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by

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1 Introduction

We consider a hypersurface V^n imbedded in a Riemannian manifold R^{n+1} of $n+1$ dimensions. Let us denote by g_{ba} and h_{ba} the first fundamental tensor and the second fundamental tensor of V^n respectively.

Now, if we denote by $k_1, k_2, \dots, k_n$ the principal curvatures of V^n , that is, the roots of the characteristic equation

$$\det(h_{ba} - kg_{ba}) = 0,$$

then the ν -th mean curvature H_ν of V^n is defined to be the ν -th elementary symmetric function of $k_1, k_2, \dots, k_n$ divided by the number of terms, that is,

$$\binom{n}{\nu} H_\nu = \sum_{c_1 < c_2 < \dots < c_\nu} k_{c_1} k_{c_2} \dots k_{c_\nu} \quad (1 \leq \nu \leq n).$$

Particularly, for a surface V^2 in an Euclidean space E^3 of three dimensions, we put

$$H = \frac{1}{2}(k_1 + k_2), \quad K = k_1 k_2,$$

and we call H and K the *mean curvature* and the *Gauss curvature* of V^2 respectively. Accordingly, H is equal to H_1 and K equal to H_2 .

When h_{ba} are proportional to g_{ba} at a point P of V^n , the point P is called an *umbilic* point. Then it is easily seen that a point P is umbilic if and only if n principal curvatures be equal at P .

The spheres are characterized by various special properties of a closed surface in an Euclidean space E^3 . In the present paper a *closed* surface means a compact connected surface without boundary. In particular, It has been proved that a closed surface in E^3 is a sphere if and only if the Gauss curvature K of the surface be constant. On the other hand, it is proved that a sphere is a closed surface with constant mean curvautre H . When $K > 0$ holds good at every point of a closed surface, the closed

surface is called an *ovaloid*. It has been proved by H. Liebmann [15]² that the only ovaloid with constant mean curvature H in E^3 is a sphere. This result was the starting point of various interesting investigations of closed surfaces. Afterward this result was generalized for a closed convex hypersurface in an n -dimensional Euclidean space E^n . Many investigators have studied characterizations of the sphere from various viewpoints, and there exist many interesting investigations, for instance, Chern's work [3] about generalizing the condition $H = \text{const.}$ in the Liebmann's Theorem. In these investigations the integral formula of Minkowski type has played one of the important role.

Let V^2 be an ovaloid in E^3 . Then, as the well-known integral formula of Minkowski, we have

$$\int_{V^2} (Kp + H) dA = 0,$$

where p denotes the oriented distance from a fixed point O in E^3 to the tangent space of V^2 at P and dA is the area element of V^2 . This formula was generalized for a closed orientable hypersurface V^n in E^{n+1} [6]. Afterward Y. Katsurada [8,9] derived the integral formulas of Minkowski type for a closed orientable hypersurface V^n in an $(n+1)$ -dimensional Riemannian manifold R^{n+1} admitting a conformal Killing vector field ξ^i and gave certain characterizations of an umbilical hypersurface in R^{n+1} . The analogous problems for a closed orientable hypersurface V^n in a Riemannian manifold R^{n+1} have been discussed by many investigators.

It is one of the interesting problem to find certain conditions for an umbilical hypersurface in a Riemannian manifold to be isometric to a sphere. If every point of a closed orientable hypersurface in E^{n+1} is umbilic, then the hypersurface is isometric to a sphere. However, in a general Riemannian manifold we can not expect the validity of the same result even if every point of a closed orientable hypersurface is umbilic. Some investigators [10,21] gave some results on this problem, making use of Obata's Theorem [16].

The *concircular* transformation is by definition a conformal transformation carrying any geodesic circle into another one. A vector field generates locally a one-parameter group of transformations. We shall say a vector field to be *isometric*, *homothetic*, *concircular* or *conformal*, if it generates a one-parameter group of corresponding transformations.

A proper conformal Killing vector field ξ with components ξ^i is characterized by the so-called conformal Killing equation

$$(1.1) \quad \mathcal{L}_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

for a non constant scalar field Ψ which is said to be *associated* with ξ^i , where G_{ji} , $\mathcal{L}_\xi G_{ji}$, and ∇_i denote the metric tensor of R^{n+1} , the Lie derivative of G_{ji} with respect to ξ , and the operator of covariant differentiation with respect to Christoffel symbols $\left\{ \begin{smallmatrix} k \\ ji \end{smallmatrix} \right\}$ formed with G_{ji} respectively.

²Numbers in brackets refer to the references at the end of the paper.

It is known that in order that ξ^i is concircular, it is necessary and sufficient that the associated scalar field Ψ satisfies the equation

$$(1.2) \quad \nabla_j \nabla_i \Psi = \phi G_{ji} \quad ([7]),$$

where ϕ is a scalar field.

We shall call a scalar field Ψ in R_{n+1} satisfying an equation of type (1.2) a *concircular* scalar field. If ϕ is linear in Ψ with constant coefficients, that is,

$$(1.3) \quad \phi = \rho \Psi + \sigma,$$

where ρ is a non zero constant and σ a constant, then Ψ is called a *special* concircular scalar field [18].

For instance, let R^{n+1} be an Einstein manifold with non zero scalar curvature R which admits a proper conformal Killing vector field ξ^i , that is, ξ^i satisfies the equation (1.1). Then it is known that the associated scalar field Ψ satisfies the equation

$$(1.4) \quad \nabla_j \nabla_i \Psi = -\frac{R}{n(n+1)} \Psi G_{ji} \quad (R = \text{non zero const.}) \quad ([23, 21, 19]),$$

that is, Ψ is a special concircular scalar field such that $\rho = -(R/n(n+1))$ and $\sigma = 0$.

The author has studied closed hypersurfaces with constant mean curvature in a Riemannian manifold R^{n+1} admitting a conformal Killing vector field, or in R^{n+1} admitting a special concircular scalar field. If H_2 is constant and the principal curvatures $k_1, k_2, \dots, k_n$ at each point of V^n are positive, then H_2 is positive constant, because

$$\binom{n}{2} H_2 = \sum_{d < c} k_d k_c.$$

This property has been the turning point of our study of a closed hypersurface V^n with $H_2 = \text{constant}$. And the author obtained the following Theorems:

Theorem 1.1 ([12]) *Let R^{n+1} ($n \geq 3$) be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) *its second mean curvature H_2 is positive constant,*
- (2) *$C^i \xi_i$ has fixed sign on V^n , where C^i denotes the normal vector to V^n .*

Then every point of V^n is umbilic.

Theorem 1.2 ([13, 14]) *Let R^{n+1} be an orientable Riemannian manifold which admits a special concircular scalar field Ψ satisfying the equation*

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}),$$

and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) its first mean curvature H_1 is non zero constant,
- (2) Ω has fixed sign on V^n , where $\Psi_i = \nabla_i \Psi$, and $\Omega = C^i \Psi_i$.

Assume further that the ambient manifold R^{n+1} satisfies one of the following conditions

- (a) R^{n+1} has the harmonic curvature, that is, $\nabla_k R_{ji} = \nabla_j R_{ki}$, where R_{ji} denotes the Ricci tensor of R^{n+1} ,
- (b) R^{n+1} has the parallel Ricci tensor, that is, $\nabla_k R_{ji} = 0$,
- (c) R^{n+1} is a conformally flat Riemannian manifold and the scalar curvature R is constant,
- (d) R^{n+1} satisfies the equation $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0^3$,
- (e) $R^{ji} R_{ji}$ is constant,
- (f) R^{n+1} is a conformally flat Riemannian manifold and there exists a point P_0 on V^n such that $S(P_0) = 0$, where S_{ji} is the symmetric tensor defined by $R_{ji} + n\rho G_{ji}$, and $S = S_{ji} G^{ji}$.

Then every point of V^n is umbilic.

Next, using these Theorems, the author proved the following ones respectively:

Theorem 1.3 ([12]) Let R^{n+1} ($n \geq 3$) be an orientable Riemannian manifold of constant curvature which admits a proper conformal Killing vector field ξ^i satisfying the equation

$$\mathcal{L}_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) H_2 is positive constant,
- (2) $C^i \Psi_i$ has fixed sign on V^n ,
- (3) $C^i \Psi_i$ is not constant along V^n , where Ψ satisfies $\nabla_j \nabla_i \Psi = -(R/n(n+1))\Psi G_{ji}$.

Then V^n is isometric to a sphere.

Theorem 1.4 ([13, 14]) Let R^{n+1} be an orientable Riemannian manifold which admits a special concircular scalar field Ψ satisfying the equation

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}),$$

and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) H_1 is non zero constant,
- (2) Ω has fixed sign on V^n ,
- (3) Ω is not constant along V^n .

Assume further that the ambient manifold R^{n+1} satisfies one of the following conditions

- (a) R^{n+1} has the harmonic curvature, that is, $\nabla_k R_{ji} = \nabla_j R_{ki}$,
- (b) R^{n+1} has the parallel Ricci tensor, that is, $\nabla_k R_{ji} = 0$,
- (c) R^{n+1} is a conformally flat Riemannian manifold and the scalar curvature R is constant,
- (d) R^{n+1} satisfies the equation $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$,

³See [4]

- (e) $R^{ji}R_{ji}$ is constant,
- (f) R^{n+1} is a conformally flat Riemannian manifold and there exists a point P_0 on V^n such that $S(P_0) = 0$.

