the stratum containing p. Since $S^{(k-1)}$ is closed in M, $X - S^{(k-1)}$ has the structure of a complex manifold. Corollary (3.10) implies that E induces a non-singular foliation on $X - S^{(k-1)}$ (whose rank must be k). Therefore there exists a family $\mathcal{L}_X$ which consists of k-dimensional complex submanifolds of $X - S^{(k-1)}$ such that $X - S^{(k-1)} = \bigcup_{L \in \mathcal{L}_X} L$ is a disjoint union and that any $L \in \mathcal{L}_X$ and $q \in L$, we have $E(q) = T_qL$. Then it is obvious that

$$\mathcal{L} = \bigcup_{k=0}^{r} \bigcup_{\substack{X \in \mathcal{S}^{(k)} \\ X - S^{(k-1)} = \emptyset}} \mathcal{L}_X$$

is the family of submanifolds of M which satisfies the conditions in the theorem.

Q.E.D.

Each element L in $\mathcal{L}$ is called a *leaf* of E.

4 The local analytical triviality along the leaves

In this section all the foliations we consider are assumed to be reduced. In the preceding section we proved the existence of the leaves for a singular foliation E on M. The following theorem says that the structure of a singular foliation E is locally analytically trivial along the leaf containing each point p in M.

Theorem 4.1 (Local Analytical Triviality) *Let $E(\subset \Theta_M)$ be a reduced foliation of rank r on a complex manifold M. Let k be an integer with $0 \leq k \leq r$ and p a point in $L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$. Then there exist a neighborhood D of 0 in $\mathbf{C}^{n-k}$, a singular foliation E' on D with $E'(0) = \{0\}$, a neighborhood U_p of p in M and a submersion $\pi : U_p \longrightarrow D$ with $\pi(p) = 0$ such that*

$$E|_{U_p} = \left(\pi^*(E'^a)\right)^a.$$

Proof. Take a coordinate neighborhood U of p with coordinates $(u_1, u_2, \ldots, u_n)$ on U such that $p = (0, 0, \ldots, 0)$. We denote by L_q the leaf of E containing each point $q \in U$ (the existence of the leaf has been proved in the preceding section). $p \in L^{(k)}$ implies $\dim_{\mathbf{C}} E(p) = k$, so L_p is a k-dimensional complex submanifold of M. Retaking the coordinates $(u_1, \ldots, u_n)$, we may assume

$$L_p \cap U = \{u_{k+1} = \cdots = u_n = 0\}.$$

Moreover, since $S^{(k-1)}$ is a closed subset of M and $L^{(k)} = S^{(k)} - S^{(k-1)}$, we may also assume that $U \cap S^{(k-1)} = \emptyset$.

At first, we take holomorphic vector fields $\gamma_1, \ldots, \gamma_k$ on U which satisfy the following two properties:

$$\text{(4.2)} \qquad \begin{aligned} &\text{(i)} \quad \gamma_i = \frac{\partial}{\partial u_i} \quad \text{on } L_p \cap U, \\ &\text{(ii)} \quad \gamma_i(q) \in E(q) \quad \text{for all } q \in U. \end{aligned} \qquad (i = 1, 2, \ldots, k)$$

Using these vector fields $\gamma_1, \ldots, \gamma_k$, we define a holomorphic vector field V_x on U for each $x = (x_1, \ldots, x_k, 0, \ldots, 0) \in L_p \cap U$ as follows:

$$\text{(4.3)} \qquad V_x = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_k\gamma_k.$$

Let $\{\varphi_{x,t} = \exp tV_x\}$ be the local 1-parameter group of transformations induced by V_x. For $\varepsilon > 0$ we set

$$\begin{aligned} U_{(\varepsilon)} &= \{(u_1, \ldots, u_n) \in U \mid |u_i| < \varepsilon \ (i = 1, 2, \ldots, n)\}, \\ L_{(\varepsilon)} &= L_p \cap U_{(\varepsilon)}. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, we may assume that $\varphi_{x,t}(q)$ stays in U for any $x \in L_{(\varepsilon)}$, $q \in U_{(\varepsilon)}$ and $t \in \mathbf{C}$ with $|t| < 2$. Moreover we set

$$\psi_x = \varphi_{x,1}.$$

The way of choice of ε tells us that $\psi_x\left(U_{(\varepsilon)}\right) \subset U$ for any $x \in L_{(\varepsilon)}$, thus we obtain a family of holomorphic maps $\{\psi_x : U_{(\varepsilon)} \to U\}_{x \in L_{(\varepsilon)}}$. We set

$$D = \{(u_1, \ldots, u_n) \in U \mid u_1 = u_2 = \cdots = u_k = 0\},$$

then (4.2) and (4.3) assures that ψ_x satisfies the following three properties:

$$\text{(4.4)} \qquad \text{for any } x \in L_{(\varepsilon)}, \qquad \psi_x(p) = x,$$

$$\text{(4.5)} \qquad \text{for any } q \in U_{(\varepsilon)}, \qquad \psi_p(q) = q,$$

$$\text{(4.6)} \qquad \text{for any } x \in L_{(\varepsilon)} \text{ and } q \in U_{(\varepsilon)}, \qquad \psi_x(q) \in L_q.$$

Let $h : L_{(\varepsilon)} \times D \to U$ be a map defined by $h(x, y) = \psi_x(y)$ for $x \in L_{(\varepsilon)}$ and $y \in D$. By the definition of $\psi_x(y)$, h is holomorphic. Moreover, if we consider $(u_1, \ldots, u_k)$ and $(u_{k+1}, \ldots, u_n)$ as coordinates on $L_{(\varepsilon)}$ and D respectively, h can be expressed explicitly as

$$h\left((u_1, \ldots, u_k), (u_{k+1}, \ldots, u_n)\right) = \left(f_1(u_1, \ldots, u_n), \ldots, f_n(u_1, \ldots, u_n)\right),$$

where $f_i(u_1, \ldots, u_n)$ are holomorphic functions. Then (4.4) implies

$$h\left((u_1, \ldots, u_k), (0, \ldots, 0)\right) = (u_1, \ldots, u_k, 0, \ldots, 0),$$

in other words,

$$\text{(4.7)} \qquad f_i(u_1, \ldots, u_k, 0, \ldots, 0) = \begin{cases} u_i & (1 \leq i \leq k) \\ 0 & (k+1 \leq i \leq n). \end{cases}$$

Similarly (4.5) implies

$$h\left((0, \ldots, 0), (u_{k+1}, \ldots, u_n)\right) = (0, \ldots, 0, u_{k+1}, \ldots, u_n),$$

in other words,

$$\text{(4.8)} \qquad f_i(0, \ldots, 0, u_{k+1}, \ldots, u_n) = \begin{cases} 0 & (1 \leq i \leq k) \\ u_i & (k+1 \leq i \leq n). \end{cases}$$

(4.7) and (4.8) tell us $\dfrac{\partial f_i}{\partial u_j}(0) = \delta_{ij}$, so we have

$$\det\left(\frac{\partial(f_1, \ldots, f_n)}{\partial(u_1, \ldots, u_n)}(0)\right) = 1 \ (\neq 0).$$

Hence, if we take $\varepsilon > 0$ sufficiently small and set $U_p = h(L_{(\varepsilon)} \times D)$ then $h : L_{(\varepsilon)} \times D \to U_p$ is a biholomorphic map. We define new coordinates $(z_1, \ldots, z_n)$ on U_p as follows:

$$\text{for any } q \in U_p, \qquad z_i(q) = \begin{cases} u_i(\mathrm{pr}_1 \circ h^{-1}(q)) & (1 \le i \le k) \\ u_i(\mathrm{pr}_2 \circ h^{-1}(q)) & (k+1 \le i \le n) \end{cases}$$

where pr_j denotes the projection to the j-th component. In other words,

$$\text{for any } x \in L_{(\varepsilon)} \text{ and } y \in D, \qquad z_i(h(x, y)) = \begin{cases} u_i(x) & (1 \le i \le k) \\ u_i(y) & (k+1 \le i \le n). \end{cases}$$

Clearly we have

$$\text{(4.9)} \qquad \bigcup_{x \in L_{(\varepsilon)}} \psi_x(y) = \bigcup_{x \in L_{(\varepsilon)}} h(x, y) = \{ q \in U_p \mid z_i(q) = z_i(y) \ (\text{for } i = k+1, \ldots, n) \}$$

On the other hand, it follows from (4.6) that

$$\text{(4.10)} \qquad \bigcup_{x \in L_{(\varepsilon)}} \psi_x(y) \subset L_y .$$

From (4.9) and (4.10) we have

$$\text{(4.11)} \qquad \frac{\partial}{\partial z_1}(y), \ \frac{\partial}{\partial z_2}(y), \ldots, \ \frac{\partial}{\partial z_k}(y) \in E(y).$$

Next let us construct the submersion $\pi : U_p \longrightarrow D$ and the singular foliation E' on D. We identify D with a neighborhood W of 0 in $\mathbf{C}^{n-k} = \{(w_{k+1}, \ldots, w_n)\}$, and set $\pi = \mathrm{pr}_2 \circ h^{-1}$. Note that, by the definition of $(z_1, \ldots, z_n)$, π is represented using the coordinates $(z_1, \ldots, z_n)$ on U_p and $(w_{k+1}, \ldots, w_n)$ on D as $\pi(z_1, \ldots, z_n) = (z_{k+1}, \ldots, z_n)$. It is clear that π is a holomorphic submersion from U_p onto D. Furthermore, let $\pi_* : TU_p \longrightarrow TD$ denote the push-forward of the vector fields form U_p to D. Then

$$\pi_* \left(\frac{\partial}{\partial z_i} \right) = \begin{cases} 0 & (1 \le i \le k) \\ \dfrac{\partial}{\partial w_i} & (k+1 \le i \le n). \end{cases}$$

Using π_* we define the coherent subsheaf $E' \subset \Theta_D$ by $(E')_y = \pi_*(E_y)$ for each point $y \in D$. Then we have $E'(p) = \{0\}$ since $E(p)$ is generated by $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_k}$.

