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Author(s)	Wachi, Akihito
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Contents

1	Introduction	3
2	Scalar generalized Verma modules	6
3	Two contravariant forms	14
4	The ratio of the two contravariant forms	17
5	b -Functions	18
6	Main theorem	22
7	Subalgebras of $\mathfrak{g}$	23
8	An expression of $q_\lambda(\mu)$ in terms of b -function	26
9	Another expression of $q_\lambda(\mu)$ in terms of b -function	28
10	Irreducibility criteria for scalar generalized Verma modules	36
11	b -Functions and L -orbits on the nilpotent radical	37
12	The unitarizability of the irreducible quotient of $M(\lambda)$	42
13	An analogue of the Capelli identity	48
13.1	Type (A_{2r-1}, r) or $GL_r \otimes GL_r$	53
13.2	Type (C_n, n) or S^2GL_n	56
13.3	Type $(D_{2r}, 2r)$ or Λ^2GL_{2r}	58
13.4	Type $(D_n, 1)$ or $O_{2n-1} \otimes GL_1$	61
13.5	Type $(B_n, 1)$ or $O_{2n} \otimes GL_1$	63
14	Factors contained in $\beta_{\lambda,i}(\mu)$	66

15 The principal symbol of $\Psi_\lambda({}^t f_r)$	68
16 Appendix—some data of commutative parabolic types	72
16.1 Type (A_n, r)	73
16.2 Type (C_n, n)	74
16.3 Type (D_n, n)	76
16.4 Type $(D_n, 1)$	78
16.5 Type $(B_n, 1)$	79
16.6 Type $(E_6, 1)$	80
16.7 Type $(E_7, 7)$	82

Generalized Verma modules induced from characters of parabolic subalgebras with commutative nilpotent radical

Akihito WACHI

Department of Mathematics, Faculty of Science,
Hokkaido University, Sapporo 060-0810, Japan

Abstract. On a scalar generalized Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$, the contravariant forms deeply relate to its structure. In this thesis, we show intrinsically that the contravariant forms have b -functions in its factor in the case where $\mathfrak{g}$ is simple and the nilpotent radical of $\mathfrak{p}$ is commutative. As an application we explain intrinsically why the b -functions control the irreducibility of $M(\lambda)$, the orbit decomposition of $\mathfrak{n}^+$ under the action of the Levi subgroup, and the unitarizability of the irreducible quotient of $M(\lambda)$. In addition, we construct an analogue on $M(\lambda)$ the Capelli identity using our main theorem.

1 Introduction

Let G be a complex simple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ its Cartan subalgebra. Let Δ and Δ^+ be the root system and the positive system, respectively. Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{h}$ and all the positive root spaces. Then the pair $(\mathfrak{g}, \mathfrak{p})$ is said to be of commutative parabolic type if the nilpotent radical $\mathfrak{n}^+$ of $\mathfrak{p}$ is commutative. In this thesis, we exclusively consider $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type.

Let $M(\lambda)$ be the scalar generalized Verma module induced from $\lambda \in \text{Hom}(\mathfrak{p}, \mathbb{C})$. Then $M(\lambda) \simeq \mathbb{C}[\mathfrak{n}^+]$ as vector spaces. We therefore obtain the representation of $U(\mathfrak{g})$ on $\mathbb{C}[\mathfrak{n}^+]$, and denote it by $\Psi_\lambda : U(\mathfrak{g}) \rightarrow \text{End } \mathbb{C}[\mathfrak{n}^+]$.

Let $\{X_\alpha, H_i\}$ be a Chevalley basis of $\mathfrak{g}$, where $X_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Delta$ and $H_i \in \mathfrak{h}$. To give the definition of contravariant forms, we define an anti-involution t on $U(\mathfrak{g})$ by $X_\alpha \mapsto X_{-\alpha}$ ($\alpha \in \Delta$) and to be the identity on $\mathfrak{h}$. For a representation (π, V) of $\mathfrak{g}$, a bilinear form $(,)$ on V is called a contravariant form or a $\pi(U(\mathfrak{g}))$ -contravariant form if

it satisfies $(\pi(X)v, w) = (v, \pi({}^tX)w)$ for $X \in \mathfrak{g}$ and $v, w \in V$. We study a canonical $\Psi_\lambda(U(\mathfrak{g}))$ -contravariant form $(\cdot, \cdot)_\lambda$ on $M(\lambda) \simeq \mathbb{C}[n^+]$. The structure of $M(\lambda)$ is deeply related to $(\cdot, \cdot)_\lambda$. For example, $M(\lambda)$ is irreducible if and only if $(\cdot, \cdot)_\lambda$ is nondegenerate (Proposition 3.3) and the irreducible quotient of $M(\lambda)$ is unitarizable if and only if a contravariant sesquilinear form $(\cdot, \cdot)_\lambda^*$ is non-negative definite (Proposition 12.10). We also study a canonical $\text{ad}(U(\mathfrak{l}))$ -contravariant form $(\cdot, \cdot)$ on $M(\lambda) \simeq \mathbb{C}[n^+]$, where $\mathfrak{l}$ is the Levi subalgebra of $\mathfrak{p}$ containing $\mathfrak{h}$. In fact, contravariant forms $(\cdot, \cdot)_\lambda$ and $(\cdot, \cdot)$ coincide up to constant on each irreducible $\text{ad}(U(\mathfrak{l}))$ -submodule of $M(\lambda)$. We denote the ratio of them by $q_\lambda(\mu)$ (see §4).

One of main purpose of this thesis is to express $q_\lambda(\mu)$ as a product of b -function $b_i(\mu)$ (Corollary 8.1). Here $b_i(\mu)$ is a b -function of a relative invariant of a prehomogeneous vector space (L_i, Ad, n_i^+) associated with $(\mathfrak{g}, \mathfrak{p})$ (see Definition 5.6 and Definition 7.4). Then we can explain why b -function control the properties of $M(\lambda)$ via the contravariant forms.

As an application we explain the reason why the b -functions control the irreducibility of $M(\lambda)$, the orbit decomposition of n^+ under the action of the Levi subgroup, and the unitarizability of $M(\lambda)$. In addition, we construct an analogue on $M(\lambda)$ the Capelli identity using our main theorem.

The contents of this thesis is as follows: In §2 to §5, we give basic definitions such as a generalized Verma module, a contravariant forms and a b -function. In §6, we state our main theorem (Theorem 6.1). In §7, we define subalgebras of $\mathfrak{g}$ and restate the main theorem. In §8, we derive an important conclusion Corollary 8.1 from the main theorem, which gives an expression of $q_\lambda(\mu)$ as a product of b -functions. In §9, we give another expression of $q_\lambda(\mu)$.

In §10 to §13, we study applications of our main result. In §10, we study the relation between the irreducibility of $M(\lambda)$ and zeros of b -functions. In §11, we consider the one-to-one correspondence between $\text{Ad}(L)$ -orbits on n^+ and zeros of b -functions. In §12, we study the relation between the unitarizability of the irreducible quotient of $M(\lambda)$ and

zeros of b -functions. This application is suggested by Professor Shuichi Suga. In §13, we construct an analogue on $M(\lambda)$ of the Capelli identity.

In §14 and §15, we prove the main theorem using Boe [1, 4.4. Theorem]. Finally, as an appendix, we give data of commutative parabolic types in §16.

Here we give some (historical) remarks. First, $q_\lambda(\mu)$ has already appeared in Wallach [32, Proposition 1.2] and is determined explicitly. Moreover the results of Kostant-Sahi [19], of Shimura [26], of Rubenthaler-Schiffmann [23] and of Faraut-Koranyi [6] are deduced from the explicit formula for $q_\lambda(\mu)$. It is, however, not mentioned in Wallach [32] that $q_\lambda(\mu)$ is a product of b -functions, and it was not known why $q_\lambda(\mu)$ is a product of them.

Second, the irreducibility of $M(\lambda)$ (§10) has been studied in several papers. In Jantzen [14], the determinant of $(\cdot, \cdot)_\lambda$ is calculated and the irreducibility criteria are described concretely. In Shapovalov [25], the determinant is calculated for the Verma module. It is observed that the values of λ which makes $M(\lambda)$ irreducible, are related to the zeros of the b -functions. The first published result which relates the irreducibility criteria and the b -functions, is Suga [27]. The necessary condition for the irreducibility is stated there in terms of b -functions in the case where $\mathfrak{g}$ is classical. Gyoja [9] and [10] conjectured an irreducibility criterion in terms of b -functions in a more general setting, and he proved this in some special cases including the commutative parabolic cases by a case study.

Third, the one-to-one correspondence between $\text{Ad}(L)$ -orbits on n^+ and zeros of b -functions (§11) was found in Tanisaki [29, Proposition 2.6] motivated by the study of hypergeometric systems (Tanisaki [28]) and a quantization of certain prehomogeneous vector spaces (Kamita-Morita-Tanisaki[16]). The proof of the correspondence was a case study.

Fourth, the unitarizability of the irreducible quotient of $M(\lambda)$ has been studied in many papers (Wallach [31], Parthasarathy [22], Garland-Zuckerman [7], Enright-Howe-Wallach [4], Enright-Joseph [5], ...). It was known only as a coincidence that the unitarizability condition was related to zeros of b -functions.

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2 Scalar generalized Verma modules

In this section, first we give basic definitions and then we give realizations of a scalar generalized Verma module as representations on a certain polynomial ring. Lastly we define several anti-involutions and prove their properties.

Let G be a connected simple algebraic group and $\mathfrak{g} = \text{Lie}(G)$ be its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the root system of $\mathfrak{g}$, Δ^+ be the positive system, $\{\alpha_1, \dots, \alpha_n\}$ be the simple system, and $\{\varpi_1, \dots, \varpi_n\}$ be the set of fundamental weights. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$ and all the positive root spaces. Let $\mathfrak{n}^+$ be the nilpotent radical of $\mathfrak{p}$ and $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{p}$ containing $\mathfrak{h}$. Let Δ_L and Δ_N^+ be the set of roots occurring in $\mathfrak{l}$ and $\mathfrak{n}^+$, respectively. We also use the notation like $\Delta_L^\pm$ or $\Delta_N^\pm$. Set $\mathfrak{n}^- = \sum_{\alpha \in \Delta_N^+} \mathfrak{g}^{-\alpha}$, where $\mathfrak{g}^\alpha$ denotes the α -root space in $\mathfrak{g}$. Let L be the connected closed subgroup of G corresponding to $\mathfrak{l}$.

Definition 2.1 For a complex simple Lie algebra $\mathfrak{g}$ and its parabolic subalgebra $\mathfrak{p}$, the pair $(\mathfrak{g}, \mathfrak{p})$ is said to be of *commutative parabolic type* or of *Hermitian symmetric type*, if the nilpotent radical $\mathfrak{n}^+$ of $\mathfrak{p}$ is nonzero and commutative.

Throughout this thesis, we assume that the pair $(\mathfrak{g}, \mathfrak{p})$ is of commutative parabolic type. In this setting, $\mathfrak{p}$ becomes a maximal parabolic subalgebra of $\mathfrak{g}$, and let i_0 be the unique number such that the simple root α_{i_0} is not a root of $\mathfrak{l}$. We denote a pair $(\mathfrak{g}, \mathfrak{p})$ also by $(\mathfrak{g}, i_0)$. We will always use Bourbaki's way of simple root numbering [2] for i_0 . All the pairs $(\mathfrak{g}, i_0)$ of commutative parabolic type are listed in Figure 1. Empty circles mean

the simple root α_{i_0} . We fix an invariant bilinear form $\langle, \rangle$ on $\mathfrak{g}$, which must be a multiple of the Killing form on $\mathfrak{g}$.

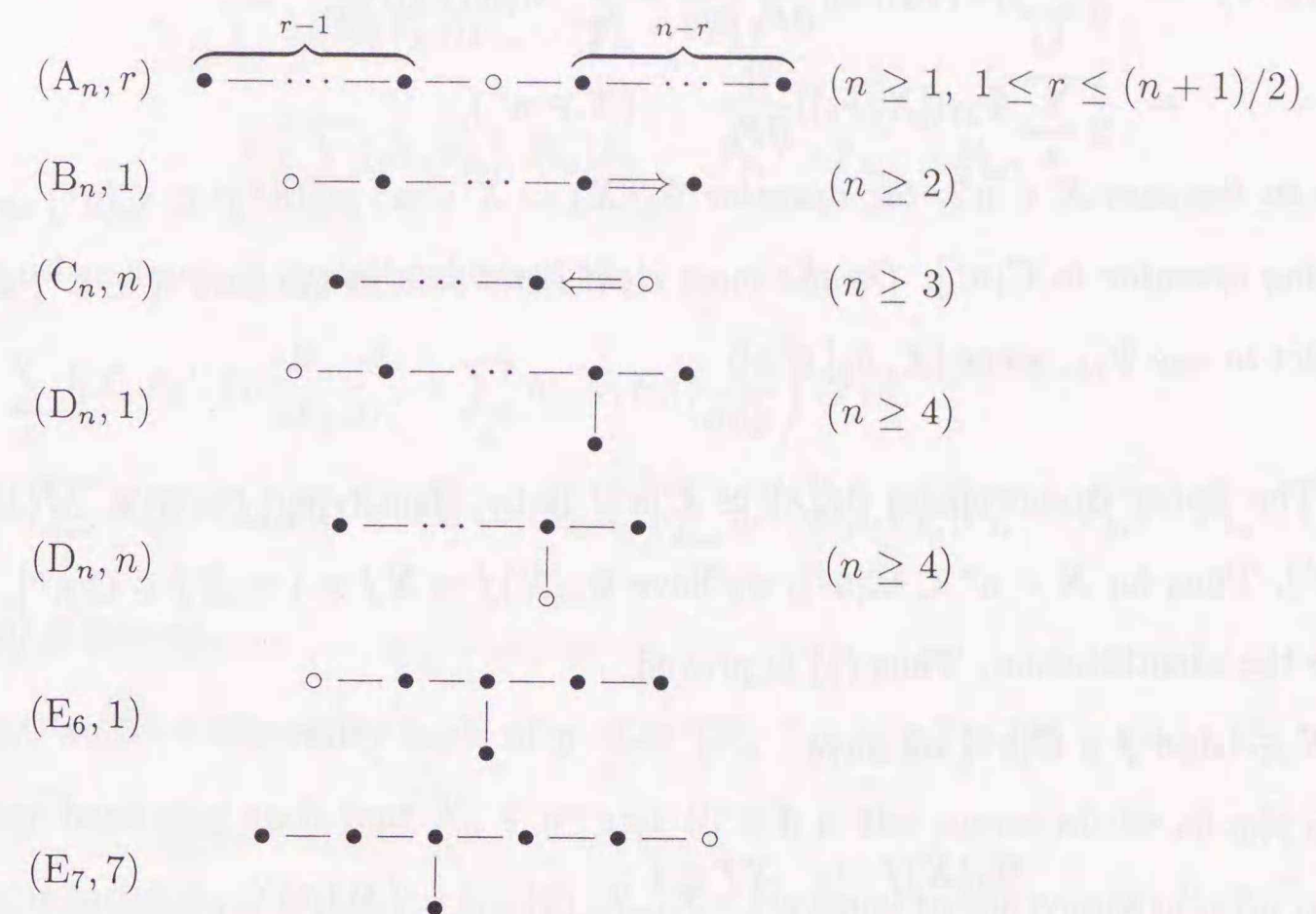


Figure 1: commutative parabolic type

For $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type, and $\lambda \in \text{Hom}(\mathfrak{p}, \mathbb{C})$, we set

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ denotes the representation space of λ . Note that $\text{Hom}(\mathfrak{p}, \mathbb{C}) \simeq \mathbb{C}\varpi_{i_0}$, since $\lambda \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ must vanish on $[\mathfrak{p}, \mathfrak{p}] = [\mathfrak{l}, \mathfrak{l}] + \mathfrak{n}^+$. We have an isomorphism of vector spaces $M(\lambda) \simeq U(\mathfrak{n}^-) = S(\mathfrak{n}^-) \simeq \mathbb{C}[\mathfrak{n}^+]$, since both $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are commutative. The last isomorphism above is obtained by identifying $\mathfrak{n}^-$ with the dual space $(\mathfrak{n}^+)^*$ of $\mathfrak{n}^+$ via the fixed invariant bilinear form. Then $U(\mathfrak{g})$ acts on $\mathbb{C}[\mathfrak{n}^+]$. We denote this representation by $\Psi_\lambda : U(\mathfrak{g}) \longrightarrow \text{End } \mathbb{C}[\mathfrak{n}^+]$.

Lemma 2.2 Let $\{F_k\}$ be a basis of $\mathfrak{n}^-$. Then we have

$$(1) \Psi_\lambda(X) = X \quad (X \in \mathfrak{n}^-),$$

$$\begin{aligned}
(2) \Psi_\lambda(X) &= \text{ad}(X) + \lambda(X) \\
&= \sum_k [X, F_k] \frac{\partial}{\partial F_k} + \lambda(X) \quad (X \in \mathfrak{l}), \\
(3) \Psi_\lambda(X) &= \frac{1}{2} \sum_{k,l} [[X, F_k], F_l] \frac{\partial}{\partial F_k} \frac{\partial}{\partial F_l} + \sum_k \lambda([X, F_k]) \frac{\partial}{\partial F_k} \\
&= \frac{1}{2} \sum_k \Psi_{2\lambda}([X, F_k]) \frac{\partial}{\partial F_k} \quad (X \in \mathfrak{n}^+).
\end{aligned}$$

Here in the case $X \in \mathfrak{n}^-$, the operator $\Psi_\lambda(X) = X \in \mathfrak{n}^- \simeq (\mathfrak{n}^+)^* \subset \mathbb{C}[\mathfrak{n}^+]$ is just a multiplying operator in $\mathbb{C}[\mathfrak{n}^+]$. On the most right hand side in the case $X \in \mathfrak{n}^+$, there is no conflict to use $\Psi_{2\lambda}$, since $[X, F_k] \in \mathfrak{l}$.

Proof. The linear isomorphism $M(\lambda) \simeq \mathbb{C}[\mathfrak{n}^+]$ is by identifying $f \otimes 1 \in M(\lambda)$ with $f \in \mathbb{C}[\mathfrak{n}^+]$. Thus for $X \in \mathfrak{n}^- \subset \mathbb{C}[\mathfrak{n}^+]$, we have $\Psi_\lambda(X)f \simeq Xf \otimes 1 \simeq Xf \in \mathbb{C}[\mathfrak{n}^+]$, where $\simeq$ means the identification. Thus (1) is proved.

For $X \in \mathfrak{l}$ and $f \in \mathbb{C}[\mathfrak{n}^+]$ we have

$$\begin{aligned}
\Psi_\lambda(X)f &\simeq Xf \otimes 1 \\
&= (fX + \text{ad}(X)f) \otimes 1 \\
&= f \otimes \lambda(X)1 + \text{ad}(X)f \otimes 1 \\
&\simeq \lambda(X)f + \text{ad}(X)f.
\end{aligned}$$

Thus we proved (2), since it is easy to show that $\text{ad}(X) = \sum_k [X, F_k] \partial / \partial F_k$.

For $X \in \mathfrak{n}^+$, the second equality follows from (2). We check the first equality. We may assume that f is a monic monomial in F_k without loss of generality. Set $f = F_{k_1} \cdots F_{k_d}$. Then we have

$$\begin{aligned}
\Psi_\lambda(X)f &\simeq Xf \otimes 1 \\
&= fX \otimes 1 + \text{ad}(X)f \otimes 1 \\
&= \text{ad}(X)f \otimes 1 \\
&= \sum_{t=1}^d F_{k_1} \cdots [X, F_{k_t}] \cdots F_{k_d} \otimes 1 \\
&= \sum_t F_{k_1} \cdots F_{k_{t-1}} \{F_{k_{t+1}} \cdots F_{k_d} [X, F_{k_t}] + [[X, F_{k_t}], F_{k_{t+1}} \cdots F_{k_d}]\} \otimes 1
\end{aligned}$$

$$\begin{aligned}
&= \sum_t F_{k_1} \cdots \widehat{F_{k_t}} \cdots F_{k_d} \otimes \lambda([X, F_{k_t}])1 \\
&\quad + \sum_{1 \leq t < u \leq d} F_{k_1} \cdots F_{k_{t-1}} \cdot F_{k_{t+1}} \cdots [[X, F_{k_t}], F_{k_u}] \cdots F_{k_d} \otimes 1 \\
&\simeq \sum_t \lambda([X, F_{k_t}]) F_{k_1} \cdots \widehat{F_{k_t}} \cdots F_{k_d} \\
&\quad + \frac{1}{2} \sum_{t \neq u} [[X, F_{k_t}], F_{k_u}] F_{k_1} \cdots \widehat{F_{k_t}} \cdots \widehat{F_{k_u}} \cdots F_{k_d}.
\end{aligned}$$

On the other hand, it is easily checked that

$$\begin{aligned}
&\left(\frac{1}{2} \sum_{kl} [[X, F_k], F_l] \frac{\partial}{\partial F_k} \frac{\partial}{\partial F_l} + \sum_k \lambda([X, F_k]) \frac{\partial}{\partial F_k} \right) (f) \\
&= \sum_t \lambda([X, F_{k_t}]) F_{k_1} \cdots \widehat{F_{k_t}} \cdots F_{k_d} + \frac{1}{2} \sum_{t \neq u} [[X, F_{k_t}], F_{k_u}] F_{k_1} \cdots \widehat{F_{k_t}} \cdots \widehat{F_{k_u}} \cdots F_{k_d}.
\end{aligned}$$

Thus (3) is proved. $\square$

Next we fix a Chevalley basis of $\mathfrak{g}$. Let $\{X_\alpha \mid \alpha \in \Delta\} \cup \{H_i \mid i = 1, \dots, n\}$ be a Chevalley basis of $\mathfrak{g}$ such that $X_\alpha \in \mathfrak{g}^\alpha$ and $H_i \in \mathfrak{h}$ is the coroot of the simple root α_i . Namely, it satisfies (1) $\varpi_j(H_i) = \delta_{ij}$, (2) $[X_\alpha, X_{-\alpha}]$ is equal to the coroot of α for $\alpha \in \Delta^+$, and (3) if we define constants $c_{\alpha\beta}$ by $[X_\alpha, X_\beta] = c_{\alpha\beta} X_{\alpha+\beta}$, then $c_{-\alpha, -\beta} = -c_{\alpha\beta}$. Since Definition 2.3 (1) and (3) depend on a choice of a Chevalley basis, we fix this Chevalley basis once and for all.

We compute $\langle X_\alpha, X_{-\alpha} \rangle$ for later use. We have $2\langle X_\alpha, X_{-\alpha} \rangle = \langle [H_\alpha, X_\alpha], X_{-\alpha} \rangle = \langle [[X_\alpha, X_{-\alpha}], X_\alpha], X_{-\alpha} \rangle = \langle H_\alpha, H_\alpha \rangle = (2\alpha/(\alpha, \alpha), 2\alpha/(\alpha, \alpha)) = 4/(\alpha, \alpha)$, where $(,)$ is the bilinear form on $\mathfrak{h}^*$ induced from the fixed invariant bilinear form $\langle, \rangle$. Thus we have

$$(2.1) \quad \langle X_\alpha, X_{-\alpha} \rangle = \frac{2}{(\alpha, \alpha)}.$$

Here we define three anti-involutions. First one is on $U(\mathfrak{g})$, and the rest are on $D_{\mathfrak{n}^+}$, the ring of polynomial coefficient differential operators on $\mathfrak{n}^+$. Naturally $D_{\mathfrak{n}^+}$ is identified with $\mathbb{C}[\mathfrak{n}^+] \otimes S(\mathfrak{n}^+)$. Indeed, $S(\mathfrak{n}^+)$ is identified with the ring of constant coefficient differential operators on $\mathfrak{n}^+$ as follows: For $P \in S(\mathfrak{n}^+)$, we define a constant coefficient differential operator $P(\partial)$ on $\mathfrak{n}^+$ as an operator such that

$$(2.2) \quad P(\partial) \exp\langle x, y \rangle = P(y) \exp\langle x, y \rangle \quad (x \in \mathfrak{n}^+, y \in \mathfrak{n}^-).$$

Here $P(y)$ on the right hand side is a polynomial function on $\mathfrak{n}^-$ by the identification $S(\mathfrak{n}^+) \simeq \mathbb{C}[\mathfrak{n}^-]$ via the fixed invariant bilinear form $\langle, \rangle$. In particular, for $F \in \mathfrak{n}^-$ and $P \in \mathfrak{n}^+$

$$(2.3) \quad P(\partial)(F) = \langle P, F \rangle.$$

Definition 2.3 Let $D_{\mathfrak{n}^+}$ denote the ring of polynomial coefficient differential operator on $\mathfrak{n}^+$.

(1) Define an anti-involution t on $U(\mathfrak{g})$ by

$$\begin{aligned} {}^tX_\alpha &= X_{-\alpha} & (\alpha \in \Delta), \\ {}^tH_i &= H_i & (i \in \{1, \dots, n\}), \end{aligned}$$

where X_α and H_i are the elements of the fixed Chevalley basis. These formulas define an anti-automorphism of the Lie algebra $\mathfrak{g}$, and we extend it to $U(\mathfrak{g})$ as an anti-homomorphism of a ring.

(2) Define an anti-involution σ on $D_{\mathfrak{n}^+}$ by

$$\begin{aligned} \sigma(F_j) &= F_j, \\ \sigma\left(\frac{\partial}{\partial F_j}\right) &= -\frac{\partial}{\partial F_j}, \end{aligned}$$

where $\{F_j\}$ is a basis of $\mathfrak{n}^-$. It is obvious that σ does not depend on a choice of a basis.

(3) Define an anti-involution τ on $D_{\mathfrak{n}^+}$ by

$$\begin{aligned} \tau(X_{-\alpha}) &= X_\alpha(\partial), \\ \tau(X_\alpha(\partial)) &= X_{-\alpha}, \end{aligned}$$

for $\alpha \in \Delta_N^+$. Here $\{X_\alpha\}$ is a part of the fixed Chevalley basis. Note that

$$X_\alpha(\partial) = \langle X_\alpha, X_{-\alpha} \rangle \frac{\partial}{\partial X_{-\alpha}} = \frac{2}{(\alpha, \alpha)} \frac{\partial}{\partial X_{-\alpha}},$$

because of (2.1) and (2.3).

Next we prove the properties of the anti-involutions defined above.

Lemma 2.4 (1) (cf. [13], §6 Exercise) *The anti-involution t on $U(\mathfrak{g})$ is the identity on $Z(\mathfrak{l})$, the center of $U(\mathfrak{l})$.*

(2) *The anti-involution σ on $D_{\mathfrak{n}^+}$ satisfies*

$$\sigma(\Psi_\lambda(u)) = \Psi_{-\lambda-2\rho}(s(u)) \quad (u \in U(\mathfrak{g})),$$

where s is an anti-involution on $U(\mathfrak{g})$ defined by

$$s(X) = \begin{cases} -X & (X \in \mathfrak{l}), \\ X & (X \in \mathfrak{n}^+ + \mathfrak{n}^-), \end{cases}$$

and $\rho \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ is half the sum of roots in Δ_N^+ , or equivalently $2\rho(X) = \text{Tr}_{\mathfrak{n}^+} \text{ad}(X)$ for $X \in \mathfrak{p}$.

(3) *The anti-involution τ on $D_{\mathfrak{n}^+}$ satisfies*

$$\tau(\Psi_\lambda(u)) = \Psi_\lambda({}^tu) \quad (u \in U(\mathfrak{l})).$$

Proof. (1) $\mathfrak{l}$ is a direct sum of the semisimple part $[\mathfrak{l}, \mathfrak{l}]$ and a one-dimensional center contained in $\mathfrak{h}$. Since the anti-involution t is the identity on $U(\mathfrak{h})$, we have only to check that t is the identity on $Z([\mathfrak{l}, \mathfrak{l}])$.

Let $\varphi : Z([\mathfrak{l}, \mathfrak{l}]) \rightarrow U(\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}])$ be the Harish-Chandra homomorphism, which maps $Z([\mathfrak{l}, \mathfrak{l}])$ isomorphically onto its image. Namely, φ is injective. The explicit form of φ is a composite of the projection from $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{g}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^+)$ to $U(\mathfrak{h})$ and an isomorphism on $U(\mathfrak{h})$ defined by $H \mapsto H - \delta(H)$. Here $\mathfrak{g}^\pm = \sum_{\alpha \in \Delta^\pm} \mathfrak{g}^\pm \alpha$ and $\delta = (1/2) \sum_{\alpha \in \Delta_L^+} \alpha$.

Since $[X, {}^tz] = -[{}^tX, z]$, the anti-involution t stabilizes $Z([\mathfrak{l}, \mathfrak{l}])$. Because of the construction of φ , it follows that $\varphi({}^tu) = \varphi(u)$ for $u \in Z([\mathfrak{l}, \mathfrak{l}])$. Then the injectivity of φ proves the assertion.

(2) It suffices to show the assertion for $u = X \in \mathfrak{g}$. It is obvious for $X \in \mathfrak{n}^-$.

Let $X \in \mathfrak{l}$ and $\{F_k\}$ be a basis of $\mathfrak{n}^-$.

$$\sigma(\Psi_\lambda(X)) = \sigma\left(\sum_k [X, F_k] \frac{\partial}{\partial F_k} + \lambda(X)\right)$$

$$\begin{aligned}
&= -\sum_k \frac{\partial}{\partial F_k} [X, F_k] + \lambda(X) \\
&= -\sum_k \left\{ [X, F_k] \frac{\partial}{\partial F_k} + \frac{\partial}{\partial F_k} ([X, F_k]) \right\} + \lambda(X) \\
&= -\text{ad}(X) + \lambda(X) - \sum_k \frac{\partial}{\partial F_k} (\text{ad}(X)(F_k)) \\
&= -\text{ad}(X) + \lambda(X) - \text{Tr}_{\mathfrak{n}^-} \text{ad}(X) \\
&= -\text{ad}(X) + \lambda(X) + 2\rho(X).
\end{aligned}$$

We proved the assertion for $X \in \mathfrak{l}$.

