Reticular Lagrangian, Legendrian Singularities and their applications

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Contents

1 Introduction

In [8] K.Jänich explained the wavefront propagation mechanism on a manifold which is completely described by a positive and positively homogeneous Hamiltonian function on the cotangent bundle and investigated the local gradient models given by the ray length function. Caustics and wavefronts generated by an initial wavefront which is a hypersurface without boundary in the manifold is investigated as Lagrangian and Legendrian singularities by V.I.Arnold (cf., [1]). In [13], I.G.Scherbak studied the case when the hypersurface has a boundary and she explained the caustics and wavefronts generated by the hypersurface with a boundary corresponds to a generalized notion of caustics and wavefronts respectively (i.e., the *boundary caustics* and *boundary fronts*).

In this paper we investigate the more general case when the hypersurface has an *r-corner*. In this case each light rays incident from each edge of the hypersurface gives a *symplectic regular r-cubic configuration* (cf., Section 3) at a point of the cotangent bundle which is a generalization of the notion of Lagrangian submanifolds and each wavefront incident from each edge of the hypersurface gives a *contact regular r-cubic configuration* (cf., Section 12) at a point of the 1-jet bundle which is a generalization of the notion of Legendrian submanifolds. The caustic generated by the hypersurface germ with an r-corner is given by the caustic of the symplectic regular r-cubic configurations (cf., Section 3) which is a generalization of the notion of *quasicaustics* given by S.Janeszko (cf., [7]). In complex analytic category, the theory of symplectic and contact regular r-cubic configurations has been developed by Nguyen Huu Duc, Nguyen Tien Dai and F.Pham (cf., [3], [6]). But their method does not work well for C^∞-category and all contact regular r-cubic configurations in their category are stable. This paper is composed of three parts.

The main purpose of Part I is the investigation of the stability of smooth symplectic regular r-cubic configurations and the classification of *stable caustics* given by stable symplectic regular r-cubic configurations in C^∞-category. In order to realize this purpose we shall define the notion of *reticular Lagrangian maps* in Section 3 which is a generalization of the notion of Lagrangian maps for our situations. We shall also prove that the equivalence relation among reticular Lagrangian maps is equivalent to a certain equivalence relation of corresponding generating families. In Section 5 we shall define the notion of *stability, homotopically stability, infinitesimal stability* of reticular Lagrangian maps and prove that these and the stability of corresponding generating families are all equivalent.

By the above results the classification of stable caustics is reduced to the classifications of function germs. In section 7 we classify unimodal function germs with respect to *reticular R-equivalence*. This gives the classification of stable caustics in manifolds of dimension$\leq$ 6. In [14], D.Siersma classified singularities with *bundle codimension*(=R-codimension−modality) $\leq$ 4 under the same equivalence relation. Hence a part of his classification list is the same as the part of our list. We shall draw the figures of stable caustics in manifolds of dimension$\leq$ 4 at the last part of this part. The investigations of this part are published in [15].

In part II we investigate the *optical stability* of caustics generated by a light source hypersurface germ with an r-corner as an application of Part I, that is the stability of caustics with respect to perturbations of the hypersurface germ under a fixed Riemannian structure. More generally, we study the optical stability of caustics with respect to perturbations of the hypersurface germ under a fixed Hamiltonian system defined in Section 2. The stability of caustics and the optical stability of caustics are generally not equivalent because the stability of caustics means the stability with respect to both of perturbations of the hypersurface germ and the Hamiltonian system. In this application, we consider the following problems, extending of the investigations by K.Jänich [8] and G.Wassermann [18]: For a fixed Hamiltonian system on the cotangent bundle;

(1) Is the optical stability of caustics equivalent to the stability of caustics?

(2) For a given function germs, when does there exist a light source hypersurface germ with an r-corner which satisfy the following conditions (a)(b)?: a) A generating family of the corresponding symplectic regular r-cubic configuration is an unfolding of the given function germ , b) The caustic generated by the hypersurface germ is optically stable.

The answer of (1) is 'Yes'. This means that stability of caustics and optical stability of caustics are equivalent.

We give a partial answer to the problem (2). The answer of (2) gives us a method to decide when the caustic defined by a function germ in the classification list can be realized as an (optically) stable caustic. The investigations of this part are published in [16].

In Part III, we investigate the two topics. The first topic is the investigation of the relation between symplectic regular r-cubic configurations and contact regular r-cubic configurations.

The second topic is the investigation of the stability of smooth contact regular r-cubic configurations and the classification of *stable wavefonts* given by stable contact regular r-cubic configurations in C^∞-category. In order to realize this purpose we shall define the notion of *reticular Legendrian maps* in Section 14 which is a generalization of the notion of Legendrian maps for our situations. We shall also prove that the equivalence relation among reticular Legendrian maps is equivalent to a certain equivalence relation of corresponding generating families. In this section we shall define the notion of *stability, homotopically stability, infinitesimal stability* of reticular Legendrian maps and prove that these and the stability of corresponding generating families are all equivalent.

By the above results the classification of stable wavefronts is reduced to the classifications of function germs. In section 7 we classify function germs with respect to *reticular K-equivalence* with *reticular K-codimension* lower than 8 . This gives the classification of stable wavefonts in manifolds of dimension$\leq$ 7. At the last of this part, we shall draw the figure of the wavefront of one of the reticular Legendrian map-germ whose generating family is a reticular versal unfolding of a function germ in the classification list.

Part I

Reticular Lagrangian Singularities

2 Preliminaries

The propagation mechanism of light rays incident from a hypersurface germ with an r-corner in a smooth manifold is described as follows (Cf., [8]): Let M be an $n(=r+k+1)$-dimensional differentiable manifold and $H : T^*M\backslash 0 \to \mathbb{R}$ be a C^∞-function, called a *Hamiltonian function*, which we suppose to be everywhere positive and positively homogeneous of degree one, that is $H(\lambda\xi) = \lambda H(\xi)$ for all $\lambda > 0$ and $\xi \in T^*M\backslash 0$, where $\pi : T^*M \to M$ is the cotangent bundle. Let X_H denote the corresponding Hamiltonian vector field on $T^*M\backslash 0$, given locally by the Hamiltonian equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where (q,p) are local canonical coordinates of T^*M.

We set $E = H^{-1}(1)$ and consider the following canonical projections $\pi : T^*M \to M$, $\pi_E : \mathbb{R}\times E \to E$, $\pi_{\mathbb{R}} : \mathbb{R}\times E \to \mathbb{R}$. We denote E_q the fiber of the spherical cotangent bundle $\pi|_E$ at $q \in M$.

Let $q_0 \in M,\ t_0 \geq 0,\ \xi_0 \in E_{q_0}$ and η_0 be the image of the phase flow of X_H at (t_0, ξ_0). Since the phase flow of X_H preserves values of H, the local phase flow $\Psi : (\mathbb{R}\times T^*M\backslash 0, (t_0,\xi_0)) \to (T^*M\backslash 0, \eta_0)$ of X_H induces the map $\Phi : (\mathbb{R}\times E, (t_0,\xi_0)) \longrightarrow (\mathbb{R}\times E, (t_0,\eta_0))$ given by $\Phi(t,\xi) = (t, \Psi(t,\xi))$.

We set $exp = \pi_M \circ \Phi : (\mathbb{R}\times E, (t_0,\xi_0)) \to (M, u_0)$, $exp_{q_0} = exp|_{\mathbb{R}\times E_{q_0}}$, $exp^- = \pi_M \circ \Phi^{-1} : (\mathbb{R}\times E, (t_0,\eta_0)) \to (M, q_0)$, $exp^-_{u_0} = exp^-|_{\mathbb{R}\times E_{u_0}}$, $\phi_1 = (\pi_M, exp) : (\mathbb{R}\times E, (t_0,\xi_0)) \to (M\times M, (q_0,u_0))$, $\phi_2 = (exp^-, \pi_M) : (\mathbb{R}\times E, (t_0,\eta_0)) \to (M\times M, (q_0,u_0))$, where $u_0 = \pi(\eta_0)$.

Then the following diagram is commutative:

$$\begin{array}{ccccc} & (\mathbb{R}\times E,(t_0,\xi_0)) & \xrightarrow{\Phi} & (\mathbb{R}\times E,(t_0,\eta_0)) & \\ \swarrow exp & & \phi_1 \searrow \ \swarrow \phi_2 & & exp^- \searrow \\ (M,u_0) & \xleftarrow{\pi_2} & (M\times M,(q_0,u_0)) & \xrightarrow{\pi_1} & (M,q_0) \end{array}$$

By [8, 2.2] we have the following proposition

Proposition 2.1 *If exp_{q_0} is regular then ϕ_1 and ϕ_2 are diffeomorphisms.*

Let exp_{q_0} be regular, then we can define the smooth function germ

$$\tau = \pi_{\mathbb{R}} \circ \phi_1^{-1} = \pi_{\mathbb{R}} \circ \phi_2^{-1} : (M \times M, (q_0, u_0)) \to (\mathbb{R}, t_0).$$

We call τ the *ray length function* associated with the regular point (t_0, ξ_0) of exp_{q_0}. Set $\xi = \pi_E \circ \phi_1^{-1} : (M \times M, (q_0, u_0)) \to (E, \xi_0)$, $\eta = \pi_E \circ \phi_2^{-1} : (M \times M, (q_0, u_0)) \to (E, \eta_0)$. By [8, Lemma 2] we have

$$d_q\tau(q,u) = -\xi(q,u),\ d_u\tau(q,u) = \eta(q,u) \quad \text{for } (q,u) \in (M \times M, (q_0, u_0)).$$

Example: Let M be a Riemannian manifold and H be the length of covectors. Then Φ maps each covector in time t a distance t along the geodesic and hence $\tau(q,u)$, $(q,u) \in (M \times M, (q_0, u_0))$, is the length of geodesic which connects q and u. In particular if M be a Euclidean space $\mathbb{E}^n$, then $\Phi(t,q,p) = (q + \frac{p}{|p|}t, p)$ and $\tau(q,u) = |q-u|$, where (q,p) are canonical coordinates of $T^*\mathbb{E}^n$ and $q, u \in \mathbb{E}^n$.

Let $\mathbb{H}^r = \{(x_1, \cdots, x_r) \in \mathbb{R}^r | x_1 \geq 0, \cdots, x_r \geq 0\}$ be an r-corner. Let V^0 be the hypersurface germ in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ with an r-corner defined as the image of an immersion $\iota : (\mathbb{H}^r \times \mathbb{R}^k, 0) \to (M, q_0)$. We parameterize V^0 by ι. For each $\sigma \subset I_r = \{1, \cdots, r\}$ we define Λ_σ^0 by the set of conormal vectors of $V_\sigma^0 := V^0 \cap \{x_\sigma = 0\}$ in (E, ξ_0) as the initial rays incident from V^0_σ. Then we regard the set L_σ the image of covectors in Λ^0_σ by $\pi_E \circ \Phi$ around time t_0, that is

$$L_\sigma = \{\pi_E \circ \Phi(t,\xi) \in (E,\eta_0) | (t,\xi) \in (\mathbb{R}, t_0) \times \Lambda^0_\sigma\},$$

as the set of rays incident from V^0_σ around t_0. We also regard the union of L_σ for all $\sigma \subset I_r$ as the set of rays incident from the hypersurface V^0 around time t_0.

Let C_σ be the critical values of $\pi|_{L_\sigma}$ for $\sigma \subset I_r$ and $Q_{\sigma,\tau} = \pi(L_\sigma \cap L_\tau)$ for $\sigma \neq \tau \subset I_r$. We define the *caustic* of the light rays incident from V^0 around q_0 by

$$\bigcup_{\sigma \subset I_r} C_\sigma \cup \bigcup_{\sigma \neq \tau} Q_{\sigma,\tau}.$$

The meaning of the caustic is the following: For example, consider the case $r = 2, k = 0$. Then $V_0 = \{(q_1,q_2,q_3) | q_1 \geq 0, q_2 \geq 0, q_3 = 0\}$ for coordinates (q_1,q_2,q_3) of (M,q_0). By the remark of the definition of the caustic in Section 3 we have $\bigcup_{\sigma \neq \tau} Q_{\sigma,\tau} = Q_{\emptyset,1} \cup Q_{\emptyset,2} \cup Q_{1,\{1,2\}} \cup Q_{2,\{1,2\}}$. The bright points generated by incident rays from V^0, $V^0 \cap \{q_1 = 0\}$, $V^0 \cap \{q_2 = 0\}$ and $V^0 \cap \{q_1 = q_2 = 0\}$ are $C_\emptyset$, C_1, C_2 and $C_{1,2}$ respectively. On the other hand the light rays incident from the boundary of V^0, $V^0 \cap \{q_1 = 0\}$ and $V^0 \cap \{q_2 = 0\}$ are $Q_{\emptyset,1} \cup Q_{\emptyset,2}$, $Q_{1,\{1,2\}}$ and $Q_{2,\{1,2\}}$ respectively. They appear as the boundary of the shadow defined by the boundary of light rays incident from V^0, $V^0 \cap \{q_1 = 0\}$ and $V^0 \cap \{q_2 = 0\}$ respectively. This definition is a natural extension of *quasicaustic* defined in [7].

The family of submanifolds $\{L_\sigma\}_{\sigma \subset I_r}$ of $(T^*M \backslash 0, \eta_0)$ is 'generated' by the ray length function τ as the following:

Proposition 2.2 *Let V^0 be the hypersurface germ in (M,q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ defined as the image of an immersion $\iota : (\mathbb{H}^r \times \mathbb{R}^k, 0) \to (M,q_0)$. Let L_σ be the set of rays incident* *from* $V^0_\sigma := V^0 \cap \{x_\sigma = 0\}$ *around time* t_0 *for* $\sigma \subset I_r$. *Define* $F := \tau \circ (\iota \times id_u) - t_0 \in \mathfrak{M}(r; k+m)$. *Then the following hold:*

(1)

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial u} \\ \dfrac{\partial^2 F}{\partial y \partial u} \end{pmatrix}_0 = r + k.$$

(2)

$$L_\sigma = \{d_u F(x,y,u) \in (T^*M\backslash 0, \eta_0) | x_\sigma = d_{x_{I_r - \sigma}} F(x,y,u) = d_y F(x,y,u) = 0\} \ \textit{for}\ \sigma \subset I_r,$$

where we identify (M, u_0) *and* $(\mathbb{R}^n, 0)$ *by coordinates* $(u_1, \cdots, u_n)$ *of* (M, u_0).

Proof. By [8, Sublemma] we have

$$\begin{pmatrix} d_u d_x F \\ d_u F \end{pmatrix} : T_{u_0} M \to T^*_{q_0} V^0 \oplus \mathbb{R}$$

is an isomorphism. This means (1).

Let $\sigma \subset I_r$ and $\eta_u \in (E, \eta_0)$. Then $\eta_u \in L_\sigma$ if and only if $\eta_u = \pi_E \circ \Phi(t, \xi_q)$ for some $\xi_q \in E_q$ and $t \in (\mathbb{R}, t_0)$ satisfying $q \in V^0_\sigma$ and $\xi_q|_{T_q V^0_\sigma} = 0$ and this holds if and only if $\eta_u = d_u \tau(q, u)$ for some $q \in V^0_\sigma$ and $u \in (M, u_0)$ satisfying $d_q \tau(q,u)|_{T_q V^0_\sigma} = 0$ and this holds if and only if $\eta_u = d_u F(x,y,u)$ for some $(x,y,u) \in (\mathbb{H}^r \times \mathbb{R}^{k+m}, 0)$ satisfying $x_\sigma = 0$ and $d_{x_{I_r - \sigma}} F(x,y,u) = d_y F(x,y,u) = 0$. ■

Let $H = \{(q_1, \cdots, q_n) \in (\mathbb{R}^n, 0) | q_1 \geq 0, \cdots, q_r \geq 0, q_{r+1} = \cdots = q_n = 0\}$ be an r-corner and L^0_σ be the conormal bundle of $H \cap \{x_\sigma = 0\}$ for $\sigma \subset I_r$. By theorem 3.2(2), proposition 2.2 implies that the family of submanifold $\{L_\sigma\}_{\sigma \subset I_r}$ of $(T^*M\backslash 0, \eta_0)$ is a *symplectic regular* r*-cubic configuration*, that is there exists a symplectomorphism $S : (T^*\mathbb{R}^n, 0) \to (T^*M\backslash 0, \eta_0)$ such that

$$L_\sigma = S(L^0_\sigma) \quad \text{for} \quad \sigma \subset I_r.$$

Hence in Part I we investigate the stability of the caustic under perturbations of the corresponding symplectomorphism. Generally the stability of the caustic under perturbations of the symplectomorphism is more stronger one of the perturbations of the immersion. Because a small perturbation of the immersion implies a small perturbation of the symplectomorphism, but for any perturbation of the immersion the corresponding submanifold L_σ is included in E for all $\sigma \subset I_r$. In order to realize our investigation we shall define *reticular Lagrangian maps* in a more general situation.

The stability of the caustic under perturbations of the immersion is studied in Part II.

3 Reticular Lagrangian maps

Here we shall define reticular Lagrangian maps, their caustics and equivalence relations.

Let (q,p) be canonical coordinates of $(T^*\mathbb{R}^n, 0)$ and $\pi : (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the cotangent bundle equipped with the canonical symplectic structure. Let $H = \{(q_1, \cdots, q_n) \in (\mathbb{R}^n, 0) | q_1 \geq 0, \cdots, q_r \geq 0, q_{r+1} = \cdots = q_n = 0\}$ be an r-corner and $H_\sigma = \{(q_1, \cdots, q_n) \in H | q_\sigma = 0\}$ be an edge of H for $\sigma \subset I_r$. We define L_σ^0 the conormal bundle of H_σ, that is

$$L_\sigma^0 = \{(q,p) \in (T^*\mathbb{R}^n, 0) | q_\sigma = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = 0, q_{I_r - \sigma} \geq 0\}.$$

We define a representative of the union of L_σ^0 for all $\sigma \subset I_r$ by

$$\mathbb{L}^0 = \{(q,p) \in T^*\mathbb{R}^n | q_1 p_1 = \cdots = q_r p_r = q_{r+1} = \cdots = q_n = 0, q_{I_r} \geq 0\}.$$

Definition 3.1 We call the map germ

$$(\mathbb{L}^0, 0) \xrightarrow{i} (T^*\mathbb{R}^n, 0) \xrightarrow{\pi} (\mathbb{R}^n, 0)$$

a *reticular Lagrangian map* if there exists a symplectomorphism S on $(T^*\mathbb{R}^n, 0)$ such that $i = S|_{\mathbb{L}^0}$. We call S an extension of i and call $\{i(L^0_\sigma)\}_{\sigma\subset I_r}$ the *symplectic regular r-cubic configuration* associated with $\pi \circ i$.

Remark: The definition of symplectic regular r-cubic configurations in the complex analytic category by Nguyen Huu Duc [6, p. 631] is as follows: If there exists a symplectomorphism S such that $L_\sigma = S(\{q_\sigma = p_{I_r-\sigma} = p_{r+1} = \cdots = p_n = 0\})$ for $\sigma \subset I_r$ then $\{L_\sigma\}_{\sigma\subset I_r}$ is called a symplectic regular r-cubic configuration.

Caustics: Let $\pi \circ i$ be a reticular Lagrangian map. Let C_σ be the caustic of the Lagrangian map $\pi \circ i|_{L^0_\sigma}$ for $\sigma \subset I_r$ (i.e., the critical value set of $\pi \circ i|_{L^0_\sigma}$) and let $Q_{\sigma,\tau} = \pi \circ i(L^0_\sigma \cap L^0_\tau)$ for $\sigma \neq \tau \subset I_r$. We define the *caustic* of $\pi \circ i$ by

$$\bigcup_{\sigma\subset I_r} C_\sigma \cup \bigcup_{\sigma\neq\tau} Q_{\sigma,\tau}.$$

We remark that for $\tau_1, \tau_2 \subset I_r$ $(\tau_1 \neq \tau_2)$ we have $Q_{\tau_1,\tau_2} \subset Q_{\sigma,\sigma\cup\{i\}}$, where $\sigma = \tau_1 \cap \tau_2$ and i be any element of $(\tau_1 - \sigma) \cup (\tau_2 - \sigma)$. This means that $\bigcup_{\sigma\neq\tau} Q_{\sigma,\tau}$ is equal to the union of $Q_{\sigma,\tau}$ for $\sigma \subset \tau \subset I_r$, $\#(\tau - \sigma) = 1$. For example, in the case $r = 2$ we have

$$\bigcup_{\sigma\neq\tau} Q_{\sigma,\tau} = Q_{\emptyset,1} \cup Q_{\emptyset,2} \cup Q_{1,\{1,2\}} \cup Q_{2,\{1,2\}}.$$

Equivalence relations: We call a homeomorphism germ $\phi : (\mathbb{L}^0, 0) \longrightarrow (\mathbb{L}^0, 0)$ a *reticular diffeomorphism* if there exists a diffeomorphism Φ on $(T^*\mathbb{R}^n, 0)$ such that $\phi = \Phi|_{\mathbb{L}^0}$ and $\phi(L^0_\sigma) = L^0_\sigma$ for $\sigma \subset I_r$. We say that reticular Lagrangian maps $\pi \circ i_1, \pi \circ i_2 : (\mathbb{L}^0, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ are *Lagrangian equivalent* if there exist a reticular diffeomorphism ϕ and a Lagrangian equivalence Θ of π such that the following diagram is commutative:

$$\begin{array}{ccccc}
(\mathbb{L}^0, 0) & \xrightarrow{i_1} & (T^*\mathbb{R}^n, 0) & \xrightarrow{\pi} & (\mathbb{R}^n, 0) \\
\phi\downarrow & & \Theta\downarrow & & g\downarrow \\
(\mathbb{L}^0, 0) & \xrightarrow{i_2} & (T^*\mathbb{R}^n, 0) & \xrightarrow{\pi} & (\mathbb{R}^n, 0),
\end{array}$$

where g is the diffeomorphism of the base space of π induced from Θ.

We remark that if reticular Lagrangian maps $\pi \circ i_1, \pi \circ i_2$ are Lagrangian equivalent then all Lagrangian maps $\pi \circ i_1|_{L_\sigma^0}, \pi \circ i_2|_{L_\sigma^0}$ are Lagrangian equivalent for each $\sigma \subset I_r$.

Here we shall define generating families of reticular Lagrangian maps and study the relations between reticular Lagrangian maps and their generating families. At first, we define several notations of function germs which are used as generating families of reticular Lagrangian maps.

Let $\mathbb{H}^r = \{(x_1, \cdots, x_r) \in \mathbb{R}^r | x_1 \geq 0, \cdots, x_r \geq 0\}$ be an r-corner. Let $\mathcal{E}(r;l)$ be the ring of smooth function germs on $(\mathbb{H}^r \times \mathbb{R}^l, 0)$ and $\mathfrak{M}(r;l) = \{f \in \mathcal{E}(r;l) | f(0) = 0\}$ be the maximal ideal of $\mathcal{E}(r;l)$. We denote simply $\mathcal{E}(l)$ for $\mathcal{E}(0;l)$ and $\mathfrak{M}(l)$ for $\mathfrak{M}(0;l)$ and denote $\mathcal{B}(r;l)$ the set of diffeomorphism germs on $(\mathbb{H}^r \times \mathbb{R}^l, 0)$ which preserve $(\mathbb{H}^r \cap \{x_\sigma = 0\}) \times \mathbb{R}^l$ for all $\sigma \subset I_r$. We remark that a diffeomorphism germ ϕ on $(\mathbb{H}^r \times \mathbb{R}^l, 0)$ is an element of $\mathcal{B}(r;l)$ if and only if ϕ is written in the form:

$$\phi(x,y) = (x_1 a_1(x,y), \cdots, x_r a_r(x,y), b_1(x,y), \cdots, b_l(x,y)) \text{ for } (x,y) \in (\mathbb{H}^r \times \mathbb{R}^l, 0),$$

where $a_1, \cdots, a_r, b_1, \cdots, b_r \in \mathcal{E}(r;l)$ and $a_1(0) > 0, \cdots, a_r(0) > 0$.

We say that function germs $f, g \in \mathfrak{M}(r;l)$ are *reticular R-equivalent* if there exists $\phi \in \mathcal{B}(r;l)$ such that $g = f \circ \phi$.

We say that function germs $F(x,y,u), G(x,y,u) \in \mathfrak{M}(r;k+n)$, where $x \in \mathbb{H}^r$, $y \in \mathbb{R}^k$ and $v \in \mathbb{R}^n$, are *reticular R^+-equivalent* (as n-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r;k+n)$ and $\alpha \in \mathfrak{M}(n)$ satisfying the following:

(1) $\Phi = (\phi, \psi)$, where $\phi : (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$ and $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.

(2) $G(x,y,u) = F(\phi(x,y,u), \psi(u)) + \alpha(u)$ for $(x,y,u) \in (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$.

We say (Φ, α) a reticular R^+-isomorphism from G to F and if $\alpha = 0$ we say that F and G are reticular R-equivalent.

We say that function germs $F(x, y_1, \cdots, y_{k_1}, u) \in \mathfrak{M}(r; k_1 + n)$ and $F(x, y_1, \cdots, y_{k_2}, u) \in \mathfrak{M}(r; k_2 + n)$ are *stably reticular R^+-equivalent* if F and G are reticular R^+-equivalent after additions of non-degenerate quadratic forms in the variables y.

A function germ $F(x, y, u) \in \mathfrak{M}^2(r; k+n)$ is called *S-non-degenerate* if

$$x_1, \cdots, x_r, \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_k}$$

are independent on $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$, that is

$$\mathrm{rank} \begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial u} \\ \dfrac{\partial^2 F}{\partial y \partial y} & \dfrac{\partial^2 F}{\partial y \partial u} \end{pmatrix}_0 = r + k.$$

We remark that $F(x, y, u) \in \mathfrak{M}^2(r; k+n)$ is S-non-degenerate only if $r \leq n$.

Let $\pi \circ i$ be a reticular Lagrangian map and $F(x, y, q) \in \mathfrak{M}(r; k+n)^2$ be a S-non-degenerate function germ. We call F a *generating family* of $\pi \circ i$ if $F|_{x_\sigma = 0}$ is a generating family of $i(L^0_\sigma)$ for $\sigma \subset I_r$, that is

$$i(L^0_\sigma) = \{(q, \frac{\partial F}{\partial q}(x, y, q)) \in (T^*\mathbb{R}^n, 0) | x_\sigma = \frac{\partial F}{\partial x_{I_r - \sigma}} = \frac{\partial F}{\partial y} = 0\} \quad \text{for } \sigma \subset I_r.$$

We also call F a generating family of $\{i(L^0_\sigma)\}_{\sigma \subset I_r}$

In the case $r = 0$, this definition is the same as that of the generating family of a Lagrangian map(cf., [1]).

Theorem 3.2 (1) *For any reticular Lagrangian map $\pi \circ i$, there exists a function germ $F \in \mathfrak{M}(r; k+n)^2$ which is a generating family of $\pi \circ i$.*

(2) *For any S-non-degenerate function germ $F \in \mathfrak{M}(r; k+n)^2$, there exists a reticular*

Lagrangian map of which F is a generating family.

(3) *Two reticular Lagrangian maps are Lagrangian equivalent if and only if their generating families are stably reticular R^+-equivalent.*

We remark that there exists an analogous result of this theorem for complex analytic category in [3, P. 13 Théoréme]. But its proof does not work well for C^∞-category because F_t in 'Preuve du lemme i' may be degenerate for some $t \in [0,1]$. Our proof is also available for complex analytic and real analytic category.

Proof. (1) Let $\pi \circ i$ be a reticular Lagrangian map and S an extension of i. Let P_S be the canonical relation associated with S, that is

$$P_S = \{(Q,P;q,p) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, 0) | (q,p) = S(Q,P)\},$$

where (Q,P) is canonical coordinates of the domain. By considering a Lagrangian equivalence of $\pi \circ i$, we may assume that there exists a generating function $T(Q,p)$ of P_S, that is

$$P_S = \{ \ (Q, -\frac{\partial T}{\partial Q}(Q,p); -\frac{\partial T}{\partial p}(Q,p), p) \ \}.$$

Define $F \in \mathfrak{M}(r;n+n)^2$ by $F(x,y,q) = T(x_1,\cdots,x_r,0,\cdots,0;y_1,\cdots,y_n) + \sum_{i=1}^n y_i q_i$. Since T is a generating function of P_S, rank $\frac{\partial^2 T}{\partial x \partial y}(0) = r$. Hence

$$\begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial q} \\ \dfrac{\partial^2 F}{\partial y^2} & \dfrac{\partial^2 F}{\partial y \partial q} \end{pmatrix}_0 = \begin{pmatrix} \dfrac{\partial^2 T}{\partial x \partial y} & 0 \\ \dfrac{\partial^2 T}{\partial y^2} & E_n \end{pmatrix}_0$$

has rank $r+n$. This means that F is S-non-degenerate.

Otherwise, we have

$$\{(q, \frac{\partial F}{\partial q}(x,y,q)) | x_\sigma = \frac{\partial F}{\partial x_{I_r - \sigma}} = \frac{\partial F}{\partial y} = 0\}$$

$$
\begin{aligned}
&= \{(q,y)|x_\sigma = \frac{\partial T}{\partial x_{I_r-\sigma}}(x,0,y) = \frac{\partial T}{\partial y}(x,0,y) + q = 0, x_{I_r-\sigma} \geq 0\} \\
&= \{(q,p)|Q_\sigma = \frac{\partial T}{\partial Q_{I_r-\sigma}}(Q,p) = Q_{r+1} = \cdots = Q_n = \frac{\partial T}{\partial p}(Q,p) + q = 0, Q_{I_r-\sigma} \geq 0\} \\
&= \{(-\frac{\partial T}{\partial p}(Q,p),p)|Q_\sigma = \frac{\partial T}{\partial Q_{I_r-\sigma}}(Q,p) = Q_{r+1} = \cdots = Q_n = 0, Q_{I_r-\sigma} \geq 0\} \\
&= S(L^0_\sigma) = i(L^0_\sigma)
\end{aligned}
$$

for $\sigma \subset I_r$. Hence F is a generating family of $\pi \circ i$.

(2) Let $F \in \mathfrak{M}(r;k+n)^2$ be a S-non-degenerate function germ. Choose an $(n-r)\times k$ matrix A and an $(n-r)\times n$ matrix B such that

$$
\begin{pmatrix}
\frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial q} \\
\frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial q} \\
A & B
\end{pmatrix}_0
$$

is invertible. Let $F'(x,y,q) \in \mathfrak{M}(r+k+n)^2$ be an extension of F and define $G \in \mathfrak{M}(k+n+n)^2$ by $G(y,x,x',q) = F'(x,y,q) + x'Ay^t + x'Bq^t$, where $y \in \mathbb{R}^k$, $(x_1,\cdots,x_r,x'_1,\cdots,x'_{n-r}) \in \mathbb{R}^n$ and $q \in \mathbb{R}^n$. Since $\frac{\partial G}{\partial x}, \frac{\partial G}{\partial x'}, \frac{\partial G}{\partial y}$ are independent,

$$
P = \{\ (x,x',-\frac{\partial G}{\partial x},-\frac{\partial G}{\partial x'};q,\frac{\partial G}{\partial q})\ |\ \frac{\partial G}{\partial y} = 0\}
$$

is the canonical relation associated with a symplectomorphism S. Hence F is a generating family of the reticular Lagrangian map $\pi \circ S|_{\mathbb{L}^0}$.

(3) By using analogous methods of the proof of [1, p.304 Theorem], it is enough to prove the following assertion:

Let $F_0(x,y,q)$, $F_1(x,y,q) \in \mathfrak{M}(r;k+n)^2$ be S-non-degenerate function germs. If F_0 and F_1 are generating families of the same reticular Lagrangian map, then F_0 and F_1 are reticular R-equivalent.

We suppose Lemma 3.3 and Lemma 3.4 and begin to prove this assertion. By using analogous methods of the proof of D.(a) $\sim$ (d) in [1, p.304 Theorem], we may assume that

$$F(y,q) := F_0(0,y,q) = F_1(0,y,q), \tag{1}$$

$$\frac{\partial^2 F}{\partial y^2}(0) = 0, \frac{\partial^2 F}{\partial y \partial q_J}(0) = E_k, J \subset \{1,2,\cdots,n\}, |J| = k. \tag{2}$$

We may assume by (1), (2) and Lemma 3.3 that

$$j^2 F_0(0) = j^2 F_1(0). \tag{3}$$

We may assume by (2), (3) and Lemma 3.4 that

$$\Sigma^\sigma := \Sigma^\sigma_{F_0} = \Sigma^\sigma_{F_1}, \frac{\partial F_0}{\partial q} - \frac{\partial F_1}{\partial q}|_{\Sigma^\sigma} \equiv 0 \text{ for all } \sigma \subset I_r, () \tag{4}$$

where $\Sigma^\sigma_{F_i} = \{\ (x,y,q) \in (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0) \mid x_\sigma = \frac{\partial F_i}{\partial x_{I_r - \sigma}} = \frac{\partial F_i}{\partial y} = 0\}$, $i = 1,2$.

Define the function germ $\bar{F}$ on $(\mathbb{H}^r \times \mathbb{R}^{k+n+1}, 0 \times [0,1])$ by $\bar{F}(x,y,q,t) = (1-t)F_0(x,y,q) + tF_1(x,y,q), t \in [0,1]$ and set $F_t \in \mathfrak{M}(r;k+n)^2$ by $F_t(x,y,q) = \bar{F}(x,y,q,t)$ for each $t \in [0,1]$. Since $j^2F_0(0) = j^2F_1(0)$, F_t is S-non-degenerate and hence $\Sigma^\sigma_{F_t} = \Sigma^\sigma$ for all $t \in [0,1]$ and $\sigma \subset I_r$ because $\Sigma^\sigma_{F_0} = \Sigma^\sigma_{F_1}$. Hence we have by hypothesis that

$$(x,y,q) \in \Sigma^\sigma \Rightarrow \frac{\partial F_t}{\partial q}(x,y,q) = \frac{\partial F_0}{\partial q}(x,y,q) \ (\ \forall t \in [0,1]\ , \forall \sigma \subset I_r\ ).$$

We now claim that $\frac{\partial \bar{F}}{\partial t}$ is written in the form:

$$\frac{\partial \bar{F}}{\partial t} = \sum_{i=1}^{r} \xi_i x_i \frac{\partial \bar{F}}{\partial x_i} + \sum_{j=1}^{k} \eta_j \frac{\partial \bar{F}}{\partial y_j}$$

for some smooth function germs $\xi_1, \cdots, \xi_r, \eta_1, \cdots, \eta_k$ on $(\mathbb{H}^r \times \mathbb{R}^{k+n+1}, 0 \times [0,1])$.

Fix $\sigma \subset I_r$, $(x,y,q) \in \Sigma^\sigma$ and let $c : [0,1] \to \Sigma^\sigma$, $t \mapsto (x(t), y(t), q(c))$, be a smooth path connects 0 and (x,y,q). Then

$$(F_1 - F_0)(x,y,q) = \int_0^1 \frac{d}{dt}(F_1 - F_0)(c(t))dt$$

$$= \int_0^1 (\sum_{j \in I_r - \sigma} \frac{\partial(F_1 - F_0)}{\partial x_j}\frac{dx_j}{dt}(t) + \sum_{j=1}^{k} \frac{\partial(F_1 - F_0)}{\partial y_j}\frac{dy_j}{dt}(t) + \sum_{i=1}^{n} \frac{\partial(F_1 - F_0)}{\partial q_i}\frac{dq_i}{dt}(t))dt$$

Since $\frac{\partial F_i}{\partial x_{I_r-\sigma}} = \frac{\partial F_i}{\partial y} = 0$ $(i = 1, 2)$, $\frac{\partial (F_1 - F_0)}{\partial q} = 0$ on Σ^σ by (4), we have $(F_1 - F_0)(x, y, q) = 0$. Therefore $(F_1 - F_0)|_{\bigcup_{\sigma \subset I_r} \Sigma^\sigma} = 0$. This means that $\frac{\partial \bar{F}}{\partial t} = 0$ on the set

$$\{(x, y, q, t) \mid x_1 \frac{\partial \bar{F}}{\partial x_1} = \cdots = x_r \frac{\partial \bar{F}}{\partial x_r} = \frac{\partial \bar{F}}{\partial y} = 0 \}.$$

Since $x_1, \cdots, x_r, \frac{\partial \bar{F}}{\partial x_1}, \cdots, \frac{\partial \bar{F}}{\partial x_r}, \frac{\partial \bar{F}}{\partial y_1}, \cdots, \frac{\partial \bar{F}}{\partial y_k}$ are independent on $(\mathbb{H}^r \times \mathbb{R}^{k+n+1}, \{0\} \times [0,1])$, we obtain the claim. Moreover since $j^2 \frac{\partial F_t}{\partial t}(0) = 0$, we have $\eta(0, t) = 0$ for $t \in [0, 1]$.

Since the time dependent vector field

$$X = \sum_{i=1}^{r} \xi_i x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{k} \eta_j \frac{\partial}{\partial y_j}$$

vanishes on $\{x = y = q = 0\}$, the flow $\Phi_t(x, y, q)$ of X with the initial condition $\Phi_0(x, y, q) = (x, y, q)$ exists for all $t \in [0, 1]$. By the uniqueness of the flow, Φ_t is written in the following form:

$$\Phi_t(x, y, q) = (x_1 a_t^1(x, y, q), \cdots, x_r a_t^r(x, y, q), h_t(x, y, q), q),$$

for each $t \in [0, 1]$. Then Φ_1 defines a reticular right equivalence from F_0 to F_1, that is $F_1 \circ \Phi_1 = F_0$. ■

Lemma 3.3 *Let $F_1, F_2 \in \mathfrak{M}(r; k + n)^2$ be S-non-degenerate function germs. Suppose that the following conditions hold:*

$$L_\sigma := L^\sigma_{F_1} = L^\sigma_{F_2} \text{ for } \sigma = I_r - \{1\}, \cdots, I_r - \{r\}, \emptyset,$$

$$F := F_1|_{x=0} = F_2|_{x=0},$$

$$\frac{\partial^2 F}{\partial y^2}(0) = 0, \frac{\partial^2 F}{\partial y \partial q_J}(0) = E_k \quad (\ J \subset I_r, |J| = k\ ),$$

where $\{L^\sigma_{F_i}\}_{\sigma \subset I_r}$ be the symplectic regular r-cubic configuration defined by F_i. Then there exist positive numbers $a_1, \cdots, a_r$ and an $r \times k$-matrix B such that $F_3(x, y, q) = F_2(a_1 x_1, \cdots, a_r x_r,$

$y+xB,q)$ *satisfies* $j^2F_1(0)=j^2F_3(0)$.

As a result, F_1 and F_3 are generating families of the same reticular Lagrangian map and the conditions (1) and (2) in the proof of Theorem 3.2 (3) hold for F_1 and F_3.

Proof. Let $I=I_n-J$. We denote $\frac{\partial^2 F_i}{\partial x\partial y}(0)$ by F^i_{xy} and denote other notations analogously. By hypothesis we have

$$L_\sigma=\{\ (q,\frac{\partial F_i}{\partial q}(x,y,q))\mid x_\sigma=\frac{\partial F_i}{\partial x_{I_r-\sigma}}(x,y,q)=\frac{\partial F_i}{\partial y}(x,y,q)=0\},$$

for $\sigma\subset I_r$, $i=1,2$. Therefore for any vector v in T_0L_σ, there exists $(a^i_\tau,b^i,c^i)\in\mathbb{R}^{|\tau|+k+n}$ (i=1,2) such that $a^i_\tau\geq 0$,

$$\begin{pmatrix} F^i_{x^2_\tau} & F^i_{x_\tau y} & F^i_{x_\tau q}\\ F^i_{yx_\tau} & F_{y^2}(=0) & F_{yq}\end{pmatrix}\begin{pmatrix} a^i_\tau\\ b^i\\ c^i\end{pmatrix}=0 \tag{5}$$

and

$$v=c^i\frac{\partial}{\partial q}|_0+\left(F^i_{qx_\tau}a^i_\tau+F_{qy}b^i+F_{qq}c^i\right)\frac{\partial}{\partial\kappa}|_0, \tag{6}$$

where $\tau=I_r-\sigma$ and (q,κ) are the canonical coordinates of $T^*\mathbb{R}^l$. Since

$$\mathrm{rank}\begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q}\\ F_{y^2} & F_{yq}\end{pmatrix}=|\tau|+k,$$

we can arbitrarily choose a^i_τ.

Fix $(a^1_\tau,b^1,c^1),(a^2_\tau,b^2,c^2)$ which define the same vector in $T^*\mathbb{R}^n$. By comparing the coefficients of $\frac{\partial}{\partial q}|_0,\frac{\partial}{\partial\kappa_I}|_0,\frac{\partial}{\partial\kappa_J}|_0$ of (6), we have

$$c^1 = c^2\ (=c)$$

$$F^1_{q_Ix_\tau}a^1_\tau+F_{q_Iy}b^1 = F^2_{q_Ix_\tau}a^2_\tau+F_{q_Iy}b^2 \tag{7}$$

$$F^1_{q_Jx_\tau}a^1_\tau+b^1 = F^2_{q_Jx_\tau}a^2_\tau+b^2. \tag{8}$$

By (5), we have $F^1_{yx_\tau}a^1_\tau + F_{yq}c = F^2_{yx_\tau}a^2_\tau + F_{yq}c$. Hence

$$F^1_{yx_\tau}a^1_\tau = F^2_{yx_\tau}a^2_\tau. \tag{9}$$

By (7)$-F_{q_Iy}$(8), we have

$$(F^1_{q_Ix_\tau} - F_{q_Iy}F^1_{q_Jx_\tau})a^1_\tau = (F^2_{q_Ix_\tau} - F_{q_Iy}F^2_{q_Jx_\tau})a^2_\tau. \tag{10}$$

By ($(9)^t$,$(10)^t$), we have

$${a^1_\tau}^t(F^1_{x_\tau y}, F^1_{x_\tau q_I} - F^1_{x_\tau q_J}F_{yq_I}) = {a^2_\tau}^t(F^2_{x_\tau y}, F^2_{x_\tau q_I} - F^2_{x_\tau q_J}F_{yq_I}). \tag{11}$$

Otherwise, since F_i is S-non-degenerate, we have

$$\operatorname{rank}\begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q} \\ F^i_{y^2} & F^i_{yq} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q_I} & F^i_{x_\tau q_J} \\ 0 & F_{yq_I} & E_k \end{pmatrix} = |\tau| + k.$$

By multiplying the invertible matrix $\begin{pmatrix} E_{|\tau|} & -F^i_{x_\tau q_J} \\ 0 & E_k \end{pmatrix}$ on the left hand side of the above, we have

$$\operatorname{rank}\begin{pmatrix} F^i_{x_\tau y} & F^i_{x_\tau q_I} - F^i_{x_\tau q_J}F_{yq_I} & 0 \\ 0 & F_{yq_I} & E_k \end{pmatrix} = |\tau| + k.$$

Hence

$$\operatorname{rank}(F^i_{x_\tau y}, F^i_{x_\tau q_I} - F^i_{x_\tau q_J}F_{yq_I}) = |\tau|. \tag{12}$$

Consider the case $\tau = \{s\}$ and $a^1_s = 1$. By (11) and (12) we have $a^2_s > 0$. Therefore if we denote $F_2(a^2_1x_1, \cdots, a^2_rx_r, y, q)$ instead of F_2, then we have

$$(F^1_{xy}, F^1_{xq_I} - F^1_{xq_J}F_{yq_I}) = (F^2_{xy}, F^2_{xq_I} - F^2_{xq_J}F_{yq_I}). \tag{13}$$

Hence $a^1 = a^2 (= a)$. Set $B = F^1_{xq_J} - F^2_{xq_J}$ and define $F_3(x, y, q) = F_2(x, y + xB, q)$. Then we need only to check that $F^1_{xx} = F^3_{xx}, F^1_{xq} = F^3_{xq}$ in order to complete the proof. We have

$$F^3_{xq_J} = F^2_{xq_J} + BF_{yq_J} = F^2_{xq_J} + B = F^2_{xq_J} + F^1_{xq_J} - F^2_{xq_J} = F^1_{xq_J},$$

$$
\begin{aligned}
F^3_{xq_I} &= F^2_{xq_I} + BF_{yq_I} = F^2_{xq_I} + F^1_{xq_J}F_{yq_I} - F^2_{xq_J}F_{yq_I} = (F^2_{xq_I} - F^2_{xq_J}F_{yq_I}) + F^1_{xq_J}F_{yq_I}, \\
&= (F^1_{xq_I} - F^1_{xq_J}F_{yq_I}) + F^1_{xq_J}F_{yq_I} = F^1_{xq_I}.
\end{aligned}
$$

Therefore $F^1_{xq} = F^3_{xq}$.

Finally repeat this proof between F_1 and F_3. In the case $\sigma = \emptyset$ let $(a^1, b^1, c^1), (a^3, b^3, c^3) \in \mathbb{R}^{r+k+n}$ define the same vector. Then we have $a^1 = a^3 (= a)$ by (11) and have $b^1 = b^3$ by (8) and have $(F^1_{xx} - F^3_{xx})a = 0$ by (5). Since a is an arbitrary real number, we have $F^1_{xx} = F^3_{xx}$. ■

Lemma 3.4 *Let $F_1, F_2 \in \mathfrak{M}(r; k+n)^2$ be S-non-degenerate function germs. Suppose that the following conditions hold:*

$$L_\sigma := L^\sigma_{F_1} = L^\sigma_{F_2} \ (\ \forall \sigma \subset I_r \ )$$

$$j^2F_1(0) = j^2F_2(0), \frac{\partial^2 F}{\partial y^2}(0) = 0, \frac{\partial^2 F}{\partial y \partial q_J}(0) = E_k \quad (\ J \subset I_r, |J| = k \ ).$$

Set

$$\Sigma^\sigma_{F_i} = \{ \ (x, y, q) \in (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0) \mid x_\sigma = \frac{\partial F_i}{\partial x_{I_r - \sigma}} = \frac{\partial F_i}{\partial y} = 0 \ \},$$

$$p^\sigma_i : \Sigma^\sigma_{F_i} \xrightarrow{\sim} L_\sigma \ (\ (x, y, q) \mapsto (q, \frac{\partial F_i}{\partial q}) \ )$$

for each $\sigma \subset I_r$. Then there exists $G \in \mathcal{B}(r; k+n)$ such that G preserves q and $G_{*0} = id|_{T_0\mathbb{H}^r \times \mathbb{R}^{k+n}}$ and for each $\sigma \subset I_r$ the following diagram is commutative:

$$
\begin{array}{ccc}
\Sigma^\sigma_{F_1} & \xrightarrow{G|_{\Sigma^\sigma_{F_1}}} & \Sigma^\sigma_{F_2} \\
 & p^\sigma_1 \searrow & \downarrow p^\sigma_2 \\
 & & L_\sigma
\end{array}
$$

As a result $F_3 = F_2 \circ G$ is reticular R-equivalent to F_2 and $\Sigma^\sigma_{F_1} = \Sigma^\sigma_{F_3} (= \Sigma^\sigma)$, $\frac{\partial F_1}{\partial q} - \frac{\partial F_3}{\partial q}|_{\Sigma^\sigma} \equiv 0$ for each $\sigma \subset I_r$.

Proof. For each $\sigma \subset I_r$, we set

$$G_\sigma = (p^\sigma_2)^{-1} \circ p^\sigma_1 : \Sigma^\sigma_{F_1} \xrightarrow{\sim} \Sigma^\sigma_{F_2}.$$

Since $L^{\sigma}_{F_1} = L^{\sigma}_{F_2}$ for each $\sigma \subset I_r$, we have

$$G_{\sigma\cap\sigma'} := G_{\sigma}|_{\Sigma^{\sigma}_{F_1}\cap\Sigma^{\sigma'}_{F_1}} = G_{\sigma'}|_{\Sigma^{\sigma}_{F_1}\cap\Sigma^{\sigma'}_{F_1}} \ (\forall\sigma,\sigma' \subset I_r\ ).$$

Since $j^2F_1(0) = j^2F_2(0)$ and F_1, F_2 are S-non-degenerate, there exist function germs $w_1, \cdots, w_{n-r}$ on $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$ such that

$$x_1, \cdots, x_r, \frac{\partial F_i}{\partial x_1}, \cdots, \frac{\partial F_i}{\partial x_r}, \frac{\partial F_i}{\partial y_1}, \cdots, \frac{\partial F_i}{\partial y_k}, w_1, \cdots, w_{n-r} \quad (\ i = 1, 2\ )$$

define coordinates of $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$. By using analogous methods of [3, lemme i], there exists a diffeomorphism G on $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$ for which the diagrams are commutative. By $j^2F_1(0) = j^2F_2(0)$, we have $G_{*0} = id|_{T_0\mathbb{H}^r\times\mathbb{R}^{k+n}}$. We have to modify G such that $G \in \mathcal{B}(r; k+n)$ and G preserves q.

Since

$$x_1 \circ G|_{x_1 = x_2\frac{\partial F_1}{\partial x_2} = \cdots = x_r\frac{\partial F_1}{\partial x_r} = \frac{\partial F_1}{\partial y} = 0} = x_1 \circ G|_{\bigcup_{1\in\sigma\subset I_r}\Sigma^{\sigma}_{F_1}} = 0,$$

$x_1 \circ G$ can be written in the form:

$$x_1 \circ G = x_1 \cdot a^1_1 + \sum_{i=2}^{r} x_i \frac{\partial F_1}{\partial x_i} \cdot a^1_i + \sum_{j=1}^{k} \frac{\partial F_1}{\partial y_j} \cdot b^1_j,$$

where $b^1_1, \cdots, b^1_k$ are independent on x_1. By $G_{*0} = id$, we have $a^1_1(0) = 1$. For each $i = 2, \cdots, r$, take $a^i_1, \cdots, a^i_r, b^i_1, \cdots, b^i_k$ which have the similar properties. Otherwise since

$$q_i \circ G|_{x_1\frac{\partial F_1}{\partial x_1} = \cdots = x_r\frac{\partial F_1}{\partial x_r} = \frac{\partial F_1}{\partial y} = 0} = q_i \circ G|_{\sum_{\sigma\subset I_r}\Sigma^{\sigma}_{F_1}} = q_i \text{ for } i = 1, \cdots, n\ ,$$

each $q_i \circ G$ can be written in the following form:

$$q_i \circ G = q_i + \sum_{j=1}^{r} x_i \frac{\partial F_1}{\partial x_j} \cdot c^i_j + \sum_{j=1}^{k} \frac{\partial F_1}{\partial y_j} \cdot d^i_j.$$

Define $G'(x, y, q) = (x_1a^1_1, \cdots, x_ra^r_r, y \circ G', q)$, then the diagrams are also commutative for G' and $G'_{*0} = id$, so that $G' \in \mathcal{B}(r; k+n)$ and G' preserves q. ■

4 Stability of unfoldings

In order to study the stabilities of reticular Lagrangian maps, we shall prepare the results of the singularity theory of function germs with respect to *reticular R^+-equivalence*. Basic techniques for the characterization of the stabilities we use in this part depend heavily on the results in this section, however the all arguments are the almost parallel along the ordinary theory of the right-equivalence (cf., [19]), so that we omit the detail.

We denote $J^l(r+k,1)$ the set of l-jets at 0 of germs in $\mathfrak{M}(r;k)$ and let $\pi_l : \mathfrak{M}(r;k) \to J^l(r+k,1)$ be the natural projection. We denote $j^l f(0)$ the l-jet of $f \in \mathfrak{M}(r;k)$.

Lemma 4.1 *Let $f \in \mathfrak{M}(r;k)$ and $O^l_{rR}(j^l f(0))$ be the submanifold of $J^l(r+k,1)$ consist of the image by π_l of the orbit of reticular R-equivalence of f. Put $z = j^l f(0)$. Then*

$$T_z(O^l_{rR}(z)) = \pi_l(\langle x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle).$$

We say that a function germ $f \in \mathfrak{M}(r;k)$ is *reticular R-l-determined* if all function germ which has same l-jet of f is reticular R-equivalent to f.

Lemma 4.2 *Let $f \in \mathfrak{M}(r;k)$ and let*

$$\mathfrak{M}(r;k)^{l+1} \subset \mathfrak{M}(r;k)(\langle x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle + \mathfrak{M}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle) + \mathfrak{M}(r;k)^{l+2},$$

then f is reticular R-l-determined. Conversely let $f \in \mathfrak{M}(r;k)$ be reticular R-l-determined, then

$$\mathfrak{M}(r;k)^{l+1} \subset \langle x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle.$$

Let $F \in \mathfrak{M}(r;k+n_1)$, $G \in \mathfrak{M}(r;k+n_2)$ be unfoldings of $f \in \mathfrak{M}(r;k)$. We say that F *is reticular R^+-f-induced from G* if there exist smooth map germs $\phi : (\mathbb{H}^r \times \mathbb{R}^{k+n_2}, 0) \to$

$(\mathbb{H}^r \times \mathbb{R}^k, 0)$, $\psi : (\mathbb{R}^{n_2}, 0) \to (\mathbb{R}^{n_1}, 0)$ and $\alpha \in \mathfrak{M}(0; n_2)$ satisfying the following conditions:

(1) $\phi((\mathbb{H}^r \cap \{x_\sigma = 0\}) \times \mathbb{R}^{k+n_2}) \subset (\mathbb{H}^r \cap \{x_\sigma = 0\}) \times \mathbb{R}^k$ for $\sigma \subset I_r$.

(2) $G(x, y, v) = F(\phi(x, y, v), \psi(v)) + \alpha(v)$ for $x \in \mathbb{H}^r$, $y \in \mathbb{R}^k$ and $v \in \mathbb{R}^{n_2}$.

Definition 4.3 Here we define several stabilities of unfoldings. Let $f \in \mathfrak{M}(r; k)$ and $F \in \mathfrak{M}(r; k + n)$ be an unfolding of f.