In order to complete the proof, we must show that E' is integrable and $\left(\pi^*(E'^a)\right)^a = E|_{U_p}$. Let $\{v'_1, \ldots, v'_s\}$ be a system of local vector fields on D which generates E', and set

$$\text{(4.12)} \qquad v'_j = \sum_{i=k+1}^{n} a^j{}_i(w_{k+1}, \ldots, w_n) \frac{\partial}{\partial w_i} \qquad (1 \le j \le s),$$

where each $a^j{}_i$ is a holomorphic function on D. By the definition of E' and (4.11), it turns out that, for any $y = (0, \ldots, 0, y_{k+1}, \ldots, y_n) \in D$, $E(y)$ is spanned by following $k+s$ vectors:

$$\text{(4.13)} \qquad \begin{aligned} & \frac{\partial}{\partial z_1}(y), \ \frac{\partial}{\partial z_2}(y), \ \ldots, \ \frac{\partial}{\partial z_k}(y), \\ \tilde{v}_1 &= \sum_{i=k+1}^{n} a^1{}_i(y_{k+1}, \ldots, y_n) \frac{\partial}{\partial z_i}(y), \\ &\ \vdots \\ \tilde{v}_s &= \sum_{i=k+1}^{n} a^s{}_i(y_{k+1}, \ldots, y_n) \frac{\partial}{\partial z_i}(y). \end{aligned}$$

On the other hand, for any $x = (x_1, \ldots, x_k, 0, \ldots, 0) \in L_{(\varepsilon)}$ we have

$$\psi_x(z_1, \ldots, z_n) = (z_1 + x_1, \ldots, z_k + x_k, z_{k+1}, \ldots, z_n),$$

which implies

$$\text{(4.14)} \qquad (\psi_x)_* \left(\frac{\partial}{\partial z_i}(y) \right) = \frac{\partial}{\partial z_i}(\psi_x(y)) \qquad (1 \le i \le n)$$

for any $y \in D$. Moreover, proposition (2.7) says that there exists $\varepsilon > 0$ such that

$$\text{(4.15)} \qquad (\psi_x)_*(E(y)) = E(\psi_x(y))$$

holds for any $x \in L_{(\varepsilon)}$ and any $y \in D$. Then it follows from (4.13), (4.14) and (4.15) that the space $E(\psi_x(y))$ is spanned by following $k+s$ vectors:

$$\text{(4.16)} \qquad \begin{aligned} & \frac{\partial}{\partial z_1}(\psi_x(y)), \ \frac{\partial}{\partial z_2}(\psi_x(y)), \ \ldots, \ \frac{\partial}{\partial z_k}(\psi_x(y)), \\ (\psi_x)_*(\tilde{v}_1) &= \sum_{i=k+1}^{n} a^1{}_i(y_{k+1}, \ldots, y_n) \frac{\partial}{\partial z_i}(\psi_x(y)), \\ &\ \vdots \\ (\psi_x)_*(\tilde{v}_s) &= \sum_{i=k+1}^{n} a^s{}_i(y_{k+1}, \ldots, y_n) \frac{\partial}{\partial z_i}(\psi_x(y)). \end{aligned}$$

This means that for every point $q \in U_p$ a system of generators of the space $E(q)$ is given by (4.16), therefore $E|_{U_p}$ is generated by the following $k+s$ vector fields:

$$\frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2},\ \dots,\ \frac{\partial}{\partial z_k},$$

$$(4.17)\qquad \begin{aligned} v_1 &= \sum_{i=k+1}^{n} a^1{}_i(z_{k+1},\dots,z_n)\frac{\partial}{\partial z_i},\\ &\ \ \vdots\\ v_s &= \sum_{i=k+1}^{n} a^s{}_i(z_{k+1},\dots,z_n)\frac{\partial}{\partial z_i}. \end{aligned}$$

(4.12) and (4.17) tell us that if E' is not integrable then $E|_{U_p}$ is not integrable either, hence E' is integrable. Similarly, it turns out that E' is reduced from (4.12), (4.17) and the reducedness of E.

Now all we have to do to complete the proof is to show that $E|_{U_p} = \left(\pi^*(E'^a)\right)^a$. We can easily check that

$$(4.18)\qquad (E')^a = \left\{ \omega' = \sum_{i=k+1}^{n} b_i(w_{k+1},\dots,w_n)\,dw_i \ \middle|\ \begin{array}{l}\sum_{i=k+1}^{n} a^j{}_i b_i \equiv 0 \quad \cdots(*)\\ (\text{for } 1 \le \forall j \le s)\end{array} \right\}.$$

By the definition of π and the coordinates $(z_1,\dots,z_n)$, $\pi^*(dw_i) = dz_i$ for all i with $k+1 \le i \le n$, hence we have

$$\pi^*\left((E')^a\right) = \left\{ \pi^*(\omega') = \sum_{i=k+1}^{n} b_i(z_{k+1},\dots,z_n)\,dz_i \ \middle|\ (b_{k+1},\dots,b_n) \text{ satisfies } (*) \right\}.$$

In order to calculate $(\pi^*((E')^a))^a$, let us consider the condition for a holomorphic vector field ξ on U_p to belong to $(\pi^*((E')^a))^a$. We set $\xi = \sum_{l=1}^{n} c_l(z_1,\dots,z_n)\frac{\partial}{\partial z_l}$ where c_l are holomorphic functions on U_p. Then we have

$$(4.19)\qquad \xi \in \left(\pi^*\left((E')^a\right)\right)^a \iff \begin{array}{l}\sum_{l=k+1}^{n} b_l(z_{k+1},\dots,z_n)\,c_l(z_1,\dots,z_n) \equiv 0\\ \text{for any } (b_{k+1},\dots,b_n) \text{ satisfying } (*).\end{array}$$

Let

$$(4.20)\qquad c_l(z_1,\dots,z_n) = \sum_{(\alpha_1,\dots,\alpha_k)\ge(0,\dots,0)} h_l^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\, z_1{}^{\alpha_1}\cdots z_k{}^{\alpha_k}$$

be the series expansion of c_l with respect to $z_1,\dots,z_k$ (all $h_l^{(\alpha_1,\dots,\alpha_k)}$ are holomorphic functions of $z_{k+1},\dots,z_n$). Substituting (4.20) to (4.19),

$$\sum_{(\alpha_1,\dots,\alpha_k)} \left(\sum_{l=k+1}^{n} b_l\, h_l^{(\alpha_1,\dots,\alpha_k)} \right) z_1{}^{\alpha_1}\cdots z_k{}^{\alpha_k} \equiv 0,$$

thus we have

$$(4.21)\qquad \sum_{l=k+1}^{n} b_l(z_{k+1},\dots,z_n)\, h_l^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n) \equiv 0$$

for every $(\alpha_1,\dots,\alpha_k) \ge (0,\dots,0)$. For each $(\alpha_1,\dots,\alpha_k)$ we define a holomorphic vector field $\xi^{(\alpha_1,\dots,\alpha_k)}$ on W by

$$(4.22)\qquad \xi^{(\alpha_1,\dots,\alpha_k)} = \sum_{l=k+1}^{n} h_l^{(\alpha_1,\dots,\alpha_k)}(w_{k+1},\dots,w_n)\frac{\partial}{\partial w_l},$$

then (4.18) and (4.21) imply $\xi^{(\alpha_1,\dots,\alpha_k)} \in ((E')^a)^a = E'$. Since E' is generated by $v'_1,\dots,v'_s$, we can express $\xi^{(\alpha_1,\dots,\alpha_k)}$ as

$$(4.23)\qquad \xi^{(\alpha_1,\dots,\alpha_k)} = \sum_{j=1}^{s} f_j^{(\alpha_1,\dots,\alpha_k)}(w_{k+1},\dots,w_n)\, v'_j,$$

where $f_j^{(\alpha_1,\dots,\alpha_k)}$ are holomorphic functions on W. From (4.22) and (4.23),

$$\sum_{l=k+1}^{n} h_l^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\frac{\partial}{\partial z_l} = \sum_{j=1}^{s} f_j^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\, v_j$$

holds for every $(\alpha_1,\dots,\alpha_k)$. Hence we obtain

$$\begin{aligned} \xi &= \sum_{l=1}^{n} c_l(z_1,\dots,z_n)\frac{\partial}{\partial z_l}\\ &= \sum_{l=k}^{n} c_l\frac{\partial}{\partial z_l} + \sum_{l=k+1}^{n} c_l(z_1,\dots,z_n)\frac{\partial}{\partial z_l}\\ &= \sum_{l=k}^{n} c_l\frac{\partial}{\partial z_l} + \sum_{l=k+1}^{n}\left(\sum_{(\alpha_1,\dots,\alpha_k)} h_l^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\cdot z_1{}^{\alpha_1}\cdots z_k{}^{\alpha_k} \right)\frac{\partial}{\partial z_l}\\ &= \sum_{l=k}^{n} c_l\frac{\partial}{\partial z_l} + \sum_{(\alpha_1,\dots,\alpha_k)}\left(z_1{}^{\alpha_1}\cdots z_k{}^{\alpha_k}\left(\sum_{l=k+1}^{n} h_l^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\frac{\partial}{\partial z_l} \right)\right)\\ &= \sum_{l=k}^{n} c_l\frac{\partial}{\partial z_l} + \sum_{(\alpha_1,\dots,\alpha_k)}\left(z_1{}^{\alpha_1}\cdots z_k{}^{\alpha_k}\left(\sum_{j=1}^{s} f_j^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\, v_j \right)\right)\\ &= \sum_{l=k}^{n} c_l\frac{\partial}{\partial z_l} + \sum_{j=1}^{s}\left(\sum_{(\alpha_1,\dots,\alpha_k)} f_j^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\cdot z_1{}^{\alpha_1}\cdots z_k{}^{\alpha_k} \right) v_j. \quad \cdots(**) \end{aligned}$$