Then V^n is isometric to a sphere.

Moreover, under the new assumption that Ψ is not constant along V^n , the author proved the following Theorems in a similar way:

Theorem 1.5 ([12]) *Let R^{n+1} ($n \geq 3$) be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i satisfying the equation*

$$\mathcal{L}_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) H_2 is positive constant,
- (2) $C^i \xi_i$ has fixed sign on V^n ,
- (3) Ψ is not constant along V^n .

Then V^n is isometric to a sphere.

Theorem 1.6 ([13, 14]) *Let R^{n+1} be an orientable Riemannian manifold which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) H_1 is non zero constant,
- (2) Ω has fixed sign on V^n ,
- (3) Ψ is not constant along V^n .

Assume further that the ambient manifold R^{n+1} satisfies one of the following conditions

- (a) R^{n+1} has the harmonic curvature, that is, $\nabla_k R_{ji} = \nabla_j R_{ki}$,
- (b) R^{n+1} has the parallel Ricci tensor, that is, $\nabla_k R_{ji} = 0$,
- (c) R^{n+1} is a conformally flat Riemannian manifold and the scalar curvature R is constant,
- (d) R^{n+1} satisfies the equation $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$,
- (e) $R^{ji}R_{ji}$ is constant,
- (f) R^{n+1} is a conformally flat Riemannian manifold and there exists a point P_0 on V^n such that $S(P_0) = 0$.

Then V^n is isometric to a sphere.

To prove Theorem 1.1, we need the following integral formulas of Minkowski type ⁴

$$(1.5) \quad \int_{V^n} H_1 \theta \, dA + \int_{V^n} \Psi \, dA = 0$$

and

$$(1.6) \quad \frac{2}{n(n-1)} \int_{V^n} \nabla_c p_a^c \beta^a \, dA + \int_{V^n} \{nH_1H_2 - (n-2)H_3\} \theta \, dA + 2 \int_{V^n} \Psi H_2 \, dA = 0,$$

⁴See the equation (3.12) and (3.13) in § 3.

where the symmetric tensor field p_{ab} is defined by $h_c^c h_{ab} - h_a^c h_{cb}$, $p_a^c = p_{ab} g^{bc}$, ∇_c denotes the operator of the van der Waerden-Bortolotti covariant differentiation along V^n , $\theta = C^i \xi_i$ and dA denotes the area element of V^n . Multiplying (1.5) by $2H_2$ (=positive constant) and subtracting the result equation from (1.6), we find that

$$\frac{2}{n(n-1)} \int_{V^n} \nabla_c p_a^c \beta^a dA + (n-2) \int_{V^n} \{H_1 H_2 - H_3\} \theta dA = 0.$$

For a hypersurface V^n with positive constant second mean curvature H_2 in a Riemannian manifold of constant curvature, it is proved that $\nabla_c p_a^c = 0$ on V^n . Accordingly, from the above equation, we have

$$(1.7) \quad \int_{V^n} \{H_1 H_2 - H_3\} \theta dA = 0.$$

Moreover, since it is proved that H_1 has fixed sign on V^n for a hypersurface V^n with H_2 =positive constant, the scalar field $H_1 H_2 - H_3$ is rewritten as follows:

$$(1.8) \quad H_1 H_2 - H_3 = \frac{1}{H_1} \{H_2(H_1^2 - H_2) + (H_2^2 - H_1 H_3)\}.$$

On the other hand, it is known that

$$(1.9) \quad H_1^2 - H_2 = \frac{1}{n^2(n-1)} \sum_{a < b} (k_b - k_a)^2 \geq 0 \quad \text{and} \quad H_2^2 - H_1 H_3 \geq 0 \quad \text{on } V^n.$$

Accordingly, from (1.8), we have

$$H_1 H_2 - H_3 \geq 0 \quad (\text{or } \leq 0) \quad \text{on } V^n.$$

Hence, using the assumption (2), from (1.7), we find that

$$H_1 H_2 - H_3 = 0, \text{ that is, } H_2(H_1^2 - H_2) + (H_2^2 - H_1 H_3) = 0 \quad \text{on } V^n,$$

from which, using (1.9) and the assumption (1), we see that

$$H_1^2 - H_2 = 0 \quad \text{on } V^n.$$

Therefore, from the former of (1.9), we conclude that

$$k_1 = k_2 = \cdots = k_n$$

at each point of V^n . This means that each point of V^n is umbilic.

Now, in R^{n+1} , we assume the existence of a special concircular scalar field Ψ which satisfies the following partial differential equation

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}),$$

where $\Psi_i = \nabla_i \Psi$. If we put

$$\Phi = \rho \Psi + \sigma,$$

then it is proved that Φ satisfies the following partial differential equation

$$\nabla_j \Phi_i = \rho \Phi G_{ji},$$

where $\Phi_i = \nabla_i \Phi$. Accordingly, to prove Theorem 1.2, we need the following integral formulas of Minkowski type ⁵

$$(1.10) \quad \int_{V^n} H_1 \Theta dA + \int_{V^n} \rho \Phi dA = 0$$

and

$$(1.11) \quad \frac{1}{n} \int_{V^n} (\nabla_a h_c^c - R_{ij} B_a^i C^j) \phi^a dA + \int_{V^n} \{nH_1^2 - (n-1)H_2\} \Theta dA + \int_{V^n} \rho \Phi H_1 dA = 0,$$

where $B_a^i = \partial x^i / \partial u^a$ and $\Theta = C^i \Phi_i$. Since H_1 is non zero constant, taking account of $H_1 = (1/n)h_a^a$, we have, from (1.11),

$$(1.12) \quad -\frac{1}{n} \int_{V^n} R_{ij} B_a^i C^j \phi^a dA + \int_{V^n} \{nH_1^2 - (n-1)H_2\} \Theta dA + H_1 \int_{V^n} \rho \Phi dA = 0.$$

Eliminating $\int_{V^n} \rho \Phi dA$ from (1.10) and (1.12), we find that

$$(1.13) \quad -\frac{1}{n} \int_{V^n} R_{ij} B_a^i C^j \phi^a dA + (n-1) \int_{V^n} (H_1^2 - H_2) \Theta dA = 0.$$

On the other hand, making use of the special concircular scalar field Φ , the assumption (1) and (2), and the condition of the ambient manifold R^{n+1} under consideration, we will show in § 6 that $S_{ji} = 0$, say, $R_{ji} = -n\rho G_{ji}$ on V^n . Accordingly, from (1.13), we obtain

$$(1.14) \quad \int_{V^n} (H_1^2 - H_2) \Theta dA = 0.$$

Moreover, it is proved that, if Ω has fixed sign on V^n , then the same holds good also for Θ . Consequently, since

$$(1.15) \quad H_1^2 - H_2 = \frac{1}{n^2(n-1)} \sum_{a < b} (k_b - k_a)^2 \geq 0,$$

we see that $H_1^2 - H_2 = 0$ on V^n by virtue of (1.14). Therefore, from (1.15), we conclude that

$$k_1 = k_2 = \dots = k_n$$

at each point of V^n , that is, every point of V^n is umbilic.

Finally, to prove that the umbilical hypersurface under consideration is isometric to a sphere, we use the Theorem ⁶ due to M. Obata [16].

⁵See the equation (3.19) and (3.20) in § 3.

⁶See Theorem A. in § 7.

For a closed hypersurface V^n with constant second mean curvature H_2 in a Riemannian manifold of constant curvature, Y. Katsurada gave Theorem 1.1 (cf., Theorem 3.2 in [8]) under the strong restriction that at each point of V^n the principal curvatures $k_1, k_2, \dots, k_n$ of V^n are positive. The purpose of the present paper is to eliminate this restriction and prove Theorem 1.1. Y. Katsurada first gave the analogous Theorem (say, Theorem 3.1 in [9]) to Theorem 1.2 under the assumption that an ambient manifold R^{n+1} be an Einstein manifold. So, in a Riemannian manifold admitting a special concircular scalar field Ψ which satisfies the following partial differential equation

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}),$$

the another purpose of the present paper is to prove Theorems 1.2 under weaker assumptions by replacing, as an ambient manifold R^{n+1} , an Einstein manifold by a more general one. § 2 is devoted to give notations and general formulas in the theory of hypersurfaces in a general Riemannian manifold R^{n+1} . In § 3 we derive some integral formulas of Minkowski type which are valid for a closed orientable hypersurface V^n in R^{n+1} admitting a conformal Killing vector field ξ^i given by (1.1), or a special concircular scalar field Φ given by $\nabla_j \nabla_i \Phi = \rho \Phi G_{ji}$, which is a special case of (1.3). In § 4, we discuss some relations between Φ^i and $\nabla_k R_{ji}$, and others in R^{n+1} admitting the scalar field Φ , where $\Phi_j = \nabla_j \Phi$ and $\Phi^i = \Phi_j G^{ji}$. Our aim of § 5 is to give a generalization without the above mentioned restriction for a closed orientable hypersurface V^n whose second mean curvature H_2 is positive constant. In R^{n+1} admitting a conformal Killing vector field ξ^i , we apply the integral formulas obtained in § 3 to a closed orientable hypersurface V^n whose H_2 is positive constant, and give a generalization of Katsurada's Theorem. Moreover, in R^{n+1} admitting a special concircular scalar field Ψ given by $\nabla_j \nabla_i \Psi = (\rho \Psi + \sigma) G_{ji}$, in § 6 we similarly apply the integral formulas obtained in § 3 to a closed orientable hypersurface V^n whose first mean curvature H_1 is non zero constant, and prove Theorem 1.2. In the last section 7, making use of the results obtained in § 5 and § 6, we give certain conditions for a closed orientable hypersurface V^n in R^{n+1} to be isometric to a sphere.