We define a smooth map germ

$$j_1^l F : (\mathbb{R}^{r+k+n}, 0) \longrightarrow (J^l(r + k, 1), j^l f(0))$$

as follow: Let $\tilde{F} : U \to \mathbb{R}$ be a representative of F. For each $(x, y, u) \in U$, We define $F_{(x,y,u)} \in \mathfrak{M}(r; k)$ by $F_{(x,y,u)}(x', y') = F(x + x', y + y', u) - F(x, y, u)$. Now define $j_1^l F(x, y, u)$ =the l-jet of $F_{(x,y,u)}$. $j_1^l F$ depends only on the germ at 0 of F. We say that F is *reticular R^+-l-transversal* if $j_1^l F|_{x=0}$ is transversal to $O_{rR}^l(j^l f(0))$. It is easy to check that F is reticular R$^+$-l-transversal if and only if

$$\mathcal{E}(r; k) = \langle x_1 \frac{\partial f}{\partial x_1}, \cdots, x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k} \rangle_{\mathcal{E}(r;k)} + V_F + \mathfrak{M}(r; k)^{l+1},$$

where $V_F = \langle 1, \frac{\partial F}{\partial u_1}|_{u=0}, \cdots, \frac{\partial F}{\partial u_n}|_{u=0} \rangle_{\mathbb{R}}$.

We say that F is *reticular R^+-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, u_0')$ is reticular R$^+$-equivalent to F for some $(0, y_0, u_0') \in U$.

We say that F is *reticular R^+-versal* if F is reticular R$^+$-f-induced from all unfolding of f.

We say that F is *reticular R^+-infinitesimal versal* if

$$\mathcal{E}(r; k) = \langle x_1 \frac{\partial f}{\partial x_1}, \cdots, x_r \frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k} \rangle_{\mathcal{E}(r;k)} + V_F.$$

We say that F is *reticular R^+-infinitesimal stable* if

$$\begin{aligned}&\mathcal{E}(r;k+n)\\ =\ &\langle x_1\frac{\partial F}{\partial x_1},\cdots,x_r\frac{\partial F}{\partial x_r},\frac{\partial F}{\partial y_1},\cdots,\frac{\partial F}{\partial y_k}\rangle_{\mathcal{E}(r;k+n)}+\langle 1,\frac{\partial F}{\partial u_1},\cdots,\frac{\partial F}{\partial u_n}\rangle_{\mathcal{E}(n)}.\end{aligned}$$

We say that F is *reticular R^+-homotopically stable* if for any smooth path-germ $(\mathbb{R},0)\to\mathcal{E}(r;k+n),t\mapsto F_t$ with $F_0=F$, there exists a smooth path-germ $(\mathbb{R},0)\to\mathcal{B}(r;k+n)\times\mathcal{E}(n),t\mapsto(\Phi_t,\alpha_t)$ with $(\Phi_0,\alpha_0)=(id,0)$ such that each (Φ_t,α_t) is a reticular R$^+$-isomorphism and $F_0=F_t\circ\Phi_t+\alpha_t$.

Theorem 4.4 (Transversality lemma) *Let U be a neighborhood of 0 in $0\in\mathbb{R}^{r+k+n}$ with the coordinates $(x_1,\cdots,x_r,y_1,\cdots,y_k,u_1,\cdots,u_n)$ and A be a submanifold of $J^l(r+k,1)$. Then the set*

$$T_A=\{F\in C^\infty(U,\mathbb{R})\mid j_1^lF|_{x=0}\ \text{is transversal to}\ A\}$$

is dense in $C^\infty(U,\mathbb{R})$ with respect to C^∞-topology, where $j_1^lF(x,y,u)$ is the l-jet of the map $(x',y')\mapsto F(x+x',y+y',u)$ at 0.

The transversality we used is a slightly different for the ordinary one [19], however we can also prove this theorem by the method which is the same as the ordinary method.

Theorem 4.5 *Let $F\in\mathfrak{M}(r;k+n)$ be an unfolding of $f\in\mathfrak{M}(r;k)$. Then the following are equivalent.*

(1) *F is reticular R^+-stable.*

(2) *F is reticular R^+-versal.*

(3) *F is reticular R^+-infinitesimal versal.*

(4) *F is reticular R^+-infinitesimal stable.*

(5) *F is reticular R^+-homotopically stable.*

For $f \in \mathfrak{M}(r;k)$ we define the *reticular R-codimension of f* by the $\mathbb{R}$-dimension of the vector space

$$\mathcal{E}(r;k)/\langle x_1\frac{\partial f}{\partial x_1},\cdots,x_r\frac{\partial f}{\partial x_r},\frac{\partial f}{\partial y_1},\cdots,\frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)}.$$

By the above theorem if $1, a_1, \cdots, a_n \in \mathcal{E}(r;k)$ is a representative of a basis of the vector space, then $f + a_1v_1 + \cdots a_nv_n \in \mathfrak{M}(r;k+n)$ is a reticular R^+-stable unfolding of f.

5 Stability of reticular Lagrangian maps

In this section we shall define several notions of stabilities for reticular Lagrangian maps and prove that they and the notion of stabilities for corresponding generating families are all equivalent.

In order to consider symplectomorphisms and symplectomorphism germs on $T^*\mathbb{R}^n$, we introduce canonical coordinates (Q,P) and (q,p) of $T^*\mathbb{R}^n$, where (Q,P) are the coordinates of the source and (q,p) are the coordinates of the target.

Stability: For any open set U in $T^*\mathbb{R}^n$ we denote $S(U,T^*\mathbb{R}^n)$ the space of symplectic embeddings from U to $T^*\mathbb{R}^n$ with C^∞-topology. We say that a reticular Lagrangian map $\pi\circ i : (\mathbb{L}^0,0) \to (T^*\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ is *stable* if the following holds: For any extension S of i and any representative $\tilde{S} \in S(U,T^*\mathbb{R}^n)$ of S, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{T} \in N_{\tilde{S}}$ the reticular Lagrangian maps $\pi\circ(\tilde{T}|_{\mathbb{L}^0}$ at $x_0)$ and $\pi\circ i$ are Lagrangian equivalent for some $x_0 = (0,\cdots,0;0,\cdots,0,P^0_{r+1},\cdots,P^0_n) \in U$.

Homotopical Stability: Let $\pi\circ i : (\mathbb{L}^0,0) \to (T^*\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ be a reticular Lagrangian map. A map germ $\bar{i} : (\mathbb{L}^0\times\mathbb{R},(0,0)) \to (T^*\mathbb{R}^n,0)((Q,P,t)\mapsto i_t(Q,P))$ is called a *reticular Lagrangian deformation* of i if $i_0 = i$ and there exists a one-parameter family of symplectomorphisms $\bar{S} : (T^*\mathbb{R}^n \times \mathbb{R}, (0,0)) \to (T^*\mathbb{R}^n, 0)((Q,P,t) \mapsto S_t(Q,P))$ such that $i_t = S_t|_{\mathbb{L}^0}$ for t around 0. We call $\bar{S}$ an *extension* of $\bar{i}$. Let $\phi : (\mathbb{L}^0, 0) \to (\mathbb{L}^0, 0)$ be a reticular diffeomorphism. A map germ $\bar{\phi} : (\mathbb{L}^0 \times \mathbb{R}, (0,0)) \to (\mathbb{L}^0, 0)((Q,P,t) \mapsto \phi_t(Q,P))$ is called a *one-parameter deformation of reticular diffeomorphisms* of ϕ if $\phi_0 = \phi$ and there exists a one-parameter family of diffeomorphisms $\bar{\Phi} : (T^*\mathbb{R}^n \times \mathbb{R}, (0,0)) \to (T^*\mathbb{R}^n, 0)((Q,P,t) \mapsto \Phi_t(Q,P))$ such that ϕ_t is a reticular diffeomorphism defined by $\phi_t = \Phi_t|_{\mathbb{L}^0}$ for t around 0. We call $\bar{\Phi}$ an *extension* of $\bar{\phi}$. We say that a reticular Lagrangian map $\pi \circ i : (\mathbb{L}^0, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is *homotopically stable* if for any reticular Lagrangian deformation $\bar{i} = \{i_t\}$ of i there exists a one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\phi_t\}$ of $id_{\mathbb{L}^0}$ and a one-parameter family of Lagrangian equivalences $\bar{\Theta} = \{\Theta_t\}$ with $\Theta_0 = id_{T^*\mathbb{R}^n}$ such that $i_t = \Theta_t \circ i \circ \phi_t$ for t around 0.

Infinitesimal Stability: A vector field v on $(T^*\mathbb{R}^n, 0)$ is said to be *tangent* to $\mathbb{L}^0$ if $v|_{L^0_\sigma}$ is tangent to L^0_σ for all $\sigma \subset I_r$. A function germ H on $(T^*\mathbb{R}^n, 0)$ is said to be *fiber preserving* if there exist function germs $h_0, \cdots, h_n$ on the base of π such that $H(q,p) = \sum_{i=1}^n h_i(q)p_i + h_0(q)$ for $(q,p) \in (T^*\mathbb{R}^n, 0)$. We say that a reticular Lagrangian map $\pi \circ i : (\mathbb{L}^0, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is *infinitesimal stable* if for any function germ f on $(T^*\mathbb{R}^n, 0)$ there exists a fiber preserving function germ H on $(T^*\mathbb{R}^n, 0)$ and a vector field v on $(T^*\mathbb{R}^n, 0)$ such that v is tangent to $\mathbb{L}^0$ and $X_f \circ i = X_H \circ i + i_* v$, where X_f and X_H are the Hamiltonian vector fields of f and H respectively and $i_* v$ is defined by $i_* v = (S_* v) \circ i$ for an extension S of i.

At first we prepare some lemmas to construct continuous maps between mapping spaces. Let U, V be open sets in $\mathbb{R}^n, \mathbb{R}^m$ respectively. We define

$$N_f(l, \varepsilon, K) = \{\ g \in C^\infty(U,V) \mid |D^\alpha(g-f)_x| < \varepsilon\ \forall x \in K, |\alpha| < l\ \}$$

for each $f \in C^\infty(U,V), l \in \mathbb{N}, \varepsilon > 0$ and compact set K in U. Then the family of sets $N_f(l,\varepsilon,K)$ forms a basis for the C^∞-topology on $C^\infty(U,V)$.

Lemma 5.1 *Let U be an open ball around 0 in $\mathbb{R}^n$. Then the map*

$$\int : C^\infty(U,\mathbb{R}) \longrightarrow C^\infty(U,\mathbb{R}) \quad (f \mapsto (x \mapsto \int_0^1 f(tx)dt))$$

is continuous.

Proof. Let $f \in C^\infty(U,\mathbb{R})$ and a neighborhood N of $\int f$ be given. We may assume that $N = N_{\int f}(l,\varepsilon,K)$ for some l, ε, K. Choose a closed ball K' around 0 in U such that $K \subset K'$ and set $N' = N_f(l,\varepsilon,K')$. Then for any $g \in N', x \in K$,

$$\begin{aligned} |D^\alpha(\int g - \int f)_x| &= |D^\alpha(\int_0^1 (g(tx) - f(tx))dt)| \\ &= |\int_0^1 (t^{|\alpha|} D^\alpha(g(tx) - f(tx)))dt| \\ &\leq \int_0^1 t^{|\alpha|} |D^\alpha(g(tx) - f(tx))| dt \\ &< \int_0^1 1 \cdot \varepsilon dt = \varepsilon \end{aligned}$$

for any $|\alpha| < l$. It follows that $\int(N') \subset N$. Hence $\int$ is continuous. ■

Proposition 5.2 *Let U, V be open sets in $\mathbb{R}^n$ satisfying $0 \in U \subset V$ and $i : U \to V$ be the inclusion map. Choose $\varepsilon > 0$ such that $\overline{U_{3\varepsilon}(0)} \subset U$ and set $U_1 = U_{3\varepsilon}(0), V_1 = U_\varepsilon(0)$. Then there exists a neighborhood N_0 of i in $C^\infty(U,V)$ such that $g|_{U_1}$ is embedding and $V_1 \subset g(U_1)$ for $g \in N_0$. Moreover*

$$N_0 \longrightarrow C^\infty(V_1, U) \quad (f \mapsto (g|_{U_1})^{-1}|_{V_1})$$

is continuous.

Proof. We define the neighborhood N_0 of i by

$$g \in N_0 \Leftrightarrow \begin{cases} \frac{\partial g_i}{\partial x_i} > \frac{1}{2}, |\frac{\partial g_i}{\partial x_j}| < \frac{1}{2n} & (i \neq j) \\ \det \frac{\partial g}{\partial x}(x) \neq 0 & \\ |g(x) - x| < \varepsilon & \text{for } x \in \overline{U_1}. \end{cases}$$

Let $g \in N_0$ and $a, b \in U_1$ $(a \neq b)$ be given, we may assume that $|a_1 - b_1| \geq |a_i - b_i| (i = 2, \cdots, n)$. Set $c(t) = (1-t)a + tb, t \in [0,1]$. Since U_1 is convex, we have $c([0,1]) \subset U_1$.

$$\begin{aligned} |g_1(b) - g_1(a)| &= |\int_0^1 \frac{d}{dt} g_1 \circ c(t) dt| \\ &= |\sum_{i=1}^n \int_0^1 \frac{\partial g_1}{\partial x_i} \circ c(t)(b_i - a_i) dt| \\ &\geq |\int_0^1 \frac{\partial g_1}{\partial x_1} \circ c(t)(b_1 - a_1) dt| - \sum_{i=2}^n |\int_0^1 \frac{\partial g_1}{\partial x_i} \circ c(t)(b_i - a_i) dt| \\ &= |b_1 - a_1| |\int_0^1 \frac{\partial g_1}{\partial x_1} \circ c(t) dt| - \sum_{i=2}^n |b_i - a_i| |\int_0^1 \frac{\partial g_1}{\partial x_i} \circ c(t) dt| \\ &> |b_1 - a_1| \frac{1}{2} - (n-1)|b_1 - a_1| \frac{1}{2n} = \frac{1}{2n} |b_1 - a_1| > 0. \end{aligned}$$

It follows that $g|_{U_1}$ is an injective. Hence $g|_{U_1}$ is an embedding. It is easy to prove that $V_1 \subset g(U_1)$ because of the definition of U_1, V_1 and the fact that $|g(x) - x| < \varepsilon$.

Let $f_0 \in N_0$ and a neighborhood N of $g_0 = (f_0|_{U_1})^{-1}|_{V_1}$ be given. We may assume that $N = N_{g_0}(l, \varepsilon', K)$ for a l, ε', K. Since the l-jet extension of $(f|_{U_1})^{-1}|_{V_1}$ is written as a continuous map of the l-jet extension of $f|_{\overline{U_1}}$ for each $f \in N_0$, it follows that there exists $\varepsilon'' > 0$ such that $(f|_{U_1})^{-1}|_{V_1} \in N$ for any $f \in N_{f_0}(l, \varepsilon'', \overline{U_1})$. ■

We have the following lemma as a corollary of Proposition 5.2.

Lemma 5.3 *Let U, V be open sets in $\mathbb{R}^n$ such that $0 \in U$ and let $f_0 : U \to V$ be a embedding. Then there exist a neighborhood U_1 of 0 in U and an open ball V around $f_0(0)$ in V and a neighborhood N_1 of f_0 in $C^\infty(U, V)$ such that $f|_{U_1}$ is embedding and $V_1 \subset f(U_1)$ for all $f \in N_1$. Moreover*

$$N_1 \longrightarrow C^\infty(V_1, U) \quad (f \mapsto (f|_{U_1})^{-1}|_{V_1})$$

is continuous.

Lemma 5.4 *For any one-parameter family of Lagrangian equivalences $\bar{\Theta} : (T^*\mathbb{R}^n \times \mathbb{R}, (0, 0$ $)) \to (T^*\mathbb{R}^n, 0)((Q, P, t) \mapsto \Theta_t(Q, P))$ with $\Theta_0 = id$, there exists a fiber preserving function germ H on $(T^*\mathbb{R}^n, 0)$ such that $X_H = \frac{d\Theta_t}{dt}|_{t=0}$. Conversely for any fiber preserving function germ H on $(T^*\mathbb{R}^n, 0)$, the flow $\bar{\Theta} = \{\Theta_t\}$ of X_H with the initial condition $\Theta_0 = id : (T^*\mathbb{R}^n, 0) \to (T^*\mathbb{R}^n, 0)$ is a one-parameter family of Lagrangian equivalences.*

Theorem 5.5 *Let $\pi \circ i : (\mathbb{L}^0, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a reticular Lagrangian map with the generating family $F(x, y, q) \in \mathfrak{M}(r; k + n)^2$. Then the following are equivalent.*

(1) F is a reticular R^+-stable unfolding of $F|_{q=0}$.

(2) $\pi \circ i$ is homotopically stable.

(3) $\pi \circ i$ is infinitesimal stable.

(4) For any function germ f on $(T^*\mathbb{R}^n, 0)$, there exists a fiber preserving function germ H on $(T^*\mathbb{R}^n, 0)$ such that $f \circ i = H \circ i$.

(5) $\pi \circ i$ is stable.

Proof. We shall prove (1)⇔(5), (1)⇒(2)⇒(3)⇒(4)⇒(1).

(1)⇒(5). Let S_0 be an extension of i and $\tilde{S}_0 \in S(U, T^*\mathbb{R}^n)$ be a representative of S_0. We shall construct the map (14) which maps a symplectic embedding around $\tilde{S}_0$ to a function around a representative of F. Define

$$\pi_{\tilde{S}} : U \longrightarrow \mathbb{R}^{2n} \quad ((Q, P) \mapsto (Q, p_{\tilde{S}}(Q, P)))$$

for each $\tilde{S} = (q_{\tilde{S}}, p_{\tilde{S}}) \in S(U, T^*\mathbb{R}^n)$. By taking some Lagrangian equivalence of $\pi \circ i$ and shrinking U if necessary, we may assume that $\pi_{\tilde{S}_0}$ is embedding. By Lemma 5.3, there exist a neighborhood $N_{\tilde{S}_0}$ of $\tilde{S}_0$ and a neighborhood U_1 of 0 in U and a convex neighborhood V of 0 in $\mathbb{R}^{2n}$ with the coordinates (Q,p) such that the map

$$N_{\tilde{S}_0} \longrightarrow C^\infty(V,U) \ (\ \tilde{S} \mapsto (\pi_{\tilde{S}}|_{U_1})^{-1}|_V = (id_Q, P_{\tilde{S}})\ )$$

is well defined and continuous. Let $\tilde{S} \in N_{\tilde{S}_0}$. Then the set

$$P_{\tilde{S}} = \{\ (Q, P_{\tilde{S}}(Q,p); q_{\tilde{S}}(Q,p), p) \in U \times T^*\mathbb{R}^n \mid (Q,p) \in V\},$$

where $q_{\tilde{S}}(Q,p) := q_{\tilde{S}}(Q, P_{\tilde{S}}(Q,p))$ for $(Q,p) \in V$, is a canonical relation associated $\tilde{S}$. Therefore there exists a smooth function $H_{\tilde{S}}$ on V such that $H_{\tilde{S}}(0,0) = 0$ and

$$P_{\tilde{S}} = \{\ (Q, -\frac{\partial H_{\tilde{S}}}{\partial Q}(Q,p), -\frac{\partial H_{\tilde{S}}}{\partial p}(Q,p), p)\ \}.$$

But

$$\begin{aligned} H_{\tilde{S}}(Q,p) &= \int_0^1 \frac{d}{dt} H_{\tilde{S}}(tQ,tp)dt \\ &= \sum_{i=1}^k Q_i \int_0^1 \frac{\partial H_{\tilde{S}}}{\partial Q_i}(tQ,tp)dt + \sum_{i=1}^k p_i \int_0^1 \frac{\partial H_{\tilde{S}}}{\partial p_i}(tQ,tp)dt \\ &= \sum_{i=1}^k Q_i(\int \frac{\partial H_{\tilde{S}}}{\partial Q_i})(Q,p) + \sum_{i=1}^k p_i(\int \frac{\partial H_{\tilde{S}}}{\partial p_i})(Q,p) \end{aligned}$$

and the maps

$$N_{\tilde{S}_0} \longrightarrow C^\infty(V,\mathbb{R}) \quad (\ \tilde{S} \mapsto \frac{\partial H_{\tilde{S}}}{\partial Q_i}(= -p^i_{\tilde{S}}), \frac{\partial H_{\tilde{S}}}{\partial p_i}(= -q^i_{\tilde{S}})\ )\ (\ i = 1, \cdots, n\ )$$

are continuous, we have by Lemma 5.1 that the map

$$N_{\tilde{S}_0} \longrightarrow C^\infty(V,\mathbb{R})\ (\ \tilde{S} \mapsto H_{\tilde{S}}\ )$$

is continuous. Let $V' = V \cap \{\ Q_{r+1} = \cdots = Q_n = 0\ \}$. Now we define the following continuous map

$$\phi : N_{\tilde{S}_0} \longrightarrow C^\infty(V' \times \mathbb{R}^n, \mathbb{R})\ (\ \tilde{S} \mapsto \tilde{F}_{\tilde{S}}(x,y,q) = H_{\tilde{S}}(x,0;y) + < y, q >\ ). \qquad (14)$$

Since $\tilde{F}_{\tilde{S}_0}|_{\mathbb{H}^r\times\mathbb{R}^{2n}}$ at 0 is a generating family of $\pi\circ i$, we may assume that $\tilde{F}_0=\tilde{F}_{\tilde{S}_0}$ is a representative of F. Since F is a reticular stable unfolding of $F|_{q=0}$, there exists a neighborhood $N_{\tilde{F}_0}$ of $\tilde{F}_0$ such that for any $\tilde{G}\in N_{\tilde{F}_0}$ the function germ $\tilde{G}$ at $(0,y^0,q^0)$ and F are reticular R^+-equivalent for some $(0,y^0,q^0)\in V'\times\mathbb{R}^n$. Set $N'_{\tilde{S}_0}=\phi^{-1}(N_{\tilde{F}_0})$. Let $\tilde{S}\in N'_{\tilde{S}_0}$. Take $(0,y^0,q^0)\in V'\times\mathbb{R}^n$ such that the above condition holds for $\tilde{F}_{\tilde{S}}$. If we denote $\{L^{\tilde{S}_0}_\sigma\}_{\sigma\subset I_r}$ the symplectic regular r-cubic configuration defined by $F_{\tilde{S}}=\tilde{F}_{\tilde{S}}|_{\mathbb{H}^r\times\mathbb{R}^{2n}}$ at $(0,y^0,q^0)$, then for each $\sigma\subset I_r$

$$\begin{aligned}
L^{\tilde{S}_0}_\sigma &= \{\ (q_0+q,\frac{\partial F_{\tilde{S}}}{\partial q}(x,y_0+y,q_0+q))|x_\sigma=\frac{\partial F_{\tilde{S}}}{\partial x_{I_r-\sigma}}=\frac{\partial F_{\tilde{S}}}{\partial y}=0,x_{I_r-\sigma}\geq 0\}\\
&= \{\ (q_0+q,y_0+y)|x_\sigma=\frac{\partial H_{\tilde{S}}}{\partial x_{I_r-\sigma}}(x,0;y_0+y)=\\
&\qquad\qquad \frac{\partial H_{\tilde{S}}}{\partial y}(x,0;y_0+y)+q_0+q=0,x_{I_r-\sigma}\geq 0\}\\
&= \{\ (-\frac{\partial H_{\tilde{S}}}{\partial p}(Q;y_0+p),y_0+p)|Q_\sigma=\frac{\partial H_{\tilde{S}}}{\partial Q_{I_r-\sigma}}(Q;y_0+p)=\\
&\qquad\qquad Q_{r+1}=\cdots=Q_n=0,Q_{I_r-\sigma}\geq 0\}\\
&= \tilde{S}((L^0_\sigma+(0;0,P_0))),
\end{aligned}$$

where $(0;0,P_0)=\tilde{S}^{-1}(q_0,y_0)$. This implies the reticular Lagrangian maps $\pi\circ(\tilde{S}|_{\mathbb{L}^0\text{ at }(0;0,P_0)})$ and $\pi\circ i$ are Lagrangian equivalent.

(5)⇒(1). Let S_0 be an extension of i. By taking some Lagrangian equivalence of $\pi\circ i$, we may assume that there exists a generating function $T_0(Q,p)$ of the canonical relation P_{S_0} associated with S_0. Then $F_0(x,y,q)=T_0(x,0;y)+<y,q>\in\mathfrak{M}^2(r;n+n)$ is a generating family of $\pi\circ i$. We prove that F_0 is reticular R^+-stable unfolding of $F_0|_{q=0}$. Let $\tilde{F}_0\in C^\infty(U,\mathbb{R})$ be a representative of F_0. We construct the map (15) which maps a function around $\tilde{F}_0$ to a symplectic embedding around a representative of S_0. The following construction is summarized in the diagram after the proof.

By shrinking U if necessary, we may assume that there exist a neighborhood U_1 of 0 in $\mathbb{R}^n$ with the coordinates Q, U_2 of 0 in $\mathbb{R}^n$ with the coordinates y, U_3 of 0 in $\mathbb{R}^n$ with the coordinates q and $\tilde{T_0}(Q,y) \in C^\infty(U_1 \times U_2, \mathbb{R})$ such that the following conditions hold:

(a) $\tilde{T_0}$ is a representative of T_0

(b) The map $U = (U_1 \cap \{Q_{r+1} = \cdots = Q_n = 0\}) \times U_2 \times U_3$

(c) $U_1 \times U_2 \times U_3 \to U_1 \times U_2 \times \mathbb{R}^n$ given by $(Q,y,q) \mapsto (Q, y, \frac{\partial \tilde{T_0}}{\partial y}(Q,y) + q)$ is an embedding.

(d) The map $U_1 \times U_2 \to U_1 \times \mathbb{R}^n$ given by $(Q,y) \mapsto (Q, -\frac{\partial \tilde{T_0}}{\partial Q}(Q,y)$ is an embedding.

Define the representative $\tilde{F_0} \in C^\infty(U, \mathbb{R})$ of F_0 by $\tilde{F}(x,y,q) = \tilde{T}(x,0;y) + \langle y, q\rangle$ and define $\bar{F_0} \in C^\infty(U_1 \times U_2 \times U_3, \mathbb{R})$ by $\bar{F_0}(Q,y,q) = \tilde{T}(Q,y) + \langle y, q\rangle$. Since the map

$$C^\infty(U,\mathbb{R}) \to C^\infty(U_1 \times U_2 \times U_3, \mathbb{R}) \quad (\ \tilde{F} \mapsto \bar{F}(Q,y,q) = \bar{F_0}(Q,y,q) + (\tilde{F} - \tilde{F_0})(Q',y,q)\ ),$$

where $Q' = (Q_1, \cdots, Q_r)$, is continuous, the map

$$C^\infty(U,\mathbb{R}) \to C^\infty(U_1 \times U_2 \times U_3, U_1 \times U_2 \times \mathbb{R}^n) \ \ (\ \tilde{F} \mapsto \phi_{\tilde{F}}(Q,y,q) = (Q, y, \frac{\partial \bar{F}}{\partial y})\ )$$

is also continuous. Since $\phi_{\tilde{F_0}}$ is embedding by (c), there exist a neighborhood $N^1_{\tilde{F_0}}$ of $\tilde{F_0}$ and a neighborhood U' of 0 in $U_1 \times U_2 \times U_3$ and a open ball V around 0 in $U_1 \times U_2 \times \mathbb{R}^n$ such that

$$N^1_{\tilde{F_0}} \to C^\infty(V, U_1 \times U_2 \times U_3) \ \ (\ \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_V\ )$$

is well defined and continuous. Let $V_1 = V \cap (U_1 \times U_2 \times \{0\})$. Then

$$N^1_{\tilde{F_0}} \to C^\infty(V_1, U_1 \times U_2 \times U_3) \ \ (\ \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}\ )$$

is also continuous. We denote $(\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}(Q,y)$ by $(Q, y, q_{\tilde{F}}(Q,y))$ for $(Q,y) \in V_1$. Then the map

$$N^1_{\tilde{F_0}} \to C^\infty(V_1, U_1 \times \mathbb{R}^n) \ \ (\ \tilde{F} \mapsto \psi_{\tilde{F}}(Q,y) = (Q, -\frac{\partial \bar{F}}{\partial Q}(Q, y, q_{\tilde{F}}(Q,y))\ )\ )$$

is also continuous. Since $\psi_{\tilde{F}_0}$ is embedding by (d), there exists a neighborhood $N^2_{\tilde{F}_0}$ of $\tilde{F}_0$ in $N^1_{\tilde{F}_0}$ and a neighborhood V_2 of 0 in V_1 and a neighborhood W of 0 in $U_1 \times \mathbb{R}^n$ such that the map

$$N^2_{\tilde{F}_0} \to C^\infty(W, V_1) \quad (\tilde{F} \mapsto (\psi_{\tilde{F}}|_{V_2})^{-1}|_W)$$

is well defined and continuous. We denote $(\psi_{\tilde{F}}|_{V_2})^{-1}|_W(Q,P)$ by $(Q, y_{\tilde{F}}(Q,P))$. Then the map

$$N^2_{\tilde{F}_0} \to C^\infty(W, U_1 \times U_2 \times U_3) \ \ (\tilde{F} \mapsto (\ (Q,P) \mapsto (Q, y_{\tilde{F}}(Q,P), q_{\tilde{F}}(Q, y_{\tilde{F}}(Q,P))) \ ) \ )$$

is also continuous. Hence the map

$$N^2_{\tilde{F}_0} \to S(W, T^*\mathbb{R}^n) \ \ (\tilde{F} \mapsto \tilde{S}_{\tilde{F}}(Q,P) = (q_{\tilde{F}}, \frac{\partial \tilde{F}}{\partial q}(Q, y_{\tilde{F}}, q_{\tilde{F}}) \ ) \ ) \tag{15}$$

is well defined and continuous. Since $\tilde{S}_{\tilde{F}_0}$ is a representative of S_0, there exists a neighborhood $N^3_{\tilde{F}_0}$ of $\tilde{F}_0$ in $N^2_{\tilde{F}_0}$ such that for any $\tilde{F} \in N^3_{\tilde{F}_0}$ the reticular Lagrangian maps $\pi \circ (\tilde{S}_{\tilde{F}}|_{\mathbb{L}^0 \text{ at } (Q^0,P^0)})$ and $\pi \circ i$ are Lagrangian equivalent for some $(Q^0, P^0) = (0, \cdots, 0; 0, \cdots, 0, P^0_{r+1}, \cdots, P^0_n) \in W$. Let $(0, y^0, q^0) = (0, y_{\tilde{F}}(Q^0, P^0), q_{\tilde{F}}(Q^0, y_{\tilde{F}}(Q^0, P^0)))$. Since $\tilde{F}$ at $(0, y^0, q^0)$ is a generating family of $\pi \circ (\tilde{S}_{\tilde{F}}|_{\mathbb{L}^0 \text{ at } (Q^0,P^0)})$, $\tilde{F}|_{\mathbb{H}^r \times \mathbb{R}^{2n}}$ at $(0, y^0, q^0)$ and F_0 is reticular R^+-equivalent.

$$\begin{array}{ccccccc}
U_1 \times U_2 \times U_3 & = & U_1 \times U_2 \times U_3 & & U_1 \times \mathbb{R}^n & \supset & W \\
(Q,y,q) & & (Q,y,q_{\tilde{F}}) & \to & (Q, -\frac{\partial \tilde{F}}{\partial Q}(Q,y,q_{\tilde{F}})) & = & (Q,P) \\
\phi_{\tilde{F}} \downarrow & & \uparrow & \nearrow \psi_{\tilde{F}} & & & \downarrow \tilde{S}_{\tilde{F}} \\
(Q, y, \frac{\partial \tilde{F}}{\partial y}) & & (Q,y,0) & & & & (q_{\tilde{F}}, \frac{\partial \tilde{F}}{\partial q}(Q, y_{\tilde{F}}, q_{\tilde{F}})) \\
U_1 \times U_2 \times \mathbb{R}^n & \supset & V_1 & & & & T^*\mathbb{R}^n
\end{array}$$

(1)⇒(2). Let $\bar{i} : (\mathbb{L}^0 \times \mathbb{R}, (0,0)) \to (T^*\mathbb{R}^n, 0)((Q,P,t) \mapsto i_t(Q,P))$ be a reticular Lagrangian deformation of i. Take a one-parameter family of symplectomorphisms $\bar{S} : (T^*\mathbb{R}^n \times \mathbb{R}, (0,0) \ ) \to (T^*\mathbb{R}^n, 0)((Q,P,t) \mapsto S_t(Q,P) = (q_t(Q,P), p_t(Q,P)))$ such that $i_t = S_t|_{\mathbb{L}^0}$ for t around 0. We may assume that there exists a function germ $\bar{T} : (\mathbb{R}^{2l} \times$

$\mathbb{R},(0,0)) \to (\mathbb{R},0)((Q,p,t) \mapsto T_t(Q,p))$ such that T_t is a generating function of the canonical relation associated with S_t for t around 0. Define $F(x,y,q,t) \in \mathcal{E}(r;n+n+1)$ by $F(x,y,q,t) = F_t(x,y,q) = T_t(x,0;y) + \langle y,q\rangle$, then F_t is a generating family of $\pi \circ i_t$ for all t. By hypothesis, there exists a one-parameter family of reticular R^+-equivalences of the form

$$F_t(x,y,q) = F(x_1a_t^1(x,y,q),\cdots,x_ra_t^r(x,y,q),h_t(x,y,q),g_t(q)) + \alpha_t(q).$$

Set a one-parameter family of Lagrangian equivalences $\bar{\Theta} = \{\Theta_t\}$ by $\Theta_t = g_t^* + d\alpha_t|_{\pi\circ g_t^*}$. Then we have $i_t(L_\sigma^0) = \Theta_t \circ i(L_\sigma^0)$ for all $\sigma \subset I_r, t$ around 0. Therefore we may define the one-parameter family of reticular diffeomorphisms $\bar{\phi} = \{\phi_t\}$ by $\phi_t = (S_0)^{-1} \circ \Theta_t^{-1} \circ S_t|_{\mathbb{L}^0}$. Then we have $i_t = \Theta_t \circ i \circ \phi_t$ for t around 0.

(2)⇒(3). Take an extension S of i. Let a function germ f on $(T^*\mathbb{R}^n,0)$ be given. Let $\bar{S} = \{\bar{S}_t\}$ be the flow of X_f with the initial condition $\bar{S}_0 = S$. Because $\bar{i} = \{\bar{i}_t = \bar{S}_t|_{\mathbb{L}^0}\}$ is a reticular Lagrangian deformation of i, there exists a one-parameter family of Lagrangian equivalences $\bar{\Theta} = \{\bar{\Theta}_t\}$ with $\bar{\Theta}_0 = id$ and a one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\bar{\phi}_t\}$ of id such that $\bar{i}_t = \bar{\Theta}_t \circ i \circ \bar{\phi}_t$ for t around 0. Let $\bar{\Phi} = \{\Phi_t\}$: $(T^*\mathbb{R}^n \times \mathbb{R},(0,0)) \to (T^*\mathbb{R}^n,0)$ be an extension of $\bar{\phi}$. Then we have

$$X_f \circ i = \frac{d\bar{S}_t}{dt}|_{t=0}|_{\mathbb{L}^0} = \frac{d\bar{\Theta}_t}{dt}|_{t=0} \circ i + (S_*\frac{d\bar{\Phi}_t}{dt}|_{t=0}) \circ i = X_H \circ i + i_*v.$$

This implies that $\pi \circ i$ is infinitesimal stable.

(3)⇒(4). Let a function germ f on $(T^*\mathbb{R}^n,0)$ be given. By hypothesis, there exists a fiber preserving function germ H on $(T^*\mathbb{R}^n,0)$ and a vector field v on $(T^*\mathbb{R}^n,0)$ such that v is tangent to $\mathbb{L}^0$ and $X_f \circ i = X_H \circ i + i_*v$. Set $i_\sigma = i|_{L_\sigma^0}, v_\sigma = v|_{L_\sigma^0}$ for each $\sigma \subset I_r$, then it is easy to prove that $(f-H)\circ i_\sigma =$ constant because $X_f \circ i_\sigma = X_H \circ i_\sigma + (i_\sigma)_*v_\sigma$. Since $\mathbb{L}^0$ is connected, we have that $(f-H)\circ i =$ constant. By replacing $H+$ constant by H if necessary, we have $f \circ i = H \circ i$.

(4)$\Rightarrow$(1). Take an extension $S = (q,p)$ of i. We may assume that there exists a generating function $T = T(Q,p)$ of the canonical relation associated with S. We define a generating family $F(x,y,q) \in \mathfrak{M}(r; n+n)^2$ of $\pi \circ i$ by $F(x,y,q) = T(x,0;y) + \langle y, q\rangle$. Since $(Q,P) \mapsto (q(Q,P), P)$ is invertible, there exists $I \subset \{1, \cdots, n\}(|I| = r)$ such that $\phi : (x,y) \mapsto (q_I(x,0;y), y), x = (x_1, \cdots, x_r)$, is also invertible. Otherwise since $(Q,p) \mapsto (Q, P(Q,p))$ is invertible, $\psi : (x,y) \mapsto (x, P(x,0;y))$ is also invertible. We define $S' = \phi \circ \psi^{-1}$.

$$\begin{array}{ccc} (Q,p) & \to & (q,p) \\ \downarrow & \nearrow S & \\ (Q,P) & & \end{array} \qquad\qquad \begin{array}{ccc} (x,y) & \overset{\phi}{\to} & (q_I,p) \\ \downarrow \psi & \nearrow S' & \\ (x,P) & & \end{array}$$

Let $f \in \mathcal{E}(r;k)$ be given. Set $g(q,y) = f \circ \phi^{-1}(q_I, y)$. Since $S(x,0;P)|_{x_1P_1=\cdots=x_rP_r=0, x\geq 0} = i$, there exists a fiber preserving function germ $H(q,p) = \sum_{i=1}^n h_i(q)p_i + h_0(q)$ on $(T^*\mathbb{R}^n, 0)$ such that

$$g \circ S(x,0;P)|_{x_1P_1=\cdots=x_rP_r=0, x\geq 0} = H \circ S(x,0;P).$$

Therefore there exist function germs $a_1, \cdots, a_r \in \mathcal{E}(r;n)$ such that

$$g \circ S(x,0;P) = H \circ S(x,0;P) + \sum_{j=1}^r x_j P_j a_j(x,P) \text{ for } (x,P) \in (\mathbb{H}^r \times \mathbb{R}^n, 0).$$

Hence

$$\begin{aligned}
f(x,y) &= (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) \circ \psi(x,y) = g \circ S' \circ \psi(x,y) \\
&= g \circ S(x,0;P(x,0;y)) = g(q(x,0,y),y) \\
&= \sum_{i=1}^n h_i(q(x,0;y))y_i + h_0(q(x,0;y)) + \sum_{j=1}^r x_j P_j(x,0;y) a'_j(x,y) \\
&= \sum_{i=1}^n h_i(-\frac{\partial T}{\partial y}(x,0;y))y_i + h_0(-\frac{\partial T}{\partial y}(x,0;y)) + \sum_{j=1}^r x_j(-\frac{\partial T}{\partial x_j}(x,0;y))a'_j(x,y) \\
&\equiv \sum_{i=1}^n h_i(0)y_i + h_0(0) \bmod \langle x_1\frac{\partial T}{\partial x_1}(x,0;y), \cdots, x_r\frac{\partial T}{\partial x_r}(x,0;y), \frac{\partial T}{\partial y}(x,0;y)\rangle_{\mathcal{E}(r;n)}.
\end{aligned}$$

, where $a'_j(x,y) = a_i(x, P(x,0,y))$ for $j = 1, \cdots, r$. This implies that F is a reticular R^+ infinitesimal versal unfolding of f. ■

6 Adjacencies of singularities

We shall study the structure of the caustics of stable symplectic regular r-cubic configurations. Firstly we investigate the adjacencies of singularities classified in Section 7 because the investigation of caustics means that of adjacencies of corresponding functions germs under reticular R-equivalence. The following list is the classification list of simple or uni-modal singularities. This includes the classification list of singularities of R-codimension≤ 7. Therefore the stable symplectic regular r-cubic configurations in manifolds of dimension≤ 6 are classified.

The classification list of simple or unimodal singularities under reticular R-equivalence

r	k	Normal form	rR-codim	Conditions	Notation
1	0	$\pm x^n$	n	$n \geq 2$	B_n
1	1	$xy \pm y^n$	n	$n \geq 3$	C_n
		$\pm x^2 + y^3$	4		F_4
		$\pm x^3 + ax^2y + y^3$	6	$4a^2 + 9 \neq 0$	$F_{1,0}$
		$ax^{n+3} \pm xy^2 + y^3$	$n+6$	$n \geq 1, a \neq 0$	$F_{1,n}$
		$\pm x^4 + y^3 + ax^3y$	8		F_8
		$\pm x^3y + y^3 + ax^2y^2$	9		F_9
		$\pm x^5 + y^3 + ax^4y$	10		F_{10}
		$\pm y^4 + axy^2 \pm x^2$	6	$a^2 \neq \pm \pm 4$	$K_{4,2}$
		$\pm y^4 \pm xy^2 + ax^n$	$n+4$	$n > 2, a \neq 0$	$K_{4,n}$
		$\varepsilon y^n \pm xy^2 + ax^m$	$n+m$	$n > 4, m \geq 2, \varepsilon^n = 1, a \neq 0$	$K_{n,m}$
		$\pm(x \pm y^2)^2 + ax^ny$	$2n+3$	$n > 1, a > 0$	$K^{\#}_{1,2n-3}$
		$\pm(x \pm y^2)^2 + ax^n$	$2n+2$	$n > 2, a \neq 0$	$K^{\#}_{1,2n-4}$
		$\pm y^4 + x^2y + ax^3$	8		K^*_8
		$\pm y^4 \pm x^3 + ax^2y^2$	9		K^*_9
		$y^5 \pm x^2 + axy^3$	8		K^{**}_8
1	2	$y_1^2y_2 \pm y_2^3 + xy_1 + axy_2$	6		$L_6 = D_{4,1}$
		$y_1^2y_2 \pm y_2^{n-1} + axy_1^m \pm xy_2$	$n+m+1$	$a^m > 0$	$D_{n,m}$
		$y_1^3 \pm y_2^4 + axy_1 + xy_2$	8		$E_{6,0}$
		$y_1^3 + y_1y_2^3 + axy_1 \pm xy_2$	9		$E_{7,0}$
		$y_1^3 + y_2^5 + axy_1 \pm xy_2$	10		$E_{8,0}$
		$y_1^2y_2 \pm y^4 + xy_1 + axy_2^2$	8		D_5^1
		$y_1^3 \pm y_2^4 \pm xy_1 + axy_2^2$	9		$E_{6,1}$
		$y_1^2y_2 \pm y_2^3 \pm x^2 + axy_1^2$	8		D_4^2
2	0	$\varepsilon x_1^2 + \delta x_2^2$	4		$B_{2,2}^{\varepsilon,\delta,0}$
		$\varepsilon x_1^2 + ax_1x_2 + \delta x_2^2$	4	$a^2 < 4,\ \varepsilon = \delta,\ a \neq 0$	$B_{2,2}^{\varepsilon,\delta,\overset{<}{\alpha}}$
		$\varepsilon x_1^2 + ax_1x_2 + \delta x_2^2$	4	$a^2 > 4,\ \varepsilon = \delta$	$B_{2,2}^{\varepsilon,\delta,\overset{>}{\alpha}}$
		$\varepsilon x_1^2 + ax_1x_2 + \delta x_2^2$	4	$\varepsilon \neq \delta, a \neq 0$	$B_{2,2}^{\varepsilon,\delta,\alpha}$
		$\varepsilon(x_1 + \delta x_2)^2 + ax_2^n$	$n+2$	$n \geq 3, a \neq 0$	$B_{2,2,n}^{\varepsilon,\delta,\alpha}$
		$\varepsilon x_1^n + ax_1x_2 + \delta x_2^m$	$n+m$	$n+m \geq 5, a \neq 0$	$B_{n,m}^{\varepsilon,\delta,\alpha}$
		$\varepsilon x_1^2 + ax_1x_2^2 + \delta x_2^3$	6		$B_{2,3'}^{\varepsilon,\delta,\alpha}$
		$\varepsilon x_1^3 + ax_1^2x_2 + \delta x_2^2$	6		$B_{3,2'}^{\varepsilon,\delta,\alpha}$
2	1	$\varepsilon y^n + x_1y + \delta x_2y + ax_2^m$	$n+m$	$n \geq 3, m \geq 2, a \neq 0$	$C_{n,m}^{\varepsilon,\delta,\alpha}$
		$\varepsilon y^3 + x_1y + ax_2y^2 + \delta x_2^2$	6		$C_{3,2,1}^{\varepsilon,\delta,\alpha}$
		$\varepsilon y^3 + x_2y + ax_1y^2 + \delta x_1^2$	6		$C_{3,2,2}^{\varepsilon,\delta,\alpha}$

In the case L_6 $a^2 \pm 1 \neq 0$, while in the case $D_{n,m}$ $a \neq 0, n \geq 4, m \geq 1, n+m > 5$. In the case $r = 2$, $\varepsilon = \pm 1$, $\delta = \pm 1$ and if $a = 0$ then $\alpha = 0$ and if $a \neq 0$ then α is the sign of a

The cases $r = 0$ and $r = 1$ were already studied as ordinary singularity and boundary singularity (for example, see [2],[9],[10]). Hence we study the case $r = 2$. From the view point of caustics, we must investigate three type of adjacencies of singularities: the first is the ordinary adjacencies, the second is the adjacency given by forgetting a boundary of the corner. For example, consider $B_{2,3}^{+,+,\alpha}$ singularity which is the orbit of $x_1^2 + ax_1x_2 + x_2^3 \in \mathfrak{M}(2;0)^2$. If we forget the boundary defined by $x_2 = 0$, this function is reticular R-equivalent to $y^3 + x_1y \in \mathfrak{M}(1;1)$. Therefore we regard $B_{2,3}^{+,+,\alpha}$ is adjacent to C_3^+. This adjacency appears as the union of the caustic $C_\emptyset$ and the quasicaustic $Q_{\emptyset,1}$ of the symplectic regular r-cubic configuration defined by a versal unfolding of $x_1^2 + ax_1x_2 + x_2^3$. The third is the adjacency given by restriction singularities to $x_1 = 0$ or $x_2 = 0$. For example, consider $C_{3,2}^{-,+,\alpha}$ singularity which is the orbit of $-y^3 + x_1y + x_2y + ax_2^2 \in \mathfrak{M}(2;1)$. If we restrict this to $x_2 = 0$, then this is equal to $-y^3 + xy \in \mathfrak{M}(1;1)$. Hence we regard $C_{3,2}^{-,+,a}$ is adjacent to C_3^-. This adjacency appears as the union of the caustic C_2 and the quasicaustic $Q_{2,\{1,2\}}$ of the symplectic regular r-cubic configuration defined by a versal unfolding of $-y^3 + x_1y + x_2y + ax_2^2$.

We shall draw the pictures of stable caustics in manifolds of dimension≤ 4 at the last part of this part. The caustics of $B_{2,2,3}$, $B_{2,3}$ and $C_{3,2}$ are diffeomorphic to (the pictures)$\times(\mathbb{R},0)$ and the caustics of $B_{3,2}^{\varepsilon,\delta,\alpha}$ are diffeomorphic to one of $B_{2,3}^{\delta,\varepsilon,\alpha}$.

The adjacencies of unimodal singularities on the 2-corner:

$$
\begin{array}{ccccccccc}
 & & & & B_{3,2'}^{\varepsilon_3,\delta_2,\alpha_{32}} & & & & \\
 & & & & \downarrow & & & & \\
 & & B_{2,2} & \longleftarrow & B_{3,2}^{\varepsilon_3,\delta_2,\alpha} & \longleftarrow & B_{4,2}^{\varepsilon_4,\delta_2,\alpha} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
B_{2,3'}^{\varepsilon_2,\delta_3,\alpha_{23}} & \longrightarrow & B_{2,3}^{\varepsilon_2,\delta_3,\alpha} & \longleftarrow & B_{3,3}^{\varepsilon_3,\delta_3,\alpha} & \longleftarrow & B_{4,3}^{\varepsilon_4,\delta_3,\alpha} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & B_{2,4}^{\varepsilon_2,\delta_4,\alpha} & \longleftarrow & B_{3,4}^{\varepsilon_3,\delta_4,\alpha} & \longleftarrow & B_{4,4}^{\varepsilon_4,\delta_4,\alpha} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & B_{2,2} & \longleftarrow & B^{\varepsilon_2,\delta,\beta_3}_{2,2,3} & \longleftarrow & B^{\varepsilon_2,\delta,\beta_4}_{2,2,4} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
C^{\varepsilon_3,\beta_2,\beta'}_{3,2,1} & \longrightarrow & C^{\varepsilon_3,\delta,\beta_2}_{3,2} & \longleftarrow & C^{\varepsilon_3,\delta,\beta_3}_{3,3} & \longleftarrow & C^{\varepsilon_3,\delta,\beta_4}_{3,4} & \longleftarrow & \cdots \\
 & \nearrow & \uparrow & & \uparrow & & \uparrow & & \\
C^{\varepsilon,\beta_2,\beta''}_{3,2,2} & & C^{\varepsilon_4,\delta,\beta_2}_{4,2} & \longleftarrow & C^{\varepsilon_4,\delta,\beta_3}_{4,3} & \longleftarrow & C^{\varepsilon_4,\delta,\beta_4}_{4,4} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & C^{\varepsilon_5,\delta,\beta_2}_{5,2} & \longleftarrow & C^{\varepsilon_5,\delta,\beta_3}_{5,3} & \longleftarrow & C^{\varepsilon_5,\delta,\beta_4}_{5,4} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots, & &
\end{array}
$$

The adjacencies $B_{2,2} \leftarrow B^{\varepsilon_3,\delta_2,\alpha}_{3,2}$ and $B_{2,2} \leftarrow B^{\varepsilon_2,\delta_3,\alpha}_{2,3}$ means that

$$
\begin{array}{ccccccc}
B^{-\delta_2,\delta_2,\pm}_{2,2} & & & \qquad & B^{\varepsilon_2,-\varepsilon_2,\pm}_{2,2} & & \\
 & \nwarrow & & & & \nwarrow & \\
B^{\delta_2,\delta_2,\overset{>}{\pm}}_{2,2} & \longleftarrow & B^{\varepsilon_3,\delta_2,\pm}_{3,2} & & B^{\varepsilon_2,\varepsilon_2,\overset{>}{\pm}}_{2,2} & \longleftarrow & B^{\varepsilon_2,\delta_3,\pm}_{2,3}.
\end{array}
$$

The adjacency $B_{2,2} \leftarrow B^{\varepsilon_2,\delta,\beta_3}_{2,2,3}$ means that

$$
\begin{array}{ccccccc}
B^{+,+,\overset{>}{\pm}}_{2,2} & & & \qquad & B^{-,-,\overset{>}{\mp}}_{2,2} & & \\
 & \nwarrow & & & & \nwarrow & \\
B^{+,+,\overset{<}{\pm}}_{2,2} & \longleftarrow & B^{+,\pm,\beta_3}_{2,2,3} & & B^{-,-,\overset{<}{\mp}}_{2,2} & \longleftarrow & B^{-,\pm,\beta_3}_{2,2,3}.
\end{array}
$$

The adjacency $B_{2,2} \leftarrow C^{\varepsilon_3,\delta,\beta_2}_{3,2}$ means that

$$
\begin{array}{ccccccc}
B^{+,+,\overset{<}{+}}_{2,2} & & & \qquad & B^{+,+,\overset{>}{+}}_{2,2} & & \\
 & \nwarrow & & & & \nwarrow & \\
B^{-,-,\overset{>}{-}}_{2,2} & \longleftarrow & C^{\pm,+,+}_{3,2} & & B^{-,-,\overset{<}{-}}_{2,2} & \longleftarrow & C^{\pm,+,-}_{3,2} \\
 & & & & & & \\
B^{+,+,\overset{<}{-}}_{2,2} & & & & B^{+,+,\overset{>}{-}}_{2,2} & & \\
 & \nwarrow & & & & \nwarrow & \\
B^{-,-,\overset{>}{+}}_{2,2} & \longleftarrow & C^{\pm,-,+}_{3,2} & & B^{-,-,\overset{<}{+}}_{2,2} & \longleftarrow & C^{\pm,-,-}_{3,2}.
\end{array}
$$

The adjacencies from singularities on the 2-corner to one on the boundary.

$$
\begin{array}{ccccccc}
B^{\varepsilon}_2 & \longleftarrow & B^{\varepsilon_3}_3 & \longleftarrow & B^{\varepsilon_4}_4 & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{\varepsilon,\delta,*}_{2,2} & \longleftarrow & B^{\varepsilon_3,\delta,\alpha}_{3,2} & \longleftarrow & B^{\varepsilon_4,\delta,\alpha}_{4,2} & \longleftarrow & \cdots \\
| & \nwarrow & & & & & \\
| & & B^{\varepsilon,\delta_3,\alpha}_{2,3} & \longleftarrow & B^{\varepsilon,\delta_4,\alpha}_{2,4} & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B^{\delta}_2 & \longleftarrow & B^{\delta_3}_3 & \longleftarrow & B^{\delta_4}_4 & \longleftarrow & \cdots,
\end{array}
$$

$$
\begin{array}{ccccccccccc}
F_4^{\alpha} & & & & & & & & & & \\
\uparrow & & & & & & & & & & \\
C_{3,2,1}^{\varepsilon_3,\alpha,\alpha'} & & C_3^{\varepsilon_3} & \longleftarrow & C_4^{\varepsilon_4} & \longleftarrow & C_5^{\varepsilon_5} & \longleftarrow & C_6^{\varepsilon_6} & \longleftarrow & \cdots \\
 & \searrow & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & C_{3,2}^{\varepsilon_3,\delta,\alpha} & \longleftarrow & C_{4,2}^{\varepsilon_4,\delta,\alpha} & \longleftarrow & C_{5,2}^{\varepsilon_5,\delta,\alpha} & \longleftarrow & C_{6,2}^{\varepsilon_6,\delta,\alpha} & \longleftarrow & \cdots \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_{3,2,2}^{\varepsilon_3,\alpha,\alpha''} & & C_3^{\varepsilon_3\delta} & \longleftarrow & C_4^{\varepsilon_4} & \longleftarrow & C_5^{\varepsilon_5\delta} & \longleftarrow & C_6^{\varepsilon_6} & \longleftarrow & \cdots, \\
\downarrow & & & & & & & & & & \\
F_4^{\alpha} & & & & & & & & & &
\end{array}
$$

The adjacencies of singularities on the 2-corner given by forgetting a boundary. The adjacency $\nwarrow$ is given by forgetting the boundary $x_1 = 0$ and $\leftarrow$ is given by forgetting $x_2 = 0$.