Note that each $\sum_{(\alpha_1,\dots,\alpha_k)} f_j^{(\alpha_1,\dots,\alpha_k)}(z_{k+1},\dots,z_n)\cdot z_1{}^{\alpha_1}\cdots z_k{}^{\alpha_k}$ appearing in $(**)$ can represent an arbitrary holomorphic function on U_p. This implies

$$\begin{aligned} \xi \in \left(\pi^*\left((E')^a\right)\right)^a &\iff \xi \text{ can be express as a linear combination}\\ &\qquad \text{of } \tfrac{\partial}{\partial z_1},\dots,\tfrac{\partial}{\partial z_l}, v_1,\dots,v_s \text{ like } (**)\\ &\iff \xi \in E|_{U_p}, \end{aligned}$$

thus we have $(\pi^*((E')^a))^a = E|_{U_p}$.

Q.E.D.

Remark 4.24 In the above proof, the fact that $E|_{U_p}$ is generated by the $k+s$ vector fields of the form (4.17) holds without assuming E is reduced ([C]). The above gives an independent proof of this in the reduced case (see prop (2.7) and the comments right before it).

As an application of theorem (4.1), we can show the following proposition.

Proposition 4.25 *If a singular foliation $E(\in \Theta_M)$ is reduced, then* $\text{codim}S(E) \geq 2$.

Remark 4.26 For the converse of this proposition, we have counterexamples. However, under the assumption that E is locally free, the converse is also true (cf. remark (1.6)).

Proof of (4.25). Suppose that E is reduced and $\text{codim}S(E) = 1$. Set $\dim_{\mathbf{C}} M = n$ and $\text{rank}E = r$. First we choose a point $p \in S(E)$ such that $p \notin \text{Sing}(S(E))$ and $\dim_p S(E) = n-1$. Take a sufficiently small neighborhood U of p and coordinates $(z_1, \dots, z_n)$ on U such that $U \cap S(E) = \{z_n = 0\}$ and $p = (0, \dots, 0)$. We set $k = \max\{\dim_{\mathbf{C}} E(q) \mid q \in U \cap S(E)\}$, then clearly $0 \leq k \leq r-1$.

Next, choose a point q in $U \cap S(E)$ such that $\dim_{\mathbf{C}} E(q) = k$. We 'shift', for simplicity, the coordinates $(z_1, \dots, z_n)$ on U so that $q = (0, \dots, 0)$. Since $S^{(k-1)} = \{x \in M \mid \dim_{\mathbf{C}} E(x) \leq k-1\}$ is a closed set, we can take a neighborhood $U_q(\subset U)$ of q so that $U_q \cap S^{(k-1)} = \emptyset$. Then we have $U_q \cap S(E) = U_q \cap L^{(k)}$ (for the definition of $L^{(k)}$, see (2.2)). Applying theorem (4.1) (or (4.17) in the proof), we can retake U_q and $(z_1, \dots, z_n)$ so that $E|_{U_q}$ is generated by the following $k+l$ holomorphic vector fields:

$$\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_k},$$

$$(4.27)\qquad \begin{aligned} v_1 &= \sum_{i=k+1}^{n} a^1{}_i(z_{k+1}, \dots, z_n)\frac{\partial}{\partial z_i}, \\ &\ \vdots \\ v_s &= \sum_{i=k+1}^{n} a^s{}_i(z_{k+1}, \dots, z_n)\frac{\partial}{\partial z_i}. \end{aligned}$$

If $s = 0$ then E gives a non-singular foliation on U_q. This contradicts $q \in S(E)$, so we have $s \geq 1$. On the other hand, $U_q \cap S(E) = U_q \cap L^{(k)}$ implies that $\dim_{\mathbf{C}} E(x) = k$

holds for every point $x \in U_q \cap S(E)$, therefore all $a^j{}_i$ appearing in (4.27) satisfy $a^j{}_i(z_{k+1}, \dots, z_{n-1}, 0) \equiv 0$. For $i = k+1, \dots, n$, we represent $a^1{}_i$ as

$$a^1{}_i(z_{k+1}, \dots, z_n) = z_n{}^{\alpha_i} \cdot b_i(z_{k+1}, \dots, z_n)$$

where $\alpha_i \in \mathbf{Z}$ and b_i are holomorphic functions such that $b_i(z_{k+1}, \dots, z_{n-1}, 0) \not\equiv 0$. Note that α_i and b_i are uniquely determined and $\alpha_i \geq 1$. We set $\alpha = \min\{\alpha_i\}$, and define a holomorphic vector field $\tilde{v_1}$ on U_q by

$$\tilde{v_1} = \sum_{i=k+1}^{n} z_n{}^{\alpha_i - \alpha}\, b_i(z_{k+1}, \dots, z_n)\frac{\partial}{\partial z_i} \quad \left(= \frac{1}{z_n{}^{\alpha}} v_1\right).$$

Then we have $\tilde{v_1}|_{U_q - S(E)} \in E|_{U_q - S(E)}$, but $\tilde{v_1} \notin E|_{U_q}$ since $\tilde{v_1} \not\equiv 0$. This contradicts that E is reduced.

Q.E.D.

Let us give some examples of singular foliations and its local analytical triviality.

Example 4.28 Let f be the holomorphic function on $M = \mathbf{C}^3$ defined by

$$f(x, y, z) = z(x^2 - y^2),$$

and ω the holomorphic 1-form on $\mathbf{C}^3$ defined by

$$\omega = df = 2xz\,dx - 2yz\,dy + (x^2 - y^2)dz.$$

The coherent subsheaf $F(\subset \Omega_M)$ generated by ω is integrable since $d\omega = ddf = 0$, so F defines a singular foliation on $\mathbf{C}^3$. $E = F^a(\subset \Theta_M)$ is generated by the following two vector fields:

$$(4.29)\qquad \begin{cases} v_1 = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} \\ v_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} - 2z\dfrac{\partial}{\partial z}. \end{cases}$$

E is reduced, and $\text{rank}E = 2$. By (4.29), $S(E) = S^{(1)} = \{xz = yz = x^2 - y^2 = 0\} = \{x = y = 0\} \cup \{z = x^2 - y^2 = 0\}$ and $S^{(0)} = \{(0,0,0)\}$. According to theorem (4.1), the structure of E is locally analytically trivial along each leaves, in particular along $\{x = y = 0\} - \{0\}$, $\{z = x - y = 0\} - \{0\}$ and $\{z = x + y = 0\} - \{0\}$.

Example 4.30 Let ω be the holomorphic 1-form on $\mathbf{C}^4 = \{(x, y, z, w)\}$ defined by

$$\omega = x(z^2 - w^2)\,dx - y(z^2 - w^2)\,dy - z(x^2 - y^2)\,dz + w(x^2 - y^2)\,dw.$$

It is easy to check that $d\omega = 4(-xz\,dx \wedge dz + xw\,dx \wedge dw + yz\,dy \wedge dz - yw\,dy \wedge dw)$. The coherent subsheaf $F(\subset \Omega_M)$ generated by ω is integrable since $\omega \wedge d\omega = 0$, so F defines a singular foliation on $\mathbf{C}^4$. $E = F^a(\subset \Theta_M)$ is generated by the following three vector fields:

$$(4.31)\qquad \begin{cases} v_1 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z} + w\dfrac{\partial}{\partial w} \\ v_2 = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} \\ v_3 = w\dfrac{\partial}{\partial z} + z\dfrac{\partial}{\partial w}. \end{cases}$$

E is reduced, and rank$E=3$. By (4.31),

$$\begin{aligned} S(E) = S^{(2)} &= \left\{x(z^2 - w^2) = y(z^2 - w^2) = z(x^2 - y^2) = w(x^2 - y^2) = 0\right\} \\ &= \{x = y = 0\} \cup \{z = w = 0\} \cup \left\{x^2 - y^2 = z^2 - w^2 = 0\right\}, \\ S^{(1)} &= \left\{x^2 - y^2 = z^2 - w^2 = xz = xw = yz = yw = 0\right\} \\ &= \left\{x^2 - y^2 = z = w = 0\right\} \cup \left\{z^2 - w^2 = x = y = 0\right\}, \\ S^{(0)} &= \{(0, 0, 0, 0)\}. \end{aligned}$$

Example 4.32 Let ω be the holomorphic 1-form on $\mathbf{C}^3$ defined by

$$\omega = y(x + y)\,dx - x(x + y)\,dy + (x^3 - y^3)\,dz.$$

It is easy to check that $d\omega = 3\{(-x - y)\,dx \wedge dy + x^2\,dx \wedge dz - y^2\,dy \wedge dz\}$. The coherent subsheaf $F(\subset \Omega_M)$ generated by ω is integrable since $\omega \wedge d\omega = 0$, so F defines a singular foliation on $\mathbf{C}^3$. $E = F^a(\subset \Theta_M)$ is generated by the following two vector fields:

$$(4.33)\qquad \begin{cases} v_1 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} \\ v_2 = y^2\dfrac{\partial}{\partial x} + x^2\dfrac{\partial}{\partial y} + (x + y)\dfrac{\partial}{\partial z}. \end{cases}$$

E is reduced, and rank$E=2$. By (4.33),

$$\begin{aligned} S(E) = S^{(1)} &= \left\{y(x + y) = x(x + y) = x^3 - y^3 = 0\right\} \\ &= \{x = y = 0\} \cup \left\{x + y = x^3 - y^3 = 0\right\} \\ &= \{x = y = 0\}, \\ S^{(0)} &= \{x = y = 0\} = S^{(1)}. \end{aligned}$$

In this case, theorem (4.1) means nothing particular about the structure of E along $S(E) = \{x = y = 0\}$, since the leaf passing through each point $p \in S(E)$ consists of only one point. In the following sections, we consider another type of the triviality of E along the singular set $S(E)$. In general, however, $S(E)$ is not always smooth, so we must stratify $S(E)$ to discuss the problem of some local triviality of E along the singular set.

