HOKKAIDO UNIVERSITY

Title	Natural Deductions for Substructural Logics
Author(s)	Ueno, Takeshi
Degree Grantor	北海道大学
Degree Name	博士(理学)
Dissertation Number	甲第4977号
Issue Date	2000-03-24
DOI	https://doi.org/10.11501/3168541
Doc URL	https://hdl.handle.net/2115/51635
Type	doctoral thesis
File Information	000000353734.pdf

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Natural Deductions for Substructural Logics

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March 2000

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Natural Deductions for Substructural Logics

PhD. thesis of

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24 March 2000

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Contents

Introduction 1

I Weakening and Contraction Rules for Natural Deduction 3

1 Introduction 3

2 Language L 3

3 A Natural Deduction System D 4

4 Sequent Calculus S 6

5 Provability 8

6 Characterization of Weakening Rule and Contraction Rule 9

II Exchange Rule for Natural Deduction 21

7 Introduction 21

8 Labeling and Ordering 21

9 Language L and its Formulas 23

10 Natural Deduction System NFL: First Formulation 24

11 Second Formulation of NFL 31

12 Sequent Calculus FL 36

13 Equivalence of FL and NFL, and How to Characterize Exchange Rules for NFL 38

References 43

Introduction

The purpose of this paper is to formalize Gentzen style natural deduction system for substructural logics, and to characterize three structural inference rules for Gentzen style natural deduction. These rules consist of weakening rule, contraction rule, and exchange rule. In substructural logics, structural inference rules are partially restricted or totally rejected. In G. Gentzen[1], structural inference rules are defined as sequent calculus inference rules which do not involve any logical constant.

The importance of substructural logics has been increasing in such field as computer science and the lattice theory which is related to quantum logics, where distributive laws and commutative laws are restricted. Substructural logics are applied to the study of logical structures in these disciplines.

So far, substructural logics have been mainly investigated in sequent calculus, because structural inference rules are introduced and characterized as the inference rules in sequent calculus. Since natural deduction systems have been introduced in G.Gentzen[1], no structural inference rules have been explicitly characterized for this system. The function of these rules is embedded in the other inference rules in natural deduction. As a result, there have been few attempts to study natural deduction for substructural logics. In a proof of natural deduction, the mutual relations of the assumptions and the relations from the assumptions to the conclusion are indicated as a tree structure. An inference rule of natural deduction is defined as an operation on the whole proof. On the other hand, an inference rule of sequent calculus is defined as an operation on a certain sequent. Between natural deduction system for intuitionistic logic, called *NJ*, and typed λ-calculus, there is a correspondence called Curry-Howard isomorphism. Due to the existence of this isomorphism, *NJ* is investigated for the foundation of constructive programming and functional programming.

Because of our result on the natural deduction systems for substructural logics, we may expect the possibility of restricting the Curry-Howard isomorphism to the correspondence between substructural logics and a "substructurized" λ-calculus. Such a restricted version of the isomorphism will provide us finer analysis of the computational process than those provided through ordinary λ-calculus.

This paper is divided into two parts.

In the first part, we present, for a system of G. Gentzen style natural deduction, characterizations of two structural rules, namely, weakening and contraction, in terms of the numbers of assumptions discharged at an application of introduction rule for logical implication. It is shown that the number is zero for the weakening rule, while the number is reduced to exactly two for the contraction rule.

In the second part, we introduce another system of G.Gentzen style natural deduction, denoted by **NFL** (Natural full Lambek), where a particular sort of labels are assigned to the formulas occurring in a tree-like proof in such a way that information concerning the location of each formula in the proof is recovered from the label attached to it. In particular, the path from the assumptions to the conclusion is recovered from the labels attached to them. The inference rules of **NFL** are formulated as an operation on proofs consisting of labeled formulas. **NFL** is equivalent to the sequent calculus **FL** (full Lambek), the basic sequent

calculus for substructural logics. Our way of labeling suggests a certain ordering on labels which enables us to define exchange rule in **NFL**, in effect, as the flipping of the two tree-like proofs situated above the two assumptions of the elimination rule for implication.

Acknowledgment

I would like to express my gratitude to Professor Toru Tsujishita, Professor Koji Nakatogawa, and Mr. Osamu Watari for their advice, encouragement, discussions, and constructive suggestions.

Part I

Weakening and Contraction Rules for Natural Deduction

1 Introduction

The purpose of this part is to characterize two structural inference rules, namely, weakening (Theorem 1) and contraction (Theorem 2), for a system D (defined below in section 3) of Gentzen-style natural deduction. Our characterizations of these two rules are given in terms of the number of assumptions discharged at an application of the introduction rule for implication. For the weakening rule, the number is zero, while it is reduced to exactly two for the contraction rule.

We start with introducing a propositional language equipped with the logical connectives, implication $\supset$, conjunction $*$, disjunction $+$, and negation $\neg$. Then, the system D is defined as a system without weakening and contraction rules. (Oide[3] introduced sequent calculus systems where the number of formulas allowed to occur on each sides of a sequent is restricted to zero or one. This system provided us of motivations and stimulations leading to the definition of D.) In this base system D, the elimination rule for implication is the ordinary modus ponens. For the introduction rule of implication , it is customary not to put any restriction on the number of discharged assumptions. In our D, however, the number of assumptions discharged at each application of the introduction rule $\supset I^1$ for implication is restricted to exactly one.

It will turn out that if an introduction rule $\supset I^0$ for implication, where the number of discharged assumptions is restricted to zero, is added to D as a new rule, it works as the weakening rule. (See *Theorem* 1 below) If the introduction rules $\supset I^n (n \geq 2)$, where the number of discharged assumptions is any positive integer strictly greater that or equal to two, are added to D, they work as the contraction rule.(See *Theorem* 2 below) For the contraction rule, the number of discharged assumptions can be reduced to two (*Lemma* 1).

Our primary concern in this paper is on neither substractural logics nor linear logics. It is mostly concerned with seeing what sort of forms the structural rules such as weakening and contraction take in a natural deduction system which follows closely as possible the original formulation of Gentzen [1]. The scope of the present part is restricted to propositional connectives.

2 Language L

The propositional connectives of our basic language L are the implication symbol $\supset$, the negation symbol $\neg$, the conjunction symbol $*$, and the disjunction symbol $+$. We also use the constant symbols $\mathbf{t}$ to denote the unit element for the conjunction, $\mathbf{f}$ to denote the unit element for the disjunction. The formulas of L are constructed by combining the constant symbols $\mathbf{t}$, $\mathbf{f}$ and the propositional constant symbols, $A, B, C, \cdots$ together with the connectives $\supset$, $\neg$, $*$, $+$, ((left parenthesis),) (right parenthesis). Some lower case

Greek alphabets ϕ, ψ , $\theta, \cdots$ denote a formula. We use capital letters such as Γ, Δ, Ψ, Φ, $\cdots$ to represent a multi-sets of formulas. In what follows we usually omit "multi-" when it is clear from the context.

The sequent of L have the following form

$$\Gamma \to \Delta \ .$$

The left hand side of a sequent may be empty. The right hand side of a sequent is either empty or consists of a single formula. To specify the element of Γ and Δ we write

$$\gamma_0, \cdots, \gamma_{m-1} \to \delta.$$

When both sides of a sequent are empty, we write

$$\to$$

and call it the empty sequent.

3 A Natural Deduction System D

We now introduce a formal system D of natural deductions which is obtained by restricting and modifying Gentzen's natural deduction system NK. Our system D has the following inference rules. The non-negative integer n used as the superscript of I, E indicates that the number of assumptions cancelled at the application of an inference rule is restricted to exactly n.

- The introduction of $\supset$ with the number of cancelled assumptions being restricted to exactly one

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \supset B} \supset I^1$$

- The elimination of $\supset$

$$\frac{A \quad A \supset B}{B} \supset E$$

- The introduction of $\neg$ with the number of cancelled assumptions being restricted to one

$$\frac{\begin{array}{c}[A] \\ \vdots \\ \mathbf{f}\end{array}}{\neg A} \neg I^1$$

- The elimination of $\neg$

$$\frac{A \quad \neg A}{\mathbf{f}} \neg E$$

- The introduction of $*$

$$\frac{A \quad B}{A * B} * I$$

- The elimination of $*$ with the number of the cancelled assumptions being restricted to one for A, and one for B respectively

$$\frac{A * B \qquad \begin{matrix}[A] \quad [B] \\ \vdots \\ C\end{matrix}}{C} * E^{1,1}$$

The 1 on the left side of the superscript of E indicates that the A used as an assumption is cancelled just once at the application of $*E^{1,1}$. The 1 on the right side of the superscript of E indicates the similar restriction for the assumption B.

- The first introduction of $+$ with the number of cancelled assumptions being restricted to one

$$\frac{\begin{matrix}[\neg A] \\ \vdots \\ B\end{matrix}}{A + B} + I_1^1$$

- The second introduction of $+$ with the the number of cancelled assumptions being restricted to one

$$\frac{\begin{matrix}[\neg B] \\ \vdots \\ A\end{matrix}}{A + B} + I_2^1$$

The above two rules are modifications of the introduction rules of the multiplicative disjunction to suite to the present circumstances . Instead of trying to give the meanings to these rules within an already available framework, their meanings are to be sought in the actual use of these rules themselves.

- The first elimination of $+$

$$\frac{\neg A \quad A + B}{B} + E_1$$

- The second elimination of $+$

$$\frac{\neg B \quad A + B}{A} + E_2$$

The above two rules are adopted from the elimination rules of the multiplicative disjunction to suite to the restrictions under which we have been working. They may be regarded as expressing a sense of logical incompatibility.

- The introduction of $\supset$ for the propositional constant $\mathbf{t}$ where the number of cancelled assumptions is restricted to zero

$$\frac{A}{\mathbf{t} \supset A} \supset I^0(\mathbf{t})$$

The superscript 0 of I indicates that the number of cancelled assumptions at the application of $\supset I^0$ is restricted to zero.

- The symbol $\mathbf{t}$, regarded as a formula, is the axiom of D.

There are other inference rules not belonging to D.

- The introduction of $\supset$ with the number of cancelled assumptions being restricted to zero

$$\frac{B}{A \supset B} \supset I^0$$

The superscript 0 of I indicates that the number of cancelled assumptions at the application of $\supset I^0$ is restricted to zero.

- The introduction of $\supset$ with the number of cancelled assumptions being restricted to $n (n \geq 2)$

$$\frac{\overbrace{[A] \cdots [A]}^{n}}{\begin{array}{c}\vdots \\ B\end{array}} \supset I^n$$

The superscript n of I indicates that the number of cancelled assumptions at the application of $\supset I^n$ is restricted to two or more than two.

4 Sequent Calculus S

Next, we introduce the sequent calculus S as follows. We say that $(\Gamma_1 \to X_1, \Gamma_2 \to X_2, \cdots, \Gamma_n \to X_n \,/\, \Gamma \to X)$ is an instance of a certain inference rule if it has the form indicated by the corresponding figure. If $(\Gamma_1 \to X_1, \Gamma_2 \to X_2, \cdots, \Gamma_n \to X_n \,/\, \Gamma \to X)$ is an instance of an inference rule α, we call $\Gamma_i \to X_i$ the i-th upper sequent of α, and $\Gamma \to X$ the lower sequent of α.

- The sequent $A \to A$ is called an axiom.
- The sequent $\to \mathbf{t}$ is called the unit axiom of $*$.
- The sequent $\mathbf{f} \to$ is called the unit axiom of $+$.
- The left rule of $\supset$

$$\frac{\Gamma \to A \quad B, \Delta \to C}{\Gamma, A \supset B, \Delta, \to C} \supset left$$

- The right rule of $\supset$

$$\frac{A, \Gamma \to B}{\Gamma \to A \supset B} \supset right$$

- The left rule of $\neg$

$$\frac{\Gamma \to A}{\neg A, \Gamma \to} \neg left$$

- The right rule of $\neg$

$$\frac{A, \Gamma \to}{\Gamma \to \neg A} \neg right$$

The right hand side upper sequent of $\neg right$ have to be empty.

- The left rule of $*$

$$\frac{A, B, \Gamma \to C}{A * B, \Gamma \to C} * left$$

- The right rule of $*$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A * B} * right$$

- The first left rule of $+$

$$\frac{A, \Gamma \to \quad B, \Delta \to C}{A + B, \Gamma, \Delta \to C} + left_1$$

The right hand side of first upper sequent of $+left_1$ has to be empty.

- The second left rule of $+$

$$\frac{A, \Gamma \to C \quad B, \Delta \to}{A + B, \Gamma, \Delta \to C} + left_2$$

The right hand side of second upper sequent of $+left_2$ has to be empty.