$\underline{B_{2,2}}$

$$
\begin{array}{ccccccccccc}
C_2^- & & & & C_2^+ & & & & C_2^+ & & \\
 & \nwarrow & & & & \nwarrow & & & & \nwarrow & \\
C_2^- & \longleftarrow & B_{2,2}^{+,+,\overset{<}{\pm}(0)} & & C_2^+ & \longleftarrow & B_{2,2}^{+,+,\overset{>}{\pm}} & & C_2^+ & \longleftarrow & B_{2,2}^{-,-,\overset{<}{\pm}(0)} \\
 & & & & & & & & & & \\
C_2^- & & & & C_2^+ & & & & C_2^- & & \\
 & \nwarrow & & & & \nwarrow & & & & \nwarrow & \\
C_2^- & \longleftarrow & B_{2,2}^{-,-,\overset{>}{\pm}} & & C_2^- & \longleftarrow & B_{2,2}^{+,-,\pm} & & C_2^+ & \longleftarrow & B_{2,2}^{-,+,\pm},
\end{array}
$$

$\underline{B_{2,2,n}}$

$$
\begin{array}{ccccccc}
B_3^{\alpha} & & & & B_n^{\alpha} & & \\
 & \nwarrow & & & & \nwarrow & \\
B_3^{-\delta\alpha} & \longleftarrow & B_{2,2,3}^{\pm,\delta,\alpha} & & B_n^{(-\delta)^n\alpha} & \longleftarrow & B_{2,2,n}^{\pm,\delta,\alpha},
\end{array}
$$

$\underline{B_{n,m}}$

$$
\begin{array}{ccccccc}
C_2^{\varepsilon} & & & & C_n^{\varepsilon\alpha^n} & & \\
 & \nwarrow & & & & \nwarrow & \\
C_3^{\delta\alpha} & \longleftarrow & B_{2,3}^{\varepsilon,\delta,\alpha} & & C_m^{\delta\alpha^m} & \longleftarrow & B_{n,m}^{\varepsilon,\delta,\alpha},
\end{array}
$$

$\underline{B_{2,3'}}$ and $\underline{B_{3,2'}}$

$$
\begin{array}{ccccccc}
B_3^{\delta} & & & & F_4^{\delta} & & \\
 & \nwarrow & & & & \nwarrow & \\
F_4^{\varepsilon} & \longleftarrow & B_{2,3'}^{\varepsilon,\delta,\alpha} & & B_3^{\varepsilon} & \longleftarrow & B_{3,2'}^{\varepsilon,\delta,\alpha}
\end{array}
$$

$\underline{C_{n,m}}$

$$
\begin{array}{ccccccc}
C_2^{-\alpha} & & & & C_m^{-\alpha} & & \\
 & \nwarrow & & & & \nwarrow & \\
C_2^{-\alpha} & \longleftarrow & C_{3,2}^{\pm,\delta,\alpha} & & C_m^{-(-\delta)^m\alpha} & \longleftarrow & C_{n,m}^{\pm,\delta,\alpha}.
\end{array}
$$

$\underline{C_{3,2,1} \text{ and } C_{3,2,2}}$

$$\begin{array}{ccc} C_2^{-\delta} & & \\ & \nwarrow & \\ C_3^{\varepsilon} & \longleftarrow & C_{3,2,1}^{\varepsilon,\delta,\alpha} \end{array} \qquad \begin{array}{ccc} C_3^{\varepsilon} & & \\ & \nwarrow & \\ C_2^{-\delta} & \longleftarrow & C_{3,2,2}^{\varepsilon,\delta,\alpha}. \end{array}$$

The adjacencies of singularities on the 2-corner given by restriction to $x_1 = 0$ or $x_2 = 0$. The adjacency $\nwarrow$ is given by restriction to $x_2 = 0$ and $\leftarrow$ is given by restriction to $x_1 = 0$.

$$\begin{array}{ccc} B_n^{\varepsilon} & & \\ & \nwarrow & \\ B_m^{\delta} & \longleftarrow & B_{n,m}^{\varepsilon,\delta,*} \end{array} \qquad \begin{array}{ccc} B_2^{\varepsilon} & & \\ & \nwarrow & \\ B_2^{\varepsilon} & \longleftarrow & B_{2,2,n}^{\varepsilon,\delta,\alpha} \end{array} \qquad \begin{array}{ccc} C_{2n}^{\varepsilon} & & \\ & \nwarrow & \\ C_{2n}^{\varepsilon} & \longleftarrow & B_{2n,m}^{\varepsilon,\delta,\alpha} \end{array}$$

$$\begin{array}{ccc} C_{2n+1}^{\varepsilon} & & \\ & \nwarrow & \\ C_{2n+1}^{\varepsilon\delta} & \longleftarrow & C_{2n+1,m}^{\varepsilon,\delta,\alpha} \end{array} \qquad \begin{array}{ccc} C_3^{\varepsilon} & & \\ & \nwarrow & \\ F_4^{\delta} & \longleftarrow & C_{3,2,1}^{\varepsilon,\delta,\alpha} \end{array} \qquad \begin{array}{ccc} F_4^{\delta} & & \\ & \nwarrow & \\ C_3^{\varepsilon} & \longleftarrow & C_{3,2,2}^{\varepsilon,\delta,\alpha}. \end{array}$$

7 Classification of function germs

In order to classify function germs we prepare the following lemmas.

Lemma 7.1 *Let $f \in \mathfrak{M}(r;k)$ be a function germ. If $\frac{\partial f}{\partial y}(0) \neq 0$ then f is reticular R-equivalent to $y_1 \in \mathfrak{M}(r;k)$.*

Lemma 7.2 *Let $f \in \mathfrak{M}(r;k)$ be a function germ satisfying $\frac{\partial f}{\partial y}(0) = 0$ and l be the corank of f. ' Then there exist a subset $\sigma \subset I_r$ and a non-degenerate quadratic form $Q(y_1, \cdots, y_{l-1})$ and a function germ $g(x', y') \in \mathfrak{M}((r - |\sigma|); l)^2$ such that the following conditions hold:*

(1) $g|_{x'=0} \in \mathfrak{M}(0;l)^3$

(2) f is reticular R-equivalent to $f_0 \in \mathfrak{M}(r;k)$ defined by

$$f_0(x_1, y) = \sum_{i \in \sigma} \pm x_i + g(x_{I_r - \sigma}, y_1, \cdots, y_l) + Q(y_{l+1}, \cdots, y_k).$$

We say a function germ $f(x,y) \in \mathfrak{M}(r;k)$ $(x \in \mathbb{H}^r, y \in \mathbb{R}^k)$ is residual if $f \in \mathfrak{M}(r;k)^2$ and $f|_{x=0} \in \mathfrak{M}(0;k)^3$.

Let $\mathcal{E}(r;k,l)$ be the set of smooth map germs $(\mathbb{H}^r \times \mathbb{R}^k, 0) \to \mathbb{R}^l$ and $\mathfrak{M}(r;k,l)$ be the set of map germs $(\mathbb{H}^r \times \mathbb{R}^k, 0) \to (\mathbb{R}^l, 0)$.

To each $\xi = (x_1\xi_1, \cdots, x_r\xi_r, \xi_{r+1}, \cdots, \xi_{r+k}) \in \mathfrak{M}(r;k,r+k)$ we define the linear map $\xi : \mathcal{E}(r;k) \to \mathcal{E}(r;k)$ by

$$\xi(f) = \sum_{i=1}^{r} x_i\xi_i \frac{\partial f}{\partial x_i} + \sum_{j=1}^{k} \xi_{r+j} \frac{\partial f}{\partial y_j}.$$

To each $\phi \in \mathcal{B}(r;k)$ we define the linear map $\phi^* : \mathcal{E}(r;k) \to \mathcal{E}(r;k)$ by $\phi^*(f) = f \circ \phi$.

We have the following four lemma's which are analogous to lemma 1.21 $\sim$ corollary 1.23 in [17].

Lemma 7.3 *Let A be a real vector subspace of $\mathcal{E}(r;k)$. Let $[0,1] \to \mathfrak{M}(r;k,r+k)$, $t \mapsto \xi_t = (x_1\xi_t^1, \cdots, x_r\xi_t^r, \xi_t^{r+1}, \cdots, \xi_t^{r+k})$ be a smooth homotopy. Suppose that $\xi_t(A) \subset A$ for all $t \in [0,1]$ and $f \in \mathcal{E}(r;k)$. Let $\Phi : (\mathbb{H}^r \times \mathbb{R}^k \times \mathbb{R}, 0 \times [0,1]) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$, $(x,y,t) \mapsto \phi_t(x,y)$, be the solution of the differential equation:*

$$\frac{d}{dt}\phi_t(x,y) \;=\; \xi_t \circ \phi_t(x,y), \;\; \phi_0(x,y) = (x,y). \tag{16}$$

Then this solution satisfies the following conditions:

(1) *$\phi_t \in \mathcal{B}(r;k)$ for all $t \in [0,1]$.*

(2) *$\phi_t^*(A) \subset A + \mathfrak{M}(r;k)^l$ for all $t \in [0,1]$, $l > 0$.*

Proof. (1) We denote $\Phi = (\Phi_1, \cdots, \Phi_{r+k})$ and $\phi_t = (\phi_t^1, \cdots, \phi_t^{r+k})$. Since $(\xi_t \circ \phi_t)_i = \phi_t^i \cdot \xi_t^i \circ \phi_t$ for $i = 1, \cdots, r$, by the uniqueness of the solution of (16) we have that $\Phi_i|_{x_i=0} \equiv 0$ for $i = 1, \cdots, r$. This means (1).

(2) Let $l > 0$ be given. For each $t \in [0,1]$ consider the map $\mathcal{E}(r;k) \to \mathcal{E}(r;k)$ given by $f \mapsto \xi_t(f)$; this map is linear and since $\xi_t \in \mathfrak{M}(r;k,r+k)$ this map maps $\mathfrak{M}(r;k)^l$ into itself. Hence this map induces the linear map $\tilde{\xi}_t : J^{l-1}(r+k,1) \to J^{l-1}(r+k,1)$. This map depends

differentiably on t. Similarly, the maps $\phi_t^* : \mathcal{E}(r;k) \to \mathcal{E}(r;k)$, $t \in [0,1]$, are linear and map $\mathfrak{M}(r;k)^l$ to itself, so they induce the linear maps $\tilde{\phi_t}^* : J^{l-1}(r+k,1) \to J^{l-1}(r+k,1)$ and $\tilde{\phi_t}^*$ depends differentiably on t.

Now choose a basis $a_1, \cdots, a_p$ for $\pi_{l-1}(A) \subset J^{l-1}(r+k,1)$ and extend this to a basis $a_1, \cdots, a_q$ of $J^{l-1}(r+k,1)$. For $t \in [0,1]$ let Q_t be the matrix of $\tilde{\xi}_t$ with respect to the basis $a_1, \cdots, a_q$ and let C_t be the matrix of $\tilde{\phi_t}^*$. Then because $\xi_t(A) \subset A$ we have that Q_t has the form:

$$\begin{array}{c} \\ p \\ q-p \end{array} \begin{array}{c} \begin{matrix} p & q-p \end{matrix} \\ \begin{pmatrix} R_t & S_t \\ 0 & T_t \end{pmatrix} \end{array}.$$

If we divide C_t into submatrices in the same way it will have some form:

$$\begin{pmatrix} W_t & X_t \\ Y_t & Z_t \end{pmatrix}.$$

What we wish to prove is that $Y_t = 0$ for all $t \in [0,1]$.

Now let $f \in \mathcal{E}(r;k)$ be given. Then for $t \in [0,1]$ we have

$$\begin{aligned} \frac{d}{dt}(\tilde{\phi_t}^*(j^{l-1}f(0))) &= j^{l-1}(\frac{\partial}{\partial t}\phi_t^* f)(0) = j^{l-1}\phi_t^*(\xi_t(f))(0) \\ &= \tilde{\phi_t}^*(\tilde{\xi}_t(j^{l-1}f(0))). \end{aligned}$$

Hence we have that $\frac{d}{dt}\tilde{\phi_t}^* = \tilde{\phi_t}^* \circ \tilde{\xi}_t$ for $t \in [0,1]$. Therefore

$$\frac{dC_t}{dt} = C_t Q_t \text{ for } t \in [0,1].$$

Because of the form of Q_t this implies

$$\frac{dY_t}{dt} = Y_t R_t \text{ for } t \in [0,1].$$

But C_0 is the identity matrix because $\phi_0^* = id_{\mathcal{E}(r;k)}$ so $Y_0 = 0$. Hence $Y_t = 0$ for all $t \in [0,1]$. ■

Lemma 7.4 *Let A be a vector subspace of $\mathcal{E}(r;k)$ and $[0,1] \to \mathcal{E}(r;k)$ $(t \mapsto f_t)$ be a smooth homotopy. Suppose that there exists a smooth homotopy $[0,1] \to \mathfrak{M}(r;k,r+k)$, $t \mapsto \xi_t = (x_1\xi_t^1, \cdots, x_r\xi_t^r, \xi_t^{r+1}, \cdots, \xi_t^{r+k})$ satisfying the following conditions:*

(1) $\xi_t(A) \subset A + \mathfrak{M}(r;k)^l$ for all $t \in [0,1]$, $l > 0$.

(2) $\frac{\partial f_t}{\partial t} - \xi_t(f_t) \in A + \mathfrak{M}(r;k)^l$ for all $t \in [0,1]$, $l > 0$.

Then for any $l > 0$ there exist $\phi \in \mathcal{B}(r;k)$ and $h \in A + \mathfrak{M}(r;k)^l$ such that $\phi^*(A + \mathfrak{M}(r;k)^l) \subset A + \mathfrak{M}(r;k)^l$ and $f_0 = f_1 \circ \phi + h$.

Proof. Let $l > 0$ be given. Consider the solution $\Phi : (\mathbb{H}^r \times \mathbb{R}^k \times \mathbb{R}, 0 \times [0,1]) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$, $(x,y,t) \mapsto \phi_t(x,y)$, of the following differential equation:

$$\frac{d}{dt}\phi_t(x,y) = -\xi_t \circ \phi_t(x,y), \ \phi_0(x,y) = (x,y).$$

we denote $\Phi = (\Phi_1, \cdots, \Phi_{r+k})$, $\phi_t = (\phi_t^1, \cdots, \phi_t^{r+k})$ and define $H : (\mathbb{H}^r \times \mathbb{R}^k \times \mathbb{R}, 0 \times [0,1]) \to \mathbb{R}$ by $H(x,y,t) = f_t \circ \phi_t(x,y)$. Then

$$\begin{aligned} \frac{\partial H}{\partial t}(x,y,t) &= (\sum_{i=1}^r \frac{\partial f_t}{\partial x_i} \circ \phi_t \frac{\partial \phi_t^i}{\partial t} + \sum_{j=1}^k \frac{\partial f_t}{\partial y_j} \circ \phi_t \frac{\partial \phi_t^{r+j}}{\partial t} + \frac{\partial f_t}{\partial t} \circ \phi_t)(x,y) \\ &= (-\sum_{i=1}^r \frac{\partial f_t}{\partial x_i} \circ \phi_t \phi_t^i \xi_t^i \circ \phi_t - \sum_{j=1}^k \frac{\partial f_t}{\partial y_j} \circ \phi_t \xi_t^{r+j} \circ \phi_t + \frac{\partial f_t}{\partial t} \circ \phi_t)(x,y) \\ &= \phi_t^*(-\xi_t(f_t) + \frac{\partial f_t}{\partial t})(x,y) \in \phi_t^*(A + \mathfrak{M}(r;k)^l) \quad \text{by (2)}. \end{aligned}$$

Now by lemma 7.3 (with $A + \mathfrak{M}(r;k)^l$ for A and $-\xi_t$ for ξ_t we have

$$\phi_t^*(A + \mathfrak{M}(r;k)^l) \subset A + \mathfrak{M}(r;k)^l \text{ for all } \ t \in [0,1], \tag{17}$$

so $\frac{\partial H}{\partial t}|_{\mathbb{R}^{r+k} \times \{t\}} \in A + \mathfrak{M}(r;k)^l$ for all $t \in [0,1]$. Therefore for $t_0 \in [0,1]$ we have

$$\frac{d}{dt}|_{t=t_0}(j^{l-1}(\phi_t^* f_t)(0)) = j^{l-1}(\frac{\partial H}{\partial t}|_{\mathbb{R}^{r+k} \times \{t_0\}})(0) \in \pi_{l-1}(A + \mathfrak{M}(r;k)^l) = \pi_{l-1}(A).$$

But since $\pi_{l-1}(A)$ is a linear subspace of $J^{l-1}(r+k,1)$ we have $j^{l-1}(\phi_1^* f_1 - \phi_0^* f_0) \in \pi_{l-1}(A)$. Hence $-h := \phi_1^* f_1 - f_0 \in A + \mathfrak{M}(r;k)^l$, so that $f_0 = f_1 \circ \phi_1 + h$. ■

Lemma 7.5 *Let $f \in \mathcal{E}(r;k)$. Let $l > 0$ be an integer and set $f_0 = j^l f(0)$ (consider as a polynomial germ in $\mathcal{E}(r;k)$). Suppose that there exist $h \in \mathfrak{M}(r;k)^q$ for some $q \geq l+1$ and $\xi = (x_1\xi_1, \cdots, x_r\xi_r, \xi_{r+1}, \cdots, \xi_{r+k}) \in \mathfrak{M}(r;k)^{p-l}\mathcal{E}(r;k,r+k)$, $p \geq q+1$ such that $\xi(f_0) - h \in \mathfrak{M}(r;k)^p$.*

Then there exists $h_1 \in \mathfrak{M}(r;k)^p$ such that $f+h \overset{\mathrm{rR}}{\sim} f+h_1$ (or in other words, $j^{p-1}(f+h)(0) \in O_{\mathrm{rR}}^{p-1}(j^{p-1}f(0))$).

Proof. Define the smooth homotopy $[0,1] \to \mathcal{E}(r;k)$, $t \mapsto f_t = f + (1-t)h$ and define the smooth homotopy $[0,1] \to \mathfrak{M}(r;k)\mathcal{E}(r;k,r+k)$, $t \mapsto \xi_t = -\xi$. Then

$$\begin{aligned} \frac{\partial f_t}{\partial t} - \xi_t(f_t) &= -h + \xi(f + (1-t)h) = -h + \xi(f_0) + \xi(f - f_0 + (1-t)h)) \\ &\in \mathfrak{M}(r;k)^p + \mathfrak{M}(r;k)^{(p-l)+l} = \mathfrak{M}(r;k)^p. \end{aligned}$$

Hence the hypotheses of lemma 7.4 are fulfilled for $A = \mathfrak{M}(r;k)^p$. Therefore there exist $\phi_1 \in \mathcal{B}(r,k)$ and $h' \in A + \mathfrak{M}(r;k)^p = \mathfrak{M}(r;k)^p$ such that $f_0 = f \circ \phi_1 + h'$. Set $h_1 = h' \circ \phi_1^{-1}$. Then we have $f_0 \circ \phi = f + h_1$ and $h_1 \in \mathfrak{M}(r;k)^p$ ■

Lemma 7.6 *Let $f \in \mathfrak{M}(r;k)^l$ and set $f_0 = j^l f(0)$ (so f_0 is a homogeneous polynomial of degree l). Let h be a homogeneous polynomial of degree $q \geq l+1$ and suppose $h \in J_{\mathrm{rR}}(f_0)$. Then there exists $h_1 \in \mathfrak{M}(r;k)^{q+1}$ such that $f+h \overset{\mathrm{rR}}{\sim} f+h_1$ (or in other words, $j^q(f+h)(0) \in O_{\mathrm{rR}}^q(j^l f(0))$), where $J_{\mathrm{rR}}(f_0)$ is the Jacobi ideal of f_0 defined by*

$$J_{\mathrm{rR}}(f_0) = \langle x_1\frac{\partial f_0}{\partial x_1}, \cdots, x_r\frac{\partial f_0}{\partial x_r}, \frac{\partial f_0}{\partial y_1}, \cdots, \frac{\partial f_0}{\partial y_k}\rangle_{\mathcal{E}(r;k)}.$$

Proof. By $h \in J_{\mathrm{rR}}(f_0)$ there exists $\xi = (x_1\xi_1, \cdots, x_r\xi_r, \xi_{r+1}, \cdots, \xi_{r+k}) \in \mathfrak{M}(r; k, r+k)$ such that $\xi(f_0) = h$. We may assume that $\xi \in \mathfrak{M}(r;k)^{q-l+1}\mathcal{E}(r;k,r+k)$ and , for if not we replace ξ by ξ' defined by $\xi' = \xi - j^{q-l}\xi(0)$. Since f_0 is homogeneous polynomial of degree l we have $(\xi - \xi')(f_0)$ is a polynomial of degree $q-1$ and $\xi'(f_0) \in \mathfrak{M}(r;k)^q$. Since $h = (\xi - \xi')(f_0) + \xi'(f_0) \in \mathfrak{M}(r;k)^q$ we have $(\xi - \xi')(f_0) = 0$ and $\xi'(f_0) = h$.

Hence the conditions of lemma 7.5 are fulfilled with $p = q + 1$, and the conclusion follows immediately from lemma 7.5. ■

We now start the classification of unimodal residual singularities in $\mathfrak{M}(r;k)^2$ $(r \geq 1)$ under reticular R-equivalence. We shall prove that this classification includes the classification of residual singularities whose reticular R-codimension is lower than 8. Firstly we introduce the following notations: $a_i, b_j, a, b, c, \cdots$ are real numbers. We say that $z \in J^l(r+k, 1)$ has modality n if the following condition holds: For any neighborhood of z there exists an element z' in this neighborhood and there exists an n-parameter family of l-jets $z'(a)$ (a in some neighborhood of 0 in $\mathbb{R}^n$) such that $z'(0) = z'$ and $z'(a) \notin O^l_{rR}(z'(b))$ if $a \neq b$. Remark that for $f \in \mathfrak{M}(r;k)^2$ if $j^l f(0)$ has modality n then f also has modality n .

Let $f \in \mathfrak{M}(r;k)^2$ be a function germ with reticular R-finite-codimension. In the procedure of the classification, we adopt the following notations:

'$\Rightarrow$' means 'see'.

'$f \overset{\mathrm{rR}}{\sim} g$' means '$f$ is reticular R-equivalent to g' ($g \in \mathcal{E}(r;k)$).

'$\overset{1}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ by lemma 7.6 by the analogous method of (2)'.

'$\overset{2}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ by lemma 7.5 by the analogous method in (6)'.

'$\overset{3}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ because g is (degree of g)-determined by Lemma 4.2'.

'$\overset{4}{\mapsto} g$' means '$f \overset{rR}{\sim} g$ by a linear coordinate change'.

The case $r = 1, k = 0$. The classification is reduced to V.I.Arnold [2] and V.I.Matov [9].

The case $r = 2, k = 0$. Let $j^2 f(0) = ax_1^2 + bx_1x_2 + cx_2^2$.

$a \neq 0, c \neq 0 \quad \Rightarrow \quad (1)$
$b \neq 0, c = 0 \quad \Rightarrow \quad (3)$
$a = 0, b \neq 0, c \neq 0 \quad \Rightarrow \quad (4)$
$a \neq 0, b = 0, c = 0 \quad \Rightarrow \quad (5)$
$a = 0, b = 0, c \neq 0 \quad \Rightarrow \quad (7)$
$a = 0, b = 0, c = 0 \quad \Rightarrow \quad (8)$

(1) $j^2 f(0)$ has the normal form $\pm x_1^2 + ax_1x_2 \pm x_2^2$ by a linear coordinate change.

$a^2 \neq \pm\pm 4 \quad \overset{3}{\mapsto} \quad \pm x_1^2 + ax_1x_2 \pm x_2^2.$
$a^2 = \pm\pm 4 \quad \Rightarrow \quad (2)$

(2) $\overset{1}{\mapsto} \quad \pm(x_1^2 \pm 2x_1x_2 + x_2^2) + \sum_{i\geq 3} a_i x_2^i$ ($^\exists i$ s.t. $a_i \neq 0$) : Let $f_0 = j^2 f(0)$. Since $x_1 \frac{\partial f_0}{\partial x_1} = \pm 2(x_1^2 \pm x_1x_2)$ and $x_2 \frac{\partial f_0}{\partial x_2} = \pm 2(\pm x_1x_2 + x_2^2)$, we may replace any term of degree ≥ 3 involving x_1 in f by terms involving less x_1's and more x_2's and terms of higher degree by lemma 7.6. As a result we have the normal form. If $a_i = 0$ for all i, then we have codimension $f = \infty$. Therefore there exists an integer i such that $a_i \neq 0$. $\overset{3}{\mapsto} \quad \pm(x_1^2 \pm 2x_1x_2 + x_2^2) + ax_2^n$ ($n \geq 3, a \neq 0$).

(3) $j^2 f(0)$ has the normal form $\pm x_1^2 \pm x_1x_2$ or $\pm x_1x_2$ by a linear coordinate change. $\overset{1}{\mapsto}$ $\sum_{i\geq 2} a_i x_1^i + x_1x_2 + \sum_{j\geq 3} b_j x_2^j$ ($^\exists i, j$ s.t. $a_i \neq 0, b_j \neq 0$) $= gx_1^n \pm x_1x_2 + hx_2^m$ (where g, h are units in $\mathcal{E}(2;0)$ and $n \geq 2, m \geq 3$ be the minimum integer satisfying $a_n \neq 0, b_m \neq 0$ respectively) $\overset{\text{rR}}{\sim} \pm x_1^n + x_1x_2 + gx_2^m$ ($n \geq 2, m \geq 3, g$:unit) $\overset{3}{\mapsto} \pm x_1^n \pm x_1x_2 + ax_2^m$ ($n \geq 2, m \geq 3, a \neq 0$).

(4) By using the analogous method of (3), we have $f \overset{\text{rR}}{\sim} \pm x_1^n \pm x_1x_2 + ax_2^2$ ($n \geq 3, a \neq 0$).

(5) $j^2 f(0)$ has the normal form $\pm x_1^2$ by a linear coordinate change. Hence $j^3 f(0)$ has the normal form $\pm x_1^2 + ax_1x_2^2 + bx_2^3$ by lemma 7.6.

$b \neq 0 \quad \overset{3}{\mapsto} \quad \pm x_1^2 + ax_1x_2^2 + bx_2^3 \ (b \neq 0)$
$\qquad \overset{4}{\mapsto} \quad \pm x_1^2 + ax_1x_2^2 \pm x_2^3.$
$b = 0 \quad \Rightarrow \quad (6)$

(6) $j^3 f(0)$ has the normal form $f_0 = \pm x_1^2 + ax_1x_2^2$. Then $j^5 f(0)$ has modality 2 in $J^5(2+1;1)$: For any neighborhood of $j^5 f(0)$ there exists an element f_1 in the neighborhood such that

$\pi_2^5(z) = \pm x_1^2 + bx_1x_2^2$ $(b \neq 0)$. Then f_1 has the normal form $\pm x_1^2 \pm x_1x_2^2 + cx_2^4 + dx_2^5$ by lemma 7.5 (since $x_1\frac{\partial f_0}{\partial x_1} = \pm 2x_1^2 \pm x_1x_2^2$ and $x_2\frac{\partial f_0}{\partial x_2} = \pm 2x_1x_2^2$, we may replace any term of degree ≥ 3 involving x_1 in f_1 by terms of higher degree). It is enough to prove that f_1 has modality 2. Suppose that $f_2 = \pm x_1^2 \pm x_1x_2^2 + c'x_2^4 + d'x_2^5 \in O^5_{\mathrm{rR}}(f_1)$. Then there exists $\phi = (x_1\phi_1, x_2\phi_2) \in \mathcal{B}(2;0)$ such that $f_2 \equiv f_1 \circ \phi \mod \mathfrak{M}(2;0)^6$, where f_1 and f_2 are considered as polynomial function germs. Let $j^2\phi_2(0) = \phi_2(0) + \phi_{21}x_1 + \phi_{22}x_2$ $(\phi_{21}, \phi_{22} \in \mathbb{R})$. By the coefficient of x_1^2 in f_2 we have $\phi_1(0) = 1$. By the coefficient of $x_1x_2^2$ we have $\phi_2(0) = 1$. By the coefficient of $x_1x_2^3$ we have $\phi_{22} = 0$. These imply that $a = a'$ by the coefficient of x_2^4 and $b = b'$ by the coefficient of $x_1x_2^5$.

On the other hand, reticular R-codimension $f\ \geq 8$: It is enough to prove that codimension of $O^5_{\mathrm{rR}}(f_1)$ in $J^5(2+0,1) \geq 8$. Set $A = \langle x_1^3, x_1^2x_2^2, x_1x_2^4, x_2^6\rangle_{\mathcal{E}(2;0)}$. Since $m^6(2;0) \subset A$ and $x_1\frac{\partial f_1}{\partial x_1} = \pm 2x_1^2 \pm x_1x_2^2, x_2\frac{\partial f_1}{\partial x_2} = \pm 2x_1x_2^2 + 4ax_2^4 + 5bx_2^5$, we have that $x_1^2\frac{\partial f_1}{\partial x_1} \equiv x_1x_2^2\frac{\partial f_1}{\partial x_1} \equiv x_1x_2\frac{\partial f_1}{\partial x_2} \equiv x_2^3\frac{\partial f_1}{\partial x_2} \equiv 0 \mod A$. Therefore

$$\begin{aligned}
&\text{codimension of } O^5_{\mathrm{rR}}(f_1) \text{ in } J^5(2+0,1)\\
&= \dim \mathcal{E}(2;0)/(\langle x_1\frac{\partial f_1}{\partial x_1}, x_2\frac{\partial f_1}{\partial x_2}\rangle + \mathfrak{M}(2;0)^6)\\
&\geq \dim \mathcal{E}(2;0)/(\langle x_1\frac{\partial f_1}{\partial x_1}, x_1x_2\frac{\partial f_1}{\partial x_1}, x_2\frac{\partial f_1}{\partial x_2}, x_2^2\frac{\partial f_1}{\partial x_2}\rangle_{\mathbb{R}} + A)\\
&= \dim \mathcal{E}(2;0)/A\ -\ 4 = 12 - 4 \geq 8.
\end{aligned}$$

(7) By using the analogous method of (5), we have $f \overset{\mathrm{rR}}{\sim} \pm x_1^3 + ax_1^2x_2 \pm x_2^2$ or f has modality 2 and codimension $f \geq 8$.

(8) $f \in \mathfrak{M}(2;0)^3$. Hence

$$\text{reticular-R-codimension } f \geq \dim \mathcal{E}(2;0)/(\langle x_1\frac{\partial f}{\partial x_1}, x_2\frac{\partial f}{\partial x_2}\rangle_{\mathbb{R}} + \mathfrak{M}(2;0)^4) \geq 10 - 2 = 8.$$

On the other hand, we can show that $j^3f(0)$ has modality 2 by analogous methods of (6): In this case, if we consider an element $z \in \pi_3(\mathfrak{M}(2;0)^3)$ such that the coefficients of x_1^3, x_2^3 are not zero. Then z has the normal form $\pm x_1^3 + ax_1^2x_2 + bx_1x_2^2 \pm x_2^3$ by a linear coordinate change. Hence z has modality 2.

<u>The case $r = 2, k = 1$.</u> Let $j^2f(0) = ax_1y + bx_2y + cx_1^2 + dx_1x_2 + ex_2^2$.

$a \neq 0, b \neq 0$	$\Rightarrow$	(9)
$a \neq 0, b = 0, e \neq 0$	$\Rightarrow$	(10)
$a \neq 0, b = 0, e = 0$	$\Rightarrow$	(12)
$a = 0, b \neq 0$	$\Rightarrow$	(13)
$a = 0, b = 0$	$\Rightarrow$	(14)

(9) $j^2f(0)$ has the normal form $x_1y \pm x_2y + ax_2^2$ by a linear coordinate change $\overset{1}{\mapsto} \sum_{i\geq 3} a_iy^i + x_1y \pm x_2y + \sum_{j\geq 2} b_ix_2^j$ ($^\exists i, j$ s.t. $a_i \neq 0, b_j \neq 0$) $\overset{\text{rR}}{\sim} \pm y^n + x_1y \pm x_2y + ax_2^m$ $(n \geq 3, m \geq 2, a \neq 0)$ (here we used the analogous method of (3)).

(10) $j^2f(0)$ has the normal form $x_1y \pm x_2^2$ by a linear coordinate change. Hence $j^3f(0)$ has the normal form $ay^3 + x_1y + bx_2y^2 \pm x_2^2$ by lemma 7.6.

$a \neq 0$	$\overset{3}{\mapsto}$	$ay^3 + x_1y + bx_2y^2 \pm x_2^2$ $(a \neq 0)$
	$\overset{4}{\mapsto}$	$\pm y^3 + x_1y + ax_2y^2 \pm x_2^2$ $(a \in \mathbb{R})$.
$a = 0$	$\Rightarrow$	(11)

(11) We can prove that $j^5f(0)$ has modality 2 and reticular R-codimension of $f \geq 8$ by analogous methods of (6): Consider an element $f_0 \in J^5(2+1,1)$ satisfying $\pi_3^5(f_0) = x_1y + ax_2y^2 \pm x_2^2$ $(a \neq 0)$. Then f_0 has the normal form $ay^5 + by^4 + x_1y \pm x_2y^2 \pm x_2^2$ and hence f_0 has modality 2. On the other hand, set $A = \langle x_1^2, x_1x_2^2, x_2^3, x_1y, x_2^2y^2, x_2y^4, y^6\rangle_{\mathcal{E}(2;1)}$. Since $\mathfrak{M}(2;1)^6 \subset A$ and $x_1\frac{\partial f_0}{\partial x_1} = x_1y, x_2\frac{\partial f_0}{\partial x_2} = \pm x_2y^2 \pm 2x_2^2, \frac{\partial f_0}{\partial y} = 5ay^4 + 4by^3 + x_1 \pm 2x_2y$, we have $x_1\frac{\partial f_0}{\partial x_1} \equiv gx_2\frac{\partial f_0}{\partial x_2} \equiv h\frac{\partial f_0}{\partial y} \equiv 0 \bmod A$ for $g = x_1, x_2, y^2$ and $h = x_1, x_2^2, x_2y, y^3$. Hence

$$\begin{aligned}
&\text{codimension of } O^5_{\text{rR}}(f_0) \text{ in } J^5(2+0,1)\\
&\geq \dim \mathcal{E}(2;1)/(\langle x_1\frac{\partial f_0}{\partial x_1}, x_2\frac{\partial f_0}{\partial x_2}, \frac{\partial f_0}{\partial y}\rangle + A)\\
&= \dim \mathcal{E}(2;1)/(\langle x_2\frac{\partial f_0}{\partial x_2}, x_2y\frac{\partial f_0}{\partial x_2}, \frac{\partial f_0}{\partial y}, x_2\frac{\partial f_0}{\partial y}, y\frac{\partial f_0}{\partial y}, y^2\frac{\partial f_0}{\partial y}\rangle_{\mathbb{R}} + A)
\end{aligned}$$

$$\geq \ \dim \mathcal{E}(2;1)/A \ - \ 6 = 14 - 6 \geq 8.$$

(12) $j^2 f(0)$ has the normal form $x_1 y$ by a linear coordinate change $\overset{1}{\mapsto} x_1 y + h(x_2, y)$ ($h \in \mathfrak{M}(x_2, y)^3$). Set $A = \langle x_1 y, x_1^2, x_1 x_2^2 \rangle + \mathfrak{M}(2;1)^4$. Then we have $x_1 \frac{\partial f}{\partial x_1} \equiv x_2 \frac{\partial f}{\partial x_2} \equiv$ mod A. Hence

$$\begin{aligned}
&\text{reticular -codimension } f\\
&\geq \ \dim \mathcal{E}(2;1)/(\langle x_1 \frac{\partial f}{\partial x_1}, x_2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y} \rangle + \mathfrak{M}(2;1)^4)\\
&\geq \ \dim \mathcal{E}(2;1)/(\langle \frac{\partial h}{\partial y}, x_1 \frac{\partial h}{\partial y}, x_2 \frac{\partial h}{\partial y}, y \frac{\partial h}{\partial y} \rangle_{\mathbb{R}} + A)\\
&\geq \ \dim \mathcal{E}(2;1)/A \ - 4 = 12 - 4 = 8.
\end{aligned}$$

On the other hand, we have $j^3 f(0)$ has modality 2 by the analogous method of (6): Consider an element $f_0 \in J^3(2+1, 1)$ which has the form $x_1 y + h(x_2, y)$ (h is a homogeneous polynomial of degree 3) and the coefficients of x_2^3 and y^3 in h are not zero, then f_0 has the normal form $x_1 y \pm x_2^3 + a x_2^2 y + b x_2 y^2 \pm y^3$ and hence f_0 has modality 2.

(13) By using the analogous method of (10) and (12), we have $f \overset{\text{rR}}{\sim} \pm y^3 + x_2 y + a x_1 y^2 \pm x_1^2$ or f has modality 2 and codimension $f \geq 8$.

(14) $j^3 f(0)$ has the form $h(x_1, x_2) + g(x_1, x_2, y)$, where h,g are homogeneous polynomials of degree 2,3 respectively. Hence

$$\begin{aligned}
&\text{reticular R-codimension } f\\
&\geq \ \dim \mathcal{E}(2;1)/(\langle x_1 \frac{\partial f}{\partial x_1}, x_2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y} \rangle + \mathfrak{M}(2;1)^4 + \mathfrak{M}(2;0)^3)\\
&= \ \dim \mathcal{E}(2;1)/(\langle x_1 \frac{\partial f}{\partial x_1}, x_2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y}, x_1 y \frac{\partial h}{\partial x_1}, x_2 y \frac{\partial h}{\partial x_2}, x_1 \frac{\partial g}{\partial y}, x_2 \frac{\partial g}{\partial y}, y \frac{\partial g}{\partial y} \rangle_{\mathbb{R}}\\
&\qquad\qquad + \mathfrak{M}(2;1)^4 + \mathfrak{M}(2;0)^3)\\
&\geq \ \dim \mathcal{E}(2;1)/\mathfrak{M}(2;1)^4 \ - 4 - 8 = 20 - 4 - 8 = 8.
\end{aligned}$$

On the other hand, $j^3 f(0)$ has modality 2 in $J^3(2+1,1)$ by analogous method of (6): Consider an element $f_0 \in J^3(2+1,1)$ which has the form $ax_1^2 + bx_1x_2 + cx_2^2 + g'(x_1, x_2, y)$ (g' is homogeneous polynomial of degree 3) and all of the coefficients of x_1^2, x_2^2, y^3 are not zero and $b^2 \neq 4ac$. Then f_0 has the normal form $\pm x_1^2 + ax_1x_2 \pm x_2^2 \pm y^3 + bx_1x_2y$ and hence has f_0 modality 2.

The case $r = 2, k \geq 2$. We prove that codimension $f \geq 8$ and $j^3 f(0)$ has modality 2. To do this, we only need to prove in the case $k = 2$ because codimension $f \geq$ codimension $f|_{y_3=\cdots=y_k=0}$ and if $f|_{y_3=\cdots=y_k=0}$ has modality 2 then f also has modality 2.

Set $A = \mathfrak{M}(2;0)^2 + \mathfrak{M}(2;0)\mathfrak{M}(0;2)^2 + \mathfrak{M}(2;2)^4$. Since $f \in \mathfrak{M}(0;2)^3 + \mathfrak{M}(2;0)\mathfrak{M}(0;2) + \mathfrak{M}(2;0)^2$ we have that $x^2\frac{\partial f}{\partial x} \equiv xy\frac{\partial f}{\partial x} \equiv xy\frac{\partial f}{\partial y} \equiv y^2\frac{\partial f}{\partial y} \equiv 0$. Hence

$$
\begin{aligned}
&\text{reticular R-codimension } f \\
&\geq \dim \mathcal{E}(2;2)/(\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle + A) \\
&\geq \dim \mathcal{E}(2;2)/(\langle x\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}, y\frac{\partial f}{\partial y}\rangle_{\mathbb{R}} + A) \\
&\geq \dim \mathcal{E}(2;2)/A - 8 \\
&= \dim \mathcal{E}(2;0)/\mathfrak{M}(2;0)^2 + \dim \mathfrak{M}(0;2)/\mathfrak{M}(0;2)^4 + 4 - 8 = 3 + 9 + 4 - 8 = 8.
\end{aligned}
$$

On the other hand, $j^3 f(0)$ has modality 2 by an analogous method of (6): Consider an element f_0 in $J^3(2+2,1)$ in which all of the coefficients of $x_1y_1, x_2y_2, y_1^3, y_2^3$ are not zero. Then f_0 has the normal form $x_1y_1 + x_2y_2 \pm y_1^3 + ay_1^2y_2 + by_1y_2^2 \pm y_2^3$ and hence f_0 has modality 2.

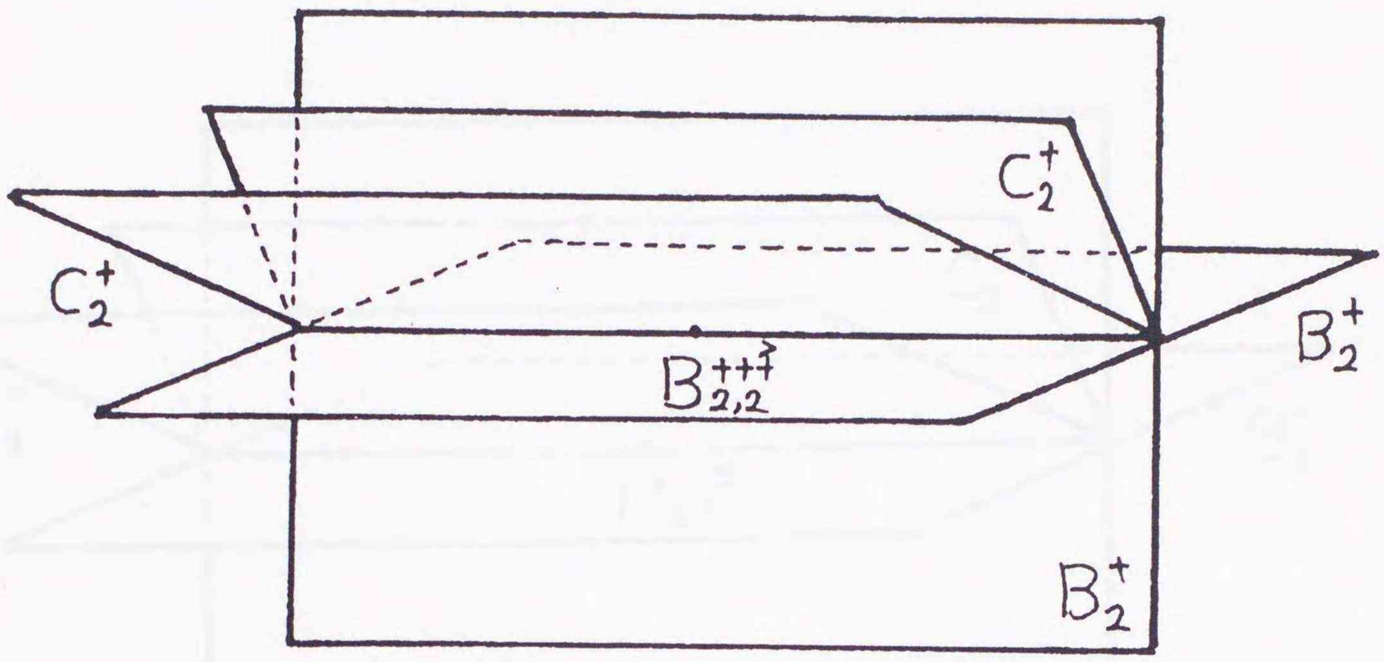
The case $r \geq 3$. We only need to prove in the case $r = 3, k = 0$ that codimension $f \geq 8$ and $j^2 f(0)$ has modality 2.

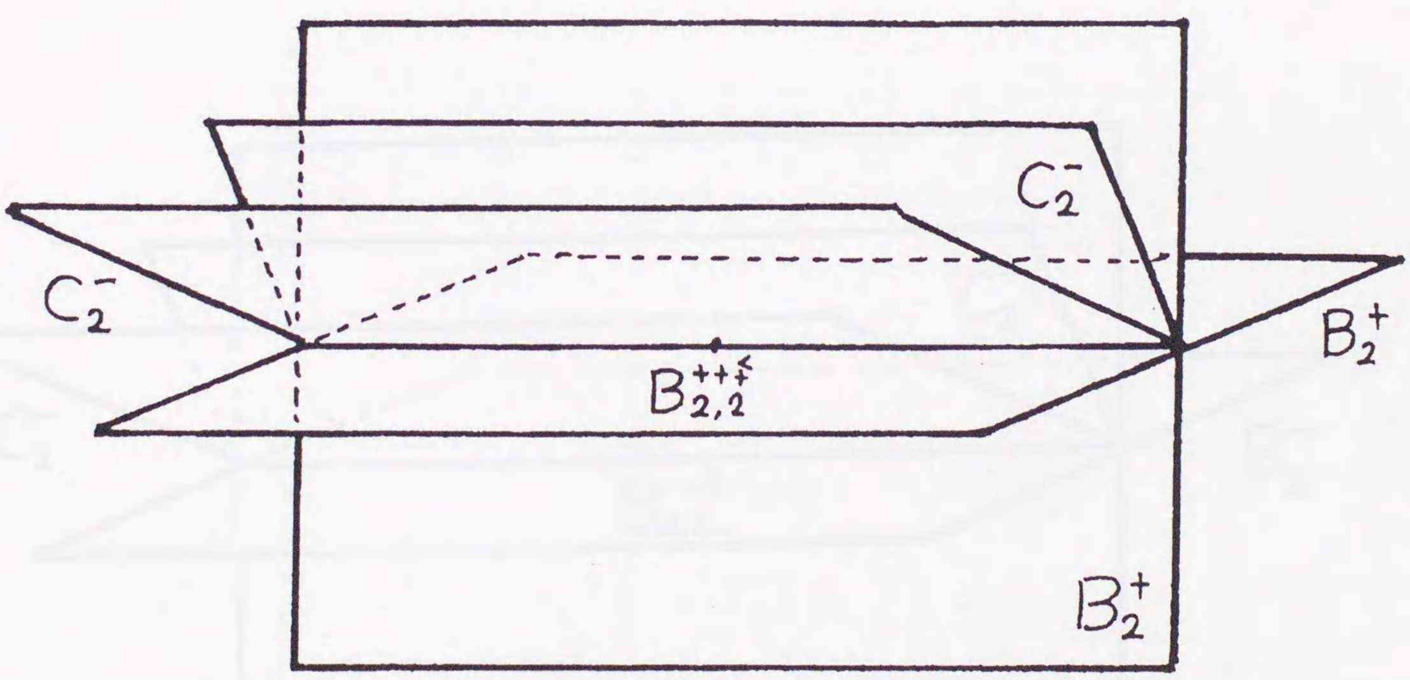
reticular R-codimension f

$$\geq \dim \mathcal{E}(3;0)/(\langle x\frac{\partial f}{\partial x}\rangle + \mathfrak{M}(3;0)^4)$$
$$\geq \dim \mathcal{E}(3;0)/(\langle x\frac{\partial f}{\partial x}, x^2\frac{\partial f}{\partial x}\rangle_{\mathbb{R}} + \mathfrak{M}(3;0)^4)$$
$$\geq \dim \mathcal{E}(3;0)/\mathfrak{M}(3;0)^4 \ -3-9 = 20-3-9 = 8.$$

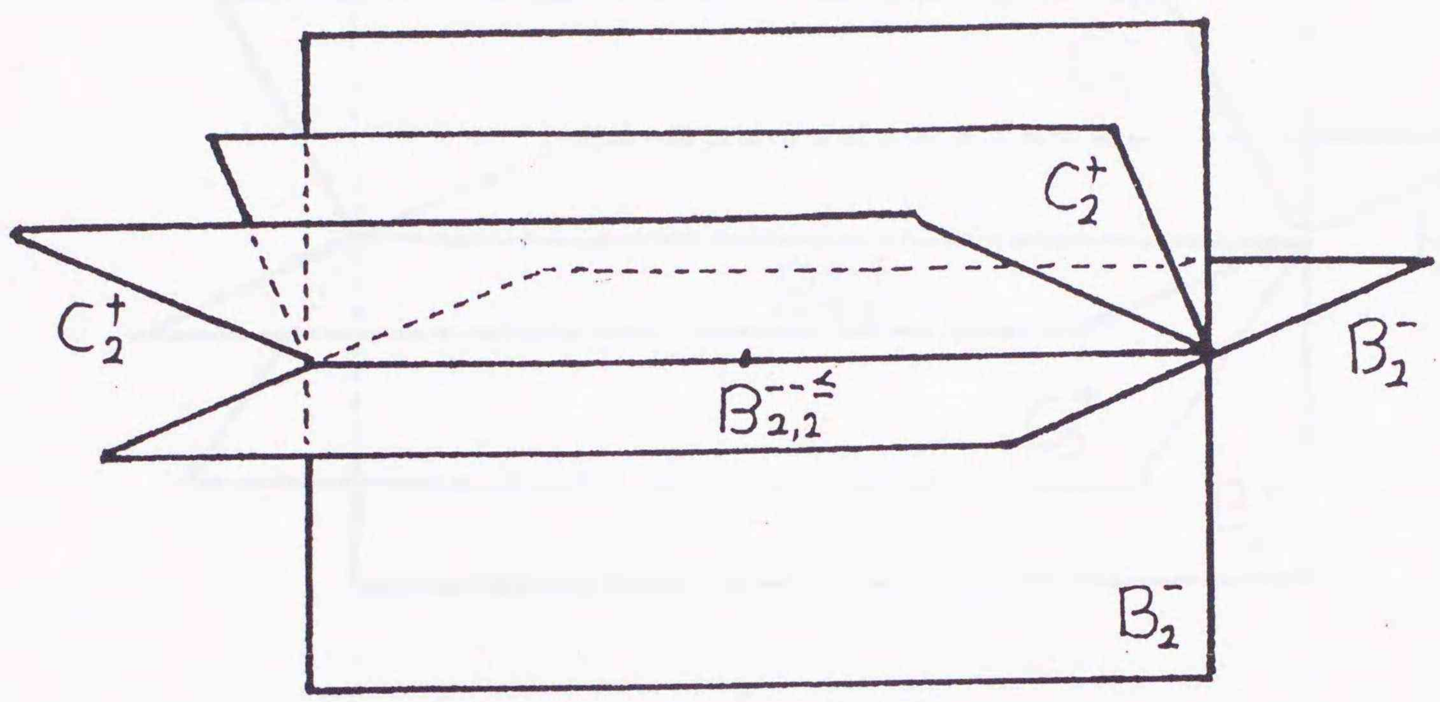
On the other hand $j^2f(0)$ has modality 2 in $J^2(3+0,1)$ by the analogous method of (6): Consider an element f_0 in $J^3(3+0,1)$ in which all of the coefficients of x_1^2, x_2^2, x_3^2 are not zero. Then f_0 has the normal form $\pm x_1^2 \pm x_2^2 \pm x_3^2 + ax_1x_2 + bx_2x_3 + cx_3x_1$ and hence f_0 has modality 3 in $J^2(3+0,1)$. ■