Lastly, let $X \in \mathfrak{n}^+$.

$$\begin{aligned}
\sigma(2\Psi_\lambda(X)) &= \sigma \left(\sum_k \Psi_{2\lambda}([X, F_k]) \frac{\partial}{\partial F_k} \right) \\
&= \sum_k \frac{\partial}{\partial F_k} \Psi_{-2\lambda-2\rho}([X, F_k]) \\
(2.4) \quad &= \sum_k \Psi_{-2\lambda-2\rho}([X, F_k]) \frac{\partial}{\partial F_k} + \sum_k \left[\frac{\partial}{\partial F_k}, \Psi_{-2\lambda-2\rho}([X, F_k]) \right],
\end{aligned}$$

where we used the (proved) assertion for $X \in \mathfrak{l}$ in the second equality. We compute the second term on the most right hand side.

$$\begin{aligned}
\sum_k \left[\frac{\partial}{\partial F_k}, \Psi_{-2\lambda-2\rho}([X, F_k]) \right] &= \sum_k \left[\frac{\partial}{\partial F_k}, \text{ad}([X, F_k]) \right] \\
&= \sum_{k,j} \left[\frac{\partial}{\partial F_k}, [[X, F_k], F_j] \frac{\partial}{\partial F_j} \right] \\
&= \sum_{k,j} \frac{\partial}{\partial F_k} (\text{ad}([X, F_j])(F_k)) \frac{\partial}{\partial F_j} \\
&= \sum_j \{ \text{Tr}_{\mathfrak{n}^-} \text{ad}([X, F_j]) \} \frac{\partial}{\partial F_j} \\
&= -\sum_j 2\rho([X, F_j]) \frac{\partial}{\partial F_j},
\end{aligned}$$

where we used the identity $[[X, F_k], F_j] = [[X, F_j], F_k]$ which is followed from $[F_k, F_j] = 0$ and the Jacobi identity. Then (2.4) is equal to

$$\begin{aligned}
\sum_k \{ \Psi_{-2\lambda-2\rho}([X, F_k]) - 2\rho([X, F_k]) \} \frac{\partial}{\partial F_k} &= \sum_k \Psi_{-2\lambda-4\rho}([X, F_k]) \frac{\partial}{\partial F_k} \\
&= 2\Psi_{-\lambda-2\rho}(X).
\end{aligned}$$

We proved (2).

(3) It suffices to show the assertion for $u = X \in \mathfrak{l}$ and $\lambda = 0$. We define $c_\alpha = \langle X_\alpha, X_{-\alpha} \rangle^{-1}$ for $\alpha \in \Delta_N^+$. Then $\partial/\partial X_{-\alpha} = c_\alpha X_\alpha(\partial)$ and $\{c_\alpha X_\alpha \mid \alpha \in \Delta_N^+\}$ is a dual basis of $\{X_{-\alpha} \mid \alpha \in \Delta_N^+\}$ with respect to $\langle, \rangle$.

$$\begin{aligned}
\tau(\text{ad}(X)) &= \tau \left(\sum_{\alpha \in \Delta_N^+} [X, X_{-\alpha}] \frac{\partial}{\partial X_{-\alpha}} \right) \\
&= \tau \left(\sum_{\alpha \in \Delta_N^+} \sum_{\beta \in \Delta_N^+} \langle c_\beta X_\beta, [X, X_{-\alpha}] \rangle X_{-\beta} \cdot c_\alpha X_\alpha(\partial) \right) \\
&= \sum_{\alpha, \beta} c_\alpha c_\beta \langle X_\beta, [X, X_{-\alpha}] \rangle X_{-\alpha} X_\beta(\partial) \\
&= \sum_{\alpha, \beta} \langle c_\alpha X_{-\alpha}, [X_\beta, X] \rangle X_{-\alpha} \cdot c_\beta X_\beta(\partial) \\
&= \sum_\beta \langle X_\beta, X \rangle c_\beta X_\beta(\partial) \\
&= \sum_\beta [{}^t X, X_{-\beta}] \frac{\partial}{\partial X_{-\beta}}.
\end{aligned}$$

We proved (3). $\square$

Remark 2.5 We note that there is another realization of $M(\lambda)$ on $\mathbf{C}[\mathfrak{n}^+]$ as follows:

First we construct a representation ψ_λ of $U(\mathfrak{g})$ on $\mathbf{C}[\mathfrak{n}^-]$. For $f \in \mathbf{C}[\mathfrak{n}^-]$, we extend f to a function on a neighborhood of the identity element of G , and we denote it also by f . We define

$$f(\exp A \exp B) = e^{-\lambda(B)} f(A) \quad (A \in \mathfrak{n}^-, B \in \mathfrak{p}).$$

In the case where λ can be lifted to a character of P , we note that f can be regarded as a restriction to $N^-P/P \simeq \mathfrak{n}^-$ of a section of a G -equivariant line bundle on G/P such that the action of P at the origin is given by λ . Here P and N^- denote the closed subgroup of G corresponding to $\mathfrak{p}$ and $\mathfrak{n}^-$, respectively.

We define ψ_λ by

$$\psi_\lambda(X)f(A) = \frac{d}{dt} f(\exp(-tX) \exp A) \Big|_{t=0} \quad (X \in \mathfrak{g}, A \in \mathfrak{n}^-).$$

Then we have by slightly hard calculations

$$\psi_\lambda(X)f(A) = \begin{cases} -X(\partial)f(A) & (X \in \mathfrak{n}^-), \\ -[X, A](\partial)f(A) + \lambda(X)f(A) & (X \in \mathfrak{l}), \\ \frac{1}{2}[A, [X, A]](\partial)f(A) + \lambda([X, A])f(A) & (X \in \mathfrak{n}^+). \end{cases}$$

Next we apply the Fourier transform to $\psi_\lambda(X)$. The Fourier transform is an isomorphism of Weyl algebras defined by

$$\begin{aligned} F: D_{\mathfrak{n}^-} &\longrightarrow D_{\mathfrak{n}^+} \\ G &\mapsto G(\partial) \quad (G \in \mathfrak{n}^+) \\ F(\partial) &\mapsto -F \quad (F \in \mathfrak{n}^-). \end{aligned}$$

We note that we do not use a usual mapping $G \mapsto \sqrt{-1}G(\partial)$ and $F(\partial) \mapsto \sqrt{-1}F$, since we want to match the coefficients of $\Psi_\lambda(X)$ with that of $F \circ \psi_\lambda(X)$. We have by harder calculations

$$F \circ \psi_\lambda = \Psi_{\lambda-2\rho}.$$

See also [28] where ψ_λ plays an important role.

3 Two contravariant forms

In this section, we give a definition of contravariant forms and then we introduce two contravariant forms on $M(\lambda)$.

Definition 3.1 Let (π, V) be a $U(\mathfrak{g})$ -module. A symmetric bilinear form $(,)$ on V is called a *contravariant form* or a $\pi(U(\mathfrak{g}))$ -*contravariant form* if $(\pi(u)v, v') = (v, \pi({}^t u)v')$ for all $u \in U(\mathfrak{g})$ and $v, v' \in V$.

The following propositions are fundamental on contravariant forms.

Proposition 3.2 (cf. Garland-Zuckerman [7, Lemma 2.5]) *Let V be a $U(\mathfrak{g})$ -module and $\mathfrak{m}$ be a reductive subalgebra of $\mathfrak{g}$ containing the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $(,)$ be an $\mathfrak{m}$ -contravariant form on V .*

- (1) *Distinct weight spaces of V are orthogonal with respect to $(,)$.*
- (2) *If W_1 and W_2 are highest weight $\mathfrak{m}$ -submodule of V with distinct highest weights, then $(W_1, W_2) = 0$.*

Proof. (1) Note that $(,)$ is also $\mathfrak{h}$ -contravariant. Let v_1 and v_2 be weight vectors of V with weight μ_1 and μ_2 , respectively. If $\mu_1 \neq \mu_2$ then there exists $H \in \mathfrak{h}$ such that $\mu_1(H) \neq \mu_2(H)$. We have $\mu_1(H)(v_1, v_2) = (H.v_1, v_2) = (v_1, H.v_2) = \mu_2(H)(v_1, v_2)$. Thus $(v_1, v_2) = 0$.

(2) Let w_i be a highest weight vector of W_i with a highest weight ν_i . Assume that $\nu_1 \neq \nu_2$ and $(W_1, W_2) \neq 0$. Then $0 \neq (U(\mathfrak{m}).w_1, U(\mathfrak{m}).w_2) = (w_1, {}^t U(\mathfrak{m})U(\mathfrak{m}).w_2) = (w_1, U(\mathfrak{m}).w_2) = (w_1, W_2)$. It follows from (1) that W_2 contains a ν_1 -weight space. Namely, $\nu_1 \leq \nu_2$. Similarly, $\nu_1 \geq \nu_2$. Thus $\nu_1 = \nu_2$. This is a contradiction. $\square$

Proposition 3.3 (cf. Humphreys [13] §6, Wallach [31]) *Let V be a $U(\mathfrak{g})$ -module. Assume that V is a highest weight module. Then we have*

- (1) *There exists a nonzero contravariant form on V , and it is unique up to constant multiple.*
- (2) *The radical of a nonzero contravariant form on V coincides with the maximal proper submodule of V .*

Proof. Let v_λ be a highest weight vector of V with a highest weight λ . First we define a $\mathbb{C}$ -linear function $\varphi_\lambda: U(\mathfrak{g}) \rightarrow \mathbb{C}$. Let $\mathfrak{g}^\pm$ denote the sum of all the positive (negative) root spaces in $\mathfrak{g}$. Define φ_λ as a composite of the projection $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{g}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^+) \rightarrow U(\mathfrak{h})$ and $\lambda: U(\mathfrak{h}) \rightarrow \mathbb{C}$. Note that $\varphi_\lambda({}^t u) = \varphi_\lambda(u)$ for $u \in U(\mathfrak{g})$. Then we define $(,)$ by

$$(p.v_\lambda, q.v_\lambda) = \varphi_\lambda({}^t qp) \quad (p, q \in U(\mathfrak{g})).$$

We must check that it is well-defined. Let V^- be the sum of weight spaces of V with weights lower than λ . Then by the definition of φ_λ , we have $u.v_\lambda \equiv \varphi_\lambda(u)v_\lambda \pmod{V^-}$.

Let Y be the unique maximal submodule of V . Obviously $Y \subset V^-$. If $v \in V$ is a cyclic vector of V , then $v_\lambda \in U(\mathfrak{g}).v$, and vice versa. In particular,

$$(3.1) \quad \varphi_\lambda(U(\mathfrak{g})u) = 0 \text{ if and only if } u.v_\lambda \in Y.$$

If $p.v_\lambda = r.v_\lambda$ for $r \in U(\mathfrak{g})$, then $(p-r).v_\lambda = 0 \in Y$. Thus $0 = \varphi_\lambda({}^tq(p-r)) = (p.v_\lambda, q.v_\lambda) - (r.v_\lambda, q.v_\lambda)$. This proves that $(,)$ is well-defined. We can easily prove that $(,)$ is a contravariant form from the definition of $(,)$ and the property of φ_λ .

Lastly, if we prove the uniqueness, then (2) follows from (3.1). Let $(,)'$ be another contravariant form on V . We have $(p.v_\lambda, q.v_\lambda)' = ({}^tqp.v_\lambda, v_\lambda)' = \varphi_\lambda({}^tqp)(v_\lambda, v_\lambda)'$, since distinct weight spaces are orthogonal by Proposition 3.2. This formula means that a contravariant form is determined only by a value $(v_\lambda, v_\lambda)'$. Thus we proved the proposition. $\square$

We introduce two contravariant forms on $M(\lambda) \simeq \mathbf{C}[n^+]$. One is a $\Psi_\lambda(U(\mathfrak{g}))$ -contravariant form and the other is an $\text{ad}(U(\mathfrak{l}))$ -contravariant form on $\mathbf{C}[n^+]$.

Definition 3.4 We define a $\Psi_\lambda(U(\mathfrak{g}))$ -contravariant form $(,)_\lambda$ by

$$(f, g)_\lambda = \varphi_\lambda({}^tgf) \quad \text{for } f, g \in \mathbf{C}[n^+] \simeq S(n^-) \subset U(\mathfrak{g}),$$

where φ_λ is the same as in the proof of Proposition 3.3.

Definition 3.5 Define symmetric $\mathbf{C}$ -bilinear form $(,)$ on $\mathbf{C}[n^+] \simeq S(n^-)$ by

$$(f, g) = ({}^tg(\partial)f)(0) \quad \text{for } f, g \in \mathbf{C}[n^+] \simeq S(n^-),$$

where ${}^tg(\partial)$ is the constant coefficient differential operator (see (2.2)) on n^+ identified with ${}^tg \in S(n^+)$, and $({}^tg(\partial)f)(0)$ means a differentiation followed by evaluation at $0 \in n^+$. This bilinear form is $\text{ad}(U(\mathfrak{l}))$ -contravariant, since the bilinear form defined by $\langle P, f \rangle = (P(\partial)f)(0)$ for $P \in S(n^+)$ and $f \in \mathbf{C}[n^+]$, is $\text{Ad}(L)$ -invariant. Moreover this contravariant form is obviously nondegenerate.

We summarize some properties of these bilinear forms.

Lemma 3.6 (1) $\Psi_\lambda(U(\mathfrak{l}))$ -contravariance and $\text{ad}(U(\mathfrak{l}))$ -contravariance are the same notion.

(2) A $\Psi_\lambda(U(\mathfrak{g}))$ -contravariant form is also $\Psi_\lambda(U(\mathfrak{l}))$ -contravariant.

$$(3) (f, gh)_\lambda = (\Psi_\lambda({}^tg)f, h)_\lambda \quad \text{for } f, g, h \in \mathbf{C}[n^+] \simeq S(n^-).$$

$$(4) (f, gh) = ({}^tg(\partial)f, h) \quad \text{for } f, g, h \in \mathbf{C}[n^+] \simeq S(n^-).$$

Proof. (1) It follows from $\Psi_\lambda(X) = \text{ad}(X) + \lambda(X)$ for $X \in \mathfrak{l}$. (2) It follows immediately from the definition of the contravariance. (3) Since $(,)_\lambda$ is $\Psi_\lambda(U(\mathfrak{g}))$ -contravariant, and since $\Psi_\lambda(u)$ is just a multiplying operator for $u \in \mathbf{C}[n^+] \simeq S(n^-)$, we obtain the identity. (4) It follows immediately from Definition 3.5. $\square$

4 The ratio of the two contravariant forms

We have defined two contravariant forms on $M(\lambda) \simeq S(n^-) \simeq \mathbf{C}[n^+]$. Since both $(,)$ and $(,)_\lambda$ are $\text{ad}(U(\mathfrak{l}))$ -contravariant by Lemma 3.6, it follows from Proposition 3.3 (1) that $(,)_\lambda$ coincides with $(,)$ up to constant multiple on each irreducible $\text{ad}(U(\mathfrak{l}))$ -submodule of $\mathbf{C}[n^+]$. In this section we define a function $q_\lambda(\mu)$ as the ratio of these two forms.

Definition 4.1 Two roots $\alpha, \beta \in \Delta$ are said to be *strongly orthogonal*, if α and β are non-proportional and both $\alpha + \beta \notin \Delta$ and $\alpha - \beta \notin \Delta$ hold.

If $\alpha, \beta \in \Delta$ and $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Delta$. Thus if α is strongly orthogonal to β then α is orthogonal to β .

We take the family of mutually strongly orthogonal roots contained in Δ_N^+ like [11]. Set $\gamma_1 = \alpha_{i_0}$. When we have taken $\gamma_1, \dots, \gamma_i$, let γ_{i+1} be the lowest root in

$$\{\alpha \in \Delta_N^+ \mid \alpha \text{ is strongly orthogonal to all } \gamma_1, \dots, \gamma_i\},$$

if this set is not empty. Let r be the index of γ_i which we could take last. Set $\lambda_i = -(\gamma_1 + \dots + \gamma_i)$ for $i \in \{1, \dots, r\}$.

Theorem 4.2 (Schmid [24]) Let I_μ be the finite dimensional irreducible $\text{ad}(U(\mathfrak{l}))$ -module with highest weight μ . We denote by $\mathbf{C}^d[n^+]$ the homogeneous component of degree

d of $\mathbb{C}[n^+]$. Then we have

$$\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{l}}(I_{\mu}, \mathbb{C}^d[n^+]) = \begin{cases} 1 & (\mu = k_1\lambda_1 + \cdots + k_r\lambda_r, \\ & \text{for some } k_j \in \mathbb{Z}_{\geq 0}, d = \sum_j jk_j), \\ 0 & (\text{otherwise}). \end{cases}$$

In particular $\mathbb{C}[n^+]$ is *multiplicity free*, that is, all the multiplicities of irreducible $\text{ad}(U(\mathfrak{l}))$ -submodules are equal to one. For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbb{Z}_{\geq 0}$), we also denote by I_{μ} the unique $\text{ad}(U(\mathfrak{l}))$ -submodule in $\mathbb{C}[n^+]$ with a highest weight μ . Then we have an irreducible decomposition

$$\mathbb{C}[n^+] = \bigoplus_{\mu \in \sum_{j=1}^r \mathbb{Z}_{\geq 0} \lambda_j} I_{\mu}.$$

Let f_i be a highest weight vector of I_{λ_i} and $f_{\mu} = f_1^{k_1} \cdots f_r^{k_r}$ for $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$. Then f_{μ} is a highest weight vector of I_{μ} .

As we stated before, two $\text{ad}(U(\mathfrak{l}))$ -contravariant forms $(\cdot, \cdot)_{\lambda}$ and $(\cdot, \cdot)$ on $\mathbb{C}[n^+]$, coincide on each irreducible submodule I_{μ} . For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbb{Z}_{\geq 0}$) and $\lambda \in \text{Hom}(\mathfrak{p}, \mathbb{C})$, we define $q_{\lambda}(\mu)$ by

$$(\cdot, \cdot)_{\lambda} = q_{\lambda}(\mu)(\cdot, \cdot) \quad \text{on } I_{\mu} \times I_{\mu}.$$

Lemma 4.3 For all μ , (f_{μ}, f_{μ}) is nonzero.

Proof. It is obvious from the definition that $(\cdot, \cdot)$ is nondegenerate on $\mathbb{C}[n^+]$. For $\mu \neq \nu$, it follows from Proposition 3.2 (2) that $(I_{\mu}, I_{\nu}) = 0$ since $\mathbb{C}[n^+]$ is multiplicity free. Thus $(\cdot, \cdot)$ is nondegenerate on each I_{μ} . We have $(f_{\mu}, f_{\mu}) \neq 0$ since the highest weight space of I_{μ} is one-dimensional. $\square$

Thanks to Lemma 4.3, we have

$$(4.1) \quad q_{\lambda}(\mu) = (f_{\mu} f_{\mu})_{\lambda} / (f_{\mu}, f_{\mu}).$$

5 b-Functions

In this section, we introduce prehomogeneous vector spaces and define b -functions of prehomogeneous vector spaces associated with $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type.

Definition 5.1 (1) A finite dimensional G -module V is called a *prehomogeneous vector space* if there exists an open G -orbit on V .

(2) A nonzero rational function f on V is called a *relative invariant* of (G, V) , if there exists a character χ of G such that $f(gv) = \chi(g)f(v)$ for all $g \in G$ and $v \in V$.

(3) A prehomogeneous vector space (G, V) is said to be *regular* if there exists a relative invariant f of (G, V) and the Hessian $\det(\partial^2 f / \partial x_i \partial x_j)$ is not identically zero, where $\{x_i\}$ is a linear coordinate system of V .

Remark 5.2 It is known that (L, Ad, n^+) is a prehomogeneous vector space and the open L -orbit contains $X_{\gamma_1} + \cdots + X_{\gamma_r}$, where X_{γ_j} is an element of our fixed Chevalley basis [21, Théorème 2.8]. The (L, Ad, n^+) is regular if and only if Hermitian symmetric space G/L is of tube type [18].

All the pairs $(\mathfrak{g}, i_0)$ of commutative parabolic type, where (L, Ad, n^+) becomes a regular prehomogeneous vector space, are listed in Figure 2. Notation is the same as before.

Let H_{γ_j} be the coroot of γ_j , that is, $H_{\gamma_j} \in [\mathfrak{g}^{\gamma_j}, \mathfrak{g}^{-\gamma_j}]$ and $\gamma_j(H_{\gamma_j}) = 2$. Set $\mathfrak{h}^- = \sum_{j=1}^r \mathbb{C}H_{\gamma_j}$.

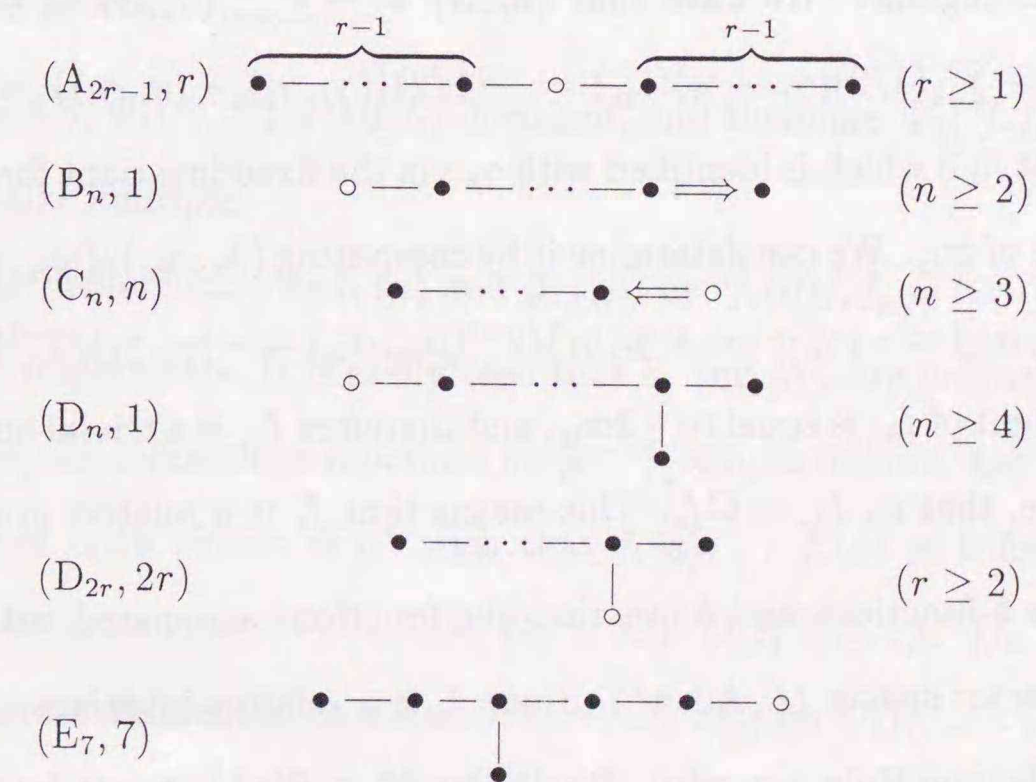


Figure 2: regular type

Theorem 5.3 (Moore [20, Theorem 2]) (1) For $\alpha \in \Delta_L^+$, the possible forms of $\alpha|_{\mathfrak{h}^-}$ are as follows:

$$\frac{1}{2}(\gamma_j - \gamma_i) \quad (1 \leq i < j \leq r), \quad -\frac{1}{2}\gamma_i \quad (1 \leq i \leq r), \quad 0.$$

(2) For $\alpha \in \Delta_N^+ \setminus \{\gamma_1, \dots, \gamma_r\}$, the possible forms of $\alpha|_{\mathfrak{h}^-}$ are as follows:

$$\frac{1}{2}(\gamma_j + \gamma_i) \quad (1 \leq i < j \leq r), \quad \frac{1}{2}\gamma_i \quad (1 \leq i \leq r).$$

(3) If $(L, \text{Ad}, \mathfrak{n}^+)$ is a regular prehomogeneous vector space, then $\pm(1/2)\gamma_i$ in the above possibilities does not occur.

We exclusively deal with the case when $(L, \text{Ad}, \mathfrak{n}^+)$ is a regular prehomogeneous vector space for the rest of this section. We can find relative invariant of a regular prehomogeneous vector space $(L, \text{Ad}, \mathfrak{n}^+)$ using Theorem 5.3.

Lemma 5.4 If $(L, \text{Ad}, \mathfrak{n}^+)$ is regular, then $\lambda_r = -2\varpi_{i_0}$. Thus f_r is a relative invariant of $(L, \text{Ad}, \mathfrak{n}^+)$.

Proof. For $\alpha \in \Delta_L$, we have $\alpha|_{\mathfrak{h}^-} = (1/2)(\gamma_j - \gamma_i)$ ($i, j \in \{1, \dots, r\}$) by Theorem 5.3, since $(L, \text{Ad}, \mathfrak{n}^+)$ is regular. We have that $(\lambda_r, \alpha) = -\sum_{k=1}^r (\gamma_k, \alpha) = -\sum_k \alpha(t_{\gamma_k}) = -\sum_k \alpha|_{\mathfrak{h}^-}(t_{\gamma_k}) = -\sum_k (1/2)(\gamma_j - \gamma_i)(t_{\gamma_k}) = -(1/2)\{(\gamma_j, \gamma_j) - (\gamma_i, \gamma_i)\} = 0$, where t_{γ_k} denotes the element in $\mathfrak{h}$ which is identified with γ_k via the fixed invariant form. Thus λ_r is a constant multiple of ϖ_{i_0} . We can determine it by computing $(\lambda_r, \alpha_{i_0})/(\varpi_{i_0}, \alpha_{i_0})$. We have that $(\lambda_r, \alpha_{i_0})/(\varpi_{i_0}, \alpha_{i_0}) = -(\gamma_1 + \dots + \gamma_r, \gamma_1)/2^{-1}(\alpha_{i_0}, \alpha_{i_0}) = -(\gamma_1, \gamma_1)/2^{-1}(\gamma_1, \gamma_1) = -2$.

The highest weight of I_{λ_r} is equal to $-2\varpi_{i_0}$, and therefore I_{λ_r} is a trivial one-dimensional $\text{ad}(U([l, l]))$ -module, that is, $I_{\lambda_r} = \mathbb{C}f_r$. This means that f_r is a relative invariant. $\square$

Here we define b -functions and b -function-like functions associated with the regular prehomogeneous vector spaces $(L, \text{Ad}, \mathfrak{n}^+)$. Since f_r is a relative invariant, $g \in L$ acts on f_r by a certain scalar multiple, say $\chi(g)$. Dually for ${}^t f_r \in S(\mathfrak{n}^+)$, g acts by $\chi(g)^{-1}$. Thus ${}^t f_r f_r \in U(\mathfrak{g})$ is $\text{Ad}(L)$ -invariant, and therefore the differential operator ${}^t f_r(\partial)f_r$ on $\mathfrak{n}^+$ is $\text{Ad}(L)$ -invariant. Then ${}^t f_r(\partial)f_r \cdot f_\mu$ has a weight μ and it is still an $\text{ad}(U(l))$ -maximal

vector. Thus ${}^t f_r(\partial)f_r$ acts on $f_\mu \in \mathbb{C}[\mathfrak{n}^+]$ ($\mu = k_1\lambda_1 + \dots + k_r\lambda_r$) by a certain scalar multiple, since $\mathbb{C}[\mathfrak{n}^+]$ is multiplicity free.

As for an $\text{Ad}(L)$ -invariance of $\Psi_\lambda({}^t f_r f_r)$, we need the following lemma.

Lemma 5.5 The representation Ψ_λ is $\text{Ad}(L)$ -equivariant. Namely, $\text{Ad}(g) \cdot \Psi_\lambda(u) := \text{Ad}(g) \circ \Psi_\lambda(u) \circ \text{Ad}(g^{-1}) = \Psi_\lambda(\text{Ad}(g)u)$, for $u \in U(\mathfrak{g})$ and $g \in L$.

Proof. We have a canonical linear isomorphism $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda \rightarrow \mathbb{C}[\mathfrak{n}^+]$. Thus we can define linear mapping $\alpha : U(\mathfrak{g}) \rightarrow \mathbb{C}[\mathfrak{n}^+]$ as a composite of the canonical surjection $U(\mathfrak{g}) \rightarrow M(\lambda)$ and the above canonical isomorphism $M(\lambda) \rightarrow \mathbb{C}[\mathfrak{n}^+]$.

First we show that α commutes with the $\text{Ad}(L)$ -action. Since $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{p})$ from PBW theorem, any $u \in U(\mathfrak{g})$ is a sum of elements such as np ($n \in U(\mathfrak{n}^-)$, $p \in U(\mathfrak{p})$). We may assume $u = np$ without loss of generality. For $g \in L$, we have $\alpha(\text{Ad}(g)(np)) = \alpha(\text{Ad}(g)n \text{Ad}(g)p) = \text{Ad}(g)n \cdot \lambda(\text{Ad}(g)p) = \text{Ad}(g)n \cdot \lambda(p) = \text{Ad}(g)\alpha(np)$. Thus α commutes with the $\text{Ad}(L)$ -action.