5 Stratifications of the singular set

Let E be a singular foliation on M. Since the singular set $S(E)$ is analytic, we can construct the "natural Whitney stratification" of $S(E)$ (for the definition of the natural Whitney stratification, see (3.4)). However this is not enough to obtain some local triviality of E along each stratum, because the dimension of the leaf of E is not always constant on each stratum.

Example 5.1 Let f be the holomorphic function on $M = \mathbf{C}^3$ defined by

$$f(x, y, z) = x^2 - y^2(y + z^2),$$

and ω the holomorphic 1-form on $\mathbf{C}^3$ defined by

$$\omega = df = 2x\,dx - y(3y + 2z^2)dy - 2y^2z\,dz.$$

The coherent subsheaf $F(\subset \Omega_M)$ generated by ω is integrable since $d\omega = ddf = 0$, so F defines a singular foliation on $\mathbf{C}^3$. $E = F^a(\subset \Theta_M)$ is generated by the following three vector fields:

$$(5.2)\qquad \begin{cases} v_1 = y(3y + 2z^2)\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y} \\ v_2 = 2yz\dfrac{\partial}{\partial y} - (3y + 2z^2)\dfrac{\partial}{\partial z} \\ v_3 = y^2z\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial z}. \end{cases}$$

E is reduced, and rank$E = 2$. By (5.2), $S(E) = S^{(1)} = \{x = yz = y(3y + 2z^2) = 0\} = \{x = y = 0\} = \{z\text{-axis}\}$ and $S^{(0)} = \{(0,0,0)\}$. Since $S(E)$ is non-singular, $S(E) = \{z\text{-axis}\}$ is the only stratum of the natural Whitney stratification of $S(E)$, but $\dim E(p)$ is not constant on the stratum.

In the above example, in order to get a Whitney stratification such that the leaf dimension is constant on each stratum, we may separate the bad point $(0,0,0)$ from the z-axis. In this section we prove that there exists a Whitney stratification of $S(E)$ such that $\dim E(p)$ is constant on each stratum.

Definition 5.3 *Let $E(\subset \Theta_M)$ be a singular foliation of dimension r on M, and let $\mathcal{S}$ be a stratification of M. We say that $\mathcal{S}$ is* adapted *to E when the leaf dimension of E is constant on each stratum $X \in \mathcal{S}$, i.e., for any stratum $X \in \mathcal{S}$ there is an integer i with $0 \le i \le r$ such that $X \subset L^{(i)}$.*

Theorem 5.4 *Let E be a singular foliation of dimension r on M. Then there exists a Whitney stratification $\mathcal{S}$ which satisfies:*

(i) *$\mathcal{S}$ consists of finitely many strata.*

(ii) *$\mathcal{S}$ is adapted to E.*

Remark 5.5 Each stratum of the stratification $\mathcal{S}$ in (5.4) is not always connected. If we need to construct a Whitney stratification $\mathcal{S}'$ which satisfies:

(i) each stratum of $\mathcal{S}'$ is connected

(ii) $\mathcal{S}'$ is adapted to E,

it is sufficient to decompose every stratum of $\mathcal{S}$ into its connected component. The number of strata of $\mathcal{S}'$ may be infinite in general, but it is finite in the case that M is compact.

Proof of (5.4). We proceed by downward induction, i.e., we show that if we have already defined a Whitney stratification $\mathcal{S}$ which satisfies the two conditions in (5.4) on $\bigcup_{i=k+1}^{r} L^{(i)}$ then we can extend it on $\bigcup_{i=k}^{r} L^{(i)}$. At first, $S^{(r-1)}$ is closed in M, so $L^{(r)} = M - S^{(r-1)}$ is a submanifold. Hence we obtain a Whitney stratification $\tilde{\mathcal{S}}^{(r)} = \{L^{(r)}\}$ of $L^{(r)}$, which clearly satisfies the two conditions in (5.4).

Next let k be an integer with $0 \le k \le r-1$, and suppose we have already defined a Whitney stratification $\tilde{\mathcal{S}}^{(k+1)}$ of $\bigcup_{i=k+1}^{r} L^{(i)}$ which satisfies the two conditions in (5.4). Let l denote the dimension of $L^{(k)}$ as an analytic set. We define a family of analytic subsets $\{V_i\}_{-1 \le i \le l}$ (inductively) as follows:

$$
(5.6)\quad \begin{cases} V_l = L^{(k)} \\ V_{l-1} = \mathrm{Cl}_{L^{(k)}}\left(\displaystyle\bigcup_{X \in \tilde{\mathcal{S}}^{(k+1)}} W(V_l, X)\right) \\ \text{For each integer with } -1 \le i \le l-2, \\ V_i = \begin{cases} V_{i+1} & (\text{ if } \dim V_{i+1} < i+1) \\ \mathrm{Cl}_{L^{(k)}}\left[\left(\displaystyle\bigcup_{j=i}^{l-2} W(V_{i+1}, V_{j+2} - V_{j+1})\right) \bigcup \left(\displaystyle\bigcup_{X \in \tilde{\mathcal{S}}^{(k+1)}} W(V_{i+1}, X)\right)\right] \\ & (\text{ if } \dim V_{i+1} = i+1), \end{cases} \end{cases}
$$

where $\mathrm{Cl}_{L^{(k)}}(\quad)$ denotes the closure in $L^{(k)}$. (For the definition of $W(\ ,\ )$, see (3.1).) The family $\{V_i\}_{-1 \le i \le l}$ is well-defined since

$$
(5.7)\qquad \dim V_i \le i
$$

holds for each integer i with $-1 \le i \le l$ by the definition of V_i, and all V_i are analytic since $\tilde{\mathcal{S}}^{(k+1)}$ has only a finite number of strata. Note that we also have $V_{-1} = \emptyset$ by (5.7). For each integer i with $-1 \le i \le l-1$, moreover, we have $V_i \subset V_{i+1}$ because V_{i+1} is closed in $L^{(k)}$. Thus we obtain a sequence of locally analytic subsets of $L^{(k)}$:

$$
(5.8)\qquad \begin{matrix} V_l & \supset V_{l-1} \supset V_{l-2} \supset \cdots\cdots \supset V_1 \supset V_0 \supset & V_{-1}. \\ \| & & \| \\ L^{(k)} & & \emptyset \end{matrix}
$$

By (5.8),

$$
L^{(k)} = \bigcup_{i=0}^{l} (V_i - V_{i-1})
$$

turns out to be a disjoint union, so we define a partition of $\bigcup_{i=k}^{r} L^{(i)}$ by

$$
(5.9)\qquad \tilde{\mathcal{S}}^{(k)} = \tilde{\mathcal{S}}^{(k+1)} \cup \{V_i - V_{i-1} \mid 0 \le i \le l,\ \ V_i - V_{i-1} \ne \emptyset\}.
$$

This partition, in fact, gives a Whitney stratification of $\bigcup_{i=k}^{r} L^{(i)}$ with our two conditions in (5.4). Obviously $\tilde{\mathcal{S}}^{(k)}$ satisfies the two conditions in (5.4) by the definition of $\tilde{\mathcal{S}}^{(k)}$ in (5.9), so all we have to do is to show that $\tilde{\mathcal{S}}^{(k)}$ is a Whitney stratification.

First, let us check that each $V_i - V_{i-1}(\neq \emptyset)$ is a submanifold of M. By (3.1) and (5.6) we have $V_{i-1} \supset \Sigma(V_i)$, hence

$$V_i - V_{i-1} \subset V_i - \Sigma(V_i)$$

holds. Since $V_i - \Sigma(V_i)$ is a submanifold of M and V_{i-1} is closed in V_i, $V_i - V_{i-1}$ is also a submanifold.

Next we check the Whitney regularity of $\tilde{\mathcal{S}}^{(k)}$. Take two strata $X, Y \in \tilde{\mathcal{S}}^{(k)}$ $(X \neq Y)$.

(Case 1) If $X, Y \in \tilde{\mathcal{S}}^{(k+1)}$, the Whitney regularity between X and Y holds by the inductive assumption.