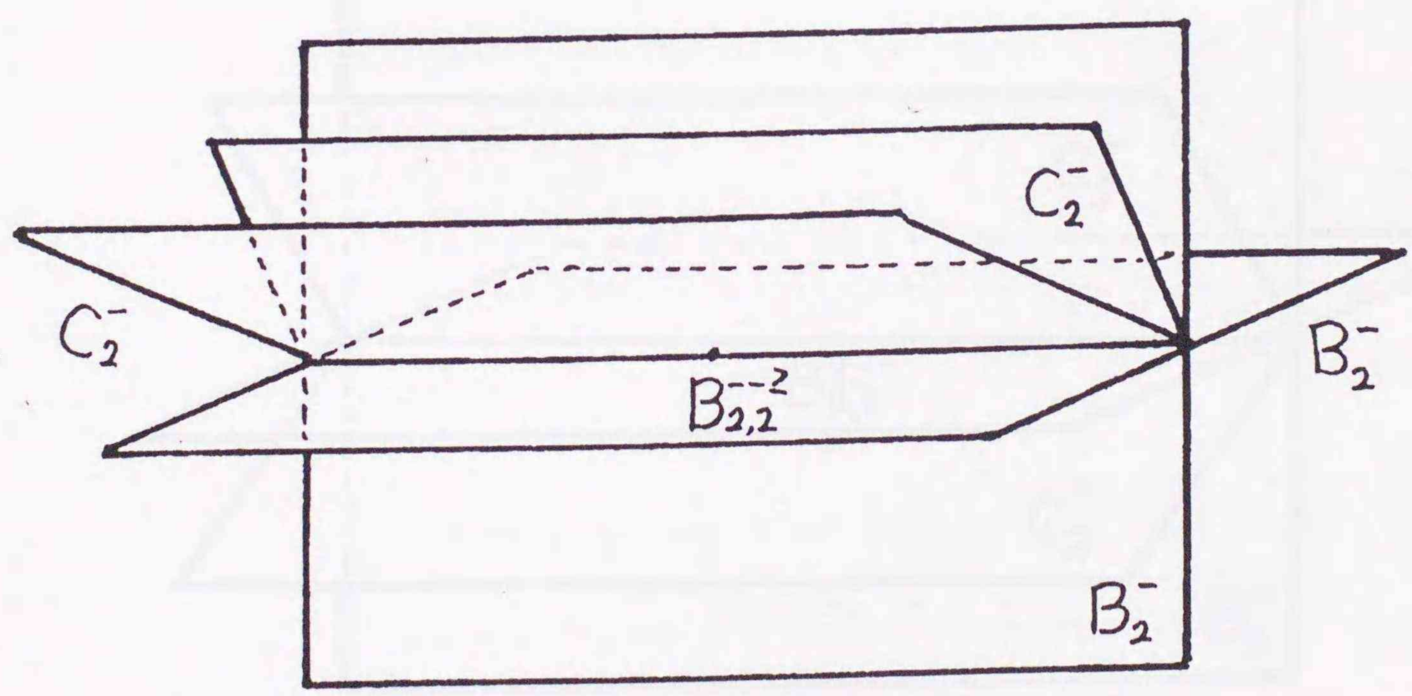
The $B_{2,2}$ caustics I



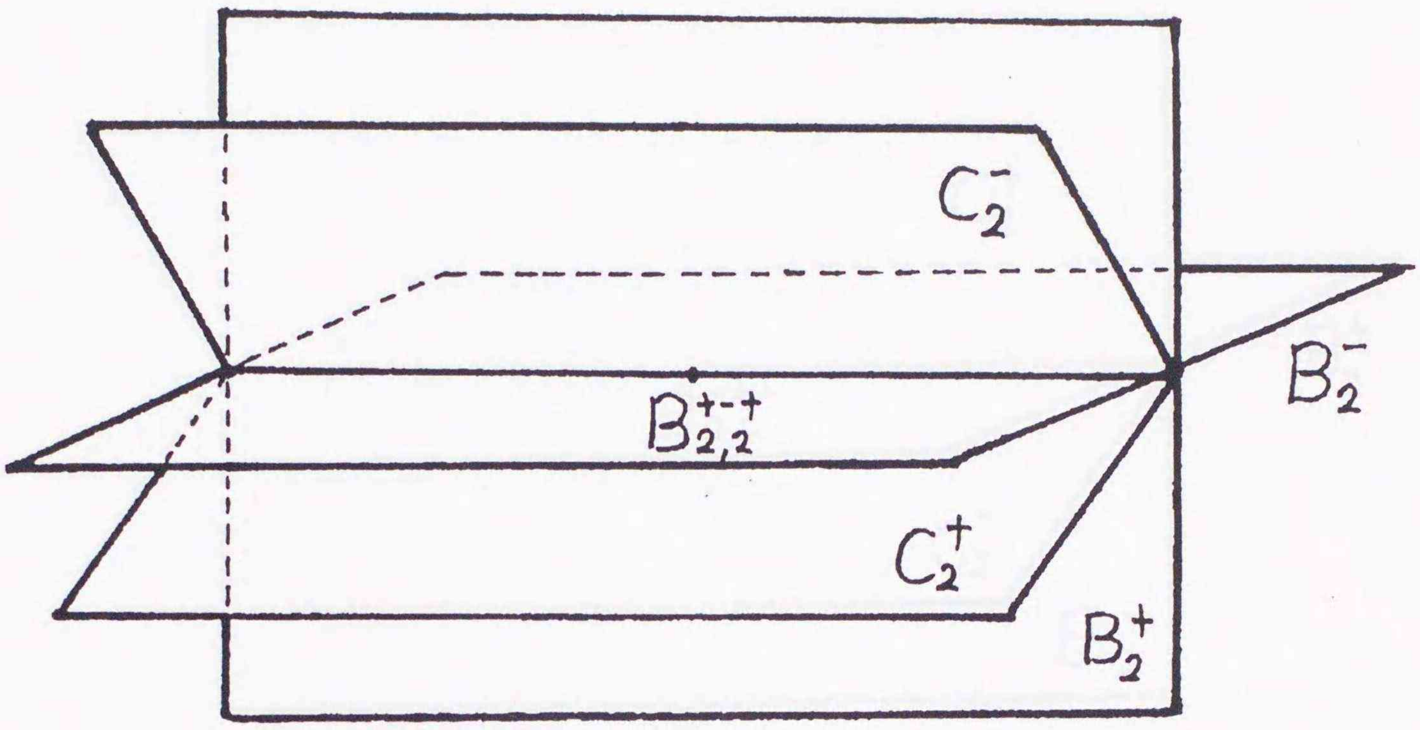


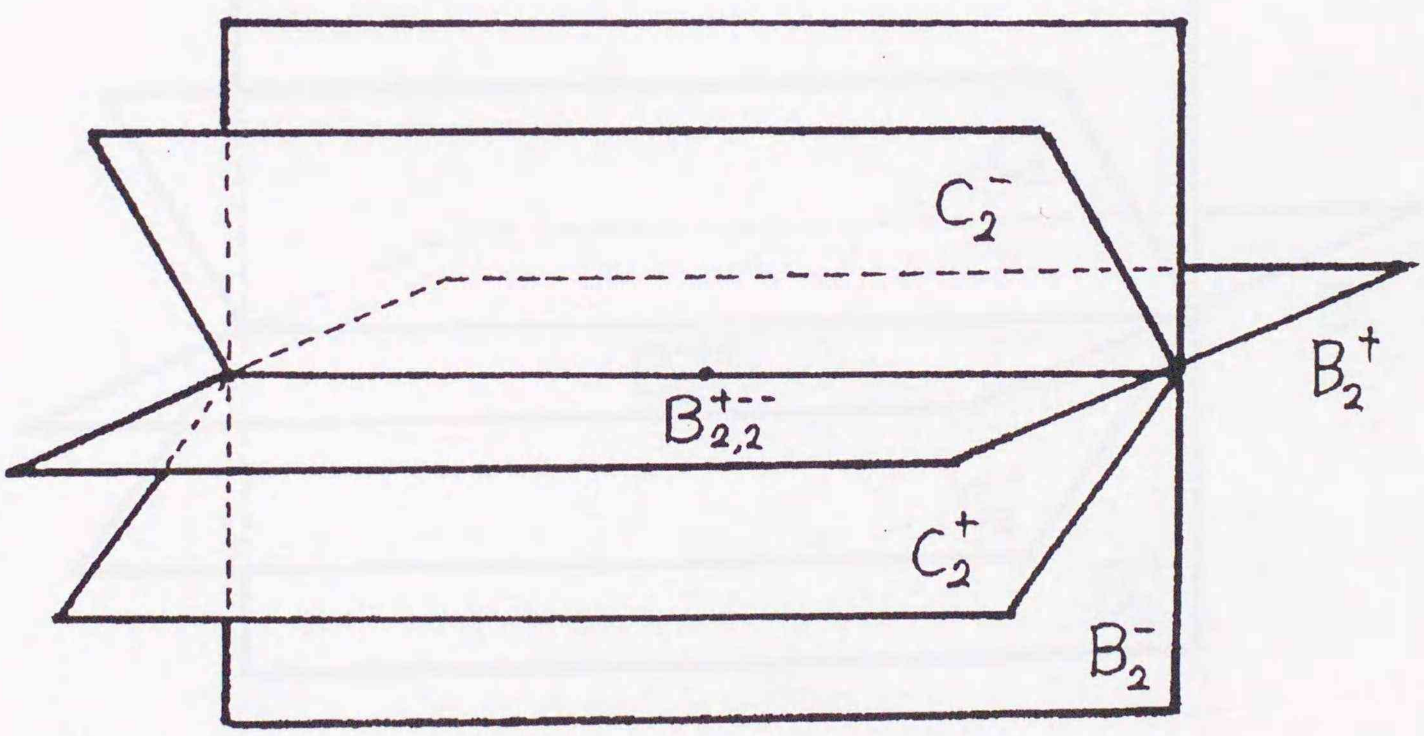
The $B_{2,2}$ caustics II



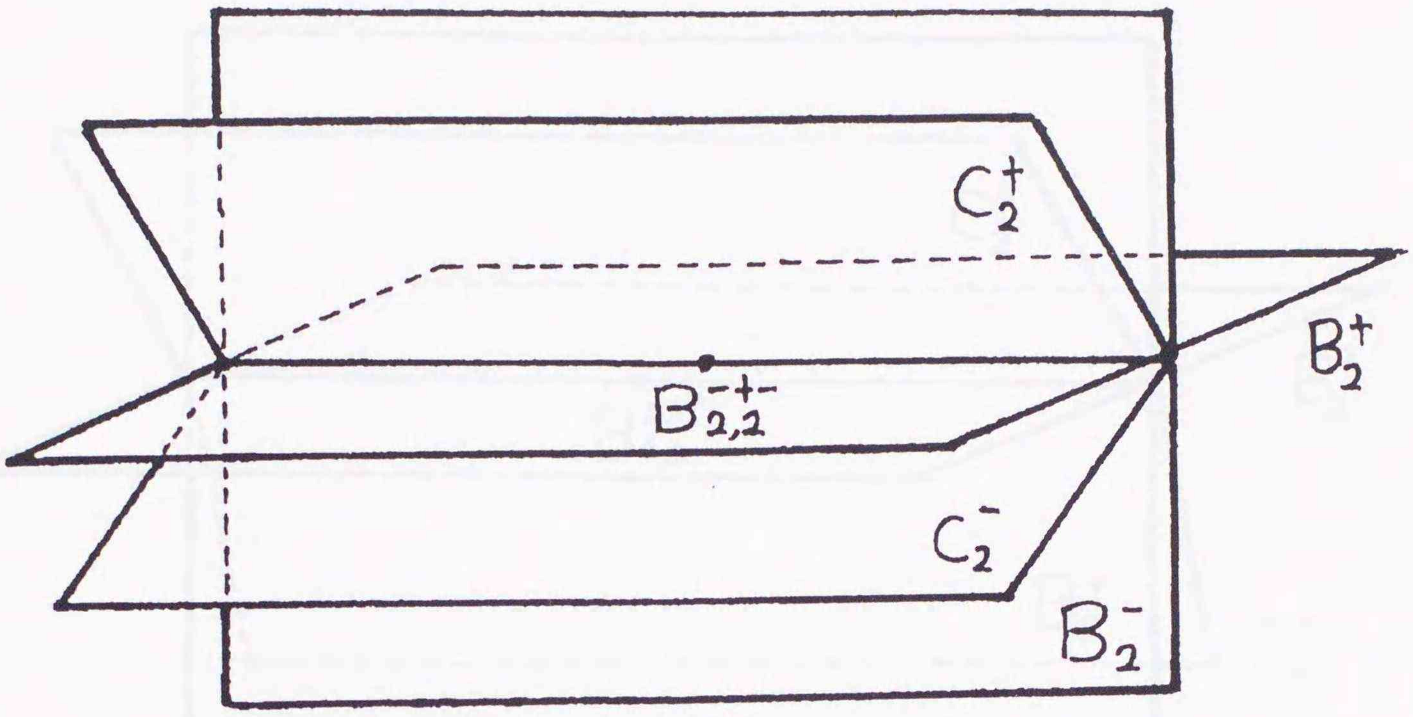


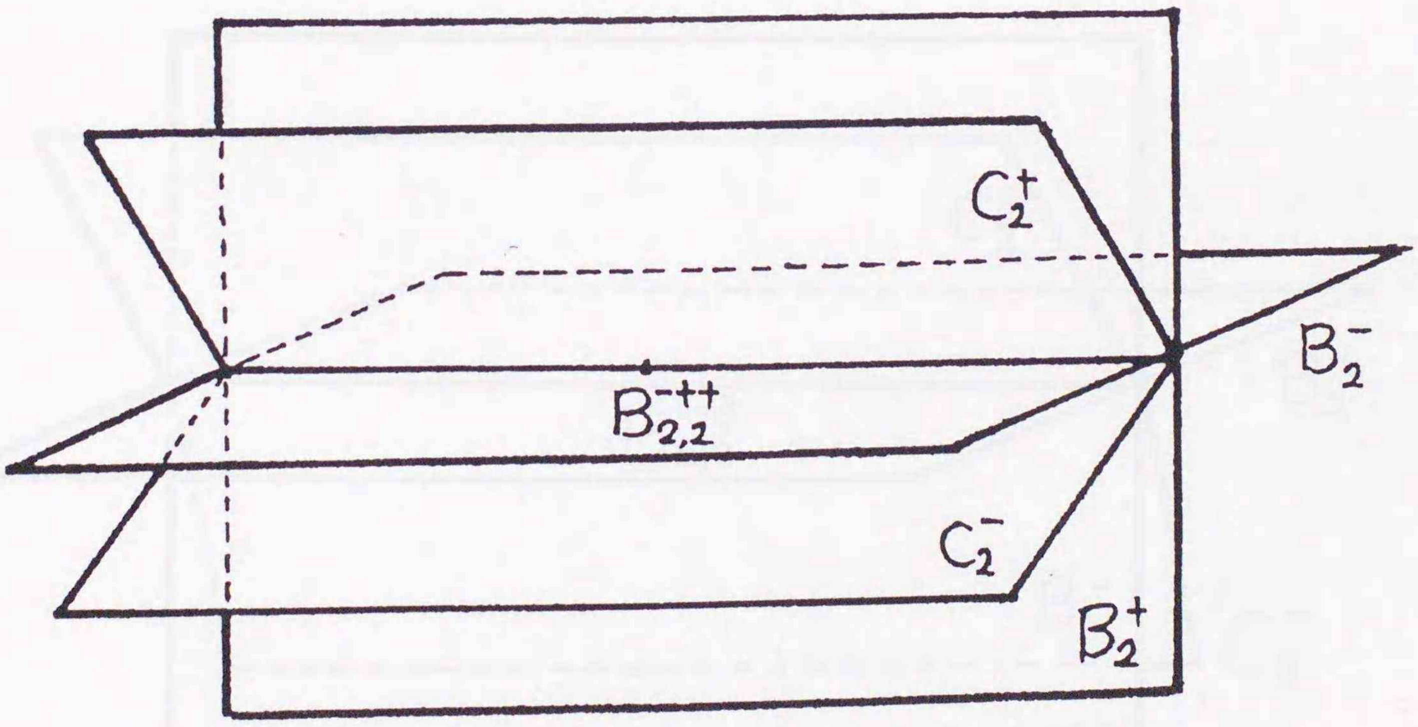
The $B_{2,2}$ caustics III



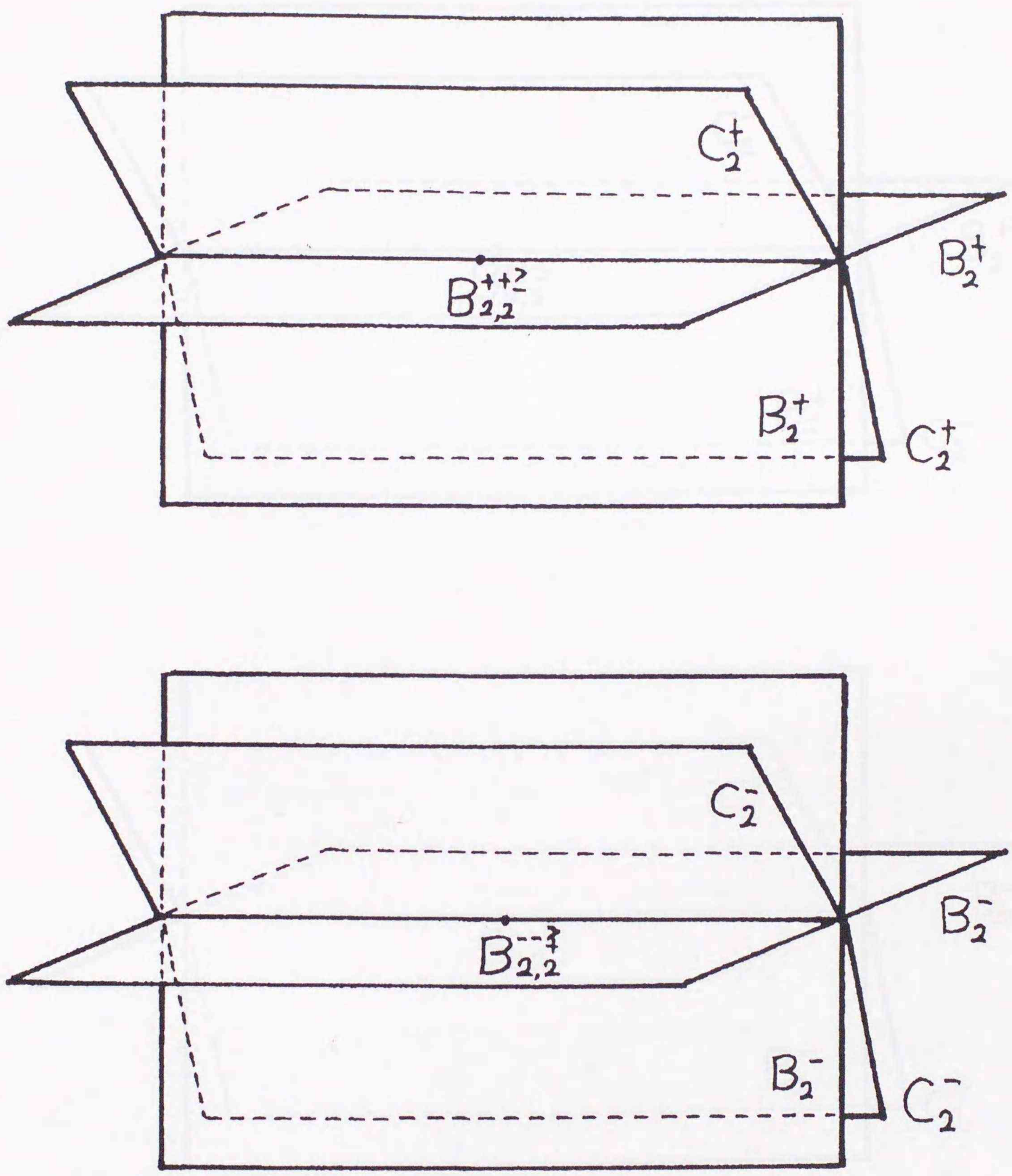


The $B_{2,2}$ caustics IV

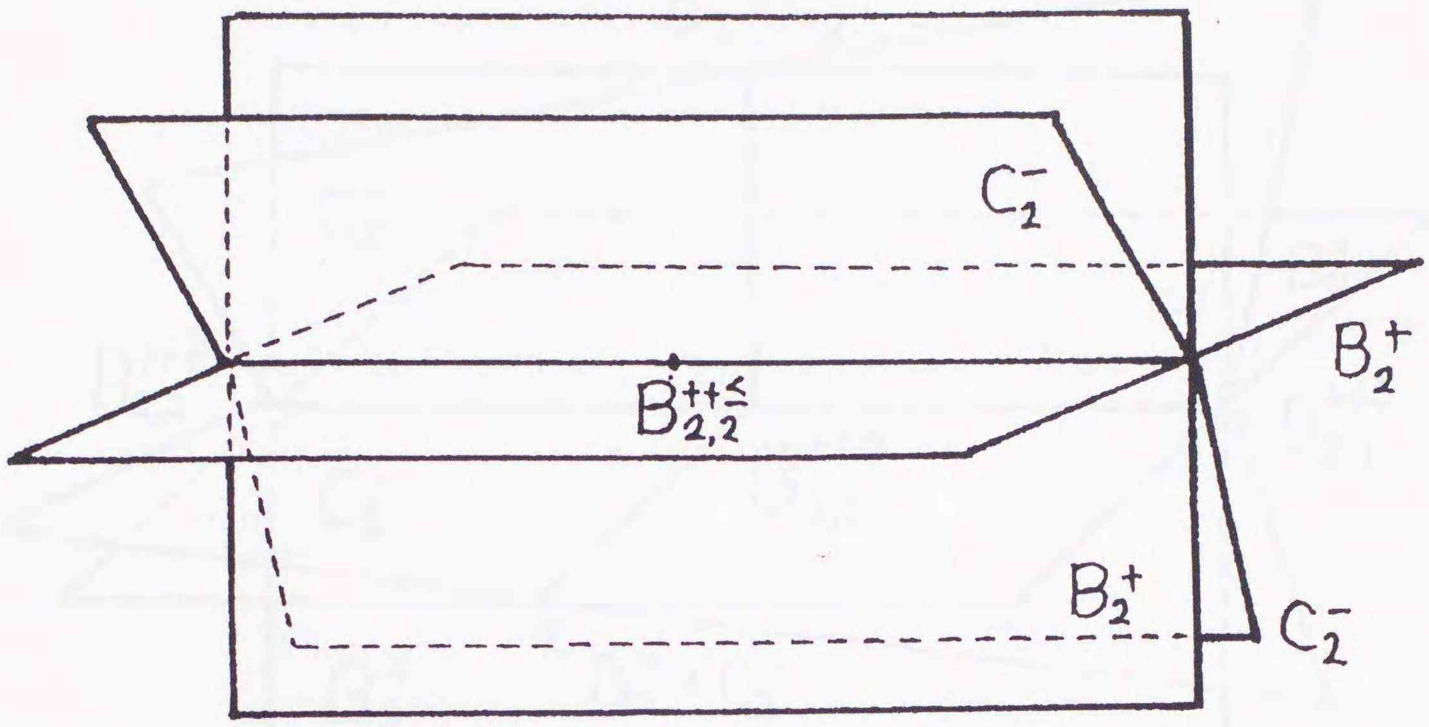


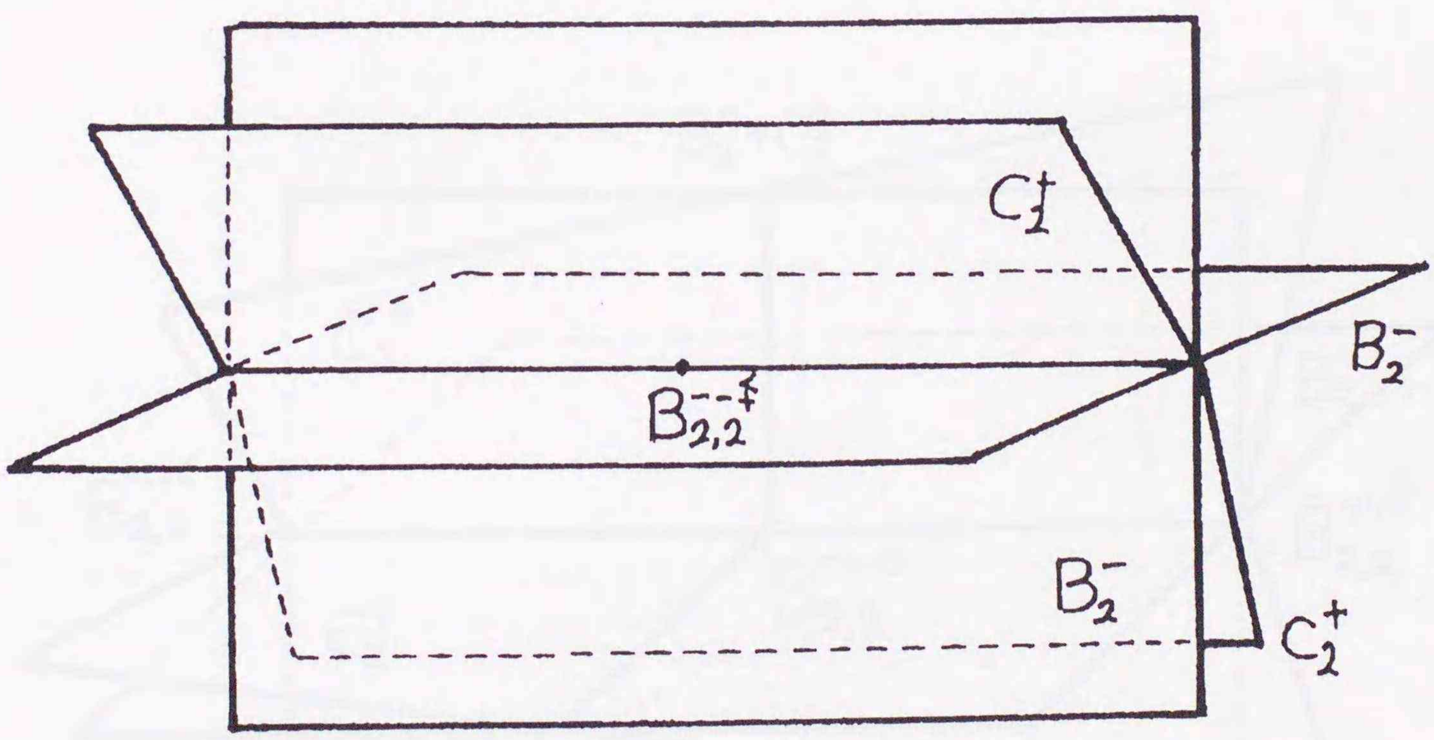


The $B_{2,2}$ caustics V

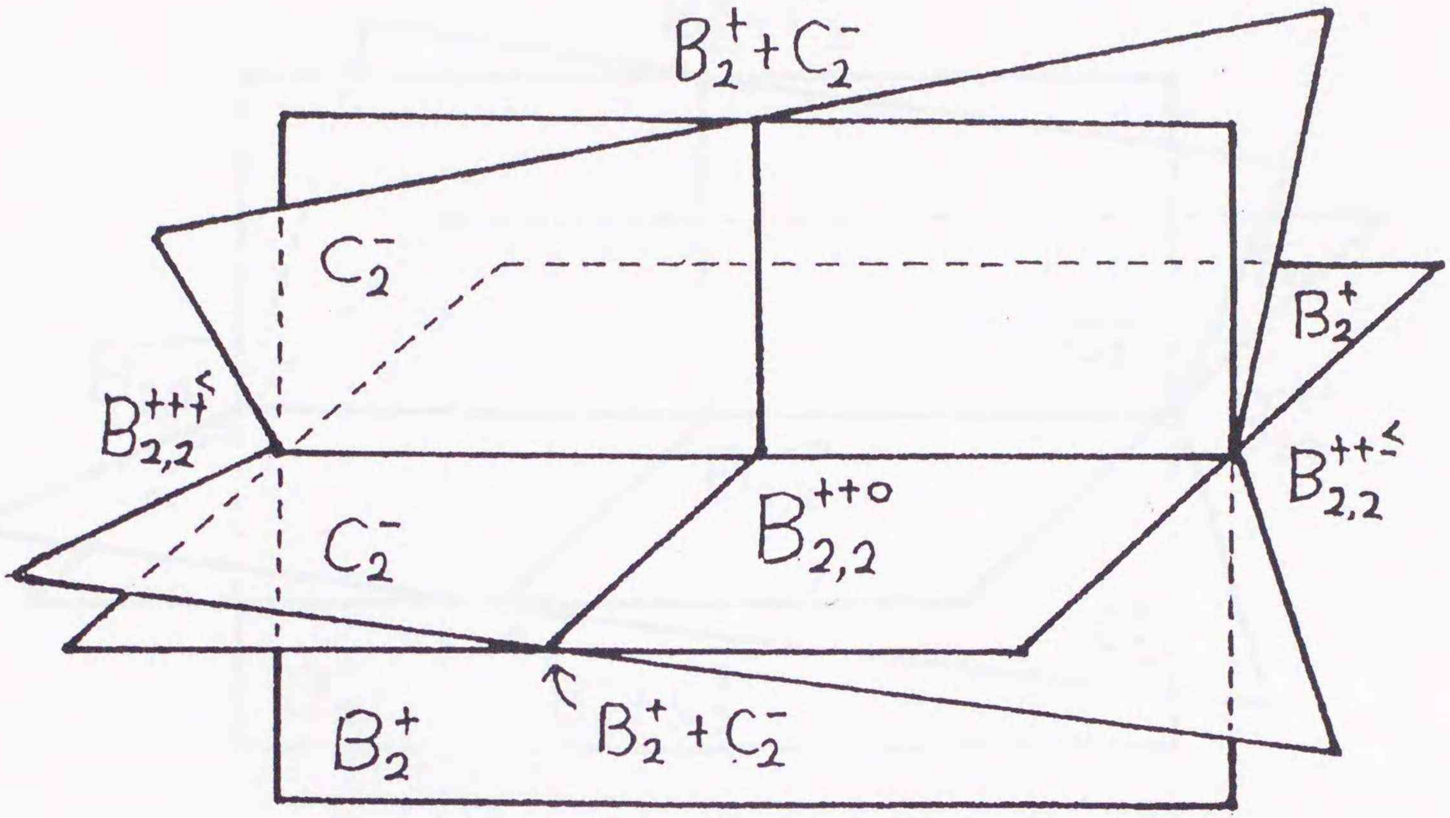


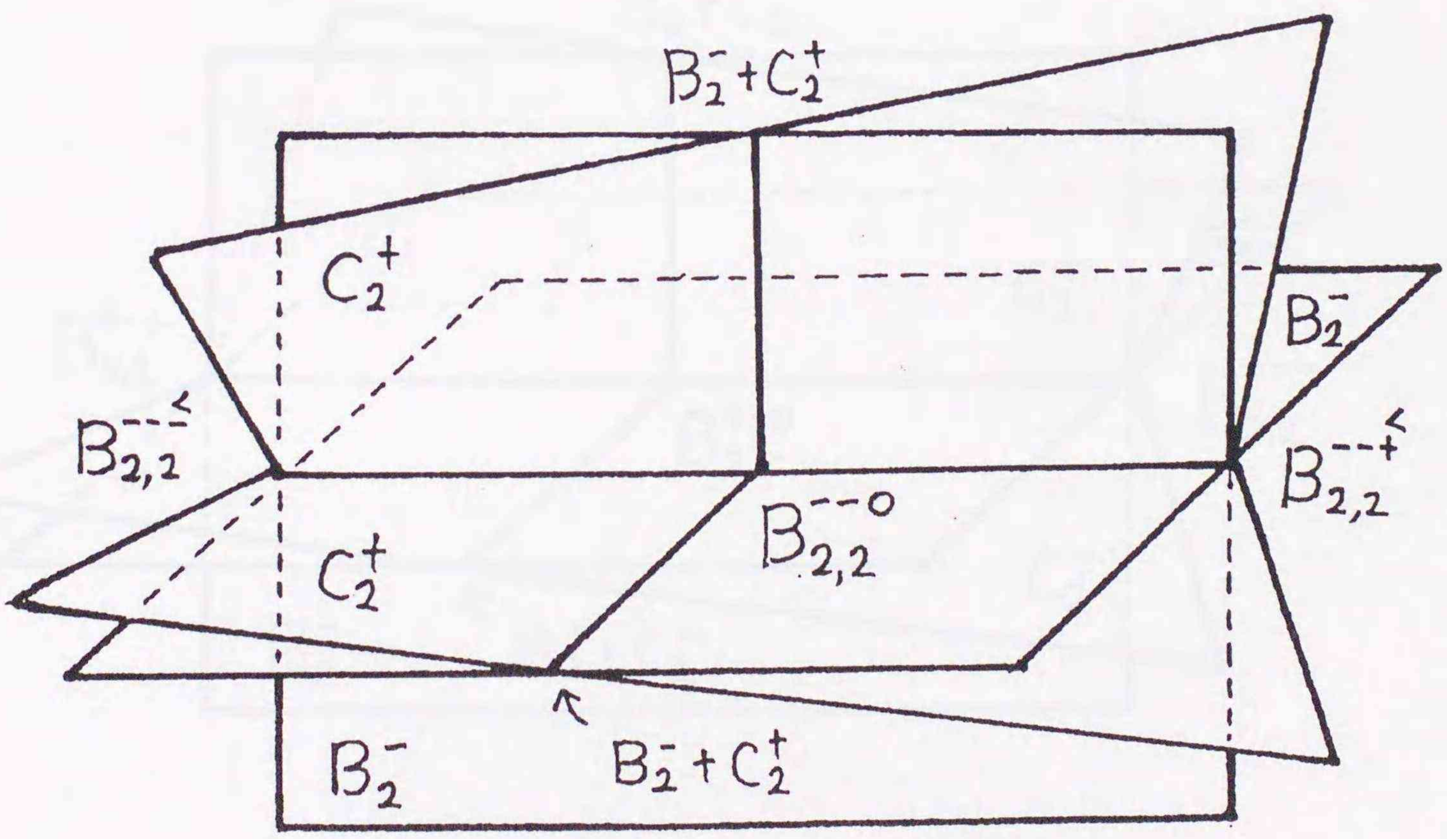
The $B_{2,2}$ caustics VI



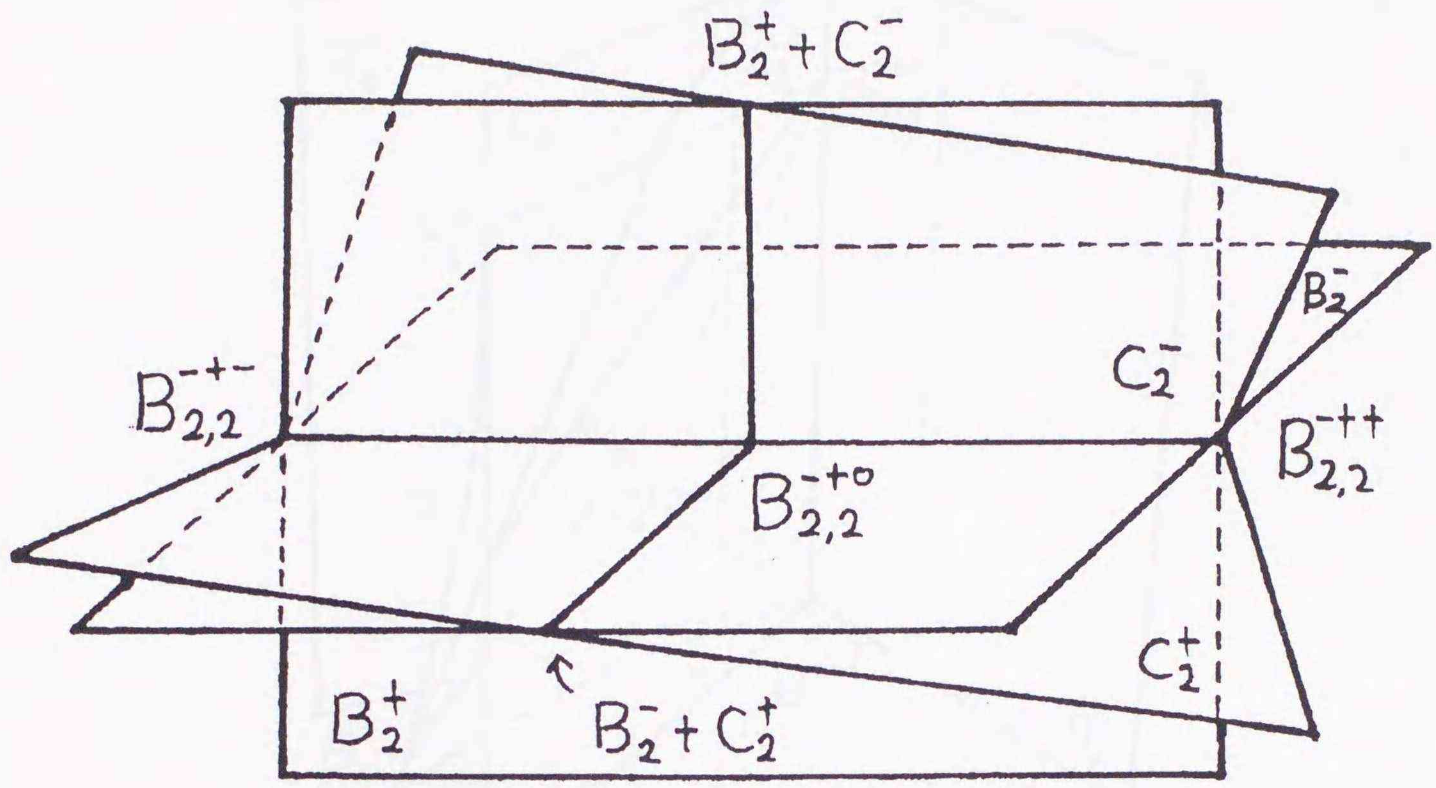


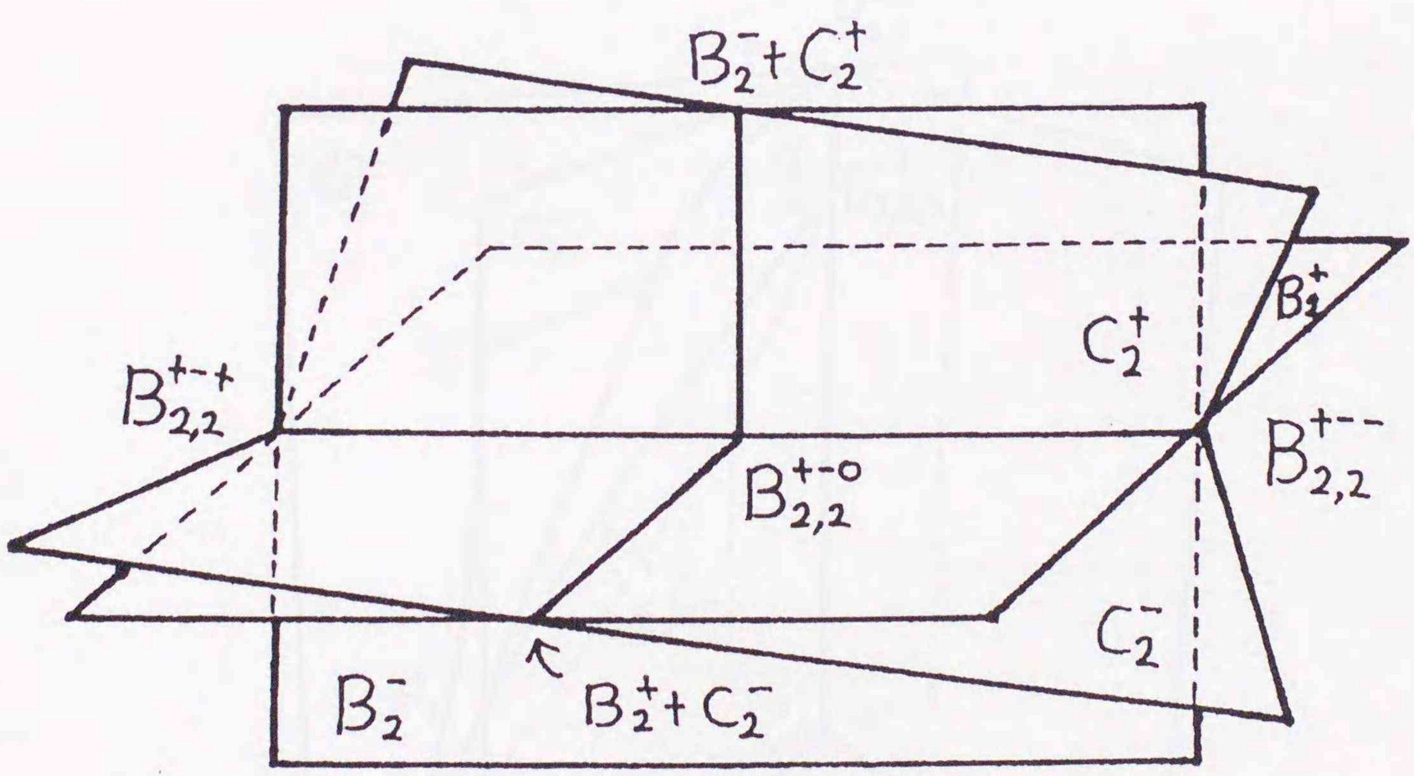
The $B_{2,2}$ caustics VII



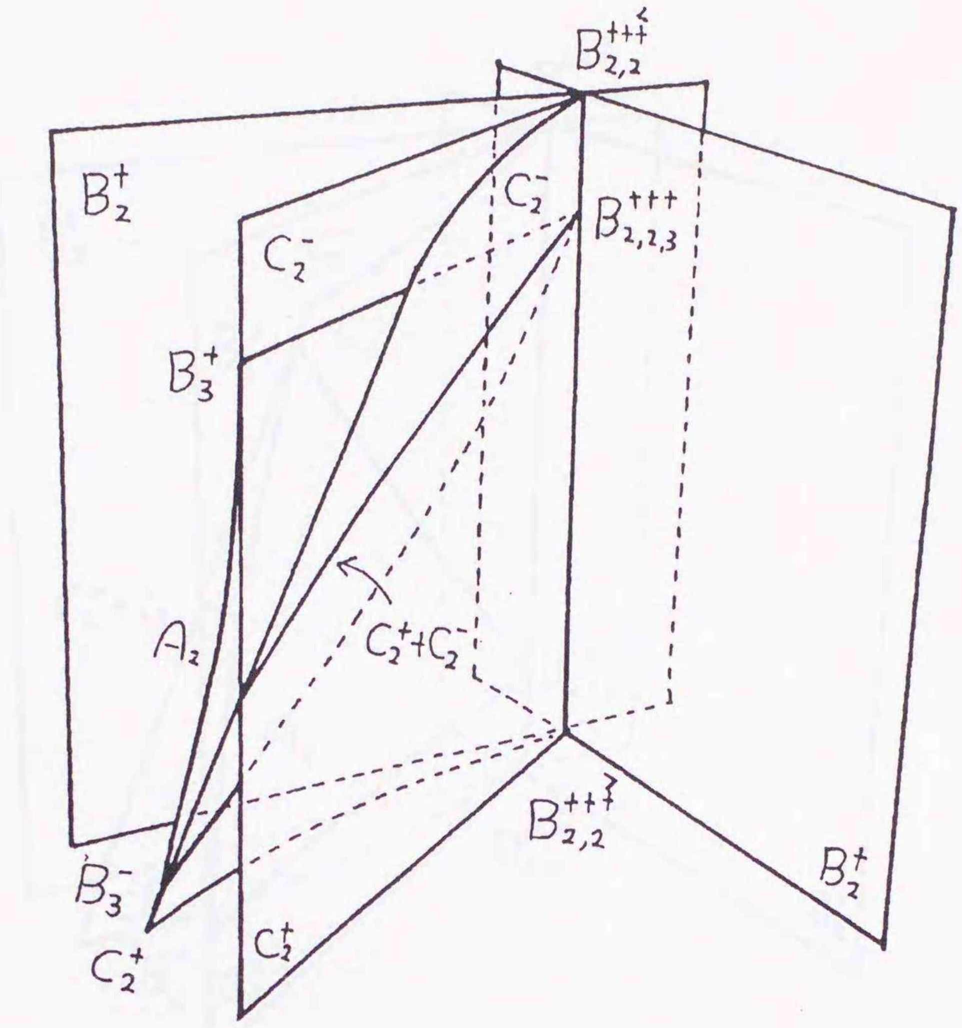


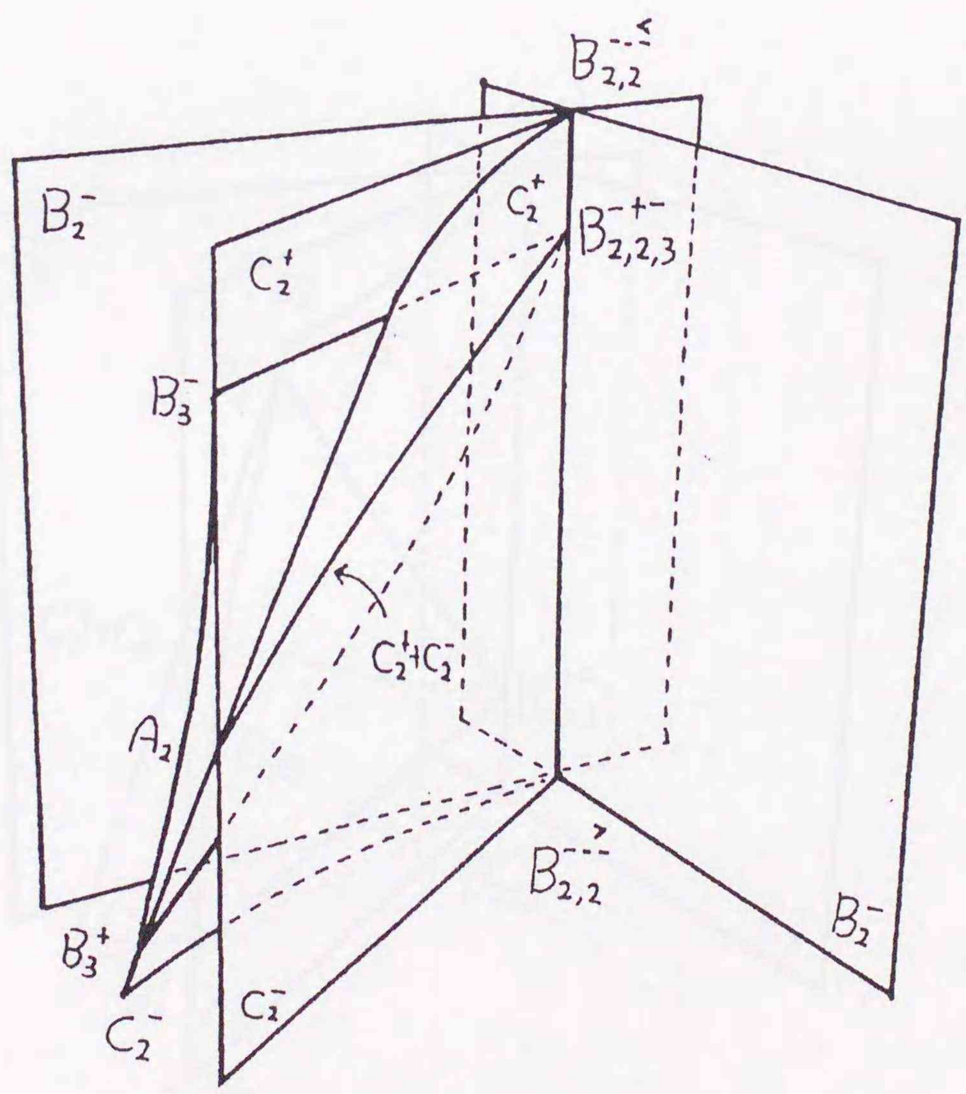
The $B_{2,2}$ caustics VIII



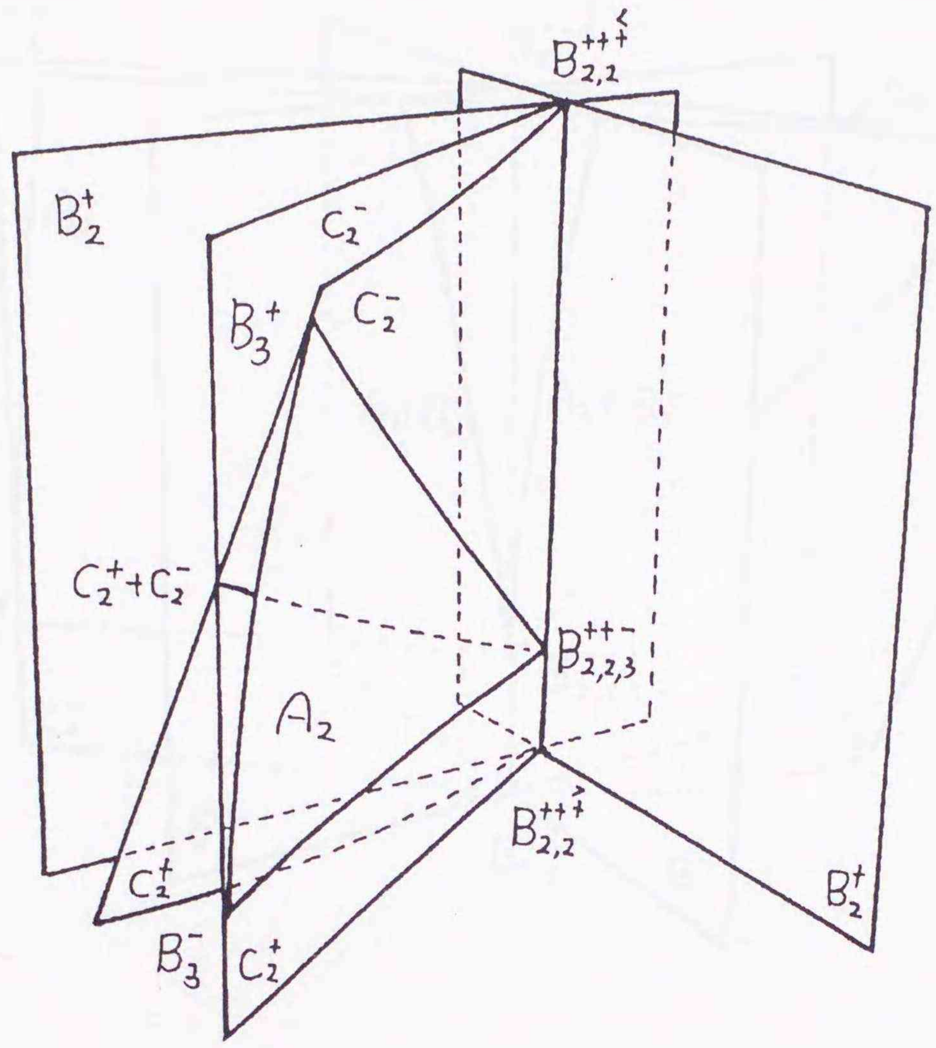


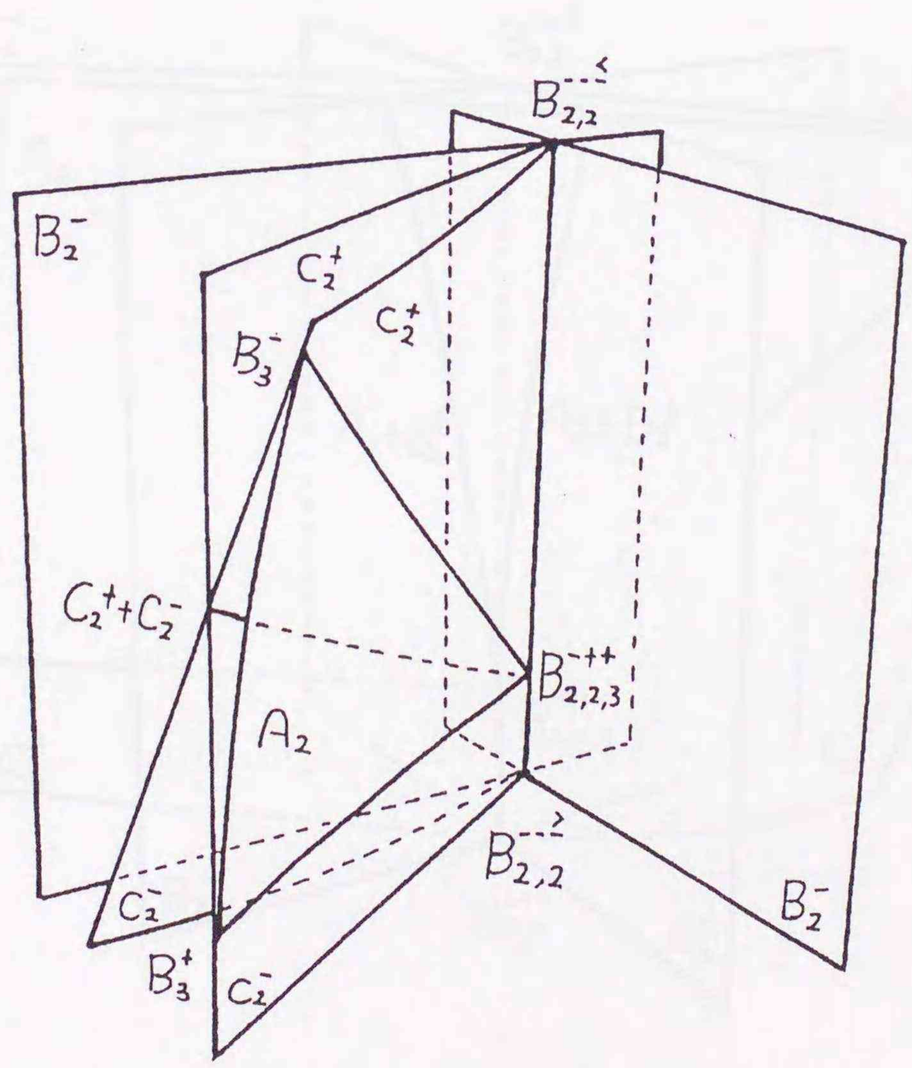
The sections of $B_{2,2,3}$ caustics I



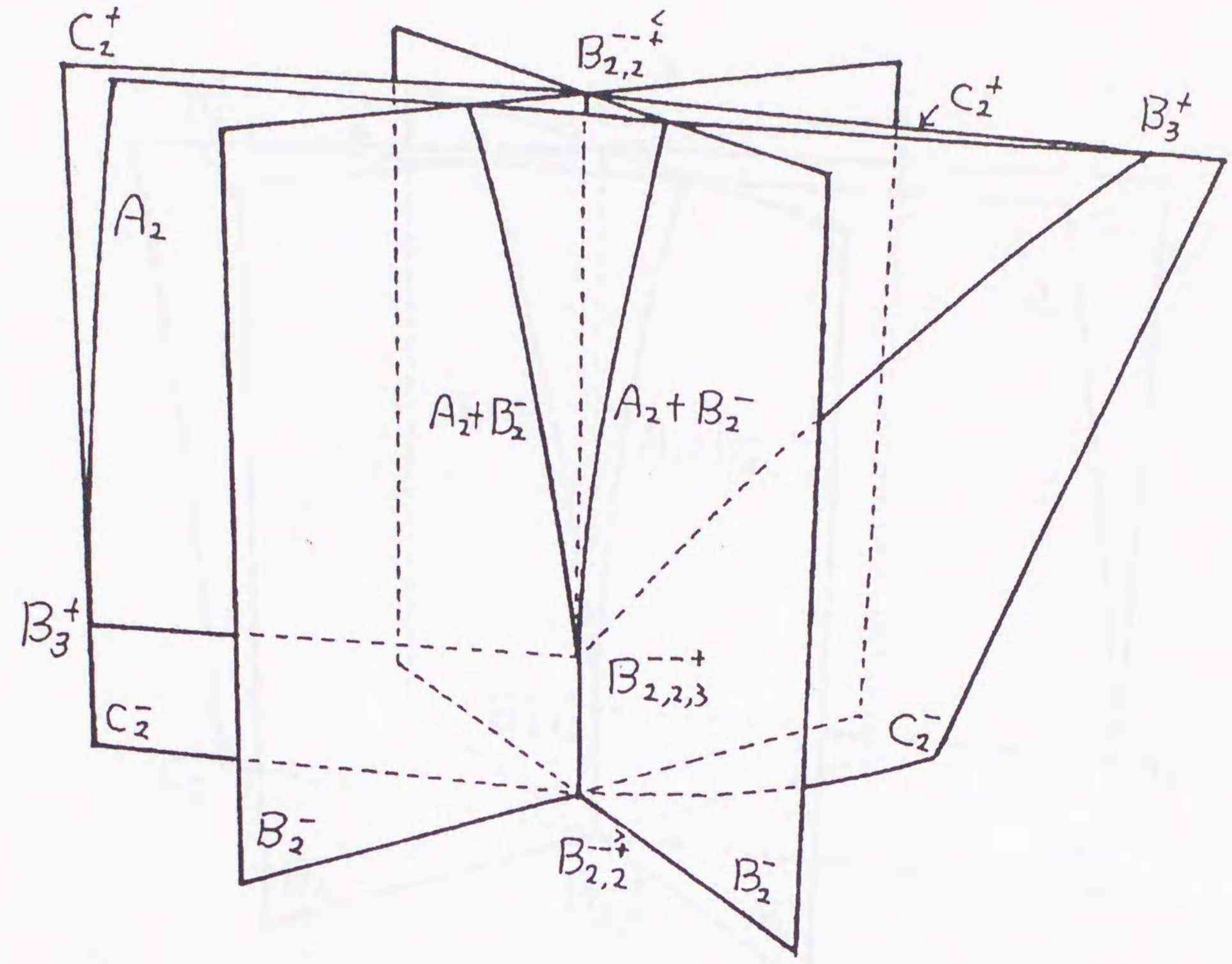


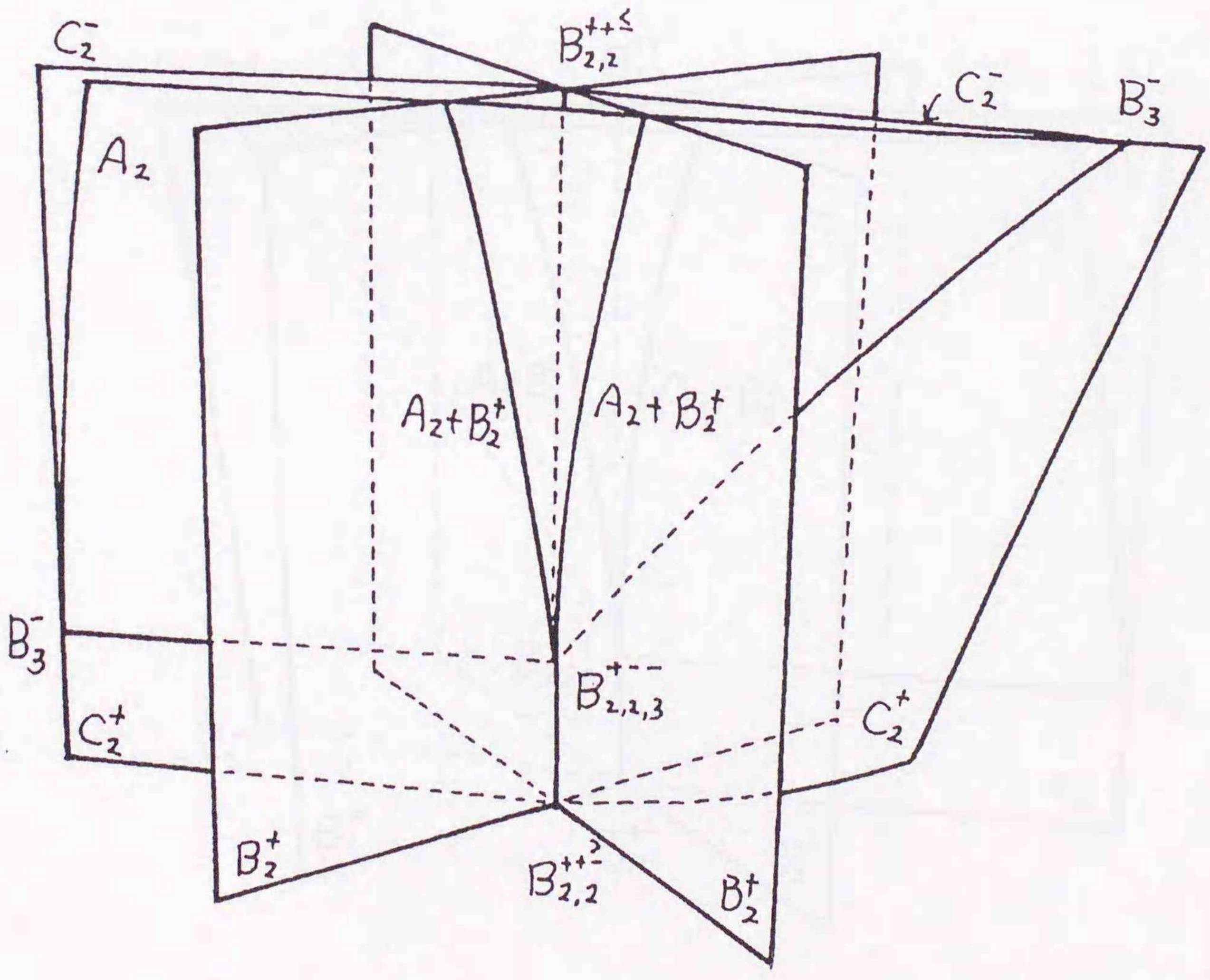
The sections of $B_{2,2,3}$ caustics II



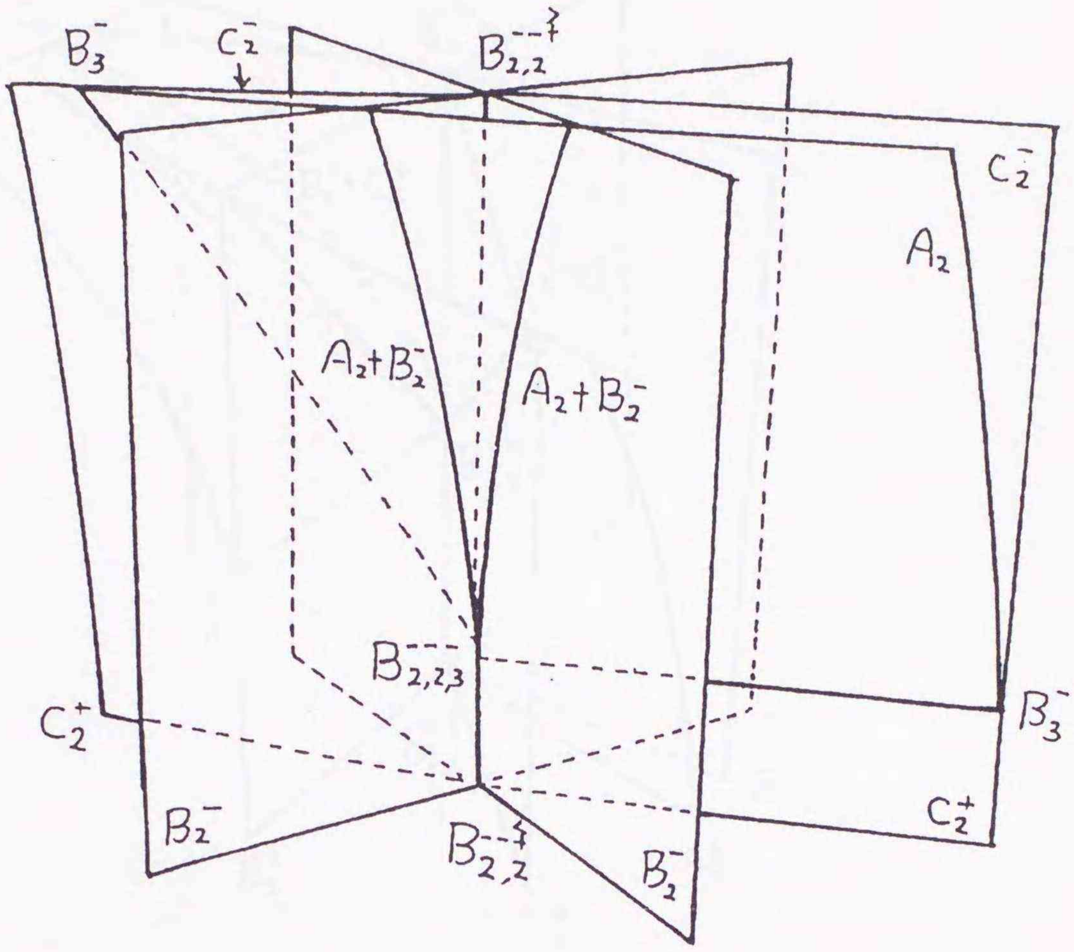


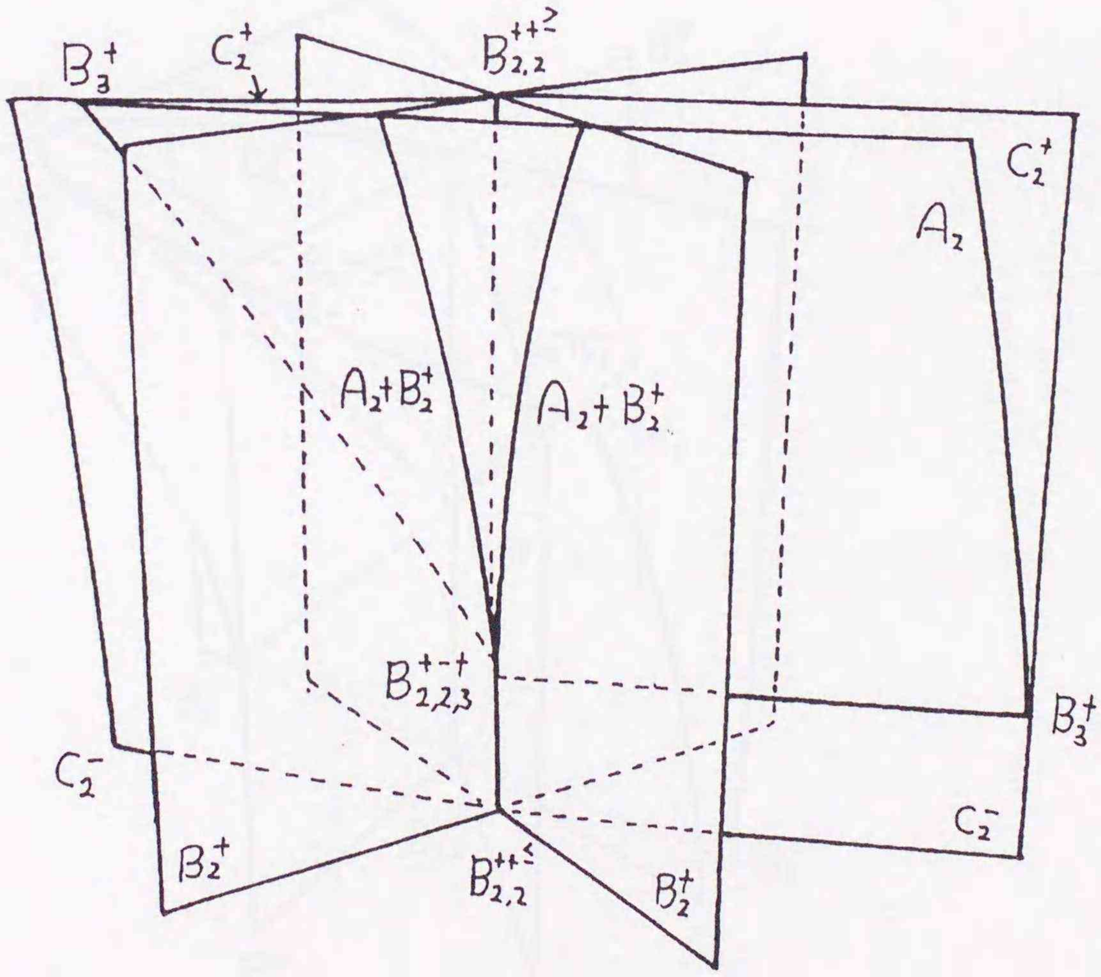
The sections of $B_{2,2,3}$ caustics III