2 Notation and general formulas

We consider an $(n + 1)$ -dimensional Riemannian manifold R^{n+1} ($n \geq 2$) of class C^r ($r \geq 3$) with local coordinates x^i , which admits a proper conformal Killing vector field ξ^i , that is, a vector field generating a local one-parameter group G of conformal transformations. In R^{n+1} , we consider a domain M , and if the domain M is simply covered by the orbits of transformations generated by ξ^i , then we call M a *regular domain* with respect to the vector field ξ^i .

The conformal Killing vector field ξ^i satisfies the conformal Killing equation (1.1), say,

$$(2.1) \quad \mathcal{L}_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

for a non constant associated scalar field Ψ in R^{n+1} , where $\xi_i = \xi^j G_{ji}$ and G_{ji} is the metric tensor of R^{n+1} .

Let us denote by V^n an n -dimensional closed orientable hypersurface imbedded in the regular domain M and locally given by

$$x^i = x^i(u^a) \quad i = 1, 2, \dots, n+1; \quad a = 1, 2, \dots, n,$$

where u^a are local coordinates of V^n . Throughout the present paper, the indices $i, j, k, \dots$ run from 1 to $n+1$ and the indices $a, b, c, \dots$ from 1 to n .

If we put

$$(2.2) \quad B_a^i = \partial x^i / \partial u^a,$$

then B_a^i ($a = 1, 2, \dots, n$) are n linearly independent vectors tangent to V^n and the first fundamental tensor g_{ba} of V^n is given by

$$(2.3) \quad g_{ba} = G_{ji} B_b^j B_a^i.$$

We assume that n vectors $B_1^i, B_2^i, \dots, B_n^i$ give the positive orientation on V^n , and we denote by C^i the unit normal vector to V^n such that

$$B_1^i, B_2^i, \dots, B_n^i, C^i$$

give the positive orientation in R^{n+1} .

Denoting by ∇_b the operator of the van der Waerden-Bortolotti covariant differentiation along V^n [17], we can write the equations of Gauss and Weingarten in the form

$$(2.4) \quad \nabla_b B_a^i = h_{ba} C^i,$$

$$(2.5) \quad \nabla_b C^i = -h_b^a B_a^i$$

respectively, where h_{ba} is the second fundamental tensor of V^n and $h_b^a = h_{bc}g^{ca}$. Also, the equations of Gauss and those of Codazzi are written as follows:

$$(2.6) \quad K_{dcba} - (h_{da}h_{cb} - h_{ca}h_{db}) = R_{kjih}B_d^k B_c^j B_b^i B_a^h,$$

$$(2.7) \quad \nabla_c h_{ba} - \nabla_b h_{ca} = R_{kjih}B_c^k B_b^j B_a^i C^h,$$

where K_{dcba} is the curvature tensor of V^n and R_{kjih} the curvature tensor of R^{n+1} . Transvecting g^{ba} to (2.7) and making use of $g^{ba}B_b^j B_a^i = G^{ji} - C^j C^i$, we find that

$$(2.8) \quad \nabla_c h_b^b - \nabla_b h_c^b = R_{kj}B_c^k C^j,$$

where $h_b^b = h_{ba}g^{ba}$ and $R_{kh} = R_{kjih}G^{ji}$.

Now, if we denote by $k_1, k_2, \dots, k_n$ the principal curvatures of V^n , that is, the roots of the characteristic equation

$$\det(h_{ba} - kg_{ba}) = 0,$$

the ν -th mean curvature H_ν of V^n is defined to be the ν -th elementary symmetric function of $k_1, k_2, \dots, k_n$ divided by the number of terms, that is,

$$\binom{n}{\nu} H_\nu = \sum_{c_1 < c_2 < \dots < c_\nu} k_{c_1} k_{c_2} \dots k_{c_\nu} \quad (1 \leq \nu \leq n).$$

Accordingly, H_1, H_2 and H_3 satisfy the following relations:

$$(2.9) \quad nH_1 = \sum_c k_c = h_a^a,$$

$$(2.10) \quad \binom{n}{2} H_2 = \sum_{d < c} k_d k_c = \frac{1}{2} \{ (h_a^a)^2 - h_a^b h_b^a \}$$

and

$$(2.11) \quad \binom{n}{3} H_3 = \sum_{f < e < d} k_f k_e k_d = \frac{1}{3!} \{ (h_a^a)^3 + 2h_a^b h_b^c h_c^a - 3h_a^a (h_b^c h_c^b) \}.$$

3 Integral formulas for a closed orientable hypersurface in R^{n+1}

We now assume that the hypersurface V^n under consideration is orientable and closed.

Let the group G be conformal, that is, ξ^i satisfy the equation

$$(3.1) \quad \mathcal{L}_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji}.$$

On the hypersurface V^n , we can put as follows

$$\xi^i = B_a^i \beta^a + \theta C^i$$

for a some vector field β^a and scalar field θ on V^n . Since $G_{ji} C^j B_a^i = 0$, it follows that

$$(3.2) \quad \theta = C^i \xi_i$$

and

$$(3.3) \quad \beta_a = B_a^i \xi_i,$$

where $\beta_a = \beta^b g_{ba}$ and $\xi_i = \xi^j G_{ji}$. We differentiate covariantly the above equation (3.3) along V^n , and, making use of (2.4) and (3.2), we get

$$\nabla_b \beta_a = h_{ba} \theta + \nabla_j \xi_i B_b^j B_a^i.$$

Transvecting g^{ba} to this equation and using (2.9) and (3.1), we have

$$(3.4) \quad \nabla_b \beta^b = n H_1 \theta + \frac{1}{2} g^{ba} B_b^j B_a^i \mathcal{L}_\xi G_{ji},$$

where $\nabla_b \beta^b = \nabla_b \beta_a g^{ba}$.

If we now put

$$\gamma_b = 2p_b^a B_a^i \xi_i,$$

where the symbols p_{ba} are the components of the symmetric tensor of V^n defined by

$$(3.5) \quad p_{ba} = h_c^c h_{ba} - h_b^c h_{ca},$$

and $p_b^a = p_{bc} g^{ca}$, then we have, by covariant differentiation along V^n and by virtue of (3.3), (2.4) and (3.2),

$$\nabla_c \gamma_b = 2\nabla_c p_b^a \beta_a + 2p_b^a h_{ac} \theta + 2p_b^a B_c^k B_a^i \nabla_k \xi_i.$$

And, transvecting g^{cb} to this equation and using (3.1), we get

$$(3.6) \quad \nabla_c \gamma^c = 2\nabla_c p_a^c \beta^a + 2p_c^a h_a^c \theta + p^{ca} B_c^k B_a^i \mathcal{L}_\xi G_{ki}.$$

On the other hand, from (2.9) and (2.10), the relation (2.11) is rewritten as follows:

$$(3.7) \quad 2p_c^a h_a^c = n(n-1) \{nH_1H_2 - (n-2)H_3\}.$$

Also, from (2.10) and the definition of p_{ba} , we can see easily that

$$(3.8) \quad p_a^a = n(n-1)H_2,$$

where $p_a^a = p_{ba}g^{ba}$. Consequently, by substituting (3.7) into (3.6), we find that

$$(3.9) \quad \nabla_c \gamma^c = 2\nabla_c p_a^c \beta^a + n(n-1) \{nH_1H_2 - (n-2)H_3\} \theta + p^{ca} B_c^k B_a^i \mathcal{L}_\xi G_{ki}.$$

Therefore, since the vector field ξ^i is conformal Killing, that is, $\mathcal{L}_\xi G_{ji} = 2\Psi G_{ji}$, from (3.4) and (3.9), we find that

$$(3.10) \quad \nabla_b \beta^b = n(H_1\theta + \Psi)$$

and

$$(3.11) \quad \nabla_c \gamma^c = 2\nabla_c p_a^c \beta^a + n(n-1) \{nH_1H_2 - (n-2)H_3\} \theta + 2n(n-1)\Psi H_2,$$

by virtue of (2.3) and (3.8). And consequently, since the hypersurface V^n is orientable and closed, we apply Green's formula [22] to (3.10) and (3.11). We then obtain

$$(3.12) \quad \int_{V^n} H_1 \theta \, dA + \int_{V^n} \Psi \, dA = 0$$

and

$$(3.13) \quad \frac{2}{n(n-1)} \int_{V^n} \nabla_c p_a^c \beta^a \, dA + \int_{V^n} \{nH_1H_2 - (n-2)H_3\} \theta \, dA + 2 \int_{V^n} \Psi H_2 \, dA = 0$$

respectively, where dA denotes the area element of V^n , and the integral formula (3.12) is due to Y. Katsurada [8, 9].