(Case 2) If $X \notin \tilde{\mathcal{S}}^{(k+1)}$ and $Y \in \tilde{\mathcal{S}}^{(k+1)}$, $X = V_i - V_{i-1}$ holds for an integer i $(0 \leq i \leq l)$. X is a subset of $S^{(k)}$ since X is contained in $L^{(k)}$, and $S^{(k)}$ is closed in M, so we have $\mathrm{Cl}_M(X) \subset S^{(k)}$. On the other hand $Y \in \tilde{\mathcal{S}}^{(k+1)}$ implies $Y \subset M - S^{(k)}$, hence we have $\mathrm{Cl}_M(X) \cap Y = \emptyset$. Therefore there is no problem about the Whitney regularity of X over Y.

Next, let p be an arbitrary point in $X = V_i - V_{i-1}$. Then we have

$$p \notin W(V_i, Y) \tag{5.10}$$

by $p \in V_i$, $p \notin V_{i-1}$ and the definition of V_{i-1}. (3.1) and (5.10) imply

$$p \notin \Sigma(V_i), \tag{5.11}$$

$$p \notin B\left(V_i - \Sigma(V_i),\, Y\right). \tag{5.12}$$

We can rewrite (5.11) as

$$p \in V_i - \Sigma(V_i), \tag{5.13}$$

so it turns out that Y is Whitney regular at p over $V_i - \Sigma(V_i)$ by (5.12) and (5.13). Since $V_i - V_{i-1}$ is a submanifold of $V_i - \Sigma(V_i)$, we also find Y to be Whitney regular at p over $V_i - V_{i-1}(= X)$. This implies the Whitney regularity of Y over X.

(Case 3) If $X, Y \notin \tilde{\mathcal{S}}^{(k+1)}$, we can take two integers i, j $(0 \leq i, j \leq l)$ such that $X = V_i - V_{i-1}$ and $Y = V_j - V_{j-1}$. We may assume $i < j$. First, Y is Whitney regular over X because (5.6) says that V_{i-1} contains all points in V_i at which $Y (= V_j - V_{j-1})$ is not Whitney regular over $V_i - \Sigma(V_i)$, so those points cannot remain on $V_i - V_{i-1}(= X)$. In order to check the Whitney regularity of X over Y, it suffices to show

$$\mathrm{Cl}_M(X) \cap Y = \emptyset. \tag{5.14}$$

Since V_i is closed in $L^{(k)}$ by (5.6), we have

$$\mathrm{Cl}_{L^{(k)}}(X) \subset \mathrm{Cl}_{L^{(k)}}(V_i) = V_i \subset V_{j-1}.$$

Moreover using $Y = V_j - V_{j-1}$ and $Y \subset L^{(k)}$ yields

$$\begin{aligned} \mathrm{Cl}_{L^{(k)}}(X) \cap Y &= \emptyset, \\ \left(M - L^{(k)}\right) \cap Y &= \emptyset, \end{aligned} \tag{5.15}$$

and on the other hand we can rewrite $\mathrm{Cl}_M(X)$ as

$$\begin{aligned} \mathrm{Cl}_M(X) &= \left(\mathrm{Cl}_M(X) \cap L^{(k)}\right) \cup \left(\mathrm{Cl}_M(X) \cap \left(M - L^{(k)}\right)\right) \\ &= \mathrm{Cl}_{L^{(k)}}(X) \cup \left(\mathrm{Cl}_M(X) \cap \left(M - L^{(k)}\right)\right). \end{aligned} \tag{5.16}$$

Then (5.14) is an immediate consequence of (5.15) and (5.16).

Q.E.D.

Now we give some examples of singular foliations and observe the stratifications we mentioned in theorem (5.4).

Example 5.17 Let f be the holomorphic function on $M = \mathbf{C}^3$ defined by

$$f(x, y, z) = z(x^2 - y^3),$$

and ω the holomorphic 1-form on $\mathbf{C}^3$ defined by

$$\omega = df = 2xz\,dx - 3y^2 z\,dy + (x^2 - y^3)dz.$$

The coherent subsheaf $F(\subset \Omega_M)$ generated by ω is integrable since $d\omega = ddf = 0$, so F defines a singular foliation on $\mathbf{C}^3$. $E = F^a(\subset \Theta_M)$ is generated by the following two vector fields:

$$\left\{ \begin{aligned} v_1 &= 3y^2\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} \\ v_2 &= 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - 6z\frac{\partial}{\partial z}. \end{aligned} \right. \tag{5.18}$$

E is reduced, and $\mathrm{rank}E = 2$. By (5.18), $S(E) = S^{(1)} = \{xz = yz = x^2 - y^3 = 0\} = \{x = y = 0\} \cup \{z = x^2 - y^3 = 0\}$ and $S^{(0)} = \{(0,0,0)\}$. In this case, a Whitney stratification $\mathcal{S}$ of M defined by

$$\mathcal{S} = \{M - S(E),\ X_1,\ X_2,\ \{0\}\} \qquad \begin{pmatrix} X_1 = \{x = y = 0\} - \{0\} \\ X_2 = \{z = x^2 - y^3 = 0\} - \{0\} \end{pmatrix}$$

meets the requirements in theorem (5.4).

The following example tells us that a Whitney stratification on M which satisfies (5.4) cannot generally be obtained by adding some strata to the natural Whitney stratification of $S^{(0)}$.

Example 5.19 Let ω be a holomorphic 1-form on $M = \mathbf{C}^3$ defined by

$$\omega = 2xz^2\,dx - 2yz\,dy + y^2\,dz,$$

and $F(\subset \Omega_M)$ a coherent subsheaf generated by ω. $E = F^a(\subset \Theta_M)$ is generated by the following two vector fields:

$$(5.20)\qquad \begin{cases} v_1 = y\dfrac{\partial}{\partial y} + 2z\dfrac{\partial}{\partial z} \\ v_2 = y\dfrac{\partial}{\partial x} + xz\dfrac{\partial}{\partial y} \end{cases}.$$

E is integrable since $[v_1, v_2] = v_2$, so E defines a singular foliation on $\mathbf{C}^3$. E is reduced, and $\mathrm{rank} E = 2$. By (5.20), $S(E) = S^{(1)} = \{y = xz = 0\} = \{x\text{-axis}\} \cup \{z\text{-axis}\}$ and $S^{(0)} = \{y = z = 0\} = \{x\text{-axis}\}$. In this situation, a stratification $\mathcal{S}'$ of M defined by

$$\mathcal{S}' = \left\{M - S(E),\ S(E) - S^{(0)},\ S^{(0)}\right\}$$

satisfies the two conditions in theorem (5.4), but this is not a Whitney stratification (and E is not trivial along $S^{(0)}$). A Whitney stratification $\mathcal{S}$ of M which meets the requirements in theorem (5.4) is given by

$$\mathcal{S} = \{M - S(E),\ X_1,\ X_2,\ \{0\}\} \qquad \left(\begin{array}{l} X_1 = \{z\text{-axis}\} - \{0\} \\ X_2 = \{x\text{-axis}\} - \{0\} \end{array}\right).$$

6 Local topological triviality of singular foliations

In this section we examine the local topological triviality of a singular foliation along its singular set. Let E be a singular foliation on M and $\mathcal{S}$ a stratification of M. For the topological triviality of E along each stratum in $\mathcal{S}$, it is necessary that $\mathcal{S}$ is adapted to E as stated in the preceding section. We consider only stratifications which is adapted to E hereafter. Note that theorem (5.4) assures that there always exists a stratification adapted to E with a stronger condition (Whitney regularity).

To tell the consequence at first, E is topologically locally trivial along each stratum in $\mathcal{S}$ if $\mathcal{S}$ satisfies the "foliated Verdier condition" which will be mentioned later. We

begin this section by recalling basic concepts to give the precise definition of the local topological triviality for singular foliations. For more details, see, for example, [GWPL] pp 41-50.

Definition 6.1 *Let X be a submanifold of M. A tubular neighborhood of X is a triple $(T,\ \pi,\ \rho)$ which satisfies:*

(i) *T is a neighborhood of X in M.*

(ii) *$\pi : T \longrightarrow X$ is a submersion (with $\pi|_X = id_X$).*

(iii) *$\rho : T \longrightarrow \mathbf{R}$ is a C^∞-function.*

(iv) *Let $\pi_X : N_X \longrightarrow X$ denote the normal bundle of X in M, and let Z denote the image of the zero section of N_X. Then there exist a neighborhood D of Z in N_X and a diffeomorphism $\varphi : D \longrightarrow T$ such that*

$$\begin{array}{ccc} D & \xrightarrow{\ \varphi\ } & T \\ & {\scriptstyle \pi_X|_D} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & X & \end{array}$$

commutes.

(v) *Let $\tau : N_X \longrightarrow \mathbf{R}$ denote the distance function to Z which is determined by a Riemannian metric induced on N_X. Then*

$$\begin{array}{ccc} D & \xrightarrow{\ \varphi\ } & T \\ & {\scriptstyle \tau|_D} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathbf{R} & \end{array}$$

commutes.

Definition 6.2 *Let M be a differentiable manifold of dimension n and A a subset of M. For a Whitney stratification $\mathcal{A}$ of A, we set*

$$A^i = \bigcup_{\substack{X\in\mathcal{A} \\ \dim X = i}} X \qquad (0 \le i \le n),$$

i.e., A^i *is the union of all strata of dimension* i. *Each* $A^i(\neq \emptyset)$ *is an* i*-dimensional submanifold in* M *since* $\mathcal{A}$ *is a Whitney stratification. A family of a tubular neighborhood of each* $A^i(\neq \emptyset)$

$$\mathcal{T} = \left\{(T^i,\ \pi^i,\ \rho^i)\right\}_{0\leq i\leq n}$$

is called a tubular neighborhood system *of* $\mathcal{A}$. *A tubular neighborhood system* $\mathcal{T} = \{(T^i,\ \pi^i,\ \rho^i)\}$ *of* $\mathcal{A}$ *is said to be* controlled *if for all integers* i, j $(i < j)$ *there exist a neighborhood* U^i *of* A^i *in* T^i *and a neighborhood* U^j *of* A^j *in* T^j *such that*

$$\pi^i \circ \pi^j = \pi^i \quad , \tag{6.3}$$

$$\rho^i \circ \pi^j = \rho^i \tag{6.4}$$

hold on $U^i \cap U^j$.