It is easy to see that $\Psi_\lambda(u)f = \alpha(uf)$ for $u \in U(\mathfrak{g})$ and $f \in \mathbb{C}[\mathfrak{n}^+] \simeq S(\mathfrak{n}^-)$. Thus we have $\text{Ad}(g) \circ \Psi_\lambda(u) \circ \text{Ad}(g^{-1})f = \text{Ad}(g)\alpha(u \text{Ad}(g^{-1})f) = \alpha((\text{Ad}(g)u)f) = \Psi_\lambda(\text{Ad}(g)u)f$. The assertion is proved. $\square$

By Lemma 5.5, $\Psi_\lambda({}^t f_r f_r)$ is $\text{Ad}(L)$ -invariant, and therefore $\Psi_\lambda({}^t f_r f_r)$ also acts on f_μ by a certain scalar multiple.

Then we can define functions $b_r(\mu)$ and $\beta_{\lambda, r}(\mu)$ by ${}^t f_r(\partial)f_r f_\mu = b_r(\mu)f_\mu$ and $\Psi_\lambda({}^t f_r f_r) f_\mu = \beta_{\lambda, r}(\mu)f_\mu$, respectively. It is easily seen that b_r and $\beta_{\lambda, r}$ are polynomials.

Moreover we can define these functions for $\mu \in \sum \mathbb{C}\lambda_i$ as follows. Let A be a connected simply connected open subset of $\mathfrak{n}^+$ such that $f_1(a), \dots, f_r(a) \neq 0$ for all $a \in A$. Set $\mathcal{O} = \mathbb{C}[\mathfrak{n}^+]$. For $\mu = k_1\lambda_1 + \dots + k_r\lambda_r$ ($k_j \in \mathbb{C}$), $\mathcal{O}[f_1^{-1}, \dots, f_r^{-1}]f_\mu$ on A becomes a D_A -module. Here a differential operator $\partial/\partial x \in D_A$ acts on $\mathcal{O}[f_1^{-1}, \dots, f_r^{-1}]f_\mu$ by

$$\begin{aligned} \frac{\partial}{\partial x}(\varphi f_\mu) &= \frac{\partial \varphi}{\partial x} f_\mu + \varphi \frac{\partial f_\mu}{\partial x} \\ &= \frac{\partial \varphi}{\partial x} f_\mu + \varphi \frac{\partial \log f_\mu}{\partial x} f_\mu \end{aligned}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial \log f_\mu}{\partial x} \right) (\varphi) f_\mu,$$

where $\varphi \in \mathcal{O}[f_1^{-1}, \dots, f_r^{-1}]$ and $\partial \log f_\mu / \partial x = \sum_i k_i f_i^{-1} \partial f_i / \partial x$. Then there exists b -function $b'_r(\mu) \in \mathbb{C}[k_1, \dots, k_r]$ such that ${}^t f_r(\partial) f_r f_\mu = b'_r(\mu) f_\mu$. Here $b_r(\mu)$ and $b'_r(\mu)$ coincide when $\mu \in \sum_i \mathbb{Z}_{\geq 0} \lambda_i$, and therefore they coincide for all $\mu \in \sum_i \mathbb{C} \lambda_i$. Similarly we can define $\beta_{\lambda,r}(\mu)$ for $\mu \in \sum_i \mathbb{C} \lambda_i$. In this way, we can define polynomials $b_r(\mu)$ and $\beta_{\lambda,r}(\mu)$ by the following definition.

Definition 5.6 Assume that $(L, \text{Ad}, \mathfrak{n}^+)$ is regular. Define polynomials b_r and $\beta_{\lambda,r}$ by

$${}^t f_r(\partial) f_r f_\mu = b_r(\mu) f_\mu,$$

$$\Psi_\lambda({}^t f_r f_r) f_\mu = \beta_{\lambda,r}(\mu) f_\mu,$$

for $\lambda \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ and $\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$ ($k_j \in \mathbb{C}$).

6 Main theorem

We continue to assume that $(L, \text{Ad}, \mathfrak{n}^+)$ is regular in this section. We state our main theorem. The theorem needs the normalization of f_r . By [21, Théorème 2.8], the open L -orbit on $(L, \text{Ad}, \mathfrak{n}^+)$ contains the vector $X_{\gamma_1} + \dots + X_{\gamma_r}$. Thus $f_r(X_{\gamma_1} + \dots + X_{\gamma_r}) \neq 0$. Then we normalize f_r by

$$(6.1) \quad f_r(X_{\gamma_1} + \dots + X_{\gamma_r}) = 1.$$

We fix this normalization for the rest of this thesis.

We define the constants which will be used in our main theorem. Let $\rho_r \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ be half the sum of the roots of $\mathfrak{n}^+$, that is,

$$\rho_r(X) = \left(\frac{1}{2} \sum_{\alpha \in \Delta_N^+} \alpha \right) (X) = \frac{1}{2} \text{Tr}_{\mathfrak{n}^+} \text{ad}(X) \quad (X \in \mathfrak{p}).$$

Since ρ_r is a constant multiple of ϖ_{i_0} , we define the complex number ρ_r^0 by

$$(6.2) \quad \rho_r = \rho_r^0 \varpi_{i_0}.$$

The following theorem and its corollary (Corollary 8.1) are our main results. This theorem suggests that the structure of a scalar generalized Verma module has a certain connection with b -functions of prehomogeneous vector spaces associated with them.

Theorem 6.1 Assume that $(L, \text{Ad}, \mathfrak{n}^+)$ is a regular prehomogeneous vector space. If f_r is normalized by (6.1), then for $\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$ ($k_i \in \mathbb{C}$),

$$\beta_{\lambda,r}(\mu) = (-1)^r b_r(\mu) b_r(\mu - (\lambda^0 + \rho_r^0) \lambda_r),$$

where λ^0 is the complex number defined by $\lambda = \lambda^0 \varpi_{i_0}$.

We will prove this theorem in §14 and §15.

7 Subalgebras of $\mathfrak{g}$

In this section, we will define certain subalgebras of $\mathfrak{g}$ and show some properties related to these subalgebras. Then we define b -functions and b -function-like functions associated with the subalgebras. We return to the situation where $(L, \text{Ad}, \mathfrak{n}^+)$ is not necessarily regular.

We define the subalgebras of $\mathfrak{g}$ following [31]. Set

$$\Delta_{N,i}^+ = \{ \alpha \in \Delta_N^+ ; \alpha|_{\mathfrak{h}^-} = (\gamma_k + \gamma_j)/2 \text{ for some } 1 \leq j < k \leq i \} \cup \{ \gamma_1, \dots, \gamma_i \},$$

$$\mathfrak{n}_i^\pm = \sum_{\alpha \in \Delta_{N,i}^+} \mathfrak{g}^{\pm \alpha}.$$

Let $\mathfrak{l}_i$ be $[\mathfrak{n}_i^+, \mathfrak{n}_i^-]$. It is easy to see that $\mathfrak{l}_i$ is a Lie algebra. Set

$$\mathfrak{p}_i = \mathfrak{l}_i + \mathfrak{n}_i^+,$$

$$\mathfrak{g}_i = \mathfrak{n}_i^- + \mathfrak{l}_i + \mathfrak{n}_i^+,$$

$$\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i,$$

and let L_i be the connected closed subgroup of G corresponding to $\mathfrak{l}_i$. Then $(\mathfrak{g}_i, \mathfrak{p}_i)$ is of commutative parabolic type, and $(L_i, \text{Ad}, \mathfrak{n}_i^+)$ is a regular prehomogeneous vector space. Obviously, the maximal family of mutually strongly orthogonal roots contained in $\Delta_{N,i}^+$,

constructed in the same way, coincides with $\{\gamma_1, \dots, \gamma_i\}$. More precisely, these roots should be restricted to $\mathfrak{h}_i$.

As in Theorem 6.1 we normalize f_i so that

$$(7.1) \quad f_i(X_{\gamma_1} + \dots + X_{\gamma_r}) = 1 \quad \text{for } i \in \{1, \dots, r\},$$

or equivalently,

$$(7.2) \quad f_i(X_{\gamma_1} + \dots + X_{\gamma_i}) = 1 \quad \text{for } i \in \{1, \dots, r\}.$$

Here we define an index set of a basis of the λ_i -weight space of $\mathbf{C}[\mathfrak{n}^+]$, and prove its properties.

Definition 7.1 For $i \in \{1, \dots, r\}$, we fix a subset

$$\mathcal{B}_i \subset \{(\beta_1, \dots, \beta_i) \mid \beta_j \in \Delta_N^+, \beta_1 + \dots + \beta_i = \gamma_1 + \dots + \gamma_i\}$$

such that the set

$$\{X_{-\beta_1} \cdots X_{-\beta_i} \in \mathbf{C}[\mathfrak{n}^+] \mid (\beta_1, \dots, \beta_i) \in \mathcal{B}_i\}$$

forms a basis of the λ_i -weight space of $\mathbf{C}[\mathfrak{n}^+]$.

We set

$$B_i = (\gamma_1, \dots, \gamma_i).$$

Obviously $\mathcal{B}_i$ contains $(\gamma_1, \dots, \gamma_i)$ up to order. We may assume that $B_i \in \mathcal{B}_i$.

The λ_i -weight space contains f_i . Thus we define $a_B \in \mathbf{C}$ ($B \in \mathcal{B}_i$) by

$$f_i = \sum_{B \in \mathcal{B}_i} a_B X_{-\beta_1} \cdots X_{-\beta_i} \quad (B = (\beta_1, \dots, \beta_i)).$$

Lemma 7.2 For $i \in \{1, \dots, r\}$ we have

- (1) If $(\beta_1, \dots, \beta_i) \in \mathcal{B}_i$, then each β_j is in $\Delta_{N,i}^+$.
- (2) $a_{B_i} = 2^{-i}(\gamma_1, \gamma_1)^i$.

Proof. (1) By Theorem 5.3, $(\beta_1 + \dots + \beta_i)|_{\mathfrak{h}^-} = (\gamma_{e_1} + \dots + \gamma_{e_m})/2$, where $e_k \in \{1, \dots, r\}$ and m is an integer such that $i \leq m \leq 2i$. On the other hand, $(\beta_1 + \dots + \beta_i)|_{\mathfrak{h}^-} =$

$(\gamma_1 + \dots + \gamma_i)|_{\mathfrak{h}^-} = \gamma_1 + \dots + \gamma_i$. Here $\gamma_1, \dots, \gamma_r$ are mutually (strongly) orthogonal, and hence linearly independent in $(\mathfrak{h}^-)^*$. Thus m must be equal to $2i$ and $(e_1, \dots, e_{2i})$ must be equal to $(1, 1, 2, 2, \dots, i, i)$ up to order. This means each β_j belongs to $\Delta_{N,i}^+$.

(2) First we note that B_i is the unique element in $\mathcal{B}_i$ which consists only of γ_j 's. Then we have

$$\begin{aligned} 1 &= f_i(X_{\gamma_1} + \dots + X_{\gamma_i}) \\ &= \sum_{B \in \mathcal{B}_i} a_B \langle X_{-\beta_1}, X_{\gamma_1} + \dots + X_{\gamma_i} \rangle \cdots \langle X_{-\beta_i}, X_{\gamma_1} + \dots + X_{\gamma_i} \rangle \\ &= a_{B_i} \langle X_{-\gamma_1}, X_{\gamma_1} + \dots + X_{\gamma_i} \rangle \cdots \langle X_{-\gamma_i}, X_{\gamma_1} + \dots + X_{\gamma_i} \rangle \\ &= a_{B_i} \frac{2}{(\gamma_1, \gamma_1)} \cdots \frac{2}{(\gamma_i, \gamma_i)} \\ &= a_{B_i} 2^i (\gamma_1, \gamma_1)^{-i}, \end{aligned}$$

where we used (2.1). Thus we have the assertion. $\square$

Lemma 7.3 For $i \in \{1, \dots, r\}$, we have $f_j \in \mathbf{C}[\mathfrak{n}_i^+]$ for $j \in \{1, \dots, i\}$ and an irreducible $\text{ad}(U(\mathfrak{l}_i))$ -module decomposition

$$\mathbf{C}[\mathfrak{n}_i^+] = \bigoplus_{\mu \in \sum_{j=1}^i \mathbf{Z}_{\geq 0} \lambda_j} (I_\mu \cap \mathbf{C}[\mathfrak{n}_i^+]).$$

Proof. First we consider the irreducible decomposition. By Theorem 4.2, the set of the possible $\text{ad}(U(\mathfrak{l}_i))$ -maximal weights in $\mathbf{C}[\mathfrak{n}_i^+]$ are exactly equal to $\sum_{j=1}^i \mathbf{Z}_{\geq 0} \lambda_j$. It is obvious that $I_\mu \cap \mathbf{C}[\mathfrak{n}_i^+]$ is an $\text{ad}(U(\mathfrak{l}_i))$ -module. Thus we have only to show that each $I_\mu \cap \mathbf{C}[\mathfrak{n}_i^+]$ contains f_μ as a highest weight vector with highest weight μ . Namely, we have only to show that f_j belongs to $\mathbf{C}[\mathfrak{n}_i^+]$ for $j \in \{1, \dots, i\}$. Lemma 7.2 (1) proves it. $\square$

We consider $\mathbf{C}[\mathfrak{n}_i^+]$ as a $U(\mathfrak{g}_i)$ -module in the following way. The restriction $\lambda|_{\mathfrak{p}_i}$ leads to the generalized Verma module $M(\lambda|_{\mathfrak{p}_i}) = U(\mathfrak{g}_i) \otimes_{U(\mathfrak{p}_i)} \mathbf{C}_{\lambda|_{\mathfrak{p}_i}}$ which is isomorphic to $\mathbf{C}[\mathfrak{n}_i^+]$ as a vector space. We denote this representation of $U(\mathfrak{g}_i)$ on $\mathbf{C}[\mathfrak{n}_i^+]$ by $\Psi_{\lambda|_{\mathfrak{p}_i}}$. Note that this representation is not equivalent to the restriction of Ψ_λ to $U(\mathfrak{g}_i)$. By the same

argument as in Lemma 5.4, we can show that $\lambda_i = -2\varpi_{i_0}$ on $\mathfrak{h}_i$, and therefore f_i is the relative invariant of $(L_i, \text{Ad}, \mathfrak{n}_i^+)$. The following definition is a generalization of Definition 5.6.

Definition 7.4 For $i \in \{1, \dots, r\}$, define polynomials b_i and $\beta_{\lambda, i}$ by

$$\begin{aligned} {}^t f_i(\partial) f_i f_\mu &= b_i(\mu) f_\mu, \\ \Psi_{\lambda|_{\mathfrak{p}_i}}({}^t f_i f_i) f_\mu &= \beta_{\lambda, i}(\mu) f_\mu \end{aligned}$$

for $\lambda \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ and $\mu = k_1 \lambda_1 + \dots + k_i \lambda_i$ ($k_j \in \mathbb{C}$).

We define a character ρ_i by

$$\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_{N, i}^+} \alpha \in \text{Hom}(\mathfrak{p}_i, \mathbb{C}).$$

Since ρ_i is a constant multiple of ϖ_{i_0} on $\mathfrak{h}_i$, we define the complex number ρ_i^0 by

$$(7.3) \quad \rho_i = \rho_i^0 \varpi_{i_0} \quad \text{on } \mathfrak{h}_i.$$

Since each $(L_i, \text{Ad}, \mathfrak{n}_i^+)$ is a regular prehomogeneous vector space, even if $(L, \text{Ad}, \mathfrak{n}^+)$ is not regular, Theorem 6.1 implies the following assertion.

Theorem 7.5 Assume that f_j ($j \in \{1, \dots, r\}$) is normalized as (7.1) and that $(\mathfrak{g}, \mathfrak{p})$ is of commutative parabolic type, where the prehomogeneous vector space $(L, \text{Ad}, \mathfrak{n}^+)$ is not necessarily regular. We fix $i \in \{1, \dots, r\}$. Then for $\mu = k_1 \lambda_1 + \dots + k_i \lambda_i$ ($k_j \in \mathbb{C}$),

$$\beta_{\lambda, i}(\mu) = (-1)^i b_i(\mu) b_i(\mu - (\lambda^0 + \rho_i^0) \lambda_i),$$

where λ^0 is the complex number defined by $\lambda|_{\mathfrak{p}_i} = \lambda^0 \varpi_{i_0}$.

8 An expression of $q_\lambda(\mu)$ in terms of b -function

In this section we give a corollary to the main theorem. The corollary is a part of our main results. It indicates that a contravariant form on a scalar generalized Verma module is deeply related to b -functions.

Corollary 8.1 If f_i 's are normalized by (7.1), then for $\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$ ($k_i \in \mathbb{Z}_{\geq 0}$),

$$q_\lambda(\mu) = (-1)^{\deg f_\mu} \prod_{i=1}^r \prod_{j=0}^{k_i-1} b_i(k_1 \lambda_1 + \dots + k_{i-1} \lambda_{i-1} + j \lambda_i - (\lambda^0 + \rho_i^0) \lambda_i).$$

Proof. We can calculate $q_\lambda(\mu)$ using (4.1): $q_\lambda(\mu) = (f_\mu, f_\mu)_\lambda / (f_\mu, f_\mu)$. First we compute $(f_\mu, f_\mu)_\lambda$.

By Definition 3.4, the construction of $(\cdot)_\lambda$, we can compute $(f_\mu, f_\mu)_\lambda$ within the subalgebra $U(\mathfrak{g}_r)$ of $U(\mathfrak{g})$, since $f_\mu \in \mathbb{C}[\mathfrak{n}_r^+]$ from Lemma 7.3. In other words, $(f_\mu, f_\mu)_\lambda = (f_\mu, f_\mu)_{\lambda|_{\mathfrak{p}_r}}$, where $(\cdot)_{\lambda|_{\mathfrak{p}_r}}$ is the $\Psi_{\lambda|_{\mathfrak{p}_r}}(U(\mathfrak{g}_r))$ -contravariant form on $\mathbb{C}[\mathfrak{n}_r^+]$ constructed in the same way as in Definition 3.4. Then we have

$$\begin{aligned} (f_\mu, f_\mu)_\lambda &= (f_\mu, f_\mu)_{\lambda|_{\mathfrak{p}_r}} \\ &= (f_r f_{\mu-\lambda_r}, f_r f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\ &= (\Psi_{\lambda|_{\mathfrak{p}_r}}(f_r) f_{\mu-\lambda_r}, \Psi_{\lambda|_{\mathfrak{p}_r}}(f_r) f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\ &= (\Psi_{\lambda|_{\mathfrak{p}_r}}({}^t f_r f_r) f_{\mu-\lambda_r}, f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\ &= \beta_{\lambda, r}(\mu - \lambda_r) (f_{\mu-\lambda_r}, f_{\mu-\lambda_r})_{\lambda|_{\mathfrak{p}_r}} \\ &= \dots \\ (8.1) \quad &= \beta_{\lambda, r}(\mu - \lambda_r) \dots \beta_{\lambda, r}(\mu - k_r \lambda_r) (f_{\mu-k_r \lambda_r}, f_{\mu-k_r \lambda_r})_{\lambda|_{\mathfrak{p}_r}}, \end{aligned}$$

by Lemma 3.6 (3). Here $f_{\mu-k_r \lambda_r} = f_1^{k_1} \dots f_{r-1}^{k_{r-1}} \in \mathbb{C}[\mathfrak{n}_{r-1}^+]$. Thus $(f_{\mu-k_r \lambda_r}, f_{\mu-k_r \lambda_r})_{\lambda|_{\mathfrak{p}_r}} = (f_{\mu-k_r \lambda_r}, f_{\mu-k_r \lambda_r})_{\lambda|_{\mathfrak{p}_{r-1}}}$ as before. Then we can apply Lemma 3.6 (3) again to (8.1), and at last we have

$$(f_\mu, f_\mu)_\lambda = \prod_{i=1}^r \prod_{j=0}^{k_i-1} \beta_{\lambda, i}(k_1 \lambda_1 + \dots + k_{i-1} \lambda_{i-1} + j \lambda_i).$$

Similarly, it follows from Lemma 3.6 (4) that

$$(f_\mu, f_\mu) = \prod_{i=1}^r \prod_{j=0}^{k_i-1} b_i(k_1 \lambda_1 + \dots + k_{i-1} \lambda_{i-1} + j \lambda_i).$$

Then we have

$$q_\lambda(\mu) = (f_\mu f_\mu)_\lambda / (f_\mu, f_\mu)$$

$$\begin{aligned}
&= \prod_{i=1}^r \prod_{j=0}^{k_i-1} \beta_{\lambda,i}(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i) / b_i(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i) \\
&= \prod_{i=1}^r \prod_{j=0}^{k_i-1} (-1)^i b_i(k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1} + j\lambda_i - (\lambda^0 + \rho_i^0)\lambda_i).
\end{aligned}$$

Here $\sum_{i=1}^r \sum_{j=0}^{k_i-1} i = \sum_{i=1}^r ik_i = \deg f_\mu$. Thus the corollary is proved. $\square$

9 Another expression of $q_\lambda(\mu)$ in terms of b -function

We give some applications in sections from 10 to 13. In this section we show some formulas for $b_i(\mu)$ and $q_\lambda(\mu)$ which will be used in the later sections.

Set

$$(9.1) \quad b_i(s) = b_i(s\lambda_i) \quad (s \in \mathbb{C}).$$

First we give a formula which expresses $b_r(\mu)$ in terms of $b_i(s)$ ($i \in \{1, \dots, r\}$) in Proposition 9.3. Note that this formula does not depend on the main theorems (Theorem 6.1, Theorem 7.5), and we will use it to prove the main theorems. We begin with the following lemma.

Lemma 9.1 *Assume that $(L, \text{Ad}, \mathfrak{n}^+)$ is regular. Let w_L be the longest element of the Weyl group of $(\mathfrak{l}, \mathfrak{h})$. Then w_L reverses $(\gamma_1, \dots, \gamma_r)$, that is, $w_L \cdot \gamma_i = \gamma_{r-i+1}$ for $i \in \{1, \dots, r\}$.*

Proof. We prove the lemma by an induction on i . For $i = 1$, we recall two facts. First, $\mathfrak{n}^+$ is an irreducible $\mathfrak{l}$ -module. Indeed, if $\mathfrak{m} \subset \mathfrak{n}^+$ is a proper $\mathfrak{l}$ -submodule, then it is easy to see that $\mathfrak{m} \oplus [\mathfrak{n}^-, \mathfrak{m}] \oplus [\mathfrak{n}^-, [\mathfrak{n}^-, \mathfrak{m}]]$ is a proper ideal of $\mathfrak{g}$. Thus the simplicity of $\mathfrak{g}$ implies that $\mathfrak{m} = 0$. (Theorem 4.2 also proves this irreducibility.) Second, w_L reverses the standard order of weights in a weight module. In particular, w_L maps the lowest weight to the highest weight.

Here, $\gamma_1 = \alpha_{i_0}$ is the lowest weight of $\mathfrak{n}^+$. We show that γ_r is the highest weight of $\mathfrak{n}^+$. If it is not the case, then there exists $\alpha \in \Delta_L^+$ such that $\gamma_r + \alpha \in \Delta_N^+$. By

Theorem 5.3, $(\gamma_r + \alpha)|_{\mathfrak{h}^-} = \gamma_r + (\gamma_l - \gamma_k)/2$ for some $1 \leq k \leq l \leq r$. On the other hand, $(\gamma_r + \alpha)|_{\mathfrak{h}^-} = (\gamma_q + \gamma_p)/2$ for some $1 \leq p < q \leq r$ again by Theorem 5.3. It follows from the linear independence of $\{\gamma_j\}$ that $k = l = p = q = r$. This contradicts $p < q$. We proved the case $i = 1$.

If we proved until i , then the sequences $w_L \cdot \gamma_i < w_L \cdot \gamma_{i-1} < \cdots < w_L \cdot \gamma_1$ and $\gamma_{r-i+1} < \gamma_{r-i+2} < \cdots < \gamma_r$ coincide. Here, from the construction of $\{\gamma_j\}$, $w_L \cdot \gamma_{i+1}$ is the highest root among the roots in Δ_N^+ which are strongly orthogonal to all $w_L \cdot \gamma_i, \dots, w_L \cdot \gamma_1$. Thus $\gamma_{r-i} \leq w_L \cdot \gamma_{i+1}$.

On the other hand, by Theorem 5.3, $w_L \cdot \gamma_{i+1}|_{\mathfrak{h}^-} = (\gamma_l + \gamma_k)/2$ for some $1 \leq k \leq l \leq r-i$, since it is orthogonal to all $\gamma_{r-i+1}, \dots, \gamma_1$. Thus $w_L \cdot \gamma_{i+1}$ must be equal to γ_{r-i} . We proved the lemma. $\square$

Lemma 9.2 (cf. Rubenthaler-Schiffmann [23, Lemme 5.6]) *Let $\tilde{f}_i$ be a lowest weight vector of $I_{\lambda_{r-i}}$ normalized by*

$$\tilde{f}_i(X_{\gamma_1} + \cdots + X_{\gamma_r}) = \tilde{f}_i(X_{\gamma_{i+1}} + \cdots + X_{\gamma_r}) = 1,$$

for $i \in \{1, \dots, r-1\}$. Then we have

$${}^t \tilde{f}_i(\partial) f_1^{k_1} \cdots f_i^{k_i} f_r^{m+1} = b_{r-i}(m) f_1^{k_1} \cdots f_{i-1}^{k_{i-1}} f_i^{k_i+1} f_r^m,$$

for $i \in \{1, \dots, r-1\}$ and $k_1, \dots, k_i, m \in \mathbb{Z}_{\geq 0}$.

Proof. Set

$$\begin{aligned}
\Delta_{M,i}^+ &= \{\alpha \in \Delta_N^+; \alpha|_{\mathfrak{h}^-} = (\gamma_k + \gamma_j)/2 \text{ for some } i < j < k \leq r\} \cup \{\gamma_{i+1}, \dots, \gamma_r\}, \\
\mathfrak{m}_i^+ &= \sum_{\alpha \in \Delta_{M,i}^+} \mathfrak{g}^\alpha.
\end{aligned}$$

First we prove

$$(9.2) \quad f_r(x+y) = f_i(x) \tilde{f}_i(y) \quad (x \in \mathfrak{n}_i^+, y \in \mathfrak{m}_i^+).$$

We have $[l_i, m_i^+] = 0$, since a sum of a root in $\Delta_{L,i}$ and a root in $\Delta_{M,i}^+$ can not be a root in Δ_N^+ by Theorem 5.3. Thus $\text{Ad}(L_i)$ acts trivially on m_i^+ and $f_r(x+y)$ is a relative invariant with respect to $\text{Ad}(L_i)$ -action as a function of $x \in n_i^+$. Its weight $\lambda_r|_{\mathfrak{h}_i^-}$ is equal to $\lambda_i|_{\mathfrak{h}_i^-}$, which is the weight of $f_i(x)$ as a function of $x \in n_i^+$. Thus $f_r(x+y)$ is a scalar multiple of $f_i(x)$ as a function of $x \in n_i^+$. Similarly, it is a scalar multiple of $\tilde{f}_i(y)$ as a function of $y \in m_i^+$. We have (9.2) by the normalization.

Let N_L be the nilpotent subgroup of L corresponding to the nilpotent subalgebra $\sum_{\alpha \in \Delta_L^+} \mathfrak{g}^\alpha$ of $\mathfrak{l}$. Then both sides are $\text{Ad}(N_L)$ -stable, and they have the same weight. Thus they coincide up to a constant multiple. Let A be an affine space $X_{\gamma_1} + \cdots + X_{\gamma_i} + m_i^+ \subset n^+$. Then we have

$$\begin{aligned} {}^t\tilde{f}_i(\partial) f_1^{k_1} \cdots f_i^{k_i} f_r^{m+1}|_A &= {}^t\tilde{f}_i(\partial) (f_1^{k_1} \cdots f_i^{k_i} f_r^{m+1}|_A) \\ &= {}^t\tilde{f}_i(\partial) \tilde{f}_i^{m+1}|_A, \end{aligned}$$

and

$$b_{r-i}(m) f_1^{k_1} \cdots f_{i-1}^{k_{i-1}} f_i^{k_i+1} f_r^m|_A = b_{r-i}(m) \tilde{f}_i^m|_A.$$

Thus we have only to show that ${}^t\tilde{f}_i(\partial) \tilde{f}_i^{m+1} = b_{r-i}(m) \tilde{f}_i^m$.

We take a representative $\dot{w}_L \in N_L(H)$ of the longest element w_L of the Weyl group of L . Since $\text{Ad}(\dot{w}_L)\mathfrak{g}^\alpha \subset \mathfrak{g}^{w_L\alpha}$ for $\alpha \in \Delta$, we can define constants $e_\alpha \in \mathbb{C}^\times$ by

$$\text{Ad}(\dot{w}_L)X_\alpha = e_\alpha X_{w_L\alpha}.$$

Then we have

$$\begin{aligned} 2/(\alpha, \alpha) &= \langle X_\alpha, X_{-\alpha} \rangle \\ &= \langle \text{Ad}(\dot{w}_L)X_\alpha, \text{Ad}(\dot{w}_L)X_{-\alpha} \rangle \\ &= \langle c_\alpha X_{w_L\alpha}, c_{-\alpha} X_{-w_L\alpha} \rangle \\ &= c_\alpha c_{-\alpha} \cdot 2/(\alpha, \alpha) \\ &= c_\alpha c_{-\alpha} \cdot 2/(\alpha, \alpha), \end{aligned}$$

and hence

$$c_\alpha c_{-\alpha} = 1 \quad (\alpha \in \Delta).$$

Since $\text{Ad}(\dot{w}_L)$ maps a highest weight vector to a lowest weight vector in $I_{\lambda_{r-i}}$, we have $\text{Ad}(\dot{w}_L)f_{r-i} = C_i \tilde{f}_i$ for some $C_i \in \mathbb{C}^\times$. The unique term in $\text{Ad}(\dot{w}_L)f_{r-i}$ which does not vanish at $X_{\gamma_1} + \cdots + X_{\gamma_r}$ is $\text{Ad}(\dot{w}_L)(a_{B_{r-i}}X_{-\gamma_1} \cdots X_{-\gamma_{r-i}}) = a_{B_{r-i}}c_{-\gamma_1} \cdots c_{-\gamma_{r-i}}X_{-\gamma_r} \cdots X_{-\gamma_{i+1}}$, where $a_{B_{r-i}}$ is defined in Definition 7.1. On the other hand, the unique term in $C_i \tilde{f}_i$ which does not vanish at $X_{\gamma_1} + \cdots + X_{\gamma_r}$ is equal to $C_i a_{B_{r-i}}X_{-\gamma_r} \cdots X_{-\gamma_{i+1}}$ thanks to the normalization. Thus we have $C_i = c_{-\gamma_1} \cdots c_{-\gamma_{r-i}}$.