- The first right rule of $+$

This and the next rules are adapted from the introduction rule of multiplicative disjunction. Our main concern here is neither the subformula property nor the nomarization of proof procedures. We regard these rules as expressing a certain meaning of logical compatibility in its own forms.

$$\frac{\neg A, \Gamma \to B}{\Gamma \to A + B} + right_1$$

- The second right rule of $+$

$$\frac{\neg B, \Gamma \to A}{\Gamma \to A + B} + right_2$$

- The left rule of $\mathbf{t}$

$$\frac{\Gamma \to A}{\mathbf{t}, \Gamma \to A}\mathbf{t}left$$

- The right rule of $\mathbf{f}$

$$\frac{\Gamma \to}{\Gamma \to \mathbf{f}}\mathbf{f}right$$

The right hand side of upper sequent of $\mathbf{f}right$ have to be empty.

- The left exchange rule

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C}\text{ex.}left$$

- The cut rule

$$\frac{\Gamma \to A \quad A, \Delta \to C}{\Gamma, \Delta \to C}\text{cut}$$

There are other structural inference rules not belong to S.

- The left contraction rule

$$\frac{A, A, \Gamma \to C}{A, \Gamma \to C}\text{c. }left$$

- The left weakening rule

$$\frac{\Gamma \to C}{A, \Gamma \to C}\text{w. }left$$

5 Provability

Our notations about provability in D and S (and in systems resulting from them by certain enlargements) are fairly standard. Thorough definitions will be found in Gentzen [1], Prawitz [6], and Troelstra [7]. A deduction Σ in D is either a single formula or a ("tree-like") figure constructed by applying inference rules of D finitely many times. ("in D" is often deleted if it is clear from the context.) When ϕ is the formula located at the bottom ("root") of a deduction Σ , we say that Σ is a deduction of ϕ in D. We distinguish proofs from deductions. A proof of ϕ in D is a deduction where all the assumptions of Σ are cancelled. ϕ is provable in D if we can construct a proof of ϕ in D. When Σ is a deduction of ϕ in D, and Γ is the set of uncancelled assumptions in Σ , we say Σ is a deduction of ϕ from Γ in D. In this case, following the common practice, we often say that Σ is a "proof" of ϕ from Γ in D, even if some assumptions Γ of ϕ are remaining uncancelled in the deduction Σ . If the context clearly tells which Σ is meant, we use the notation, $\Gamma \vdash \phi$ in D. Since the formula $\mathbf{t}$ is an axiom of D, the formula $\mathbf{t}$ appearing in Σ is regarded as

being cancelled. Thus, one might feel inclined to think that $\mathbf{t}$ is not listed in the assumptions Γ of Σ. But we explicitly display $\mathbf{t}$ among the assumptions Γ whenever $\mathbf{t}$ appears in Σ.

A formula ϕ is provable from Γ in D if we can write a deduction Σ of ϕ from Γ in D.

Provability for a sequent is also defined in a usual way.

A proof Π in S is a list of sequents defined inductively as follows.

(1) A sequent axioms $A \to A$, $\to \mathbf{t}$, $\mathbf{f} \to$ are a proof in S.

(2) If Π is a proof in S, then the figure obtained from Π by applying an inference rule of S to the last sequent of Π is also a proof in S.

(3) The proofs in S are only those obtained by applying of (1) and (2) finitely many times.

Let α be an inference rule or axiom. Let X be a formal system. $X \cup \{\alpha\}$ means a new enlarged system resulting from the addition of an inference rule or axiom to a system X. For example, $D \cup \{\supset I^0\}$ is a system obtained from D by adding a new inference rule $\supset I^0$. The notions and phrases about provability carry over to those enlarged systems.

6 Characterization of Weakening Rule and Contraction Rule

In Gentzen [1], structural inference rules are formulated in classical and intuitionistic sequent calculuses called LK and LJ, respectively . In the corresponding Gentzen-style systems of natural deductions, namely NK and NJ in Gentzen [1], there is no structural inference rules. The powers of the structural inference rules in sequent calculuses are distributed unevenly over the logical inference rules of natural deduction systems, and can not easily be detected. Our strategy here is then, first, to introduce a system S of sequent calculus which is small enough not to include any structural inference rules and any effect resulting from their presence, but, large enough to capture basic characteristics of logical connectives . (The idea of removing and restricting structural inference rules in sequent calculuses is found in Oide [3].) Second, we carefully construct a natural deduction system, here called D, in such a way that the provability in D is the same as S, so that D is free of the effect of structural inference rules. Third, the effect of structural rules such as weakening and contraction will be captured in the corresponding axioms, and then the equivalence of the systems enlarged by these axioms are proved. In this way, the effects stemming from structural rules in sequent calculuses are isolated and added to natural deduction systems in such a way that these two systems posses the equal proving capabilities. Equivalence proof of S and D is obtained by adapting to our situation, the standard equivalence proof of LK and NK , which is carried out in detail in Gentzen [1]. Since the lack of structural inference rules require some care, we present the inductive proof of the equivalence of S and D.

Lemma 1

Let ϕ be a formula of the language L. Let Γ be a set of formulas of L. Then, ϕ is provable from Γ in D if and only if the sequent $\Gamma \to \phi$ is provable in S.

Proof First, we assume that ϕ is provable from Γ in D. Then, we can construct a deduction of ϕ from Γ in D

Let A, B *and* C be formulas of L.

Let Σ be a deduction of ϕ from Γ in D.

We use induction on the number $\sharp\Sigma$ of the application of inference rules in Σ .

If the number of the application of inference rules in Σ is zero, then Σ consisted of only one formula ϕ. So we make a proof in S as follows .

$$\phi \to \phi$$

We assume the result for the case where $\sharp\Sigma$ is less than n, and prove it for n.

Case 1 The bottom inference rule in Σ is $\supset I^1$ and the form of ϕ is $A \supset B$.

By induction hypothesis, the sequent $A, \Gamma \to B$ can be proved in S. So we deduce the sequent $\Gamma \to A \supset B$ by $\supset$ *right* rule.

Case 2 The bottom inference rule in Σ is $\supset E$ and ϕ is B.

By induction hypothesis, the sequent $\Gamma_1 \to A$ and $\Gamma_2 \to A \supset B$ can be proved in S $(\Gamma_1 \cup \Gamma_2 = \Gamma)$.

We deduce the sequent $A \supset B, A \to B$ from $A \to A$ and $B \to B$ by $\supset left$ rule. Then we deduce $\Gamma_2, A \to B$ from

$A \supset B, A \to B$ and $\Gamma_2 \to A \supset B$ by *cut* rule. So we deduce $\Gamma_1, \Gamma_2 \to B$ from $\Gamma_1 \to A$ and $\Gamma_2, A \to B$ by *cut* rule.

Case 3 The cases where the bottom inference rules in Σ are $\neg I^1$, $\neg E$, $*I$ are omitted.

Case 4 The bottom inference rule in Σ is $*E^{11}$ and ϕ is C.

By induction hypothesis, the sequent $\Gamma_1 \to A * B$ and $A, B, \Gamma_2 \to C$ can be proved in S $(\Gamma_1 \cup \Gamma_2 = \Gamma)$.

We deduce the sequent $A * B, \Gamma_2 \to C$ from $A, B, \Gamma_2 \to C$ by $*left$ rule. Then we deduce $\Gamma_1, \Gamma_2 \to C$ from $\Gamma_1 \to A * B$ and $A * B, \Gamma_2 \to C$ by *cut* rule.

Case 5 The bottom inference rule in Σ is $+I_1^1$ and ϕ is C.

By induction hypothesis, the sequent

$\neg A, \Gamma \to B$ can be proved in S .

Then we deduce the sequent $\Gamma \to A + B$ from $\neg A, \Gamma \to B$ by $+right_1$ rule.

Case 6 The cases where the bottom inference rules in Σ are $+I_2^1, +E_1, +E_2$ are omitted.

Case 7 The bottom inference rule in Σ is $\supset I^0(\mathbf{t})$ and ϕ is $\mathbf{t} \supset A$.

By induction hypothesis, the sequent $\Gamma \to A$ can be proved in S .

We deduce the sequent $\mathbf{t}, \Gamma \to A$ by $\mathbf{t}left$ rule.

Then we deduce the sequent $\Gamma \to \mathbf{t} \supset A$ from $\mathbf{t}, \Gamma \to A$ by $\supset right$ rule.

Next we assume that we can prove a sequent $\Gamma \to \phi$ in S.

$\Gamma \to$ in S is defined to correspond to $\Gamma \vdash \mathbf{f}$ in D.

Let Π be proof of the sequent $\Gamma \to \phi$ in S.

We use induction on the number of the applications of inference rules in S. This number is denoted by the notation $\sharp\Pi$.

If $\sharp\Pi$ is zero, then Π consists of one of the axioms, $A \to A$, $\to \mathbf{t}$, $\mathbf{f} \to$. These proofs in S correspond to deductions or proof; A, $\mathbf{t}$, $\mathbf{f}$ in D respectively.

We assume the result for the cases where $\sharp\Pi$ is less than n , and prove it for n.

Case 1 the bottom inference rule in Π is $\supset$ *right* rule and the form of $\Gamma \to \phi$ is $\Gamma \to A \supset B$.

By induction hypothesis, the formula B is provable from $\Gamma \cup \{A\}$ in D. So we deduce the formula $A \supset B$ from B by applying the inference rule $\supset I^1$. Then $A \supset B$ is provable from Γ in D.

Case 2 The bottom inference rule in Π is $\supset left$ rule and the form of $\Gamma \to \phi$ is $\Gamma_1, A \supset B, \Gamma_2 \to C$.

By induction hypothesis, the formula A is provable from Γ_1 in D and the formula C is provable from $\Gamma_2 \cup \{B\}$ in D. So we deduce B from A and $A \supset B$ by $\supset E$ rule. Then C is provable from $\Gamma_1 \cup \Gamma_2 \cup \{A \supset B\}$ in D.

Case 3 The cases where the bottom inference rules in Π are $\neg right$, $\neg left$, $*right$, $*left$, $+right_1$, $+right_2$ are omitted.

Case 4 The bottom inference rule in Π is $+left_1$ rule and the form of $\Gamma \to \phi$ is $A+B, \Gamma_1, \Gamma_2 \to C$. By induction hypothesis, the formula $\mathbf{f}$ is provable from $\Gamma_1 \cup \{A\}$ in D and the formula C is provable from $\Gamma_2 \cup \{B\}$ in D. We deduce $\neg A$ from $\mathbf{f}$ by applying $\neg I^1$ rule. So we deduce B from $\neg A$ and $A+B$ by applying $+E_1$ rule. Then C is provable from $\Gamma_1 \cup \Gamma_2 \cup \{A+B\}$ in D.

Case 5 The bottom inference rule in Π is $1left$ rule and the form of $\Gamma \to \phi$ is $1, \Gamma_1 \to A$.

By induction hypothesis, the formula A is provable from Γ_1 in D. We deduce $1 \supset A$ from A by applying $\supset I^0(\mathbf{t})$ rule. So we deduce A from $\mathbf{t}$ and $\mathbf{t} \supset A$ by applying $\supset E$ rule. Then A is provable from $\Gamma_1 \cup \{\mathbf{t}\}$ in D.

Case 6 The cases where the bottom inference rules in Π are $+left_2$, $\mathbf{f}right$, ex.$left$, *cut* are omitted.

q.e.d.

To prove the following two theorems, we do not have to introduce notions such as labeling for the formulas appearing in a proof and the anti-lexicographic ordering on the labels . However, for an inductive argument to provide a concrete guide in re-writing a given proof to obtain another proof possessing an intended properties, we need labels and anti-lexicographic ordering. (Our definitions of these notions are not given for the general cases, but, limited to the cases relevant to our proofs.)

Let R be the set of all finite sequences consisting of zero's and one's. This R is a countably infinite set and is called the set of labels. If two sequences s and t in R are distinct form each other, there must be the first place, namely the i-th terms s_i and t_i, where the two sequences take different values for the first time, so that $s_i \neq t_i$. If $s_i < t_i$ (in the order of the usual magnitude of natural numbers), we put $s >_R t$. If $s_i > t_i$, we set $s <_R t$. Thus, we put the following definition of the anti-lexicographic ordering $>_R$ on R. For any two mutually distinct labels s and t, $s >_R t$ if and only if there is some i in $\mathbf{N}$ (the set of natural numbers including zero) such that $s_i < t_i$ and that $s_j = t_j$ for any $j < i$, where $<$ is the usual order of magnitude among the natural numbers. It is not difficult to see that this $<_R$ is a well ordering on the countable set R. We call $<_R$ the anti-lexicographic ordering on R.