It is generally known that every Whitney stratification admits a controlled tubular neighborhood system.

Now we give the definition of the local topological triviality for singular foliations. In the following we shall fix a Riemannian metric of M.

Definition 6.5 *Let* M *be a complex manifold of dimension* n *and* $E(\subset \Theta_M)$ *a singular foliation on* M. *Also, let* X *be a submanifold in* M *and set* $l = \dim_{\mathbf{C}} X$. *Suppose* $X \subset L^{(k)}$, *i.e., the leaf dimension of* E *is constant on* X. E *is said to be* topologically locally trivial *along* X *when for any point* $p \in X$ *there exist*

$$\begin{array}{rcl} (T,\ \pi,\ \rho) & : & \text{a tubular neighborhood of } X, \\ U_p & : & \text{a sufficiently small neighborhood of } p \text{ in } M, \\ D & : & \text{a small neighborhood around } 0 \text{ in } \mathbf{C}^{n-l}, \\ E' & : & \text{a singular foliation on } D, \\ h & : & \text{a homeomorphism from } U_p \text{ onto } (X \cap U_p) \times D \end{array}$$

such that

(i) $h(x) = (x, 0)$ *holds for any* $x \in X \cap U_p$.

(ii) $h|_{U_p - X}$ *transforms the leaves defined by* E *into the product of* $X \cap U_p$ *and the leaves defined by* E'.

(iii) *Let* $pr_1 : (X \cap U_p) \times D \longrightarrow X$ *denote the natural projection to the first component, then*

$$\begin{array}{ccc} U_p & \xrightarrow{\quad h \quad} & (X \cap U_p) \times D \\ & {\scriptstyle \pi|_{U_p}} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & X & \end{array}$$

commutes.

Remark 6.6 In this paper we consider only complex analytic foliations, so the trivialization $h : U_p \longrightarrow (X \cap U_p) \times D$ is in practice a diffeomorphism.

Next, let us introduce the foliated Verdier condition for a stratification of $S(E)$. In order to define the condition, it is necessary to refer to the notion of the distance between two vector subspaces. The distance is generally defined by measuring the angle between V and W.

Definition 6.7 *Let* V, W *be two vector subspaces of a finite-dimensional inner product space. We define the* distance *between* V *and* W *by*

$$\delta(W, V) = \sup_{\substack{u \in W^{\perp} - \{0\} \\ v \in V - \{0\}}} \frac{|\langle u, v\rangle|}{\|u\| \cdot \|v\|}$$

where $\|\ \cdot\ \|$ *denotes the norm induced by the inner product* $\langle\ \ ,\ \ \rangle$.

Note that $\delta(W, V)$ is not always equal to $\delta(V, W)$. Clearly we have $\delta(W, V) \in [0, 1]$, and we can also express $\delta(W, V)$ as follows:

$$\delta(W, V) = \sup_{v \in V - \{0\}} \inf_{w \in W - \{0\}} \sin\langle\langle v, w\rangle\rangle,$$

where $\langle\langle v, w\rangle\rangle$ denotes the angle between v and w.

Remark 6.8 It is easy to check that $\delta(W, V)$ satisfies the following properties.

(i) $\quad \delta(W, V) = 0 \iff V \subset W$.

(ii) $\quad \dim V = \dim W \implies \delta(V, W) = \delta(W, V)$.

(iii) $\delta(W,V)=1 \iff$ there exists a non-zero vector $v\in V$ such that $\langle v,w\rangle=0$ for all $w\in W$

$\iff V\cap W^{\perp}\neq\{0\}$.

(iv) $\dim W<\dim V \implies \delta(W,V)=1$.

Now, we are ready to define the foliated Verdier condition for a stratification adapted to E. In the following we identify tangent spaces of nearby points by parallel translation determined by the Riemannian metric.

Definition 6.9 *Let $E(\subset\Theta_M)$ be a singular foliation on M and let X be a submanifold in M such that $X\subset L^{(k)}$, i.e., the leaf dimension of E is constant on X. Let p be a point in X. We say E* satisfies the foliated Verdier condition at p over X *when there exist a tubular neighborhood $(T,\ \pi,\ \rho)$ of X, a neighborhood U_p around p contained in T, and a real number $\lambda>0$ such that*

$$\delta\left(E(y),\ T_pX\right)\le\lambda\cdot\rho(y)$$

hold for all $y\in U_p-X$. If E satisfies the foliated Verdier condition over X at every point $p\in X$, then E is simply said to satisfy the foliated Verdier condition over X. *Moreover, a stratification $\mathcal{S}$ adapted to E is called a* foliated Verdier stratification *if E satisfies the foliated Verdier condition over all strata $X\in\mathcal{S}$.*

We have the following "isotopy lemma" for singular foliations.

Theorem 6.10 *Let $E(\subset\Theta_M)$ be a singular foliation of dimension r on M. Suppose $\mathcal{S}$ is a foliated Verdier stratification. Then E is topologically locally trivial along each stratum $X\in\mathcal{S}$.*

We introduce here a lemma for the proof of theorem (6.10). This lemma, which is the most essential part in the proof of (6.10), says that the foliated Verdier condition assures that any continuous vector field on a stratum X can be "lifted" onto each stratum sufficiently close to X.

Lemma 6.11 *Let $E(\subset\Theta_M)$ be a singular foliation of dimension r on M and let $\mathcal{S}$ be a foliated Verdier stratification. Let X,Y be two strata of $\mathcal{S}$ such that $X\cap\overline{Y}\neq\emptyset$. Given a real continuous vector field $v:X\longrightarrow TX$ such that $v(x)\neq 0$ for all $x\in X$. Then for any tubular neighborhood $(T,\ \pi,\ \rho)$ of X, we can construct a continuous extension ξ of v on $U\cap Y$ (where U is a sufficiently small neighborhood of X) so that the following conditions are fulfilled:*

(i) *$\pi_*\circ\xi=v\circ\pi$ holds on $U\cap Y$.*

(ii) *$\xi(y)\in E(y)$ hold for all $y\in U\cap Y$, i.e., ξ is tangent to the leaves defined by E at all point y in $U\cap Y$.*

(iii) *Let $\{\varphi_t=\exp t\xi\}$ be the local 1-parameter group of transformations induced by ξ. Then for all t sufficiently close to 0 and all point $y\in U\cap Y$, we have $\rho\left(\varphi_t(y)\right)>0$, i.e., the (local) integral curve containing y does not meet X.*

In the proof of this lemma, we use some basic facts about linear algebra. Let $\mathcal{V}$ be a finite-dimensional vector space with an inner product $\langle\ ,\ \rangle$, and let $V,\ W$ be two vector subspaces of $\mathcal{V}$. We set $K=W\cap V^{\perp}$ and $J=W\cap K^{\perp}$, and we denote by $pr_V:\mathcal{V}\longrightarrow V$ the orthogonal projection to V. Then we have the following properties.

(i) $pr_V|_J:J\longrightarrow V$ is injective.

(ii) $pr_V|_J:J\longrightarrow V$ is surjective $\iff V\cap W^{\perp}=\{0\}$.

(iii) $\delta(W,V)=\delta(J,V)$.

Proof of (6.11). Set $\dim_{\mathbf{R}}X=2l$. We will give below how to determine $\xi(y)$ for all point $y\in Y$ around a fixed point $p\in X$. Since we may only consider y sufficiently close to X, we may discuss under the following situation:

$$\begin{array}{rcl}
U_p &:& \text{a coordinate neighborhood around } p\ ,\\
(x_1,\ldots,x_{2n}) &:& \text{real coordinates on } U_p \text{ such that } x(p)=(0,0,\ldots,0)\\
&& \text{and } X\cap U_p=\{x_{2l+1}=x_{2l+2}=\cdots=x_{2n}=0\},\\
\pi &:& U_p \longrightarrow X\\
&& (x_1,x_2,\ldots,x_{2n})\longmapsto(x_1,x_2,\ldots,x_{2l},0,0,\ldots,0),\\
\rho &:& U_p \longrightarrow \mathbf{R}\\
&& (x_1,x_2,\ldots,x_{2n})\longmapsto\sqrt{{x_{2l+1}}^2+{x_{2l+2}}^2+\cdots+{x_{2n}}^2}\ .
\end{array}$$

For any point $y\in U_p$, the vectors $\{(\frac{\partial}{\partial x_1})_y,\ (\frac{\partial}{\partial x_2})_y,\ldots,\ (\frac{\partial}{\partial x_{2n}})_y\}$ form an orthonormal basis of T_yM. We may also assume that there exists $\lambda>0$ such that

$$\delta\left(E(y),\ T_pX\right)\le\lambda\cdot\rho(y) \tag{6.12}$$

hold for all $y\in U_p\cap Y$. For the sake of simplicity, we put $q=\pi(y)$.