Similarly, if we define $C'_i \in \mathbb{C}^\times$ by $\text{Ad}(\dot{w}_L){}^t\tilde{f}_{r-i} = C'_i {}^t\tilde{f}_i$, then we have $C'_i = c_{\gamma_1} \cdots c_{\gamma_{r-i}} = C_i^{-1}$. Thus we have

$$\begin{aligned} \tilde{f}_i &= C_i^{-1} \text{Ad}(\dot{w}_L)f_{r-i}, \\ {}^t\tilde{f}_i &= C_i \text{Ad}(\dot{w}_L){}^t f_{r-i}. \end{aligned}$$

Then we have

$$\begin{aligned} {}^t\tilde{f}_i(\partial) \tilde{f}_i^{m+1} &= (C_i \text{Ad}(\dot{w}_L){}^t f_{r-i})(\partial) (C_i^{-1} \text{Ad}(\dot{w}_L)f_{r-i})^{m+1} \\ &= C_i^{-m} \text{Ad}(\dot{w}_L)({}^t f_{r-i}(\partial) f_{r-i}^{m+1}) \\ &= C_i^{-m} \text{Ad}(\dot{w}_L)b_{r-i}(m) f_{r-i}^m \\ &= b_{r-i}(m) \tilde{f}_i^m. \end{aligned}$$

We proved the lemma. $\square$

Proposition 9.3 For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbb{C}$),

$$b_r(\mu) = \frac{b_r(k_1 + \cdots + k_r)}{b_{r-1}(k_1 + \cdots + k_r)} \cdots \frac{b_2(k_{r-1} + k_r)}{b_1(k_{r-1} + k_r)} b_1(k_r).$$

In particular, $b_i(s)$ divides $b_{i+1}(s)$ and therefore $b_r(\mu)$ is a polynomial of degree i in k_i . Moreover the total degree of $b_r(\mu)$ in $k_1, \dots, k_r$ is equal to r .

Proof. If we know the above equality, then it is easy to see that $b_i(s)$ divides $b_{i+1}(s)$, since $b_r(\mu)$ and $b_i(s)$ are polynomials. Then, in addition, it is obvious that $b_r(\mu)$ is a polynomial

of degree i in k_i ($i \in \{1, \dots, r\}$), since the degree of $b_i(s)$ is equal to i . Moreover it follows that the total degree of $b_r(\mu)$ in $k_1, \dots, k_r$ is equal to r .

We therefore have only to prove the equality. Applying Lemma 9.2 repeatedly, we have

$${}^t\tilde{f}_{r-1}^{k_{r-1}}(\partial) \cdots {}^t\tilde{f}_1^{k_1}(\partial) f_r^{k_1+\cdots+k_r+1} = \prod_{i=1}^{r-1} \prod_{j=1}^{k_i} b_{r-i}(j + k_{i+1} + \cdots + k_r) \times f_1^{k_1} \cdots f_r^{k_r} f_r.$$

Applying ${}^t f_r(\partial)$ to this equality and using Lemma 9.2 again to the left hand side, we have

$$\begin{aligned} & b_r(k_1 + \cdots + k_r) \prod_{i=1}^{r-1} \prod_{j=0}^{k_i-1} b_{r-i}(j + k_{i+1} + \cdots + k_r) \times f_\mu \\ &= b_r(\mu) \prod_{i=1}^{r-1} \prod_{j=1}^{k_i} b_{r-i}(j + k_{i+1} + \cdots + k_r) \times f_\mu \end{aligned}$$

Then the proposition is proved by comparing both sides. $\square$

Second we show the explicit formula for the b -function $b_i(s)$ in Proposition 9.9, although [21, Théorème 4.6] gives an intrinsic proof of the formula. We need several known lemmas to show the formula.

Lemma 9.4 For $i \in \{1, \dots, r-1\}$, $I_{\lambda_1} I_{\lambda_i} \supset I_{\lambda_{i+1}}$.

Proof. Let $\mathbf{C}^d[\mathbf{n}^+]$ denote the homogeneous component of degree d of $\mathbf{C}[\mathbf{n}^+]$. Then $I_{\lambda_{i+1}} \subset \mathbf{C}^{i+1}[\mathbf{n}^+] = \mathbf{C}^i[\mathbf{n}^+] \mathbf{C}^1[\mathbf{n}^+] = \mathbf{C}^i[\mathbf{n}^+] I_{\lambda_1}$. Thus there exists $\mu = k_1 \lambda_1 + \cdots + k_r \lambda_r$ such that $I_\mu \subset \mathbf{C}^i[\mathbf{n}^+]$ and $I_{\lambda_{i+1}} \subset I_\mu I_{\lambda_1}$. Assume that $\mu \neq \lambda_i$. Then $\mu = k_1 \lambda_1 + \cdots + k_{i-1} \lambda_{i-1} = -(m_1 \gamma_1 + \cdots + m_{i-1} \gamma_{i-1})$, where $m_j = k_j + \cdots + k_r$. Any $\text{ad}(U(\mathfrak{l}))$ -maximal weight occurring in $I_\mu I_{\lambda_1}$ is a sum of μ and a weight of I_{λ_1} . Thus $\lambda_{i+1} = \mu + \alpha$ for some $\alpha \in \Delta_N^-$. Here $(\lambda_{i+1} - \mu)|_{\mathfrak{h}^-} = \{(m_1 - 1)\gamma_1 + \cdots + (m_{i-1} - 1)\gamma_{i-1} - \gamma_i - \gamma_{i+1}\}|_{\mathfrak{h}^-}$. This can not be equal to $\alpha|_{\mathfrak{h}^-}$ by Theorem 5.3. Thus μ must be λ_i and we proved the lemma. $\square$

Lemma 9.5 Let Y be the maximal submodule of $M(\lambda)$. For $\mu = k_1 \lambda_1 + \cdots + k_r \lambda_r$ ($k_j \in \mathbf{Z}_{\geq 0}$), I_μ occurs as a component of the irreducible decomposition of Y regarded as an $\text{ad}(U(\mathfrak{l}))$ -module, if and only if $q_\lambda(\mu) = 0$.

Proof. It follows from Proposition 3.2 (2) and Proposition 3.3 (2) that $I_\mu \subset Y$ if and only if $(I_\mu, I_\mu)_\lambda = 0$. Since I_μ is an irreducible $\text{ad}(U(\mathfrak{l}))$ -module, the nonzero contravariant form $(,)$ defined in Definition 3.5 is nondegenerate on I_μ . Thus it follows from the definition of $q_\lambda(\mu)$ that $I_\mu \subset Y$ if and only if $q_\lambda(\mu) = 0$. $\square$

Definition 9.6 For $1 \leq i < j \leq r$, define $c = \#\{\alpha \in \Delta_L \cap \Delta^+; \alpha|_{\mathfrak{h}^-} = (\gamma_j - \gamma_i)/2\}$.

We can show a property of c which justifies the definition above. We need a lemma.

Lemma 9.7 (cf. Enright-Joseph [5, Lemma 2.3]) For any $\sigma \in \mathfrak{S}_r$, there exists $w_\sigma \in W_L$ such that $w_\sigma(\gamma_i) = \gamma_{\sigma(i)}$ for $i \in \{1, \dots, r\}$, where W_L denotes the Weyl group of L .

Proof. We first note that the proof in [5] is a case study. For $i \in \{1, \dots, r\}$, there exists a (longest) element w_i in the Weyl group of L_i such that $w_i(\gamma_j) = \gamma_{i-j+1}$ ($1 \leq j \leq i$) by Lemma 9.1. We regard w_i as an element of W_L by expressing in a composite of simple reflections.

By the construction of $\Delta_{L,i}$, we have $w_i(\gamma_j) = \gamma_j$ ($i < j \leq r$). Then it follows that $\{w_1, \dots, w_r\}$ generates $\mathfrak{S}_r$, since we can easily show that $\{w_1, \dots, w_i\}$ generates $\mathfrak{S}_i$ inductively. We proved the lemma. $\square$

Lemma 9.8 c is independent of i or j and $c = \#\{\alpha \in \Delta_N^+; \alpha|_{\mathfrak{h}^-} = (\gamma_j + \gamma_i)/2\}$ for $1 \leq i < j \leq r$. Moreover, if $(L, \text{Ad}, \mathbf{n}^+)$ is a regular prehomogeneous vector space, then

$$(9.3) \quad c = \frac{2}{r(r-1)}(\dim \mathbf{n}^+ - r)$$

Proof. Set $C_{ij} = \{\alpha \in \Delta_L; \alpha|_{\mathfrak{h}^-} = (\gamma_j - \gamma_i)/2\}$ for $1 \leq i < j \leq r$. For $1 \leq i < j < k \leq r$, we can take $w_{(jk)} \in W_L$ by Lemma 9.7, where (jk) is a transposition in $\mathfrak{S}_r$. Then it is easy to show that $w_{(jk)} C_{ij} = C_{ik}$. Thus $\#C_{ij}$ is independent of j and similarly independent of i . We proved that c is independent of i or j .

Next we set $D_{ij} = \{\alpha \in \Delta_N^+; \alpha|_{\mathfrak{h}^-} = (\gamma_j + \gamma_i)/2\}$ for $1 \leq i < j \leq r$. We take a simple reflection $s_{\gamma_i} \in W_L$ corresponding to γ_i . Then $s_{\gamma_i}(C_{ij}) = D_{ij}$ and the second assertion is proved. We also note that $D_{ij} = C_{ij} + \gamma_i$.

Lastly, if $(L, \text{Ad}, \mathfrak{n}^+)$ is regular, then it follows from Theorem 5.3 that

$$\#(\Delta_N^+ \setminus \{\gamma_1, \dots, \gamma_r\}) = c \times \binom{r}{2}.$$

Thus

$$c = \binom{r}{2}^{-1} (\dim \mathfrak{n}^+ - r) = \frac{2}{r(r-1)} (\dim \mathfrak{n}^+ - r).$$

□

Note that c is defined by (9.3) in [21, Théorème 4.6].

Here we can determine the constant ρ_i^0 defined in (7.3) in the same way as in Lemma 5.4. We know $\rho_i^0 \in \text{Hom}(\mathfrak{p}_i, \mathbb{C}) \simeq \mathbb{C}\varpi_{i_0}$. Thus we have $\rho_i^0 = (\rho_i, \alpha_{i_0})/(\varpi_{i_0}, \alpha_{i_0})$. By Lemma 9.8,

$$\begin{aligned} (\rho_i, \alpha_{i_0}) &= \left(\frac{1}{2} \sum_{\alpha \in \Delta_{N,i}^+} \alpha, \gamma_1 \right) \\ &= \frac{1}{2} \sum_{\alpha} (\alpha|_{\mathfrak{h}^-}, \gamma_1) \\ &= \frac{1}{2} \left\{ \left(\sum_{1 \leq j < k \leq i} \frac{c}{2} (\gamma_k + \gamma_j), \gamma_1 \right) + (\gamma_1 + \dots + \gamma_i, \gamma_1) \right\} \\ &= \frac{c}{4} (i-1)(\gamma_1, \gamma_1) + \frac{1}{2} (\gamma_1, \gamma_1) \\ &= \frac{(\gamma_1, \gamma_1)}{2} \left\{ \frac{i-1}{2} c + 1 \right\}. \end{aligned}$$

Since $(\varpi_{i_0}, \alpha_{i_0}) = (\gamma_1, \gamma_1)/2$, we have

$$(9.4) \quad \rho_i^0 = \frac{i-1}{2} c + 1.$$

Proposition 9.9 ([21, Théorème 4.6], [32, Theorem 3.3]) *For $i \in \{1, \dots, r\}$*

$$b_i(s) = d_i \prod_{j=0}^{i-1} \left(s + 1 + \frac{j}{2} c \right),$$

where $d_i \in \mathbb{C}^\times$ is a constant.

Proof. Let Y be the maximal submodule of $M(\lambda)$. By Lemma 9.4 if $I_{\lambda_i} \subset Y$ then $I_{\lambda_{i+1}} \subset Y$ since $\Psi_\lambda(X)$ is a multiplying operator for $X \in \mathfrak{n}^-$. Then it follows from Lemma 9.5 that $q_\lambda(\lambda_i) = 0$ implies $q_\lambda(\lambda_{i+1}) = 0$ for $i \in \{1, \dots, r-1\}$. By Corollary 8.1, if $b_i(-\lambda^0 - \rho_i^0) = 0$ then $b_{i+1}(-\lambda^0 - \rho_{i+1}^0) = 0$. Obviously $b_j(s)$ has $s+1$ in its factors for any j and it is known that $b_j(s)$ is of degree j in s . Since $b_1(s) = s+1$ up to constant, if $b_1(-\lambda^0 - \rho_1^0) = 0$ then $-\lambda^0 - \rho_1^0 + 1 = 0$. In this case, we have $b_2(-\lambda^0 - \rho_2^0) = 0$. Since $-\lambda^0 - \rho_2^0 + 1 \neq 0$, $b_2(-\lambda^0 - \rho_2^0)$ has a factor $-\lambda^0 - \rho_1^0 + 1$. Thus we have $b_2(-\lambda^0 - \rho_2^0) = (-\lambda^0 - \rho_1^0 + 1)(-\lambda^0 - \rho_2^0 + 1)$ up to constant. Inductively we have $b_i(-\lambda^0 - \rho_i^0) = (-\lambda^0 - \rho_1^0 + 1) \cdots (-\lambda^0 - \rho_{i-1}^0 + 1)$ for $i \in \{1, \dots, r\}$ up to constant multiple. The proposition is proved by replacing $-\lambda^0 - \rho_i^0$ by s and using (9.4) for ρ_j^0 . □

Proposition 9.10 *For $i \in \{1, \dots, r\}$ and $\mu = k_1 \lambda_1 + \dots + k_i \lambda_i$ ($k_j \in \mathbb{C}$) we have*

$$(9.5) \quad b_i(\mu) = d_i \prod_{j=0}^{i-1} \left(k_{i-j} + \dots + k_i + 1 + \frac{j}{2} c \right),$$

where d_i 's are defined in Proposition 9.9.

Proof. It immediately follows from Proposition 9.3 and Proposition 9.9. □

Proposition 9.11 *For $\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$ ($k_j \in \mathbb{Z}_{\geq 0}$) we have*

$$(9.6) \quad q_\lambda(\mu) = \left(\prod_{i=1}^r d_i^{k_i} \right) \prod_{i=0}^{r-1} \prod_{m=0}^{k_{i+1} + \dots + k_r - 1} \left(\frac{i}{2} c + \lambda^0 - m \right)$$

$$(9.7) \quad q_\lambda(\mu) = \prod_{i=1}^r \prod_{j=1}^{k_i} b_i(\lambda^0 - (j + k_{i+1} + \dots + k_r)),$$

where d_i 's are defined in Proposition 9.9.

Proof. By Corollary 8.1, Proposition 9.10 and (9.4), we have

$$q_\lambda(\mu) = (-1)^{\deg f_\mu} \prod_{i=1}^r \prod_{j=0}^{k_i-1} b_i(k_1 \lambda_1 + \dots + k_{i-1} \lambda_{i-1} + j \lambda_i - (\lambda^0 + \rho_i^0) \lambda_i)$$

$$\begin{aligned}
&= (-1)^{\deg f_\mu} \prod_{i=1}^r \prod_{j=0}^{k_i-1} d_i \prod_{s=0}^{i-1} \left(k_{i-s} + \cdots k_{i-1} + (j - \lambda^0 - \rho_i^0) + 1 + \frac{s}{2}c \right) \\
&= (-1)^{\deg f_\mu} \left(\prod_{i=1}^r d_i^{k_i} \right) \\
&\quad \times \prod_{i=1}^r \prod_{j=0}^{k_i-1} \prod_{s=0}^{i-1} \left(k_{i-s} + \cdots k_{i-1} + j - \lambda^0 - \left(\frac{i-1}{2}c + 1 \right) + 1 + \frac{s}{2}c \right) \\
&= (-1)^{\deg f_\mu} \left(\prod_{i=1}^r d_i^{k_i} \right) \prod_{i=1}^r \prod_{j=0}^{k_i-1} \prod_{s=0}^{i-1} \left(k_{i-s} + \cdots k_{i-1} + j - \lambda^0 - \frac{i-s-1}{2}c \right).
\end{aligned}$$

Replacing s by $t = i - s - 1$, this is equal to

$$\begin{aligned}
&(-1)^{\deg f_\mu} \left(\prod_{i=1}^r d_i^{k_i} \right) \prod_{i=1}^r \prod_{j=0}^{k_i-1} \prod_{t=0}^{i-1} \left(k_{t+1} + \cdots + k_{i-1} + j - \lambda^0 - \frac{t}{2}c \right) \\
&= (-1)^{\deg f_\mu} \left(\prod_{i=1}^r d_i^{k_i} \right) \prod_{t=0}^{r-1} \prod_{i=t+1}^r \prod_{j=0}^{k_i-1} \left(k_{t+1} + \cdots + k_{i-1} + j - \lambda^0 - \frac{t}{2}c \right) \\
&= (-1)^{\deg f_\mu} \left(\prod_{i=1}^r d_i^{k_i} \right) \prod_{t=0}^{r-1} \prod_{m=0}^{k_{i+1}+\cdots+k_r-1} \left(m - \lambda^0 - \frac{t}{2}c \right).
\end{aligned}$$

We proved (9.6).

Next we check that the right hand side of (9.7) is equal to the right hand side of (9.6).

By Proposition 9.9, we have

$$\prod_{i=1}^r \prod_{j=1}^{k_i} b_i(\lambda^0 - (j + k_{i-1} + \cdots + k_r)) = \prod_{i=1}^r \prod_{j=1}^{k_i} d_i \prod_{t=0}^{i-1} \left(\lambda^0 - j - k_{i-1} - \cdots - k_r + 1 + \frac{t}{2}c \right).$$

It is easy to see that this is equal to the third last formula in the calculation above. We proved (9.7). $\square$

10 Irreducibility criteria for scalar generalized Verma modules

In this section we consider two irreducibility criteria for scalar generalized Verma modules. One is in terms of b -functions and the other is in terms of contravariant forms. We see how these two criteria relate to each other through Corollary 8.1 or Proposition 9.11.

Proposition 10.1 For $\lambda = \lambda^0 \varpi_{i_0} \in \text{Hom}(\mathfrak{p}, \mathbb{C})$, the following are equivalent:

- (1) The scalar generalized Verma module $M(\lambda)$ with a highest weight λ is irreducible.
- (2) $\prod_{i=1}^r \prod_{j=1}^{k_i} b_i(\lambda^0 - (j + k_{i+1} + \cdots + k_r)) \neq 0$ for all $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$.
- (3) $b_r(\lambda^0 - m) \neq 0$ for all $m \in \mathbb{Z}_{>0}$.

Proof. By Lemma 9.5 and (9.7), it follows that (1) and (2) are equivalent. Next we assume (2). Then $b_i(\lambda^0 - m) \neq 0$ for all $i \in \{1, \dots, r\}$ and $m \in \mathbb{Z}_{>0}$ and therefore (3) follows. Conversely if we assume (3), then $b_i(\lambda^0 - m) \neq 0$ for all $i \in \{1, \dots, r\}$ and $m \in \mathbb{Z}_{>0}$, since b_i divides b_r . Then (2) follows. $\square$

11 b -Functions and L -orbits on the nilpotent radical

In this section we consider $\text{Ad}(L)$ -orbits on $\mathfrak{n}^+$. The set of $\text{Ad}(L)$ -orbits on $\mathfrak{n}^+$ and the set of zeros of the b -function of the relative invariant are in one-to-one correspondence [29]. We give another proof of the correspondence and it explains why there exists a correspondence between the orbits and the zeros of the b -function. Our proof is intrinsic.

We begin with a lemma on a polynomial ring.

Lemma 11.1 For $i \in \{1, \dots, r\}$, set $R_i = \bigoplus_{\mu} I_{\mu}$ ($\mu = k_1 \lambda_1 + \cdots + k_r \lambda_r, k_j \in \mathbb{Z}_{\geq 0}, k_i + \cdots + k_r > 0$). Then we have

- (1) $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_i} = R_i$.
- (2) R_i is a radical ideal.

Proof. (1) $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_i}$ contains $f_i, \dots, f_r$ by Lemma 9.4. Thus $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_i}$ contains all the maximal weight vectors occurring in R_i , and we have $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_i} \supset R_i$, since both of them are $\text{ad}(U(\mathfrak{l}))$ -stable.

Conversely, assume that μ is an $\text{ad}(U(\mathfrak{l}))$ -maximal weight occurring in $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_i}$. Then we can write

$$\begin{aligned}\mu &= k_1\lambda_1 + \cdots + k_r\lambda_r \quad (k_j \in \mathbb{Z}_{\geq 0}) \\ &= -m_1\gamma_1 - \cdots - m_r\gamma_r \quad (m_j = k_j + \cdots + k_r)\end{aligned}$$

by Theorem 4.2. In addition, μ is a sum of λ_i and a weight occurring in $\mathbb{C}[\mathfrak{n}^+]$, say ν . Here ν is a sum of roots in Δ_N^- , and therefore $\nu|_{\mathfrak{h}^-}$ is a certain sum of $-(1/2)\gamma_j$'s ($1 \leq j \leq r$) by Theorem 5.3. Then we have $m_i > 0$, since μ has $\lambda_i = -(\gamma_1 + \cdots + \gamma_i)$ as its summand. This means $k_i + \cdots + k_r > 0$. Thus μ occurs in R_i .

(2) We show R_i is a radical ideal. Since R_i is $\text{ad}(U(\mathfrak{l}))$ -stable and this action is derived from $\text{Ad}(L)$ -action, its radical is also $\text{ad}(U(\mathfrak{l}))$ -stable. If R_i is not a radical ideal then there exists $\nu = k_1\lambda_1 + \cdots + k_{i-1}\lambda_{i-1}$ such that I_ν is contained in the radical of R_i . Then there exists $m \in \mathbb{Z}_{>0}$ such that $(f_\nu)^m \in R_i$, where f_ν is the highest weight vector in I_ν defined before. However $(f_\nu)^m \in I_{m\nu}$, indeed $(f_\nu)^m$ is the highest weight vector in $I_{m\nu}$. Since $I_{m\nu} \cap R_i = 0$, this is a contradiction, and we proved (2). $\square$

We define $\text{Ad}(L)$ -stable subset C_i of $\mathfrak{n}^+$, which will turn out to be $\text{Ad}(L)$ -orbits on $\mathfrak{n}^+$.

$$\begin{aligned}C_0 &= \mathbf{V}(I_{\lambda_1}) = \{0\}, \\ C_i &= \mathbf{V}(I_{\lambda_{i+1}}) \setminus \mathbf{V}(I_{\lambda_i}) \quad (i \in \{1, \dots, r-1\}), \\ C_r &= \mathfrak{n}^+ \setminus \mathbf{V}(I_{\lambda_r}),\end{aligned}$$

where $\mathbf{V}(I) = \{x \in \mathfrak{n}^+ \mid f(x) = 0 \text{ for all } f \in I\}$ for a subset $I \subset \mathbb{C}[\mathfrak{n}^+]$. Here $\mathbf{V}(I_{\lambda_j})$'s are distinct (Zariski) closed non-empty subset of $\mathfrak{n}^+$, since their defining ideals $\mathbf{I}(\mathbf{V}(I_{\lambda_j})) = \sqrt{\mathbb{C}[\mathfrak{n}^+]I_{\lambda_j}} = \mathbb{C}[\mathfrak{n}^+]I_{\lambda_j}$, are distinct. We know $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_i} \supset \mathbb{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$ by Lemma 9.4, and therefore we have $\mathbf{V}(I_{\lambda_i}) \subset \mathbf{V}(I_{\lambda_{i+1}})$. Thus C_i ($i \in \{0, \dots, r\}$) is not empty.

Moreover C_j is $\text{Ad}(L)$ -stable, since I_{λ_j} is $\text{Ad}(L)$ -stable and therefore $\mathbf{V}(I_{\lambda_j})$ is $\text{Ad}(L)$ -stable. Thus we have a disjoint $\text{Ad}(L)$ -stable non-empty decomposition of $\mathfrak{n}^+$:

$$\mathfrak{n}^+ = C_0 \amalg \cdots \amalg C_r.$$

We will show that this decomposition is actually an orbit decomposition. The following lemma describes all the $\text{Ad}(L)$ -stable radical ideal of $\mathbb{C}[\mathfrak{n}^+]$.

Lemma 11.2 For $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbb{Z}_{\geq 0}$), set $t = \max\{j \mid k_j \neq 0\}$. Then we have $\mathbf{V}(I_\mu) = \mathbf{V}(I_{\lambda_t})$. In particular, all the non-trivial $\text{Ad}(L)$ -stable radical ideals of $\mathbb{C}[\mathfrak{n}^+]$ are $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_1}, \dots, \mathbb{C}[\mathfrak{n}^+]I_{\lambda_r}$.

Proof. First, I_{λ_t} contains f_t as a highest weight vector and then $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_t}$ contains $f_\mu = f_1^{k_1} \cdots f_r^{k_r}$, since $k_t \neq 0$. Thus we have $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_t} \supset \mathbb{C}[\mathfrak{n}^+]I_\mu$, and therefore $\mathbf{V}(I_{\lambda_t}) \subset \mathbf{V}(I_\mu)$.

Next we take $x \in \mathbf{V}(I_\mu)$ and set $W = \overline{\text{Ad}(L) \cdot x}$. Since $\mathbf{V}(I_\mu)$ is an $\text{Ad}(L)$ -stable closed set, $W \subset \mathbf{V}(I_\mu)$. Then for any $y \in W$, we have

$$f_1^{k_1}(y) \cdots f_r^{k_r}(y) = f_\mu(y) = 0.$$

Thus we have

$$W = \bigcup_{k_i \neq 0} W_i, \quad W_i = W \cap \mathbf{V}(f_i).$$

If $W_i \neq W$ for some i , then W_i is a hypersurface of W and $\dim W_i = \dim W - 1$. Thus there exists i ($k_i \neq 0$) such that $W_i = W$. We have

$$(g \cdot f_i)(x) = f_i(g^{-1}x) = 0 \quad (g \in L).$$

Since $\{g \cdot f_i \mid g \in L\}$ spans I_{λ_i} , we have $x \in \mathbf{V}(I_{\lambda_i})$. Finally, $\mathbb{C}[\mathfrak{n}^+]I_{\lambda_i} \supset \mathbb{C}[\mathfrak{n}^+]I_{\lambda_t}$ by $i \leq t$ and we have $x \in \mathbf{V}(I_{\lambda_i}) \subset \mathbf{V}(I_{\lambda_t})$. We proved that $\mathbf{V}(I_\mu) \subset \mathbf{V}(I_{\lambda_t})$.

We show the last assertion of the lemma. Let I be an $\text{Ad}(L)$ -stable radical ideal of $\mathbb{C}[\mathfrak{n}^+]$. Then $I = \bigoplus_{\mu \in M} I_\mu$ for some $M \subset \{k_1\lambda_1 + \cdots + k_r\lambda_r \mid k_j \in \mathbb{Z}_{\geq 0}\}$. We have

$$\begin{aligned}I &= \mathbf{I}(\mathbf{V}(I)) \\ &= \mathbf{I}\left(\mathbf{V}\left(\bigoplus_{\mu \in M} I_\mu\right)\right) \\ &= \mathbf{I}\left(\mathbf{V}\left(\sum_{\mu} \mathbb{C}[\mathfrak{n}^+]I_\mu\right)\right) \\ &= \mathbf{I}\left(\bigcap_{\mu} \mathbf{V}(I_\mu)\right).\end{aligned}$$

Here $\mathbf{V}(I_\mu) = \mathbf{V}(I_{\lambda_i})$ for some i , and their intersection is $\mathbf{V}(I_{\lambda_j})$ for some j , since $\mathbf{V}(I_{\lambda_1}) \subset \dots \subset \mathbf{V}(I_{\lambda_r})$. Thus

$$I = \mathbf{I}(\mathbf{V}(I_{\lambda_j})) = \mathbf{C}[\mathfrak{n}^+]I_{\lambda_j}.$$

We proved the lemma. $\square$

In fact, these $\text{Ad}(L)$ -stable radical ideals are prime.