Now, let Σ be any deduction of ϕ from Γ in D, and let $F(\Sigma)$ be the set of all formulas in Σ. We construct a labeling function $lb()$ from $F(\Sigma)$ to R as follows. First the conclusions ϕ (the "root") of the deduction Σ is assigned to $\langle 0 \rangle$, the sequence of length one, consisting of just one zero. Next, suppose ϕ is the conclusion of (the application of) a certain inference rule, say IR, which has one assumption ψ. Then, ψ is assigned to $\langle 0,0 \rangle$. In case IR has two assumptions ϕ_1 on the left and ϕ_2 on the right, then ψ_1 is assigned to $\langle 0,0 \rangle$, while ψ_2 is assigned to $\langle 0,1 \rangle$. In general, if a formula θ in Σ is assigned to s, i.e., $lb(\theta) = s$, then the assumptions χ of the inference rule IR having θ as its conclusion is assigned to $\hat{s}\langle 0 \rangle$, provide that IR has one assumption. In case IR has two assumptions χ_1 on the left and χ_2 on the right, then $lb(\chi_1) = \hat{s}\langle 0 \rangle$, and $lb(\chi_2) = \hat{s}\langle 1 \rangle$. The symbol $\hat{}$ indicates the concatenation of two sequences. The set of all labels assigned to the formulas in Σ is ordered by $<_R$ as a subset of R. In a similar way, we can define labeling function from the sequences in a given proof of S to$R$.

Theorem 1 (Characterization of weakening)
Let ϕ, X and Y be formulas of L. Let Γ be a set of formulas of L.
Then, the following four statements are logically equivalent.

(1) We can prove ϕ from Γ in $D \cup \{\supset I^0\}$.

(2) We can prove ϕ from Γ in $D \cup \{$axiom $X \supset (Y \supset X)\}$.

(3) We can prove a sequent $\Gamma \to \phi$ in $S \cup \{$sequent axiom $\to X \supset (Y \supset X)\}$.

(4) We can prove a sequent $\Gamma \to \phi$ in $S \cup \{$the left weakening rule $\}$.

Proof

(1) $\Rightarrow$ (2)

Let Σ be a deduction of ϕ in $D \cup \{\supset I^0\}$.
We use induction on the number of the applications of $\supset I^0$ in Σ.
If Σ contains no application of $\supset I^0$, then we are done.
We assume the result for the case where the number of the
applications of $\supset I^0$ in Σ is less than n, and we prove it for n.
The set P of all formulas appearing in Σ are ordered by the anti-lexicographic ordering $<_R$, which we defined earlier.

Let Q be the set of formulas which is the consequence of application of $\supset I^0$ in Σ. Then, Q , as a subset of P, is ordered by $<_R$.

We denote the elements of Q by,

$$q_1 <_R q_2 <_R \cdots <_R q_n.$$

Now, we consider a deduction Ξ of q_1 inside Σ , which has q_1 as its last formula. By the leastness of q_1 , Ξ contains no application of $\supset I^0$ except at the very bottom. Then Ξ can be re-written to Ξ' in the following way, so that Ξ' does not contain any application of $\supset I^0$ at all.

$$\dfrac{\begin{array}{c}\Xi\\ B\end{array}}{A \supset B} \supset I^0 \quad q_1 = A \supset B$$

The above deduction Ξ can be re-written as follows.

$$\dfrac{\begin{array}{cc}\Xi & \\ B & B \supset (A \supset B)\end{array}}{A \supset B} \supset E$$

Let Σ' the deduction resulting from Σ by replacing Ξ with Ξ'.

We then notice that the number of applications of $\supset I^0$ in Σ' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the proof Σ_0 of ϕ having no application of $\supset I^0$.

(2) $\Rightarrow$ (1)

Let Σ be a deduction of ϕ in $D \cup \{axiom\ X \supset (Y \supset X)\}$.

We use induction on the number of the axioms of $X \supset (Y \supset X)$ in Σ.

If Σ contain no axiom of $X \supset (Y \supset X)$ in Σ , then we are done.

We assume the result for the case where the number of the axioms of $X \supset (Y \supset X)$ in Σ is less than n, and we prove it for n.

The set P of all formulas appearing in Σ are ordered by the anti-lexicographic ordering $<_R$, which we defined earlier.

Let Q be the set of axioms $X \supset (Y \supset X)$ in Σ. Q , as a subset of P, is ordered by $<_R$.

We denote the elements of Q by

$$q_1 <_R q_2 <_R \cdots <_R q_n.$$

Now, the axiom q_1 in Σ can be proved as follows:

$$\dfrac{\dfrac{A}{B \supset A} \supset I^0}{A \supset (B \supset A)} \supset I^1$$

where q_1 is $A \supset (B \supset A)$.

Let Σ' be the deduction resulting from Σ by replacing the axiom q_1 by the above proof.

We then notice that the number of the axioms $X \supset (Y \supset X)$ in Σ' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the deduction Σ_0 of ϕ having no axioms of $X \supset (Y \supset X)$.

(2) $\Leftrightarrow$ (3)

This is an immediate consequence of *Lemma* 1, together with the fact that we can prove the axiom of $X \supset (Y \supset X)$ if and only if we can prove the sequent axiom of $\rightarrow X \supset (Y \supset X)$.

(3) $\Rightarrow$ (4)

Let Π be a proof of $\Gamma \rightarrow \phi$ in $S \cup \{\text{sequent axiom } \rightarrow X \supset (Y \supset X)\}$.
We use induction on the number of the sequent axioms of $\rightarrow X \supset (Y \supset X)$.
If Π contain no sequent axiom of $\rightarrow X \supset (Y \supset X)$, then we are done.
We assume the result for the case where the number of the sequent axioms of $\rightarrow X \supset (Y \supset X)$ in Π is less than n, and we prove it for n.
The set T of all sequents in Π are ordered by the anti-lexicographic ordering $<_R$, which we define earlier.
Let U be the set of sequent axioms of $\rightarrow X \supset (Y \supset X)$ in Π. Then, U, as a subset of T, is ordered by $<_R$.
We denote the elements of U by,

$$u_1 <_R u_2 <_R \cdots <_R u_n.$$

The axiom u_1 can be proved in $S \cup \{$ the left weakening rule $\}$ as follows.

$$\dfrac{\dfrac{\dfrac{A \rightarrow A}{B, A \rightarrow A}\ \text{w}.left}{A \rightarrow B \supset A}\ \supset right}{\rightarrow A \supset (B \supset A)}\ \supset right$$

Let Π' be the proof resulting from Π by replacing u_1 by its above proof.
We then notice that the number of the sequent axioms $\rightarrow X \supset (Y \supset X)$ in Π is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the proof Π_0 of $\Gamma \rightarrow \phi$ having no sequent axiom $\rightarrow X \supset (Y \supset X)$.

(4) $\Rightarrow$ (3)

Let Π be a proof of $\Gamma \rightarrow \phi$ in $S \cup \{$the left weakening rule$\}$.
We use induction on the number of the applications of w.$left$ in Π.
If Π contain no application of w.$left$, then we are done.
We assume the result for the case where the number of the applications of w.$left$ rule in Π is less than n, and we prove it for n.
The set T of all sequents in Π are ordered by the anti-lexicographic ordering $<_R$, which we define earlier.
Let U be the set of sequents which is the lower sequent of w.$left$ rule in Π. U, as a subset of T, is ordered by $<_R$.
We denote the elements of U by,

$$u_1 <_R u_2 <_R \cdots <_R u_n.$$

Now, we consider a proof Θ of u_1 inside Π, which has u_1 as its last sequent. By the leastness of u_1, Θ contains no application of w.$left$ rule except at the very bottom. Then Θ can be re-written to Θ' in the following way so that Θ' does not contain any application of w.$left$ rule at all.

$$\dfrac{\begin{array}{c}\Theta\\ A,\Gamma_1,\rightarrow C\end{array}}{B,A,\Gamma_1\rightarrow C}\mathrm{w}.left \quad u_1 = B,A,\Gamma_1\rightarrow A$$

The above proof Θ can be re-written as follows.

$$\dfrac{\dfrac{\dfrac{}{A\rightarrow A}\quad \dfrac{\dfrac{}{B\rightarrow B}\quad\dfrac{}{A\rightarrow A}}{B\supset A,B\rightarrow A}\supset left}{A\supset(B\supset A),A,B\rightarrow A}\supset left}{A,B\rightarrow A}\,\chi\rightarrow \quad cut \quad \dfrac{\dfrac{\Theta}{A,\Gamma_1\rightarrow C}}{A,B,\Gamma_1\rightarrow C}\,\mathrm{cut}$$
$$\dfrac{A,B,\Gamma_1\rightarrow C}{B,A,\Gamma_1\rightarrow C}\,ex.left$$

where χ is a formula $A\supset(B\supset A)$.

When the set $\{A\}\cup\Gamma_1$ of formulas is empty, we use $\mathbf{t}$ as the cut formula instead. This proof is as follows.

$$\dfrac{\dfrac{\dfrac{}{\rightarrow\mathbf{t}\supset(B\supset\mathbf{t})}\quad \dfrac{\dfrac{\dfrac{}{B\rightarrow B}\quad\dfrac{}{\mathbf{t}\rightarrow\mathbf{t}}}{B\supset\mathbf{t},B\rightarrow\mathbf{t}}\supset left}{\mathbf{t}\supset(B\supset\mathbf{t}),B\rightarrow\mathbf{t}}\supset left}{B\rightarrow\mathbf{t}}\,cut \quad \dfrac{\dfrac{\Theta}{\rightarrow C}}{\mathbf{t}\rightarrow C}}{B\rightarrow C}\,\mathrm{cut}$$

We then notice that the number of applications of w.$left$ in Π' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the proof Π_0 of $\Gamma\rightarrow\phi$ having no application of w.$left$.

q.e.d.

Theorem 2 (Characterization of contraction)
Let ϕ, X and Y be formulas of L. Let Γ be a set of formula of L.
Then the following four statements are logically equivalent.

(1) We can prove ϕ from Γ in $D\cup\{\supset I^n(n\geq 2)\}$.

(2) We can prove ϕ from Γ in $D\cup\{$axiom $(X\supset(X\supset Y))\supset(X\supset Y)\}$.

(3) We can prove a sequent $\Gamma\rightarrow\phi$ in $S\cup\{$sequent axiom $\rightarrow(X\supset(X\supset Y))\supset(X\supset Y)\}$.

(4) We can prove a sequent $\Gamma\rightarrow\phi$ in $S\cup\{$ the left contraction rule $\}$.

Lemma 2
Let ϕ, X and Y be formulas of L. Let Γ be a set of formula of L.
We can prove ϕ from Γ in $D\cup\{\supset I^2\}$ if and only if we can prove ϕ from Γ in $D\cup\{\supset I^n(n\geq 2)\}$.

Proof of Theorem **2**

(1) $\Rightarrow$ (2)

We assume *Lemma* 2.

By *Lemma* 2, we can prove ϕ from Γ in $D \cup \{\supset I^2\}$ if and only if we can prove ϕ from Γ in $D \cup \{\supset I^n (n \geq 2)\}$. So, it is sufficient to show that ϕ from Γ in $D \cup \{\supset I^2\}$ can be proved if and only if ϕ can be proved from Γ in $D \cup \{\text{axiom}(A \supset (A \supset B)) \supset (A \supset B)\}$. Let Σ be a deduction of ϕ in $D \cup \{\supset I^2\}$.

We use induction on the number of the applications of $\supset I^2$ in Σ.

If Σ contain no application of $\supset I^2$, then we are done.

We assume the result for the case where the number of the applications of $\supset I^2$ in Σ is less than n, and we prove it for n.

The set P of all formulas appearing in Σ are ordered by the anti-lexicographic ordering $<_R$, which we defined earlier.

Let Q be the set of formulas which is the consequence of application of $\supset I^2$ in Σ. Q, as a subset of P, is ordered by $<_R$.

We denote the elements of Q by,

$$q_1 <_R q_2 <_R \cdots <_R q_n.$$

Now, we consider a deduction Ξ of q_1 inside Σ , which has q_1 as its last formula. By the leastness of q_1 , Ξ contains no application of $\supset I^2$ except at the very bottom. Then Ξ can be re-written to Ξ' in the following way, so that Ξ' does not contain any application of $\supset I^2$ at all.

$$\dfrac{\begin{array}{c}[A]\ [A] \\ \Xi \\ B\end{array}}{A \supset B} \supset I^2 \quad q_1 = A \supset B$$

The above proof Ξ can be re-written as follows.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\ [A] \\ \Xi \\ B\end{array}}{A \supset B} \supset I^1}{A \supset (A \supset B)} \supset I^1 \qquad (A \supset (A \supset B)) \supset (A \supset B)}{A \supset B} \supset E$$

Let Σ' be the poof resulting from Σ by replacing Ξ with Ξ'.

We then notice that the number of applications of $\supset I^2$ in Σ' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the deduction Σ_0 of ϕ having no application of $\supset I^2$.

(2) $\Rightarrow$ (1)

Let A and B be formulas of L. Let Σ be a deduction of ϕ in $D \cup \{\text{axiom } (X \supset (X \supset Y)) \supset (X \supset Y)\}$.