Let ψ_y be the linear map from T_qX to T_yM defined by

$$\psi_y\left(\left(\frac{\partial}{\partial x_i}\right)_q\right) = \left(\frac{\partial}{\partial x_i}\right)_y \qquad (i = 1, 2, \ldots, 2l),$$

i.e., $\psi_y(u)$ is the parallel displacement of u. Then the vector field η on $U_p \cap Y$ determined by $\eta(y) = \psi_y(v(q))$ is clearly a continuous extension of v and satisfies the first and third conditions in (6.11), but does not meet the requirement of the second condition. So we will modify η so that $\eta(y)$ is tangent to the leaf defined by E at each y.

We also define two sub-vector spaces of T_yM by

$$\begin{aligned} K(y) &= E(y) \cap \ker\left(\pi_*|_{T_yM}\right) \quad \left(= E(y) \cap (\psi_y(T_qX))^\perp\right) \\ J(y) &= E(y) \cap K(y)^\perp. \end{aligned}$$

By (6.12), we have

$$\delta\left(E(y),\, T_qX\right) < \frac{1}{2} \tag{6.13}$$

for all y sufficiently close to X, so we may assume (6.13) hold for all $y \in U_p \cap Y$. Then we have $T_qX \cap E(y)^\perp = \{0\}$ from (6.8), thus $\pi_*|_{J(y)} : J(y) \longrightarrow T_qX$ is a linear isomorphism. We define here a linear map $L_y : T_qX \longrightarrow T_yM$, which gives the modification for η, by

$$L_y = \left(\pi_*|_{J(y)}\right)^{-1} - \psi_y.$$

Then we have

$$\begin{aligned} \delta(E(y), T_qX) &= \delta(J(y), T_qX) \\ &= \delta(T_qX, J(y)) \\ &= \sup_{w \in J(y)-\{0\}} \inf_{u \in T_qX - \{0\}} \sin\langle\langle w,\, u\rangle\rangle \\ &= \sup_{w \in J(y)-\{0\}} \sin\langle\langle w,\, \pi_*(w)\rangle\rangle \\ &= \sup_{w \in J(y)-\{0\}} \tan\langle\langle w,\, \pi_*(w)\rangle\rangle \cdot \cos\langle\langle w,\, \pi_*(w)\rangle\rangle \\ &= \sup_{w \in J(y)-\{0\}} \frac{\|w - \psi_y(\pi_*(w))\|}{\|\psi_y(\pi_*(w))\|} \cdot \cos\langle\langle w,\, \pi_*(w)\rangle\rangle \\ &= \sup_{w \in J(y)-\{0\}} \frac{\left\|L_y(\pi_*(w))\right\|}{\|\pi_*(w)\|} \cdot \cos\langle\langle w,\, \pi_*(w)\rangle\rangle . \end{aligned} \tag{6.14}$$

Now we define the vector field $\xi(y)$, which we are asking for, by

$$\xi(y) = \psi_y(v(q)) + L_y(v(q)) \qquad \left(= \left(\pi_*|_{J(y)}\right)^{-1}(v(q))\right)$$

for all $y \in U_p \cap Y$. It is clear that ξ satisfies (i) and (ii) in (6.11), so let us check (iii) and that ξ is a continuous extension of v.

Take a sufficiently small $\varepsilon > 0$ and set $I = (-\varepsilon, \varepsilon)$. Suppose that there exists $y \in U_p \cap Y$ such that $\rho(\varphi_t(y)) = 0$ for some $t \in I$. Since $\rho(\varphi_0(y)) = \rho(y) > 0$, we can take a real number $t_0 \neq 0$ such that $\rho(\varphi_t(y)) > 0$ for all $t \in (-|t_0|, |t_0|)$ and $\rho(\varphi_{t_0}(y)) = 0$. We may assume $t_0 > 0$. For the sake of simplicity, we put $f(t) = \rho(\varphi_t(y))$.

Set $\varphi_t(y) = (y_1(t), y_2(t), \ldots, y_{2n}(t))$. Since $\varphi_t = \exp t\xi$, we have

$$\xi(\varphi_t(y)) = \sum_{i=1}^{2n} {y_i}'(t) \frac{\partial}{\partial x_i}.$$

Then for all $t \in (0, t_0)$ we obtain

$$\begin{aligned} \left|\frac{df}{dt}(t)\right| &= \left|\sum_{i=1}^{2n} \frac{\partial \rho}{\partial x_i}(\varphi_t(y)) \cdot \frac{dy_i}{dt}(t)\right| \\ &= \left|\sum_{i=2l+1}^{2n} \frac{y_i(t)}{\sqrt{y_{2l+1}(t)^2 + \cdots + y_{2n}(t)^2}} \cdot {y_i}'(t)\right| \\ &= \left|\sum_{i=2l+1}^{2n} \frac{y_i(t)}{\rho(\varphi_t(y))} \cdot {y_i}'(t)\right| \\ &\leq \sqrt{\sum_{i=2l+1}^{2n} \left(\frac{y_i(t)}{\rho(\varphi_t(y))}\right)^2} \cdot \sqrt{\sum_{i=2l+1}^{2n} {y_i}'(t)^2} \\ &= 1 \cdot \left\| L_{\varphi_t(y)}\left(v(\pi(\varphi_t(y)))\right) \right\| \\ &= \left\| L_{\varphi_t(y)}(v(q_t)) \right\|, \end{aligned} \tag{6.15}$$

where $q_t = \pi(\varphi_t(y))$. On the other hand, (6.14) tells us that for all y we have

$$\delta(E(y), T_qX) \;\geq\; \frac{\|L_y(v(q))\|}{\|v(q)\|} \cdot \cos\left\langle\left\langle \left(\pi_*|_{J(y)}\right)^{-1}(v(q)),\, v(q)\right\rangle\right\rangle, \tag{6.16}$$

and the foliated Verdier condition implies

$$\left\langle\left\langle \left(\pi_*|_{J(y)}\right)^{-1}(v(q)),\, v(q)\right\rangle\right\rangle \longrightarrow 0 \quad (\text{as } \rho(y) \longrightarrow 0).$$

Moreover we may assume U_p is relatively compact, thus

$$M_0 = \sup_{x \in X \cap U_p} \|v(x)\| < +\infty.$$

Hence (6.16) implies that

$$\begin{aligned} \left\| L_y\left(v(q)\right) \right\| &\leq \delta(E(y), T_q X) \cdot \frac{\|v(q)\|}{\cos \left\langle\!\left\langle \left(\pi_*|_{J(y)}\right)^{-1}(v(q)),\ v(q) \right\rangle\!\right\rangle} \\ &\leq 2M_0 \cdot \delta(E(y), T_q X) \\ &\leq 2\lambda M_0 \cdot \rho(y) \end{aligned} \tag{6.17}$$

hold for all $y \in U_p \cap Y$. By (6.15) and (6.17), we obtain

$$\left| \frac{df}{dt}(t) \right| \leq 2\lambda M_0 \cdot \rho(\varphi_t(y)) = \exists \lambda_0 \cdot f(t)$$

for all $t \in (0, t_0)$. Thus we have $-\lambda_0 \cdot f(t) \leq \frac{df}{dt}(t)$. Integrating the both sides from 0 to t, it turns out that

$$\rho(y) \cdot e^{-\lambda_0 t} \leq f(t)$$

hold for all $t \in (0, t_0)$. This contradicts $f(t_0) = 0$, thus (iii) in (6.11) holds.

Next let us check the continuity of ξ constructed above. E induces a non-singular foliation on Y because $\mathcal{S}$ is adapted to E. Hence it is clear that ξ is continuous on Y by the way of construction. The fact that ξ is a continuous extension of v is an immediate consequence of (6.17).

Q.E.D.

Proof of (6.10). Let $\mathcal{S}$ be a foliated Verdier stratification. Look upon all strata in $\mathcal{S}$ as real differentiable submanifolds and take a controlled tubular neighborhood system $\mathcal{T} = \{(T^{2i}, \pi^{2i}, \rho^{2i})\}$ of $\mathcal{S}$. Recall that M^{2i} denotes the union of all strata of dimension $2i$ and each $(T^{2i}, \pi^{2i}, \rho^{2i})$ is a tubular neighborhood of M^{2i}. Choose a stratum $X \in \mathcal{S}$ arbitrarily, and set $\dim_{\mathbf{R}} X = 2l$.

By the definition of the local topological triviality, it suffices to show that for every point $p \in X$ E is trivial along X on a sufficiently small neighborhood U of p. This is a local assertion at p, so we may assume

U : a coordinate neighborhood of p contained in T^{2l},

$(x_1, \ldots, x_{2n})$: real coordinates on U such that $x(p) = (0, 0, \ldots, 0)$ and $X \cap U = \{x_{2l+1} = x_{2l+2} = \cdots = x_{2n} = 0\}$.

Hereafter we argue only on U. By shrinking all T^{2i} $(l \leq i < n)$, we can take (sufficiently small) closed disk bundles $F^{2i} \subset T^{2i}$ so that

$$\mathrm{Cl}_U(F^{2i}) - F^{2i} \subset \bigcup_{l \leq j \leq i-1} \left(M^{2j} \cap U\right). \tag{6.18}$$

In order to get a local trivialization $h : U \longrightarrow (U \cap X) \times D$, it is sufficient to integrate continuous vector fields $\xi_1, \xi_2, \ldots, \xi_{2l}$ on U such that for every j $(1 \leq j \leq 2l)$ we have

(6.19)

(i) $\left(\pi^{2l}\right)_* \circ \xi_j = \dfrac{\partial}{\partial x_j} \circ \pi^{2l}$ holds on U.