Lemma 11.3 (cf. Joseph [15, Proposition 2.3]) *For $i \in \{1, \dots, r\}$, the ideal $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i}$ is prime. Thus all the non-trivial $\text{Ad}(L)$ -stable prime ideals of $\mathbf{C}[\mathfrak{n}^+]$ are $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_1}, \dots, \mathbf{C}[\mathfrak{n}^+]I_{\lambda_r}$.*

Proof. First, $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_1}$ is a maximal ideal, and therefore it is prime. For $i \in \{2, \dots, r\}$, we show that $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i}$ is prime. An $\text{Ad}(L)$ -stable set $C_{i-1} = \mathbf{V}(I_{\lambda_i}) \setminus \mathbf{V}(I_{\lambda_{i-1}})$ is non-empty. Thus there exists an $\text{Ad}(L)$ -orbit in C_{i-1} , say O . All the $\text{Ad}(L)$ -stable closed subset of $\mathfrak{n}^+$ are of the form $\mathbf{V}(I_{\lambda_j})$, $\emptyset$ or $\mathfrak{n}^+$. Since $\overline{O}$ is also $\text{Ad}(L)$ -stable, we have $\overline{O} = \mathbf{V}(I_{\lambda_i})$. Here O is an irreducible set, since O is a continuous image of a connected (therefore irreducible as a variety) group L . Thus $\overline{O} = \mathbf{V}(I_{\lambda_i})$ is irreducible and $\mathbf{C}[\mathfrak{n}^+]I_{\lambda_i}$ is prime. $\square$

In general, an orbit of an algebraic group is open in its closure and two non-empty open sets always intersect in an irreducible variety. Thus it follows from Lemma 11.3 that each C_i must consist of a single $\text{Ad}(L)$ -orbit and $\overline{C_i} = \mathbf{V}(I_{\lambda_{i+1}})$ for $i \in \{0, \dots, r-1\}$ and $\overline{C_r} = \mathfrak{n}^+$.

In this way, we proved that the $\text{Ad}(L)$ -orbit decomposition of $\mathfrak{n}^+$ and the closure relation:

$$\begin{aligned} \mathfrak{n}^+ &= C_0 \amalg \dots \amalg C_r, \\ \{0\} &= \overline{C_0} \subset \overline{C_1} \subset \dots \subset \overline{C_r} = \mathfrak{n}^+. \end{aligned}$$

The open orbit of a prehomogeneous vector space $(L, \text{Ad}, \mathfrak{n}^+)$ is C_r . Moreover we can find a representative of orbits.

Lemma 11.4 (cf. Muller-Rubenthaler-Schiffmann [21, Théorème 2.8]) $X_{\gamma_1} + \dots + X_{\gamma_i} \in C_i$ ($i \in \{1, \dots, r\}$).

Proof. First we show that $X_{\gamma_1} + \dots + X_{\gamma_i} \notin \mathbf{V}(I_{\lambda_i})$ for $i \in \{1, \dots, r\}$. It suffices to show that $f_i(X_{\gamma_1} + \dots + X_{\gamma_i}) \neq 0$, and (7.2) proves it.

Next we show that $X_{\gamma_1} + \dots + X_{\gamma_i} \in \mathbf{V}(I_{\lambda_{i+1}})$ for $i \in \{1, \dots, r-1\}$. If we assume that $X_{\gamma_1} + \dots + X_{\gamma_i} \notin \mathbf{V}(I_{\lambda_{i+1}})$, then there exists $f \in I_{\lambda_{i+1}}$ such that $f(X_{\gamma_1} + \dots + X_{\gamma_i}) \neq 0$. We expand f into monomials $X_{-\beta_1} \dots X_{-\beta_{i+1}}$ ($\beta_j \in \Delta_N^+$). We can take a monomial which does not vanish at $X_{\gamma_1} + \dots + X_{\gamma_i}$ and such a monomial has a form $X_{-\gamma_1}^{m_1} \dots X_{-\gamma_i}^{m_i}$ ($m_j \in \mathbb{Z}_{\geq 0}$), where $-m_1\gamma_1 - \dots - m_i\gamma_i$ is a weight occurring in $I_{\lambda_{i+1}}$. Namely, $m_1 + \dots + m_i = i+1$ and $\lambda_{i+1} - (-m_1\gamma_1 - \dots - m_i\gamma_i)$ is a certain sum of roots in Δ_N^+ .

Here a weight

$$(11.1) \quad \lambda_{i+1} - (-m_1\gamma_1 - \dots - m_i\gamma_i) = (m_1 - 1)\gamma_1 + \dots + (m_i - 1)\gamma_i - \gamma_{i+1}$$

can not be any sum of roots in Δ_N^+ . Indeed, it follows from Theorem 5.3 that

$$\alpha|_{\mathfrak{h}^-} = \begin{cases} (\gamma_t - \gamma_s)/2 & (1 \leq s \leq t \leq r), \\ -\gamma_t/2 & (1 \leq t \leq r), \end{cases}$$

for $\alpha \in \Delta_N^+$. If (11.1) is a sum of roots in Δ_L^+ , then $-\gamma_t/2$ does not occur as a summand of (11.1), since the sum of coefficient in (11.1) is equal to zero, when we restrict (11.1) to $\mathfrak{h}^-$. Then the term $-\gamma_{i+1}$ in (11.1) implies a contradiction. Thus (11.1) can not be any sum of roots in Δ_L^+ . We proved the lemma. $\square$

Proposition 11.5 (Tanisaki [29, Proposition 2.6]) *For a non-open orbit C_i ($i \in \{0, \dots, r-1\}$), the defining ideal $\mathbf{I}(\overline{C_i}) = \mathbf{C}[\mathfrak{n}^+]I_{\lambda_{i+1}}$ is equal to the unique maximal submodule of $M(\lambda) \simeq \mathbf{C}[\mathfrak{n}^+]$ if and only if $\lambda = (a_{i+1} + 1)\varpi_{i_0}$, where a_i is the unique zero of $b_i(s)$ which is not a zero of $b_{i-1}(s)$.*

Thus we have a one-to-one correspondence between non-open $\text{Ad}(L)$ -orbits on $\mathfrak{n}^+$ and the zeros of $b_r(s)$:

$$\begin{array}{ccc} \{\text{non-open } \text{Ad}(L)\text{-orbits on } \mathfrak{n}^+\} & \longrightarrow & \{\text{zeros of } b_r(s)\} \\ C_i & \mapsto & a_{i+1} \quad (i \in \{0, \dots, r-1\}). \end{array}$$

Proof. First we assume that $C[n^+]I_{\lambda_{i+1}}$ is the unique maximal submodule of $M(\lambda)$ for some $\lambda = \lambda^0 \varpi_{i_0}$. Then it follows from Lemma 9.5 and Lemma 11.1 that $q_\lambda(\mu) = 0$ if and only if $k_{i+1} + \dots + k_r > 0$ for $\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$. In particular, we have $q_\lambda(\lambda_i) \neq 0$ and $q_\lambda(\lambda_{i+1}) = 0$. Then by (9.7) we have $b_i(\lambda^0 - 1) \neq 0$ and $b_{i+1}(\lambda^0 - 1) = 0$, and therefore it follows that there exists at most one λ such that $C[n^+]I_{\lambda_{i+1}}$ becomes the maximal submodule of $M(\lambda)$. Namely, the unique possibility for such λ is given by $\lambda^0 = a_{i+1} + 1$.

Conversely we assume that $\lambda = (a_{j+1} + 1)\varpi_{i_0}$ for $j \in \{0, \dots, r-1\}$. We find $a_{j+1} = -jc/2 - 1$ from Proposition 9.9. For $\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$, if $k_{j+1} + \dots + k_r > 0$ then $q_\lambda(\mu)$ in (9.6) has a factor in which $i = j$ and $m = 0$, that is, $(jc/2 - jc/2 - 0)$. Thus $q_\lambda(\mu) = 0$. On the other hand, if $k_{j+1} = \dots = k_r = 0$ then a factor $(ic/2 - jc/2 - m)$ in (9.6) occurs only when $i < j$. Thus all these factors are negative and therefore $q_\lambda(\mu) \neq 0$. Then we have that $q_\lambda(\mu) = 0$ if and only if $k_{j+1} + \dots + k_r > 0$ for $\lambda^0 = a_{j+1} + 1$. In other words, the maximal submodule of $M(\lambda^0 \varpi_{i_0})$ is equal to $C[n^+]I_{\lambda_{i+1}}$ for $\lambda^0 = a_{i+1} + 1$ by Lemma 11.1 (1).

In this way we obtain the desired correspondence. $\square$

12 The unitarizability of the irreducible quotient of $M(\lambda)$

Let $L(\lambda)$ be the irreducible quotient of $M(\lambda)$. In this section, we consider the unitarizability of $L(\lambda)$. This application is suggested by Professor Shuichi Suga. This problem is considered in many articles ([31], [22], [7], [4], [5], ...). It is known from these articles that the values of λ such that $L(\lambda)$ is unitarizable, are related to the zeros of b -functions. We will explain in terms of our main theorem, the reason why there exists such a relation.

We must consider real Lie algebras. We take the real form $\mathfrak{g}_0$ such that

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{l}_0 \oplus \mathfrak{n}_0 \quad (\text{Cartan decomposition}), \\ (\mathfrak{g}_0)_C &= \mathfrak{g}, \end{aligned}$$

$$(\mathfrak{l}_0)_C = \mathfrak{l},$$

$$(\mathfrak{n}_0)_C = \mathfrak{n}^+ \oplus \mathfrak{n}^-,$$

where subscript C means the complexification.

When we work in the 'real' situation, the definitions in preceding sections must be slightly modified.

Definition 12.1 Define a conjugate linear anti-involution $\cdot^*$ of $U(\mathfrak{g})$ by

$$\begin{aligned} H_i^* &= H_i & (i \in \{1, \dots, n\}), \\ X_\alpha^* &= X_{-\alpha} & (\alpha \in \Delta_L), \\ X_\alpha^* &= -X_{-\alpha} & (\alpha \in \Delta_N). \end{aligned}$$

We extend $\cdot^*$ conjugate linearly to $\mathfrak{g}$ and anti-automorphically to $U(\mathfrak{g})$. Note that $\cdot^*$ is a composite of t and the Cartan involution. Note also that $\cdot^*$ is a complex conjugation by regarding $\mathfrak{g}_0$ as a purely imaginary part and $\sqrt{-1}\mathfrak{g}_0$ as a real part. See also [32, p.79], [7, p.878], and [25, §2].

Lemma 12.2 Under the normalization (7.1), $f_\mu^* = (-1)^{\deg f_\mu} {}^t f_\mu$ for $\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$ ($k_j \in \mathbb{Z}_{\geq 0}$).

Proof. We have only to prove the lemma for $\mu = \lambda_i$ ($i \in \{1, \dots, r\}$), since $f_\mu = f_1^{k_1} \dots f_r^{k_r}$. For $\alpha \in \Delta_L^+$ we have $[X_{-\alpha}, f_i^*] = -[X_{-\alpha}^*, f_i] = -[X_\alpha, f_i] = 0$. Clearly the weight of f_i^* is equal to $-\lambda_i$ and therefore $f_i^* \in C {}^t f_i$.

Set $X_+ = X_{\gamma_1} + \dots + X_{\gamma_r}$ and $X_- = {}^t X_+ = X_{-\gamma_1} + \dots + X_{-\gamma_r}$. We compute $f_i^*(X_-)$, where $f_i^* \in S(\mathfrak{n}^+)$ is regarded as a function on $\mathfrak{n}^-$. We expand f_i as in Definition 7.1:

$$f_i = \sum_{B \in \mathcal{B}_i} a_B X_{-\beta_1} \dots X_{-\beta_i} \quad (B = (\beta_1, \dots, \beta_i)).$$

We recall that $a_{B_i} \in \mathbb{R}$ for $B_i = (\gamma_1, \dots, \gamma_i)$ by Lemma 7.2. Then we have

$$\begin{aligned} f_i^*(X_-) &= \sum_{B \in \mathcal{B}_i} (a_B X_{-\beta_1} \dots X_{-\beta_i})^*(X_-) \\ &= a_{B_i} (-X_{\beta_1}) \dots (-X_{\beta_i})(X_-) \\ &= (-1)^i a_{B_i} X_{\beta_1} \dots X_{\beta_i}(X_-) \\ &= (-1)^i {}^t f_i(X_-). \end{aligned}$$

We proved the lemma. $\square$

We define 'real' analogues of b -functions $b_i(\mu)$ and $\beta_{\lambda,i}(\mu)$. See §7 for the notation.

Definition 12.3 For $i \in \{1, \dots, r\}$, define polynomials b_i^* and $\beta_{\lambda,i}^*$ by

$$\begin{aligned} f_i^*(\partial) f_i f_\mu &= b_i^*(\mu) f_\mu, \\ \Psi_{\lambda|p_i}(f_i^* f_i) f_\mu &= \beta_{\lambda,i}^*(\mu) f_\mu, \end{aligned}$$

for $\lambda \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ and $\mu = k_1 \lambda_1 + \dots + k_i \lambda_i$ ($k_j \in \mathbb{C}$).

Lemma 12.4 For $i \in \{1, \dots, r\}$ and $\mu = k_1 \lambda_1 + \dots + k_i \lambda_i$ ($k_j \in \mathbb{C}$),

$$\begin{aligned} b_i^*(\mu) &= (-1)^i b_i(\mu), \\ \beta_{\lambda,i}^*(\mu) &= (-1)^i \beta_{\lambda,i}(\mu). \end{aligned}$$

Proof. The lemma immediately follows from Lemma 12.2. $\square$

Next we consider a 'real' analogue of $q_\lambda(\mu)$.

Definition 12.5 Let (π, V) be a representation of $\mathfrak{g}$.

(1) A Hermitian form $(,)$ on V is called a *contravariant sesquilinear form* or a $\pi(U(\mathfrak{g}))$ -*contravariant sesquilinear form* if $(,)$ satisfies

$$(12.1) \quad (\pi(u)v, w) = (v, \pi(u^*)w),$$

for $u \in U(\mathfrak{g})$ and $v, w \in V$.

(2) We say V to be $\mathfrak{g}_0$ -infinitesimally unitary if there exists a positive definite Hermitian form $(,)$ on V such that

$$(12.2) \quad (\pi(X)v, w) = (v, -\pi(X)w),$$

for $X \in \mathfrak{g}_0$ and $v, w \in V$.

For a positive definite Hermitian form $(,)$, the conditions (12.1) and (12.2) are equivalent.

Lemma 12.6 Let V be a highest weight $U(\mathfrak{g})$ -module with a highest weight $\nu \in \mathfrak{h}^*$.

- (1) Contravariant sesquilinear forms on V are unique up to constant multiples.
- (2) If there exists a nonzero contravariant sesquilinear form $(,)$ on V , then $\nu \in \mathbb{R}\varpi_1 + \dots + \mathbb{R}\varpi_n$, where ϖ_j 's are fundamental weights.

Proof. (1) can be proved in the similar way to Proposition 3.3 (1).

(2) Let $v_+ \in V$ be a highest weight vector. Then $\nu(H_i)(v_+, v_+) = (H_i \cdot v_+, v_+) = (v_+, H_i^* \cdot v_+) = (v_+, H_i \cdot v_+) = \overline{\nu(H_i)}(v_+, v_+)$, where H_i is the coroot of a simple root α_i , that is, an element of our fixed Chevalley basis. If $(v_+, v_+) \neq 0$ then $\nu(H_i) = \overline{\nu(H_i)}$ and we have $\nu \in \mathbb{R}\varpi_1 + \dots + \mathbb{R}\varpi_n$. $\square$

We assume that $\lambda \in \mathbb{R}\varpi_1 + \dots + \mathbb{R}\varpi_n$, that is, $\lambda \in \mathbb{R}\varpi_{i_0}$.

Definition 12.7 Define two bilinear form $(,)_\lambda^*$ and $(,)^*$ on $\mathbb{C}[\mathfrak{n}^+]$ by

$$\begin{aligned} (f, g)_\lambda^* &= \varphi_\lambda(g^* f) \\ (f, g)^* &= g^*(\partial) f(0) \end{aligned}$$

for $f, g \in \mathbb{C}[\mathfrak{n}^+]$, where φ_λ is the same as in Definition 3.4.

The following lemma gives important properties of these forms.

Lemma 12.8 (1) The bilinear form $(,)^*$ is an $\text{ad}(U(\mathfrak{l}))$ -contravariant sesquilinear form.

(2) For $\lambda \in \mathbb{R}\varpi_{i_0}$, $(,)_\lambda^*$ is a $\Psi_\lambda(U(\mathfrak{g}))$ -contravariant sesquilinear form.

(3) The bilinear form $(,)^*$ is positive definite on $\mathbb{C}^d[\mathfrak{n}^+]$ if d is even, and is negative definite if d is odd, where $\mathbb{C}^d[\mathfrak{n}^+]$ is a homogeneous component of degree d . Therefore $(-1)^{\deg f_\mu} (,)^*$ is positive definite on I_μ .

(4) The radical of $(,)_\lambda^*$ is equal to the maximal submodule of $M(\lambda)$.

(5) $(fg, h)^* = (g, f^*(\partial)h)^*$ for $f, g, h \in \mathbb{C}[\mathfrak{n}^+]$.

Proof. Set $\mathfrak{l}_{Re} = \sum_{i=1}^n \mathbb{R}H_i + \sum_{\alpha \in \Delta_L} \mathbb{R}X_\alpha$ and $\mathfrak{n}_{Im}^+ = \sum_{\alpha \in \Delta_N^+} \sqrt{-1}\mathbb{R}X_\alpha$. Then we have $X^* = {}^tX$ for $X \in \mathfrak{l}_{Re} + \mathfrak{n}_{Im}^+$ by the definition. It follows that $\text{ad}(X)f \in \mathbb{R}[\mathfrak{n}_{Im}^+]$ for $X \in \mathfrak{l}_{Re}$

and $f \in \mathbf{R}[\mathfrak{n}_{Im}^+]$, since the structure constants of $\mathfrak{g}$ with respect to a Chevalley basis are integral.

(1) It is enough to show that $(\text{ad}(X)f, g)^* = (f, \text{ad}(X^*)g)^*$ for $X \in \mathfrak{l}$ and $f, g \in \mathbf{C}[\mathfrak{n}^+]$. This holds for $X \in \mathfrak{l}_{Re}$ and $f, g \in \mathbf{R}[\mathfrak{n}_{Im}^+]$, since in this case the above formula becomes $(\text{ad}(X)f, g) = (f, \text{ad}(X)g)$. It is easy to see that $(,)^*$ is a Hermitian form and $\cdot^*$ is conjugate linear by the definition. These proves (1).

(2) It is easy to see that $(,)_\lambda^*$ is a Hermitian form when $\lambda \in \mathbf{R}\varpi_{i_0}$. Then the assertion follows from the same argument as (1).

(3) The first assertion is clear from the definition of $\cdot^*$. By the following (4), $(,)^*$ is nondegenerate on each I_μ , since it is nonzero on each I_μ . Then the second assertion is proved, since I_μ is a homogeneous subspace of $\mathbf{C}[\mathfrak{n}^+]$.

(4) follows from a similar argument to (1) and Proposition 3.3. (5) is a direct consequence of the definition. $\square$

Here we shall define the analogue of $q_\lambda(\mu)$. Define $q_\lambda^*(\mu)$ by

$$q_\lambda^*(\mu) = (f_\mu, f_\mu)_\lambda^* / (f_\mu, f_\mu)^*,$$

where $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbf{Z}_{\geq 0}$).

Lemma 12.9 $q_\lambda^*(\mu) = q_\lambda(\mu)$.

Proof.

$$\begin{aligned} q_\lambda^*(\mu) &= (f_\mu, f_\mu)_\lambda^* / (f_\mu, f_\mu)^* \\ &= \varphi_\lambda(f_\mu^* f_\mu) / f_\mu^*(\partial) f_\mu(0) \\ &= (-1)^{\deg f_\mu} \varphi_\lambda({}^t f_\mu f_\mu) / (-1)^{\deg f_\mu} {}^t f_\mu(\partial) f_\mu(0) \\ &= (f_\mu, f_\mu)_\lambda / (f_\mu, f_\mu) \\ &= q_\lambda(\mu). \end{aligned}$$

$\square$

Proposition 12.10 The following conditions are equivalent:

- (1) The irreducible quotient $L(\lambda)$ of $M(\lambda)$ is infinitesimally unitary.
- (2) $(,)_\lambda^*$ is nonnegative definite.
- (3) For all $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbf{Z}_{\geq 0}$), $(-1)^{\deg f_\mu} q_\lambda^*(\mu) \geq 0$.
- (4) $\lambda^0 = a_i + 1$ ($i \in \{1, \dots, r\}$) or $\lambda^0 < a_r + 1$, where λ^0 is the complex number determined by $\lambda = \lambda^0 \varpi_{i_0}$ and a_i is the unique zero of $b_i(s)$ which is not a zero of $b_{i-1}(s)$.

Proof. The contravariant sesquilinear form $(,)_\lambda^*$ on $M(\lambda)$ induces a nonzero contravariant sesquilinear form on $L(\lambda)$. This induced form on $L(\lambda)$ can not be negative definite, since $(1, 1)_\lambda^* = 1$. By Lemma 12.6 (1), any contravariant sesquilinear form on $L(\lambda)$ is its constant multiple. Thus $L(\lambda)$ is infinitesimally unitary if and only if the induced form on $L(\lambda)$ is positive definite. Then by Lemma 12.8 (4), (1) and (2) are equivalent.

By Lemma 12.8 (3) and the definition of $q_\lambda^*(\mu)$, (2) and (3) are equivalent.

Next we show that (3) implies (4). By Lemma 12.8 (3) we have $(-1)^i (f_i, f_i)^* > 0$. Then $0 < (-1)^i (f_i^*(\partial) f_i, 1)^* = (-1)^i b_i^*(0) = b_i(0) = d_i \prod_{j=0}^{i-1} (1 + jc/2)$, by Proposition 9.9. Here $c \in \mathbf{Z}_{\geq 0}$ and therefore we have $d_i > 0$.

Thus (3) says

$$(-1)^{\deg f_\mu} \prod_{i=1}^r \prod_{j=1}^{k_i} b_i(\lambda^0 - (j + k_{i+1} + \cdots + k_r)) \geq 0$$

for all $\mu = k_1\lambda_1 + \cdots + k_r\lambda_r$ ($k_j \in \mathbf{Z}_{\geq 0}$). In particular, if $\mu = \lambda_t$ then $(-1)^t b_t(\lambda^0 - 1) \geq 0$ for $t \in \{1, \dots, r\}$. Thus if $\lambda^0 \neq a_i + 1$ for all $i \in \{1, \dots, r\}$ then

$$0 < (-1)^t d_t \prod_{j=0}^{t-1} \left(\lambda^0 + \frac{j}{2} c \right) = d_t \prod_{j=0}^{t-1} \left(-\lambda^0 - \frac{j}{2} c \right)$$

for all t . Here $d_i > 0$ and $c \in \mathbf{Z}_{\geq 0}$, and therefore we have $\lambda^0 < -(r-1)c/2 = a_r + 1$. We proved that (3) implies (4).

Lastly we show that (4) implies (3). By (9.6)

$$(12.3) \quad (-1)^{\deg f_\mu} q_\lambda(\mu) = \prod_{i=0}^{r-1} \prod_{m=0}^{k_{i+1}+\dots+k_r-1} \left(-\lambda^0 - \frac{i}{2}c + m \right)$$

up to positive constant multiple. If $\lambda^0 < a_r + 1 = -(r-1)c/2$ then $-\lambda^0 > (r-1)c/2$ and therefore all the factors appearing on the right hand side of (12.3) is nonnegative. Namely (3) is satisfied. Next we assume that $\lambda^0 = a_t + 1$ for some $t \in \{1, \dots, r\}$. If a factor $(-\lambda^0 - ic/2 + m)$ occurs in (12.3) for $i \geq t-1$ then a factor $(-\lambda^0 - (t-1)c/2 + 0)$ also occurs. In this case (12.3) is equal to zero and (3) is satisfied. If all the factors $(-\lambda^0 - ic/2 + m)$ in (12.3) satisfy that $i < t-1$, then every factor is positive, and then (3) is satisfied. Thus we have shown that (3) implies (4). We proved the proposition. $\square$

13 An analogue of the Capelli identity

The Capelli identity plays an important role in invariant theory. In [12], it is studied in the context of multiplicity-free actions, and analogues of the Capelli identity are given for these actions if it is possible. A polynomial ring $\mathbb{C}[n^+]$ is multiplicity free with respect to the action of $\text{ad}(\mathfrak{l})$. Thus the Capelli identity for the prehomogeneous vector space (L, Ad, n^+) is investigated in [12]. The main purpose of this section is to give an analogue of the Capelli identity on $M(\lambda)$.

From now on, to avoid confusion, we call it the *classical* Capelli identities that the Capelli identity [3] in the nineteenth century and its analogue on (L, Ad, n^+) by [12], and call its analogue on $M(\lambda)$ a Ψ_λ -analogue of the (classical) Capelli identity.

This Ψ_λ -analogue of the Capelli identity was originally motivated by the study on an irreducibility criterion of $M(\lambda)$, and at first the Ψ_λ -analogue was proved by type-by-type calculation. By using the classical Capelli identity, we can easily calculate the b -function $b_r(\mu)$. Similarly, we can calculate $\beta_{\lambda,r}(\mu)$ by using the Ψ_λ -analogue of the Capelli identity. Moreover we find that $\beta_{\lambda,r}(\mu)$ is a product of $b_r(\mu)$ and $b_r(\mu + \text{shift})$.

This property for $\beta_{\lambda,r}(\mu)$, however, can be proved type-freely (Theorem 6.1), and, in fact, we use it conversely to prove Proposition 13.2.

We also give examples of the Ψ_λ -analogue of the Capelli identity for the type (A_{2r-1}, r) , $(B_n, 1)$, (C_n, n) , $(D_n, 1)$ and $(D_{2r}, 2r)$ in the notation of Figure 2, or $GL_r \otimes GL_r$, $O_{2n} \otimes GL_1$, $S^2 GL_n$, $O_{2n-1} \otimes GL_1$ and $A^2 GL_{2r}$ in the notation of [12, (11.0.1)], respectively. These examples are the only arguments in this thesis which contain type-by-type study.

We assume that $(\mathfrak{g}, \mathfrak{p})$ is of commutative parabolic type and (L, Ad, n^+) is a regular prehomogeneous vector space until the end of this section. In this case, f_r is a relative invariant.

Lemma 13.1 *Let $f = f_r \in \mathbb{C}[n^+]$ be the relative invariant of (L, Ad, n^+) with a weight $-2\varpi_{i_0}$. Then we have*

$$f^\alpha \Psi_\lambda(u) f^{-\alpha} = \Psi_{\lambda+2\alpha\varpi_{i_0}}(u) \quad (\alpha \in \mathbb{C}, u \in U(\mathfrak{l})).$$

Proof. We have only to show the equality for $u = X \in \mathfrak{l}$. We note that $-2\varpi_{i_0}$ is not only a weight, but a character of $\mathfrak{p}$. We have

$$\begin{aligned} [\Psi_\lambda(X), f^{-\alpha}] &= [\text{ad}(X), f^{-\alpha}] \\ &= \text{ad}(X)(f^{-\alpha}) \\ &= 2\alpha\varpi_{i_0}(X)f^{-\alpha}. \end{aligned}$$

Thus

$$\Psi_\lambda(X)f^{-\alpha} = f^{-\alpha}(\Psi_\lambda(X) + 2\alpha\varpi_{i_0}(X)) = f^{-\alpha}\Psi_{\lambda+2\alpha\varpi_{i_0}}(X).$$

This proves the equality. $\square$

Now we can state the main result of this section.

Proposition 13.2 *Let $(\mathfrak{g}, \mathfrak{p})$ be of commutative parabolic type, where (L, Ad, n^+) is a regular prehomogeneous vector space. Let $f = f_r \in \mathbb{C}[n^+]$ be the relative invariant of (L, Ad, n^+) with a weight $-2\varpi_{i_0}$, and r be the degree of f .*

(1) For $k \in \mathbb{Z}_{>0}$, if there exists $u \in U(\mathfrak{l})$ such that

$$(13.1) \quad f^k {}^t f^k (\partial) = \text{ad}(u),$$

then we have

$$(13.2) \quad \Psi_\lambda(f^k {}^t f^k) = (-1)^{kr} \Psi_{2\lambda+2\rho}(u) \Psi_0(u).$$

(2) For $k \in \mathbb{Z}_{>0}$, if there exists $v \in U(\mathfrak{l})$ such that

$$(13.3) \quad {}^t f^k (\partial) f^k = \text{ad}(v),$$

then we have

$$(13.4) \quad \Psi_\lambda({}^t f^k f^k) = (-1)^{kr} \Psi_{2\lambda+2\rho}(v) \Psi_0(v).$$

Here $\rho \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ is half the sum of roots in Δ_N^+ .

Remark 13.3 (1) In the proposition above, the classical Capelli identity is the $k = 1$ case of (13.1). Among the pairs listed in Figure 2, only in the case $(E_7, 7)$, the classical Capelli identity does not exist [12, §§11.13]. There may be, however, a possibility that $\Psi_\lambda({}^t f^k f^k)$ is in the image $\Psi_\lambda(Z(\mathfrak{l}))$ for the case $(E_7, 7)$.

(2) Even in the case where the classical Capelli identity exists, either u or v in the theorem above is not unique. In the following subsections, we give several choices for u and v .

Proof. We prove only (2). We can prove (1) similarly, and we can prove (1) also by using Lemma 13.6.