We use introduction on the number of the axioms of $(X \supset (X \supset Y)) \supset (X \supset Y)$ in Σ.

If Σ contain no axiom of $(X \supset (X \supset Y)) \supset (X \supset Y)$ in Σ, then we are done.

We assume the result for the case where the number of the axioms of $(X \supset (X \supset Y)) \supset (X \supset Y)$ in Σ is less than n, and we prove it for n.

The set P of all formulas appearing in Σ are ordered by the anti-lexicographic ordering $<_R$, which we defined earlier.

Let Q be the set of axiom formulas of $(X \supset (X \supset Y)) \supset (X \supset Y)$ in Σ. Q, as a subset of P, is ordered by $<_R$.

We denote the elements of Q by,

$$q_1 <_R q_2 <_R \cdots <_R q_n.$$

The axiom of q_1 can be re-written as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{A \quad \dfrac{A \quad A \supset (A \supset B)}{A \supset B} \supset E}{B} \supset E}{A \supset B} \supset I^2}{(A \supset (A \supset B)) \supset (A \supset B)} \supset I^1$$

Let Σ' be the deduction resulting from Σ by re-writing the axiom q_1.

We then notice that the number of the axioms of $(X \supset (X \supset Y)) \supset (X \supset Y)$ in Σ' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the deduction Σ_0 of ϕ having no axioms of $(X \supset (X \supset Y)) \supset (X \supset Y)$.

(2) $\Leftrightarrow$ (3)

This is an immediate consequence of *Lemma* 1, together with the fact that we can prove the axiom of $(X \supset (X \supset Y)) \supset (X \supset Y)$ in $D \cup \{\text{axiom } (X \supset (X \supset Y)) \supset (X \supset Y)\}$ if and only if we can prove the sequent axiom of $\rightarrow (X \supset (X \supset Y)) \supset (X \supset Y)$ in $S \cup \{\text{sequent axiom } \rightarrow (X \supset (X \supset Y)) \supset (X \supset Y)\}$.

(3) $\Rightarrow$ (4)

Let Π be a proof of $\Gamma \rightarrow \phi$ in $S \cup \{\text{sequent axiom } \rightarrow (X \supset (X \supset Y)) \supset (X \supset Y)\}$.

We use induction on the number of the sequent axioms of $\rightarrow (X \supset (X \supset Y)) \supset (X \supset Y)$.

If Π contain no sequent axiom of $\rightarrow (X \supset (X \supset Y)) \supset (X \supset Y)$, then we are done.

We assume the result for the case where the number of the sequent axioms of $\rightarrow (X \supset (X \supset Y)) \supset (X \supset Y)$ in Π is less than n, and we prove it for n.

The set T of all sequents in Π are ordered by the anti-lexicographic ordering $<_R$, which we define earlier.

Let U be the set of sequent axioms of $\rightarrow (X \supset (X \supset Y)) \supset (X \supset Y)$ in Π. U, as a subset of T, is ordered by $<_R$.

We denote the elements of U by,

$$u_1 <_R u_2 <_R \cdots <_R u_n.$$

The axiom of u_1 can be re-written as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A\quad B\to B}{A\supset B, A\to B}\supset left}{A\supset(A\supset B), A, A\to B}\supset left}{A, A, A\supset(A\supset B)\to B}\text{ex.}left\ \ n\text{ times}}{A, A\supset(A\supset B)\to B}\text{c.}left}{A\supset(A\supset B)\to A\supset B}\supset right}{\to A\supset(A\supset B)\supset(A\supset B)}\supset right}{A\to A}$$

Let Π' be the proof resulting from Π by re-writing the sequent axiom u_1.

We then notice that the number of the sequent axioms of $\to (X\supset(X\supset Y))\supset(X\supset Y)$ in Π' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the proof Π_0 of $\Gamma\to\phi$ having no sequent axiom of $\to (X\supset(X\supset Y))\supset(X\supset Y)$.

(4) $\Rightarrow$ (3)

Let Π be a proof of $\Gamma\to\phi$ in $S\cup\{$ the left contraction rule $\}$.

We use induction on the number of the applications of c.$left$ in Π.

If Π contain no application of c.$left$, then we are done.

We assume the result for the case where the number of the applications of c.$left$ rule in Π is less than n , and we prove it for n.

The set T of all sequents in Π are ordered by the anti-lexicographic ordering $<_R$, which we define earlier.

Let U be the set of sequents which is the lower sequent of c.$left$ rule in Π. U , as a subset of T, is ordered by $<_R$.

We denote the elements of U by,

$$u_1 <_R u_2 <_R \cdots <_R u_n.$$

Now, we consider a proof Θ of u_1 inside Π , which has u_1 as its last sequent. By the leastness of u_1 , Θ contains no application of c.$left$ rule except at the very bottom. Then Θ can be re-written to Θ' in the following way so that Θ' does not contain any application of c.$left$ rule at all.

$$\dfrac{\begin{matrix}\Theta\\ A, A, \Gamma\to C\end{matrix}}{A, \Gamma_1\to C}\text{c.}left\quad (u_1 = A, \Gamma_1\to C)$$

The above proof Θ can be re-written as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\Theta\\ A, A, \Gamma_1\to C\end{matrix}}{A, \Gamma_1\to A\supset C}\supset right}{\Gamma_1\to A\supset(A\supset C)}\supset right \qquad \dfrac{A\to A\quad C\to C}{A\supset C, A\to C}\supset left}{\Gamma_1, (A\supset(A\supset C))\supset(A\supset C), A\to C}\supset left}{(A\supset(A\supset C))\supset(A\supset C), \Gamma_1, A\to C}\text{ex.}left\ \ \text{n times}}{\Gamma_1, A\to C}\text{cut}}{A, \Gamma_1\to C}\text{ex.}left\ \ \text{n times}}{\to\chi}$$

where χ is a formula $(A\supset(A\supset C))\supset(A\supset C)$.

When the formula of C is not present, we use $\mathbf{f}$ as the cut formula instead. The proof in this case is as follows.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Theta}{A,A,\Gamma_1 \to}}{A,A,\Gamma_1 \to \mathbf{f}}\,\mathbf{f}right}{A,\Gamma_1 \to A \supset \mathbf{f}}\,\supset right}{\Gamma_1 \to A \supset (A \supset \mathbf{f})}\,\supset right \qquad \dfrac{A \to A \quad \mathbf{f} \to}{A \supset \mathbf{f}, A \to}\,\supset left}{\Gamma_1, (A \supset (A \supset \mathbf{f})) \supset (A \supset \mathbf{f}), A \to}\,\supset left}{(A \supset (A \supset \mathbf{f})) \supset (A \supset \mathbf{f}), \Gamma_1, A \to}\,\mathrm{ex}.left \quad \mathrm{n}\ times}{\Gamma_1, A \to}\,\mathrm{cut}}{A, \Gamma_1 \to}\,\mathrm{ex}.left \quad \mathrm{n}\ times \qquad \to \chi
$$

where χ is a formula $(A \supset (A \supset \mathbf{f})) \supset (A \supset \mathbf{f})$.

We then notice that the number of applications of c.$left$ in Π' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the proof Π_0 of $\Gamma \to \phi$ having no application of c.$left$.

q.e.d.

Proof of Lemma **2** First, we assume that ϕ is provable from Γ in $D \cup \{\supset I^2\}$. Then it is obvious that we can prove ϕ from Γ in $D \cup \{\supset I^2 (n \geq 2\}$.

Next, we shall assume that we prove ϕ from Γ in $D \cup \{\supset I^2 (n \geq 2)\}$.

Let Σ a deduction of ϕ in $D \cup \{\supset I^2 (n \geq 2)\}$.

We use induction on the number of the applications of $\supset I^n (n \geq 2)$ in Σ.

If Σ contain no application of $\supset I^n (n \geq 3)$, then we are done.

We assume the result for the case where the number of the applications of $\supset I^n (n \geq 3)$ in Σ is less than n, and we prove it for n.

The set P of all formulas appearing in Σ are ordered by the anti-lexicographic ordering $<_R$, which we defined earlier.

Let Q be the set of formulas which is the consequence of application of $\supset I^n (n \geq 3)$ in Σ. Q, as a subset of P, is ordered by $<_R$.

We denote the elements of Q by,

$$q_1 <_R q_2 <_R \cdots <_R q_n.$$

Now, we consider a deduction Ξ of q_1 inside Σ, which has q_1 as its last formula. By the leastness of q_1, Ξ contains no application of $\supset I^n (n \geq 3)$ except at the very bottom. Then Ξ can be re-written to Ξ' in the following way, so that Ξ' does not contain any application of $\supset I^n (n \geq 3)$ at all.

$$
\dfrac{\begin{array}{c}\overbrace{[A] \cdots [A]}^{\mathrm{n}} \\ \Xi \\ B\end{array}}{A \supset B}\ \supset I^n (n \geq 3)
$$

The above deduction of Ξ can be re-written as follows.

$$
\begin{array}{c}
\overbrace{[A]\cdots[A]}^{\mathrm{n}} \\
\Xi \\
\dfrac{\dfrac{B}{A \supset B}\supset I^2}{\dfrac{B}{A \supset B}\supset I^2}\supset E \\
\vdots \\
\dfrac{\dfrac{A \supset B}{B}\supset E}{A \supset B}\supset I^2
\end{array}
$$

After these re-writing, the number of the application of $\supset I^2$ becomes $n-1$, and the number of the application of $\supset E$ becomes $n-2$.

Let Σ' be the deduction resulting from Σ by replacing Ξ with Ξ'.

We then notice that the number of applications of $\supset I^n (n \geq 3)$ in Σ' is strictly less than n (in fact $n-1$). By our induction hypothesis, we obtain the deduction Σ_0 of ϕ having no application of $\supset I^n (n \geq 3)$. So we can prove ϕ from Γ in $D \cup \{\supset I^2\}$.

q.e.d.

Part II
Exchange Rule for Natural Deduction

7 Introduction

In Part I, which is based on [2], we introduced systems of natural deduction in the sense of G.Gentzen which explicitly excludes two structural inference rules, i.e., contraction and weakening. In this part, we proceed further to eliminate the exchange rule, to define a natural deduction system called **NFL**, which is free of all the three structural inference rules. A novel feature of **NFL** is that all (the occurrences of) the formulas in a proof is labeled in a certain way so that a "natural" ordering can be introduced on the set of these labeled formulas in the proof. This ordering, when restricted to the set of uncanceled assumptions in the proof, yields an ordering among them, which, in turn, enables us to characterize the exchange rule. After showing the equivalence of **NFL** and **FL**, which is the sequent calculus for substructural logics, we will characterize the exchange rule for **NFL**. **FL** expanded with the ordinary exchange rule for sequents is equivalent to **NFL** endowed with this newly formalized exchange rule for natural deduction.

8 Labeling and Ordering

Our method of labeling of the formulas occurring in a proof and our definition of ordering on the set of those labeled formulas are general enough to be introduced to any tree-like structure. Therefore, we will explain how to label the nodes of a tree before we introduce the formal language.

A tree consists of nodes and edges. It must have the root, a particular kind of node in the sense that there is no edges stretching downward from it. Edges can grow from a node only in the upward direction. These descriptions of a tree is made precise by the following inductive definition. We consider only finitely branching tree with countable nodes. (Namely, each node may have only finitely many edges stretching upward from it.)

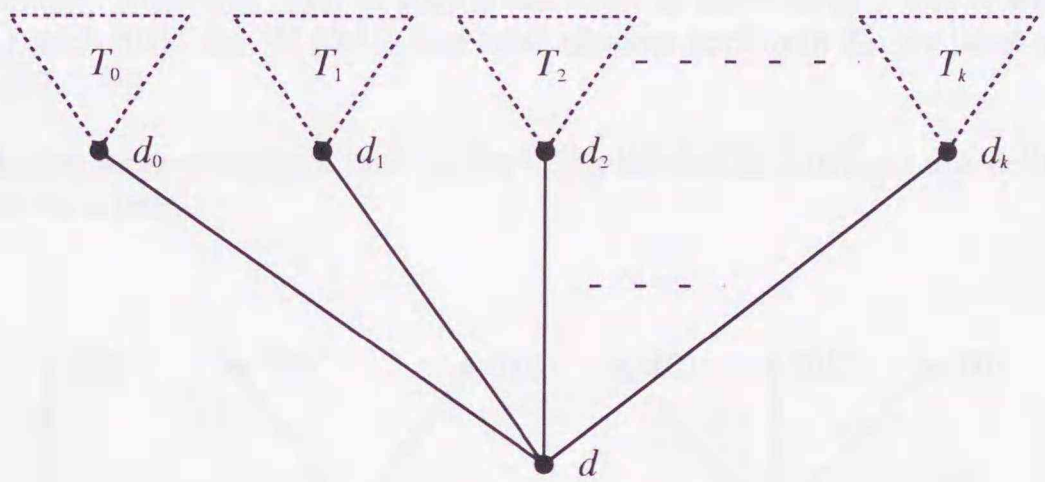
Let D be a (countable) set of nodes. We use $c, d, e, ..., d_0, d_1, d_2, \cdots$ to denote nodes. A tree is defined inductively as follows:

(1) Each single node is a tree. The root of this tree consisting of one node is that node itself.