Next, we assume the existence of a special concircular scalar field, say, a non constant scalar one Φ which satisfies the partial differential equation defined by

$$(3.14) \quad \nabla_j \Phi_i = \rho \Phi G_{ji} \quad (\rho = \text{constant} \neq 0),$$

where $\Phi_i = \nabla_i \Phi$. For example, in an Einstein manifold with scalar curvature R ($\neq 0$) admitting a conformal Killing vector field ξ^i , we know the fact that the associated scalar field Ψ satisfies the equation (1.4), say,

$$\nabla_j \nabla_i \Psi = -\frac{R}{n(n+1)} \Psi G_{ji} \quad (R = \text{non zero const.}).$$

If $\Phi = 0$ on V^n , since the second covariant derivative of $\Phi = 0$ along V^n is given by

$$\nabla_j \Phi_i B_b^j B_a^i + \Phi_i \nabla_b B_a^i = 0,$$

substituting (3.14) and (2.4) into this equation and transvecting g^{ba} to the resulting equation, we can see that $H_1 \Theta = 0$ on V^n , where $\Theta = C^i \Phi_i$. Hence we have the following

Lemma 3.1 *Let R^{n+1} be a Riemannian manifold which admits the special concircular scalar field Φ . If, on a hypersurface V^n in R^{n+1} , $H_1\Theta$ is not identically zero, then Φ is not identically zero on the V^n .*

Now, on the hypersurface V^n , we can put

$$\Phi^j = B_b^j \phi^b + \Theta C^j,$$

where $\Phi^j = \Phi_i G^{ji}$. Transvecting $G_{ji} B_a^i$ to this equation and making use of (2.3), we get

$$(3.15) \quad \phi_a = B_a^i \Phi_i,$$

from which, by covariant differentiation along V^n and by virtue of (2.4), (3.14) and (2.3), we obtain

$$\nabla_b \phi_a = \Theta h_{ba} + \rho \Phi g_{ba}.$$

And, transvecting g^{ba} to this equation and making use of (2.9), we get

$$(3.16) \quad \nabla_b \phi^b = n(H_1\Theta + \rho\Phi),$$

where $\nabla_b \phi^b = \nabla_b \phi_a g^{ba}$.

We now put

$$\omega_b = h_b^a B_a^i \Phi_i,$$

from which, by covariant differentiation along V^n , we obtain, by virtue of (2.4), (3.14) and (2.3),

$$\nabla_c \omega_b = \nabla_c h_b^a B_a^i \Phi_i + h_b^a h_{ca} \Theta + \rho \Phi h_{bc}.$$

And, transvecting g^{bc} to this equation, we get

$$(3.17) \quad \nabla_c \omega^c = \nabla_c h_a^c \phi^a + h_c^a h_a^c \Theta + \rho \Phi h_c^c,$$

by virtue of (3.15). On the other hand, since

$$h_c^c = nH_1 \quad \text{and} \quad h_c^a h_a^c = n^2 H_1^2 - n(n-1)H_2,$$

by virtue of (2.9) and (2.10), we have, from (3.17),

$$\nabla_c \omega^c = \nabla_c h_a^c \phi^a + n\{nH_1^2 - (n-1)H_2\}\Theta + n\rho\Phi H_1,$$

and consequently, using (2.8), we obtain,

$$(3.18) \quad \nabla_c \omega^c = (\nabla_a h_c^c - R_{ij} B_a^i C^j) \phi^a + n\{nH_1^2 - (n-1)H_2\}\Theta + n\rho\Phi H_1,$$

And, since the hypersurface V^n under consideration is orientable and closed, we apply Green's formula [22] to (3.16) and (3.18). Then we obtain

$$(3.19) \quad \int_{V^n} H_1 \Theta dA + \int_{V^n} \rho \Phi dA = 0$$

and

$$(3.20) \quad \frac{1}{n} \int_{V^n} (\nabla_a h_c^c - R_{ij} B_a^i C^j) \phi^a dA + \int_{V^n} \{nH_1^2 - (n-1)H_2\} \Theta dA + \int_{V^n} \rho \Phi H_1 dA = 0$$

respectively [9], where dA denotes the area element of V^n .

4 Relations in a Riemannian manifold admitting the special concircular scalar field Φ

Let R^{n+1} be a Riemannian manifold which admits a special concircular scalar field Φ defined by (3.14). Substituting (3.14) into the Ricci identity

$$\nabla_k \nabla_j \Phi_i - \nabla_j \nabla_k \Phi_i = -R_{kji}^l \Phi_l,$$

we find that

$$(4.1) \quad R_{kji}^l \Phi_l = \rho(\Phi_j G_{ki} - \Phi_k G_{ji}),$$

from which, by covariant differentiation and (3.14), we obtain

$$(4.2) \quad \nabla_h R_{kji}^l \Phi_l = -\rho \Phi \{ R_{kjih} - \rho(G_{ki} G_{jh} - G_{kh} G_{ji}) \}.$$

So, transvecting G^{ji} to this equation, we obtain

$$(4.3) \quad \nabla_h R_{kl} \Phi^l = -\rho \Phi (R_{kh} + n \rho G_{kh}),$$

where R_{kh} is the Ricci tensor of R^{n+1} , and if we put

$$(4.4) \quad S_{kh} = R_{kh} + n \rho G_{kh},$$

then the tensor S_{kh} is symmetric in k and h , and consequently, (4.3) is rewritten as follows:

$$(4.5) \quad \nabla_h R_{kl} \Phi^l = -\rho \Phi S_{hk}.$$

Moreover, transvecting G^{hk} to this equation and making use of $\nabla_h R^h_l = (1/2) \nabla_l R$, we get

$$(4.6) \quad \nabla_l R \Phi^l = -2\rho \Phi S,$$

where $S = S_{hk} G^{hk}$, and $R = R_{hk} G^{hk}$, say, R is the scalar curvature of R^{n+1} . Also, transvecting G^{hk} to (4.4), we obtain

$$(4.7) \quad S = R + n(n+1)\rho.$$

Therefore, using (4.6), (4.7) and Lemma 3.1, we have

Lemma 4.1 *Let R^{n+1} be a Riemannian manifold which admits a special concircular scalar field Φ such that $\nabla_j \Phi_i = \rho \Phi G_{ji}$ ($\rho = \text{constant} \neq 0$). If its scalar curvature R is non zero constant and $H_1 \Theta$ has fixed sign on V^n , we have*

$$\rho = -\frac{R}{n(n+1)}.$$

Next, transvecting G^{ji} to (4.1), we get

$$R_{kl}\Phi^l + n\rho\Phi_k = 0.$$

Thus, from (4.4), we have

$$(4.8) \quad S_{kl}\Phi^l = 0.$$

Now, from $R_{kji}l = R_{lijk}$, the left-hand side of (4.2) is equal to $\nabla_h R_{lijk}\Phi^l$. Thus, transvecting G^{hk} to (4.2), we get, from (4.4),

$$(4.9) \quad \nabla_h R_{lij}{}^h \Phi^l = -\rho\Phi S_{ij}.$$

On the other hand, transvecting G^{hk} to the Bianchi's identity, say, $\nabla_h R_{lijk} + \nabla_l R_{ihjk} + \nabla_i R_{hljk} = 0$, we find that

$$(4.10) \quad \nabla_h R_{lij}{}^h = \nabla_l R_{ij} - \nabla_i R_{lj},$$

and consequently, transvecting Φ^l to this equation, we get, from (4.5) and (4.9),

$$(4.11) \quad \nabla_l R_{ij}\Phi^l = -2\rho\Phi S_{ij}.$$

Lemma 4.2 *Let R^{n+1} be a Riemannian manifold which admits a special consircular scalar field Φ . If the scalar field Φ satisfies the following equation:*

$$(4.12) \quad \Phi S_{kj} = 0,$$

then we have

$$(\Phi_l\Phi^l)S_{kj} = 0 \quad \text{in } R^{n+1}.$$

PROOF. Covariantly differentiating (4.12), we get, from (4.4),

$$\Phi_l S_{kj} + \Phi \nabla_l R_{kj} = 0.$$

Transvecting Φ^l to this equation and using of (4.10), we obtain

$$(\Phi_l\Phi^l)S_{kj} - 2\rho\Phi^2 S_{kj} = 0,$$

from which, taking account of the assumption (4.12), we conclude that Lemma 4.2 holds good.

5 A closed hypersurface with $H_2 = \text{const.} > 0$ in R^{n+1} admitting a conformal Killing vector field ξ^i

From (2.9) and (2.10), it follows that

$$(5.1) \quad n^2 H_1^2 = n(n-1)H_2 + h_a^b h_b^a.$$

If we assume that the second mean curvature H_2 is positive constant, then the left hand member of (5.1) is positive. Hence, there exists no point P on V^n such that $H_1(P) = 0$. Accordingly, H_1 must have fixed sign on V^n . Therefore we have

Lemma 5.1 *If the second mean curvature H_2 of V^n is positive constant, then the first mean curvature H_1 of V^n has fixed sign on V^n .*