We can take $P_\lambda \in D_{n^+}^L$ such that $\Psi_\lambda({}^t f^k) = P_\lambda {}^t f^k (\partial)$. Indeed, we prove the existence of P_λ by modifying the proof of Lemma 14.1, that is, we take f^k instead of f_r in the proof. Then we have only to show that $P_\lambda = (-1)^{kr} \Psi_{2\lambda+2\rho}(v)$. In the following calculation, we set $b(\mu) = b_r(\mu)$ and $\beta_\lambda(\mu) = \beta_{\lambda,r}(\mu)$ for simplicity. We have

$$\begin{aligned} \Psi_\lambda({}^t f^k f^k) f_\mu &= \Psi_\lambda({}^t f^{k-1}) \Psi_\lambda({}^t f f) f^{k-1} f_\mu \\ &= \Psi_\lambda({}^t f^{k-1}) \Psi_\lambda({}^t f f) f_{\mu+(k-1)\lambda_r} \end{aligned}$$

$$\begin{aligned} &= \beta_\lambda(\mu + (k-1)\lambda_r) \Psi_\lambda({}^t f^{k-1} f^{k-1}) f_\mu \\ &= \dots \\ &= \beta_\lambda(\mu + (k-1)\lambda_r) \beta_\lambda(\mu + (k-2)\lambda_r) \dots \beta_\lambda(\mu) f_\mu. \end{aligned}$$

Similarly we have

$${}^t f^k (\partial) f^k \cdot f_\mu = b(\mu + (k-1)\lambda_r) b(\mu + (k-2)\lambda_r) \dots b(\mu) f_\mu.$$

It follows from the above two equalities and Theorem 6.1 that

$$(13.5) \quad P_\lambda f_\mu = (-1)^{kr} b(\mu + (k-1-\lambda^0-\rho^0)\lambda_r) b(\mu + (k-2-\lambda^0-\rho^0)\lambda_r) \dots b(\mu + (-\lambda^0-\rho^0)\lambda_r) f_\mu.$$

On the other hand, by Lemma 13.1,

$$\begin{aligned} &\Psi_{2\lambda+2\rho}(v) f_\mu \\ &= f^{\lambda^0+\rho^0} \Psi_0(v) f^{-\lambda^0-\rho^0} f_\mu \\ &= f^{\lambda^0+\rho^0} {}^t f^k (\partial) f^k \cdot f_{\mu-(\lambda^0+\rho^0)\lambda_r} \\ &= f^{\lambda^0+\rho^0} \times b(\mu - (\lambda^0+\rho^0)\lambda_r + (k-1)\lambda_r) \dots b(\mu - (\lambda^0+\rho^0)\lambda_r) f_{\mu-(\lambda^0+\rho^0)\lambda_r} \\ (13.6) \quad &= b(\mu + (k-1-\lambda^0-\rho^0)\lambda_r) \dots b(\mu + (-\lambda^0-\rho^0)\lambda_r) f_\mu. \end{aligned}$$

Here, two operators P_λ and $\Psi_{2\lambda+2\rho}(v)$ are $\text{Ad}(L)$ -invariant. Thus they are scalar operators on each $\text{ad}(U(\mathfrak{l}))$ -irreducible submodule of $\mathbb{C}[n^+]$. The values of the scalars are given by (13.5) and (13.6), and they coincide (up to sign). Thus we have $P_\lambda = (-1)^{kr} \Psi_{2\lambda+2\rho}(v)$ and we proved the proposition. $\square$

Remark 13.4 Under the notation in the proof of Proposition 13.2, let $\{g_j\}$ be a basis of I_μ and $\{g_j^*\}$ be the dual basis of I_μ with respect to the invariant perfect pairing of $\mathbb{C}[n^+]$ and $S(n^+)$ defined by $(g, p) \mapsto (p(\partial)g)(0)$. Set $z_\mu = \sum_j g_j g_j^*(\partial)$. Then z_μ is in $D_{n^+}^L$ and $\{z_\mu\}$ spans $D_{n^+}^L$ thanks to Schur's lemma. It is expected that Proposition 13.2 is generalized in such a way that z_μ replaces $f^k {}^t f^k (\partial)$.

In the classical Capelli identity case, such a generalization holds [12, (15.1) Table], that is, ad maps $Z(\mathfrak{l})$ onto $D_{n^+}^L$. Although the generalization in the classical case holds, its Ψ_λ -analogue does not hold in general.

We can describe the relation between u and v in Proposition 13.2.

Definition 13.5 Define an automorphism ι_ν on $U(\mathfrak{l})$ for $\nu \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ by

$$\iota_\nu(X) = X + \nu(X) \quad (X \in \mathfrak{l}).$$

Then we have

$$\Psi_\lambda(\iota_\nu(u)) = \Psi_{\lambda+\nu}(u) \quad (u \in U(\mathfrak{l})),$$

since $\Psi_\lambda(X) = \text{ad}(X) + \lambda(X)$ for $X \in \mathfrak{l}$.

The next lemma is useful in calculating the following examples.

Lemma 13.6 Let $f \in \mathbb{C}[n^+]$ be the relative invariant with a weight $-2\varpi_{i_0}$. The anti-involution s is the same as in Lemma 2.4(2). Then we have

(1) For $k \in \mathbb{Z}_{>0}$, if there exists $u \in U(\mathfrak{l})$ such that

$$f^k {}^t f^k(\partial) = \text{ad}(u),$$

then we have

$${}^t f^k(\partial) f^k = \text{ad}(\iota_{-2k\varpi_{i_0}}(u)) = (-1)^{kr} \text{ad}(\iota_{-2\rho}(s({}^t u))).$$

(2) For $k \in \mathbb{Z}_{>0}$, if there exists $v \in U(\mathfrak{l})$ such that

$${}^t f^k(\partial) f^k = \text{ad}(v),$$

then we have

$$f^k {}^t f^k(\partial) = \text{ad}(\iota_{2k\varpi_{i_0}}(v)) = (-1)^{kr} \text{ad}(\iota_{-2\rho}(s({}^t v))).$$

Proof. If $f^k {}^t f^k(\partial) = \text{ad}(u)$ for some $u \in U(\mathfrak{l})$, then it follows from Lemma 13.1 that

$${}^t f^k(\partial) f^k = f^{-k} \text{ad}(u) f^k = \Psi_{-2k\varpi_{i_0}}(u) = \Psi_0(\iota_{-2k\varpi_{i_0}}(u)).$$

Next, an automorphism $\sigma \circ \tau$ (see Definition 2.3) maps $f^k {}^t f^k(\partial)$ to $(-1)^{kr} {}^t f^k(\partial) f^k$ by the definition. Moreover $\sigma \circ \tau(\text{ad}(u)) = \sigma(\text{ad}({}^t u)) = \Psi_{-2\rho}(s({}^t u)) = \text{ad}(\iota_{-2\rho}(s({}^t u)))$, by Lemma 2.4. Thus we proved (1). We can prove (2) similarly. $\square$

In the following subsections, we give examples of Ψ_λ -analogues of the Capelli identity. We deal with $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type, where (L, Ad, n^+) is a regular prehomogeneous vector space, and, in addition, the classical Capelli identity for the relative invariant holds. Such a pair $(\mathfrak{g}, \mathfrak{p})$ is (A_{2r-1}, r) , $(B_n, 1)$, (C_n, n) , $(D_n, 1)$ or $(D_{2r}, 2r)$. These pairs correspond to $GL_r \otimes GL_r$, $O_{2n} \otimes GL_1$, $S^2 GL_n$, $O_{2n-1} \otimes GL_1$ and $\Lambda^2 GL_{2r}$, respectively, in the notation of [12, (11.0.1)].

For each type, we first give a realization of $\mathfrak{g}$, give an explicit form of the relative invariant f (f^2 in $(D_{2r}, 2r)$), and state the classical Capelli identity for the relative invariant and its Ψ_λ -analogue.

In the following subsections, E_{ij} denotes a matrix unit where (i, j) -entry is equal to one, Π and Π_L denote simple system of $\mathfrak{g}$ and $\mathfrak{l}$, respectively, and the other notation is the same as in preceding sections.

For a determinant of a matrix where entries do not necessarily commute, we recall the definition of a determinant.

Definition 13.7 Let $A = (a_{ij})_{ij}$ be a matrix. Define $\det A$ by

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n},$$

where $\varepsilon(\sigma)$ is a sign of σ .

13.1 Type (A_{2r-1}, r) or $GL_r \otimes GL_r$

We can not discuss the classical Capelli identity within $\mathfrak{sl}(2r, \mathbb{C})$. We therefore set $\mathfrak{g} = \mathfrak{gl}(2r, \mathbb{C})$. Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. We define $\varepsilon_i \in \mathfrak{h}^*$ ($i \in \{1, \dots, 2r\}$) by $\varepsilon_i(E_{jj}) = \delta_{ij}$.

We list the information about a root system, a Chevalley basis (C. B.) which we fix,

an invariant bilinear form which we fix, and so on.

$$\begin{aligned}
\Pi &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{2r-1} - \varepsilon_{2r}\}, \\
\Delta^+ &= \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq 2r\}, \\
E_{ij} &: (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \neq j, \\
\Pi_L &= \Pi \setminus \{\varepsilon_r - \varepsilon_{r+1}\}, \\
\Delta_L^+ &= \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq r\} \cup \{\varepsilon_i - \varepsilon_j \mid r+1 \leq i < j \leq 2r\}, \\
\varpi_{i_0} &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_r - \varepsilon_{r+1} - \dots - \varepsilon_{2r}), \\
\rho &= r\varpi_{i_0}, \\
\langle X, Y \rangle &= \text{Tr}(XY) \quad (X, Y \in \mathfrak{g}), \\
\text{C. B.} &: \{E_{ij} \mid i \neq j\} \cup \{E_{ii} - E_{i+1, i+1} \mid 1 \leq i < 2r\},
\end{aligned}$$

Forms of $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\begin{aligned}
\mathfrak{p} &= \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathfrak{g} \mid A, B, D \in \mathfrak{gl}(r, \mathbb{C}) \right\}, \\
\mathfrak{n}^+ &= \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \mid B \in \mathfrak{gl}(r, \mathbb{C}) \right\}, \\
\mathfrak{l} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{g} \mid A, D \in \mathfrak{gl}(r, \mathbb{C}) \right\}.
\end{aligned}$$

Set $x_{ij} = E_{r+j, i}$ and $\partial_{ij} = \partial/\partial x_{ij}$ for $i, j \in \{1, \dots, r\}$. Then $\{x_{ij}\}$ is a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbb{C}[\mathfrak{n}^+]$ with a weight $-2\varpi_{i_0}$ is given by

$$f = \det(x_{ij}).$$

The famous classical Capelli identity ([3], [12, (1.12) or (11.1.9)]) says

$$(13.7) \quad \det(x_{ij}) \det(\partial_{ij}) = \det \left[\sum_{k=1}^r x_{ki} \partial_{kj} + (r-j) \delta_{ij} \right]_{1 \leq i, j \leq r}.$$

It follows from Lemma 2.2 (2) that

$$\begin{aligned}
\text{ad}(E_{ij}) &= - \sum_{k=1}^r x_{jk} \partial_{ik}, \\
\text{ad}(E_{r+i, r+j}) &= \sum_{k=1}^r x_{ki} \partial_{kj}.
\end{aligned}$$

Thus we can translate (13.7) into our setting, and obtain

$$(13.8) \quad f {}^t f(\partial) = \text{ad}(\det[E_{r+i, r+j} + (r-j)\delta_{ij}]_{ij}).$$

Moreover we obtain the following proposition.

Proposition 13.8 Set

$$\begin{aligned}
u_r^R &= \det[E_{r+i, r+j} + (r-j)\delta_{ij}]_{ij}, \\
u_r^{RT} &= \det[E_{r+j, r+i} + (j-1)\delta_{ij}]_{ij}, \\
v_r^R &= \det[E_{r+i, r+j} + (r+1-j)\delta_{ij}]_{ij}, \\
v_r^{RT} &= \det[E_{r+j, r+i} + j\delta_{ij}]_{ij}, \\
u_r^{LT} &= \det[-E_{ji} + (r-j)\delta_{ij}]_{ij}, \\
u_r^L &= \det[-E_{ij} + (j-1)\delta_{ij}]_{ij}, \\
v_r^{LT} &= \det[-E_{ji} + (r+1-j)\delta_{ij}]_{ij}, \\
v_r^L &= \det[-E_{ij} + j\delta_{ij}]_{ij}.
\end{aligned}$$

Then we have

$$f {}^t f(\partial) = \text{ad}(u),$$

$${}^t f(\partial) f = \text{ad}(v),$$

where u is $u_r^R, u_r^{RT}, u_r^{LT}$ or u_r^L and v is similar. Thus we have

$$\Psi_\lambda(f {}^t f) = (-1)^r \Psi_{2\lambda+2\rho}(u) \Psi_0(u),$$

$$\Psi_\lambda({}^t f f) = (-1)^r \Psi_{2\lambda+2\rho}(v) \Psi_0(v).$$

Proof. The last assertion is just a result of Proposition 13.2. Thus we have only to prove the first assertion.

In the equality (13.7), we change variables x_{ij} to x_{ji} , and then we obtain

$$\det(x_{ij}) \det(\partial_{ij}) = \det \left[\sum_{k=1}^r x_{ik} \partial_{jk} + (r-j) \delta_{ij} \right]_{1 \leq i, j \leq r}.$$

This can be translated into

$$f {}^t f(\partial) = \text{ad}(\det[-E_{ji} + (r-j)\delta_{ij}]_{ij}).$$

In this way we can obtain 'L-version' from 'R-version'. Thus we have only to show the assertions of the 'R-version'.

We use Lemma 13.6. We have

$$\iota_{-2\varpi_{i_0}}(u_r^R) = \det[E_{r+i, r+j} + \delta_{ij} + (r-j)\delta_{ij}]_{ij} = v_r^R,$$

and also $\iota_{-2\varpi_{i_0}}(u_r^{RT}) = v_r^{RT}$. Moreover

$$(-1)^r \iota_{-2\rho}(s({}^t v_r^R)) = (-1)^r \det[-E_{r+j, r+i} - r\delta_{ij} + (r+1-j)\delta_{ij}]_{ij} = u_r^{RT},$$

and also $(-1)^r \iota_{-2\rho}(s({}^t v_r^{RT})) = u_r^R$. Then by Lemma 13.6 we proved all the assertions of the 'R-version', since we already know the equality (13.8) for u_r^R . $\square$

13.2 Type (C_n, n) or S^2GL_n

We set

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathfrak{gl}(2n, \mathbb{C}) \mid \begin{matrix} A, B, C \in \mathfrak{gl}(n, \mathbb{C}) \\ {}^tB = B, {}^tC = C \end{matrix} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Set

$$H_{ij} = E_{ij} - E_{n+j, n+i},$$

$$G_{ij} = E_{i, n+j} + E_{j, n+i},$$

$$F_{ij} = E_{n+i, j} + E_{n+j, i},$$

for $i, j \in \{1, \dots, n\}$. We define $\varepsilon_i \in \mathfrak{h}^*$ ($i \in \{1, \dots, n\}$) by $\varepsilon_i(H_{jj}) = \delta_{ij}$.

We list the information about a root system, a Chevalley basis (C. B.) which we fix, an invariant bilinear form which we fix, and so on.

$$\begin{aligned} \Pi &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}, \\ \Delta^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\varepsilon_i\}, \\ H_{ij} &: (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \neq j, \\ G_{ij} &: (\varepsilon_i + \varepsilon_j)\text{-root vector}, \\ F_{ij} &: -(\varepsilon_i + \varepsilon_j)\text{-root vector}, \\ \Pi_L &= \Pi \setminus \{2\varepsilon_n\}, \\ \Delta_L^+ &= \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}, \\ \varpi_{i_0} &= \varepsilon_1 + \dots + \varepsilon_n, \\ \rho &= \frac{n+1}{2}\varpi_{i_0}, \\ \langle X, Y \rangle &= \frac{1}{2}\text{Tr}(XY) \quad (X, Y \in \mathfrak{g}). \\ \text{C. B.} &: \{H_{ij}\} \cup \{G_{ij} \mid i < j\} \cup \{\frac{1}{2}G_{ii}\} \cup \{F_{ij} \mid i < j\} \cup \{\frac{1}{2}F_{ii}\}, \end{aligned}$$

Forms of $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\begin{aligned} \mathfrak{p} &= \left\{ \begin{pmatrix} A & B \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{g} \mid A, B \in \mathfrak{gl}(n, \mathbb{C}), {}^tB = B \right\}, \\ \mathfrak{n}^+ &= \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \mid B \in \mathfrak{gl}(n, \mathbb{C}), {}^tB = B \right\}, \\ \mathfrak{l} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{gl}(n, \mathbb{C}) \right\}. \end{aligned}$$

Set $x_{ij} = F_{ij}$ and $\partial_{ij} = \partial/\partial x_{ij}$ for $i, j \in \{1, \dots, n\}$. Then $\{x_{ij} \mid i \leq j\}$ is a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbb{C}[\mathfrak{n}^+]$ with a weight $-2\varpi_{i_0}$ is given by

$$f = \det(x_{ij})$$

For $i \neq j$, ${}^tF_{ij}(\partial) = G_{ij}(\partial) = \langle G_{ij}, F_{ij} \rangle \partial_{ij} = \partial_{ij}$. On the other hand ${}^tF_{ii}(\partial) = \langle G_{ii}, F_{ii} \rangle \partial_{ii} = 2\partial_{ii}$. Thus we set $\tilde{\partial}_{ij} = (1 + \delta_{ij})\partial_{ij}$ and obtain

$${}^t f(\partial) = \det(\tilde{\partial}_{ij}).$$

The classical Capelli identity for the type (C_n, n) ([30], [12, (11.2.7)]) is

$$(13.9) \quad \det(x_{ij}) \det(\tilde{\partial}_{ij}) = \det \left[\sum_{k=1}^n x_{ik} \tilde{\partial}_{jk} + (n-j)\delta_{ij} \right]_{1 \leq i, j \leq n}.$$

It follows from Lemma 2.2 (2) that

$$\text{ad}(H_{ij}) = - \sum_{k=1}^n x_{jk} \tilde{\partial}_{ik}.$$

Thus we translate (13.9) into our setting, and obtain

$$(13.10) \quad f {}^t f(\partial) = \text{ad}(\det[-H_{ji} + (n-j)\delta_{ij}]_{ij}).$$

Moreover we obtain the following proposition.

Proposition 13.9 Set

$$u_n^T = \det[-H_{ji} + (n-j)\delta_{ij}]_{ij},$$

$$u_n = \det[-H_{ij} + (j-1)\delta_{ij}]_{ij},$$

$$v_n^T = \det[-H_{ji} + (n+2-j)\delta_{ij}]_{ij},$$

$$v_n = \det[-H_{ij} + (j+1)\delta_{ij}]_{ij}.$$

Then we have

$$f {}^t f(\partial) = \text{ad}(u),$$

$${}^t f(\partial) f = \text{ad}(v),$$

where u is u_n^T or u_n and v is similar. Thus we have

$$\Psi_\lambda(f {}^t f) = (-1)^n \Psi_{2\lambda+2\rho}(u) \Psi_0(u),$$

$$\Psi_\lambda({}^t f f) = (-1)^n \Psi_{2\lambda+2\rho}(v) \Psi_0(v).$$

Proof. The equalities for Ψ_λ -analogues are the result of Proposition 13.2. We will prove the first assertion.

We use Lemma 13.6 as in §13.1. We easily have

$$\iota_{-2\varpi_{i_0}}(u_n^T) = v_n^T,$$

$$\iota_{-2\varpi_{i_0}}(u_n) = v_n,$$

and also

$$(-1)^n \iota_{-2\rho}(s({}^t v_n^T)) = u_n,$$

$$(-1)^n \iota_{-2\rho}(s({}^t v_n)) = u_n^T.$$

Then by Lemma 13.6 we proved the proposition, since we already know the equality (13.10) for u_n^T . $\square$

13.3 Type $(D_{2r}, 2r)$ or $A^2 GL_{2r}$

We set $n = 2r$ and

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix} \in \mathfrak{gl}(2n, \mathbb{C}) \mid \begin{matrix} A, B, C \in \mathfrak{gl}(n, \mathbb{C}) \\ {}^t B = -B, {}^t C = -C \end{matrix} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Set

$$H_{ij} = E_{ij} - E_{n+j, n+i},$$

$$G_{ij} = E_{i, n+j} - E_{j, n+i},$$

$$F_{ij} = E_{n+j, i} - E_{n+i, j},$$

for $i, j \in \{1, \dots, n\}$. We define $\varepsilon_i \in \mathfrak{h}^*$ ($i \in \{1, \dots, n\}$) by $\varepsilon_i(H_{jj}) = \delta_{ij}$. We list the information about a root system, a Chevalley basis (C. B.) which we fix, an invariant bilinear form which we fix, and so on.

$$\begin{aligned} \Pi &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}, \\ \Delta^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}, \\ H_{ij} &: (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \neq j, \\ G_{ij} &: (\varepsilon_i + \varepsilon_j)\text{-root vector for } i \neq j, \\ F_{ij} &: -(\varepsilon_i + \varepsilon_j)\text{-root vector for } i \neq j, \\ \Pi_L &= \Pi \setminus \{\varepsilon_{n-1} + \varepsilon_n\}, \\ \Delta_L^+ &= \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}, \\ \varpi_{i_0} &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n), \\ \rho &= (n-1)\varpi_{i_0}, \\ \langle X, Y \rangle &= \frac{1}{2} \text{Tr}(XY) \quad (X, Y \in \mathfrak{g}), \\ \text{C. B.} &: \{H_{ij}\} \cup \{G_{ij} \mid i < j\} \cup \{F_{ij} \mid i < j\}, \end{aligned}$$

Forms of $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\begin{aligned} \mathfrak{p} &= \left\{ \begin{pmatrix} A & B \\ 0 & -{}^t A \end{pmatrix} \in \mathfrak{g} \mid A, B \in \mathfrak{gl}(n, \mathbb{C}), {}^t B = -B \right\}, \\ \mathfrak{n}^+ &= \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \mid B \in \mathfrak{gl}(n, \mathbb{C}), {}^t B = -B \right\}, \\ \mathfrak{l} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^t A \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{gl}(n, \mathbb{C}) \right\}. \end{aligned}$$

Set $x_{ij} = F_{ij}$ and $\partial_{ij} = \partial/\partial x_{ij}$ for $i \neq j$ in $\{1, \dots, n\}$. Then $\{x_{ij} \mid i < j\}$ is a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbb{C}[\mathfrak{n}^+]$ with a weight $-2\varpi_{i_0}$ is given by Pfaffian

$$f = \text{Pf}(x_{ij}).$$

Then we have

$${}^t f(\partial) = \text{Pf}(\partial_{ij}).$$

Since the classical Capelli identity for the Pfaffian [12, (11.3.19) Corollary] is complicated to state, we give an example only for the determinant which is the square of the Pfaffian and has a weight $-4\varpi_{i_0}$.

The classical Capelli identity for the determinant in the type $(D_{2r}, 2r)$ [12, (11.3.20)] is

$$(13.11) \quad f^2 {}^t f^2(\partial) = \text{ad}(\det[-H_{ji} + (n-1-j)\delta_{ij}]_{1 \leq i, j \leq n}).$$

Here from Lemma 2.2 (2),

$$\text{ad}(H_{ij}) = - \sum_{k \neq i,j} x_{jk} \partial_{ik}.$$

Moreover we obtain the following proposition.

Proposition 13.10 *Set*

$$u_n^T = \det[-H_{ji} + (n-1-j)\delta_{ij}]_{ij},$$

$$u_n = \det[-H_{ij} + (j-2)\delta_{ij}]_{ij},$$

$$v_n^T = \det[-H_{ji} + (n+1-j)\delta_{ij}]_{ij},$$

$$v_n = \det[-H_{ij} + j\delta_{ij}]_{ij}.$$

Then we have

$$f^2 {}^t f^2(\partial) = \text{ad}(u),$$

$${}^t f^2(\partial) f^2 = \text{ad}(v),$$

where u is u_n^T or u_n and v is similar. Thus we have

$$\Psi_\lambda(f^2 {}^t f^2) = \Psi_{2\lambda+2\rho}(u) \Psi_0(u),$$

$$\Psi_\lambda({}^t f^2 f^2) = \Psi_{2\lambda+2\rho}(v) \Psi_0(v).$$

Proof. The equalities for Ψ_λ -analogues are the result of Proposition 13.2. We will prove the first assertion.

We use Lemma 13.6 as in §§13.1. We easily have

$$\iota_{-4\varpi_{i_0}}(u_n^T) = v_n^T,$$

$$\iota_{-4\varpi_{i_0}}(u_n) = v_n,$$

and also

$$\iota_{-2\rho}(s({}^t v_n^T)) = u_n,$$

$$\iota_{-2\rho}(s({}^t v_n)) = u_n^T.$$

Then by Lemma 13.6 we proved the proposition, since we already know the equality (13.11) for u_n^T . $\square$

13.4 Type $(D_n, 1)$ or $O_{2n-1} \otimes GL_1$

We set

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix} \in \mathfrak{gl}(2n, \mathbb{C}) \mid \begin{matrix} A, B, C \in \mathfrak{gl}(n, \mathbb{C}) \\ {}^t B = -B, {}^t C = -C \end{matrix} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Here we define basis elements of $\mathfrak{g}$ in a slightly tricky way. Set

$$H_{ij} = E_{\bar{i}\bar{j}} - E_{\overline{n+j}, \overline{n+i}} \quad (i, j \in \mathbb{Z}_{>0}),$$

where $\bar{i}$ denotes the integer such that $1 \leq \bar{i} \leq 2n$ and $i \equiv \bar{i} \pmod{2n}$. Then every H_{ij} belongs to $\mathfrak{g}$ and it follows that

$$H_{n+i, n+j} = -{}^t H_{ij} = -H_{ji},$$

$$H_{i, n+i} = 0,$$

where ${}^t H_{ij}$ means just a transposed matrix of H_{ij} , not the image of the anti-involution on $U(\mathfrak{g})$, although we will find that they coincide when we fix a certain Chevalley basis.

We define $\varepsilon_i \in \mathfrak{h}^*$ ($i \in \{1, \dots, n\}$) by $\varepsilon_i(H_{jj}) = \delta_{ij}$ ($j \in \{1, \dots, n\}$).

We list the information about a root system, a Chevalley basis (C. B.) which we fix, an invariant bilinear form which we fix, and so on.

$$\begin{aligned} \Pi &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}, \\ \Delta^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}, \\ H_{ij} &: (\varepsilon_i - \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n, i \neq j), \\ H_{i, n+j} &: (\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n, i \neq j), \\ H_{n+j, i} &: -(\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n, i \neq j), \\ \Pi_L &= \Pi \setminus \{\varepsilon_1 - \varepsilon_2\}, \\ \Delta_L^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 < i < j \leq n\}, \\ \varpi_{i_0} &= \varepsilon_1, \\ \rho &= (n-1)\varpi_{i_0}, \\ \langle X, Y \rangle &= \frac{1}{2} \text{Tr}(XY) \quad (X, Y \in \mathfrak{g}). \\ \text{C. B.} &: \{H_{ij} \mid 1 \leq i, j \leq n\} \cup \{H_{i, n+j} \mid 1 \leq i < j \leq n\} \cup \{H_{n+j, i} \mid 1 \leq i < j \leq n\}, \end{aligned}$$

Subalgebras $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\mathfrak{l} = \text{span}_{\mathbb{C}}\{H_{11}, H_{ij}(1 < i, j \leq n), H_{i,n+j}(1 < i, j \leq n), H_{n+j,i}(1 < i, j \leq n)\},$$

$$\mathfrak{n}^+ = \text{span}_{\mathbb{C}}\{H_{1j}(1 < j \leq n), H_{1,n+j}(1 < j \leq n)\},$$

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+.$$

Set $x_i = H_{i1}$ and $\partial_i = \partial/\partial x_i$ for $i \in \{2, \dots, n, n+2, \dots, 2n\}$. Then $\{x_i \mid 1 < i \leq n \text{ or } n+1 < i \leq 2n\}$ forms a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbb{C}[\mathfrak{n}^+]$ with a weight $-2\varpi_{i_0}$ is given by

$$f = x_2 x_{n+2} + x_3 x_{n+3} + \dots + x_n x_{2n}.$$

Moreover we have

$${}^t f(\partial) = \partial_2 \partial_{n+2} + \dots + \partial_n \partial_{2n}.$$

The classical Capelli identity for the type $(D_n, 1)$ [12, (11.4.12)] is

$$(13.12) \quad f {}^t f(\partial) = \text{ad} \left(\frac{1}{4} H_{11} (H_{11} - (2n - 4)) - \frac{1}{4} c \right),$$

where c is the (universal) Casimir element of $[\mathfrak{l}, \mathfrak{l}]$, the semisimple part of $\mathfrak{l}$, with respect to the invariant bilinear form $\langle, \rangle$, and

$$\text{ad}(H_{11}) = - \sum_j x_j \partial_j \quad (j \in \{2, \dots, n, n+2, \dots, 2n\}).$$

Note that this formula differs from [12, (11.4.12)] in a multiple $1/4$ on the right hand side because of the difference of settings. Moreover we obtain the following proposition.

Proposition 13.11 *Set*

$$\begin{aligned} u_{2n} &= \frac{1}{4} H_{11} (H_{11} - (2n - 4)) - \frac{1}{4} c, \\ v_{2n} &= \frac{1}{4} (H_{11} - 2) (H_{11} - (2n - 2)) - \frac{1}{4} c. \end{aligned}$$

Then we have

$$f {}^t f(\partial) = \text{ad}(u_{2n}),$$

$${}^t f(\partial) f = \text{ad}(v_{2n}),$$

and hence,

$$\Psi_{\lambda}(f {}^t f) = \Psi_{2\lambda+2\rho}(u_{2n}) \Psi_0(u_{2n}),$$

$$\Psi_{\lambda}({}^t f f) = \Psi_{2\lambda+2\rho}(v_{2n}) \Psi_0(v_{2n}).$$

Proof. As in the preceding case, we have only to show the classical equalities. We already know the equality (13.12) for u_{2n} . We have $\iota_{-2\varpi_{i_0}}(u_{2n}) = v_{2n}$, since the fact that Casimir element c belongs to $U([\mathfrak{l}, \mathfrak{l}])$ implies $\varpi_{i_0}(c) = 0$. Then the proposition is proved by Lemma 13.6. $\square$

13.5 Type $(B_n, 1)$ or $O_{2n} \otimes GL_1$

We set

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b \\ -{}^t b & A & B \\ -{}^t a & C & -{}^t A \end{pmatrix} \in \mathfrak{gl}(2n+1, \mathbb{C}) \mid \begin{array}{l} A, B, C \in \mathfrak{gl}(n, \mathbb{C}) \\ a, b \in \mathbb{C}^n \\ {}^t B = -B, {}^t C = -C \end{array} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Here we define basis elements of $\mathfrak{g}$ by extending that in the type $(D_n, 1)$. We use the convention that indices of rows and columns of matrices in $\mathfrak{g}$ begin with zero. Set

$$H_{ij} = E_{i\bar{j}} - E_{\overline{n+j}, \overline{n+i}} \quad (i, j \in \mathbb{Z}_{>0}),$$

$$g_i = E_{0\bar{i}} - E_{\overline{n+i}, 0} \quad (i \in \mathbb{Z}_{>0}),$$

where $\bar{i}$ is the same as in the type $(D_n, 1)$. Then every H_{ij} and g_i belongs to $\mathfrak{g}$ and it follows that

$$H_{n+i, n+j} = -{}^t H_{ij} = -H_{ji},$$

$$H_{i, n+i} = 0,$$

$$g_{n+i} = -{}^t g_i.$$

We define $\varepsilon_i \in \mathfrak{h}^*$ ($i \in \{1, \dots, n\}$) by $\varepsilon_i(H_{jj}) = \delta_{ij}$ ($j \in \{1, \dots, n\}$).