(2) If $T_0, \cdots, T_{k-1}$ are already know to be a tree, each $d_i (1 \leq i \leq k-1)$ is the root of the corresponding tree T_i, and d is a node, then a figure obtained by putting the T_i's above d and connecting d with the d_i's is also a tree, the root of which is d.

(3) Only those figures that are obtained through finitely many applications of (1) and (2) are to be regarded as a tree.

The inductive step involved in the clause (2) can be schematically represented by the following figure:



In order to avoid the reliance on the visual intuitions such as 'left' and 'right' in the definition of a tree, a linear ordering could be introduced locally on the set of nodes located immediately above each node. Such a presentation would be too rigorous for our purposes in the present paper. We would like to keep our expositions reasonably intuitive and visual, as far as confusions can be avoided. We note that for our purpose of applying trees to the proofs of a formal system considered here, it is sufficient to take turnery branching trees into account. Therefor, we define the notion of labels only for turnery trees, i.e., each node is allowed to have, at most, three edges growing upward from it.

Labeling for a given tree T is defined by induction on the construction of trees in the following way:

(1) If T consist of a single node d, we label d by $\langle 0 \rangle$. Note that the root of T is d itself, and it is labeled by $\langle 0 \rangle$. As the induction hypothesis, we may assume that all the "subtrees" which are obtained during the construction process of the whole tree T are already labeled.

(2) If T does not consists of one node, there are three cases depending on the number of nodes above the root d of T.

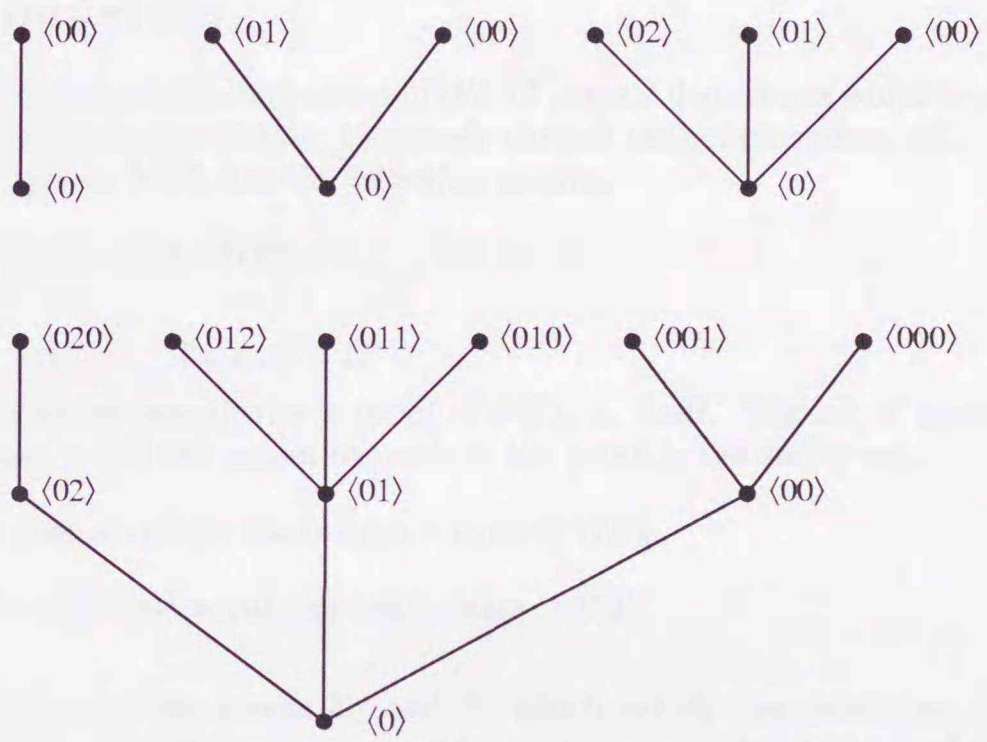
(2-1) If there is one nodes d_0 above the root d of T, and if d_0 is the root of the (sub)tree T_0 of T then d is labeled by $\langle 0 \rangle$ and d_0 is labeled by $\langle 00 \rangle$. Furthermore, if $\langle 0p \rangle$ is the label of some node e in T_0, we label e with $\langle 00 \rangle^\frown \langle p \rangle$, where p is a finite sequence consisting of 0,1, and 2, and $^\frown$ means the concatenation of two sequences.

(2-2) If there are two nodes d_0 and d_1 above the root d of T, and if d_0, d_1 (resp.) are the roots of subtrees T_0, T_1 (resp.) of T, then d is labeled by $\langle 0 \rangle$, d_0 is labeled by $\langle 01 \rangle$, d_1 is labeled by $\langle 00 \rangle$. Furthermore, if $\langle 0p \rangle$ is the label of some node e in T_0, we label e with $\langle 01 \rangle^\frown \langle p \rangle$. If $\langle 0q \rangle$ is the label of some node f in T_1, we label f with $\langle 00 \rangle^\frown \langle q \rangle$.

(2-3) If there are three nodes d_0, d_1 and d_2 above the root d of T, and if d_0, d_1, d_2 (resp.) are the roots of subtrees T_0, T_1, T_2 (resp.) of T, then d

is labeled by $\langle 0 \rangle$, d_0 is labeled by $\langle 02 \rangle$, d_1 is labeled by $\langle 01 \rangle$, and d_0 is labeled by $\langle 00 \rangle$. Furthermore, if $\langle 0p \rangle$ is the label of some node e in T_0, we label e with $\langle 02 \rangle^\frown \langle p \rangle$. If $\langle 0q \rangle$ is the label of some node f in T_1, we label f with $\langle 01 \rangle^\frown \langle q \rangle$. If $\langle 0r \rangle$ is the label of some node g in T_0, we label g with $\langle 00 \rangle^\frown \langle r \rangle$.

We give some trees with their nodes being labeled to illustrate our definition of labeling a tree.



Now, let us define an ordering relation R on the set B of finite sequences (labels) consisting of 0, 1, and 2. We restrict our definition of the ordering to those labels which are to be attached to the occurrences of formulas in a proof of our logical system.

Let x and y be the labels in B. Let R be a subset of $B \times B$. Then, we define the ordered pair $\langle x, y \rangle$ to be an element of R if and only if x and y are the same sequent, or else they are not the same, there must be the first place, say the i-th place, where these two sequences x and y begin to take different values. We use $(x)i$ for the i-th value of the sequent x, and $(y)i$ for the i-th value of the sequent y. Then, we set $\langle x, y \rangle \in R$ iff $(x)i < (y)i$, where the ordering $<$ on the right side means the order of magnitude among natural numbers. In case the order of $(x)i$ and $(y)i$ is reversed, we set $\langle y, x \rangle \in R$ iff $(y)i < (x)i$.

9 Language L and its Formulas

Our language L has propositional constant symbols $A, B, C, \cdots$. As for logical connectives, it has the implication symbols $\supset$, $\supset'$, the negation symbols $\neg$, $\neg'$, the multiplicative conjunction symbol $*$, the additive conjunction symbol $\wedge$, the

additive disjunction symbol $\vee$. In L, there are constant symbols **t** to denote the unit element for the multiplicative conjunction, **f** to denote the unit element for the multiplicative disjunction which is not introduced in this paper, $\top$ to denote the unit element for the additive conjunction, and $\bot$ to denote the unit element for the additive disjunction.

The formulas of L are defined inductively as a finite sequence of these symbols together with parenthesizes.

10 Natural Deduction System NFL: First Formulation

We now introduce a formal system **NFL** of natural deductions which is obtained by restricting and modifying Gentzen's natural deduction system *NK*.

Our system **NFL** has the following axioms.

- The axioms for **NFL** :

  t, $A \supset \top$, $(\bot * A) \supset C$

  An axiom constitutes a proof of **NFL** in itself. The set of assumptions upon which the axiom depends in the proof is the empty set.

Next, we introduce the inference rules of **NFL**.

- The elimination rule for implication $\supset$ E F

  Given any two proofs Σ_1 and Σ_2 which satisfy the conditions specified below, this inference rule enables us to construct a new proof Σ which includes the two proofs as its parts.

  Assume that two proofs Σ_1 and Σ_2 are given which satisfy the following conditions: The conclusion of Σ_1 is a formula A. The set of assumptions on which the conclusion A depends in Σ_1 is Γ_1. Note that Γ_1 is a set of labeled formulas.

  Each element of Γ_1 is a formula occurring at the "top" of some branch. Therefore, the order defined on the set of all formulas occurring in Σ_1 induces a subordering on Γ_1,

  A list of labeled formulas is obtained by listing the labeled formulas in Γ_1. If we take off the labels attached to the formulas in this list, we obtain another list consisting of plain (unlabeled) formulas. The latter list is expressed in symbols:

  $$\Gamma_1 = \langle \gamma_0^1, \gamma_1^1, \gamma_2^1, \cdots, \gamma_{m-1}^1 \rangle$$

  The Γ_1 on the left side is used in two way. In one sense, it means a linearly ordered set Γ_1 of labeled formulas. In the other sense, it means a list Γ_1 of those unlabeled formulas obtained by taking off the label of each formula appearing in the list of the labeled formulas which constitute the linearly ordered set in the first sense. It is understood that each unlabeled formula

in the list Γ_1 keeps its original position in the linearly ordered set Γ_1 of labeled formulas. We do not mention explicitly in which sense the term Γ_1 is used, whenever it is clear from the context.

The conclusion of the proof Σ_2 is a formula $A \supset B$. Let Γ_2 be the set of assumptions upon which the conclusion $A \supset B$ depends in the proof Σ_2. Notice that this assumption set Γ_2 is the set of labeled formulas in the proof Σ_2.

The linear ordering of the set Σ_2 induces a linear ordering on its subset Γ_2. Taking off the labels from the formulas in this linearly ordered set Γ_2, while keeping unchanged the position of each formula in that linear ordering, we obtain a list Γ_2 of unlabeled formulas, which is expressed in symbols as follows:

$$\Gamma_2 = \langle \gamma_0^2, \gamma_1^2, \gamma_2^2, \cdots, \gamma_{n-1}^2 \rangle$$

where Γ_2 has two meanings, just as Γ_1 has two.

Now then, one obtains a new proof Σ through the application of the elimination rule $\supset$ E of implication to two given proofs Σ_1 and Σ_2. This new proof Σ just constructed at the stage of the above application is schematically represented in the following way:

$$\frac{\Sigma_1 \quad \Sigma_2}{B} \supset \mathrm{E}$$

In this scheme, B is the conclusion of Σ. The set of assumptions (labeled in Σ) upon which B depends in S is the union of the set Γ_1 of formulas labeled in Σ and the set Γ_2 of formulas labeled in Σ. $\supset$ E in the above scheme indicates the place where the elimination rule of implication is applied.

Since every formula in this set Γ is a formula located at the "top" of some "branch" in Σ, the ordering on the set of formulas which are occurring and labeled in Σ induces a linear ordering on the set of formulas in Γ. The list Γ of unlabeled formulas which are obtained from the labeled formulas in the linearly ordered set Γ by taking off their labels but keeping their positions in the linear ordering is expressed in the following way.

$$\Gamma = \langle \gamma_0^1, \gamma_1^1, \gamma_2^1, \cdots, \gamma_{m-1}^1, \gamma_0^2, \gamma_1^2, \gamma_2^2, \cdots, \gamma_{n-1}^2 \rangle$$

The Γ in the left side of the above equation is used in two ways, namely as the linearly ordered set Γ of labeled formulas, and as the list of unlabeled formulas which are obtained by taking off their labels but keeping their positions in the linear ordering. We do not explicitly mention which of these two usages are meant whenever it is clear from the context.

- The introduction rule for implication $\supset$ I F

  Given a given proof Γ satisfying the conditions specified below, this inference rule enables us to construct a new proof Σ which include Γ as its part.

Assume that a proof Σ_1 is given and that it is satisfying the following conditions:

The conclusion of Σ_1 is a formula B. The set of assumptions on which the conclusion B depends in Σ_1 is $\langle A, \Gamma_1 \rangle$. Note that Γ_1 is a set of labeled formulas. This set $\langle A, \Gamma_1 \rangle$ is obtained as a subset of the formulas occurring and labeled in Σ, so that it is linearly ordered by the ordering on the set of labels attached to the formula occurring in Σ.