We now assume that R^{n+1} is an Einstein manifold, say, $R_{kj} = (R/(n+1))G_{kj}$, where $R_{kj} (= R_{lkji}G^{li})$ and $R (= R_{kj}G^{kj})$ are the Ricci tensor and the scalar curvature of R^{n+1} respectively. Transvecting g^{da} to the equations of Gauss (2.6), we have

$$K_{cb} = (h_a^a h_{cb} - h_c^a h_{ab}) + R_{kjih} B_d^k B_c^j B_b^i B_a^h g^{da},$$

where K_{cb} is the Ricci tensor of V^n . Making use of (3.5) and $B_d^k B_a^h g^{da} = G^{kh} - C^k C^h$, we can write in the form

$$K_{cb} = p_{cb} + R_{ji} B_c^j B_b^i - R_{kjih} C^k B_c^j B_b^i C^h.$$

Accordingly, since R^{n+1} is Einstein, we obtain

$$K_{cb} = p_{cb} + \frac{R}{n+1} g_{cb} - R_{kjih} C^k B_c^j B_b^i C^h.$$

Moreover, transvecting g^{cb} to this equation, we have

$$K = p_b^b + \frac{nR}{n+1} - R_{kh} C^k C^h,$$

where K is the scalar curvature of V^n and $p_b^b = p_{cb} g^{cb}$. Consequently, since R^{n+1} is an Einstein manifold and $p_b^b = n(n-1)H_2$, we get

$$K = n(n-1)H_2 + \frac{n-1}{n+1}R.$$

Therefore, making use of the fact that the scalar curvature R in an Einstein manifold R^{n+1} is constant, we finally reach the following

Lemma 5.2 *Let V^n be a hypersurface in an Einstein manifold R^{n+1} . Then, necessary and sufficient condition that the second mean curvature H_2 be constant is that the scalar curvature K of V^n be constant.*

Next, as a special case of the Lemma, let R^{n+1} be a Riemannian manifold of constant curvature κ , say, $R_{kjih} = \kappa(G_{kh}G_{ji} - G_{ki}G_{jh})$, where $\kappa = R/n(n+1)$. Then, using (2.3), the equations of Gauss (2.6) is written in the form

$$K_{dcba} = h_{da}h_{cb} - h_{ca}h_{db} + \kappa(g_{da}g_{cb} - g_{ca}g_{db}).$$

Similarly, transvecting g^{da} to this equation, we have

$$K_{cb} = p_{cb} + (n-1)\kappa g_{cb},$$

from which, by covariant differentiation along V^n , we get

$$\nabla_a K_{cb} = \nabla_a p_{cb} + (n-1)\nabla_a \kappa g_{cb},$$

and consequently, since κ is constant, we obtain

$$\nabla_a K_{cb} = \nabla_a p_{cb}.$$

Moreover, transvecting g^{ab} to this equation, we have

$$(5.2) \quad \nabla_b K_c^b = \nabla_b p_c^b.$$

On the other hand, as is well-known, the following equation is valid for any Riemannian manifolds:

$$(5.3) \quad \nabla_b K_c^b = \frac{1}{2} \nabla_c K.$$

Accordingly, from (5.2) and (5.3), we get

$$\nabla_b p_c^b = \frac{1}{2} \nabla_c K.$$

Consequently, from Lemma 5.2, we have

Lemma 5.3 *Let V^n be a hypersurface in a Riemannian manifold of constant curvature. If the second mean curvature H_2 of V^n is constant, then $\nabla_b p_c^b = 0$ on V^n .*

Now we can prove the following

Theorem 5.4 *Let R^{n+1} ($n \geq 3$) be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) H_2 is positive constant,
 - (2) $\theta (= C^i \xi_i)$ has fixed sign on V^n ,
- then every point of V^n is umbilic.*

PROOF. Multiplying (3.12) by $2H_2$ (=positive constant), we obtain

$$\int_{V^n} 2H_1H_2\theta \, dA + 2 \int_{V^n} \Psi H_2 \, dA = 0,$$

and subtracting the above formula from (3.13), we find that

$$\frac{2}{n(n-1)} \int_{V^n} \nabla_c p_a^c \beta^a \, dA + (n-2) \int_{V^n} \{H_1H_2 - H_3\} \theta \, dA = 0.$$

So, from Lemms 5.3 and $n \geq 3$, we have

$$(5.4) \quad \int_{V^n} \{H_1H_2 - H_3\} \theta \, dA = 0.$$

Moreover, from Lemma 5.1, that is, from the fact that H_1 has fixed sign on V^n , the scalar field on V^n defined by $H_1H_2 - H_3$ is rewritten as follows:

$$(5.5) \quad H_1H_2 - H_3 = \frac{1}{H_1} \{H_2(H_1^2 - H_2) + (H_2^2 - H_1H_3)\}.$$

On the other hand, we know the fact that

$$H_a^2 - H_{a-1}H_{a+1} \geq 0 \quad (a = 1, 2, \dots, n-1) \quad ([5, 1]),$$

where $H_0 = 1$. In particular, we see that

$$(5.6) \quad H_1^2 - H_2 \geq 0 \quad \text{and} \quad H_2^2 - H_1H_3 \geq 0 \quad \text{on } V^n.$$

Accordingly, making use of the assumption (1) and Lemma 5.1, we have, from (5.5),

$$H_1H_2 - H_3 \geq 0 \quad (\text{or } \leq 0) \quad \text{on } V^n.$$

Hence, using the assumption (2), from (5.4), we find that

$$H_1H_2 - H_3 = 0 \quad \text{on } V^n.$$

Consequently, from (5.5), it follows that

$$H_2(H_1^2 - H_2) + (H_2^2 - H_1H_3) = 0 \quad \text{on } V^n,$$

from which, from (5.6) and the assumption (1), we see that

$$H_1^2 - H_2 = 0 \quad \text{on } V^n.$$

Therefore, since

$$H_1^2 - H_2 = \frac{1}{n^2(n-1)} \sum_{a < b} (k_b - k_a)^2,$$

we conclude that

$$k_1 = k_2 = \dots = k_n$$

at each point of V^n . (Then we have $H_1 = k_1$, $H_2 = k_1^2$ and $H_3 = k_1^3$, from which we obtain $H_2^2 - H_1H_3 = 0$.) This means that each point of V^n is umbilic.

6 Closed hypersurfaces with $H_1 = \text{const.} \neq 0$ in R^{n+1} admitting a special concircular scalar field Ψ

Now, in R^{n+1} , we assume the existence of a special concircular scalar field, say, a non constant scalar one Ψ which satisfies the following partial differential equation

$$(6.1) \quad \nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}),$$

where $\Psi_i = \nabla_i \Psi$.

If we put

$$\bar{\Phi} = \rho \Psi + \sigma,$$

then (6.1) becomes

$$(6.2) \quad \nabla_j \Psi_i = \bar{\Phi} G_{ji},$$

and, by covariant differentiation, we have

$$(6.3) \quad \bar{\Phi}_i = \rho \Psi_i,$$

where $\bar{\Phi}_i = \nabla_i \bar{\Phi}$. Moreover, by covariant differentiation of (6.3), from (6.2), we find that

$$\nabla_j \bar{\Phi}_i = \rho \bar{\Phi} G_{ji},$$

that is, the scalar field $\bar{\Phi}$ satisfies the same partial differential equation as Φ given by (3.14). Therefore, if we put anew

$$(6.4) \quad \Phi = \rho \Psi + \sigma,$$

then, transvecting C^i to (6.3), we have

$$(6.5) \quad C^i \Phi_i = \rho C^i \Psi_i \quad \text{on } V^n,$$

from which we get the following

Lemma 6.1 *If $C^i \Psi_i (= \Omega)$ has fixed sign on V^n , and is not constant along V^n , then the same holds good also for $C^i \Phi_i (= \Theta)$ respectively.*

Thus, without loss of generality we may discuss the properties of a hypersurface V^n in R^{n+1} admitting a concircular scalar field Φ defined by (3.14), that is,

$$\nabla_j \Phi_i = \rho \Phi G_{ji} \quad (\rho = \text{const.} \neq 0).$$

Now, if we assume that the first mean curvature of V^n is non zero constant, say, $H_1 = \text{const.} \neq 0$, then, taking account of (2.9), we obtain, from (3.20),

$$(6.6) \quad -\frac{1}{n} \int_{V^n} R_{ij} B_a^i C^j \phi^a dA + \int_{V^n} \{nH_1^2 - (n-1)H_2\} \Theta dA + H_1 \int_{V^n} \rho \Phi dA = 0.$$

Eliminating $\int_{V^n} \rho \Phi dA$ from (3.19) and (6.6), we find that

$$(6.7) \quad -\frac{1}{n} \int_{V^n} R_{ij} B_a^i C^j \phi^a dA + (n-1) \int_{V^n} (H_1^2 - H_2) \Theta dA = 0.$$

First, we shall prove the following Theorem:

Theorem 6.2 *Let R^{n+1} be an orientable Riemannian manifold with harmonic curvature, say, $\nabla_k R_{ji} = \nabla_j R_{ki}$ which admits a special concircular scalar field Ψ satisfying the equation*

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}),$$

and V^n a closed orientable hypersurface in R^{n+1} such that

$$(1) \quad H_1 = \text{const.} \neq 0,$$

$$(2) \quad \Omega \text{ has fixed sign on } V^n, \text{ where } \Omega = C^i \Psi_i.$$

Then every point of V^n is umbilic.

PROOF. Transvecting Φ^k to the assumption $\nabla_k R_{ji} = \nabla_j R_{ki}$, we get, from (4.11) and (4.5), $\Phi S_{ji} = 0$. Thus, using Lemma 4.2, we have $(\Phi_k \Phi^k) S_{ji} = 0$ in R^{n+1} . Accordingly, making use of the assumption (2) and Lemma 6.1, we can see that $S_{ji} = 0$ on V^n , say, $R_{ji} = -n\rho G_{ji}$ on V^n . Consequently, taking account of the assumption (1), we obtain, from (6.7),

$$(6.8) \quad \int_{V^n} (H_1^2 - H_2) \Theta dA = 0.$$

Also, since

$$(6.9) \quad H_1^2 - H_2 = \frac{1}{n^2(n-1)} \sum_{a < b} (k_b - k_a)^2,$$

we can see easily that $H_1^2 - H_2 \geq 0$. Thus, using the assumption (2) and Lemma 6.1, we conclude that $H_1^2 - H_2 = 0$ by virtue of (6.8), and therefore, because of (6.9), that

$$k_1 = k_2 = \dots = k_n$$

at each point of V^n , that is, every point of V^n is umbilic.