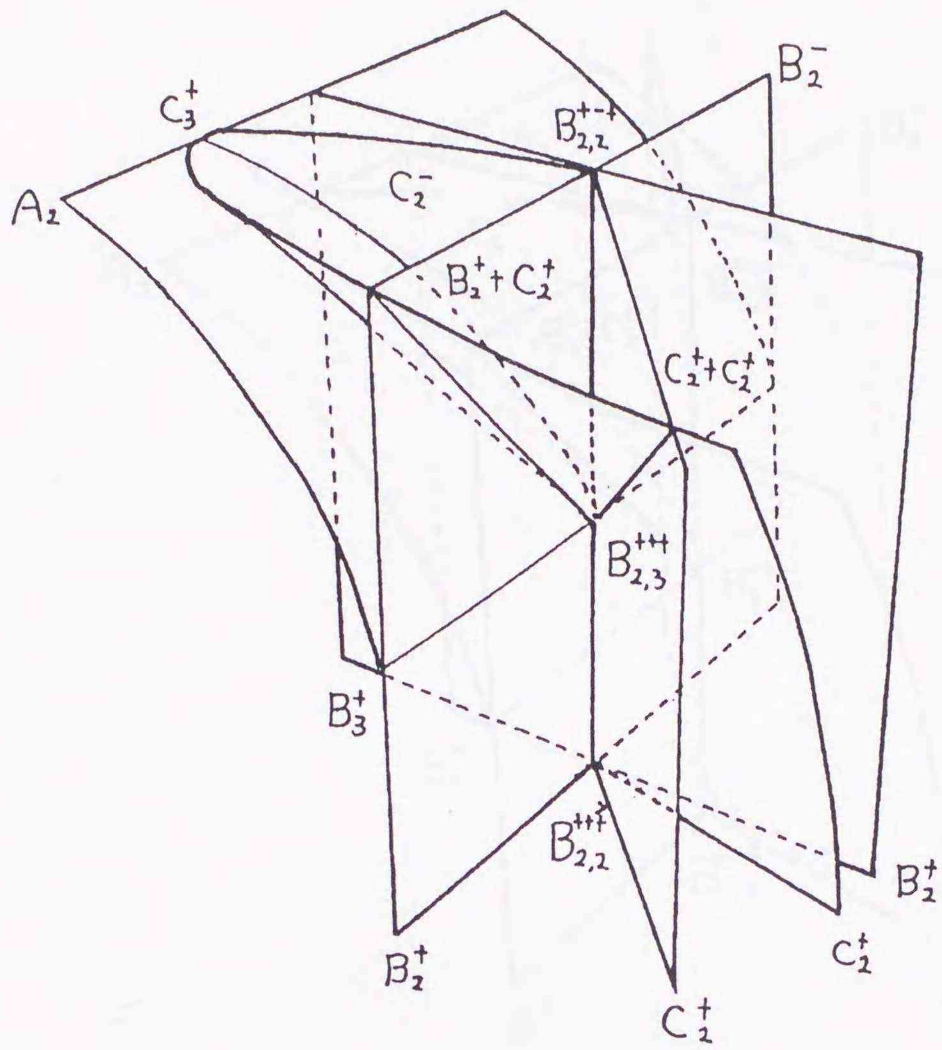


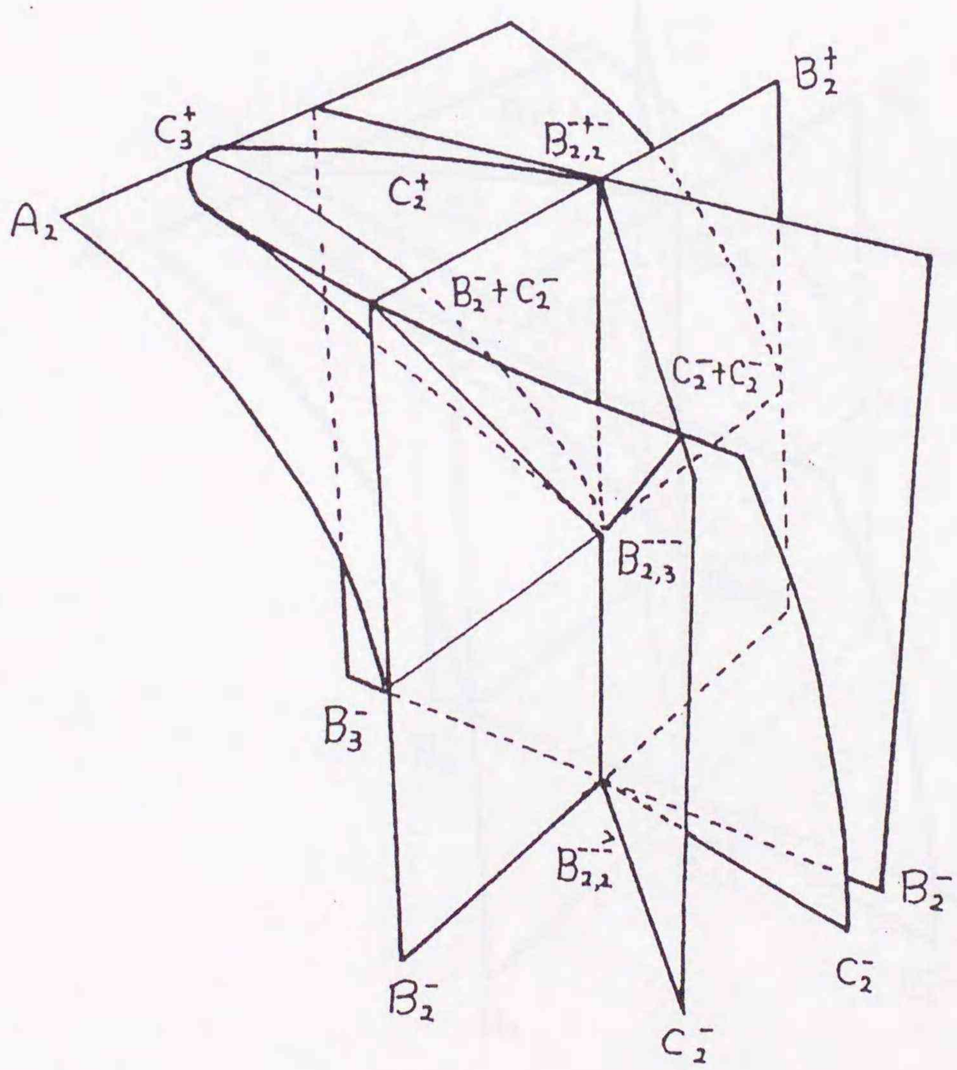
The sections of $B_{2,2,3}$ caustics IV



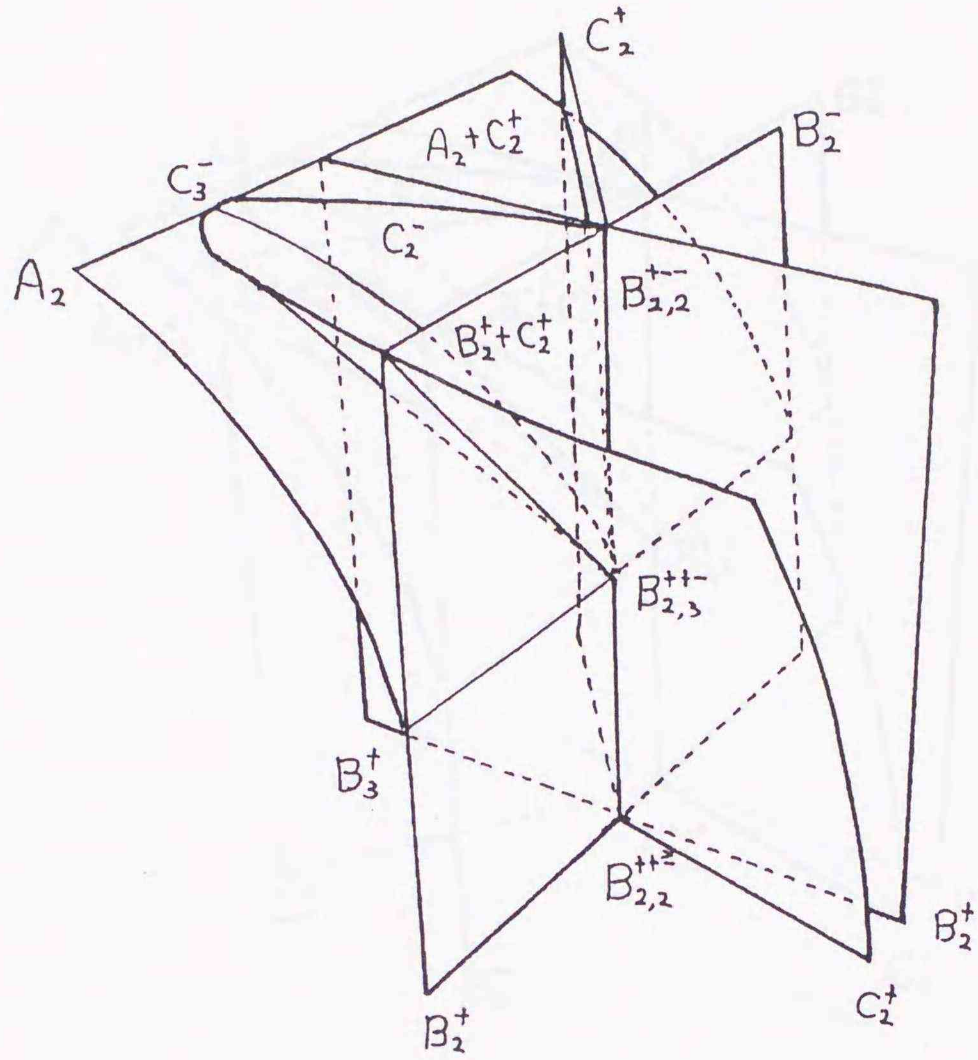


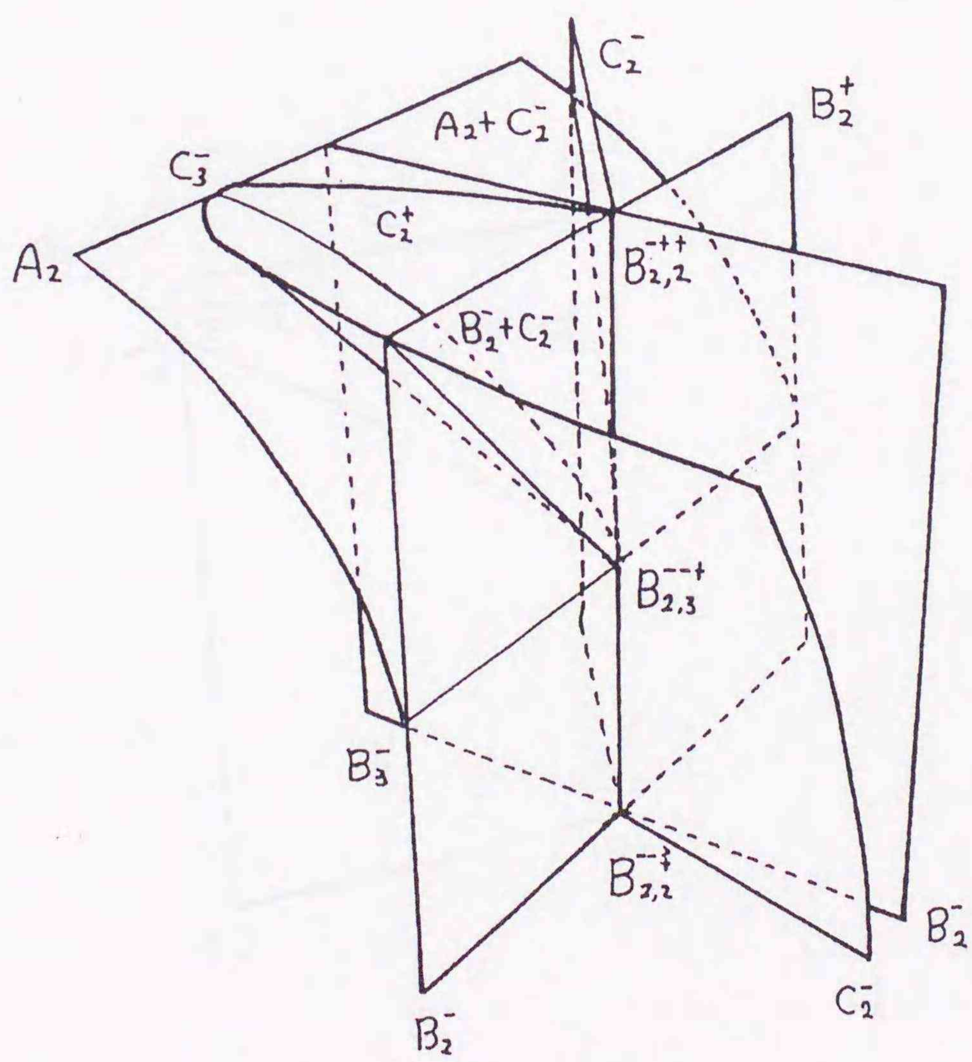
The sections of $B_{2,3}$ caustics I



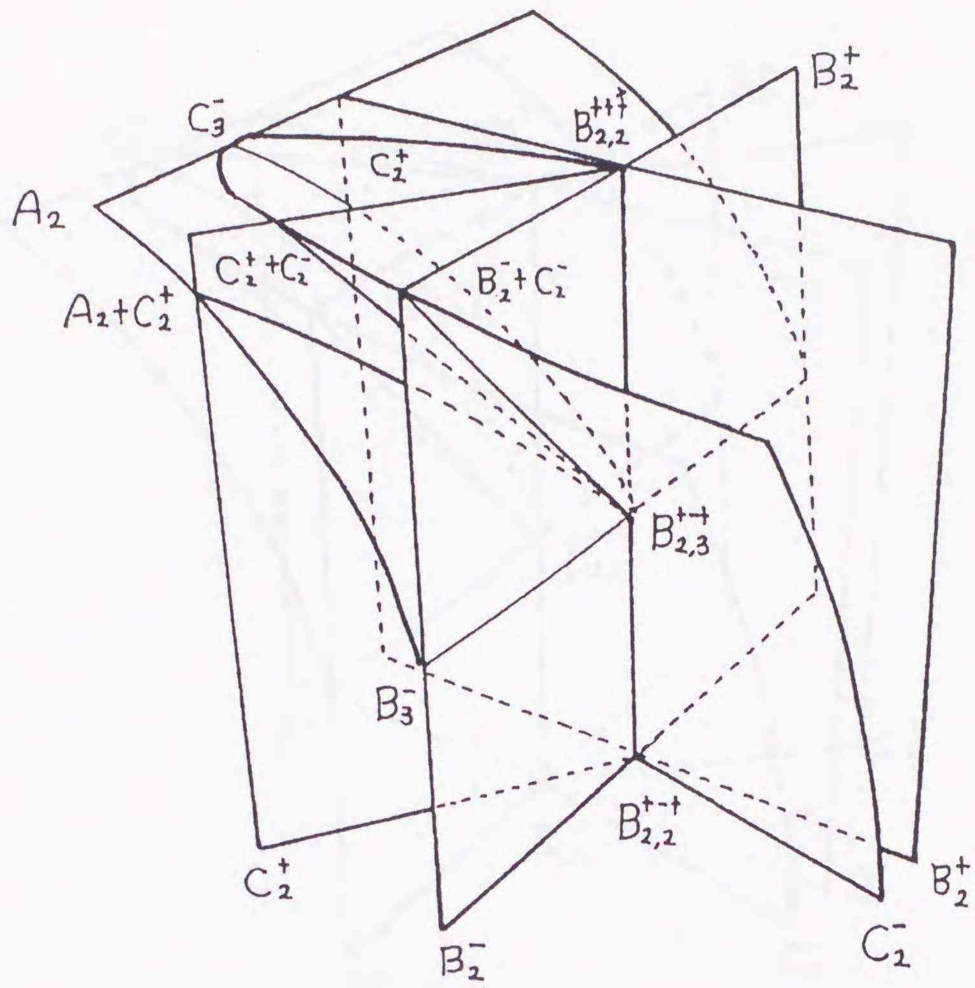


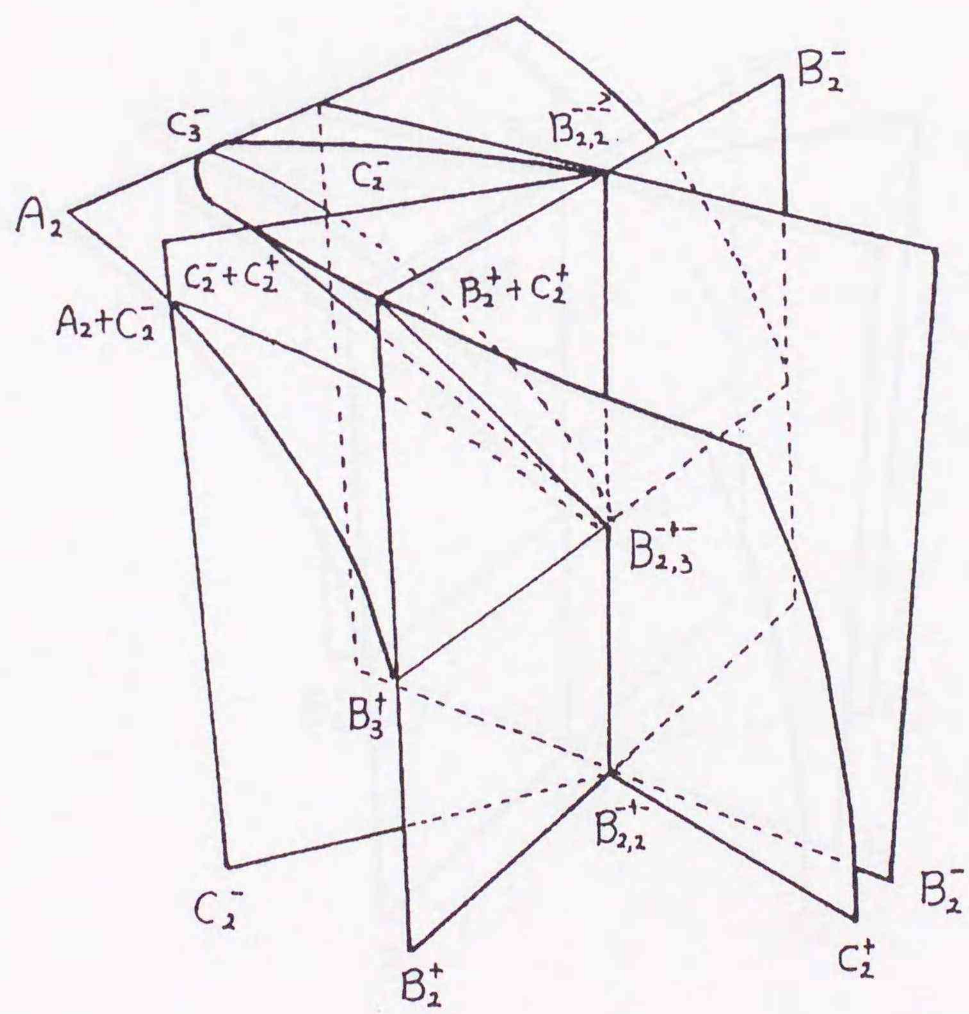
The sections of $B_{2,3}$ caustics II



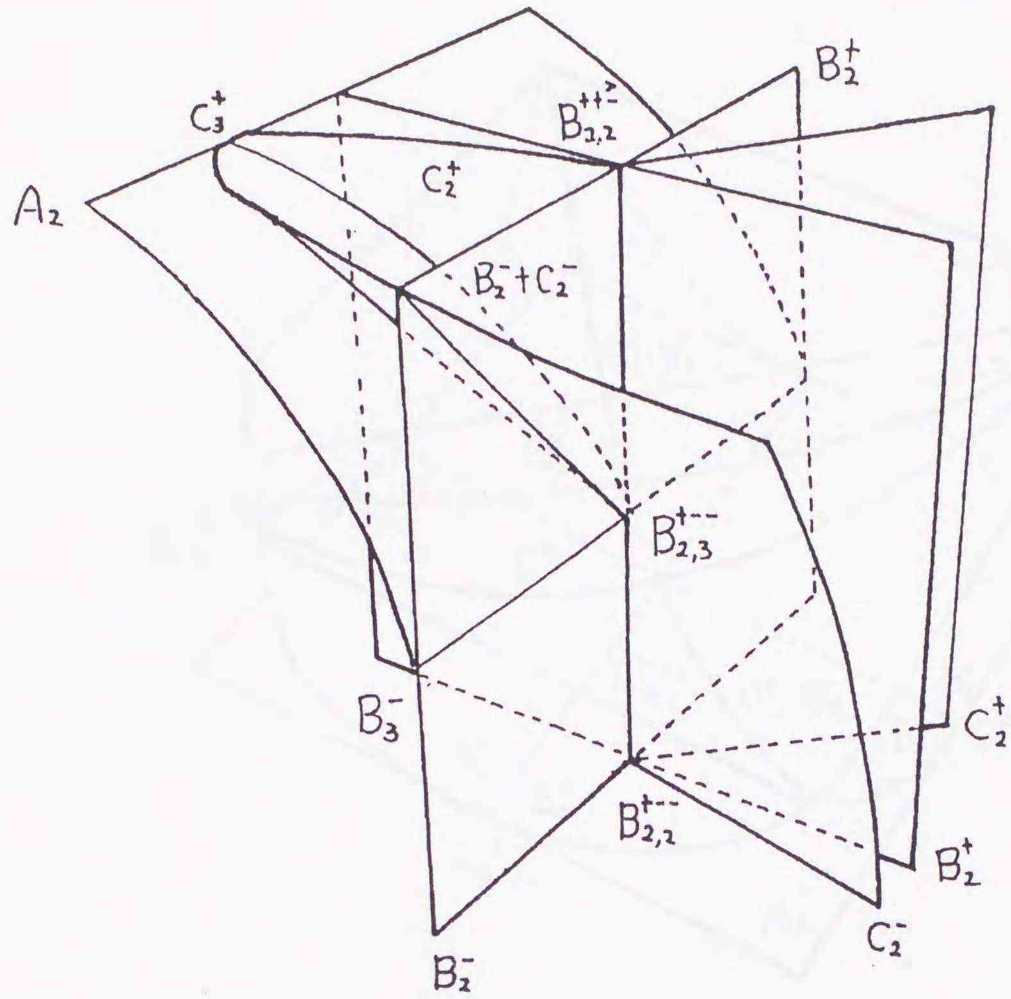


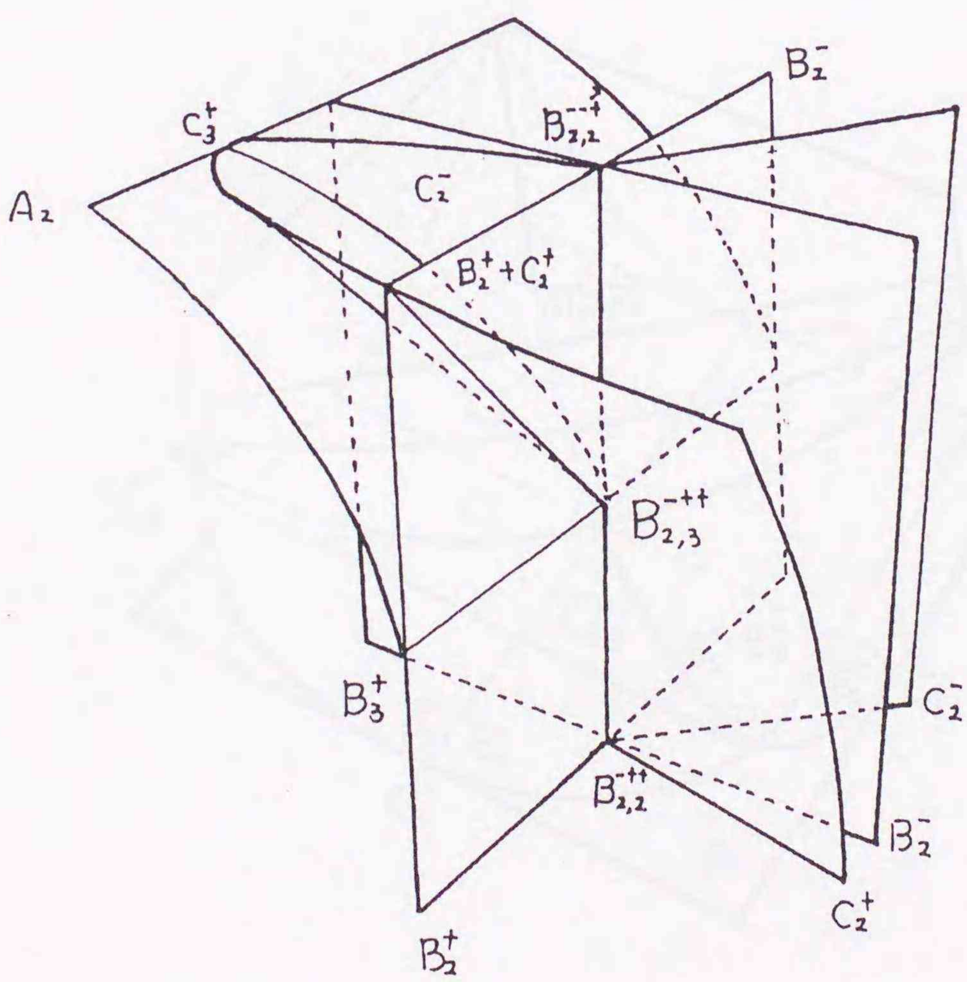
The sections of $B_{2,3}$ caustics III



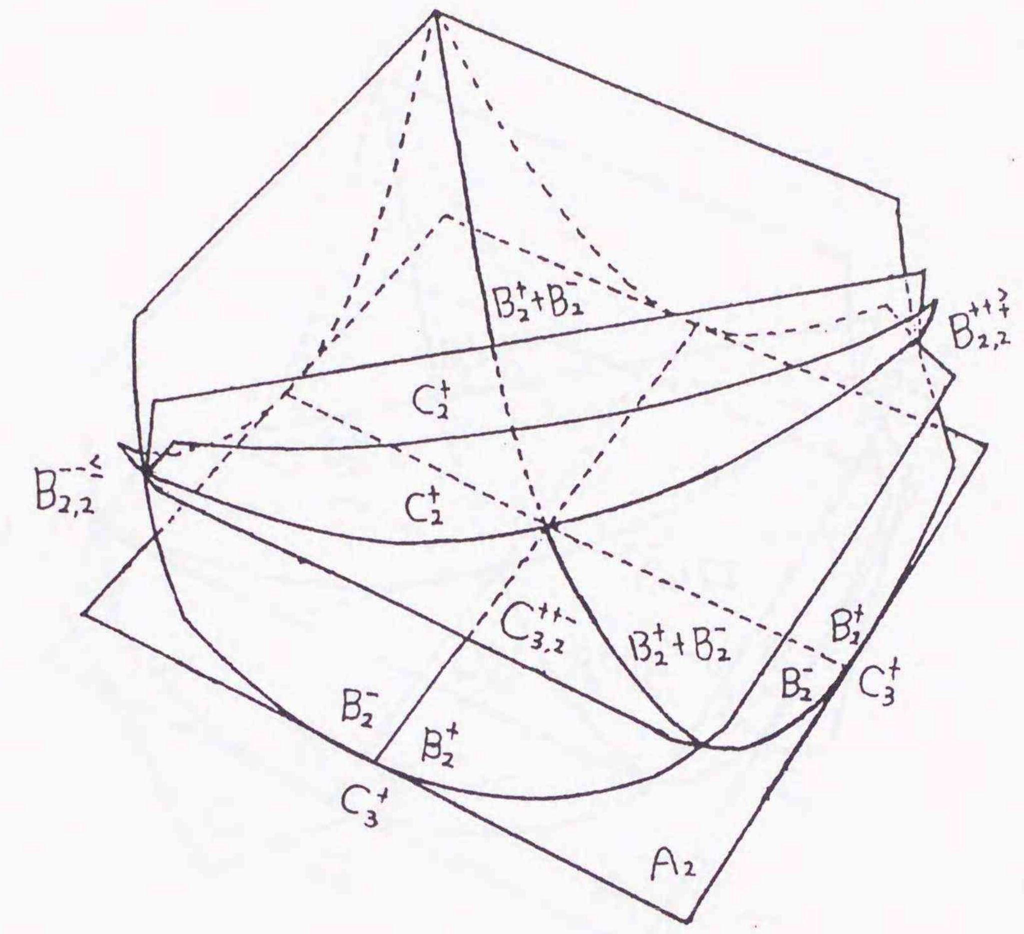


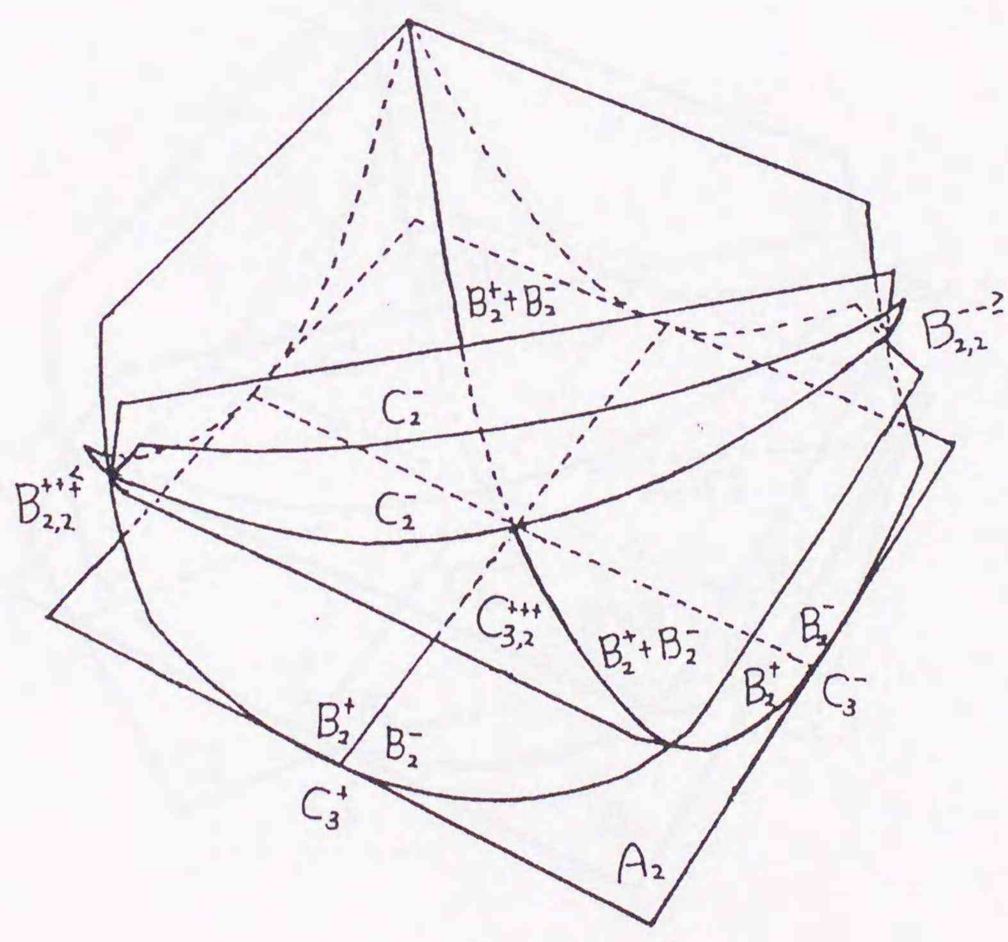
The sections of $B_{2,3}$ caustics IV



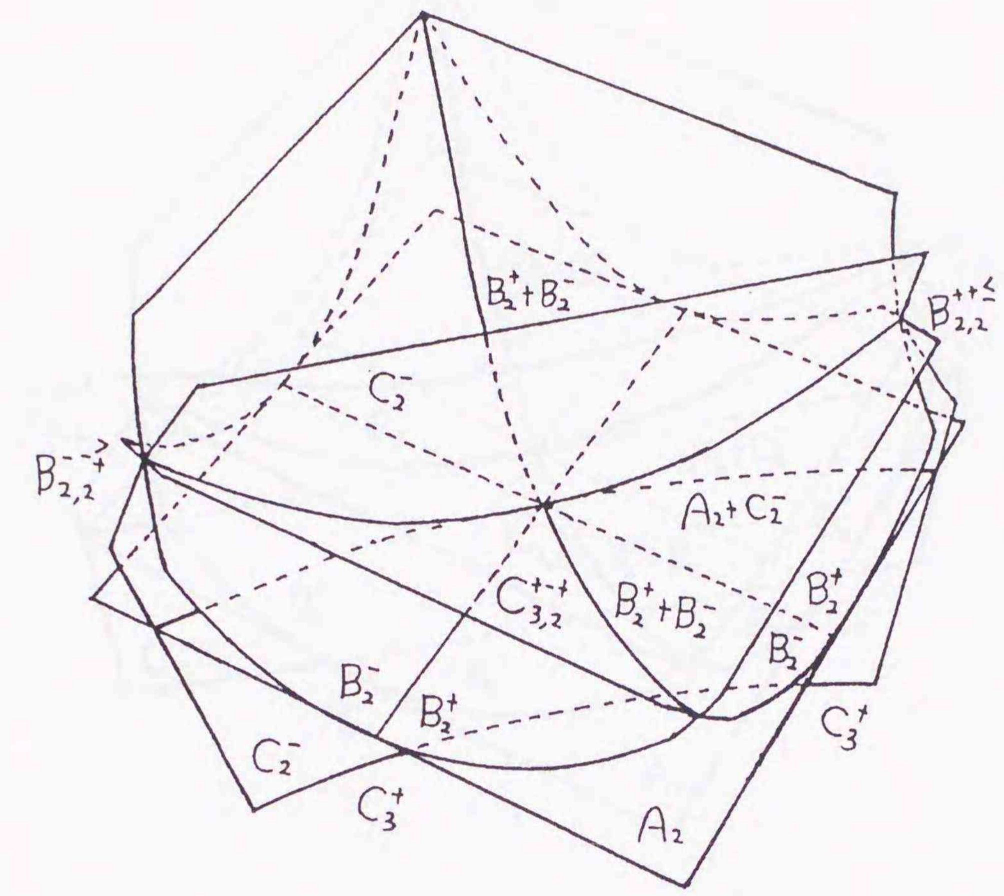


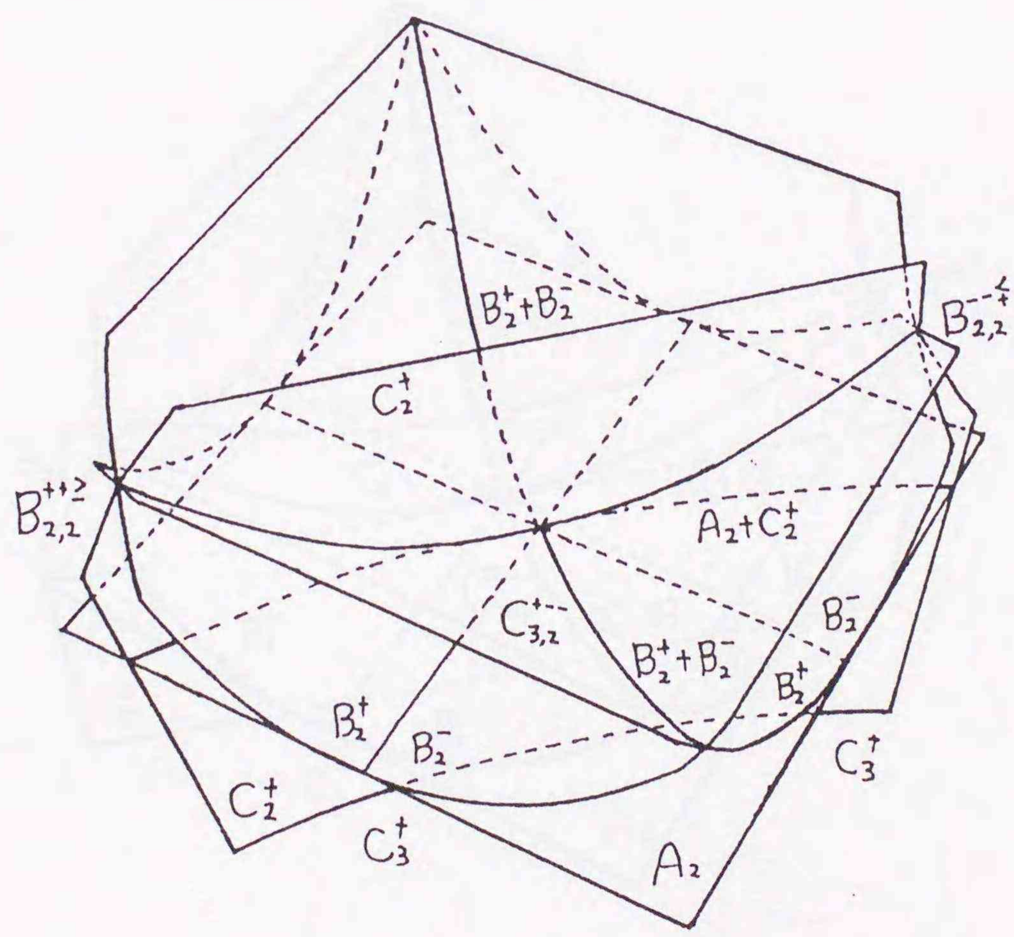
The sections of $C_{3,2}$ caustics I



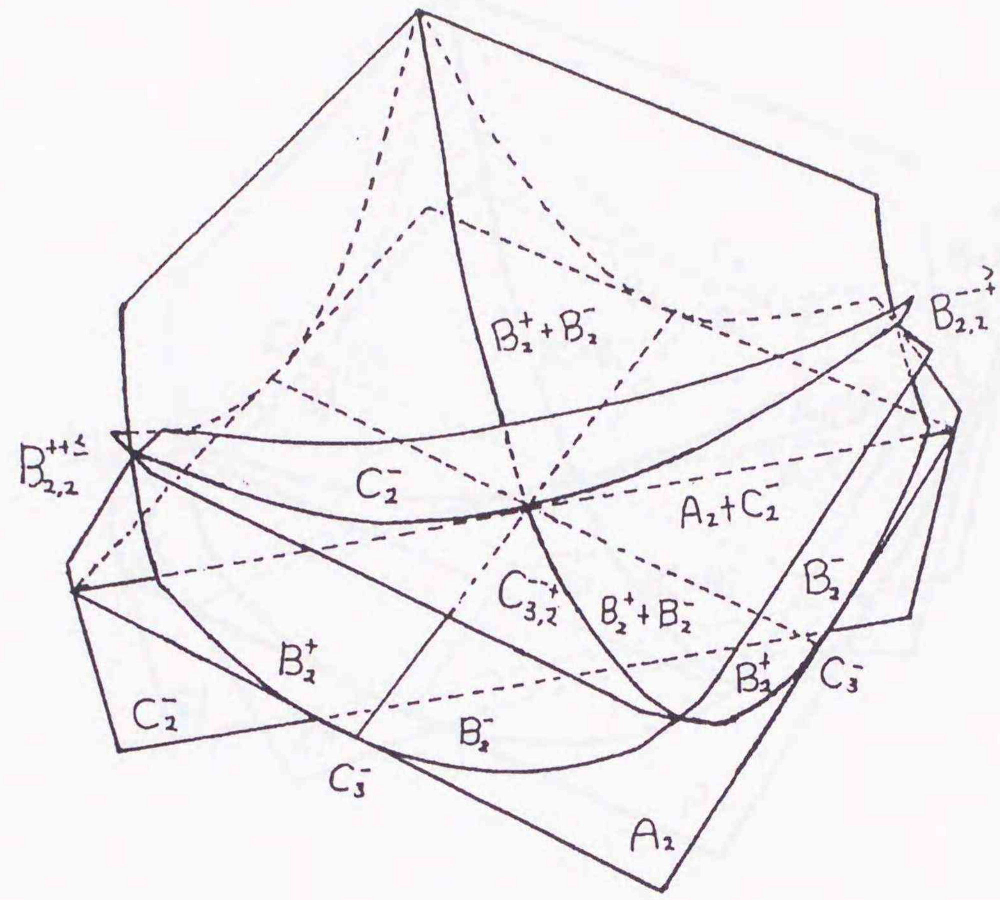


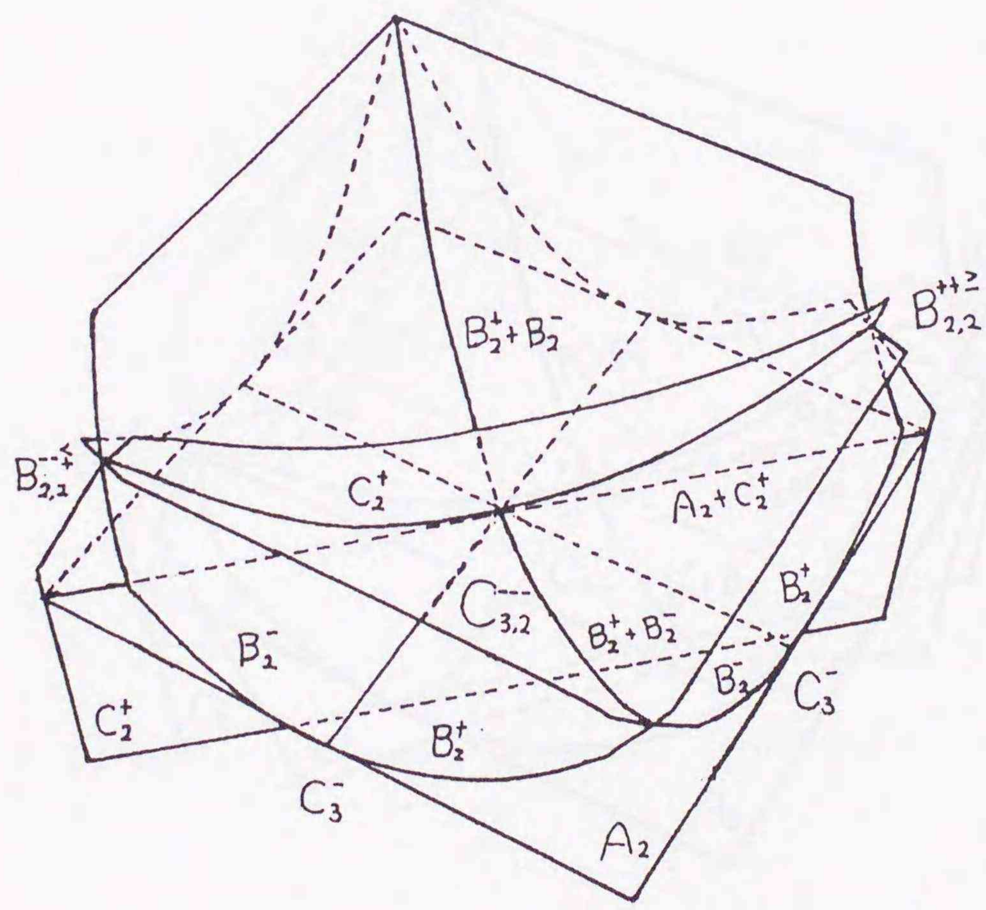
The sections of $C_{3,2}$ caustics II



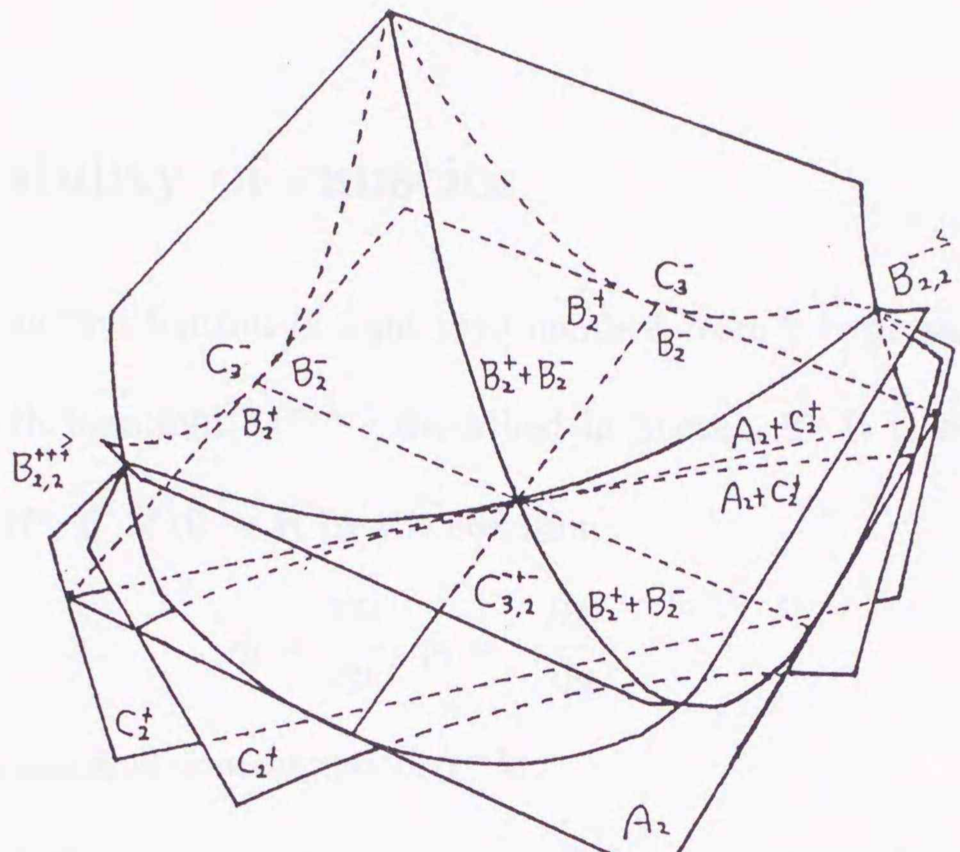


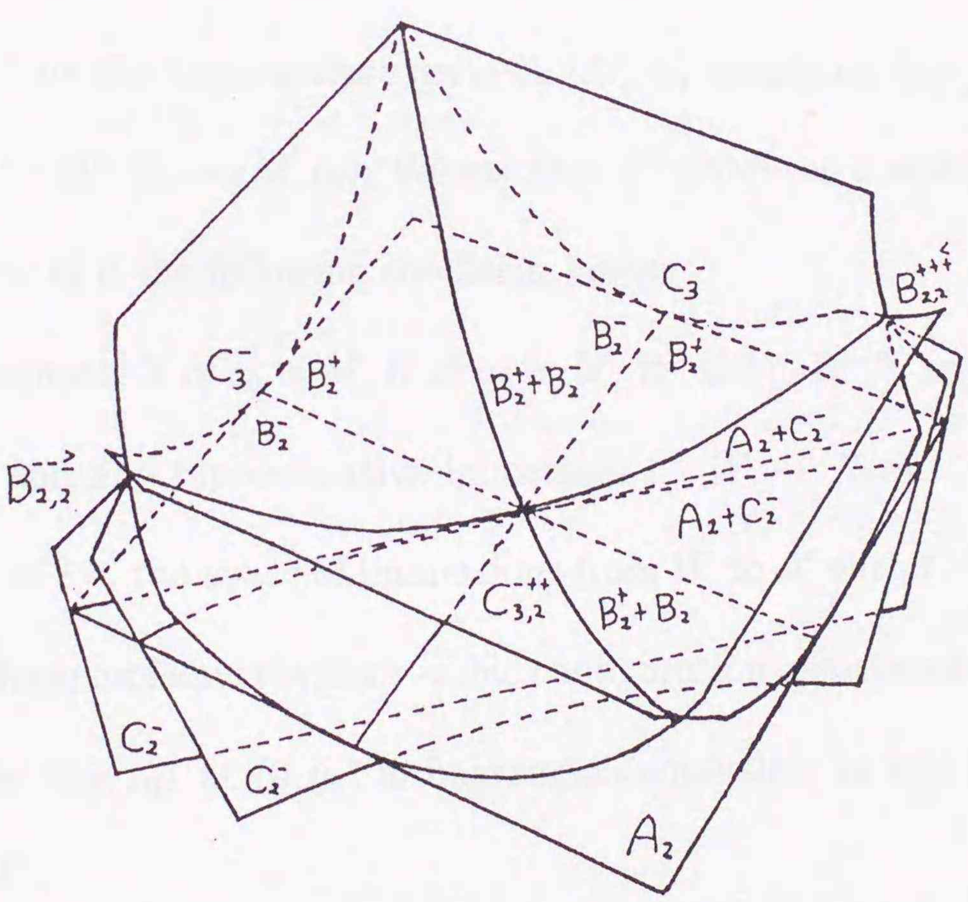
The sections of $C_{3,2}$ caustics III





The sections of $C_{3,2}$ caustics IV





Part II

Optical stability of caustics with r-corners

8 Optical stability of caustics

Recall the propagation mechanism of light rays incident from a hypersurface germ with an r-corner in a smooth manifold M^{r+k+1} described in Section 2. It is controlled by the Hamiltonian function $H : T^*M\backslash 0 \to \mathbb{R}$ by the equation:

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where (q,p) are local canonical coordinates of T^*M.

In this section we shall investigate the *optical stability* of caustic under perturbations of a light source hypersurface with an r-corner under a fixed Hamiltonian function.

Definition 8.1 Let V^0 be the hypersurface germ in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ defined by an immersion $\iota : (\mathbb{H}^r \times \mathbb{R}^k, 0) \to (M, q_0)$. We say that V^0 produces *a stable optical caustic with an r-corner at time t_0* if the following condition holds:

For any open neighborhood X of q_0 in M, U of u_0 in M, W of 0 in $\mathbb{R}^{r+k}$, any representative $\tilde{\tau} : X \times U \to \mathbb{R}$ of τ and any representative immersion $\tilde{\iota} : W \to X$ of ι, there exists an open neighborhood $N_{\tilde{\iota}}$ of $\tilde{\iota}$ in the space of immersions from W to X with C^∞-topology such that for every $\tilde{\kappa} \in N_{\tilde{\iota}}$ the symplectic regular r-cubic configuration associated the light source hypersurface defined by $\tilde{\kappa}|_{\mathbb{H}^r\times\mathbb{R}^k}$ at $(0, y_0)$ is Lagrangian equivalent to one associated with V^0 for some $(0, y_0) \in W$.

We remark that, by Theorem 3.2 (3), the condition defined by changing the part 'the symplectic regular r-cubic $\cdots$ for some $(0, y_0) \in W$' in Definition 8.1 to '$(\tilde{\tau} \circ (\tilde{\kappa} \times id_u) -$

$t_0)|_{\mathbb{H}^r \times \mathbb{R}^{k+m}}$ at $(0, y_0, u_0')$ is reticular R^+-equivalent to $\tau \circ (\iota \times id_u) - t_0$ for some $(0, y_0, u_0') \in W \times U$' is equivalent to the original.

Now we give the affirmative answer to the problem (1).

Theorem 8.2 *Let M be an $m(= r + k + 1)$-dimensional differentiable manifold, $H : T^*M \backslash 0 \to \mathbb{R}$ a positive and positively homogeneous Hamilton function, $q_0 \in M$, $\xi_0 \in E_{q_0}$, $t_0 \geq 0$ and τ the ray length function associated with the regular point (t_0, ξ_0) of exp_{q_0}. Let V^0 be the hypersurface germ in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ defined by an immersion $\iota : (\mathbb{H}^r \times \mathbb{R}^k, 0) \to (M, q_0)$. Then V^0 produces a stable optical caustic with an r-corner at time t_0 if and only if $F := \tau \circ (\iota \times id_u) - t_0$ is a reticular R^+-versal unfolding of $F|_{u=u_0}$.*

By the above remark, this theorem asserts that the stability of F with respect to perturbations of ι is sufficient to one of F as an m-dimensional unfolding. However generally these stabilities are not equivalent. Since the stability as an unfolding means the stability with respect to both of perturbations of the corresponding light source hypersurface and the Hamiltonian function.

Proof. ($\Leftarrow$) Let $\tilde{\iota} : W \to X$ be a representative immersion of ι and $\tilde{\tau} : X \times U \to \mathbb{R}$ be a representative of τ. By shrinking X and U if necessary, we may assume that $\tilde{\tau}|_{X \times u}$ is submersion for every $u \in U$. We denote $\mathrm{Imm}(W, X)$ the set of immersions from W to X and define the continuous map

$$\begin{array}{rccc} \Phi : & \mathrm{Imm}(W, X) & \longrightarrow & C^\infty(W \times U, \mathbb{R}) \\ & \tilde{\kappa} & \longmapsto & \tilde{\tau} \circ (\tilde{\kappa} \times id_u) - t_0 \end{array} .$$

Set $\tilde{F} = \Phi(\tilde{\iota})$. Since F is a reticular R^+-stable unfolding of f, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that, for every function $\tilde{G} \in N_{\tilde{F}}$, the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+m}}$ at $(0, y_0, u_0')$ and F are reticular R^+-equivalent for some $(0, y_0, u_0') \in W \times U$. Then $\Phi^{-1}(N_{\tilde{F}})$ is a neighborhood of $\tilde{\iota}$

for which the condition in definition 8.1 holds.

($\Rightarrow$) We suppose lemma 8.3. Let $\tilde{\iota}' : W' \to X$ be a representative immersion of ι and $\tilde{\tau} : X \times U \to \mathbb{R}$ be a representative of τ. Choose a relative compact neighborhood W of 0 in $\mathbb{R}^{r+k}$ such that $\overline{W} \subset W'$ and choose a neighborhood $N_{\tilde{\iota}}$ of $\tilde{\iota} := \tilde{\iota}'|_W$ for which the condition in definition 8.1 holds. We define

$$B_l = \{\tilde{\kappa} \in C^\infty(W', X) | j_1^l(\tilde{\tau} \circ (\tilde{\kappa} \times id_u) - t_0)|_{x=0} \text{ is transversal to } O_{rR}^l(j^l f(0))\}$$

for each $l \in \mathbb{N}$, Then B_l is a residual set in $C^\infty(W', X)$ by lemma 8.3. Since $C^\infty(W', X)$ is a Baire space, $B := \bigcap_{l \in \mathbb{N}} B_l$ is dense.

Set the open set $O = \{\tilde{\kappa} \in C^\infty(W', X) | \tilde{\kappa}|_{\overline{W}}$ is an immersion $\}$. Then the map $O \to \mathrm{Imm}(W, X)$ given by $\tilde{\kappa} \mapsto \tilde{\kappa}|_W$ is continuous. Therefore the inverse image $N_{\tilde{\iota}'}$ of $N_{\tilde{\iota}}$ by the above map is open neighborhood of $\tilde{\iota}'$.

Fix $\tilde{\kappa} \in N_{\tilde{\iota}'} \cap B$ sufficiently close to $\tilde{\iota}'$ such that $(\tilde{\tau} \circ (\tilde{\kappa} \times id_u) - t_0)|_{\mathbb{H}^r \times \mathbb{R}^{r+k}}$ at $(0, y_0, u_0')$ and F are reticular R^+-equivalent at $(0, y_0, u_0') \in W \times U$. Define $G \in \mathfrak{M}(r; k+m)$ by $G(x, y, u) := \tilde{\tau}(\tilde{\kappa}(x, y + y_0), u + u_0') - t_0$. Then G is reticular R^+-l-transversal unfolding of $g := G|_{u=0}$ for all $l \in \mathbb{N}$. Hence G is a reticular R^+-versal unfolding of g. Therefore F is also a reticular R^+-versal unfolding of f. ■

The following completes the proof the Theorem 8.2.

Lemma 8.3 *Let W, X and U be neighborhoods of 0 in $\mathbb{R}^{r+k}, \mathbb{R}^m$ and $\mathbb{R}^n$ respectively and we denote their coordinates $(x_1, \cdots, x_r, y_1, \cdots, y_k), (q_1, \cdots, q_m)$ and $(u_1, \cdots, u_n)$ respectively. Let $H : X \times U \to \mathbb{R}$ be a smooth map such that $H|_{X \times u}$ is a submersion for all $u \in U$ and A be a submanifold of $J^l(r+k, 1)$. Then the set*

$$B = \{f \in C^\infty(W, X) | j_1^l H \circ (f \times id_u)|_{x=0} \text{ is transversal to } A\}$$

is residual.

Proof. Let $V = W \cap \{x = 0\}$. Then the map

$$\gamma : C^\infty(W, X) \to C^\infty(V \times U, J^l(r+k,1)) \ (\ f \mapsto j_1^l(H \circ (g \times id_u))|_{x=0} \ )$$

is continuous. If $K \subset A$ is a compact subset, then $C = \{F \in C^\infty(V \times U, J^l(r+k,1))|F$ is transversal to A on $K\}$ is open. Therefore $B = \gamma^{-1}(C)$ is open.

Choose relatively compact open covering $\{W_i\}_{i \in \mathbb{N}}$ and $\{W'_i\}_{i \in \mathbb{N}}$ of W such that $\overline{W_i} \subset W'_i$ for $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ set

$$B_i = \{f \in C^\infty(W, X)|j_1^l H \circ (f \times id_u)|_{x=0} \text{ is transversal to } A \text{ on } \overline{W_i} \cap \{x = 0\} \ \}.$$

Since $B = \bigcap_{i \in \mathbb{N}} B_i$ and each B_i is open by an analogous proof of the above, it is enough to prove that every B_i is dense in order to complete the proof.

The proof is analogous to that of ordinary transversal lemma. Fix $i \in \mathbb{N}$ and $f \in C^\infty(W, X)$. Let P be the set of all n-tuples of polynomial maps of degree $\leq l$ on x, y. Choose a smooth function $\rho : W \to [0,1]$ such that $\rho = 1$ on $\overline{W_i}$ and $\rho = 0$ on $W - W'_i$. Put $P' = \{\alpha \in P|(f + \rho \cdot \alpha)(W) \subset X\}$. Since $P' = \{\alpha \in P|(f + \rho \cdot \alpha)(W'_i) \subset X\}$ and $\overline{W}_i$ is compact, P' is a neighborhood of 0. We define the following maps for $\alpha \in P'$:

$$\iota_\alpha : V \times U \longrightarrow W' \times U \times P' \ (\ (y,u) \mapsto (y,u,\alpha) \ )$$

$$\mu : V \times U \times P' \longrightarrow J^l(r+k,1) \ (\ (y,u,\alpha) \mapsto j_1^l(H \circ ((f + \rho \cdot \alpha) \times id_u))(0,y,u)) \ ).$$

Let $\alpha \in P'$. Then $(f + \rho \cdot \alpha) \in B_i$ if and only if $j_1^l(H \circ ((f + \rho \cdot \alpha) \times id_u))|_{x=0}$ is transversal to A on $\overline{W_i} \cap \{x = 0\}$, and this holds if and only if $\mu \circ \iota_\alpha$ is transversal to A on $\overline{W_i} \cap \{x = 0\}$, Since $\rho = 1$ on W_i and $H|_{X \times u}$ is a submersion, μ is submersion and hence this holds if ι_α is transversal to $A' := \mu^{-1}(A)$. Hence ι_α is transversal to A at $(0,y,u,\alpha) \in V \times U \times P'$ if and only if $(0, y, u, \alpha) \notin A'$ or the projection $\pi : A' \to P'$ is regular at $(0, y, u, \alpha)$.

Since the set of critical values of π has measure 0 in P' by the Sard-Brown theorem, there exists α arbitrarily near 0 such that $j_1^l(H \circ ((f + \rho \cdot \alpha) \times id_u))|_{x=0}$ is transversal to A on $\overline{W}_i \cap \{x = 0\}$. This means that there exists $g \in C^\infty(W, X)$ arbitrarily close f such that $j_1^l(H \circ (g \times id_u))|_{x=0}$ is transversal to A on $\overline{W}_i \cap \{x = 0\}$. Hence B_i is dense. ∎

9 Optical versality of caustics

In this section we shall investigate our second problem. Recall that $\tau : (M \times M, (q_0, u_0)) \to (\mathbb{R}, t_0)$ denotes the ray length function. We say that a function germ $f \in \mathfrak{M}(r; k)^2$ occur as an *organizer* of an optical versal caustic at (t_0, ξ_0) if there exists the hypersurface germ V^f in (M, q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ defined by an immersion $\iota_f : (\mathbb{H}^r \times \mathbb{R}^k, 0) \to (M, q_0)$ such that $\tau \circ (\iota_f \times id_u) - t_0$ is a reticular R^+-versal unfolding of f.

Lemma 9.1 *Let a function germ $f \in \mathfrak{M}(r; k)^2$ occur as an organizer of a optical versal caustic at (t_0, ξ_0). If a function germ $g \in \mathfrak{M}(r; k)^2$ is reticular R-equivalent to f, then g also does occur as an organizer of a optical versal caustic at (t_0, ξ_0).*

Proof. By the hypothesis, there exists a hypersurface germ V^f and an immersion ι_f to which above condition holds. Since f is reticular R-equivalent to g, there exists $\phi \in \mathcal{B}(r, k)$ such that $g = f \circ \phi$. Consider the coordinate change $(x, y) \mapsto \phi^{-1}(x, y)$ on V^f. Let V^g be the hypersurface germ of (M, q_0) parameterized by ι_g:

$$\begin{array}{ccccc} & \iota_g & V^g & g & \\ & \swarrow & & \searrow & \\ M & & \downarrow \phi & & \mathbb{R} \\ & \nwarrow & & \nearrow & \\ & \iota_f & V^f & f & \end{array} .$$

By the above diagram we have

$$G(x,y,u) := \tau(\iota_g(x,y),u) - t_0 = \tau(\iota_f(\phi(x,y)),u) - t_0 = F(\phi(x,y),u).$$

Since F is reticular R^+-versal unfolding of f, G is reticular R^+-versal unfolding of $G|_{u=0} = f \circ \phi = g$. ■.

Definition 9.2 [8, 3.2] Let $u \in M$ and $\eta \in E_u$. Then we say that the Hamiltonian function *H has rank s at u in direction η* if the following condition holds: Let L_η be the line in T_u^*M spanned by η. If we introduce affine coordinates $v_1, \cdots, v_m$ in T_u^*M such that $T_\eta E_u$ is given by $v_m = 1$, the v_m-axis is L_η, and if we represent E locally at η as $v_m = 1 + h(v_1, \cdots, v_{m-1})$, then the Hessian of h at η has rank s.

Theorem 9.3 *Let M be an $m(= r+k+1)$-dimensional differentiable manifold, $H : T^*M \backslash 0 \to \mathbb{R}$ a positive and positively homogeneous Hamilton function, $q_0 \in M$, $\xi_0 \in E_{q_0}$ and $t_0 \geq 0$. Assume that (t_0, ξ_0) is a regular point of exp_{q_0}, put $u_0 = exp_{q_0}(t_0, \xi_0)$ and suppose $\eta_0 \in E_{u_0}$ be the image of ξ_0 under the local flow of H at time t_0. Then each of following conditions (1), (2) is sufficient for $f \in \mathfrak{M}(r;k)^2$ to occur as an organizer of a optical versal caustic at (t_0, ξ_0):*

(1) The reticular R-codimension $f \leq m$.

(2) The reticular R-codimension $f = m+1$, corank $f \geq 1$ and the rank s of H at u_0 in direction $\eta_0 \geq 1$.

Proof. Choose coordinates $(u_1, \cdots, u_m)$ of M at u_0 such that, with respect to the corresponding fiber coordinates $(v_1, \cdots, v_m)$ in $T_{u_0}^*M$, H satisfies the conditions in definition 9.2. By a linear coordinate change of $(u_1, \cdots, u_{m-1})$, we may assume that h has the form $h(v_1, \cdots, v_m) = \sum_{i=1}^{r} \varepsilon_i v_i^2 + \sum_{j=1}^{k} \delta_j v_{r+j}^2 + a$, where $\varepsilon_i, \delta_j = 0$ or ± 1, $a \in \mathfrak{M}(r;k)^3$ and in the case (2) $\delta_1 \neq 0$.

Let $f \in \mathfrak{M}(r;k)^2$ satisfy the condition (1) or (2). By Splitting lemma 7.2, there exists a function germ $f_0 \in \mathfrak{M}(r;l)^2$ such that f is reticular R-equivalent to $f_0(x_1,\cdots,x_r,y_1,\cdots,y_l)\pm y_{l+1}^2\pm\cdots\pm y_k^2$ and $f_0|_{x=0} \in \mathfrak{M}(0;l)^3$. We may assume that $f = f_0 \pm y_{l+1}^2 \pm \cdots \pm y_k^2$ by lemma 9.1. Then we have $\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \mathcal{E}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$.

First, We prove that $x_1,\cdots,x_r,y_1,\cdots,y_l$ are linearly independent in $\mathcal{E}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$ over $\mathbb{R}$. Let $\alpha_1x_1+\cdots+\alpha_rx_r+\beta_1y_1+\cdots+\beta_ly_l = 0 \in \mathcal{E}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$ for $\alpha_1,\cdots,\alpha_r,\beta_1,\cdots,\beta_l \in \mathbb{R}$. Since $\langle \beta, y\rangle = 0$ in $\mathcal{E}(0;l)/\langle \frac{\partial f_0}{\partial y}|_{x=0}\rangle$ and $f_0|_{x=0} \in m^3(0;l)$, we have $\beta = 0$. Suppose that $\alpha_1 \neq 0$. Then there exist $\gamma_0,\cdots,\gamma_l \in \mathfrak{M}(1;l)$ such that

$$x_1 + \gamma_0(x_1,y)x_1\frac{\partial f_0}{\partial x_1}(x_1,0,y) + \gamma_1(x_1,y)\frac{\partial f_0}{\partial y_1}(x_1,0,y) + \gamma_l(x_1,y)\frac{\partial f_0}{\partial y_l}(x_1,0,y) = 0.$$

Therefore we have $\gamma_i(0) \neq 0$ for some $i \geq 1$, for $x_1\frac{\partial f_0}{\partial x_1}(x_1,0,y) \in \mathfrak{M}(1;l)^2$. We may assume that $i = 1$. Then this means that $\frac{\partial f_0}{\partial y_1}|_{x=0} \in \langle \frac{\partial f_0}{\partial y_2}|_{x=0},\cdots,\frac{\partial f_0}{\partial y_l}|_{x=0}\rangle$ and contradicts that $\dim_{\mathbb{R}} \mathfrak{M}(0;l)/\langle \frac{\partial f_0}{\partial y}|_{x=0}\rangle < \infty$.

Consider the case (2). The vector space

$$\mathfrak{M}(r;l)/(\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle + \langle x,y\rangle_{\mathbb{R}} + \mathfrak{M}(r;l)^3)$$

must have a positive dimension because if not we have reticular R-codimension $f \leq m$. Therefore we may assume by lemma 9.1 and lemma 9.4 below that

$$b_{k+1} := \sum_{i=1}^{r}\varepsilon_i x_i^2 + \sum_{j=1}^{l}\delta_j y_j^2 \neq 0 \text{ in } \mathfrak{M}(r;l)/(\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle + \langle x,y\rangle_{\mathbb{R}}).$$

Now choose $b_{l+1},\cdots,b_t \in \mathfrak{M}(r;l)^2$ and $b_{t+1},\cdots,b_k \in \mathfrak{M}(r;l)^3$ such that $x_1,\cdots,x_r,y_1,\cdots,y_l$, $b_{l+1},\cdots,b_t,b_{k+1}$ is a basis of $\mathfrak{M}(r;l)/(\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle + \mathfrak{M}(r;l)^3)$ and $x_1,\cdots,x_r,y_1,\cdots,y_l,b_{l+1},\cdots,$ b_{k+1} is a basis of $\mathfrak{M}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$.

In the case (1), choose $b_{l+1}, \cdots, b_k$ such that $x_1, \cdots, x_r, y_1, \cdots, y_l, b_{l+1}, \cdots, b_k$ generate $\mathfrak{M}(r;l)/\langle x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle$ over $\mathbb{R}$.

Now define $\phi \in \mathcal{B}(r;k)$ by

$$\phi(x_1, \cdots, x_r, y_1, \cdots, y_k) = (x_1, \cdots, x_r, y_1, \cdots, y_l, y_{l+1} + b_{l+1}, \cdots, y_k + b_k).$$

Since $exp^-_{u_0}$ is invertible, the map $\iota_f : (\mathbb{H}^r \times \mathbb{R}^k, 0) \to (M, q_0)$ given by $\iota_f(x,y) = exp^-_{u_0}(f(x,y) + t_0, (\phi(x,y), 1 + h \circ \phi(x,y)))$ defines a hypersurface germ V_f in (M, q_0). Then we have

$$F(x,y,0) := \tau(\iota_f(x,y), u_0) - t_0 = (f(x,y) + t_0) - t_0 = f(x,y),$$

$$\xi(q_0, u_0)|_{T_{q_0}V^f} = -d_{(x,y)}(\tau \circ \iota_f)((0,0), u_0) = -d_{(x,y)} f(0,0) = 0,$$

$$\begin{aligned} d_u F(x,y,0) &= d_u \tau(\iota_f(x,y), u_0) = \eta(\iota_f(x,y), u_0) = (\phi(x,y), 1 + h \circ \phi(x,y)) \\ &= (x_1, \cdots, x_r, y_1, \cdots, y_l, y_{l+1} + b_{l+1}, \cdots, y_k + b_k, b_{k+1} + a + 1), \end{aligned}$$

where $a \in \mathfrak{M}(r;k)^3$.

In the case (1), we have

$$\begin{aligned} \mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle &= \langle 1, x_1, \cdots, x_r, y_1, \cdots, y_l, b_{l+1}, \cdots, b_k\rangle_{\mathbb{R}} \\ &= \langle 1, x_1, \cdots, x_r, y_1, \cdots, y_l, y_{l+1} + b_{l+1}, \cdots, y_k + b_k\rangle_{\mathbb{R}}. \end{aligned}$$

Hence the proof of the case (1) is completed.

In the case (2). since $1, x_1, \cdots, x_r, y_1, \cdots, y_l, b_{l+1}, \cdots, b_k, b_{k+1}$ is a basis of $\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$, there exist $\alpha_0, \alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_{k+1} \in \mathbb{R}$ such that

$$a \equiv \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_r x_r + \beta_1 y_1 + \cdots + \beta_l y_l + \beta_{l+1} b_{l+1} + \cdots + \beta_{k+1} b_{k+1} \mod \mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle.$$

Hence

$$0 \equiv \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_r x_r + \beta_1 y_1 + \cdots + \beta_l y_l + \beta_{l+1} b_{l+1} + \cdots + \beta_t b_t + \beta_{k+1} b_{k+1}$$

$$\mod \mathcal{E}(r;k)/(\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle + \mathfrak{M}(r;k)^3).$$

Since $x_1, \cdots, x_r, y_1, \cdots, y_l, b_{l+1}, \cdots, b_t, b_{k+1}$ is a basis of $\mathcal{E}(r;k)/(\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle + \mathfrak{M}(r;k)^3)$,

$$\alpha_0 = \alpha_1 = \cdots = \alpha_r = \beta_1 = \cdots = \beta_t = \beta_{k+1} = 0.$$

Hence $a \in \langle b_{t+1}, \cdots, b_k\rangle_{\mathbb{R}}$ in $\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$. This means that

$$\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \langle 1, x_1, \cdots, x_r, y_1, \cdots, y_l, b_{l+1}, \cdots, b_k, b_{k+1} + a\rangle_{\mathbb{R}}.$$

Therefore

$$\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \langle 1, x_1, \cdots, x_r, y_1, \cdots, y_l, y_{l+1} + b_{l+1}, \cdots, y_k + b_k, b_{k+1} + a + 1\rangle_{\mathbb{R}}.$$