(ii) $\xi_j(y) \in E(y)$ hold for all $y \in U - X$, i.e., ξ_j is tangent to the leaves defined by E at all point y in $U - X$.

(iii) For any point $y \in U$, the integral curve of ξ_j passing through y stays in the stratum including y.

We do this work by constructing $\xi_1^{(2i)}, \xi_2^{(2i)}, \ldots, \xi_{2l}^{(2i)}$ on each $M^{2i} \cap U$ successively for $i = l, l+1, \ldots, n$.

We define $\xi_1^{(2l)}, \xi_2^{(2l)}, \ldots, \xi_{2l}^{(2l)}$ on $M^{2l} \cap U$ $(= X \cap U = \{x_{2l+1} = x_{2l+2} = \cdots = x_{2n} = 0\})$ to be $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{2l}}$ respectively. It is clear that $\xi_1^{(2l)}, \xi_2^{(2l)}, \ldots, \xi_{2l}^{(2l)}$ satisfy the three conditions in (6.19).

Suppose we have already constructed all $\xi_1^{(2i)}, \xi_2^{(2i)}, \ldots, \xi_{2l}^{(2i)}$ which satisfy (i)-(iii) in (6.19) on each $M^{2i} \cap U$ for $l \leq i \leq a-1$. In order to define $\xi_1^{(2a)}, \xi_2^{(2a)}, \ldots, \xi_{2l}^{(2a)}$ on $M^{2a} \cap U$, we construct them on each $T^{2i} \cap M^{2a} \cap U$ $(l \leq i \leq a-1)$ and glue the pieces together by means of a partition of unity on $M^{2a} \cap U$.

We may assume that each T^{2i} $(l \leq i \leq a-1)$ is so small that we could apply lemma (6.11) to $M^{2i} \cap U$ and $T^{2i} \cap U$. Note that $\xi_j^{(2i)} \neq 0$ on $M^{2i} \cap U$ for all i with $l \leq i \leq a-1$ and for all j with $1 \leq j \leq 2l$ because $(\pi^{2l})_* \circ \xi_j^{(2i)} = \frac{\partial}{\partial x_j} \circ \pi^{2l}$ holds on $M^{2i} \cap U$ by the inductive assumption. Thus, applying lemma (6.11), we obtain continuous extensions η_j^{2i} of $\xi_j^{(2i)}$ on $T^{2i} \cap M^{2a} \cap U$ which satisfy (i)–(iii) in (6.11) for all i, j with $l \leq i \leq a-1$, $1 \leq j \leq 2l$.

Now we define $\xi_j^{(2a)}$ on $M^{2a} \cap U$ as follows. First, we set

$$Q^{2(a-1)} = T^{2(a-1)} \cap M^{2a} \cap U,$$

$$Q^{2i} = \left(T^{2i} - \bigcup_{m=i+1}^{a-1} \mathrm{Cl}_U(F^{2m}) \right) \cap M^{2a} \cap U \quad (\text{for } i = a-2,\ a-3, \ldots, l+1,\ l).$$

Note that $\{Q^{2i}\}_{l \le i \le a-1}$ is an open covering of $M^{2a} \cap U$ by (6.18). Then glue all η_j^{2i} on Q^{2i} together by means of a partition of unity on $M^{2a} \cap U$ subordinate to $\{Q^{2i}\}$, and define $\xi_j^{(2a)}$ to be the resulting vecter field. Let us check below that $\xi_j^{(2a)}$ meets the three requirements in (6.19) for each j.

At first, we will show that all η_j^{2k} $(l \le k \le a-1)$ satisfy (i). The following equation holds on Q^{2k}:

$$\begin{aligned}\left(\pi^{2l}\right)_* \circ \eta_j^{2k} &= \left(\pi^{2l} \circ \pi^{2k}\right)_* \circ \eta_j^{2k} = \left(\pi^{2l}\right)_* \circ \left(\pi^{2k}\right)_* \circ \eta_j^{2k} = \left(\pi^{2l}\right)_* \circ \xi_j^{(2k)} \circ \pi^{2k} \\ &= \xi_j^{(2l)} \circ \pi^{2l} \circ \pi^{2k} = \frac{\partial}{\partial x_j} \circ \pi^{2l},\end{aligned}$$

thus η_j^{2k} fulfills (i). It is obvious that all η_j^{2k} satisfy (ii). For (iii), it suffices to show that if $y \in \mathrm{Int}(F^{2i}) \cap M^{2a} \cap U$ then the integral curve of $\xi_j^{(2a)}$ containing y does not meet $M^{2i} \cap U$ (for every integer i with $l \le i \le a-1$). Since F^{2i} does not intersect Q^{2k} with $l \le k \le i-1$ by the definition of Q^{2k}, $\xi_j^{(2a)}$ has been determined on $\mathrm{Int}(F^{2i}) \cap M^{2a} \cap U$ using only η_j^{2k} with $i \le k \le a-1$. Let $\{(\psi_j^{2k})_t = \exp t\eta_j^{2k}\}$ and $\{(\varphi_j^{(2k)})_t = \exp t\xi_j^{(2k)}\}$ denote the local 1-parameter groups of transformations respectively. By the construction of η_j^{2i}, $(\psi_j^{2i})_t(y)$ does not meet M^{2i}. For each integer k with $i < k \le a-1$, we have $\rho^{2i} = \rho^{2i} \circ \pi^{2k}$ by (6.4). Hence

$$\rho^{2i} \circ (\psi_j^{2k})_t(y) = \rho^{2i} \circ \pi^{2k} \circ (\psi_j^{2k})_t(y) = \rho^{2i} \circ (\varphi_j^{(2k)})_t(y) > 0$$

hold for all t sufficiently close to 0 by the inductive assumption. This implies that $(\psi_j^{2k})_t(y)$ does not meet M^{2i} for all t sufficiently close to 0, thus (iii) holds.

This completes the induction and the proof of this theorem.

Q.E.D.

Let us close this paper by giving some examples about foliated Verdier stratifications.

Example 6.20 Let v_1, v_2 be holomorphic vector fields on $M = \mathbf{C}^3$ defined by

$$\text{(6.21)} \quad \begin{cases} v_1 = y\dfrac{\partial}{\partial x} - xyz\dfrac{\partial}{\partial y} + xy^2\dfrac{\partial}{\partial z} \\ v_2 = z\dfrac{\partial}{\partial x} - xz^2\dfrac{\partial}{\partial y} + xyz\dfrac{\partial}{\partial z} \end{cases}$$

Let $E(\subset \Theta_M)$ be the coherent subsheaf generated by v_1, v_2. E is integrable since $[v_1, v_2] = xyv_1 + xzv_2$, so E defines a singular foliation on $\mathbf{C}^3$. The rank of E is one,

and by (6.21), $S(E) = S^{(0)} = \{y = z = 0\} = \{x\text{-axis}\}$. Set $X = \{x\text{-axis}\}$ and $Y = \mathbf{C}^3 - \{x\text{-axis}\}$, then $\mathcal{S} = \{X, Y\}$ gives a foliated Verdier stratification of $\mathbf{C}^3$. Therefore E is topologically locally trivial along X by (6.10).

Example 6.22 Let v_1, v_2, v_3 be holomorphic vector fields on $M = \mathbf{C}^3$ defined by

$$\text{(6.23)} \quad \begin{cases} v_1 = 3y^2\dfrac{\partial}{\partial x} + 2x\dfrac{\partial}{\partial y} \\ v_2 = (x^2 - y^3)\dfrac{\partial}{\partial y} + 3y^2 z\dfrac{\partial}{\partial z} \\ v_3 = (x^2 - y^3)\dfrac{\partial}{\partial x} - 2xz\dfrac{\partial}{\partial z}\,. \end{cases}$$

Let $E(\subset \Theta_M)$ be the coherent subsheaf generated by v_1, v_2, v_3. We can easily check that E is integrable, so E defines a singular foliation on $\mathbf{C}^3$. The rank of E is two, and by (6.23), $S(E) = S^{(1)} = \{xz = yz = x^2 - y^3 = 0\} = \{x = y = 0\} \cup \{z = x^2 - y^3 = 0\}$ and $S^{(0)} = \{x = y = 0\}$. Set $X_1 = \{x = y = 0\} - \{0\}$ and $X_2 = \{z = x^2 - y^3 = 0\} - \{0\}$, then $\mathcal{S} = \{\mathbf{C}^3 - S(E),\ X_1,\ X_2,\ \{0\}\}$ gives a foliated Verdier stratification of $\mathbf{C}^3$. Therefore E is topologically locally trivial along each stratum of $\mathcal{S}$ by (6.10).

Example 6.24 Let us recall the singular foliation E on $\mathbf{C}^3$ given in (5.19). If we take the stratification $\mathcal{S}' = \left\{\mathbf{C}^3 - S(E),\ S(E) - S^{(0)},\ S^{(0)}\right\}$, the structure of E is not trivial at 0 along the strarum $S^{(0)}$. So it is neccesary to separate the bad point 0 from $S^{(0)}$ to obtain the local triviality along each stratum. The stratification $\mathcal{S} = \{\mathbf{C}^3 - S(E),\ X_1,\ X_2,\ \{0\}\}$ is adapted to E, but this is not a foliated Verdier stratification (E does not satisfy the foliated Verdier condition at $p = (x, 0, 0)$ $(x \neq 0)$ over $X_2 = \{x\text{-axis}\} - \{0\}$. See the directions of the leaves passing through $(x, 0, z) \in \mathbf{C}^3$ for $z \in \mathbf{C}$ sufficiently close to 0). We cannot take a foliated Verdier stratification for this type of singular foliations.

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