Each assumption of B is a formula occurring at the "top" of some branch. Therefore, the ordering defined on the set of all formulas occurring in Σ_1 induces a linear ordering on the assumptions of B. A is the least labeled formula in the linearly ordered set of assumptions of B. Note that A is not an element of Γ_1.

A list of labeled formulas is obtained by listing the labeled formulas in $\langle A, \Gamma_1 \rangle$. If we take off the labels attached to the formulas in that list while keeping their positions in that list, we obtain another list consisting of plain (unlabeled) formulas. The latter list is expressed in symbols:

$$\langle A, \Gamma_1 \rangle = \langle A, \gamma_0^1, \gamma_1^1, \gamma_2^1, \cdots, \gamma_{m-1}^1 \rangle$$

(In more precise terms, the concatenation of the two lists can be written as $\langle A \rangle^\frown \Gamma_1$.) Here, again, the notation $\langle A, \Gamma_1 \rangle$ is used in two ways. According to the first use, it means a linearly ordered set of labeled formulas. According to the second use, it means a list of unlabeled formulas which are obtained from the labeled formulas in the linearly ordered set $\langle A, \Gamma_1 \rangle$ by taking off their labels, but keeping their positions in that linear order.

Now then, one obtains a new proof Σ through the application of the introduction rule $\supset$ I of implication to the given proof Σ_1. This new proof Σ just constructed at the stage of the above application is schematically represented in the following way:

$$\frac{\Sigma_1}{A \supset B} \supset \mathrm{I}$$

where $A \supset B$ is the conclusion of Σ. The set Σ of assumptions upon which $A \supset B$ depends in Σ is defined to be the linearly ordered set Γ_1 of formulas which are re-labeled in Σ. $\supset$ I in the above scheme indicates the place where the introduction rule of implication is applied.

- The introduction rule for multiplicative conjunction $*$IF

  Given any two proofs Σ_1 and Σ_2 which satisfy the conditions specified below, this inference rule enables us to construct a new proof Σ which includes the two proofs as its parts.

  Assume that two proofs S_1 and Σ_2 are given which satisfy the following conditions:

  The conclusion of Σ_1 is a formula A. The set of assumptions on which the conclusion A depends in Σ_1 is Γ_1. Note that Γ_1 is a set of formulas which are labeled in Σ_1.

The conclusion of Σ_2 is a formula B. The set of assumptions on which the conclusion B depends in Σ_2 is Γ_2. Note that Γ_2 is a set of formulas which are labeled in Σ_1.

Now then, one obtains a new proof Σ through the application of the introduction rule $*\mathrm{I}$ of multiplicative conjunction to those two given proofs Σ_1 and Σ_2. This new proof Σ just constructed at the stage of the above application is schematically represented in the following way:

$$\frac{\Sigma_1 \quad \Sigma_2}{A * B} *\mathrm{I}$$

where $A * B$ is the conclusion of Σ. $*\mathrm{I}$ in the above scheme indicates the place where the introduction rule of multiplicative conjunction is applied.

The set of assumptions upon which $A * B$ depends in Σ is defined to be the union of the two linearly ordered sets Γ_1 and Γ_2 of formulas which are labeled in Σ.

Simply taking the union of Γ_1 and Γ_2, the elements of which are labeled in Σ_1 and Σ_2 respectively, does not yields the set of assumptions upon which the conclusion $A * B$ depend in the newly constructed proof Σ.

After the application of $*\mathrm{I}$ to the two proofs Σ_1 and Σ_2, the new proof Σ is constructed, and then, this Σ includes Σ_1 and Σ_2 as its parts. Therefore, Σ_1 and Σ_2 are now re-labeled in Σ after the application has taken a place. From this new labeling of (all the occurrences of formulas in) Γ_1, a new ordering, actually a linear ordering, is induced on Γ_1. This new linear ordering on Γ_1 as a subordering of that on Σ is the same as (order isomorphic to) the linear ordering on Γ_1 as a subordering of that on Σ_1. The same remark holds for the labels and orderings of Γ_2 in Σ and Σ_2.

The labeled formulas in Γ_1 can be listed according to the ordering defined on the set of labels for the occurrences of formulas in Σ. Taking off the labels from the formulas in the list, while keeping their ordering in the list, we obtain another list of formulas without labels. From Γ_2, a similar list is obtained through the same procedure of removing the labels. These two list can be written as follows:

$$\Gamma_1 = \langle \gamma_0^1, \gamma_1^1, \gamma_2^1, \cdots, \gamma_{m-1}^1 \rangle$$

$$\Gamma_2 = \langle \gamma_0^2, \gamma_1^2, \gamma_2^2, \cdots, \gamma_{n-1}^2 \rangle$$

A list of labeled formulas is obtained by ordering the labeled formulas in Γ. The labeled formulas in Γ can be listed according to the ordering of their labels, which are attached to them as a subset of Σ. Taking off the labels from the formulas in Γ, while keeping their ordering in the list, we obtain another list of un-labeled formulas. This second list is expressed as follows:

$$\Gamma = \langle \gamma_0^1, \gamma_1^1, \gamma_2^1, \cdots, \gamma_{m-1}^1, \gamma_0^2, \gamma_1^2, \gamma_2^2, \cdots, \gamma_{n-1}^2 \rangle$$

- The introduction rule fro additive conjunction $\wedge$IF

  Given any two proofs Σ_1 and Σ_2 which satisfy the conditions specified below, this inference rule enables us to construct a new proof Σ which includes the two proofs as its parts.

  Assume that two proofs Σ_1 and Σ_2 are given which satisfy the following conditions:

  The conclusion of Σ_1 is a formula A. The set of assumptions on which the conclusion A depends in Σ_1 is Γ. Note that the assumption set Γ of A in Σ_1 is a set of formulas which are labeled in Σ_1.

  The conclusion of Σ_2 is a formula B. The set of assumptions on which the conclusion B depends in Σ_2 is Γ. Note that the assumption set Γ of B in Σ_2 is a set of formulas which are labeled in Σ_2.

  Now then, one obtains a new proof Σ through the application of the introduction rule $\wedge$I of additive conjunction to those two given proofs Σ_1 and Σ_2. This new proof Σ just constructed at the stage of the above application is schematically represented in the following way:

$$\frac{\Sigma_1 \quad \Sigma_2}{A \wedge B} \wedge\text{I}$$

  where $A \wedge B$ is the conclusion of Σ. $\wedge$E in the above scheme indicates the place where the elimination rule of additive conjunction is applied.

  The set of assumptions upon which $A * B$ depends in Σ is defined to be the union of two assumption sets, the assumption set of A in Σ and the assumption set of of B in Σ. These two assumption sets are obtained from the assumption set of A in Σ_1 and that of B in Σ_2 by re-labeling them in the newly constructed proof Σ

  It goes without saying that simply taking the union of Γ_1 and Γ_2, the elements of which are labeled in Σ_1 and Σ_2 respectively, does not yields the set of assumptions upon which the conclusion $A * B$ depend in the newly constructed proof Σ. The occurrences of formulas in Γ_1 and Γ_2 are to be re-labeled in Σ. Our method of labeling keep the ordering of the occurrences of formulas in these two sets unchanged.

  We can list the elements of the assumption set Γ of A in Σ_1 according to the ordering defined on the labels attached to them in Σ_1. Taking off the label from each item in this list, we obtain another list of formulas, not having labels but keeping the ordering in the previous list.

  In a similar procedure, we obtain the list of formulas, not having labels but keeping the ordering in Γ_2.

  The two lists of un-labeled formulas obtained as the final outcomes of the above procedures are exactly the same, In other words, they share the terms in the list and the order among them.

  Therefore, the assumption set (the union set) of the conclusion $A \wedge B$ in the proof Σ can be described, when taking off the labels from the formulas contained in it as elements, in the following way:

$$\Gamma = \langle \gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_{m-1} \rangle$$

where the Γ is used as a list of uncabled formulas.

The assumption set of $A \wedge B$ in Σ must satisfy the extra condition: if one of the two sets Γ in Σ_1 and Γ in Σ_2, together constituting the assumption set of $A \wedge B$ in Σ and both labeled in Σ, happens to be changed in some way or other during the course of constructing proof, the other one must be changed exactly in the same way. (To make this condition as part of inductive definition of inference rules and proofs, it should be attached to all of those inference rules which change assumption sets through their applications.)

More specifically, if some element, say the i-th element, of Γ in Σ_1 or Γ in Σ_2, happen to be canceled in an application of a certain inference rule, then the corresponding i-th element of the other Γ in Σ_2 or Σ_1 must be canceled in the same application of the same inference rule. Furthermore, if some element is added to one set, then the same change must be made in the corresponding position in the other set.

We call the above condition the parallel change requirement on the assumption sets.

- The elimination rule for additive disjunction $\vee$EF

  Given any three proofs Σ_1, Σ_2, and Σ_3 which satisfy the conditions specified below, this inference rule enables us to construct a new proof Σ which includes the three proofs as its parts.

  Assume that three proofs Σ_1, Σ_2, and Σ_3 are given which satisfy the following conditions:

  The conclusion of Σ_1 is a formula $A \vee B$. The set of assumptions on which the conclusion $A \vee B$ depends in Σ_1 is Δ. The assumption set Δ of $A \vee B$ in Σ_1 is a set of formulas which are labeled in Σ_1.

  The conclusion of Σ_2 is a formula C. The set of assumptions on which the conclusion C depends in Σ_2 is expressed as $\langle A, \Gamma \rangle$. This set consists of formulas labeled in Σ_2. Therefore, it is linearly ordered as a subset of Σ_2. Then we use A for the least element in this linear ordering. (A is not an element of Γ.)

  We can arrange the labeled formulas in $\langle A, \Gamma_1 \rangle$ into a list according to this linear ordering, and then, taking off the label from each term in the list, we obtain another list of formulas, this time, without labels, which is expressed as:

  $$\langle A, \Gamma_1 \rangle = \langle A, \gamma_0^1, \gamma_1^1, \gamma_2^1, \cdots \gamma_{m-1}^1 \rangle$$

  (More precisely, the left hand side can be written as $\langle A \rangle^\frown \Gamma_1$.)

  The conclusion of Σ_3 is a formula C. The set of assumptions on which the conclusion C depends in Σ_3 is expressed as $\langle B, \Gamma \rangle$. This set consists of formulas labeled in Σ_3. Therefore, it is linearly ordered as a subset of

Σ_3. Then we use B for the least element in this linear ordering. (B is not the element of Γ.)

We can arrange the labeled formulas in $\langle B, \Gamma_2 \rangle$ into a list according to this linear ordering, and then, taking off the label from each term in this list, we obtain another list of formulas, this time, without labels, which is expressed as:

$$\langle B, \Gamma_2 \rangle = \langle B, \gamma_0^2, \gamma_1^2, \gamma_2^2, \cdots \gamma_{m-1}^2 \rangle$$

(More precisely, the left hand side can be written as $\langle B \rangle^\frown \Gamma_2$.)

Now then, one obtains a new proof Σ through the application of the elimination rule $\vee$E of additive disjunction to those three given proofs Σ_1, Σ_2, and Σ_3. This new proof Σ just constructed at the stage of the above application is schematically represented in the following way:

$$\frac{\Sigma_1 \quad \Sigma_2 \quad \Sigma_3}{C} \vee\text{E}$$

where C is the conclusion of Σ. $\vee$E in the above scheme indicates the place where the elimination rule of additive disjunction is applied.

The assumption set of $A \vee B$ in Σ is the union of the three sets, (1) the assumption set Δ, which is re-labeled in Σ. (2) a set which is obtained from $\langle A, \Gamma_1 \rangle$ by canceling off A, and which is re-labeled in Σ. (3)a set which is obtained from $\langle B, \Gamma_2 \rangle$ by canceling off B, and which is re-labeled in Σ.

The assumption sets in the closes (2) and (3) above must satisfy the parallel change requirement on the assumption sets: whenever some inference rule acts on one of the assumption set Γ_1 and Γ_2, it must act on the other set exactly in the same way. If an elimination of some element occurs in one set, the corresponding element in the other set must be eliminated, and vice versa. In such a case, we regard just one single formula is eliminated.

If we take off the labels from the elements of Γ_1 and Γ_2, they become precisely the same list of formulas. This list is expressed as:

$$\Gamma = \langle \gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_{m-1} \rangle$$

The above heuristic expositions of some of the representative inference rules will suffice to show how the notion of proofs in natural deduction can be extended by our way of labeling and ordering. Therefore, we will not include the explanations of other inference rules in the present paper. For the analysis of exchange rule and normalization processes, however, it is worth having a somewhat handy and a bit more simplified version of inference rules. The second formulation of **NFL** presented in the next section is essentially the same as its first formulation. In the second formulation, convenience comes first, and the motivation is pushed behind. Each inference rule is introduced with a schematic figure. Its uncanceled assumptions are represented by "sequents", which are not part of tree-like proofs, but, auxiliary and handy tools attached to the proofs.