So, we have the following

Corollary 6.3 *Let R^{n+1} be an orientable Riemannian manifold with parallel Ricci tensor, say, $\nabla_k R_{ji} = 0$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

$$(1) \quad H_1 = \text{const.} \neq 0,$$

$$(2) \quad \Omega \text{ has fixed sign on } V^n.$$

Then every point of V^n is umbilic.

Next, in a conformally flat Riemannian manifold R^{n+1} ,

$$R_{kji}^h = -\frac{1}{n-1}(R_{ki}\delta_j^h - R_{ji}\delta_k^h + G_{ki}R_j^h - G_{ji}R_k^h) + \frac{R}{n(n-1)}(G_{ki}\delta_j^h - G_{ji}\delta_k^h).$$

By covariant differentiation, we have

$$\nabla_l R_{kji}^h = -\frac{1}{n-1}(\nabla_l R_{ki}\delta_j^h - \nabla_l R_{ji}\delta_k^h + G_{ki}\nabla_l R_j^h - G_{ji}\nabla_l R_k^h) + \frac{\nabla_l R}{n(n-1)}(G_{ki}\delta_j^h - G_{ji}\delta_k^h),$$

from which, replacing l by h and summing for h , we have

$$\nabla_h R_{kji}^h = -\frac{1}{n-1}(\nabla_j R_{ki} - \nabla_k R_{ji} + G_{ki}\nabla_h R_j^h - G_{ji}\nabla_h R_k^h) + \frac{1}{n(n-1)}(G_{ki}\nabla_j R - G_{ji}\nabla_k R).$$

And, making use of (4.10) and $\nabla_h R_j^h = (1/2)\nabla_j R$, we find that

$$(6.10) \quad \nabla_j R_{ki} - \nabla_k R_{ji} - \frac{1}{2n}(G_{ki}\nabla_j R - G_{ji}\nabla_k R) = 0 \quad (n > 2).$$

Also, in case $n = 2$, a conformally flat Riemannian manifold is defined by (6.10). Therefore, assuming the scalar curvature R of R^{n+1} to be constant, we see easily, from (6.10), that $\nabla_j R_{ki} - \nabla_k R_{ji} = 0$. Thus, using Theorem 6.2, we have the following

Corollary 6.4 *Let R^{n+1} ($n \geq 2$) be an orientable conformally flat Riemannian manifold with $R = \text{const.}$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .

Then every point of V^n is umbilic,

Moreover, making use of (4.5) and (4.11), we can prove the following Theorem by an argument similar to that used in the proof of Theorem 6.2:

Theorem 6.5 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .

Then every point of V^n is umbilic.

We now assume that the length of the Ricci tensor R_{ji} is constant in R^{n+1} , that is,

$$(6.11) \quad R^{ji}R_{ji} = \lambda \quad (\lambda = \text{const.}).$$

By covariant differentiation, we have

$$(6.12) \quad \nabla_k R^{ji}R_{ji} = 0.$$

On the other hand, from (4.4), (4.7) and (6.11), we can see that

$$(6.13) \quad S^{ji}S_{ji} = 2n\rho S - n^2(n+1)\rho^2 + \lambda.$$

Also, transvecting $\nabla_k R^{ji}$ to $S_{ji} = R_{ji} + n\rho G_{ji}$, say, (4.4) and using (6.12), we have

$$\nabla_k R^{ji}S_{ji} = n\rho \nabla_k R,$$

and, moreover, transvecting Φ^k to this equation, from (4.11) and (4.6), we find that

$$(6.14) \quad \Phi S^{ji}S_{ji} = n\rho \Phi S.$$

Consequently, substituting (6.13) into (6.14), we can see easily that

$$(6.15) \quad \Phi \{n\rho S - n^2(n+1)\rho^2 + \lambda\} = 0.$$

So, covariantly differentiating and using the fact that $\nabla_i S$ is equal to $\nabla_i R$, we have

$$\Phi_i \{n\rho S - n^2(n+1)\rho^2 + \lambda\} + n\rho \Phi \nabla_i R = 0,$$

from which, multiplying by Φ and using (6.15), we obtain

$$(6.16) \quad \Phi^2 \nabla_i R = 0.$$

Lemma 6.6 *Let R^{n+1} be a Riemannian manifold with $R^{ji}R_{ji} = \text{const.}$ which admits a special concircular scalar field Ψ , V^n a hypersurface in R^{n+1} and Ω fixed sign on V^n . If $\mu \neq 0$, then we have $\nabla_i R = 0$ in R^{n+1} , and if $\mu = 0$, then we have $\nabla_i R = 0$ on V^n , where μ is a constant number defined by*

$$(6.17) \quad \mu = \rho\Phi^2 - \Phi_j\Phi^j.$$

PROOF. Since $\nabla_i(\rho\Phi^2 - \Phi_j\Phi^j) = 0$, we see that μ is constant in R^{n+1} .

Now, by covariant differentiation of (6.16), we get

$$2\Phi\Phi_j\nabla_i R + \Phi^2\nabla_j\nabla_i R = 0,$$

from which, using the fact that the tensor $\nabla_j\nabla_i R$ is symmetric, we obtain $\Phi(\Phi_j\nabla_i R - \Phi_i\nabla_j R) = 0$. Moreover, covariantly differentiating and using (3.14), we have

$$\Phi_k(\Phi_j\nabla_i R - \Phi_i\nabla_j R) + \rho\Phi^2(G_{kj}\nabla_i R - G_{ki}\nabla_j R) + \Phi(\Phi_j\nabla_k\nabla_i R - \Phi_i\nabla_k\nabla_j R) = 0,$$

and accordingly, using (6.16), we obtain

$$\Phi_k(\Phi_j\nabla_i R - \Phi_i\nabla_j R) + \Phi(\Phi_j\nabla_k\nabla_i R - \Phi_i\nabla_k\nabla_j R) = 0.$$

And consequently, transvecting G^{kj} to this equation, we have, from (4.6), (6.17) and (6.16),

$$(6.18) \quad -\mu\nabla_i R + \Phi\{\Phi^j\nabla_j\nabla_i R + \Phi_i(2\rho S - \nabla^j\nabla_j R)\} = 0,$$

where $\nabla^j\nabla_j R = G^{kj}\nabla_k\nabla_j R$. Thus, if $\mu \neq 0$, then we have, from (6.16) at a point P_1 such that $\Phi(P_1) \neq 0$, or from (6.18) at a point P_2 such that $\Phi(P_2) = 0$, $\nabla_i R = 0$. And if $\mu = 0$, then we have, from (6.17), $\rho\Phi^2 = \Phi_j\Phi^j$. On the other hand, using the assumption that Ω has fixed sign on V^n , we can see, from Lemma 6.1, that Θ has fixed sign on V^n , and accordingly, $\Phi_j\Phi^j > 0$ on V^n . Thus, it follows from (6.16) that $\nabla_i R = 0$ on V^n .

Lemma 6.7 *Let R^{n+1} be an orientable Riemannian manifold which admits a special concircular scalar field Ψ and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .

If $\nabla_i R = 0$ on V^n , then we have $S = 0$ on V^n .

PROOF. Transvecting B_a^i to $\nabla_i R = 0$, we get $\nabla_a R = 0$, that is, $R = \text{const.}$ on V^n . Consequently, because of (4.7), we have

$$(6.19) \quad S = \text{const.} \quad \text{on } V^n.$$

Moreover, using the assumption that $\nabla_i R = 0$ on V^n , we get, from (4.6),

$$\Phi S = 0 \quad \text{on } V^n.$$

Here Φ is not identically zero on V^n . Because, using the assumption (2) and Lemma 6.1, we can see that Θ has fixed sign on V^n , and accordingly, $H_1\Theta$ is not identically zero on V^n , which shows that this fact holds because of Lemma 3.1. Thus, taking account of (6.19), we can prove that $S = 0$ on V^n .

Next, making use of these Lemmas, we shall prove the following

Theorem 6.8 *Let R^{n+1} be an orientable Riemannian manifold with $R^{ji}R_{ji} = \text{const.}$ which admits a special concircular scalar field Ψ and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .

Then every point of V^n is umbilic.