Hence the proof for the case (2) is completed, supposing Lemma 9.4. ■

It remains to show:

Lemma 9.4 *Let* $A = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_r, \delta_1, \cdots, \delta_l) \in M(r+l, r+l; \mathbb{R})$, $\varepsilon_1, \cdots, \varepsilon_r, \delta_1, \cdots, \delta_l$ *are* 0 *or* ± 1 *and* $\delta_1 \neq 0$. *Then the set* $\mathcal{F}$ *of matrices linearly generated by* $D\phi(0)^t A D\phi(0)$ *for all* $\phi \in \mathcal{B}(r;l)$ *is equal to that of symmetric matrices in* $M(r+l, r+l; \mathbb{R})$.

Proof. We denote $\varepsilon = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_r)$ and $\delta = \mathrm{diag}(\delta_1, \cdots, \delta_l)$. At first we remark that

$$\{B\delta C \in M(s,t;\mathbb{R}) | B \in M(s,l;\mathbb{R}), C \in M(l,t;\mathbb{R})\} = M(s,t;\mathbb{R})$$

for any integer s and t. Let $\phi \in \mathcal{B}(r;l)$ be given. We denote $\phi(x,y) = (x_1 a_1(x,y), \cdots, x_r a_r(x,y), b_1(x,y), \cdots, b_l(x,y))$. Then we have by immediately calculation that

$$D\phi(0)^t A D\phi(0) = \begin{pmatrix} \mathrm{diag}(a_1^2(0)\varepsilon_1, \cdots, a_r^2(0)\varepsilon_r) + 2(\frac{\partial b}{\partial x}(0))^t \delta (\frac{\partial b}{\partial x}(0)) & 2(\frac{\partial b}{\partial x}(0))^t \delta (\frac{\partial b}{\partial y}(0)) \\ 2(\frac{\partial b}{\partial y}(0))^t \delta (\frac{\partial b}{\partial x}(0)) & 2(\frac{\partial b}{\partial y}(0))^t \delta (\frac{\partial b}{\partial y}(0)) \end{pmatrix}.$$

By considering the case $\frac{\partial b}{\partial x}(0) = 0$ we have

$$\{\begin{pmatrix} \mathrm{diag}(a_1^2(0)\varepsilon_1, \cdots, a_r^2(0)\varepsilon_r) & 0 \\ 0 & B \end{pmatrix} \in M(r+l, r+l; \mathbb{R}) | B \in M(l, l; \mathbb{R})\} \subset \mathcal{F}.$$

This means that

$$\{\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \in M(r+l, r+l; \mathbb{R}) | B \in M(l, l; \mathbb{R})\} \subset \mathcal{F}.$$

Let $\phi, \phi' \in \mathcal{B}(r; l)$ satisfy the conditions that $a(0) = a'(0), \frac{\partial b}{\partial x}(0) = \frac{\partial b'}{\partial x}(0)$, where $\phi(x, y) = (x_1 a_1(x, y), \cdots, x_r a_r(x, y), b_1(x, y), \cdots, b_l(x, y))$ and $\phi'(x, y) = (x_1 a_1'(x, y), \cdots, x_r a_r'(x, y), b_1'(x, y), \cdots, b_l'(x, y))$. Then

$$\begin{aligned} & D\phi(0)^t A D\phi(0) - D\phi'(0)^t A D\phi'(0) \\ = & \begin{pmatrix} 0 & 2(\frac{\partial b}{\partial x}(0))^t \delta(\frac{\partial b}{\partial y}(0) - \frac{\partial b'}{\partial y}(0)) \\ 2(\frac{\partial b}{\partial y}(0) - \frac{\partial b'}{\partial y}(0))^t \delta(\frac{\partial b}{\partial x}(0)) & * \end{pmatrix}. \end{aligned}$$

Therefore

$$\{\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \in M(r+l, r+l; \mathbb{R}) | B \in M(r, l; \mathbb{R})\} \subset \mathcal{F}.$$

Similarly we have

$$\{\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M(r+l, r+l; \mathbb{R}) | B \in M(r, r; \mathbb{R}),\ B^t = B\} \subset \mathcal{F}.$$

■

Part III
Reticular Legendrian Singularities

10 Preliminaries

Here we shall define several notations and recall basic facts. Let $\mathbb{H}^r = \{(x_1, \cdots, x_r) \in \mathbb{R}^r | x_1 \geq 0, \cdots, x_r \geq 0\}$ be an r-corner. Let $\mathcal{E}(r; l)$ be the ring of smooth function germ at 0 on $\mathbb{H}^r \times \mathbb{R}^l$ for $r, l \in \mathbb{N}$ and $m(r;l) = \{f \in \mathcal{E}(r,l) | \ f(0) = 0\}$ be the maximal ideal of $\mathcal{E}(r;l)$. Let $\mathcal{B}(r;l)$ the set of diffeomorphism germs on $(\mathbb{H}^r \times \mathbb{R}^l, 0)$ preserving $\mathbb{H}^r \cap \{x_\sigma = 0\} \times \mathbb{R}^l$ for all $\sigma \subset I_r = \{1, \cdots, r\}$.

A function germ $\bar{F}(x_1, \cdots, x_r, y_1, \cdots, y_k, \lambda_1, \cdots, \lambda_{n+1}) \in \mathfrak{M}(r; k+n+1)$ is called *C-non-degenerate* if $\frac{\partial \bar{F}}{\partial x}(0) = 0$, $\frac{\partial \bar{F}}{\partial y}(0) = 0$ and

$$x_1, \cdots, x_r, \bar{F}, \frac{\partial \bar{F}}{\partial x_1}, \cdots, \frac{\partial \bar{F}}{\partial x_r}, \frac{\partial \bar{F}}{\partial y_1}, \cdots, \frac{\partial \bar{F}}{\partial y_k}$$

are independent on $(\mathbb{H}^r \times \mathbb{R}^{k+n+1}, 0)$, that is

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial \bar{F}}{\partial y} & \dfrac{\partial \bar{F}}{\partial \lambda} \\ \dfrac{\partial^2 \bar{F}}{\partial x \partial y} & \dfrac{\partial^2 \bar{F}}{\partial x \partial \lambda} \\ \dfrac{\partial^2 \bar{F}}{\partial y \partial y} & \dfrac{\partial^2 \bar{F}}{\partial y \partial \lambda} \end{pmatrix}_0 = r + k + 1.$$

We remark that $\bar{F}(x, y, \lambda) \in \mathfrak{M}(r; k+n+1)$ is C-non-degenerate only if $r \leq n$.

Let $\bar{\pi} : PT^*\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the projective cotangent bundle equipped with the contact structure defined in [1, p.310]. By the trivialization

$$\begin{array}{ccc} PT^*\mathbb{R}^{n+1} & \cong & \mathbb{R}^{n+1} \times P(\mathbb{R}^{n+1}) \\ [\xi_1 d\lambda_1|_\lambda + \cdots + \xi_{n+1} d\lambda_{n+1}|_\lambda] & & ((\lambda_1, \cdots, \lambda_{n+1}), [\xi_1; \cdots; \xi_{n+1}]), \end{array}$$

we call $(\lambda, [\xi])$ *a homogeneous coordinate*, where λ is coordinates of the base space of $\bar{\pi}$.

Let $\tilde{\pi} : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^{n+1}((q, z; p) \mapsto (q, z))$ be the canonical Legendrian bundle equipped with the contact structure defined by the canonical 1-form $\alpha = dz - pdq$, where $(q_1, \cdots, q_n, z; p_1, \cdots, p_n)$ are canonical coordinates of $J^1(\mathbb{R}^n, \mathbb{R})$.

We fix $[\xi^0] \in PT^*\mathbb{R}^{n+1}$. Choose coordinates $(q_1, \cdots, q_n, z)$ of $\mathbb{R}^{n+1}$ (the base space of $\bar{\pi}$ and $\tilde{\pi}$) such that $[\xi^0] = (0, [0; \cdots; 0; 1])$. Set the affine chart of $PT^*\mathbb{R}^{n+1}$:$U_z = \{((q_1, \cdots, q_n, z), [\xi_1; \cdots; \xi_n; \eta]) | \eta \neq 0\}$. Then

$$\psi_z : U_z \xrightarrow{\sim} J^1(\mathbb{R}^n, \mathbb{R})((q, z), [\xi; \eta]) \mapsto (q, z, -\frac{\xi_1}{\eta}, \cdots, -\frac{\xi_n}{\eta}))$$

is a Legendrian equivalence. We define

$$p_1 : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n \ (\ (q,z) \mapsto q \ ),$$

$$\tilde{p_1} : J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow T^*\mathbb{R}^n \ (\ (q,z,p) \mapsto (q,p) \ ).$$

Then the following diagram is commutative:

$$\begin{array}{ccccc} U_z & \xrightarrow{\psi_z} & J^1(\mathbb{R}^n,\mathbb{R}) & \xrightarrow{\tilde{p_1}} & T^*\mathbb{R}^n \\ \bar{\pi}\downarrow & & \tilde{\pi}\downarrow & & \downarrow\pi \\ \mathbb{R}^{n+1} & \xrightarrow{id} & \mathbb{R}^{n+1} & \xrightarrow{p_1} & \mathbb{R}^n. \end{array}$$

We say that function germs $F(x,y,u), G(x,y,u) \in \mathfrak{M}(r;k+l)$, where $x \in \mathbb{H}^r$, $y \in \mathbb{R}^k$ and $u \in \mathbb{R}^l$, are *reticular K-equivalent* (as l-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r;k+l)$ and a unit $a \in \mathcal{E}(r;k+l)$ satisfying the following:

(1) $\Phi = (\phi, \psi)$, where $\phi : (\mathbb{H}^r \times \mathbb{R}^{k+l}, 0) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$ and $\psi : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$.

(2) $G(x,y,u) = a(x,y,u) \cdot F(\phi(x,y,u), \psi(u))$ for $(x,y,u) \in (\mathbb{H}^r \times \mathbb{R}^{k+l}, 0)$.

Lemma 10.1 *Let $\bar{F}(x,y,q,z) \in \mathfrak{M}(r;k+n+1)$ be a C-non-degenerate function germ. Then $\bar{F}$ is reticular K-equivalent to $-z+F(x,y,q)$, where $F \in \mathfrak{M}^2(r;k+n)$ is S-non-degenerate.*

Proof. By taking some coordinate change of (q,z), we may assume that $\frac{\partial \bar{F}}{\partial q}(0) = 0, \frac{\partial \bar{F}}{\partial z}(0) \neq 0$. By implicit function theorem, there exists $F \in \mathfrak{M}(r;k+n)$ such that $\bar{F}(x,y,q,F(x,y,q)) \equiv 0$.It is easy to check that $F \in \mathfrak{M}^2(r;k+n)$. Since $\bar{F}|_{\{-z+F=0\}} = 0$, there exists $a \in \mathcal{E}(r;k+n+1)$ such that $\bar{F} = a \cdot (-z+F)$. Since $\frac{\partial \bar{F}}{\partial z}(0) = -a(0)$, a is an unit. By Proposition 12.5, F is S-non-degenerate. ■

By [1, p.313 Proposition and p.323 Proposition] and [20], we obtain the following Lemma.

Lemma 10.2 *Let $\mathcal{C}^n$ be the set of Legendrian submanifolds of $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ and $\mathcal{S}^n$ be the set of Lagrangian submanifolds of $(T^*\mathbb{R}^n,0)$. Then $\mathcal{C}^n$ and $\mathcal{S}^n$ have the following relations:*

(1) $\tilde{p_1}$ *gives a bijection from* $\mathcal{C}^n$ *to* $\mathcal{S}^n$.

(2) *Let* $\bar{F}(y,q,z) = -z + F(y,q) \in \mathcal{E}(k+n+1)(F \in \mathfrak{M}^2(k+n))$ *and* $\tilde{L} \in \mathcal{C}^n$. *Then* $\bar{F}$ *is a generating family of* $\tilde{L}$ *if and only if* F *is a generating family of* $\tilde{p_1}(\tilde{L})$.

Indeed let $\bar{L}$ be a Legendrian submanifold germ of $(PT^*\mathbb{R}^{n+1}, [\xi^0])$ and $\bar{F}(y,q,z) = -z + F(y,q_1,\cdots,q_n) \in \mathcal{E}(k+n+1)(F \in \mathfrak{M}^2(k+n))$be a generating family of $\bar{L}$. Then

$$\begin{aligned}
\bar{L} &= \{(q_1,\cdots,q_n,z,[\frac{\partial \bar{F}}{\partial q_1};\cdots;\frac{\partial \bar{F}}{\partial q_n};-1])|\frac{\partial \bar{F}}{\partial y} = \bar{F} = 0\},\\
\tilde{L} = \psi_z(\bar{L}) &= \{(q_1,\cdots,q_n,F,\frac{\partial F}{\partial q_1},\cdots,\frac{\partial F}{\partial q_n})|\frac{\partial F}{\partial y} = 0\},\\
L = \tilde{p_1}(\tilde{L}) &= \{(q_1,\cdots,q_n,\frac{\partial F}{\partial q_1},\cdots,\frac{\partial F}{\partial q_n})|\frac{\partial F}{\partial y} = 0\}.
\end{aligned}$$

Under these fact, we identify $(PT^*\mathbb{R}^{n+1}, [\xi^0])$ and $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ and identify Legendrian submanifold of $(PT^*\mathbb{R}^{n+1}, [\xi^0])$ and that of $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ respectively.

11 Propagation mechanism of wavefronts

Recall the propagation mechanism of wavefronts incident from a hypersurface germ with an r-corner in a smooth manifold introduced in Section 2.

Let τ be the ray length function associated with the regular point (t_0,ξ_0) of exp_{q_0}. Then the following diagram is commutative:

$$\begin{array}{ccccc}
 & (\mathbb{R}\times E,(t_0,\xi_0)) & \xrightarrow{\Phi} & (\mathbb{R}\times E,(t_0,\eta_0)) & \\
 & \swarrow (\pi_{\mathbb{R}},exp) & \phi_1 \searrow \quad \swarrow \phi_2 & (\pi_{\mathbb{R}},exp^-)\searrow & \\
(\mathbb{R}\times M,(t_0.u_0)) & \xleftarrow{(\tau,\pi_2)} & (M\times M,(q_0,u_0)) & \xrightarrow{(\tau,\pi_1)} & (\mathbb{R}\times M,(t_0,q_0))
\end{array}$$

Let V^0 be the hypersurface germ in (M,q_0) satisfying $\xi_0|_{T_{q_0}V^0} = 0$ with an r-corner defined as the image of an immersion $\iota : (\mathbb{H}^r\times\mathbb{R}^k,0) \to (M,q_0)$. We parameterize V^0 by ι. For each $\sigma \subset I_r = \{1,\cdots,r\}$ we define Λ^0_σ by the set of conormal vectors of $V^0_\sigma := V^0 \cap \{x_\sigma = 0\}$ in (E,ξ_0) as the lift of the initial wavefront incident from V^0_σ. Then we regard the set $\tilde{L_\sigma}$ the image of covectors in Λ^0_σ by Φ around time t_0, that is

$$\tilde{L_\sigma} = \{\Phi(t,\xi) \in (\mathbb{R}\times E,(t_0,\eta_0))|(t,\xi)\in(\mathbb{R},t_0)\times\Lambda^0_\sigma\},$$

as the set of the lift of the wavefronts incident from V^0_σ around time t_0. We also regard the union of $\tilde{L_\sigma}$ for all $\sigma \subset I_r$ as the set of the lift of wavefront incident from the hypersurface V^0 around time t_0. We define the wavefront incident from V^0 by

$$\bigcup_{\sigma\subset I_r}(\pi_{\mathbb{R}},\pi_M)(\tilde{L_\sigma}).$$

The family of submanifolds $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ of $(\mathbb{R}\times E,(t_0,\eta_0))$ is 'generated' by the ray length function τ as the following:

Proposition 11.1 *Let V^0 be the hypersurface germ in (M,q_0) satisfying $\xi_0|_{T_{q_0}V^0}=0$ which is the image of an immersion $\iota:(\mathbb{H}^r\times\mathbb{R}^k,0)\to(M,q_0)$. Let $\tilde{L_\sigma}$ be the set of the lift of the wavefronts incident from $V^0_\sigma := V^0\cap\{x_\sigma=0\}$ around time t_0 for $\sigma\subset I_r$. Define $\bar{F}(x,y,u,t) := -t+\tau\circ(\iota(x,y),u)\in\mathcal{E}(r;k+m+1)$. Then the following hold:*

(1) $\bar{F}$ is C-non-degenerate, that is $\frac{\partial\bar{F}}{\partial x}(0)=0$, $\frac{\partial\bar{F}}{\partial y}(0)=0$ and

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial\bar{F}}{\partial y} & \dfrac{\partial\bar{F}}{\partial u} \\ \dfrac{\partial^2\bar{F}}{\partial x\partial y} & \dfrac{\partial^2\bar{F}}{\partial x\partial u} \\ \dfrac{\partial^2\bar{F}}{\partial y\partial y} & \dfrac{\partial^2\bar{F}}{\partial y\partial u}\end{pmatrix}_0 = r+k+1.$$

(2)

$$\tilde{L_\sigma} = \{(t,d_u\bar{F}(x,y,u))\in(\mathbb{R}\times T^*M\backslash 0,(t_0,\eta_0))| \\ x_\sigma = d_{x_{I_r-\sigma}}\bar{F}(x,y,u) = d_y\bar{F}(x,y,u) = \bar{F} = 0\}$$

for $\sigma\subset I_r$, where we identify (M,u_0) and $(\mathbb{R}^n,0)$ by coordinates $(u_1,\cdots,u_n)$ of (M,u_0).

Proof. This is immediately followed by Proposition 2.2.

By Theorem 12.6 (2), Proposition 11.1 means that $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ is a *contact regular r-cubic configuration* of $(\mathbb{R}\times T^*M\backslash 0,(t_0,\eta_0))$ with the contact structure defined by the canonical 1-form $dt - pdu$, where p are the fiber coordinates corresponding to u. Hence there exists a contact diffeomorphism $C:(J^1(\mathbb{R}^n,\mathbb{R}),0)\longrightarrow(\mathbb{R}\times T^*M\backslash 0,(t_0,\eta_0))$ such that

$$\bar{L_\sigma} = C(\tilde{L_0^\sigma}) \qquad \text{for} \qquad \sigma\subset I_r,$$

where $\tilde{L_\sigma^0} = \{(q,z,p)\in(J^1(\mathbb{R}^n,\mathbb{R}),0)|q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = z = 0, q_{I_r-\sigma}\geq 0\}$ (cf., Section 12).

Small perturbations of the immersion ι implies small perturbations of contact diffeomorphism C. Therefore we investigate the stabilities of contact regular r-cubic configurations with respect to perturbations of corresponding contact diffeomorphisms in a more general situation in Section 14.

12 Contact regular r-cubic configurations

In this section we shall define *Contact regular r-cubic configurations* and investigate the relations between symplectic and contact regular r-cubic configurations.

Let $(q_1,\cdots,q_n,z,p_1,\cdots,p_n)$ be canonical coordinates of $J^1(\mathbb{R}^n,\mathbb{R})$. Set $\tilde{L_\sigma^0} = \{(q,z,p)\in(J^1(\mathbb{R}^n,\mathbb{R}),0)|q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = z = 0, q_{I_r-\sigma}\geq 0\}$ for each $\sigma\subset I_r$.

Definition 12.1 Let $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ be a family of 2^r Legendrian submanifold germs of $(J^1(\mathbb{R}^n,\mathbb{R}),0)$. Then $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ is called *a contact regular r-cubic configuration* if there exists a contact diffeomorphism C on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that $\tilde{L}_\sigma = C(\tilde{L_\sigma^0})$ for all $\sigma\subset I_r$.

Two contact regular r-cubic configurations $\{\tilde{L_\sigma^1}\}_{\sigma\subset I_r}$ and $\{\tilde{L_\sigma^2}\}_{\sigma\subset I_r}$ are said to be *Legendrian equivalent* if there exist Legendrian equivalence Θ of $\tilde{\pi}$(or $\bar{\pi}$) such that $\tilde{L_\sigma^2} = \Theta(\tilde{L_\sigma^1})$ for all $\sigma \subset I_r$.

Remark: The definition of contact regular r-cubic configuration by Nguyen Huu Duc [6, p. 631] is that there exists a contact diffeomorphism C such that $L_\sigma = C(\{q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = z = 0\})$ for all $\sigma \subset I_r$. Then $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ is called a contact regular r-cubic configuration.

Definition 12.2 Let $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ be a contact regular r-cubic configuration in $(J^1(\mathbb{R}^n,\mathbb{R}),0)$. Then $\tilde{F}(x,y,q,z) \in \mathfrak{M}(r;k+n+1)$ is called *a generating family* of $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ if the following conditions hold:

(1) $\tilde{F}$ is C-non-degenerate.

(2) For each $\sigma \subset I_r$, $\tilde{F}|_{x_\sigma=0}$ is a generating family of $\tilde{L}_\sigma$, that is

$$\tilde{L_\sigma} = \{(q, z, \frac{\partial \tilde{F}}{\partial q}/(-\frac{\partial \tilde{F}}{\partial z}))|\ x_\sigma = \frac{\partial \tilde{F}}{\partial x_{I_r-\sigma}} = \frac{\partial \tilde{F}}{\partial y} = \tilde{F} = 0\}.$$

We now consider contact diffeomorphisms and contact diffeomorphism germs on $J^1(\mathbb{R}^n,\mathbb{R})$ and $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ respectively. Let (Q,Z,P) be canonical coordinates on the source and (q,z,p) be canonical coordinates of the target. We define the following notations:

$\imath : (J^1(\mathbb{R}^n,\mathbb{R}) \cap \{Z=0\},0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0)$ be the inclusion map on the domain.

$$\begin{aligned}
C(J^1(\mathbb{R}^n,\mathbb{R}),0) &= \{C : (J^1(\mathbb{R}^n,\mathbb{R}),0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0) | C : \text{ contact diffeomorphism}\} \\
C^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0) &= \{C \in C(J^1(\mathbb{R}^n,\mathbb{R}),0)|\ C \text{ preserves the canonical 1-form }\} \\
C_Z(J^1(\mathbb{R}^n,\mathbb{R}),0) &= \{C \circ \imath\ | C \in C(J^1(\mathbb{R}^n,\mathbb{R}),0)\} \\
C_Z^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0) &= \{C \circ \imath\ | C \in C^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0)\}
\end{aligned}$$

Let U be an open set in $J^1(\mathbb{R}^n, \mathbb{R})$ and $V = U \cap \{Z = 0\}$. Let $\tilde{\imath} : V \to U$ be the inclusion map.

$$\begin{aligned}
C(U, J^1(\mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} : U \to J^1(\mathbb{R}^n, \mathbb{R}) | \tilde{C} : \text{contact embedding } \} \\
C^\alpha(U, J^1(\mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} \in C(U, J^1(\mathbb{R}^n, \mathbb{R})) \mid C \text{ preserves the canonical 1-form } \} \\
C_Z(V, J^1(\mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} \circ \tilde{\imath} \,|\tilde{C} \in (U, J^1(\mathbb{R}^n, \mathbb{R}))\} \\
C_Z^\alpha(V, J^1(\mathbb{R}^n, \mathbb{R})) &= \{\tilde{C} \circ \tilde{\imath} \,|\tilde{C} \in C^\alpha(U, J^1(\mathbb{R}^n, \mathbb{R}))\}
\end{aligned}$$

Lemma 12.3 *Let $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$ be a contact regular r-cubic configuration in $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ defined by $C \in C(J^1(\mathbb{R}^n, \mathbb{R}), 0)$. Then there exists $C' \in C^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ that also defines $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$.*

Proof. Let $C = (q_C, z_C, p_C)$. Define the function a on $C(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ by the relation $C^*(dz - pdq) = a(dZ - PdQ)$. Define

$$\phi : (J^1(\mathbb{R}^n, \mathbb{R}) \cap \{Z = 0\}, 0) \tilde{\to} (J^1(\mathbb{R}^n, \mathbb{R}) \cap \{Z = 0\}, 0) \;\; (\; (Q, P) \mapsto (Q, a \circ \imath(Q, P)P) \;),$$

$$\begin{aligned}
C' : (J^1(\mathbb{R}^n, \mathbb{R}), 0) &\tilde{\to} \quad (J^1(\mathbb{R}^n, \mathbb{R}), 0) \\
(Q, Z, \bar{P}) &\mapsto (q_C \circ \imath \circ \phi^{-1}(Q, \bar{P}), Z + z_C \circ \imath \circ \phi^{-1}(Q, \bar{P}), p_C \circ \imath \circ \phi^{-1}(Q, \bar{P})).
\end{aligned}$$

Then

$$C'^*(dz - pdq) = dZ + ((C \circ \imath) \circ \phi^{-1})^*(dz - pdq) = dZ - (\phi^{-1})^*(a \circ \imath(Q, P)PdQ) = dZ - \bar{P}dQ.$$

Therefore $C' \in C^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0)$. Since $C'(Q, 0, a(Q, P)P) = C(Q, 0, P)$, C' also defines $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$.