Applying these rules of the second formulation mechanically, one cannot help constructing a proof in accord with the ordering we defined on labels for the first formulation of **NFL**.

11 Second Formulation of NFL

There is another formulation of **NFL**, whose inference rules are more convenient to manage than those of the formulation of **NFL** presented in the previous section.

In this new formulation, labeling and ordering are put behind. Instead of dealing with labeled formulas, we follows instructions provided by each inference rule in a somewhat mechanical manner so that the "order" of assumptions are almost automatically modified in the same way as in the original formulation of **NFL**. These two formulations share the same set of axioms and theorems, when labels are neglected. They differ only in the presentations of inference rules.

- The axioms for **NFL** :

  $\mathbf{t}, \quad A \supset \top, \quad (\bot * A) \supset C$

- The inference rule of the introduction of $\supset$:

$$\dfrac{\begin{array}{c}[A] \quad \Gamma \\ \Sigma_1 \\ B\end{array}}{A \supset B} \supset \mathrm{I}$$

  Σ_1 is a proof of B from $\langle A \rangle^\frown \Gamma$. By applying the $\supset$ I, one constructs a proof Σ of $A \supset B$ from Γ.

- The inference rule of the elimination of $\supset$:

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2 \\ \Sigma_1 & \Sigma_2 \\ A & A \supset B\end{array}}{B} \supset \mathrm{E}$$

  Σ_1 is a proof of A from Γ_1, Σ_2 is a proof of $A \supset B$ from Γ_2. By applying the $\supset$ E, one constructs a proof Σ of B from $\Gamma_1^\frown \Gamma_2$.

- The inference rule of the introduction of $\supset'$:

$$\dfrac{\begin{array}{c}\Gamma \quad [A] \\ \Sigma_1 \\ B\end{array}}{A \supset' B} \supset' \mathrm{I}$$

Σ_1 is a proof of B from $\Gamma^\frown\langle A\rangle$. By applying the $\supset'$ I, one constructs a proof Σ of $A \supset' B$ from Γ.

- The inference rule of the elimination of $\supset'$:

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \sum_1 & \sum_2\\ A \supset' B & A\end{array}}{B}\ \supset' \mathrm{E}$$

Σ_1 is a proof of $A \supset' B$ from Γ_1, Σ_2 is a proof of A from Γ_2. By applying the $\supset'$ E, one constructs a proof Σ of B from $\Gamma_1^\frown\Gamma_2$.

- The inference rule of the introduction of $\neg$:

$$\dfrac{\begin{array}{c}[A]\ \ \Gamma\\ \sum_1\\ \mathbf{f}\end{array}}{\neg A}\ \neg\mathrm{I}$$

Σ_1 is a proof of $\mathbf{f}$ from $\langle A\rangle^\frown\Gamma$. By applying the $\neg$I, one constructs a proof Σ of $\neg A$ from Γ.

- The inference rule of the elimination of $\neg$:

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \sum_1 & \sum_2\\ A & \neg A\end{array}}{\mathbf{f}}\ \neg\mathrm{E}$$

Σ_1 is a proof of A from Γ_1, Σ_2 is a proof of $\neg A$ from Γ_2. By applying the $\neg$E, one constructs a proof Σ of $\mathbf{f}$ from $\Gamma_1^\frown\Gamma_2$.

- The inference rule of the introduction of $\neg'$:

$$\dfrac{\begin{array}{c}\Gamma\ \ [A]\\ \sum_1\\ \mathbf{f}\end{array}}{\neg A}\ \neg'\mathrm{I}$$

Σ_1 is a proof of $\mathbf{f}$ from $\Gamma^\frown\langle A\rangle$. By applying the $\neg'$I, one constructs a proof Σ of $\neg' A$ from Γ.

- The inference rule of the elimination of $\neg'$:

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \sum_1 & \sum_2\\ \neg' A & A\end{array}}{\mathbf{f}}\ \neg'\mathrm{E}$$

Σ_1 is a proof of $\neg' A$ from Γ_1, Σ_2 is a proof of A from Γ_2. By applying the $\neg'$E, one constructs a proof Σ of $\mathbf{f}$ from $\Gamma_1 \frown \Gamma_2$.

- The inference rule of the introduction of $*$:

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \sum_1 & \sum_2\\ A & B\end{array}}{A * B}\ *\mathrm{I}$$

Σ_1 is a proof of A from Γ_1, Σ_2 is a proof of B from Γ_2. By applying the $*$I, one constructs a proof Σ of $A * B$ from $\Gamma_1 \frown \Gamma_2$.

- The inference rule of the elimination of $*$:

$$\dfrac{\begin{array}{cc}\Gamma_1 & [A][B]\Gamma_2\\ \sum_1 & \sum_2\\ A * B & C\end{array}}{C}\ *\mathrm{E}$$

Σ_1 is a proof of $A * B$ from Γ_1, Σ_2 is a proof of C from $\langle A\rangle \frown \langle B\rangle \frown \Gamma_2$. By applying the $*$E, one constructs a proof Σ of C from $\Gamma_1 \frown \Gamma_2$.

- The inference rule of the introduction of $\wedge$:

$$\dfrac{\begin{array}{cc}\Gamma & \Gamma\\ \sum_1 & \sum_2\\ A & B\end{array}}{A \wedge B}\ \wedge\mathrm{I}$$

Σ_1 is a proof of A from Γ, Σ_2 is a proof of B from Γ. By applying the $\wedge$I, one constructs a proof Σ of $A \wedge B$ from Γ. As in $\wedge$I, we identifies <u>two</u> Γ's as <u>one</u> list Γ. If an occurrence of α in the left Γ and an occurrence of α in the right Γ, which is occurring in the same place as that occurrence of α in the left Γ, happen to be canceled along the way of "longer proof", these two α's are to be regarded as one formula, and to be canceled as "one" assumptions of that proof.

- The inference rule of the first elimination of $\wedge$:

$$\dfrac{\begin{array}{c}\Gamma\\ \sum_1\\ A \wedge B\end{array}}{A}\ \wedge \mathrm{E}_1$$

Σ_1 is a proof of $A \wedge B$ from Γ. By applying the $\wedge \mathrm{E}_1$, one constructs a proof Σ of A from Γ.

- The inference rule of the second elimination of $\wedge$:

$$\dfrac{\begin{array}{c}\Gamma\\ \sum_1\\ A \wedge B\end{array}}{B}\ \wedge \mathrm{E}_2$$

Σ_1 is a proof of $A \wedge B$ from Γ. By applying the $\wedge \mathrm{E}_2$, one constructs a proof Σ of B from Γ.

- The inference rule of the first introduction of $\vee$:

$$\dfrac{\begin{array}{c}\Gamma\\ \sum_1\\ A\end{array}}{A \vee B}\ \vee \mathrm{I}_1$$

Σ_1 is a proof of A from Γ. By applying the $\vee \mathrm{I}_1$, one constructs a proof Σ of $A \vee B$ from Γ.

- The inference rule of the second introduction of $\vee$:

$$\dfrac{\begin{array}{c}\Gamma\\ \sum_1\\ B\end{array}}{A \vee B}\ \vee \mathrm{I}_2$$

Σ_1 is a proof of B from Γ. By applying the $\vee \mathrm{I}_2$, one constructs a proof Σ of $A \vee B$ from Γ.

- The inference rule of elimination of $\vee$:

$$\dfrac{\begin{array}{ccc}\Delta & [A]\Gamma & [B]\Gamma\\ \sum_1 & \sum_2 & \sum_3\\ A \vee B & C & C\end{array}}{C}\ \vee \mathrm{E}$$

Σ_1 is a proof of $A \vee B$ from Δ, Σ_2 is a proof of A from $\langle A \rangle^\frown \Gamma$, and Σ_3 is a proof of B from $\langle B \rangle^\frown \Gamma$. By applying the $\vee$E to Σ_1, Σ_2, and Σ_3 one constructs a proof Σ of C from $\Delta^\frown \Gamma$. As in $\vee$E, we identifies <u>two</u> Γ's as <u>one</u> list Γ. If an occurrence of α in the left Γ and an occurrence of α in the right Γ, which is occurring in the same place as that occurrence of α in the left Γ, happen to be canceled along the way of "longer proof", these two α's are to be regarded as one formula, and to be canceled as "one" assumptions of that proof.

- The inference rule of the introduction of $\supset$ for the propositional constant $\mathbf{t}$:

$$\dfrac{\begin{array}{c}\Gamma \\ \Sigma_1 \\ B\end{array}}{\mathbf{t} \supset B} \supset \mathrm{I}^0(\mathbf{t})$$

Σ_1 is a proof of B from Γ. By applying the $\supset \mathrm{I}^0(\mathbf{t})$, one constructs a proof Σ of $\mathbf{t} \supset B$ from Γ. As in $\supset \mathrm{I}^0(\mathbf{t})$, the superscript 0 of I indicates that the number of canceled assumptions at the application of $\supset \mathrm{I}^0(\mathbf{t})$ is restricted to zero.

There are other inference rules not belonging to **NFL**.

- The inference rule of the inversed introduction of $\supset$:

$$\dfrac{\begin{array}{c}\Gamma \quad [A] \\ \Sigma_1 \\ B\end{array}}{A \supset B} \supset \mathrm{I}^{\circlearrowleft}$$

Σ_1 is a proof of B from $\Gamma^\frown \langle A \rangle$. By applying the $\supset \mathrm{I}^{\circlearrowleft}$, one constructs a proof Σ of $A \supset B$ from Γ.

- The inference rule of the inversed elimination of $\supset'$:

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2 \\ \Sigma_1 & \Sigma_2 \\ A \supset B & A\end{array}}{B} \supset \mathrm{E}^{\circlearrowleft}$$

Σ_1 is a proof of $A \supset B$ from Γ_1, Σ_2 is a proof of A from Γ_2. By applying the $\supset \mathrm{E}^{\circlearrowleft}$, one constructs a proof Σ of B from $\Gamma_1^\frown \Gamma_2$.

- The inference rule of the inversed introduction of $\supset'$:

$$\frac{\begin{array}{c}[A]\quad \Gamma \\ \sum_1 \\ B\end{array}}{A \supset' B}\ \supset' \mathrm{I}^{\circlearrowleft}$$

Σ_1 is a proof of B from $\langle A\rangle^\frown\Gamma$. By applying the $\supset' \mathrm{I}^{\circlearrowleft}$, one constructs a proof Σ of $A \supset' B$ from Γ.

- The inference rule of the inversed elimination of $\supset'$:

$$\frac{\begin{array}{cc}\Gamma_1 & \Gamma_2 \\ \sum_1 & \sum_2 \\ A & A \supset' B\end{array}}{B}\ \supset' \mathrm{E}^{\circlearrowleft}$$

Σ_1 is a proof of A from Γ_1, Σ_2 is a proof of $A \supset' B$ from Γ_2. By applying the $\supset' \mathrm{E}^{\circlearrowleft}$, one constructs a proof Σ of B from $\Gamma_1^\frown\Gamma_2$.

12 Sequent Calculus FL

The sequent of the language L have the following form

$$\Gamma \to \Delta\ .$$

The left hand side of a sequent may be empty. The right hand side of a sequent is either empty or consists of a single formula. To specify the element of Γ and Δ we write

$$\gamma_0, \cdots, \gamma_{m-1} \to \delta.$$

When both sides of a sequent are empty, we write

$$\to$$

Next, we introduce the sequent calculus **FL** as follows. We say that $(\Gamma_1 \to X_1, \Gamma_2 \to X_2, \cdots, \Gamma_n \to X_n \ / \ \Gamma \to X)$ is an instance of a certain inference rule if it has the form indicated by the corresponding figure. If $(\Gamma_1 \to X_1, \Gamma_2 \to X_2, \cdots, \Gamma_n \to X_n \ / \ \Gamma \to X)$ is an instance of an inference rule α , we call $\Gamma_i \to X_i$ the i-th upper sequent of α , and $\Gamma \to X$ the lower sequent of α. (The origin of **FL** goes back to a classical paper written by J. Lambek in 1950's. Our presentation of **FL** is based on Ono[4].)