PROOF. Transvectin Φ^k to (6.12), and using (4.11) and (4.4), we obtain

$$\Phi(S^{ji}S_{ji} - n\rho S) = 0.$$

Now, by covariant differentiation, we get, from (4.4) and (4.7),

$$\Phi_k(S^{ji}S_{ji} - n\rho S) + \Phi(2\nabla_k R^{ji}S_{ji} - n\rho\nabla_k R) = 0,$$

and, substituting (4.4) into the second term of the left-hand side of this equation and using (6.12), we can see that

$$\Phi_k(S^{ji}S_{ji} - n\rho S) + n\rho\Phi\nabla_k R = 0.$$

Thus, making use of Lemma 6.6 and Lemma 6.7, and transvecting Φ^k to the result equation, it follows that

$$(\Phi_k\Phi^k)(S^{ji}S_{ji}) = 0 \quad \text{on } V^n,$$

and, since, taking account of the assumption that Ω has fixed sign on V^n , we can prove from Lemma 6.1 that $\Phi_k \Phi^k > 0$ on V^n , we conclude that

$$S_{ji} = 0, \quad \text{that is,} \quad R_{ji} = -n\rho G_{ji} \quad \text{on } V^n.$$

Therefore, from (6.7), we obtain

$$\int_{V^n} (H_1^2 - H_2) \Theta dA = 0,$$

and consequently, by the argument similar to that used in the proof of Theorem 6.2, we can prove that Theorem 6.8 holds.

REMARK 1. Theorem 6.8 is a generalization of Corollary 6.3, because of the fact that $\nabla_k(R_{ji}R^{ji}) = 2\nabla_k R_{ji}R^{ji}$.

Moreover, as a generalization of Corollary 6.4, we have the following

Theorem 6.9 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*
- (3) Ω has fixed sign on V^n .

Then every point of V^n is umbilic.

PROOF. In a conformally flat Riemannian manifold R^{n+1} ,

$$R_{kji}^h = -\frac{1}{n-1}(R_{ki}\delta_j^h - R_{ji}\delta_k^h + G_{ki}R_j^h - G_{ji}R_k^h) + \frac{R}{n(n-1)}(G_{ki}\delta_j^h - G_{ji}\delta_k^h).$$

Using the same calculations as the proof of Corollary 6.4, we obtain (6.10), that is,

$$\nabla_j R_{ki} - \nabla_k R_{ji} - \frac{1}{2n}(G_{ki}\nabla_j R - G_{ji}\nabla_k R) = 0. \quad (n \geq 2)$$

Now, transvecting $2n\Phi^k$ to this equation and making use of (4.5), (4.11) and (4.6), we get

$$(6.20) \quad 2n\rho\Phi S_{ji} - (\nabla_j R\Phi_i + 2\rho\Phi S G_{ji}) = 0.$$

Moreover, transvecting Φ^i to this equation and making use of (4.8), we have

$$(6.21) \quad \nabla_j R\Phi_i\Phi^i + 2\rho\Phi S\Phi_j = 0.$$

And consequently, making use of (3.14) and (4.7), (6.21) is rewritten as follows:

$$(6.22) \quad \nabla_j(S\Phi_i\Phi^i) = 0,$$

from which, by the assumptions that there exists a point P_0 on V^n such that $S(P_0) = 0$, and the hypersurface V^n is closed, we find that

$$(6.23) \quad S\Phi_i\Phi^i = 0$$

on V^n .

On the other hand, transvecting S^{ji} to (6.20) and making use of (4.8), we obtain

$$(6.24) \quad \Phi(nS_{ji}S^{ji} - S^2) = 0.$$

By covariant differentiation, we have, from (4.4) and (4.7),

$$\Phi_h(nS_{ji}S^{ji} - S^2) + 2\Phi(n\nabla_h R_{ji}S^{ji} - \nabla_h RS) = 0.$$

And, transvecting Φ^h to this equation, from (4.11), (4.6) and (6.24), we find that

$$(6.25) \quad \Phi_h\Phi^h(nS_{ji}S^{ji} - S^2) = 0.$$

Thus, making use of (6.23), we have $\Phi_h\Phi^h S_{ji}S^{ji} = 0$ on V^n . And, from the assumption (3), we find that $S_{ji} = 0$ on V^n , that is, $R_{ji} = -n\rho G_{ji}$ on V^n . Consequently, from (6.7), we obtain

$$(6.26) \quad \int_{V^n} \{H_1^2 - H_2\} \Theta dA = 0.$$

Also, we can see that $H_1^2 - H_2 \geq 0$, because

$$(6.27) \quad H_1^2 - H_2 = \frac{1}{n^2(n-1)} \sum_{b < a} (k_b - k_a)^2,$$

Thus, using the assumption (3) and Lemma 6.1, we find that $H_1^2 - H_2 = 0$ by virtue of (6.26), and consequently, because of (6.27), we conclude that $k_1 = k_2 = \dots = k_n$ at each point of V^n . This means that every point of V^n is umbilic.

REMARK 2. Theorem 6.9 is a generalization of Corollary 6.4. Because, using the assumption that $R = \text{const.}$ and (4.6), we have $\Phi S = 0$, from which, taking account of Lemma 3.1, we can see that there exists a point P_0 on V^n such that $S(P_0) = 0$.

7 Some characterizations of a hypersurface to be isometric to a sphere

To prove that the hypersurface under consideration is isometric to a sphere, we need the following Theorem due to M. Obata [16].

Theorem A. (Obata) *Let V^n ($n \geq 2$) be a complete Riemannian manifold which admits a non-null function ψ such that $\nabla_b \nabla_a \psi = -c^2 \psi g_{ba}$ ($c = \text{const.}$), where g_{ba} and ∇_a denote the metric tensor of V^n and the operator of covariant differentiation with respect to Christoffel symbols $\left\{ \begin{smallmatrix} c \\ ba \end{smallmatrix} \right\}$ formed with g_{ba} respectively. Then V^n is isometric to a sphere of radius $1/c$.*

Now, using Theorem 6.2, we can prove the following

Theorem 7.1 *Let R^{n+1} be an orientable Riemannian manifold with harmonic curvature, say, $\nabla_k R_{ji} = \nabla_j R_{ki}$ which admits a special concircular scalar field Ψ satisfying the equation*

$$\nabla_j \Psi_i = (\rho \Psi + \sigma) G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}),$$

and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n , where $\Omega = C^i \Psi_i$,
- (3) Ω is not constant along V^n .

Then V^n is isometric to a sphere.

PROOF. By covariant differentiation of the definition $\Theta = C^i \Phi_i$ along V^n , we have, from (2.5) and (3.14),

$$(7.1) \quad \nabla_b \Theta = -h_b^a B_a^i \Phi_i.$$

Also, by virtue of Theorem 6.2, every point of V^n is umbilic, that is,

$$(7.2) \quad h_{bc} = H_1 g_{bc}.$$

Transvecting g^{ca} to this equation, we see that $h_b^a = H_1 \delta_b^a$. So, substituting this equation into (7.1), we have

$$(7.3) \quad \nabla_b \Theta = -H_1 B_b^i \Phi_i,$$

that is,

$$(7.4) \quad \nabla_b \Theta + H_1 \nabla_b \Phi = 0.$$

Accordingly, under the assumption that $H_1 = \text{const.}$, we can see that

$$(7.5) \quad \Theta + H_1 \Phi = C \quad (C = \text{const.})$$

on V^n .

Now, by covariant differentiation of (7.3) along V^n , we get

$$(7.6) \quad \nabla_c \nabla_b \Theta = -H_1(\rho \Phi g_{cb} + \Theta h_{cb}),$$

by virtue of (2.4), (3.14) and (2.3). Thus, from (7.2) and (7.5), we find that

$$(7.7) \quad \nabla_c \nabla_b \Theta = -\{(H_1^2 - \rho)\Theta + \rho C\}g_{cb}.$$

Here we claim $H_1^2 - \rho \neq 0$. In fact, if $H_1^2 - \rho = 0$, then (7.7) becomes $\nabla_c \nabla_b \Theta = -\rho C g_{cb}$, from which $\Delta \Theta = -n\rho C$, that is, $\Delta \Theta = \text{const.}$, where $\Delta \Theta = g^{cb} \nabla_c \nabla_b \Theta$. However this is impossible, unless $\Theta = \text{const.}$ on V^n [22, 2]. But, using the assumption (3) and Lemma 6.1, we can see that Θ is not constant along V^n . Thus, $H_1^2 - \rho$ being different from zero, (7.7) is rewritten as follows:

$$(7.8) \quad \nabla_c \nabla_b \left(\Theta + \frac{\rho C}{H_1^2 - \rho} \right) = -(H_1^2 - \rho) \left(\Theta + \frac{\rho C}{H_1^2 - \rho} \right) g_{cb},$$

from which we get

$$\Delta \left(\Theta + \frac{\rho C}{H_1^2 - \rho} \right) = -n(H_1^2 - \rho) \left(\Theta + \frac{\rho C}{H_1^2 - \rho} \right),$$

and consequently, it follows that $H_1^2 - \rho > 0$ [20]. Therefore, using Theorem A, the equation (7.8) shows that the hypersurface V^n under consideration is isometric to a sphere [10, 11].

And, from this Theorem 7.1, we have

Corollary 7.2 *Let R^{n+1} be an orientable Riemannian manifold with parallel Ricci tensor, say, $\nabla_k R_{ji} = 0$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .
- (3) Ω is not constant along V^n .

Then V^n is isometric to a sphere.

Corollary 7.3 *Let R^{n+1} be an orientable conformally flat Riemannian manifold with $R = \text{const.}$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .
- (3) Ω is not constant along V^n .

Then V^n is isometric to a sphere.

Moreover, using Theorem 6.5, Theorem 6.8 and Theorem 6.9 respectively, we obtain

Theorem 7.4 Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .
- (3) Ω is not constant along V^n .

Then V^n is isometric to a sphere.

Theorem 7.5 Let R^{n+1} be an orientable Riemannian manifold with $R^{ji} R_{ji} = \text{const.}$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n .
- (3) Ω is not constant along V^n .