Lemma 12.4 *Let $S(T^*\mathbb{R}^n, 0)$ be the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n, 0)$. We define the following maps:*

$$\begin{aligned}
C_Z^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0) &\to \qquad S(T^*\mathbb{R}^n, 0) \\
C = (q_C, z_C, p_C) &\mapsto \;\; (\; S^C : (Q, P) \mapsto (q_C, p_C)(Q, P) \;)
\end{aligned}$$

$$\begin{array}{rcl} S(T^*\mathbb{R}^n,0) & \to & C^\alpha_Z(J^1(\mathbb{R}^n,\mathbb{R}),0) \\ S=(q_S,p_S) & \mapsto & (\ C^S:(Q,P)\mapsto(q_S,f^S,p_S)(Q,P)\ ), \end{array}$$

where $f^S(Q,P)$ is uniquely defined by the relation that $S^*(pdq)-PdQ=df^S$, $f^S(0,0)=0$. Then these maps are well defined and inverse to each other(that is $S^{C^S}=S, C^{S^C}=C$).

Proof. Let $C\in C^\alpha_Z(J^1(\mathbb{R}^n,\mathbb{R}),0)$ be given. Take $\bar{C}\in C^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that $\bar{C}\circ\imath=C$. Since $S^C=(q_C,p_C)$, we have

$$\begin{aligned}(S^C)^*(dp\wedge dq) &= C^*(dp\wedge dq)=C^*(-d(dz-pdq))=-d((\bar{C}\circ\imath)^*(dz-pdq)) \\ &= -d(\imath^*(dZ-PdQ))=-d(-PdQ)=dP\wedge dQ\end{aligned}$$

Hence $S^C\in S(T^*\mathbb{R}^n,0)$.

Conversely let $S=(q_S,p_S)\in S(T^*\mathbb{R}^n,0)$ be given . We define the diffeomorphism $\bar{C}^S$ on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ by $\bar{C}^S(Q,Z,P)=(q_S(Q,P),Z+f^S(Q,P),p_S(Q,P))$. Then $\bar{C}^S\circ\imath=C^S$ and

$$(\bar{C}^S)^*(dz-pdq)=dZ+df^S-S^*(pdq)=dZ+(S^*(pdq)-PdQ)-S^*(pdq)=dZ-PdQ.$$

Hence $C^S\in C^\alpha_Z(J^1(\mathbb{R}^n,\mathbb{R}),0)$. On the other hand, by definition, we have

$$S^{C^S}=(q_{C^S},p_{C^S})=(q_S,p_S)\ ,\ C^{S^C}=(q_{S^C},f^{S^C},p_{S^C})=(q_C,f^{S^C},p_C).$$

Since f^{S^C} and z_C satisfy the equation of $z(Q,P)$ that $dz=p_Cdq_C-PdQ$ and $z(0,0)=0$, we have that $f^{S^C}=z_C$. ∎

Proposition 12.5 *Let $\mathcal{C}^n_r$ be the set of contact regular r-cubic configurations in $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ and $\mathcal{S}^n_r$ be the set of symplectic regular r-cubic configurations in $(T^*\mathbb{R}^n,0)$. We define*

$$T_S:\mathcal{C}^n_r\to\mathcal{S}^n_r\ (\ \{C(\tilde{L}^0_\sigma)\}_{\sigma\subset I_r}\mapsto\{S^C(L^0_\sigma)\}_{\sigma\subset I_r}\ ),\text{where } C\in C^\alpha_Z(J^1(\mathbb{R}^n,\mathbb{R}),0)$$

$$T_C:\mathcal{S}^n_r\to\mathcal{C}^n_r\ (\ \{S(L^0_\sigma)\}_{\sigma\subset I_r}\mapsto\{C^S(\tilde{L}^0_\sigma)\}_{\sigma\subset I_r}\ ),\text{where } S\in S(T^*\mathbb{R}^n,0)$$

Then (1) T_S *and* T_C *are well defined and inverse to each other.*

(2) *A function germ* $F(x,y,q) \in \mathfrak{M}^2(r;k+n)$ *is S-non-degenerate if and only if* $-z+F$ *is C-non-degenerate.*

(3) *A function germ* $F(x,y,q) \in \mathfrak{M}^2(r;k+n)$ *is a generating family of a symplectic regular r-cubic configuration if and only if* $-z+F$ *is a generating family of the corresponding contact regular r-cubic configuration.*

Proof.(1) Let $C=(q_C,z_C,p_C)\in C_Z^{\alpha}(J^1(\mathbb{R}^n,\mathbb{R}),0)$ and $S\in S(T^*\mathbb{R}^n,0)$ satisfy that $S=S^C$ (hence $C=C^S$). Since $S=(q_C,p_C)$, we have $S(L_\sigma^0)=\tilde{p_1}(C(\tilde{L}_\sigma^0))$ for all $\sigma\subset I_r$. Since $S(L_\sigma^0)$ and $C(\tilde{L_\sigma^0})$ are uniquely determined by each other under $\tilde{p_1}$ by Lemma 10.2, we have (1).

(2) Let $F(x,y,q)\in\mathfrak{M}^2(r;k+n)$. If we define $\bar{F}\in\mathfrak{M}(r;k+n+1)$ by $\bar{F}(x,y,q,z)=-z+F(x,y,q)$. Then $\frac{\partial\bar{F}}{\partial x}(0)=\frac{\partial F}{\partial x}(0)=0$, $\frac{\partial\bar{F}}{\partial y}(0)=\frac{\partial F}{\partial y}(0)=0$ and

$$\begin{pmatrix} \frac{\partial\bar{F}}{\partial y} & \frac{\partial\bar{F}}{\partial q} & \frac{\partial\bar{F}}{\partial z}\\ \frac{\partial^2\bar{F}}{\partial x\partial y} & \frac{\partial^2\bar{F}}{\partial x\partial q} & \frac{\partial^2\bar{F}}{\partial x\partial z}\\ \frac{\partial^2\bar{F}}{\partial y\partial y} & \frac{\partial^2\bar{F}}{\partial y\partial q} & \frac{\partial^2\bar{F}}{\partial y\partial z}\end{pmatrix}_0 = \begin{pmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial q} & -1\\ \frac{\partial^2 F}{\partial x\partial y} & \frac{\partial^2 F}{\partial x\partial q} & 0\\ \frac{\partial^2 F}{\partial y\partial y} & \frac{\partial^2 F}{\partial y\partial q} & 0\end{pmatrix}_0.$$

This implies (2).

(3) By (2),we have

$$\begin{aligned} & \bar{F}=-z+F \text{ is a generating family of } \{C(\tilde{L}_\sigma^0)\}_{\sigma\subseteq I_r}\\ \Leftrightarrow\ & \bar{F} \text{ is C-non-degenerate and } \bar{F}|_{x_\sigma=0} \text{ generates } C(\tilde{L}_\sigma^0) \text{ for all } \sigma\subset I_r\\ \Leftrightarrow\ & F \text{ is S-non-degenerate and } F|_{x_\sigma=0} \text{ generates } S(L_\sigma^0) \text{ for all } \sigma\subset I_r\\ \Leftrightarrow\ & F \text{ is a generating family of } \{S(L_\sigma^0)\}_{\sigma\subset I_r}. \end{aligned}$$

■

The relation between contact and symplectic regular r-cubic configurations is given in the following diagram:

$$\begin{array}{ccccccc} (J^1(\mathbb{R}^n,\mathbb{R})\cap\{Z=0\},0) & \overset{C=C^S}{\rightarrow} & (J^1(\mathbb{R}^n,\mathbb{R}),0) & & \{\tilde{L}_\sigma^0\}_{\sigma\subset I_r} & \mapsto & \{\tilde{L}_\sigma\}_{\sigma\subset I_r}\\ \tilde{p_1}|_{Z=0}\downarrow & & \downarrow\tilde{p_1} & & \downarrow & & T_S\downarrow\uparrow T_C\\ (T^*\mathbb{R}^n,0) & \overset{S=S^C}{\rightarrow} & (T^*\mathbb{R}^n,0) & & \{L_\sigma^0\}_{\sigma\subset I_r} & \mapsto & \{L_\sigma\}_{\sigma\subset I_r} \end{array}$$

We say that function germs $F(x,y_1,\cdots,y_{k_1},u)\in\mathfrak{M}(r;k_1+m)$ and $F(x,y_1,\cdots,y_{k_2},u)\in$

$\mathfrak{M}(r; k_2 + m)$ are *stably reticular K-equivalent* if F and G are reticular K-equivalent after additions of non-degenerate quadratic forms in the variables y.

The relations between contact regular r-cubic configurations and their generating families are given in the following theorem.

Theorem 12.6 (1) *For any contact regular r-cubic configuration $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$ in $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$, there exists a function germ $\bar{F} \in \mathfrak{M}(r; k+n+1)$ which is a generating family of $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$.*

(2) *For any C-non-degenerate function $\bar{F} \in \mathfrak{M}(r; k+n+1)$, there exists a contact regular r-cubic configuration in $(PT^*\mathbb{R}^{n+1}, (0, [\frac{\partial \bar{F}}{\partial \lambda}(0)]))$ (or in $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$) of which $\bar{F}$ is a generating family.*

(3) *Two contact regular r-cubic configurations are Legendrian equivalent if and only if their generating families are stably reticular K-equivalent.*

Proof. (1) Let a contact regular r-cubic configuration $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$ in $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ be given. Set $\{L_\sigma\}_{\sigma \subset I_r} = T_S(\{\tilde{L_\sigma}\}_{\sigma \subset I_r})$ and let $F \in \mathfrak{M}^2(r; k+n)$ be a generating family of $\{L_\sigma\}_{\sigma \subset I_r}$. Then $-z + F \in \mathfrak{M}(r; k+n+1)$ is a generating family of $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$ by Proposition 12.5 (3).

(2) Let a C-non-degenerate function $\bar{F} \in \mathfrak{M}(r; k+n+1)$ be given. By Lemma 10.1 and (3)a, we may assume that $\bar{F}$ has the form $\bar{F}(x, y, q, z) = -z + F(x, y, q)(F \in \mathfrak{M}^2(r; k+n))$. Then F is a generating family of a symplectic regular r-cubic configuration $\{L_\sigma\}_{\sigma \subset I_r}$ in $(T^*\mathbb{R}^n, 0)$ by Proposition 12.5 (2) and Theorem 3.2 (2). Hence $\bar{F}$ is a generating family of $T_C(\{L_\sigma\}_{\sigma \subset I_r})$ by Proposition 12.5 (3).

(3) We suppose the assertions (3)a, (3)b, (3)c, (3)d and prove (3). Let $\{\bar{L^1_\sigma}\}_{\sigma \subset I_r}$ and $\{\bar{L^2_\sigma}\}_{\sigma \subset I_r}$ be contact regular r-cubic configurations in $(PT^*\mathbb{R}^{n+1}, (0, [\eta_1]))$ and $(PT^*\mathbb{R}^{n+1}, (0, [\eta_2]))$ respectively and $\bar{F} \in \mathfrak{M}(r; k_1 + n + 1)$ and $\bar{G} \in \mathfrak{M}(r; k_2 + n + 1)$ be generating families of $\{\bar{L^1_\sigma}\}_{\sigma \subset I_r}$ and $\{\bar{L^2_\sigma}\}_{\sigma \subset I_r}$ respectively.

($\Leftarrow$) Let $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent. By (3)b, we may assume that $k_1 = k_2$. Hence by (3)a, $\{\bar{L_\sigma^1}\}_{\sigma\subset I_r}$ and $\{\bar{L_\sigma^2}\}_{\sigma\subset I_r}$ are Legendrian equivalent.

($\Rightarrow$) Let $\{\bar{L_\sigma^1}\}_{\sigma\subset I_r}$ and $\{\bar{L_\sigma^2}\}_{\sigma\subset I_r}$ be Legendrian equivalent. By (3)c, we may assume that $\bar{F}$ and $\bar{G}$ are generating families of the same contact regular r-cubic configuration. By Lemma 10.1, we may assume that $\bar{F}$ and $\bar{G}$ are written in the form: $\bar{F} = -z + F, \bar{G} = -z + G$, where $F \in \mathfrak{M}^2(r; k_1 + n), G \in \mathfrak{M}^2(r; k_2 + n)$. Hence by (3)d, $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent.

a. *Let $\bar{F} \in \mathfrak{M}(r; k + n + 1)$ be a generating family of a contact regular r-cubic configuration $\{\bar{L}_\sigma^1\}_{\sigma\subset I_r}$ of $(PT^*\mathbb{R}^{n+1}, (0, [\frac{\partial \bar{F}}{\partial \lambda}(0)]))$ and suppose that $\bar{F}$ and $\bar{G} \in \mathfrak{M}(r; k + n + 1)$ are reticular K-equivalent. Then $\bar{G}$ is a generating family of a contact regular r-cubic configuration $\{\bar{L}_\sigma^2\}_{\sigma\subset I_r}$ to which $\{\bar{L}_\sigma^1\}_{\sigma\subset I_r}$ is Legendrian equivalent.*

We write the reticular K-equivalence between $\bar{F}$ and $\bar{G}$ by

$$\bar{G}(x, y, \lambda) = \alpha(x, y, \lambda)\bar{F}(x_1a_1(x, y, \lambda), \cdots, x_ra_r(x, y, \lambda), h(x, y, \lambda), g(\lambda)).$$

Since $\frac{\partial \bar{G}}{\partial x}(0) = \alpha(0)(\frac{\partial \bar{F}}{\partial x} + \frac{\partial \bar{F}}{\partial y}\frac{\partial h}{\partial x})(0) = 0$, $\frac{\partial \bar{G}}{\partial y}(0) = \alpha(0)(\frac{\partial \bar{F}}{\partial y}\frac{\partial h}{\partial y})(0) = 0$ and

$$\begin{pmatrix} \frac{\partial \bar{G}}{\partial y} & \frac{\partial \bar{G}}{\partial \lambda} \\ \frac{\partial^2 \bar{G}}{\partial x \partial y} & \frac{\partial^2 \bar{G}}{\partial x \partial \lambda} \\ \frac{\partial^2 \bar{G}}{\partial y \partial y} & \frac{\partial^2 \bar{G}}{\partial y \partial \lambda} \end{pmatrix}_0 = \alpha(0) \begin{pmatrix} 1 & \frac{\partial \alpha}{\partial x}/\alpha & \frac{\partial \alpha}{\partial y}/\alpha \\ & a_1 & \\ 0 & \ddots & 0 \\ & a_r & \\ 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}_0^t \begin{pmatrix} \frac{\partial \bar{F}}{\partial y} & \frac{\partial \bar{F}}{\partial \lambda} \\ \frac{\partial^2 \bar{F}}{\partial x \partial y} & \frac{\partial^2 \bar{F}}{\partial x \partial \lambda} \\ \frac{\partial^2 \bar{F}}{\partial y \partial y} & \frac{\partial^2 \bar{F}}{\partial y \partial \lambda} \end{pmatrix}_0 \begin{pmatrix} \frac{\partial h}{\partial y} & \frac{\partial h}{\partial \lambda} \\ 0 & \frac{\partial g}{\partial \lambda} \end{pmatrix}_0,$$

$\bar{G}$ is C-non-degenerate. Set

$(x', y', \lambda') = H(x, y, \lambda) = (x_1a_1(x, y, \lambda), \cdots, x_ra_r(x, y, \lambda), h(x, y, \lambda), g(\lambda))$. Then

$$\frac{\partial \bar{G}}{\partial x_l} = \frac{\partial \alpha}{\partial x_l}\bar{F} \circ H + \alpha(\sum_{i=1}^{r} \frac{\partial \bar{F}}{\partial x_i} \circ H \cdot x_i \frac{\partial a_i}{\partial x_l} + \sum_{j=1}^{k} \frac{\partial \bar{F}}{\partial y_j} \circ H \cdot \frac{\partial h_j}{\partial x_l} + \frac{\partial \bar{F}}{\partial x_l} \circ H \cdot a_l)$$

$(l = 1, \cdots, r)$,

$$\frac{\partial \bar{G}}{\partial y_l} = \frac{\partial \alpha}{\partial y_l}\bar{F} \circ H + \alpha(\sum_{i=1}^{r} \frac{\partial \bar{F}}{\partial x_i} \circ H \cdot x_i \frac{\partial a_i}{\partial y_l} + \sum_{j=1}^{k} \frac{\partial \bar{F}}{\partial y_j} \circ H \cdot \frac{\partial h_j}{\partial y_l})$$

$(l=1,\cdots,k)$,

$$\frac{\partial \bar{G}}{\partial \lambda_l}=\frac{\partial \alpha}{\partial \lambda_l}\bar{F}\circ H+\alpha(\sum_{i_1=1}^{r}\frac{\partial \bar{F}}{\partial x_{i_1}}\circ H\cdot x_{i_1}\frac{\partial a_{i_1}}{\partial \lambda_l}+\sum_{i_2=1}^{k}\frac{\partial \bar{F}}{\partial y_{i_2}}\circ H\cdot\frac{\partial h_{i_2}}{\partial \lambda_l}+\sum_{i_3=1}^{n}\frac{\partial \bar{F}}{\partial \lambda_{i_3}}\circ H\cdot\frac{\partial g_{i_3}}{\partial \lambda_l})$$

$(l=1,\cdots,n)$. Therefore for $\sigma\subset I_r$ we have

$$\begin{aligned}
&\{\ (\lambda,[\frac{\partial \bar{G}}{\partial \lambda}])\mid x_\sigma=\frac{\partial \bar{G}}{\partial x_\tau}=\frac{\partial \bar{G}}{\partial y}=\bar{G}=0\ \}\\
=&\ \{\ (\lambda,[\alpha\sum_{i=1}^{n}\frac{\partial \bar{F}}{\partial \lambda_i}(x',y',\lambda')\cdot\frac{\partial g_i}{\partial \lambda}(\lambda)])\mid\\
&\quad x'_\sigma=\frac{\partial \bar{F}}{\partial x_\tau}(x',y',\lambda')=\frac{\partial \bar{F}}{\partial y}(x',y',\lambda')=\bar{F}(x',y',\lambda')=0\ \}\\
=&\ \{\ (g^{-1}(\lambda'),[\sum_{i=1}^{n}\frac{\partial \bar{F}}{\partial \lambda_i}(x',y',\lambda')\cdot\frac{\partial g_i}{\partial \lambda}(g^{-1}(\lambda'))])\mid\\
&\quad x'_\sigma=\frac{\partial \bar{F}}{\partial x_\tau}(x',y',\lambda')=\frac{\partial \bar{F}}{\partial y}(x',y',\lambda')=\bar{F}(x',y',\lambda')=0\ \}\\
=&\ \bar{g}^*(\bar{L}^1_\sigma)
\end{aligned}$$

$(\tau=I_r-\sigma)$. It follows that $\bar{G}$ is a generating family of $\{\bar{g}^*(\bar{L}^1_\sigma)\}_{\sigma\subset I_r}$.

b. *Let $\bar{F}(x,y,\lambda)\in\mathfrak{M}(r;k+n+1)$ be a generating family of a contact regular r-cubic configuration $\{\bar{L_\sigma}\}_{\sigma\subset I_r}$ in $(PT^*\mathbb{R}^{n+1},(0,[\frac{\partial \bar{F}}{\partial \lambda}(0)]))$. If $\bar{F}$ and $\bar{G}$ are stably equivalent, then $\bar{G}$ is also a generating family of $\{\bar{L_\sigma}\}_{\sigma\subset I_r}$.*

c. *Let $\{\bar{L}^1_\sigma\}_{\sigma\subset I_r}$ and $\{\bar{L}^2_\sigma\}_{\sigma\subset I_r}$ be contact regular r-cubic configurations in $(PT^*\mathbb{R}^{n+1},[\eta_1])$ and $(PT^*\mathbb{R}^{n+1},[\eta_2])$ respectively and let $\bar{F}\in\mathfrak{M}(r;k+n+1)$ be a generating family of $\{\bar{L}^1_\sigma\}_{\sigma\subset I_r}$. If these configurations are Legendrian equivalent, then there exists a generating family $\bar{G}$ of $\{\bar{L}^2_\sigma\}_{\sigma\subset I_r}$ to which $\bar{F}$ is reticular K-equivalent.*

Take a diffeomorphism g on the base of $\bar{\pi}$ such that $\bar{L}^2_\sigma=\bar{g}^*(\bar{L}^1_\sigma)$ for all $\sigma\subset I_r$. Define $\bar{G}(x,y,\lambda)=\bar{F}(x,y,g(\lambda))$. Then we can prove that $\bar{G}$ is a generating family of $\{\bar{L}^2_\sigma\}_{\sigma\subset I_r}$ by the same method of (3)a.

d. *Let $\bar{F}(x,y^1,q,z)=-z+F(x,y^1,q)$ and $\bar{G}(x,y^2,q,z)=-z+G(x,y^2,q)$ $(F\in\mathfrak{M}^2(r;k_1+$*

$n), G \in \mathfrak{M}^2(r; k_2 + n))$ *are two generating families of the same contact regular r-cubic configuration. Then $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent.*

By Proposition 12.5, F and G are generating families of the same symplectic regular r-cubic configuration. Therefore F and G are stably reticular R-equivalent by the remark after Theorem 3.2. Hence $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent. ■

13 Stability of unfoldings

In order to investigate the stabilities of smooth contact regular r-cubic configurations, we shall prepare the results of the singularity theory of function germs with respect to *reticular K-equivalence.* Basic techniques for the characterization of the stabilities we use in this part depend heavily on the results in this section, however the all arguments are almost parallel along the ordinary theory of the right-equivalence (cf., [19]), so that we omit the detail.

We say $f, g \in \mathfrak{M}(r; l)$ are *reticular K-equivalent* if there exists $\phi \in \mathcal{B}(k; l)$ such that $g = f \circ f$.

Lemma 13.1 *Let $f \in \mathfrak{M}(r; k)$ and $O^l_K(j^l f(0))$ be the submanifold of $J^l(r + k, 1)$ consist of the image by π_l of the orbit of reticular K-equivalence of f. Put $z = j^l f(0)$. Then*

$$T_z(O^l_{rK}(z)) = \pi_l(\langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\langle \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle).$$

We say that a function germ $f \in \mathfrak{M}(r; k)$ is *reticular K-l-determined* if all function germ which has same l-jet of f is reticular K-equivalent to f.

Lemma 13.2 *Let $f \in \mathfrak{M}(r; k)$ and let*

$$\mathfrak{M}(r;k)^{l+1} \subset \mathfrak{M}(r;k)(\langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle + \mathfrak{M}(r;k)\langle \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle) + \mathfrak{M}(r;k)^{l+2},$$

then f is reticular K-l-determined. Conversely let $f \in \mathfrak{M}(r;k)$ be reticular K-l-determined, then

$$\mathfrak{M}(r;k)^{l+1} \subset \langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\langle \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle.$$

Let $F \in \mathfrak{M}(r;k+n_1)$, $G \in \mathfrak{M}(r;k+n_2)$ be unfoldings of $f \in \mathfrak{M}(r;k)$. We say that F *is reticular K-f-induced from G* if there exist smooth map germs $\phi : (\mathbb{H}^r \times \mathbb{R}^{k+n_2}, 0) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$, $\psi : (\mathbb{R}^{n_2}, 0) \to (\mathbb{R}^{n_1}, 0)$ and $\alpha \in \mathcal{E}(0;n_2)$ satisfying the following conditions:

(1) $\phi((\mathbb{H}^r \cap \{x_\sigma = 0\}) \times \mathbb{R}^{k+n_2}) \subset (\mathbb{H}^r \cap \{x_\sigma = 0\}) \times \mathbb{R}^k$ for $\sigma \subset I_r$.

(2) $G(x,y,v) = \alpha(x,y,v) \cdot F(\phi(x,y,v), \psi(v))$ for $x \in \mathbb{H}^r$, $y \in \mathbb{R}^k$ and $v \in \mathbb{R}^{n_2}$.

Definition 13.3 Here we give several definitions of the stabilities of unfoldings. Let $f \in \mathfrak{M}(r;k)$ and $F \in \mathfrak{M}(r;k+n)$ be an unfolding of f.

We say that F is *reticular K-l-transversal* if $j_1^l F|_{x=0}$ is transversal to $O_K^l(j^l f(0))$. It is easy to check that F is reticular K-l-transversal if and only if

$$\mathcal{E}(r;k) = \langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)} + W_F + \mathfrak{M}(r;k)^{l+1},$$

where $W_F = \langle \frac{\partial F}{\partial u_1}|_{u=0}, \cdots, \frac{\partial F}{\partial u_n}|_{u=0}\rangle_{\mathbb{R}}$.

We say that F is *reticular K-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, u_0')$ is reticular K-equivalent to F for some $(0, y_0, u_0') \in U$.

We say that F is *reticular K-versal* if F is reticular K-f-induced from all unfolding of f.

We say that F is *reticular K-infinitesimal versal* if

$$\mathcal{E}(r;k) = \langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)} + W_F.$$

We say that F is *reticular K-infinitesimal stable* if

$$\begin{aligned}&\mathcal{E}(r;k+n)\\ =\ &\langle F, x_1\frac{\partial F}{\partial x_1},\cdots,x_r\frac{\partial F}{\partial x_r},\frac{\partial F}{\partial y_1},\cdots,\frac{\partial F}{\partial y_k}\rangle_{\mathcal{E}(r;k+n)}+\langle\frac{\partial F}{\partial u_1},\cdots,\frac{\partial F}{\partial u_n}\rangle_{\mathcal{E}(n)}.\end{aligned}$$

We say that F is *reticular K-homotopically stable* if for any smooth path-germ $(\mathbb{R},0)\to\mathcal{E}(r;k+n), t\mapsto F_t$ with $F_0=F$, there exists a smooth path-germ $(\mathbb{R},0)\to\mathcal{B}(r;k+n)\times\mathcal{E}(n), t\mapsto(\Phi_t,\alpha_t)$ with $(\Phi_0,\alpha_0)=(id,1)$ such that each (Φ_t,α_t) is a reticular K-isomorphism and $F_0=\alpha_t\cdot F_t\circ\Phi_t$.

Theorem 13.4 *Let $F\in\mathfrak{M}(r;k+n)$ be an unfolding of $f\in\mathfrak{M}(r;k)$. Then the following are equivalent.*

(1) *F is reticular K-stable.*

(2) *F is reticular K-versal.*

(3) *F is reticular K-infinitesimal versal.*

(4) *F is reticular K-infinitesimal stable.*

(5) *F is reticular K-homotopically stable.*

For $f\in\mathfrak{M}(r;k)$ we define the *reticular K-codimension of f* by the $\mathbb{R}$-dimension of the vector space

$$\mathcal{E}(r;k)/\langle f, x_1\frac{\partial f}{\partial x_1},\cdots,x_r\frac{\partial f}{\partial x_r},\frac{\partial f}{\partial y_1},\cdots,\frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)}.$$

By the above theorem if $a_1,\cdots,a_n\in\mathcal{E}(r;k)$ is a representative of a basis of the vector space, then $f+a_1v_1+\cdots a_nv_n\in\mathfrak{M}(r;k+n)$ is a reticular K-stable unfolding of f.

14 Reticular Legendrian maps

Our purpose in this section is to investigate the stabilities of smooth contact regular r-cubic configurations. At first, we define the reticular Legendrian maps and their equivalence relation.

Let $\tilde{\mathbb{L}^0} = \{(q,z,p) \in J^1(\mathbb{R}^n,\mathbb{R}) | q_1p_1 = \cdots = q_rp_r = q_{r+1} = \cdots = q_n = z = 0, q_{I_r} \geq 0\}$ be a representative of the union of $\tilde{L^0_\sigma}$ for all $\sigma \subset I_r$. We call the map germ

$$(\tilde{\mathbb{L}^0}, 0) \xrightarrow{i} (J^1(\mathbb{R}^n,\mathbb{R}),0) \xrightarrow{\tilde{\pi}} (\mathbb{R}^n \times \mathbb{R}, 0)$$

a *reticular Legendrian map* if there exists a contact diffeomorphism C on $(J^1(\mathbb{R}^n,\mathbb{R}),\ 0)$ such that $i = C|_{\tilde{\mathbb{L}^0}}$. C is called an *extension* of i. We call $\{i(\tilde{L^0_\sigma})\}_{\sigma \subset I_r}$ the contact regular r-cubic configuration associated with $\tilde{\pi} \circ i$. We call F a *generating family* of $\tilde{\pi} \circ i$ if F is a generating family of $\{i(\tilde{L^0_\sigma})\}_{\sigma \subset I_r}$. A homeomorphism germ $\phi : (\tilde{\mathbb{L}^0}, 0) \longrightarrow (\tilde{\mathbb{L}^0}, 0)$ is called a *reticular diffeomorphism* if there exists a diffeomorphism Φ on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that $\phi = \Phi|_{\tilde{\mathbb{L}^0}}$ and $\phi(\tilde{L^0_\sigma}) = \tilde{L^0_\sigma}$ for all $\sigma \subset I_r$. Two reticular Legendrian maps $\tilde{\pi} \circ i_1, \tilde{\pi} \circ i_2$: $(\tilde{\mathbb{L}^0}, 0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ are called *Legendrian equivalent* if there exists a reticular diffeomorphism ϕ and a Legendrian equivalence Θ on $\tilde{\pi}$ such that the following diagram is commutative:

$$\begin{array}{ccccc} (\tilde{\mathbb{L}^0}, 0) & \xrightarrow{i_1} & (J^1(\mathbb{R}^n,\mathbb{R}),0) & \xrightarrow{\tilde{\pi}} & (\mathbb{R}^n \times \mathbb{R}, 0) \\ \phi \downarrow & & \Theta \downarrow & & g \downarrow \\ (\tilde{\mathbb{L}^0}, 0) & \xrightarrow{i_2} & (J^1(\mathbb{R}^n,\mathbb{R}),0) & \xrightarrow{\tilde{\pi}} & (\mathbb{R}^n \times \mathbb{R}, 0) \end{array}$$

where g is the diffeomorphism of the base of $\tilde{\pi}$ induced from Θ.

Under this equivalence relation, we have the following theorem as a corollary of Theorem 12.6.

Theorem 14.1 (1) *For any reticular Legendrian map $\tilde{\pi} \circ i$, there exists a function germ $F \in \mathfrak{M}(r; k+n+1)$ which is a generating family of $\tilde{\pi} \circ i$.*

(2) *For any C-non-degenerate function germ $F \in \mathfrak{M}(r; k+n+1)$, there exists a reticular Legendrian map of which F is a generating family.*

(3) *Two reticular Legendrian maps are Legendrian equivalent if and only if their generating families are stably reticular K-equivalent.*

Here we give several definitions of the stabilities of reticular Legendrian maps.

Stability: Let $\tilde{\pi} \circ i : (\tilde{\mathbb{L}^0}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ be a reticular Legendrian map. $\tilde{\pi} \circ i$ is called *stable* if the following condition holds: For any extension $C_0 \in C(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ of i and any representative $\tilde{C}_0 \in C(U, J^1(\mathbb{R}^n, \mathbb{R}))$, there exists a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ in C^∞-topology such that for all $\tilde{C} \in N_{\tilde{C}_0}$ $\tilde{\pi} \circ \tilde{C}|_{\mathbb{L}^0}$ *at* x_0 and $\tilde{\pi} \circ i$ are Legendrian equivalent for some $x_0 = (0; 0; 0, \cdots, 0, P^0_{r+1}, \cdots, P^0_n) \in U$.

Let $\tilde{\pi} \circ i$ is a reticular Legendrian map. By Lemma 12.3, we may assume that there exists an extension $C \in C^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ of i_0. Therefore we may consider the following other definitions of stabilities of reticular Legendrian maps: (1) The definition given by replacing $C(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ and $C(U, J^1(\mathbb{R}^n, \mathbb{R}))$ to $C^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ and $C^\alpha(U, J^1(\mathbb{R}^n, \mathbb{R}))$ of the original definition respectively. (2) The definition given by replacing to $C_Z(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ and $C_Z(V, J^1(\mathbb{R}^n, \mathbb{R}))$ respectively. (3) The definition given by replacing to $C^\alpha_Z(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ and $C^\alpha_Z(V, J^1(\mathbb{R}^n, \mathbb{R}))$ respectively.

Lemma 14.2 *The original definition and these definitions of stabilities of reticular Legendrian maps are all equivalent.*

Proof. (original)$\Rightarrow$(1). Let $C_0 \in C^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ be an extension of i_0 and $\tilde{C}_0 \in C^\alpha(U,$

$J^1(\mathbb{R}^n,\mathbb{R}))$ be a representative of C_0. Take a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ in $C(U, J^1(\mathbb{R}^n,\mathbb{R}))$ for which the hypothesis of the original definition holds. Set $N'_{\tilde{C}_0} = N_{\tilde{C}_0} \cap C^\alpha(U, J^1(\mathbb{R}^n,\mathbb{R}))$. Then the hypothesis of the definition of (1) holds for $N'_{\tilde{C}_0}$.

(1)⇒(3). Let $C_0 \in C^\alpha_Z(J^1(\mathbb{R}^n,\mathbb{R}),0)$ be an extension of i_0 and $\tilde{C}_0 \in C^\alpha_Z(V, J^1(\mathbb{R}^n,\mathbb{R}))$ be a representative of C_0. We construct the continuous map $C^\alpha_Z(V, J^1(\mathbb{R}^n,\mathbb{R})) \to C^\alpha(V \times \mathbb{R}, J^1(\mathbb{R}^n,\mathbb{R}))$ $(\tilde{C} \mapsto \tilde{C}')$ by the following: Let $\tilde{C} = (z_{\tilde{C}}, q_{\tilde{C}}, p_{\tilde{C}}) \in C^\alpha_Z(V, J^1(\mathbb{R}^n,\mathbb{R}))$. Then $\tilde{C}'$ is defined by $\tilde{C}'(Q,Z,P) = (q_{\tilde{C}}(Q,P), Z + z_{\tilde{C}}(Q,P), p_{\tilde{C}}(Q,P))$. Then $\tilde{C}'^*(dz - pdq) = dZ + \tilde{C}^*(dz - pdq) = dZ - PdQ$. Hence this map is well defined. Take a neighborhood $N_{\tilde{C}_0}{}'$ of $\tilde{C}_0{}'$ in $C(V \times \mathbb{R}, J^1(\mathbb{R}^n,\mathbb{R}))$ for which the hypothesis of the definition of (1) holds. Let $N'_{\tilde{C}_0}$ be the inverse image of $N_{\tilde{C}_0}{}'$ by the preceding map. Then the hypothesis of the definition of (3) holds for $N'_{\tilde{C}_0}$.

(3)⇒(2). Let $C_0 \in C_Z(J^1(\mathbb{R}^n,\mathbb{R}),0)$ be an extension of i_0 and $\tilde{C}_0 \in C_Z(V, J^1(\mathbb{R}^n,\mathbb{R}))$ be a representative of C_0. Define

$$C_Z(V, J^1(\mathbb{R}^n,\mathbb{R})) \to C^\infty(V, J^1(\mathbb{R}^n,\mathbb{R}) \cap \{Z=0\})\ (\tilde{C} \mapsto \phi_{\tilde{C}} : (Q,P) \mapsto (Q, f_{\tilde{C}}P)\ ),$$

where $f^{\tilde{C}} \in C^\infty(V,\mathbb{R})$ is defined by $\tilde{C}^*(dz - pdq) = -f^{\tilde{C}}PdQ$. Then this map is continuous because $f^{\tilde{C}}P_i = (f^{\tilde{C}}(PdQ))(\frac{\partial}{\partial Q_i}) = -(dz - pdq)(\tilde{C}_*\frac{\partial}{\partial Q_i}) = -\frac{\partial z_{\tilde{C}}}{\partial Q_i} + p_{\tilde{C}}\frac{\partial q_{\tilde{C}}}{\partial Q_i}$ $(i = 1, \cdots, n)$. We may assume $\phi_{\tilde{C}_0}$ is embedding by shrinking V if necessary. By Lemma 5.3 there exists a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ and a neighborhood V_1 of 0 in V and a neighborhood W of 0 in $J^1(\mathbb{R}^n,\mathbb{R}) \cap \{Z=0\}$ such that

$$N_{\tilde{C}_0} \to \mathrm{Emb}(W,V)\ (\tilde{C} \mapsto (\phi_{\tilde{C}}|_{V_1})^{-1}|_W\ )$$

is well defined and continuous. Therefore we may define the following continuous map:

$$N_{\tilde{C}_0} \to C^\alpha_Z(W, J^1(\mathbb{R}^n,\mathbb{R}))\ (\tilde{C} \mapsto \tilde{C} \circ (\phi_{\tilde{C}}|_{V_1})^{-1}|_W\ ).$$

Take a neighborhood N of $\tilde{C}_0 \circ (\phi_{\tilde{C}_0}|_{V_1})^{-1}|_W$ for which the hypothesis of the definition of (3) holds. Let $N'_{\tilde{C}_0}$ be the inverse image of N by the preceding map. Then the hypothesis of the definition of (2) holds for $N'_{\tilde{C}_0}$.

(2)$\Rightarrow$(original). Let $C_0 \in C(J^1(\mathbb{R}^n,\mathbb{R}),0)$ be an extension of i_0 and $\tilde{C}_0 \in C(U, J^1(\mathbb{R}^n,\mathbb{R}))$ be a representative of C_0. Let $V = U \cap \{Z=0\}$ and $\tilde{C}_0{}' = \tilde{C}_0|_{Z=0}$. Take a neighborhood $N_{\tilde{C}_0{}'}$ of $\tilde{C}_0{}'$ in $C_Z(V, J^1(\mathbb{R}^n,\mathbb{R}))$ for which the hypothesis of the definition of (2) holds. Because $C(U, J^1(\mathbb{R}^n,\mathbb{R})) \to C_Z(V, J^1(\mathbb{R}^n,\mathbb{R}))$ ($\tilde{C} \mapsto \tilde{C}|_{Z=0}$) is continuous, if we set $N'_{\tilde{C}_0}$ the inverse image of $N_{\tilde{C}_0{}'}$ by the preceding map then the hypothesis of the original definition holds for $N'_{\tilde{C}_0}$. ■

<u>**Homotopical Stability**</u>: Let $\bar{\pi} \circ i : (\tilde{\mathbb{L}^0}, 0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ be a reticular Legendrian map. A map germ $\bar{i} : (\tilde{\mathbb{L}^0} \times \mathbb{R}, 0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0)((Q,P,t) \mapsto \bar{i}_t(Q,P))(\bar{i}_0 = i)$ is called a *reticular Legendrian deformation* of i if there exists a one-parameter family of contact diffeomorphisms $\bar{C} : (J^1(\mathbb{R}^n,\mathbb{R}) \times \mathbb{R}, 0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0)((Q,Z,P,t) \mapsto \bar{C}_t(Q,Z,P))$ such that $\bar{i}_t = \bar{C}_t|_{\tilde{\mathbb{L}^0}}$ for t near 0. We call $\bar{C}$ an *extension* of $\bar{i}$. Let $\phi : (\tilde{\mathbb{L}^0}, 0) \to (\tilde{\mathbb{L}^0}, 0)$ be a reticular diffeomorphism. A map germ $\bar{\phi} : (\tilde{\mathbb{L}^0} \times \mathbb{R}, 0) \to (\tilde{\mathbb{L}^0}, 0)((Q,P,t) \mapsto \bar{\phi}_t(Q,P))(\bar{\phi}_0 = \phi)$ is called a *one-parameter deformation of reticular diffeomorphisms* of ϕ if there exists a one-parameter family of diffeomorphisms $\bar{\Phi} : (J^1(\mathbb{R}^n,\mathbb{R}) \times \mathbb{R}, 0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0)((Q,Z,P,t) \mapsto \bar{\Phi}_t(Q,Z,P))$ such that $\bar{\phi}_t = \bar{\Phi}_t|_{\tilde{\mathbb{L}^0}}$ for t near 0 and each $\bar{\phi}_t$ is a reticular diffeomorphism. We call $\bar{\Phi}$ an *extension* of $\bar{\phi}$. A reticular Legendrian map $\bar{\pi} \circ i : (\tilde{\mathbb{L}^0}, 0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ is called *homotopically stable* if for any reticular Legendrian deformation $\bar{i} = \{\bar{i}_t\}$ of i there exist a one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\bar{\phi}_t\}$ of $id_{(\tilde{\mathbb{L}^0},0)}$ and a one-parameter family of Legendrian equivalences $\bar{\Theta} = \{\bar{\Theta}_t\}$ with $\bar{\Theta}_0 = id_{(J^1(\mathbb{R}^n,\mathbb{R}),0)}$ such that $\bar{i}_t = \bar{\Theta}_t \circ i \circ \bar{\phi}_t$ for t near 0.

Infinitesimal Stability: A vector field v on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ is called *tangent* to $(\tilde{\mathbb{L}^0},0)$ if $v|_{\tilde{L^0_\sigma}}$ is tangent to $\tilde{L^0_\sigma}$ for all $\sigma \subset I_r$. A function germ H on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ is called fiber preserving if there exists function germs $h_0,\cdots,h_n$ on the base of $\tilde{\pi}$ such that $H(q,z,p) = \sum_{i=1}^n h_i(q,z)p_i + h_0(q,z)$. A reticular Legendrian map $\tilde{\pi}\circ i : (\tilde{\mathbb{L}^0},0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0) \to (\mathbb{R}^n \times \mathbb{R},0)$ is called *infinitesimal stable* if for any function germ f on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ there exists a fiber preserving function germ H on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ and a vector field v on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that v is tangent to $(\tilde{\mathbb{L}^0},0)$ and $X_f \circ i = X_H \circ i + i_* v$, where X_f and X_H are the contact hamiltonian vector field of f and H respectively and $i_* v$ is defined by $i_* v = (C_* v)\circ i$ for an extension $C \in C(J^1(\mathbb{R}^n,\mathbb{R}),0)$ of i.

Lemma 14.3 *For any one-parameter family of Legendrian equivalences $\bar{\Theta} : (J^1(\mathbb{R}^n,\mathbb{R}) \times \mathbb{R},0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0)$ with $\bar{\Theta}_0 = id$, there exists a fiber preserving function germ H on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that $X_H = \frac{d\bar{\Theta}}{dt}|_{t=0}$. Conversely for any fiber preserving function germ H on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$, the flow $\bar{\Theta}$ of X_H with the initial condition $\bar{\Theta}_0 = id$ is a one-parameter family of Legendrian equivalences.*

Theorem 14.4 *Let $\tilde{\pi}\circ i : (\tilde{\mathbb{L}^0},0) \to (J^1(\mathbb{R}^n,\mathbb{R}),0) \to (\mathbb{R}^n \times \mathbb{R},0)$ be a reticular Legendrian map with the generating family $F(x,y,q,z) \in \mathfrak{M}(r;k+n+1)$. Let $f = F|_{\{q=z=0\}}$. Then the following are equivalent.*

(1) *F is a reticular K-stable unfolding of f.*

(2) *$\tilde{\pi}\circ i$ is homotopically stable.*

(3) *$\tilde{\pi}\circ i$ is infinitesimal stable.*

(4) *For any function germ f on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$, there exists a fiber preserving function germ H on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that $f\circ i = H\circ i$.*

(5) *$\tilde{\pi}\circ i$ is stable.*

Proof. We shall prove (1)⇔(5), (1)⇒(2)⇒(3)⇒(4)⇒(1).

(1)⇒(5). Let $C_0 \in C_Z^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0)$ be an extension of i_0 and $\tilde{C}_0 = (q_{\tilde{C}_0}, z_{\tilde{C}_0}, p_{\tilde{C}_0}) \in C_Z^\alpha(U, J^1(\mathbb{R}^n,\mathbb{R}))$ be a representative of C_0. We shall construct the map (18) which maps a contact embedding near $\tilde{C}_0$ to a function near a representative of F.

For each $\tilde{C} \in C_Z^\alpha(U, J^1(\mathbb{R}^n,\mathbb{R}))$ we define

$$P_{\tilde{C}} = \{(Q,P;q,z,p) \in U \times J^1(\mathbb{R}^n,\mathbb{R}) \,|(z,q,p) = \tilde{C}(Q,P)\}$$

$$\pi_{\tilde{C}} : U \to \mathbb{R}^{2n} \ (\ (Q,P) \mapsto (Q, p_{\tilde{C}}(Q,P))\ ).$$

By taking some Legendrian equivalence of $\tilde{\pi} \circ i$ and shrinking U if necessary, we may assume $\pi_{\tilde{C}_0}$ is embedding. By Lemma 5.3 there exists a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ and a neighborhood U_1 of 0 in U and a convex neighborhood V of 0 in $\mathbb{R}^{2n}$ such that

$$N_{\tilde{C}_0} \to \mathrm{Emb(V,U)}\ (\ \tilde{\mathrm{C}} \mapsto (\pi_{\tilde{\mathrm{C}}}|_{\mathrm{U}_1})^{-1}|_{\mathrm{V}}\ )$$

is well defined and continuous. Let $\tilde{C} \in N_{\tilde{C}_0}$. Since $dz - pdq = -PdQ$ on $P_{\tilde{C}}$, there exists $H_{\tilde{C}} \in C^\infty(V,\mathbb{R})$ $(H_{\tilde{C}}(0,0) = 0)$ such that

$$P_{\tilde{C}}|_{(Q,p)\in V} = \{(Q, -\frac{\partial H_{\tilde{C}}}{\partial Q}(Q,p); -\frac{\partial H_{\tilde{C}}}{\partial p}(Q,p), H_{\tilde{C}} - \langle \frac{\partial H_{\tilde{C}}}{\partial p}(Q,p), p\rangle, p)\}.$$

Since $H_{\tilde{C}}(Q,p) = z - \langle q,p\rangle$ on $P_{\tilde{C}}$, the map

$$N_{\tilde{C}_0} \to C^\infty(V,\mathbb{R})\ (\tilde{C} \mapsto H_{\tilde{C}})$$

is continuous. Let $V' = V \cap \{Q_{r+1} = \cdots = Q_n = 0\}$. Now we consider the continuous map

$$\phi : N_{\tilde{C}_0} \to C^\infty(V' \times \mathbb{R}^{n+1},\mathbb{R})\ (\tilde{C} \mapsto \tilde{F}_{\tilde{C}}(x,y,q,z) = -z + H_{\tilde{C}}(x,0;y) + \langle y,q\rangle), \tag{18}$$

where $x = (x_1,\cdots,x_r), y = (y_1,\cdots,y_n), q = (q_1,\cdots,q_n)$. This map maps a contact embedding near $\tilde{C}_0$ to a function near $\tilde{F}_0 = \tilde{F}_{\tilde{C}_0}$. We may assume that $\tilde{F}_0$ is a representative of F. Since F is stable by hypothesis, there exists a neighborhood $N_{\tilde{F}_0}$ of $\tilde{F}_0$ such that for any $\tilde{G} \in N_{\tilde{F}_0}$ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{2n+1}}$ at $(0, y^0, q^0, z^0)$ and F are reticular K-equivalent for some $(0, y^0, q^0, z^0) \in V' \times \mathbb{R}^{n+1}$. Set $N'_{\tilde{C}_0} = \phi^{-1}(N_{\tilde{F}_0})$. Let $\tilde{C} \in N'_{\tilde{C}}$ be given. Take $(0, y^0, q^0, z^0) \in V' \times \mathbb{R}^{n+1}$ such that the preceding condition holds for $\tilde{F}_{\tilde{C}}$. If we denote $\{\tilde{L^{\tilde{C}}_\sigma}\}_{\sigma \subset I_r}$ the contact regular r-cubic configuration defined by $F_{\tilde{C}} = \tilde{F}_{\tilde{C}}|_{\mathbb{H}^r \times \mathbb{R}^{2n+1}}$ at $(0, y^0, q^0, z^0)$, then for each $\sigma \subset I_r$

$$
\begin{aligned}
\tilde{L^{\tilde{C}}_\sigma} &= \{(q_0 + q, z_0 + z, \frac{\partial F_{\tilde{C}}}{\partial q}(x, y^0 + y, q^0 + q, z^0 + z))| \\
&\qquad\qquad\qquad x_\sigma = \frac{\partial F_{\tilde{C}}}{\partial x_{I_r - \sigma}} = \frac{\partial F_{\tilde{C}}}{\partial y} = F_{\tilde{C}} = 0\} \\
&= \{(q_0 + q, H_{\tilde{C}}(x, 0; y_0 + y) + \langle y_0 + y, q_0 + q\rangle, y_0 + y)| \\
&\qquad\qquad\qquad x_\sigma = \frac{\partial H_{\tilde{C}}}{\partial x_{I_r - \sigma}} = \frac{\partial H_{\tilde{C}}}{\partial y} + q_0 + q = 0\} \\
&= \{(-\frac{\partial H_{\tilde{C}}}{\partial y}, H_{\tilde{C}} - \langle y_0 + y, \frac{\partial H_{\tilde{C}}}{\partial y}\rangle, y_0 + y)| x_\sigma = \frac{\partial H_{\tilde{C}}}{\partial x_{I_r - \sigma}}(x, 0; y_0 + y) = 0\} \\
&= \{(-\frac{\partial H_{\tilde{C}}}{\partial p}, H_{\tilde{C}} - \langle y_0 + p, \frac{\partial H_{\tilde{C}}}{\partial p}\rangle, y_0 + p)| \\
&\qquad\qquad Q_\sigma = \frac{\partial H_{\tilde{C}}}{\partial Q_{I_r - \sigma}}(Q; y_0 + p) = Q_{r+1} = \cdots = Q_n = 0, Q_{I_r - \sigma} \geq 0\} \\
&= \tilde{C}(\tilde{\tilde{L}}_\sigma \ at \ (0; 0, P^0)),
\end{aligned}
$$

where $(0; 0, P^0) = \tilde{C}^{-1}(q_0, z_0, y_0)$ and $\tilde{\tilde{L}}_\sigma = \{(Q, Z, P) \in J^1(\mathbb{R}^n, \mathbb{R}) | Q_\sigma = P_{I_r - \sigma} = Q_{r+1} = \cdots = Q_n = Z = 0, Q_{I_r - \sigma} \geq 0\}$. Since $F_{\tilde{C}}$ and F_0 are reticular K-equivalent, this implies that $\tilde{\pi} \circ \tilde{C}|_{\tilde{\mathbb{L}}^0 \ at \ (0;0,P^0)}$ and $\tilde{\pi} \circ i$ are Legendrian equivalent.

(5)⇒(1). Let a reticular Legendrian map $\tilde{\pi} \circ i$ be given and C_0 be an extension of i. Then we may assume that $C_0 \in C^\alpha_Z(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ by Lemma 14.2. By considering some Legendrian equivalence of $\tilde{\pi} \circ i$, we may assume that there exists a function germ $T_0 \in \mathfrak{M}^2(2n)$ such that

$$\{(Q, P; C_0(Q, P))\} = \{(Q, -\frac{\partial T_0}{\partial Q}(Q, p); -\frac{\partial T_0}{\partial p}(Q, p), T_0(Q, p) - \langle \frac{\partial T_0}{\partial p}(Q, p), p\rangle, p)\}$$

Define a generating family $F_0(x, y, q, z) \in \mathfrak{M}(r; n + n + 1)$ of $\tilde{\pi} \circ i$ by $F_0(x, y, q, z) = -z + T(x, 0; y) + \langle y, q\rangle$. It is enough to prove that F_0 is a reticular K-stable unfolding of $F_0|_{\{q=z=0\}}$.