- Axioms and rule for logical constants:

$$A \to A$$

$$\to \mathbf{t} \qquad \mathbf{f} \to$$

$$\frac{\Gamma \to C}{\mathbf{t}, \Gamma \to C}\ \mathbf{t}\mathrm{w} \qquad \frac{\Gamma \to}{\Gamma \to \mathbf{f}}\ \mathbf{f}\mathrm{w}$$

- Structural inference rule:

$$\frac{\Gamma_1 \to A \quad \Gamma_2, A, \Gamma_3 \to C}{\Gamma_2, \Gamma_1, \Gamma_3 \to C}\ (\mathrm{cut})$$

- Logical inference rule:

$$\frac{\Gamma_1 \to A \quad B, \Gamma_2 \to C}{\Gamma_1, A \supset B, \Gamma_2 \to C}\ (\supset \text{ left}) \qquad \frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ (\supset \text{ right})$$

$$\frac{\Gamma_1 \to A \quad B, \Gamma_2 \to C}{A \supset' B, \Gamma_1, \Gamma_2 \to C}\ (\supset' \text{ left}) \qquad \frac{\Gamma, A \to B}{\Gamma \to A \supset' B}\ (\supset' \text{ right})$$

$$\frac{\Gamma \to A}{\Gamma, \neg A \to}\ (\neg \text{ left}) \qquad \frac{A, \Gamma \to}{\Gamma \to \neg A}\ (\neg \text{ right})$$

$$\frac{\Gamma \to A}{\neg' A, \Gamma \to}\ (\neg' \text{ left}) \qquad \frac{\Gamma, A \to}{\Gamma \to \neg' A}\ (\neg' \text{ right})$$

$$\frac{A, B, \Gamma \to C}{A * B, \Gamma \to C}\ (* \text{ left}) \qquad \frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma_1, \Gamma_2 \to A * B}\ (* \text{ right})$$

$$\frac{A, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge_1 \text{ left}) \qquad \frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ (\wedge \text{ right})$$

$$\frac{B, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge_2 \text{ left})$$

$$\frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Gamma \to C}\ (\vee \text{ left}) \qquad \frac{\Gamma \to A}{\Gamma \to A \vee B}\ (\vee_1 \text{ right})$$

$$\frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\vee_2 \text{ right})$$

There are other structural inference rule not belong to **FL**.

$$\frac{\Gamma_1, A, B, \Gamma_2 \to C}{\Gamma_1, B, A, \Gamma_2 \to C}\ (\mathrm{e} \text{ left})$$

13 Equivalence of FL and NFL, and How to Characterize Exchange Rules for NFL

Theorem 3 (Equivalence of FL and NFL)
Let ϕ *be a formula of the language L. Let* Γ *be a list of formulas of L.*

Then, ϕ *is provable from* Γ *in* **NFL** *if and only if the sequent* $\Gamma \to \phi$ *is provable in* **FL**.

Proof We prove one direction, namely, that of sequent $\Gamma \to \phi$ is provable with proof Π in **FL**, then ϕ is provable from Γ with a proof Σ in **NFL**. To prove this, we use induction on the number $\sharp(\Pi)$ of the application of inference rules in the proof Π.

If $\sharp(\Pi)$ is zero, $\Gamma \to \phi$ must be an axiom of **FL**.

Case 1 The sequent $\Gamma \to \phi$ is $A \to A$. We can prove A from A with the proof A in **NFL**.

Case 2 The sequent $\Gamma \to \phi$ is $\to \mathbf{t}$. We can prove $\mathbf{t}$ with the proof $\mathbf{t}$ in **NFL**.

Case 3 The sequent $\Gamma \to \phi$ is $\mathbf{f} \to$. We can prove $\mathbf{f}$ from $\mathbf{f}$ with the proof $\mathbf{f}$ in **NFL**.

Case 4 The cases where the sequents $\Gamma \to \phi$ are $\Gamma_1 \to \top$ and $\bot, \Gamma_1 \to C$ are omitted.

Now, we can assume the theorem for $\sharp(\Pi) < n$, and pove it for $\sharp\ (\Pi) = n$.

Our proof is devited into cases according to which inference rule is used as the "bottom" inference rule in Π. We use α to denote the name of the "bottom" inference rule in Π.

Case 1 The bottom inference rule α in Π is ($\supset$ right) and the form of $\Gamma \to \phi$ is $\Gamma \to A \supset B$, so that the bottom part of Π has the following form:

$$\dfrac{\begin{array}{c}\vdots\\ A, \Gamma \to B\end{array}}{\Gamma \to A \supset B}\ (\supset \text{ right})$$

By induction hypothesis, the formula B is provable from $\langle A \rangle^\frown \Gamma$ in **NFL**, namely we have a proof of B:

$$\begin{array}{c} A \quad \Gamma \\ \sum_1 \\ B \end{array}$$

Then we can construct a proof of B in **NFL** as follows:

$$\dfrac{\begin{array}{c}[A] \quad \Gamma \\ \sum_1 \\ B\end{array}}{A \supset B}\ \supset \mathrm{I}$$

Case 2 α is ($\supset$ left), so that the bottom part of Π has the following form:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma_1 \to A & B, \Gamma_2 \to C\end{array}}{\Gamma_1, A \supset B, \Gamma_2 \to C}\ (\supset \text{ left})$$

By induction hypothesis, we have a proof of A and a proof of C:

$$\begin{array}{ccc}\Gamma_1 & B & \Gamma_2 \\ \sum_1 & & \sum_2 \\ A & & C\end{array}$$

Then we can construct a proof of C in **NFL** as follows:

$$\dfrac{\begin{array}{cc}\Gamma_1 & \\ \sum_1 & \\ A & A \supset B\end{array}}{B}\supset \text{E} \quad \begin{array}{c}\Gamma_2 \\ \sum_2 \\ C\end{array}$$

Case 3 α is ($*$ left), so taht the bottom part of Π has the following form:

$$\dfrac{\begin{array}{c}\vdots \\ A, B, \Gamma \to C\end{array}}{A * B, \Gamma \to C}\ (* \text{ left})$$

By induction hypothesis, we have a proof of C:

$$\begin{array}{ccc}A & B & \Gamma \\ & \sum_1 & \\ & C & \end{array}$$

Then we can construct a proof of C in **NFL** as follows:

$$\dfrac{\begin{array}{cc}A * B & \begin{array}{c}[A][B]\Gamma \\ \sum_1 \\ C\end{array}\end{array}}{C}\ *\text{E}$$

Case 4 α is ($\vee$ left), so that the bottom part of Π has the following form:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ A, \Gamma \to C & B, \Gamma \to C\end{array}}{A \vee B, \Gamma \to C}\ (\vee \text{ left})$$

By induction hypothesis, we have two proofs of C:

$$\begin{array}{cccc}A & \Gamma & B & \Gamma \\ \sum_1 & & \sum_2 & \\ C & & C & \end{array}$$

Then we can construct a proof of C in **NFL** as follows:

$$\dfrac{A \vee B \qquad \begin{array}{c}[A]\Gamma \\ \sum_1 \\ C\end{array} \qquad \begin{array}{c}[B]\Gamma \\ \sum_2 \\ C\end{array}}{C} \vee \mathrm{E}$$

q.e.d.

Theorem 4 (Characterization of Exchange)
Let ϕ be a formula of the language L. Let Γ be a list of formulas of L.
Then, the following five conditions are equivalent:

(1) the sequent $\Gamma \to \phi$ is provable in **FL** $\oplus \{exchange\}$ *which is obtained from* **FL** *by addition of the exchange rule.*

(2) ϕ is provable from Γ in **NFL** $\oplus \{\supset \mathrm{I}^{\circlearrowleft}\}$ *which results from* **NFL** *through the addition of the inverted rule, $\supset \mathrm{I}^{\circlearrowleft}$.*

(3) ϕ is provable from Γ in **NFL** $\oplus \{\supset \mathrm{E}^{\circlearrowleft}\}$ *which results from* **NFL** *through the addition of the inverted rule, $\supset \mathrm{E}^{\circlearrowleft}$.*

(4) ϕ is provable from Γ in **NFL** $\oplus \supset' \mathrm{I}^{\circlearrowleft}$ *which results from* **NFL** *through the addition of the inverted rule, $\supset' \mathrm{I}^{\circlearrowleft}$.*

(5) ϕ is provable from Γ in **NFL** $\oplus \supset' \mathrm{E}^{\circlearrowleft}$ *which results from* **NFL** *through the addition of the inverted rule, $\supset' \mathrm{E}^{\circlearrowleft}$.*

Proof First, we prove that if a sequent $\Gamma \to \phi$ is provable with proof Π in **FL** $\oplus \{exchange\}$, then ϕ is provable from Γ with a proof Σ in **NFL** $\oplus \{\supset \mathrm{I}^{\circlearrowleft}\}$. To prove this, we use induction on the number $\sharp(\Pi)$ of the application of inference rules in the proof Π.

If $\sharp(\Pi)$ is zero, we are done as in the proof of Theorem 1.

Now, we can assume the theorem for $\sharp\,(\Pi) > n$, and prove it for $\sharp\,(\Pi) = n$. As for the cases for the inference rules of **FL** , proofs in Theorem 1 carries over. We then proceed to consider the cases where the bottom inference rule α is the exchange rule.

Then Π is the following form:

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma_1, A, B, \Gamma_2 \to C\end{array}}{\Gamma_1, B, A, \Gamma_2 \to C}\ (\text{e left})$$

Without loss of the generality, Γ_1 and Γ_2 can be single formula. By the induction hypothesis, we have a proof of C from $\Gamma_1^\frown\langle A\rangle^\frown\langle B\rangle^\frown\Gamma_2$ in **NFL** $\oplus$ $\{\supset \mathrm{I}^{\circlearrowleft}\}$.

Σ_1 has the following form:

$$\begin{array}{c}\Gamma_1\ A\ B\ \Gamma_2 \\ \sum_1 \\ C\end{array}$$

We can now construct a proof of C from $\Gamma_1^\frown\langle B\rangle^\frown\langle A\rangle^\frown\Gamma_2$ in **NFL** $\oplus$ $\{\supset \mathrm{I}^{\circlearrowleft}\}$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma_1]\ A\ [B]\ [\Gamma_2]}{\sum_1 \atop C}}{\Gamma_1 \supset C}\supset \mathrm{I}}{\Gamma_2 \supset' \Gamma_1 \supset C}\supset' \mathrm{I}}{B \quad B \supset G_2 \supset' \Gamma_1 \supset C}\supset \mathrm{I}^{\circlearrowleft}}{\Gamma_2 \supset' \Gamma_1 \supset C}\supset \mathrm{E} \qquad \Gamma_2}{\Gamma_1 \qquad \Gamma_1 \supset C}\supset' \mathrm{E}}{C}\supset \mathrm{E}$$

This figure is indeed a proof Σ of C from $\Gamma_1{}^\frown\langle B\rangle{}^\frown\langle A\rangle{}^\frown\Gamma_2$ in **NFL** $\oplus \{\supset \mathrm{I}^{\circlearrowleft}\}$. The list $\Gamma_1{}^\frown\langle B\rangle{}^\frown\langle A\rangle{}^\frown\Gamma_2$ of hypothesis occurring in Σ is ordered according to the unti-lexicographic ordering on the labels on Σ.

Notice that the other three possible pair of inference rules, namely ($\supset$ I, $\supset$ $\mathrm{E}^{\circlearrowleft}$), ($\supset'$ $\mathrm{I}^{\circlearrowleft}$, $\supset'$ E), ($\supset'$ I, $\supset'$ $\mathrm{E}^{\circlearrowleft}$), can work as good as the above pair ($\supset$ $\mathrm{I}^{\circlearrowleft}$, $\supset$ E). It is these four pairs of inference rules that play the roll of "exchange rule".

The proof of the other direction of the Theorem 4 is indicated by the following proof figures.

Given a proof :

$$\dfrac{\dfrac{\Gamma \quad [A]}{\sum_1 \atop B}}{A \supset B}\supset \mathrm{I}^{\circlearrowleft}$$

By induction hypothesis, we have a proof of $\Gamma, A \to B$. Then, we can construct a proof of $\Gamma \to A \supset B$ as follows:

$$\dfrac{\dfrac{\vdots}{\Gamma, A \to B}}{A, \Gamma \to B}\ (\text{e left}) \times n$$
$$\Gamma \to A \supset B\ (\supset \text{ right})$$

Given a proof:

$$\dfrac{\dfrac{\Gamma_1 \qquad \Gamma_2}{\sum_1 \qquad \sum_2}}{A \supset B \qquad A}\supset \mathrm{E}^{\circlearrowleft}$$
$$B$$

By induction hypothesis, we have a proof of $\Gamma_1 \to A \supset B$ and $\Gamma_2 \to A$. Then we can construct a proof of $\Gamma_1, \Gamma_2 \to B$ as follows:

$$\dfrac{\Gamma_1 \to A \supset B \qquad \dfrac{\dfrac{\vdots}{\Gamma_2 \to A \quad B \to B}}{\Gamma_2, A \supset B \to B}\ (\supset \text{ left})}{A \supset B, \Gamma_2 \to B}\ (\text{e left}) \times n$$
$$\Gamma_1, \Gamma_2 \to B\ (\text{cut})$$

Cases for $\supset'$ $I^{\supset}$ and $\supset'$ $E^{\supset}$ can be treated in a similar way. All the other inference rules are already taken care of in the proof of Theorem 3.

q.e.d.

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