We list the information about a root system, a Chevalley basis (C. B.) which we fix, an invariant bilinear form which we fix, and so on.

$$\begin{aligned}
\Pi &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}, \\
\Delta^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_i \mid 1 \leq i \leq n\}, \\
H_{ij} &: (\varepsilon_i - \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n, i \neq j), \\
H_{i,n+j} &: (\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n, i \neq j), \\
H_{n+j,i} &: -(\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n, i \neq j), \\
g_{n+i} &: \varepsilon_i\text{-root vector } (1 \leq i \leq n), \\
g_i &: -\varepsilon_i\text{-root vector } (1 \leq i \leq n), \\
\Pi_L &= \Pi \setminus \{\varepsilon_1 - \varepsilon_2\}, \\
\Delta_L^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 < i < j \leq n\} \cup \{\varepsilon_i \mid 1 < i \leq n\}, \\
\varpi_{i_0} &= \varepsilon_1, \\
\rho &= \frac{2n-1}{2} \varpi_{i_0}, \\
\langle X, Y \rangle &= \frac{1}{2} \text{Tr}(XY) \quad (X, Y \in \mathfrak{g}). \\
\text{C. B.} &: \{H_{ij} \mid 1 \leq i, j \leq n\} \cup \{H_{i,n+j} \mid 1 \leq i < j \leq n\} \\
&\quad \cup \{H_{n+j,i} \mid 1 \leq i < j \leq n\} \cup \{\sqrt{-2}g_i \mid 1 \leq i \leq 2n\},
\end{aligned}$$

Subalgebras $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\begin{aligned}
\mathfrak{l} &= \text{span}_{\mathbb{C}}\{H_{11}, H_{ij}(1 < i, j \leq n), H_{i,n+j}(1 < i, j \leq n), H_{n+j,i}(1 < i, j \leq n), \\
&\quad g_i(1 < i \leq n), g_{n+i}(1 < i \leq n)\}, \\
\mathfrak{n}^+ &= \text{span}_{\mathbb{C}}\{H_{1j}(1 < j \leq n), H_{1,n+j}(1 < j \leq n), g_{n+1}\}, \\
\mathfrak{p} &= \mathfrak{l} + \mathfrak{n}^+.
\end{aligned}$$

Set $x_i = H_{i1}$, $x_0 = g_1$, $\partial_i = \partial/\partial x_i$ and $\partial_0 = \partial/\partial x_0$ for $i \in \{2, \dots, n, n+2, \dots, 2n\}$. Then $\{x_i \mid i = 0, 1 < i \leq n \text{ or } n+1 < i \leq 2n\}$ forms a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbb{C}[\mathfrak{n}^+]$ with a weight $-2\varpi_{i_0}$ is given by

$$f = x_2 x_{n+2} + x_3 x_{n+3} + \dots + x_n x_{2n} + \frac{1}{2} x_0^2.$$

Moreover we have

$${}^t f(\partial) = \partial_2 \partial_{n+2} + \dots + \partial_n \partial_{2n} + \frac{1}{2} \partial_0 \partial_0.$$

The classical Capelli identity for the type $(B_n, 1)$ [12, (11.4.12)] is

$$(13.13) \quad f {}^t f(\partial) = \text{ad} \left(\frac{1}{4} H_{11} (H_{11} - (2n-3)) - \frac{1}{4} c \right),$$

where c is the Casimir element of $[\mathfrak{l}, \mathfrak{l}]$ with respect to the invariant bilinear form $\langle, \rangle$, and

$$\text{ad}(H_{11}) = - \sum_j x_j \partial_j \quad (j \in \{0, 2, \dots, n, n+2, \dots, 2n\}).$$

This formula differs from [12, (11.4.12)] again in a multiple $1/4$ on the right hand side because of the difference of settings. Moreover we obtain the following proposition.

Proposition 13.12 Set

$$\begin{aligned}
u_{2n+1} &= \frac{1}{4} H_{11} (H_{11} - (2n-3)) - \frac{1}{4} c, \\
v_{2n+1} &= \frac{1}{4} (H_{11} - 2) (H_{11} - (2n-1)) - \frac{1}{4} c.
\end{aligned}$$

Then we have

$$\begin{aligned}
f {}^t f(\partial) &= \text{ad}(u_{2n+1}), \\
{}^t f(\partial) f &= \text{ad}(v_{2n+1}),
\end{aligned}$$

and hence,

$$\begin{aligned}
\Psi_\lambda(f {}^t f) &= \Psi_{2\lambda+2\rho}(u_{2n+1}) \Psi_0(u_{2n+1}), \\
\Psi_\lambda({}^t f f) &= \Psi_{2\lambda+2\rho}(v_{2n+1}) \Psi_0(v_{2n+1}).
\end{aligned}$$

Proof.

We can prove the proposition in a similar way to Proposition 13.11. $\square$

Remark 13.13 We can state examples of type $(D_n, 1)$ and $(B_n, 1)$ together. Define a complex number ρ^0 by $\rho = \rho^0 \varpi_{i_0}$. Namely, $2\rho^0 = 2n-2$ in $(D_n, 1)$ and $2\rho^0 = 2n-1$ in $(B_n, 1)$. We also define $m \in \mathbb{Z}_{>0}$ as the number of rows of matrices in $\mathfrak{g}$. Namely, $m = 2n$ in $(D_n, 1)$ and $m = 2n+1$ in $(B_n, 1)$. Then we can combine Proposition 13.11 and Proposition 13.12 to define u and v .

$$\begin{aligned}
u_m &= \frac{1}{4} H_{11} (H_{11} - 2\rho^0 + 2) - \frac{1}{4} c, \\
v_m &= \frac{1}{4} (H_{11} - 2) (H_{11} - 2\rho^0) - \frac{1}{4} c.
\end{aligned}$$

14 Factors contained in $\beta_{\lambda, i}(\mu)$

The sections 14 and 15 are devoted to proving the main theorems (Theorem 6.1 and Theorem 7.5). We have only to prove Theorem 6.1 since Theorem 7.5 is a direct consequence of Theorem 6.1. Thus we may assume that the prehomogeneous vector space $(L, \text{Ad}, \mathfrak{n}^+)$ is regular.

The proof of the theorem requires several steps. First, we show that $b_r(\mu)$ divides $\beta_{\lambda, r}(\mu)$ in the ring $\mathbb{C}[k_1, \dots, k_r, \lambda^0]$. Second, we show $\beta_{\lambda, r}(\mu) = \beta_{-\lambda-2\rho_r, r}(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ by using Boe's theorem. Then we know $b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ divides $\beta_{\lambda, r}(\mu)$ in the ring $\mathbb{C}[k_1, \dots, k_r, \lambda^0]$. Thus $b_r(\mu)b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ divides $\beta_{\lambda, r}(\mu)$. Third, we show that $\beta_{\lambda, r}(\mu) = b_r(\mu)b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ up to constant multiple. Lastly, we calculate the principal symbol of $\Psi_\lambda({}^t f_r f_r)$ to determine the constant. This last step is in §15.

Lemma 14.1 *Let $D_{\mathfrak{n}^+}$ be the ring of polynomial coefficient differential operators on $\mathfrak{n}^+$. Then $\Psi_\lambda({}^t f_r) = P_\lambda {}^t f_r(\partial)$ for some $P_\lambda \in D_{\mathfrak{n}^+}^L$, where $D_{\mathfrak{n}^+}^L$ is the subspace of L -invariant elements in $D_{\mathfrak{n}^+}$.*

Proof. In general, let $g^* \in S(\mathfrak{n}^+)$ and $h \in D_{\mathfrak{n}^+} \simeq \mathbb{C}[\mathfrak{n}^+] \otimes S(\mathfrak{n}^+)$ be relative invariants with respect to the $\text{Ad}(L)$ -action, and we assume that they correspond to the same character $\chi \in \text{Hom}(L, \mathbb{C}^\times)$. Then h is a sum of several relative invariants $h_i \in I_{\mu_i} \otimes {}^t I_{\nu_i}$ which correspond to the same character χ , where ${}^t I_\mu = \{{}^t f \mid f \in \mathbb{C}[\mathfrak{n}^+]\} \subset S(\mathfrak{n}^+)$.

Set $g = {}^t(g^*) \in \mathbb{C}[\mathfrak{n}^+]$. Then $\mathbb{C}gh_i \subset gI_{\mu_i} \otimes {}^t I_{\nu_i}$ is a trivial $\text{Ad}(L)$ -submodule. Namely, $gI_{\mu_i} \otimes {}^t I_{\nu_i}$ contains an $\text{Ad}(L)$ -invariant nonzero element. Thus it follows from Schur's lemma that gI_{μ_i} is the dual module of ${}^t I_{\nu_i}$, since both gI_{μ_i} and ${}^t I_{\nu_i}$ are irreducible. Therefore we have ${}^t I_{\nu_i} = {}^t(gI_{\mu_i}) = {}^t I_{\mu_i} g^*$, since $\mathbb{C}[\mathfrak{n}^+]$ is multiplicity free.

Thus $h_i \in I_{\mu_i} \otimes {}^t I_{\nu_i} = I_{\mu_i} \otimes {}^t I_{\mu_i} g^*$ and there exists $P_i \in D_{\mathfrak{n}^+}$ such that $h_i = P_i g^*$. Here h_i and g^* have the same character, and therefore P_i is $\text{Ad}(L)$ -invariant. Then $h = (\sum P_i)g^*$, where $\sum P_i$ is an $\text{Ad}(L)$ -invariant.

Finally, we take ${}^t f_r(\partial)$ and $\Psi_\lambda({}^t f_r)$ as g^* and h , respectively, to prove the lemma. $\square$

We have the following proposition from Lemma 14.1 and the definition of $\beta_{\lambda, r}$ and b_r . This is the goal of the first step.

Proposition 14.2 *In $\mathbb{C}[k_1, \dots, k_r, \lambda^0]$, $b_r(\mu)$ divides $\beta_{\lambda, r}(\mu)$ ($\mu = k_1 \lambda_1 + \dots + k_r \lambda_r$).*

Next, we show that $\beta_{\lambda, r}$ have another factor related to b_r . We use the theorem of Boe [1].

Theorem 14.3 (Boe [1, Theorem 4.4]) *If $(\mathfrak{g}, \mathfrak{p})$ is of commutative parabolic type and $(L, \text{Ad}, \mathfrak{n}^+)$ is a regular prehomogeneous vector space, then for $\lambda, \lambda' \in \text{Hom}(\mathfrak{p}, \mathbb{C})$ the necessary and sufficient condition for $\text{Hom}_{U(\mathfrak{g})}(M(\lambda'), M(\lambda))$ to be nonzero is that $\lambda = \lambda'$ or $\lambda = l\varpi_{i_0} - \rho_r$ and $\lambda' = -l\varpi_{i_0} - \rho_r$ for some $l \in \mathbb{Z}_{>0}$.*

If $\lambda^0 + \rho_r^0 \in \mathbb{Z}_{\geq 0}$ then $\text{Hom}(M(-\lambda - 2\rho_r), M(\lambda))$ contains the mapping

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{n}^+] & \longrightarrow & \mathbb{C}[\mathfrak{n}^+] \\ f & \longmapsto & f f_r^{\lambda^0 + \rho_r^0} \end{array}$$

by the proof of the theorem. Thus it follows that

$$(14.1) \quad f_r^{\lambda^0 + \rho_r^0} \Psi_{-\lambda - 2\rho_r}(u) = \Psi_\lambda(u) f_r^{\lambda^0 + \rho_r^0} \quad (u \in U(\mathfrak{g})),$$

if $\lambda^0 + \rho_r^0 \in \mathbb{Z}_{\geq 0}$. In fact, this equality holds even if $\lambda^0 + \rho_r^0$ is a complex number. We note that this formula is a generalization of Lemma 13.1 in some sense.

Now we can prove that $\beta_{\lambda, r}$ have another factor related to b_r .

Proposition 14.4 *The equality $\beta_{\lambda, r}(\mu) = \beta_{-\lambda-2\rho_r}(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ holds, and therefore $b_r(\mu - (\lambda^0 + \rho_r^0)\lambda_r)$ divides $\beta_{\lambda, r}(\mu)$ in $\mathbb{C}[k_1, \dots, k_r, \lambda^0]$.*

Proof. We have $\Psi_\lambda({}^t f_r f_r) f_\mu = \beta_{\lambda, r}(\mu) f_\mu$ by the definition of $\beta_{\lambda, r}$. On the other hand, by (14.1), $\Psi_\lambda({}^t f_r f_r) f_\mu = f_r^{\lambda^0 + \rho_r^0} \Psi_{-\lambda-2\rho_r}({}^t f_r f_r) f_r^{-\lambda^0 - \rho_r^0} f_\mu = \beta_{-\lambda-2\rho_r}(\mu - (\lambda^0 + \rho_r^0)\lambda_r) f_\mu$. This proves the first statement of the proposition. The second statement follows from Proposition 14.2. $\square$

The proposition above is the goal of the second step. Next we obtain a proposition which proves the third step.

Lemma 14.5 $\beta_{\lambda,r}(\mu) = b_r(\mu)b_r(\mu - (\lambda^0 + \rho^0)\lambda_r)$ (up to constant multiple).

Proof. We have that $\beta_{\lambda,r}(\mu)$ has factors $b_r(\mu)$ and $b_r(\mu - (\lambda^0 + \rho^0)\lambda_r)$ from Proposition 14.2 and Proposition 14.4. Here $b_r(\mu)$ and $b_r(\mu - (\lambda^0 + \rho^0)\lambda_r)$ are relatively prime, since all irreducible factors in $b_r(\mu)$ are different from that in $b_r(\mu - (\lambda^0 + \rho^0)\lambda_r)$. Thus $\beta_{\lambda,r}(\mu)$ is divisible by $b_r(\mu)b_r(\mu - (\lambda^0 + \rho^0)\lambda_r)$. Then by Proposition 9.3 the total degree of $\beta_{\lambda,r}(\mu)$ in $k_1, \dots, k_r$ is at least $2r$, and the degree in λ^0 is at least r .

On the other hand, it follows from Lemma 2.2 that the operator $\Psi_\lambda({}^t f_r)$ is of order at most $2r$ and of degree at most r in λ^0 , since ${}^t f_r$ is of degree r . Thus we have that the total degree of $\beta_{\lambda,r}(\mu)$ in $k_1, \dots, k_r$ is at most $2r$, and the degree of λ^0 is at most r . Then $\beta_{\lambda,r}(\mu)$ must be a constant multiple of $b_r(\mu)b_r(\mu - (\lambda^0 + \rho^0)\lambda_r)$, and we obtain the proposition. $\square$

15 The principal symbol of $\Psi_\lambda({}^t f_r)$

In this section, we determine the constant which appeared in Lemma 14.5 and prove our main theorem. To determine it, we show that the principal symbol of $\Psi_\lambda({}^t f_r)$ and that of $f_r {}^t f_r(\partial) {}^t f_r(\partial)$ coincide up to a certain constant multiple.

First, we introduce the principal symbol of a differential operator. Let D_{n^+} be the ring of polynomial coefficient differential operators, and $D_{n^+}^d$ the subspace of D_{n^+} consisting of operators of order at most d . We define a linear mapping $\sigma_d : D_{n^+}^d \rightarrow \mathbb{C}[n^+ \oplus n^-] \simeq \mathbb{C}[n^+] \otimes \mathbb{C}[n^-]$ as follows: If the d th order part of $P \in D_{n^+}^d$ is a certain sum of $g\partial/\partial X_{-\beta_1} \cdots \partial/\partial X_{-\beta_d}$ ($g \in \mathbb{C}[n^+]$, $\beta_j \in \Delta_N^+$), then $\sigma_d(P)$ is the sum of $g\xi_{\beta_1} \cdots \xi_{\beta_d}$. Here $\xi_\beta \in \mathbb{C}[n^-]$ is the linear mapping defined by

$$\xi_\beta(X_{-\delta}) = \frac{\partial}{\partial X_{-\beta}}(X_{-\delta}) = \begin{cases} 1 & (\delta = \beta) \\ 0 & (\delta \neq \beta) \end{cases} \quad \text{for } \delta \in \Delta_N^+.$$

In particular, for $P \in D_{n^+}$ of order d , $\sigma_d(P)$ is the principal symbol of P .

Proposition 15.1 $\sigma(\Psi_\lambda({}^t f_r)) = (-1)^r \sigma(f_r {}^t f_r(\partial) {}^t f_r(\partial))$, under the normalization (7.1).

Proof. [Step 1] First we show that $\Psi_\lambda({}^t f_r)$ is a differential operator of order $2r$, namely that $\sigma_{2r}(\Psi_\lambda({}^t f_r)) \neq 0$.

Set $X_+ = X_{\gamma_1} + \cdots + X_{\gamma_r}$ and $X_- = {}^t X_+ = X_{-\gamma_1} + \cdots + X_{-\gamma_r}$. Principal symbols can be considered as polynomial functions on the cotangent bundle of n^+ , which can be identified with $n^+ \times n^-$. Then we evaluate $\sigma_{2r}(\Psi_\lambda({}^t f_r))$ at (X_+, X_-) by using the expansion of f_r in Definition 7.1 and we have

$$\begin{aligned} & \sigma_{2r}(\Psi_\lambda({}^t f_r))(X_+, X_-) \\ &= \sum_{B=(\beta_1, \dots, \beta_r) \in B_r} a_B \sigma_{2r}(\Psi_\lambda(X_{\beta_1} \cdots X_{\beta_r}))(X_+, X_-) \\ &= \sum_{B \in B_r} a_B \left\{ \frac{1}{2} \sum_{\delta, \eta \in \Delta_N^+} [[X_{\beta_1}, X_{-\delta}], X_{-\eta}] \xi_\delta \xi_\eta \right\} \cdots \\ (15.1) \quad & \cdots \left\{ \frac{1}{2} \sum_{\delta, \eta \in \Delta_N^+} [[X_{\beta_r}, X_{-\delta}], X_{-\eta}] \xi_\delta \xi_\eta \right\} (X_+, X_-), \end{aligned}$$

where we adopted the basis $\{X_{-\delta} \mid \delta \in \Delta_N^+\}$ as $\{F_k\}$ in Lemma 2.2, since $\{F_k\}$ is any basis of n^- there.

Here we compute the j th factor of (15.1).

$$\begin{aligned} & \left\{ \frac{1}{2} \sum_{\delta, \eta \in \Delta_N^+} [[X_{\beta_j}, X_{-\delta}], X_{-\eta}] \xi_\delta \xi_\eta \right\} (X_+, X_-) \\ &= \frac{1}{2} \sum_{\delta, \eta} \langle [[X_{\beta_j}, X_{-\delta}], X_{-\eta}], X_+ \rangle \xi_\delta(X_-) \xi_\eta(X_-) \\ &= \frac{1}{2} \sum_{k,l=1}^r \langle [[X_{\beta_j}, X_{-\gamma_k}], X_{-\gamma_l}], X_+ \rangle \\ &= \frac{1}{2} \sum_{k,l=1}^r \langle [X_{\beta_j}, X_{-\gamma_k}], -H_{\gamma_l} \rangle \\ &= - \sum_{k=1}^r \langle X_{\beta_j}, X_{-\gamma_k} \rangle, \end{aligned}$$

where H_{γ_l} is the coroot of γ_l , that is, $H_{\gamma_l} = [X_{\gamma_l}, X_{-\gamma_l}]$. The formula above is equal to zero if $\beta_j \notin \{\gamma_1, \dots, \gamma_r\}$. If $\beta_j = \gamma_m$ for some m , then $\langle X_{\beta_j}, X_{-\gamma_m} \rangle = 2/(\gamma_m, \gamma_m) = 2/(\gamma_1, \gamma_1)$.

Therefore the summand of (15.1) which does not vanish, is given only by $B = B_r =$

$(\gamma_1, \dots, \gamma_r)$, and we have

$$\begin{aligned} \sigma_{2r}(\Psi_\lambda({}^t f_r))(X_+, X_-) &= a_{B_r} \left\{ -\sum_{k=1}^r \langle X_{\beta_1}, X_{-\gamma_k} \rangle \right\} \cdots \left\{ -\sum_{k=1}^r \langle X_{\beta_r}, X_{-\gamma_k} \rangle \right\} \\ &= 2^{-r} (\gamma_1, \gamma_1)^r \cdot \{-2/(\gamma_1, \gamma_1)\}^r \\ (15.2) \quad &= (-1)^r. \end{aligned}$$

Moreover we find that $\delta = \eta = \gamma_j$ in the j th factor of the summand of (15.1) which does not vanish.

[Step 2] By Lemma 14.1, there exists $\text{Ad}(L)$ -invariant operator $P_\lambda \in D_{n^+}^L$ such that $\Psi_\lambda({}^t f_r) = P_\lambda {}^t f_r(\partial)$. Here P_λ is of order r by Step 1. In this step, we show that $\sigma_r(P_\lambda)$ is a certain constant multiple of $\sigma_r(f_r {}^t f_r(\partial))$.

Let $(I_\mu \otimes {}^t I_\mu)^L$ denote the subspace of $I_\mu \otimes {}^t I_\mu$ consisting of $\text{Ad}(L)$ -invariant elements. By Schur's lemma, it is one-dimensional and spanned by $g_1 \otimes g_1^* + g_2 \otimes g_2^* + \cdots$, where $\{g_j\}$ is a basis of I_μ and $\{g_j^*\}$ is the dual basis with respect to $\langle, \rangle$ which appeared in Definition 3.5. Here we may assume that all g_j 's are weight vectors, and that g_1 is the highest weight vector f_μ , and we put $f_\mu^* = g_1^*$. We embed $I_\mu \otimes {}^t I_\mu$ in D_{n^+} by $g \otimes P \mapsto gP(\partial)$ as before. Then $D_{n^+}^L = \bigoplus_\mu (I_\mu \otimes {}^t I_\mu)^L$ and we can write $P_\lambda = \sum_\mu z_\mu$ ($z_\mu \in (I_\mu \otimes {}^t I_\mu)^L$). Since the order of P_λ is equal to r , we have

$$P_\lambda = \sum_{\deg f_\mu \leq r} z_\mu.$$

Moreover

$$\sigma_r(P_\lambda) = \sum_{\deg f_\mu = r} \sigma_r(z_\mu).$$

We have only to show that $\sigma_r(z_\mu)$ is equal to zero if $\deg f_\mu = r$ and $\mu \neq \lambda_r$, since $\dim I_{\lambda_r} = 1$ implies that $z_{\lambda_r} \in \mathbb{C} f_r {}^t f_r(\partial)$.

A summand of $\sigma_r(P_\lambda)$ has a form $gQ(\partial)$, where $g \in \mathbb{C}[n^+]$ is a polynomial of degree r and $Q \in \mathbb{C}[n^-]$ is a monomial of degree r . We may assume that g and Q are weight vectors. We call such g a coefficient polynomial. Let ν_0 be a maximal element among the weights of coefficient polynomials in $\sigma_r(P_\lambda)$. Then ν_0 is an $\text{Ad}(L)$ -maximal weight occurring in $\mathbb{C}[n^+]$ by the maximality, since $\sigma_r(P_\lambda)$ is a sum of $\sigma_r(z_\mu) = f_\mu f_\mu^*(\partial) + g_2 g_2^*(\partial) + \cdots$. In

particular, all the terms of $\sigma_r(P_\lambda)$ in which the weight of the coefficient polynomial is ν_0 , come from $\mathbb{C} f_{\nu_0} f_{\nu_0}^* = \mathbb{C} f_{\nu_0} {}^t f_{\nu_0}$. Let $P_0 \in D_{n^+}$ be the sum of terms of P_λ in which the weight of the coefficient polynomial is equal to ν_0 . Here $P_0 \neq 0$ by the property of ν_0 . Then we have $P_0 = s f_{\nu_0} {}^t f_{\nu_0}(\partial)$ ($s \in \mathbb{C}^\times$). Here we compute $\sigma(P_0 {}^t f_r(\partial))(X_+, X_-)$ in two different ways.

First, we define $h_j \in \mathbb{Z}_{\geq 0}$ by $\nu_0 = h_1 \lambda_1 + \cdots + h_r \lambda_r$. Then $f_{\nu_0} = f_1^{h_1} \cdots f_r^{h_r}$ and it follows from the normalization (7.1) that

$$\begin{aligned} (15.3) \quad \sigma_{2r}(P_0 {}^t f_r(\partial))(X_+, X_-) &= \sigma_{2r}(s f_{\nu_0} {}^t f_{\nu_0}(\partial) {}^t f_r(\partial))(X_+, X_-) \\ &= s f_{\nu_0}(X_+) {}^t f_{\nu_0}(X_-) {}^t f_r(X_-) \\ &= s \neq 0, \end{aligned}$$

where ${}^t f_{\nu_0}$ and ${}^t f_r \in S(n^+) \simeq \mathbb{C}[n^-]$ are regarded as functions on n^- .

Second, we compute $\sigma(P_0 {}^t f_r(\partial))(X_+, X_-)$ by using the explicit formula for $\Psi_\lambda({}^t f_r)$.

In the formula

$$\begin{aligned} (15.4) \quad \sigma_{2r}(\Psi_\lambda({}^t f_r)) &= \sum_{B=(\beta_1, \dots, \beta_r) \in B_r} a_B \left\{ \frac{1}{2} \sum_{\delta_1, \eta_1 \in \Delta_N^+} [[X_{\beta_1}, X_{-\delta_1}], X_{-\eta_1}] \xi_{\delta_1} \xi_{\eta_1} \right\} \cdots \\ &\quad \cdots \left\{ \frac{1}{2} \sum_{\delta_r, \eta_r \in \Delta_N^+} [[X_{\beta_r}, X_{-\delta_r}], X_{-\eta_r}] \xi_{\delta_r} \xi_{\eta_r} \right\}, \end{aligned}$$

the terms of $\sigma_{2r}(P_0 {}^t f_r(\partial))$ precisely correspond to the sum of the terms in (15.4) in which the coefficient polynomials have the weight ν_0 , that is, in which $\sum_j (\beta_j - \delta_j - \eta_j) = \nu_0$ or equivalently $\sum_j (\delta_j + \eta_j) = -\nu_0 - \lambda_r$. Thus we have

$$\sigma_{2r}(P_0 {}^t f_r(\partial)) = \sum_B a_B 2^{-r} \sum_{\substack{\delta_j, \eta_j \in \Delta_N^+ \\ \sum (\delta_j + \eta_j) = -\nu_0 - \lambda_r}} [[X_{\beta_1}, X_{-\delta_1}], X_{-\eta_1}] \cdots [[X_{\beta_r}, X_{-\delta_r}], X_{-\eta_r}] \xi_{\delta_1} \xi_{\eta_1} \cdots \xi_{\delta_r} \xi_{\eta_r}.$$

As is stated in the last paragraph of Step 1, the nonzero summand of $\sigma_{2r}(\Psi_\lambda({}^t f_r))(X_+, X_-)$ is given only by $B = B_r$ and $\delta_j = \eta_j = \gamma_j$ for all j . Thus the nonzero summand of $\sigma_{2r}(P_0 {}^t f_r(\partial))(X_+, X_-)$ also satisfies $B = B_r$ and $\delta_j = \eta_j = \gamma_j$. Such a summand occurs only if $\nu_0 = \lambda_r$, since $\sum_j (\delta_j + \eta_j) = -\nu_0 - \lambda_r$. By (15.3) $\sigma_{2r}(P_0 {}^t f_r(\partial))(X_+, X_-)$ is nonzero

and ν_0 must be equal to λ_r . Then we proved this step, since all the $\text{Ad}(L)$ -maximal weight satisfying $\deg f_\mu = r$ are equal to or higher than λ_r .

[Step 3] At last, we can prove the proposition. We have

$$\sigma_{2r}(f_r {}^t f_r(\partial) {}^t f_r(\partial))(X_+, X_-) = f_r(X_+) {}^t f_r(X_-) {}^t f_r(X_-) = 1.$$

We combine this with (15.2), and obtain the proposition. $\square$

Now we can prove Theorem 6.1. For $\mu = k_1 \lambda_1 + \cdots + k_r \lambda_r$, there exists a complex number a by Lemma 14.5 such that

$$(15.5) \quad \Psi_\lambda({}^t f_r f_r) f_\mu = \beta_{\lambda, r}(\mu) f_\mu = a b_r(\mu) b_r(\mu - (\lambda^0 + \rho_r^0) \lambda_r) f_\mu \quad (a \in \mathbb{C}),$$

and obviously we have

$$(15.6) \quad ({}^t f_r(\partial) f_r {}^t f_r(\partial) f_r) f_\mu = b_r(\mu)^2 f_\mu.$$

Here $a \neq 0$. Indeed, if $a = 0$, then $\Psi_\lambda({}^t f_r f_r) f_\mu = 0$ for all μ , and $\Psi_\lambda({}^t f_r f_r) = 0$ as an operator, since $\Psi_\lambda({}^t f_r f_r)$ commutes with the $\text{Ad}(L)$ -action and f_μ 's generate $\mathbb{C}[n^+]$ as an $\text{Ad}(L)$ -module. This contradicts the fact that the order of $\Psi_\lambda({}^t f_r)$ is equal to $2r$. Thus $a \neq 0$.