Let $\tilde{F}_0 \in C^\infty(U, \mathbb{R})$ be a representative of F_0. By shrinking U if necessary, we may assume that there exist neighborhood U_1, U_2, U_3 of 0 in $(\mathbb{R}^n; Q), (\mathbb{R}^n; y), (\mathbb{R}^{n+1}; (q, z))$ respectively and $\tilde{T}_0(Q, y) \in C^\infty(U_1 \times U_2, \mathbb{R})$ such that the following conditions hold:

(a) $\tilde{T}_0$ is a representative of T_0,

(b) $U = (U_1 \cap \{Q_{r+1} = \cdots = Q_n = 0\}) \times U_2 \times U_3$ and $\tilde{F}_0(x, y, q, z) = -z + \tilde{T}_0(x, 0; y) + \langle y, q \rangle$,

(c) The map $U_1 \times U_2 \times U_3 \to U_1 \times U_2 \times \mathbb{R}^{n+1}$ $((Q, y, q, z) \mapsto (Q, y, \frac{\partial \tilde{T}_0}{\partial y}(Q, y) + q,\ -z + \tilde{T}(Q, y) + \langle y, q \rangle))$ is an embedding,

(d) The map $U_1 \times U_2 \to U_1 \times \mathbb{R}^n$ $((Q, y) \mapsto (Q, -\frac{\partial \tilde{T}_0}{\partial Q}(Q, y))$ is an embedding.

Define $\bar{F}_0 \in C^\infty(U_1 \times U_2 \times U_3, \mathbb{R})$ by $\bar{F}_0(Q, y, q, z) = -z + \tilde{T}(Q, y) + \langle y, q \rangle$. Since the map

$$C^\infty(U, \mathbb{R}) \to C^\infty(U_1 \times U_2 \times U_3, \mathbb{R}) \ (\ \tilde{F} \mapsto \bar{F}(Q, y, q, z) = \bar{F}_0(Q, y, q, z) + (\tilde{F} - \tilde{F}_0)(Q', y, q, z)\ )$$

$(Q' = (Q_1, \cdots, Q_{r+1}))$ is continuous, the map

$$C^\infty(U, \mathbb{R}) \to C^\infty(U_1 \times U_2 \times U_3, U_1 \times U_2 \times \mathbb{R}^{n+1}) \ (\ \tilde{F} \mapsto \phi_{\tilde{F}}(Q, y, q, z) = (Q, y, \frac{\partial \bar{F}}{\partial y}, \bar{F})\ )$$

is also continuous. Since $\phi_{\tilde{F}_0}$ is embedding by (c), there exist a neighborhood $N^1_{\tilde{F}_0}$ of $\tilde{F}_0$ and a neighborhood U' of 0 in $U_1 \times U_2 \times U_3$ and an open ball V around 0 in $U_1 \times U_2 \times \mathbb{R}^{n+1}$ such that

$$N^1_{\tilde{F}_0} \to \mathrm{Emb}(V, U_1 \times U_2 \times U_3) \ (\ \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_V\ )$$

is well defined and continuous. Let $V_1 = V \cap (U_1 \times U_2 \times \{0\})$. Then the map

$$N^1_{\tilde{F}_0} \to \mathrm{Emb}(V_1, U_1 \times U_2 \times U_3) \ (\ \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}\ )$$

is also continuous. We denote $(\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}(Q, y)$ by $(Q, y, q_{\tilde{F}}(Q, y), z_{\tilde{F}}(Q, y))$. Then the map

$$N^1_{\tilde{F}_0} \to C^\infty(V_1, U_1 \times \mathbb{R}^n) \ (\ \tilde{F} \mapsto \psi_{\tilde{F}}(Q, y) = (Q, -\frac{\partial \bar{F}}{\partial Q}(Q, y, q_{\tilde{F}}(Q, y), z_{\tilde{F}}(Q, y)))\ )$$

is also continuous. Since $\psi_{\tilde{F}_0}$ is embedding by (d), there exist a neighborhood $N^2_{\tilde{F}_0}$ of $\tilde{F}_0$ and a neighborhood V_2 of 0 in V_1 and an open ball W around 0 in $U_1 \times \mathbb{R}^n$ such that

$$N^2_{\tilde{F}_0} \to \mathrm{Emb}(W, V_1) \ (\ \tilde{F} \mapsto (\psi_{\tilde{F}}|_{V_2})^{-1}|_W\ )$$

is well defined and continuous. We denote $(\psi_{\tilde{F}}|_{V_2})^{-1}|_W(Q,P)$ by $(Q, y_{\tilde{F}}(Q,P))$. Then the map

$$N^2_{\tilde{F}_0} \to \mathrm{Emb}(W, U_1 \times U_2 \times U_3) \ (\ \tilde{F} \mapsto (\ (Q,P) \mapsto (Q, y_{\tilde{F}}(Q,P), q_{\tilde{F}}(Q, y_{\tilde{F}}), z_{\tilde{F}}(Q, y_{\tilde{F}}))\ )$$

is also continuous. Hence the map

$$N^2_{\tilde{F}_0} \to C^{\alpha}_Z(W, J^1(\mathbb{R}^n, \mathbb{R}))\ (\tilde{F} \mapsto \tilde{C}_{\tilde{F}}(Q,P) = (q_{\tilde{F}}, z_{\tilde{F}}, \frac{\partial \bar{F}}{\partial q}(Q, y_{\tilde{F}}, q_{\tilde{F}}, z_{\tilde{F}}))\ ) \tag{19}$$

is well defined and continuous. This map maps a function near $\tilde{F}_0$ to a contact embedding near a representative $\tilde{C}_{\tilde{F}_0}$ of C_0. Hence by hypothesis, there exists a neighborhood $N^3_{\tilde{F}_0}$ of $\tilde{F}_0$ such that for any $\tilde{F} \in N^3_{\tilde{F}_0}$ $\tilde{\pi} \circ (\tilde{C}_{\tilde{F}}|_{\tilde{\mathbb{L}^0} \text{ at } (Q^0,P^0)})$ and $\tilde{\pi} \circ i$ are Legendrian equivalent for some $(Q^0, P^0) = (0; 0, \cdots, 0, P^0_{r+1}, \cdots, P^0_n) \in W$. Therefore if we set $(Q^0, y^0, q^0, z^0) = (Q^0, y_{\tilde{F}}(Q^0, P^0), q_{\tilde{F}}(Q^0, y_{\tilde{F}}), z_{\tilde{F}}(Q^0, y_{\tilde{F}}))$ then $\tilde{F}$ at $(0, y^0, q^0, z^0)$ and F_0 are reticular K-equivalent. ■

The construction of (19) is summarized in the following diagram:

$$\begin{array}{cccccc}
U_1 \times U_2 \times U_3 & = U_1 \times U_2 \times U_3 & & U_1 \times \mathbb{R}^n & \supset & W \\
(Q,y,q,z) & (Q, y, q_{\tilde{F}}, z_{\tilde{F}}) & \to & (Q, -\frac{\partial \bar{F}}{\partial Q}(Q, y, q_{\tilde{F}}, z_{\tilde{F}})) & = & (Q,P) \\
\phi_{\tilde{F}} \downarrow & \uparrow & \nearrow & & & \downarrow \tilde{C}_{\tilde{F}} \\
(Q, y, \frac{\partial \bar{F}}{\partial y}, \bar{F}) & (Q, y, 0, 0) & \psi_{\tilde{F}} & & & (q_{\tilde{F}}, z_{\tilde{F}}, \frac{\partial \bar{F}}{\partial q}(Q, y_{\tilde{F}}, q_{\tilde{F}}, z_{\tilde{F}})) \\
U_1 \times U_2 \times \mathbb{R}^{n+1} \supset & V_1 \times \{0\} & & & & J^1(\mathbb{R}^n, \mathbb{R})
\end{array}$$

(1)⇒(2). Let $\bar{i} : (\tilde{\mathbb{L}^0} \times \mathbb{R}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)((Q,P,t) \mapsto \bar{i}_t(Q,P))$ be a reticular Legendrian deformation of i. Take a one-parameter family of contact diffeomorphisms $\bar{C} : (J^1(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)((Q,Z,P,t) \mapsto \bar{C}_t(Q,Z,P))$ such that $\bar{i}_t = \bar{C}_t|_{\tilde{\mathbb{L}^0}}$ for t near 0. By using analogous methods of the proof of Lemma 12.3, we may assume that $\bar{C}_t \in C^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ for t near 0. Moreover we may assume that there exists a smooth function germ $\bar{T} : (\mathbb{R}^{2n} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)((Q, p, t) \mapsto \bar{T}_t(Q, p))$ such that

$$\{(Q, P; \bar{C}_t(Q, 0, P); t)\} = \{(Q, -\frac{\partial \bar{T}_t}{\partial Q}(Q, p); -\frac{\partial \bar{T}_t}{\partial p}(Q, p), \bar{T}_t(Q, p) - \langle \frac{\partial \bar{T}_t}{\partial p}(Q, p), p\rangle, p; t)\}.$$

Define $F(x, y, q, z, t) \in \mathcal{E}(r; n + n + 1 + 1)$ by $F(x, y, q, z, t) = F_t(x, y, q, z) = -z + \bar{T}_t(x, 0; y) + \langle y, q\rangle$. Then F_t is a generating family of $\bar{i}_t$ for t near 0. By hypothesis there exists a one-parameter family of reticular K-equivalences $\{(a_t, \Psi_t)\}$ such that $F_t = a_t \cdot F_0 \circ \Psi_t$ for t near 0. If Ψ_t is written in the form $\Psi_t(x, y, q, z) = (*, *, g_t^1(q, z), \cdots, g_t^{n+1}(q, z))$ then we can prove that $\bar{i}_t(\tilde{L_\sigma^0}) = \bar{g}_t^* \circ \bar{i}_0(\tilde{L_\sigma^0})$ for all $\sigma \subset I_r$ by the same method of the proof of theorem 12.6-(3)a. Define the one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\bar{\phi}_t\}$ by $\bar{\phi}_t = C_0^{-1} \circ \bar{g}_t^{*-1} \circ C_t|_{\tilde{\mathbb{L}^0}}$. Then we have $\bar{i}_t = \bar{g}_t^* \circ \bar{i}_0 \circ \bar{\phi}_t$ for t near 0.

(2)⇒(3). Let a function germ f on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ be given. Take an extension $C \in C(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ of i. Let X_f be the contact hamiltonian vector field of f on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ and $\bar{C} = \{\bar{C}_t\}$ be the flow of X_f with the initial condition $\bar{C}_0 = C$. Since $\bar{i} = \{\bar{i}_t = \bar{C}_t|_{\tilde{\mathbb{L}^0}}\}$ is a reticular Legendrian deformation of i, there exists a one-parameter family of Legendrian equivalences $\bar{\Theta} = \{\bar{\Theta}_t\}$ with $\bar{\Theta}_0 = id_{(J^1(\mathbb{R}^n, \mathbb{R}), 0)}$ and a one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\bar{\phi}_t\}$ of $id_{(\tilde{\mathbb{L}^0}, 0)}$ such that $\bar{i}_t = \bar{\Theta}_t \circ i \circ \bar{\phi}_t$ for t near 0. Let $\bar{\Phi} : (J^1(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)((Q, Z, P, t) \mapsto \bar{\Phi}_t(Q, Z, P))$ be an extension of $\bar{\phi}$ and set $v = \frac{d\bar{\Phi}_t}{dt}|_{t=0}$. Then v is tangent to $(\tilde{\mathbb{L}^0}, 0)$ and by Lemma 14.3 there exists a fiber preserving function germ H on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ such that $\frac{d\bar{\Theta}_t}{dt}|_{t=0} = X_H$. Then

$$X_f \circ i = \frac{d\bar{C}_t}{dt}|_{t=0} \circ i = \frac{d\bar{\Theta}_t}{dt}|_{t=0} \circ i + (C_* \frac{d\bar{\Phi}_t}{dt}|_{t=0}) \circ i = X_H \circ i + i_* v.$$

This implies that $\bar{\pi} \circ i$ is infinitesimal stable.

(3)⇒(4). Let a function germ f on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ be given. By hypothesis, there exists a fiber preserving function germ H on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ and a vector field v on $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that v is tangent to $\tilde{\mathbb{L}^0}$ and $X_f \circ i = X_H \circ i + i_* v$. Set $i_\sigma = i|_{\tilde{L^0_\sigma}}, v_\sigma = v|_{\tilde{L^0_\sigma}}$ for each $\sigma \subset I_r$. Then it is easy to prove that $(f-H)\circ i_\sigma = 0$ because $X_f \circ i_\sigma = X_H \circ i_\sigma + (i_\sigma)_* v_\sigma$. Therefore $f \circ i = H \circ i$.

(4)⇒(1)Take an extension $C=(q,z,p) \in C^\alpha_Z(J^1(\mathbb{R}^n,\mathbb{R}),0)$ of i. We may assume that there exists a function germ $T(Q,p)$ on $(\mathbb{R}^{2n},0)$ such that

$$\{(Q,P;C(Q,P))\} = \{(Q,-\frac{\partial T}{\partial Q}(Q,p);-\frac{\partial T}{\partial p}(Q,p),T(Q,p)-\langle\frac{\partial T}{\partial p}(Q,p),p\rangle,p)\}$$

Define a generating family $F(x,y,q,z) \in \mathcal{E}(r;n+n+1)$ of $\tilde{\pi}\circ i$ by $F(x,y,q,z) = -z + T(x,0;y)+\langle y,q\rangle$. Since $(Q,p)\mapsto(q(Q,p),p)$ is invertible, there exists $I \subset \{1,\cdots,n\}(|I|=r)$ such that $\phi:(x,y)\mapsto(q_I(x,0;y),y)$ is also invertible. On the other hand since $(Q,p)\mapsto(Q,P(Q,p))$ is invertible, $\psi:(x,y)\mapsto(x,P(x,0;y))$ is also invertible. Set $C'=\phi\circ\psi^{-1}$. Let $f := F|_{q=z=0}$. Set $g(q,z,y)=g'(q,y) := f\circ\phi^{-1}(q_I,y)$. Since $C(x,0;P)|_{x_1P_1=\cdots=x_rP_r=0} = i(x,0;P)$, there exists a fiber preserving function germ $H(q,z,p)=\sum_{i=1}^n h_i(q,z)p_i + h_0(q,z)$ such that

$$g\circ C(x,0;P)|_{x_1P_1=\cdots=x_rP_r=0} = H\circ C(x,0;P).$$

Therefore there exist $a_1,\cdots,a_r \in \mathcal{E}(r;n)$ such that

$$g\circ C(x,0;P) = H\circ C(x,0;P)+\sum_{j=1}^r x_jP_ja_j(x,P).$$

Hence

$$\begin{aligned}
& f(x,y) \\
= \ & (f\circ\phi^{-1})\circ(\phi\circ\psi^{-1})\circ\psi = g'\circ C'\circ\psi = g\circ C(x,0;P(x,0;y))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} h_i(q(x,0;y), z(x,0;y))y_i + h_0(q(x,0;y), z(x,0;y)) + \sum_{j=1}^{r} x_j P_j(x,0;y)a'_j(x,y) \\
&= \sum_{i=1}^{n} h_i(-\frac{\partial T}{\partial y}, T - \langle \frac{\partial T}{\partial y}, y\rangle)(x,0;y)y_i + h_0(-\frac{\partial T}{\partial y}, T - \langle \frac{\partial T}{\partial y}, y\rangle)(x,0;y) \\
&\qquad + \sum_{j=1}^{r} x_j(-\frac{\partial T}{\partial x_j}(x,0;y))a'_j(x,y) \\
&\equiv \sum_{i=1}^{n} h_i(0)y_i + h_0(0) \bmod \langle \frac{\partial T}{\partial y}(x,0;y), (T - \langle \frac{\partial T}{\partial y}, y\rangle)(x,0;y), x\frac{\partial T}{\partial x}(x,0;y)\rangle_{\mathcal{E}(r;n)}, \\
&\equiv \sum_{i=1}^{n} h_i(0)y_i + h_0(0) \bmod \langle T(x,0;y), x\frac{\partial T}{\partial x}(x,0;y), \frac{\partial T}{\partial y}(x,0;y)\rangle_{\mathcal{E}(r;n)},
\end{aligned}
$$

where $a'_j(x,0;y) = a_j(x,0;-\frac{\partial T}{\partial Q}(x,0;y))$ $(j = 1,\cdots,r)$. This implies that F is a reticular K-infinitesimal versal unfolding of f. Hence F is a reticular K-stable unfolding of f. ■

15 Classification of function germs

We now start the classification of function germs with reticular K-codimension lower than 8 with respect to reticular K-equivalence. By Lemma 7.1 and Lemma 7.2 in Section 7, we have only to classify residual singularities, that is function germs in $\mathfrak{M}(r;k)^2$ whose restriction to $x = 0$ is an element of $\mathfrak{M}(0;k)^3$. $j_{y^\alpha,x^\beta} f(0) \approx g$ denotes quasihomogeneous equivalence of jets and $f \approx g$ means f is reticular K-equivalent to g and $\Rightarrow$ means 'see' or 'implies'.

Let $f \in \mathfrak{M}(r;k)^2$ be a residual singularity with the reticular K-codimension lower than 8. We set $\phi(y) = f(0,y) \in \mathfrak{M}(0;k)^3$.

<u>The case $r = 1, k = 0$.</u> $f \approx x^n$ $(n = 2,\cdots,7)$.

<u>The case $r = 1, k = 1$.</u> One of the five:

$$
\begin{array}{lll}
j^2 f(0) \approx xy + x^2 \text{ or } xy & \Rightarrow & f \approx xy + \varepsilon y^n \ (\varepsilon^{n+1} = 1, n = 3,\cdots,7), \\
j_{y^3,x^2} f(0) \approx y^3 + x^2 & \Rightarrow & f \approx y^3 + x^2, \\
j_{y^3,x^2} f(0) \approx x^2 & \Rightarrow & (15), \\
j_{y^3,x^2} f(0) \approx y^3 & \Rightarrow & (17), \\
j_{y^3,x^2} f(0) \approx 0 & \Rightarrow & (19).
\end{array}
$$

(15) $j_{y^3,x^2} f(0) = x^2 \Rightarrow$ one of the five:

$$\begin{array}{lll}
j_{y^4,x^2}f(0)\approx y^4+axy^2\pm x^2(a^2\neq 4) & \Rightarrow & f\approx y^4+axy^2\pm x^2(a^2\neq 4),\\
j_{y^4,x^2}f(0)\approx (y^2\pm x)^2 & \Rightarrow & f\approx y^5+(y^2\pm x)^2 \text{ or } y^6\pm(y^2\pm x)^2,\\
j_{xy^2,x^2}f(0)\approx xy^2\pm x^2 & \Rightarrow & f\approx y^5+xy^2\pm x^2,\\
j_{y^5,x^2}f(0)\approx y^5+x^2 & \Rightarrow & f\approx y^5\pm xy^3+x^2,\\
j_{y^5,x^2}f(0)\approx x^2 \text{ or } 0 & \Rightarrow & (16).
\end{array}$$

(16) $j_{y^6,x^2}f(0)$ is adjacent to $y^6+axy^3\pm x^2(a^2\neq\pm 4)$ and hence

the codimension of $f \geq \dim\mathcal{E}(1;1)/(\langle x\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},y\frac{\partial f}{\partial y}\rangle_{\mathbb{R}}+\langle x^3,x^2y,xy^4,y^7\rangle_{\mathcal{E}(1;1)})\geq 12-3=9.$

(17) $j_{y^3,x^2}f(0)=y^3 \Rightarrow$ one of the five:

$$\begin{array}{lll}
j^3f(0)\approx y^3+ax^2y+2x^3(a\neq -3) & \Rightarrow & f\approx y^3+ax^2y+2x^3(a\neq -3),\\
j^3f(0)\approx y^3+xy^2 & \Rightarrow & f\approx y^3+xy^2\pm x^4 \text{ or } y^3+xy^2\pm x^5,\\
j^3f(0)\approx y^3+x^2y & \Rightarrow & f\approx y^3+x^2y,\\
j_{y^3,x^4}f(0)\approx y^3+x^4 & \Rightarrow & f\approx y^3\pm x^3y+x^4,\\
j_{y^3,x^4}f(0)\approx y^3 & \Rightarrow & (18).
\end{array}$$

(18) f is adjacent to $y^3\pm x^4y+x^5$ and this has codimension 9.

(19) $j_{y^3,x^2}f(0)=0 \Rightarrow$ one of the four:

$$\begin{array}{lll}
j^3f(0)\approx xy^2\pm x^3 & \Rightarrow & f\approx y^4\pm xy^2\pm x^3 \text{ or } y^5\pm xy^2\pm x^3,\\
j^3f(0)\approx xy^2 & \Rightarrow & f\approx y^4\pm xy^2\pm x^4,\\
j^3f(0)\approx x^2y & \Rightarrow & f\approx y^4+xy^3\pm x^2y,\\
j^3f(0)\approx x^3 \text{ or } 0 & \Rightarrow & (20).
\end{array}$$

(20) $j^3f(0)=x^3$ or $0 \Rightarrow f$ is adjacent to $y^4+xy^3\pm x^3$ and this has codimension 8.

The case $r=1,k=2$ One of the two:

$$\begin{array}{lll}
j^3\phi\neq 0 & \Rightarrow & (21),\\
j^3\phi=0 & \Rightarrow & (36).
\end{array}$$

(21). $j^3\phi\neq 0 \Rightarrow$ on of the four:

$$\begin{array}{lll}
\phi\in D_4 & \Rightarrow & (22),\\
\phi\in D_5 & \Rightarrow & (26),\\
\phi\in D_6 & \Rightarrow & (30),\\
\phi\in E_6 & \Rightarrow & (33).
\end{array}$$

(22). $\phi=y_1^2y_2\pm y_2^3 \Rightarrow$ one of the four:

$$\begin{array}{lllll}
j_{y_1^2y_2,y_2^3,xy_2}f(0) & \approx & y_1^2y_2\pm y_2^3+xy_1+axy_2, & & \\
 & & a^2\pm 1\neq 0 & \Rightarrow & f\approx y_1^2y_2\pm y_2^3+xy_1+axy_2,\\
j_{y_1^2y_2,y_2^3,xy_2}f(0) & \approx & y_1^2y_2\pm y_2^3\pm xy_2 & \Rightarrow & (23),\\
j_{y_1^2y_2,y_2^3,x^2}f(0) & \approx & y_1^2y_2\pm y_2^3+x^2 & \Rightarrow & (24),\\
j_{y_1^2y_2,y_2^3,x^2}f(0) & \approx & y_1^2y_2\pm y_2^3 & \Rightarrow & (25).
\end{array}$$

(23) $j_{y_1^2y_2,y_2^3,xy_2}f(0) = y_1^2y_2 \pm y_2^3 \pm xy_2 \Rightarrow f \approx y_1^2y_2 \pm y_2^3 \pm xy_2 + xy_1^2$ or $y_1^2y_2 \pm y_2^3 \pm xy_2 + xy_1^3$.

(24). $j_{y_1^2y_2,y_2^3,x^2}f(0) = y_1^2y_2 \pm y_2^3 + x^2 \Rightarrow f \approx y_1^2y_2 \pm y_2^3 + x^2 \pm xy_2^2$.

(25) $f \in \mathfrak{M}^3(1;2)$. Therefore the codimension of $f \geq \mathcal{E}(1;2)/(\langle \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}\rangle_{\mathbb{R}} + \mathfrak{M}^3(1;2)) \geq$

$10 - 2 = 8$.

(26). $\phi = y_1^2y_2 + y_2^4 \Rightarrow$ one of the three:

$$\begin{array}{lcll} j_{y_1^2y_2,y_2^4,xy_2}f(0) & \approx & y_1^2y_2 + y_2^4 \pm xy_2 & \Rightarrow (27), \\ j_{y_1^2y_2,y_2^4,xy_1}f(0) & \approx & y_1^2y_2 + y_2^4 + xy_1 & \Rightarrow (28), \\ j_{y_1^2y_2,y_2^4,xy_1}f(0) & \approx & y_1^2y_2 + y_2^4 & \Rightarrow (29). \end{array}$$

(27). $j_{y_1^2y_2,y_2^4,xy_2}f(0) = y_1^2y_2 + y_2^4 \pm xy_2 \Rightarrow f \approx y_1^2y_2 + y_2^4 \pm xy_2 + xy_1$ or $y_1^2y_2 + y_2^4 \pm xy_2 + xy_1^2$.

(28) $j_{y_1^2y_2,y_2^4,xy_1}f(0) = y_1^2y_2 + y_2^4 + xy_1 \Rightarrow f \approx y_1^2y_2 + y_2^4 + xy_1 \pm xy_2^2$.

(29) $j_{y_1^2y_2,y_2^4,xy_1}f(0) = y_1^2y_2 + y_2^4$. Then f is adjacent to $y_1^2y_2 + y_2^4 + \varepsilon x^2 + xy_2^2(a + \delta y_2)$ $(a^2 \neq 4\varepsilon)$

and this has codimension 9.

(30) $\phi = y_1^2y_2 \pm y_2^5 \Rightarrow$ one of the two:

$$\begin{array}{lcll} j_{y_1^2y_2,y_2^4,xy_2}f(0) & \approx & y_1^2y_2 \pm xy_2 & \Rightarrow (31), \\ j_{y_1^2y_2,y_2^4,xy_2}f(0) & \approx & y_1^2y_2 & \Rightarrow (32). \end{array}$$

(31). $j_{y_1^2y_2,y_2^4,xy_2}f(0) = y_1^2y_2 \pm xy_2 \;\Rightarrow\; f \approx y_1^2y_2 \pm y_2^5 \pm xy_2 + xy_1$.

(32) $j_{y_1^2y_2,y_2^4,xy_2}f(0) = y_1^2y_2$. Then f is adjacent to $y_1^2y_2 + \varepsilon y_2^5 + xy_1 + xy_2^2(a + \delta y_2)$ $(a^2 \neq -\varepsilon)$

and this has codimension 10

(33). $\phi = y_1^3 + y_2^4. \Rightarrow$ one of the two:

$$\begin{array}{lcll} j_{y_1^3,y_2^4,xy_2}f(0) & \approx & = y_1^3 + y_2^4 \pm xy_2 & \Rightarrow (34), \\ j_{y_1^3,y_2^4,xy_2}f(0) & \approx & = y_1^3 + y_2^4 & \Rightarrow (35). \end{array}$$

(34). $j_{y_1^3,y_2^4,xy_2}f(0) = y_1^3 + y_2^4 \pm xy_2 \;\Rightarrow\; f \approx y_1^3 + y_2^4 \pm xy_1 + xy_2$

(35). $j_{y_1^3,y_2^4,xy_2}f(0) = y_1^3 + y_2^4$. Then f is adjacent to $y_1^3 + y_2^4 \pm xy_1 \pm xy_2^2$ and this has

codimension 8.

(36) Since $\phi \in \mathfrak{M}^4(0;2)$, we have the codimension of $\phi \geq \mathcal{E}(0;2)/(\langle \frac{\partial \phi}{\partial y_1}, \frac{\partial \phi}{\partial y_2}\rangle_{\mathbb{R}} + \mathfrak{M}^4(0;2)) \geq$

$10-2=8$. Therefore f has codimension ≥ 9.

<u>**The case** $r=1, k\geq 3$.</u> We need only to prove that the codimension of $f\geq 8$ in the case $r=1, k=3$. Since the codimension of $\phi \geq \mathcal{E}(0;3)/(\langle \frac{\partial\phi}{\partial y_1}, \frac{\partial\phi}{\partial y_2}, \frac{\partial\phi}{\partial y_3}\rangle_{\mathbb{R}} + \mathfrak{M}^3(0;3)) \geq 10-3=7$.

Therefore the codimension of $f\geq 7+1=8$.

<u>**The case** $r=2, k=0$.</u> One of the five:

$$\begin{array}{lll} j^2f(0)\approx x_1^2+ax_1x_2\pm x_2^2(a^2\neq \pm 4) & \Rightarrow & f\approx x_1^2+ax_1x_2\pm x_2^2(a^2\neq\pm 4),\\ j^2f(0)\approx (x_1\pm x_2)^2 & \Rightarrow & f\approx (x_1\pm x_2)^2\pm x_2^n(n=3,\cdots,6),\\ j^2f(0)\approx x_1^2\pm x_1x_2 \text{ or } \pm x_1x_2+x_2^2 \text{ or } x_1x_2 & \Rightarrow & f\approx x_1^n\pm x_1x_2\pm x_2^m\\ & & (n,m\geq 2,\ 5\leq n+m\leq 8),\\ j^2f(0)\approx x_1^2 \text{ or } x_2^2 & \Rightarrow & (37),\\ j^2f(0)\approx 0 & \Rightarrow & (40). \end{array}$$

(37) We investigate only the case $j^2f(0)=x_1^2$. But the case $j^2f(0)=x_2^2$ is calculated analogously.

One of the two:

$$\begin{array}{lll} j_{x_1^2,x_2^3}f(0)\approx x_1^2\pm x_2^3 & \Rightarrow & f\approx x_1^2\pm x_1x_2^2\pm x_2^3 \text{ or } x_1^2\pm x_2^3,\\ j_{x_1^2,x_2^3}f(0)\approx x_1^2 & \Rightarrow & (38). \end{array}$$

(38) One of the three:

$$\begin{array}{lll} j_{x_1^2,x_2^4}f(0)\approx x_1^2+ax_1^2x_2\pm x_2^4 & \Rightarrow & f\approx x_1^2+ax_1^2x_2\pm x_2^4\pm x_1x_2^3,\\ j_{x_1^2,x_2^4}f(0)\approx x_1^2\pm x_1^2x_2 & \Rightarrow & f\approx x_1^2\pm x_1^2x_2\pm x_2^5,\\ j_{x_1^2,x_2^4}f(0)\approx x_1^2 & \Rightarrow & (39). \end{array}$$

(39) $j_{x_1^2,x_2^4}f(0)=x_1^2 \Rightarrow f$ is adjacent to $x_1^2\pm x_1x_2^3\pm x_2^5$ and this has codimension 8.

(40) $j^2f(0)=0 \Rightarrow$ Since $f\in\mathfrak{M}^3(2,0)$, the codimension of $f\geq \mathcal{E}(2,0)/(\langle x_1\frac{\partial f}{\partial x_1}, x_2\frac{\partial f}{\partial x_2}\rangle_{\mathbb{R}} + \mathfrak{M}^4(2;0))\geq 10-2=8$.

<u>**The case** $r=2, k=1$.</u> One of the five:

$$\begin{array}{lll} j^2f(0)\approx x_1y\pm x_2y\pm x^2 \text{ or } x_1y\pm x_2y & \Rightarrow & f\approx y^n\pm x_1y\pm x_2y+x_2^2\\ & & (n\geq 3, m\geq 2, m+n\leq 8),\\ j^2f(0)\approx x_1y+x^2 \text{ or } x_2y+x_1^2 & \Rightarrow & (41),\\ j^2f(0)\approx x_1y \text{ or } x_2y & \Rightarrow & (43),\\ j_{x_1^2,x_2^2,y^3}f(0)\approx x_1^2+ax_1x_2\pm x_2^2(a^2\neq\pm 4) & \Rightarrow & f\approx y^3+\varepsilon x_2^2y+x_1^2+ax_1x_2+\delta x_2^2\\ & & (a^2\neq 4\delta),\\ \text{others} & \Rightarrow & (44). \end{array}$$

(41) We investigate only the case $j^2f(0) = x_1y + x^2$. But the case $j^2f(0) = x_2y + x_1^2$ is calculated analogously.

One of the four:

$$\begin{array}{lll} j_{y^3,x_1y,x_2^2}f(0)f \approx y^3 \pm x_1y + x_2^2 & \Rightarrow & f \approx y^3 \pm x_1y \pm x_2y^2 + x_2^2, \\ j_{y^4,x_1y,x_2^2}f(0) \approx y^4 + ax_2y^2 + x_1y \pm x_2^2 & \Rightarrow & f \approx y^4 + ax_2y^2 \pm x_2^2y + x_1y \pm x_2^2, \\ j_{y^4,x_1y,x_2^2}f(0) \approx x_2y^2 \pm x_1y \pm x_2^2 & \Rightarrow & y^5 \pm x_2y^2 \pm x_1y + x_2^2, \\ j_{y^4,x_1y,x_2^2}f(0) \approx \pm x_1y + x_2^2 & \Rightarrow & (42). \end{array}$$

(42) $j_{y^4,x_1y,x_2^2}f(0) = \pm x_1y + x_2^2 \Rightarrow f$ is adjacent to $y^5 \pm x_2y^3 \pm x_1y + x_2^2$ and this has codimension 8.

(43) $j^2f(0) = x_1y \Rightarrow f$ is adjacent to $y^3 + ax_2^2y + 2x_2^3 \pm x_2^2y^2 \pm x_1y (a \neq -3)$ and this has codimension 8.

$j^2f(0) = x_2y \Rightarrow f$ is adjacent to $y^3 + ax_1^2y + 2x_1^3 \pm x_1^2y^2 \pm x_2y (a \neq -3)$ and this has codimension 8.

(44) $j^2f(0)$ is adjacent to $f_0 = (x_1 \pm x_2)^2$ or $x_1^2 \pm x_1x_2$ or $\pm x_1x_2 + x_2^2 \Rightarrow f$ is adjacent to $f_0 + y^3 + ax_2^2y \pm 2x_2^3 (a \neq -3)$ and this has codimension 8.

The classification list of singularities with reticular K-codimension lower than 8

$r = 1$

k	Normal form	codim	Conditions	Notation
0	x^n	n	$n = 2, \cdots, 7$	B_n
1	$xy + \varepsilon y^n$	n	$\varepsilon^{n+1} = 1, n = 3, \cdots, 7$	C_n^{ε}
	$y^3 + x^2$	4		F_4
	$y^4 + axy^2 \pm x^2$	6	$a^2 \neq \pm 4$	$K_{4,2}^{\pm,a}$
	$y^5 + (y^2 \pm x)^2$	6		$K_{1,1}^{\#,\pm}$
	$y^6 + \varepsilon(y^2 + \delta x)^2$	7		$K_{1,2}^{\#,\varepsilon,\delta}$
	$y^5 \pm xy^3 + x^2$	7		$K_{5,3}^{1,\pm}$
	$y^3 + ax^2y + 2x^3$	6	$a \neq -3$	$F_{1,0}^{a}$
	$y^3 + xy^2 \pm x^4$	6		$F_6^{\pm}$
	$y^3 + xy^2 \pm x^5$	7		$F_7^{\pm}$
	$y^3 \pm x^2y$	6		$F_{1,0}^{',\pm}$
	$y^3 \pm x^3y + x^4$	7		$F_7^{',\pm}$
	$y^4 + \varepsilon xy^2 + \delta x^3$	6		$K_{4,3}^{\varepsilon,\delta}$
	$y^5 + xy^2 \pm x^2$	6		$K_{5,2}^{\pm}$
	$y^5 + xy^2 \pm x^3$	7		$K_{5,3}^{\pm}$
	$y^4 + \varepsilon xy^2 + \delta x^4$	7		$K_{4,4}^{\varepsilon,\delta}$
	$y^4 + xy^3 \pm x^2y$	7		$K_{4,2}^{2,\pm}$
2	$y_1^2y_2 \pm y_2^3 + xy_1 + axy_2$	6	$a^2 \pm 1 \neq 0$	$D_{4,1}^{\pm,a}$
	$y_1^2y_2 + \varepsilon y_2^3 + \delta xy_2 + xy_1^2$	6		$D_{4,2}^{\varepsilon,\delta}$
	$y_1^2y_2 + \varepsilon y_2^3 + \delta xy_2 + xy_1^3$	7		$D_{4,3}^{\varepsilon,\delta}$
	$y_1^2y_2 + \varepsilon y_2^3 + \delta xy_2^2 + x^2$	7		$D_4^{2,\varepsilon,\delta}$
	$y_1^2y_2 + y_2^4 + xy_1 \pm xy_2$	6		$D_{5,1}^{\pm}$
	$y_1^2y_2 + y_2^4 + \varepsilon xy_1^2 + \delta xy_2$	7		$D_{5,2}^{\varepsilon,\delta}$
	$y_1^2y_2 + y_2^4 + xy_1 \pm xy_2^2$	7		$D_5^{1,\pm}$
	$y_1^2y_2 + \varepsilon y_2^5 + xy_1 + \delta xy_2$	7		$D_{6,1}^{\varepsilon,\delta}$
	$y_1^3 + y_2^4 \pm xy_1 + xy_2$	7		$E_{6,0}^{\pm}$

where $\varepsilon = \pm 1, \delta = \pm 1$.

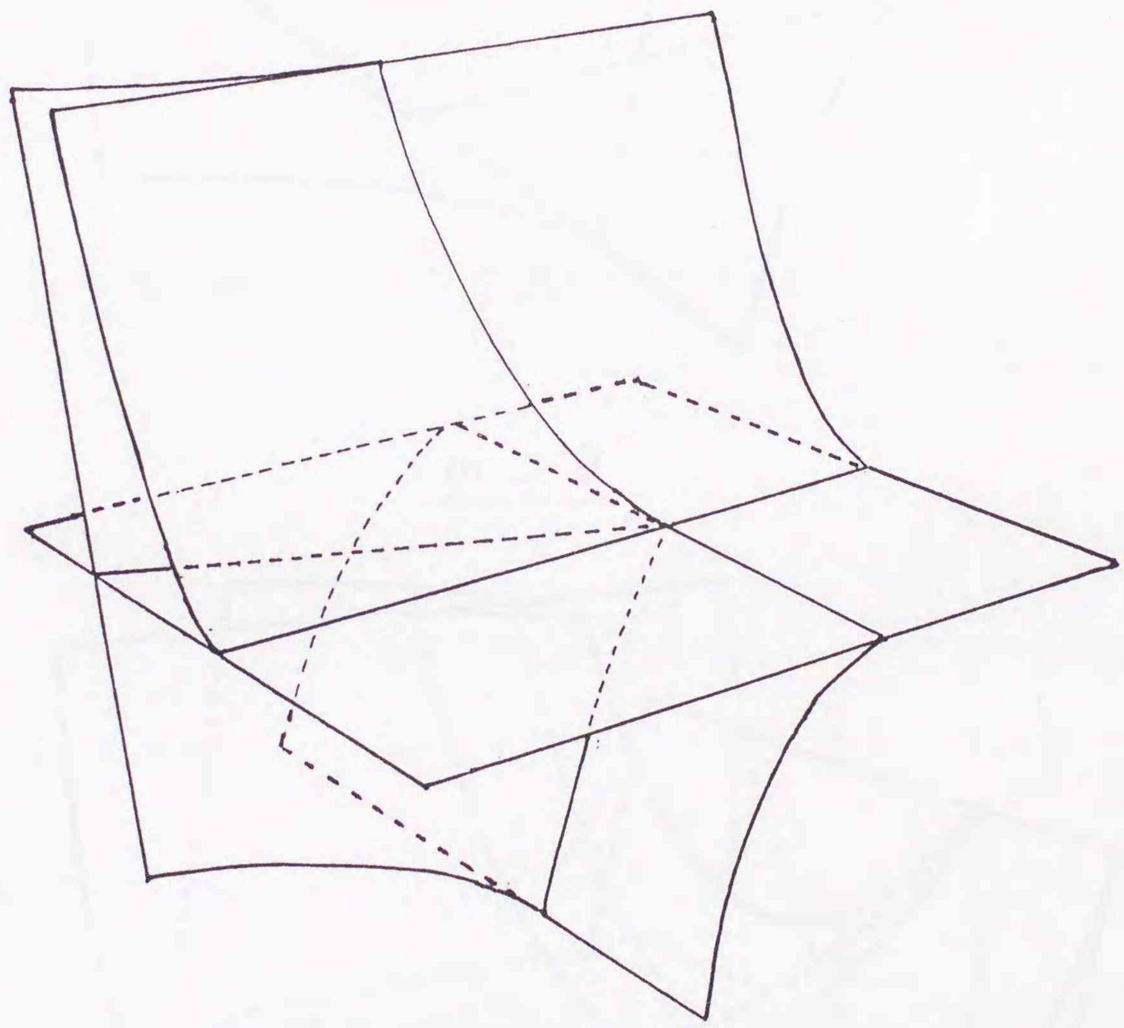
$r = 2$

k	Normal form	codim	Conditions	Notation
0	$x_1^2 + ax_1x_2 \pm x_2^2$	4	$a^2 \neq \pm 4$	$B_{2,2}^{\pm,a}$
	$(x_1 + \varepsilon x_2)^2 + \delta x_2^n$	$n+1$	$n = 3, \cdots, 6$	$B_{2,2,n}^{\varepsilon,\delta}$
	$x_1^n + \varepsilon x_1x_2 + \delta x_2^m$	$n+m-1$	$n, m \geq 2,\ 5 \leq n+m \leq 8$	$B_{n,m}^{\varepsilon,\delta}$
	$x_1^2 + \varepsilon x_1x_2^2 + \delta x_2^3$	5		$B_{2,3'}^{\varepsilon,\delta}$
	$x_2^2 + \varepsilon x_1^2x_2 + \delta x_1^3$	5		$B_{3,2'}^{\varepsilon,\delta}$
	$x_1^2 \pm x_2^3$	6		$B_{2,3,0}^{\pm}$
	$x_2^2 \pm x_1^3$	6		$B_{3,2,0}^{\pm}$
	$x_1^2 + ax_1^2x_2 + \varepsilon x_2^4 + \delta x_1x_2^3$	7		$B_{2,4'}^{\varepsilon,\delta,a}$
	$x_2^2 + ax_1x_2^2 + \varepsilon x_1^4 + \delta x_1^3x_2$	7		$B_{4,2'}^{\varepsilon,\delta,a}$
	$x_1^2 + \varepsilon x_1^2x_2 + \delta x_2^5$	7		$B_{2,5'}^{\varepsilon,\delta}$
	$x_2^2 + \varepsilon x_1x_2^2 + \delta x_1^5$	7		$B_{5,2'}^{\varepsilon,\delta}$
1	$y_1^n + \varepsilon x_1y + \delta x_2y + x_2^m$	$n+m-1$	$n \geq 3, m \geq 2,\ m+n \leq 8$	$C_{n,m}^{\varepsilon,\delta}$
	$y^3 + \varepsilon x_1y + \delta x_2y^2 + x_2^2$	5		$C_{3,2,1}^{\varepsilon,\delta}$
	$y^3 + \varepsilon x_2y + \delta x_1y^2 + x_1^2$	5		$C_{3,2,2}^{\varepsilon,\delta}$
	$y^4 + ax_2y^2 + \varepsilon x_2^2y + x_1y + \delta x_2^2$	7		$C_{4,2,1}^{\varepsilon,\delta,a}$
	$y^4 + ax_1y^2 + \varepsilon x_1^2y + x_2y + \delta x_1^2$	7		$C_{4,2,2}^{\varepsilon,\delta,a}$
	$y^5 + \varepsilon x_2y^2 + \delta x_1y + x_2^2$	7		$C_{5,2,1}^{\varepsilon,\delta}$
	$y^5 + \varepsilon x_1y^2 + \delta x_2y + x_1^2$	7		$C_{5,2,2}^{\varepsilon,\delta}$
	$y^3 + \varepsilon x_2^2y + x_1^2 + ax_1x_2 + \delta x_2^2$	7	$a^2 \neq 4\delta$	$C_{3,2'}^{\varepsilon,\delta,a}$

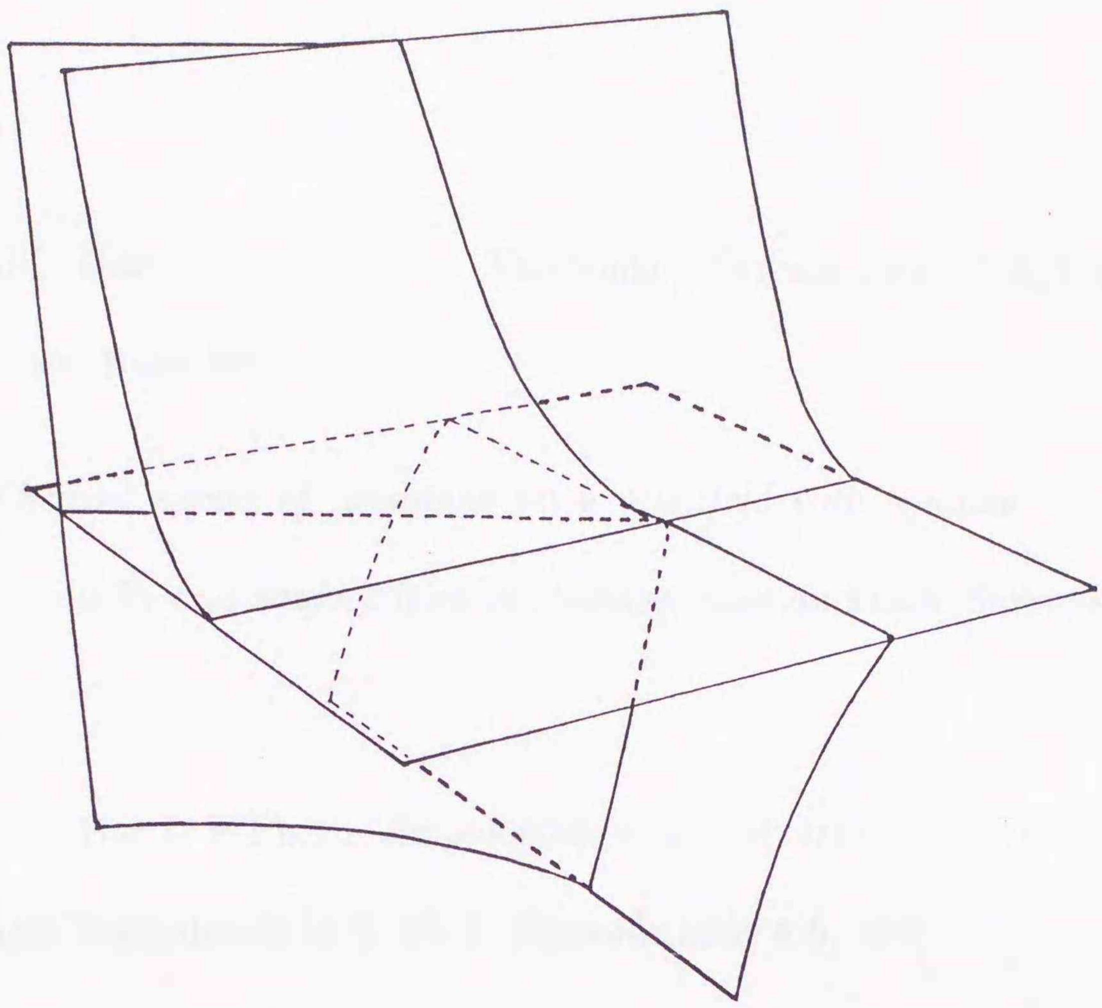
where $\varepsilon = \pm 1, \delta = \pm 1$.

Here, we draw the figure of the wavefront of one of the reticular Legendrian map-germ whose generating family is a reticular versal unfolding of $B^{-;-}_{2,3}$-singularity in the classification list, that is $x_1^2 - x_1x_2 - x_2^3 + q_1x_2^2 + q_2x_2 + q_3x_1 + q_4$. The wavefront given by this generating family is a subset in (q_1, q_2, q_3, q_4)-space around 0. Hence we draw the sections of this wavefront in (q_2, q_3, q_4)-space given by cutting at $q_1 < 0, q_1 = 0, q_1 > 0$ respectively.

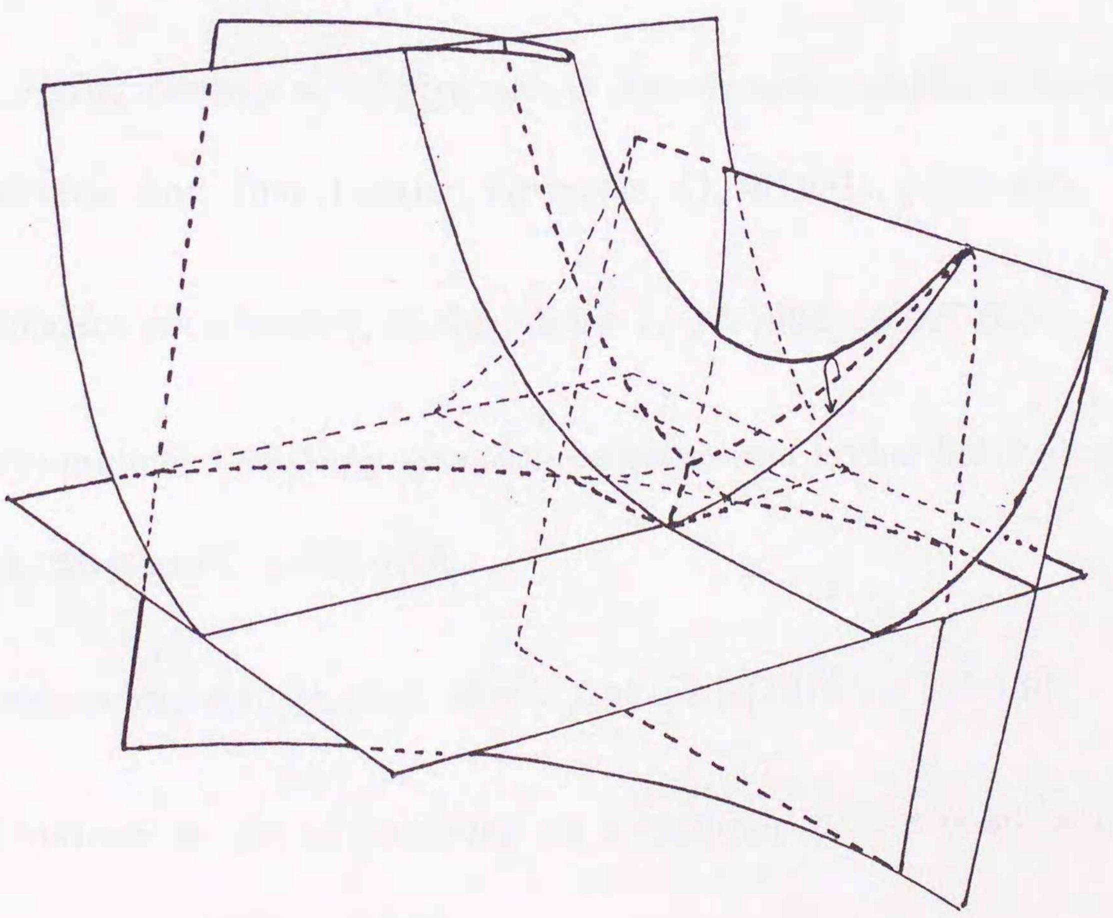
$\underline{q_1 < 0}$



$\underline{q_1 = 0}$



$\underline{q_1 > 0}$



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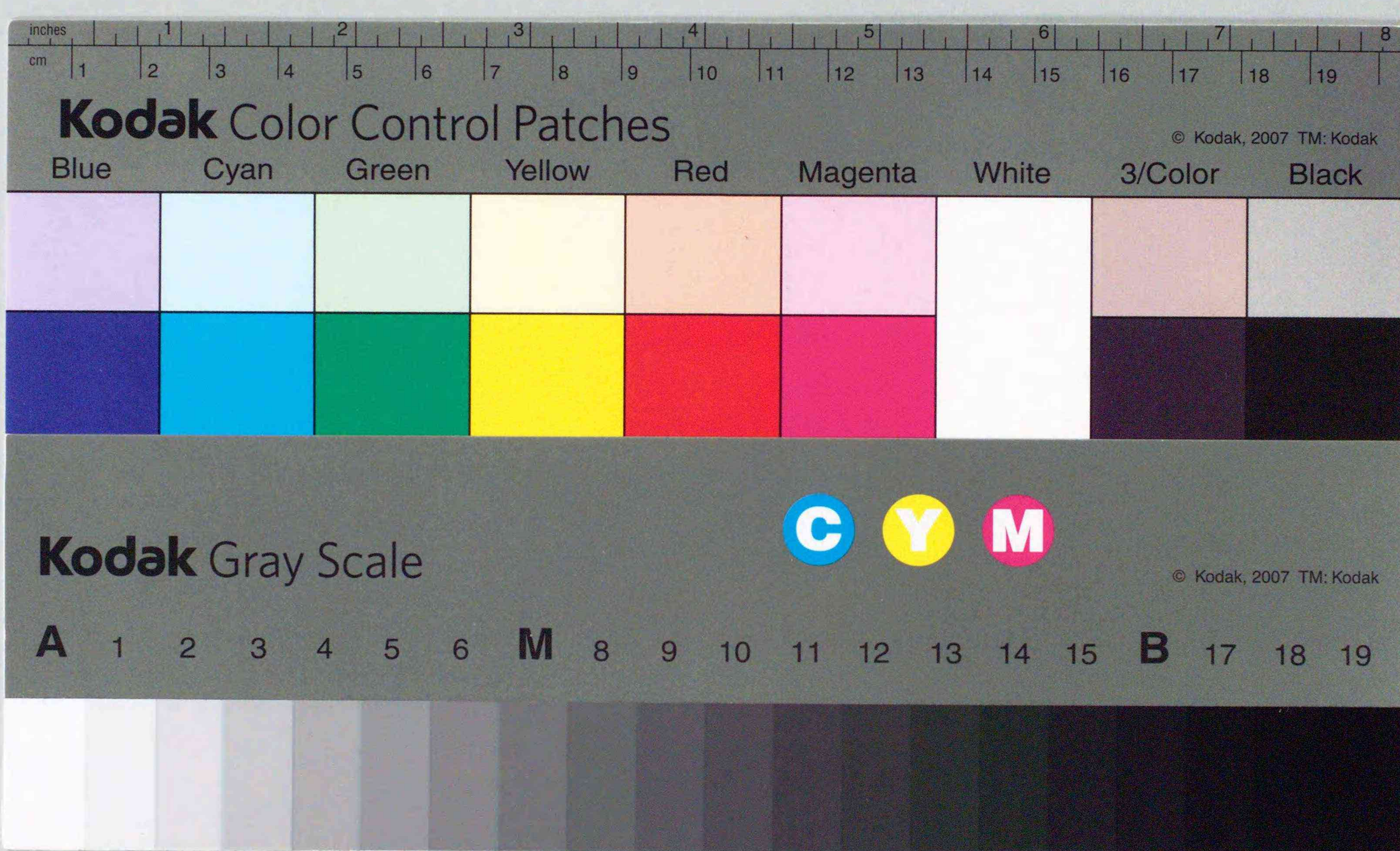
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