When we consider the top degree parts of $a b_r(\mu) b_r(\mu - (\lambda^0 + \rho_r^0) \lambda_r)$ and of $b_r(\mu)^2$ in (15.5) and (15.6), they come from the top order parts of $\Psi_\lambda({}^t f_r f_r)$ and of ${}^t f_r(\partial) f_r {}^t f_r(\partial) f_r$, respectively. The relation between these top order parts is described in Proposition 15.1. Thus we have $a = (-1)^r$, since the top degree parts of $b_r(\mu) b_r(\mu - (\lambda^0 + \rho_r^0) \lambda_r)$ and $b_r(\mu)^2$ coincide. We have proved Theorem 6.1.

16 Appendix—some data of commutative parabolic types

In the body of this thesis, we have not given examples depending on a type of a Lie algebra, except for examples in §13. Here we summarize data for convenience of the author such as root systems, fundamental weights, γ_i and λ_i (below Definition 4.1), ρ_i (below Definition

7.4), c (Definition 9.6) or $\mathfrak{g}_i$ (§7) for all the commutative parabolic types. In addition, we set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_N^+} \alpha \in \text{Hom}(\mathfrak{p}, \mathbb{C}),$$

which is equal to ρ_r if $(\mathfrak{g}, \mathfrak{p})$ is of regular commutative parabolic type. We use Bourbaki's rules of simple root numbering [2] in the following Dynkin diagrams and $\{\varepsilon_i\}$ denotes an orthonormal basis of an Euclidean space $\mathbb{R}^m$.

16.1 Type (A_n, r)

The root system is realized in a hyperplane in $\mathbb{R}^{n+1}$ through the origin orthogonal to $\varepsilon_1 + \cdots + \varepsilon_{n+1}$. We may assume that $r \leq [(n+1)/2]$ without loss of generality.

16.1.1 Root system

$$\begin{aligned} \Pi &= \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n\}, \\ \Lambda &= \left\{ \varpi_i = \frac{n+1-i}{n+1}(\varepsilon_1 + \cdots + \varepsilon_i) - \frac{i}{n+1}(\varepsilon_{i+1} + \cdots + \varepsilon_{n+1}) \mid 1 \leq i \leq n \right\}, \\ \Delta^+ &= \{\varepsilon_i - \varepsilon_j \mid i < j\}, \\ \Delta_N^+ &= \{\varepsilon_i - \varepsilon_j \mid i \leq r < j\}, \\ \sharp \Delta^+ &= n(n+1)/2, \\ \sharp \Delta_N^+ &= r(n+1-r). \end{aligned}$$

In particular,

$$\begin{aligned} \alpha_{i_0} &= \alpha_r = \varepsilon_r - \varepsilon_{r+1}, \\ \varpi_{i_0} &= \varpi_r = \frac{n+1-r}{n+1}(\varepsilon_1 + \cdots + \varepsilon_r) - \frac{r}{n+1}(\varepsilon_{r+1} + \cdots + \varepsilon_{n+1}). \end{aligned}$$

16.1.2 γ_i

$$\begin{aligned} \gamma_1 &= \varepsilon_r - \varepsilon_{r+1} = \alpha_r, \\ \gamma_2 &= \varepsilon_{r-1} - \varepsilon_{r+2} = \alpha_{r-1} + \alpha_r + \alpha_{r+1}, \\ &\vdots \\ \gamma_i &= \varepsilon_{r-i+1} - \varepsilon_{r+i} = \alpha_{r-i+1} + \cdots + \alpha_{r+i-1}, \\ &\vdots \\ \gamma_r &= \varepsilon_1 - \varepsilon_{2r} = \alpha_1 + \cdots + \alpha_{2r-1}. \end{aligned}$$

16.1.3 λ_i

$$\begin{aligned}\lambda_1 &= -\varepsilon_r + \varepsilon_{r+1}, \\ \lambda_2 &= -(\varepsilon_{r-1} + \varepsilon_r) + (\varepsilon_{r+1} + \varepsilon_{r+2}), \\ &\vdots \\ \lambda_i &= -(\varepsilon_{r-i+1} + \cdots + \varepsilon_r) + (\varepsilon_{r+1} + \cdots + \varepsilon_{r+i}), \\ &\vdots \\ \lambda_r &= -(\varepsilon_1 + \cdots + \varepsilon_r) + (\varepsilon_{r+1} + \cdots + \varepsilon_{2r}).\end{aligned}$$

16.1.4 c, ρ_i and ρ

$$\begin{aligned}c &= 2, \\ \rho_i &= i\varpi_r|_{\mathfrak{h}_i}, \\ \rho &= \frac{n+1}{2}\varpi_r.\end{aligned}$$

16.1.5 Dynkin diagrams

$$\begin{array}{lll}\mathfrak{g} : & \overset{1}{\bullet} \cdots \cdots \cdots \overset{r}{\circ} \cdots \cdots \cdots \overset{2r-1}{\bullet} \cdots \cdots \cdots \overset{n}{\bullet} & (A_n, r) \\ \mathfrak{g}_r : & \bullet \cdots \cdots \cdots \circ \cdots \cdots \cdots \bullet & (A_{2r-1}, r) \\ \mathfrak{g}_{r-1} : & \bullet \cdots \cdots \cdots \circ \cdots \cdots \cdots \bullet & (A_{2r-3}, r-1) \\ \vdots & \vdots & \vdots \\ \mathfrak{g}_2 : & \bullet \cdots \circ \cdots \bullet & (A_3, 2) \\ \mathfrak{g}_1 : & \circ & (A_1, 1)\end{array}$$

16.2 Type (C_n, n)

The root system is realized in $\mathbf{R}^n$.

16.2.1 Root system

$$\Pi = \left\{ \begin{array}{ll} \alpha_1 &= \varepsilon_1 - \varepsilon_2, \\ \vdots & \\ \alpha_{n-1} &= \varepsilon_{n-1} - \varepsilon_n, \\ \alpha_n &= 2\varepsilon_n \end{array} \right\},$$

$$\Lambda = \{\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \mid 1 \leq i \leq n\},$$

$$\Delta^+ = \left\{ \begin{array}{ll} \varepsilon_i - \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1} \\ \varepsilon_i + \varepsilon_j &= (\alpha_i + \cdots + \alpha_{j-1}) + 2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n \\ 2\varepsilon_i &= 2(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n \end{array} \begin{array}{l} (i < j), \\ (i < j), \\ \end{array} \right\},$$

$$\Delta_N^+ = \{\varepsilon_i + \varepsilon_j \mid i < j\} \cup \{2\varepsilon_i\},$$

$$\#\Delta^+ = n^2,$$

$$\#\Delta_N^+ = n(n+1)/2.$$

In particular,

$$\alpha_{i_0} = \alpha_n = 2\varepsilon_n,$$

$$\varpi_{i_0} = \varpi_n = \varepsilon_1 + \cdots + \varepsilon_n.$$

16.2.2 γ_i

$$\begin{aligned}\gamma_1 &= 2\varepsilon_n = \alpha_n, \\ \gamma_2 &= 2\varepsilon_{n-1} = 2\alpha_{n-1} + \alpha_n, \\ &\vdots \\ \gamma_i &= 2\varepsilon_{n-i+1} = 2(\alpha_{n-i+1} + \cdots + \alpha_{n-1}) + \alpha_n, \\ &\vdots \\ \gamma_n &= 2\varepsilon_1 = 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n.\end{aligned}$$

16.2.3 λ_i

$$\begin{aligned}\lambda_1 &= -2\varepsilon_n, \\ \lambda_2 &= -2(\varepsilon_{n-1} + \varepsilon_n), \\ &\vdots \\ \lambda_i &= -2(\varepsilon_{n-i+1} + \cdots + \varepsilon_n), \\ &\vdots \\ \lambda_n &= -2(\varepsilon_1 + \cdots + \varepsilon_n).\end{aligned}$$

16.2.4 c, ρ_i and ρ

$$c = 1,$$

$$\begin{aligned}\rho_i &= \frac{i+1}{2} \varpi_n|_{\mathfrak{h}_i}, \\ \rho &= \frac{n+1}{2} \varpi_n.\end{aligned}$$

16.2.5 Dynkin diagrams

$$\begin{aligned}\mathfrak{g} = \mathfrak{g}_r &: \overset{1}{\bullet} - \cdots - \bullet \leftarrow \overset{n}{\circ} \quad (C_n, n) \\ \mathfrak{g}_{r-1} &: \bullet - \cdots - \bullet \leftarrow \circ \quad (C_{n-1}, n-1) \\ &\vdots \\ \mathfrak{g}_2 &: \bullet \leftarrow \circ \quad (C_2, 2) \\ \mathfrak{g}_1 &: \circ \quad (A_1, 1)\end{aligned}$$

16.3 Type (D_n, n)

The root system is realized in $\mathbf{R}^n$.

16.3.1 Root system

$$\begin{aligned}\Pi &= \left\{ \begin{array}{l} \alpha_1 = \varepsilon_1 - \varepsilon_2, \\ \vdots \\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \\ \alpha_n = \varepsilon_{n-1} + \varepsilon_n \end{array} \right\}, \\ \Lambda &= \left\{ \begin{array}{l} \varpi_1 = \varepsilon_1, \\ \vdots \\ \varpi_{n-2} = \varepsilon_1 + \cdots + \varepsilon_{n-2}, \\ \varpi_{n-1} = (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2, \\ \varpi_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2 \end{array} \right\}, \\ \Delta^+ &= \left\{ \begin{array}{l} \varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1} \quad (i < j), \\ \varepsilon_i + \varepsilon_j = (\alpha_i + \cdots + \alpha_{j-1}) \\ \quad + (2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n) \quad (i < j) \end{array} \right\}, \\ \Delta_N^+ &= \{\varepsilon_i + \varepsilon_j \mid i < j\}, \\ \sharp \Delta^+ &= n(n-1), \\ \sharp \Delta_N^+ &= n(n-1)/2.\end{aligned}$$

In particular,

$$\begin{aligned}\alpha_{i_0} &= \alpha_n = \varepsilon_{n-1} + \varepsilon_n, \\ \varpi_{i_0} &= \varpi_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n).\end{aligned}$$

16.3.2 γ_i

$$\begin{aligned}\gamma_1 &= \varepsilon_{n-1} + \varepsilon_n = \alpha_n, \\ \gamma_2 &= \varepsilon_{n-3} + \varepsilon_{n-2} = \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \\ &\vdots \\ \gamma_i &= \varepsilon_{n-2i+1} + \varepsilon_{n-2i+2} = \alpha_{n-2i+1} + 2\alpha_{n-2i+2} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \\ &\vdots \\ \gamma_{n/2} &= \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \quad (n : \text{even}), \\ \gamma_{(n-1)/2} &= \varepsilon_2 + \varepsilon_3 = \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \quad (n : \text{odd}).\end{aligned}$$

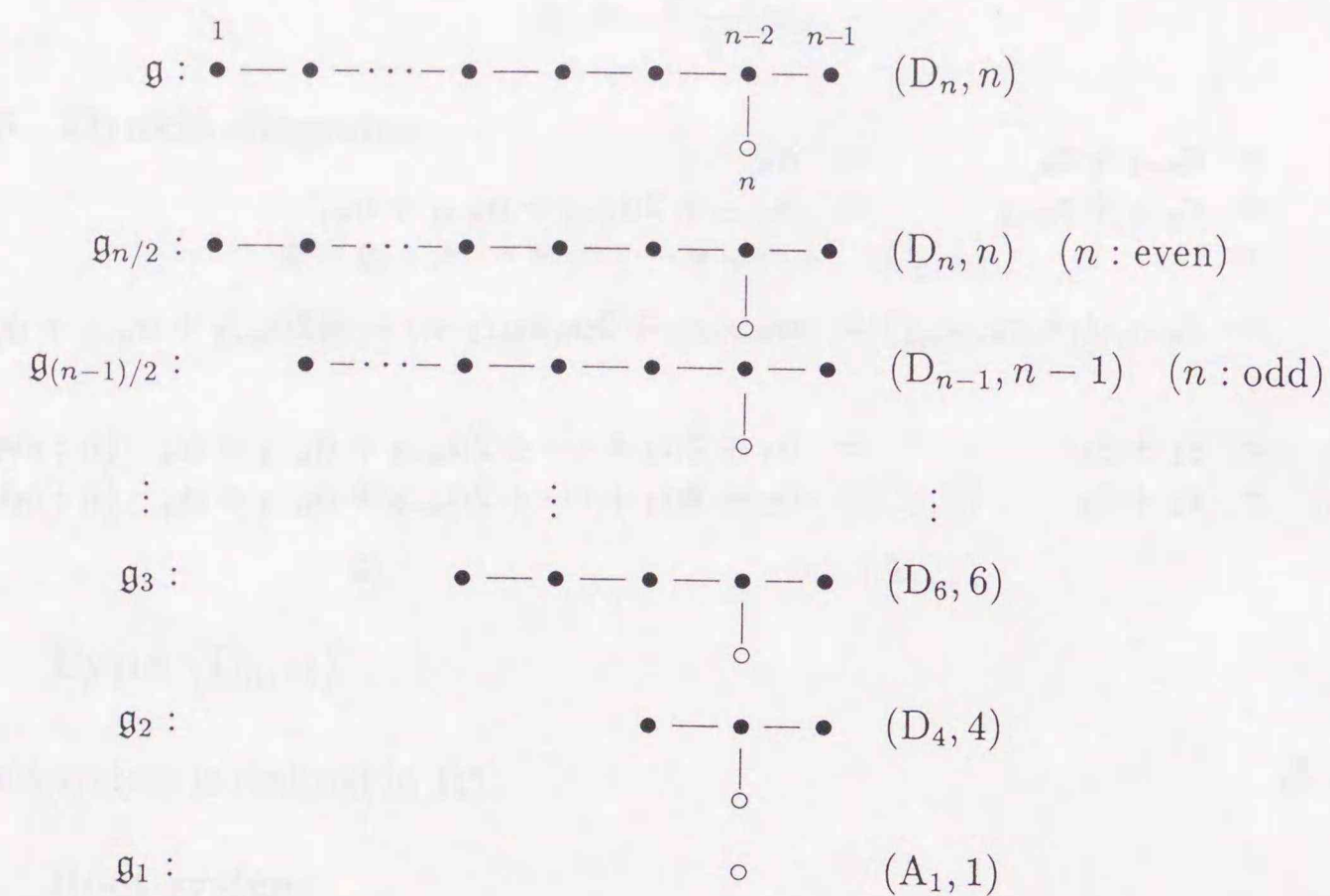
16.3.3 λ_i

$$\begin{aligned}\lambda_1 &= -(\varepsilon_{n-1} + \varepsilon_n), \\ \lambda_2 &= -(\varepsilon_{n-3} + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n), \\ &\vdots \\ \lambda_{n/2} &= -(\varepsilon_1 + \cdots + \varepsilon_n) \quad (n : \text{even}), \\ \lambda_{(n-1)/2} &= -(\varepsilon_2 + \cdots + \varepsilon_n) \quad (n : \text{odd}).\end{aligned}$$

16.3.4 c, ρ_i and ρ

$$\begin{aligned}c &= 4, \\ \rho_i &= (2i-1)\varpi_n|_{\mathfrak{h}_i}, \\ \rho &= (n-1)\varpi_n.\end{aligned}$$

16.3.5 Dynkin diagrams



16.4 Type $(D_n, 1)$

The root system is realized in $\mathbf{R}^n$.

16.4.1 Root system

Π , Λ and Δ^+ are the same as §§16.3.

$$\Delta_N^+ = \{\varepsilon_1 \pm \varepsilon_j \mid 1 < j\},$$

$$\#\Delta_N^+ = 2(n-1),$$

$$\alpha_{i_0} = \alpha_1 = \varepsilon_1 - \varepsilon_2,$$

$$\varpi_{i_0} = \varpi_1 = \varepsilon_1.$$

16.4.2 γ_i

$$\gamma_1 = \varepsilon_1 - \varepsilon_2 = \alpha_1,$$

$$\gamma_2 = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

16.4.3 λ_i

$$\lambda_1 = -\varepsilon_1 + \varepsilon_2,$$

$$\lambda_2 = -2\varepsilon_1.$$

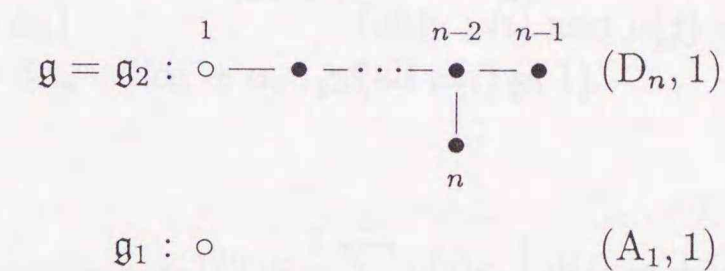
16.4.4 c, ρ_i and ρ

$$c = 2(n-2),$$

$$\rho_1 = \varpi_1|_{\mathfrak{h}_1},$$

$$\rho_2 = \rho = (n-1)\varpi_1.$$

16.4.5 Dynkin diagrams



16.5 Type $(B_n, 1)$

The root system is realized in $\mathbf{R}^n$.

16.5.1 Root system

$$\Pi = \left\{ \begin{array}{ll} \alpha_1 & = \varepsilon_1 - \varepsilon_2, \\ \vdots & \\ \alpha_{n-1} & = \varepsilon_{n-1} - \varepsilon_n, \\ \alpha_n & = \varepsilon_n \end{array} \right\},$$

$$\Lambda = \left\{ \begin{array}{ll} \varpi_1 & = \varepsilon_1, \\ \vdots & \\ \varpi_{n-1} & = \varepsilon_1 + \cdots + \varepsilon_{n-1}, \\ \varpi_n & = (\varepsilon_1 + \cdots + \varepsilon_n)/2 \end{array} \right\},$$

$$\Delta^+ = \left\{ \begin{array}{ll} \varepsilon_i - \varepsilon_j & = \alpha_i + \cdots + \alpha_{j-1} \quad (i < j), \\ \varepsilon_i + \varepsilon_j & = (\alpha_i + \cdots + \alpha_{j-1}) + 2(\alpha_j + \cdots + \alpha_n) \quad (i < j), \\ \varepsilon_i & = \alpha_i + \cdots + \alpha_n \end{array} \right\},$$

$$\Delta_N^+ = \{\varepsilon_1 \pm \varepsilon_j \mid 1 < j\}, \quad \varepsilon_1\},$$

$$\#\Delta^+ = n^2,$$

$$\#\Delta_N^+ = 2n-1.$$

In particular,

$$\alpha_{i_0} = \alpha_1 = \varepsilon_1 - \varepsilon_2,$$

$$\varpi_{i_0} = \varpi_1 = \varepsilon_1.$$

16.5.2 γ_i

$$\gamma_1 = \varepsilon_1 - \varepsilon_2 = \alpha_1,$$

$$\gamma_2 = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n).$$

16.5.3 λ_i

$$\lambda_1 = -\varepsilon_1 + \varepsilon_2,$$

$$\lambda_2 = -2\varepsilon_1.$$

16.5.4 c, ρ_i and ρ

$$c = 2n - 3,$$

$$\rho_1 = \varpi_1|_{\mathfrak{h}_1},$$

$$\rho_2 = \rho = \frac{2n-1}{2} \varpi_1.$$

16.5.5 Dynkin diagrams

$$\begin{array}{ccc} \mathfrak{g} = \mathfrak{g}_2 : \circ \overset{1}{\text{---}} \bullet \cdots \cdots \bullet \overset{n}{\text{---}} \bullet & (B_n, 1) \\ \mathfrak{g}_1 : \circ & (A_1, 1) \end{array}$$

16.6 Type $(E_6, 1)$

The root system is realized in the subspace of $\mathbf{R}^8$ spanned by the following simple roots.

16.6.1 Root system

$$\Pi = \left\{ \begin{array}{l} \alpha_1 = (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2 + (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)/2, \\ \alpha_2 = \varepsilon_1 + \varepsilon_2, \\ \alpha_3 = \varepsilon_2 - \varepsilon_1, \\ \alpha_4 = \varepsilon_3 - \varepsilon_2, \\ \alpha_5 = \varepsilon_4 - \varepsilon_3, \\ \alpha_6 = \varepsilon_5 - \varepsilon_4 \end{array} \right\},$$

$$\Lambda = \left\{ \begin{array}{l} \varpi_1 = (2/3)(\varepsilon_8 - \varepsilon_7 - \varepsilon_6), \\ \varpi_2 = (\varepsilon_1 + \cdots + \varepsilon_5)/2 + (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2, \\ \varpi_3 = (-\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_5)/2 + (5/6)(\varepsilon_8 - \varepsilon_7 - \varepsilon_6), \\ \varpi_4 = \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + (\varepsilon_8 - \varepsilon_7 - \varepsilon_6), \\ \varpi_5 = \varepsilon_4 + \varepsilon_5 + (2/3)(\varepsilon_8 - \varepsilon_7 - \varepsilon_6), \\ \varpi_6 = \varepsilon_5 + (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/3 \end{array} \right\},$$

$$\Delta^+ = \left\{ \begin{array}{ll} \varepsilon_i - \varepsilon_j = \alpha_{j+2} + \cdots + \alpha_{i+1} & (1 \leq j < i \leq 5), \\ \varepsilon_i + \varepsilon_j = (\alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{j+1}) & (1 \leq j < i \leq 5), \\ \quad + (\alpha_{j+2} + \cdots + \alpha_{i+1}) & (1 \leq j < i \leq 5), \\ (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2 + (1/2) \sum_{i=1}^5 \nu(i) \varepsilon_i & (\nu(i) = \pm 1, \prod_{i=1}^5 \nu(i) = 1) \end{array} \right\}.$$

Here the positive root in the last line is expressed as a linear combination of the simple roots as follows:

$$\left\{ \begin{array}{ll} \alpha_1 + (\alpha_3 + \cdots + \alpha_{i+1}) & (\text{only } \nu(i) = 1, 1 \leq i \leq 5), \\ (\alpha_1 + \cdots + \alpha_4) + (\alpha_{j+2} + \cdots + \alpha_5) & \\ \quad + (\alpha_{i+2} + \cdots + \alpha_6) & (\text{only } \nu(i) \text{ and } \nu(j) = -1, 1 \leq j < i \leq 5), \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 & (\text{all } \nu(i) = 1). \end{array} \right.$$

$$\Delta_N^+ = \left\{ (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2 + \frac{1}{2} \sum_{i=1}^5 \nu(i) \varepsilon_i \mid \nu(i) = \pm 1, \prod_{i=1}^5 \nu(i) = 1 \right\},$$

$$\#\Delta^+ = 36,$$

$$\#\Delta_N^+ = 16.$$

In particular,

$$\alpha_{i_0} = \alpha_1 = (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2 + (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5),$$

$$\varpi_{i_0} = \varpi_1 = (2/3)(\varepsilon_8 - \varepsilon_7 - \varepsilon_6).$$

16.6.2 γ_i

$$\gamma_1 = (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2 + (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)/2 = \alpha_1,$$

$$\gamma_2 = (\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2 + (-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5)/2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5.$$

16.6.3 λ_i

$$\lambda_1 = -(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)/2 - (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)/2,$$

$$\lambda_2 = -(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5.$$

16.6.4 ρ_i and ρ

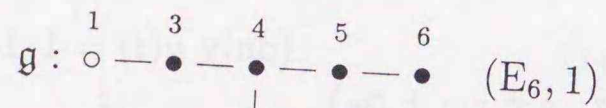
$$c = 6,$$

$$\rho_1 = \varpi_1|_{\mathfrak{h}_1},$$

$$\rho_2 = 4\varpi_1|_{h_2},$$

$$\rho = 6\varpi_1.$$

16.6.5 Linkin diagrams



16.7 $\text{pe } (E_7, 7)$

The root system is realized in the hyperplane in $\mathbf{R}^8$ through the origin orthogonal to $\varepsilon_7 + \varepsilon_8$.

16.7.1 ot system

$$\left. \begin{aligned} \alpha_1 &= (\varepsilon_8 - \varepsilon_7)/2 + (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6), \\ \alpha_2 &= \varepsilon_1 + \varepsilon_2, \\ \alpha_3 &= \varepsilon_2 - \varepsilon_1, \\ \alpha_4 &= \varepsilon_3 - \varepsilon_2, \\ \alpha_5 &= \varepsilon_4 - \varepsilon_3, \\ \alpha_6 &= \varepsilon_5 - \varepsilon_4, \\ \alpha_7 &= \varepsilon_6 - \varepsilon_5 \end{aligned} \right\},$$

$$\Delta^+ = \left\{ \begin{array}{ll} \varepsilon_i - \varepsilon_j = \alpha_{j+2} + \cdots + \alpha_{i+1} & (1 \leq j < i \leq 6), \\ \varepsilon_i + \varepsilon_j = (\alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{j+1}) & \\ \quad + (\alpha_{j+2} + \cdots + \alpha_{i+1}) & (1 \leq j < i \leq 6), \\ \varepsilon_8 - \varepsilon_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 & \\ \quad + 3\alpha_5 + 2\alpha_6 + \alpha_7, & \\ (\varepsilon_8 - \varepsilon_7)/2 + (1/2) \sum_{i=1}^6 \nu(i) \varepsilon_i & (\nu(i) = \pm 1, \prod_{i=1}^6 \nu(i) = -1) \end{array} \right\}.$$

Here the positive root in the last line is expressed as a linear combination of the simple roots as follows:

$$\left\{ \begin{array}{ll} \alpha_1 + (\alpha_3 + \cdots + \alpha_{i+1}) & (\text{only } \nu(i) = 1, 1 \leq i \leq 6), \\ (\alpha_1 + \cdots + \alpha_4) + (\alpha_3 + \cdots + \alpha_{k+1}) \\ \quad + (\alpha_4 + \cdots + \alpha_{j+1}) + (\alpha_5 + \cdots + \alpha_{i+1}) & (\text{only } \nu(i), \nu(j) \text{ and } \nu(k) = 1, \\ & 1 \leq k < j < i \leq 6), \\ (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) \\ \quad + (\alpha_{i+2} + \cdots + \alpha_7) & (\text{only } \nu(i) = -1, 1 \leq i \leq 6). \end{array} \right.$$

$$\Delta_N^+ = \left\{ \begin{array}{l} \varepsilon_6 \pm \varepsilon_j \quad (1 \leq j \leq 5), \\ \varepsilon_8 - \varepsilon_7, \\ (\varepsilon_8 - \varepsilon_7)/2 + \frac{1}{2}\varepsilon_6 + \frac{1}{2}\sum_{i=1}^5 \nu(i)\varepsilon_i \quad (\nu(i) = \pm 1, \prod_{i=1}^5 \nu(i) = -1) \end{array} \right\},$$

$$\#\Delta^+ = 63,$$

$$\sharp \Delta_N^+ = 27.$$

In particular,

$$\alpha_{i_0} = \alpha_7 = \varepsilon_6 - \varepsilon_5,$$

$$\varpi_{i_0} = \varpi_7 = \varepsilon_6 + (\varepsilon_8 - \varepsilon_7)/2.$$

16.7.2 γ_i

$$\begin{aligned}\gamma_1 &= \varepsilon_6 - \varepsilon_5 = \alpha_7, \\ \gamma_2 &= \varepsilon_6 + \varepsilon_5 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\ \gamma_3 &= \varepsilon_8 - \varepsilon_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7.\end{aligned}$$

16.7.3 λ_i

$$\lambda_1 = -\varepsilon_6 + \varepsilon_5,$$

$$\lambda_2 = -2\varepsilon_6,$$

$$\lambda_3 = -2\varepsilon_6 + \varepsilon_7 - \varepsilon_8.$$

16.7.4 c, ρ_i and ρ

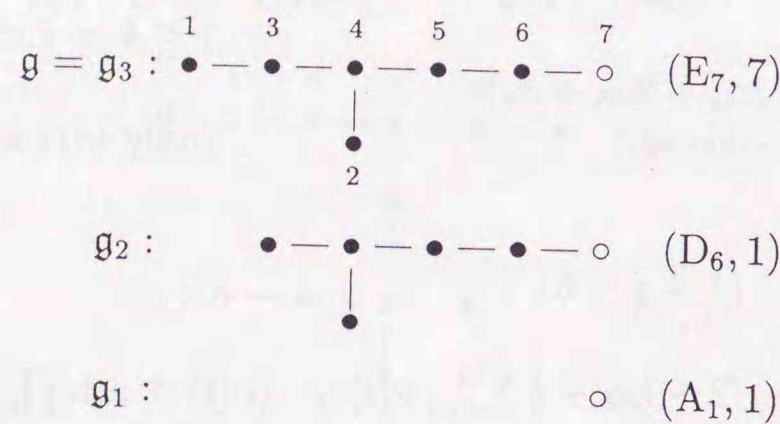
$$c = 8,$$

$$\rho_1 = \varpi_1|_{\mathfrak{h}_1},$$

$$\rho_2 = 5\varpi_1|_{h_2}$$

$$\rho_3 = \rho = 9\varpi_1.$$

16.7.5 Dynkin diagrams



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