Then V^n is isometric to a sphere.

Theorem 7.6 Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) $H_1 = \text{const.} \neq 0$,
- (2) there exists a point P_0 on V^n such that $S(P_0) = 0$,
- (3) Ω has fixed sign on V^n .
- (4) Ω is not constant along V^n .

Then V^n is isometric to a sphere.

Lemma 7.7 Let V^n be a hypersurface with $H_2 = \text{positive const.}$ in R^{n+1} . If V^n is an umbilical hypersurface, then H_1 is non zero constant.

PROOF. Since V^n is an umbilical hypersurface, we have

$$k_1 = k_2 = \dots = k_n,$$

from which, taking account of (2.10), we obtain

$$H_2 = k_1^2 = H_1^2.$$

Accordingly, since H_2 is positive constant, we can see that H_1 is non zero constant.

Thus, using Theorem 5.4 and Lemma 7.7, we have the following

Theorem 7.8 Let R^{n+1} ($n \geq 3$) be an orientable Riemannian manifold of constant curvature which admits a proper conformal Killing vector field ξ^i satisfying the equation

$$\mathcal{L}_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n a closed orientable hypersurface in R^{n+1} such that

- (1) H_2 is positive constant,

(2) $\Omega (= C^i \Psi_i)$ has fixed sign on V^n ,

(3) Ω is not constant along V^n .

Then V^n is isometric to a sphere, where Ψ satisfies $\nabla_j \nabla_i \Psi = -(R/n(n+1))\Psi G_{ji}$ and $\Psi_i = \nabla_i \Psi$.

Next, under the new assumption that Ψ is not constant along V^n , instead of the condition of Ω , we prove the following Theorem in a similar way:

Theorem 7.9 *Let R^{n+1} be an orientable Riemannian manifold with harmonic curvature which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

(1) $H_1 = \text{const.} \neq 0$,

(2) Ω has fixed sign on V^n ,

(3) Ψ is not constant along V^n .

Then V^n is isometric to a sphere.

PROOF. Since, $\nabla_b(\Phi_i B_a^i) = \nabla_j \Phi_i B_b^j B_a^i + \Phi_i \nabla_b B_a^i$, we see, from (3.14), (2.4) and $\Theta = C^i \Phi_i$, that

$$(7.9) \quad \nabla_b \nabla_a \Phi = \rho \Phi g_{ba} + \Theta h_{ba}.$$

Also, by virtue of Theorem 6.2, every point of V^n is umbilic, that is, $h_{ba} = H_1 g_{ba}$. Consequently, from (7.9), we have

$$\nabla_b \nabla_a \Phi = (\rho \Phi + H_1 \Theta) g_{ba}.$$

So, substituting (7.5) into this equation, we find that

$$(7.10) \quad \nabla_b \nabla_a \Phi = \left\{ -(H_1^2 - \rho) \Phi + CH_1 \right\} g_{ba}.$$

Here, using the assumption (3) and (6.4), it follows that Φ is not constant along V^n , and accordingly, we can prove that $H_1^2 - \rho \neq 0$, by an argument similar to that used in the proof of Theorem 7.1. Thus, (7.10) is rewritten as follows:

$$(7.11) \quad \nabla_b \nabla_a \left(\Phi - \frac{CH_1}{H_1^2 - \rho} \right) = -(H_1^2 - \rho) \left(\Phi - \frac{CH_1}{H_1^2 - \rho} \right) g_{ba},$$

from which we get

$$\Delta \left(\Phi - \frac{CH_1}{H_1^2 - \rho} \right) = -n(H_1^2 - \rho) \left(\Phi - \frac{CH_1}{H_1^2 - \rho} \right),$$

and consequently, it follows that $H_1^2 - \rho > 0$. Therefore, using Theorem A, the hypersurface V^n is isometric to a sphere [10, 21], by virtue of (7.11).

And, from this Theorem 7.9, we obtain

Corollary 7.10 *Let R^{n+1} be an orientable Riemannian manifold with parallel Ricci tensor which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n ,
- (3) Ψ is not constant along V^n .

Then V^n is isometric to a sphere.

Corollary 7.11 *Let R^{n+1} be an orientable conformally flat Riemannian manifold with $R = \text{const.}$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n ,
- (3) Ψ is not constant along V^n .

Then V^n is isometric to a sphere.

Moreover, using Theorem 6.5, Theorem 6.8 and Theorem 6.9 respectively, we obtain similarly the following

Theorem 7.12 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n ,
- (3) Ψ is not constant along V^n .

Then V^n is isometric to a sphere.

Theorem 7.13 *Let R^{n+1} be an orientable Riemannian manifold with $R^j{}_i R_{ji} = \text{const.}$ which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) Ω has fixed sign on V^n ,
- (3) Ψ is not constant along V^n .

Then V^n is isometric to a sphere.

Theorem 7.14 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ , and V^n a closed orientable hypersurface in R^{n+1} such that*

- (1) $H_1 = \text{const.} \neq 0$,
- (2) there exists a point P_0 on V^n such that $S(P_0) = 0$,
- (3) Ω has fixed sign on V^n ,
- (4) Ψ is not constant along V^n .

Then V^n is isometric to a sphere.

Finally, using Theorem 5.4 and Lemma 7.7, we have the following

Theorem 7.15 Let R^{n+1} ($n \geq 3$) be an orientable Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i satisfying the equation

$$\mathcal{L}_\xi G_{ji} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 2\Psi G_{ji},$$

and V^n a closed orientable hypersurface in R^{n+1} such that

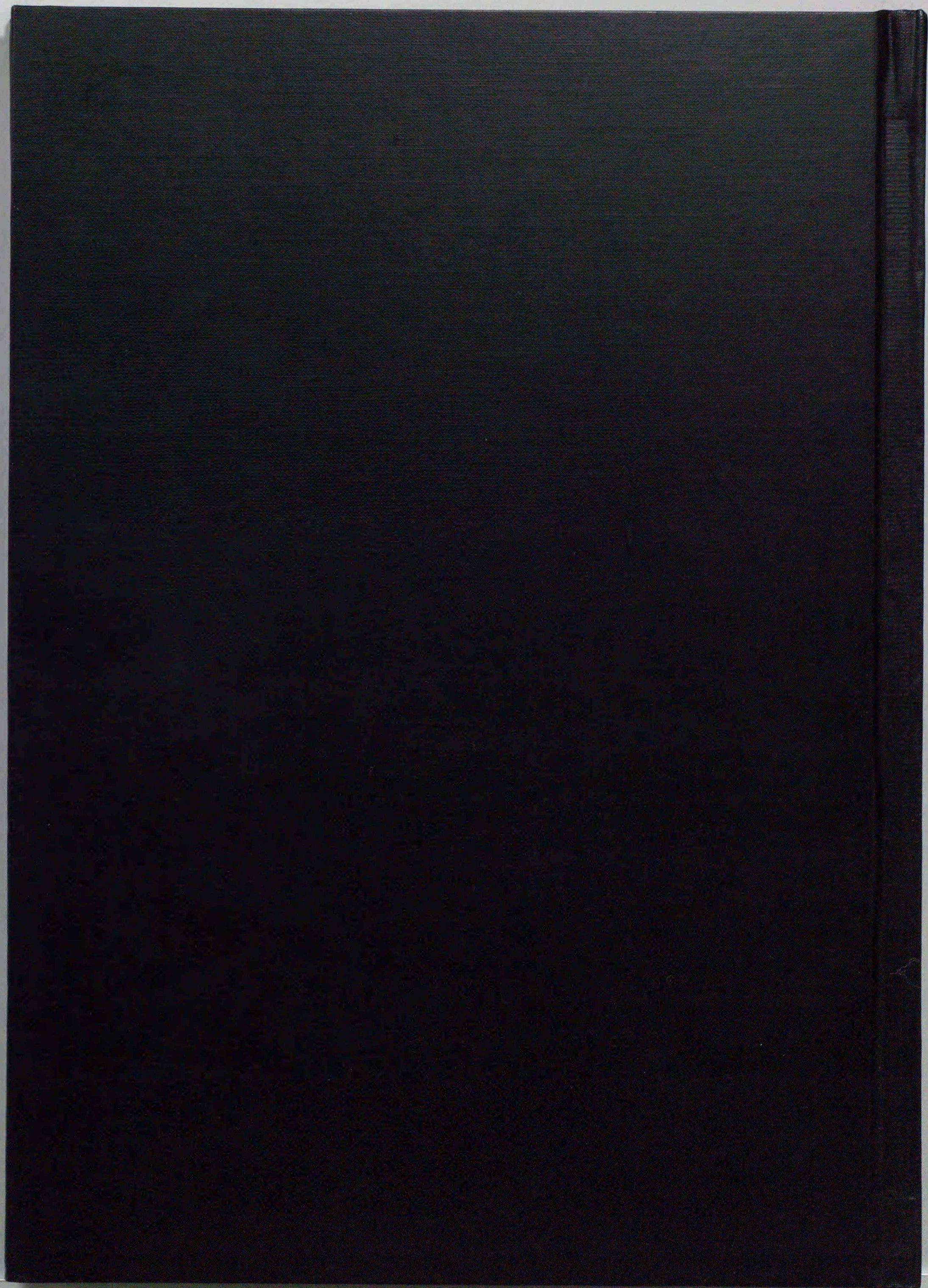
- (1) H_2 is positive constant,
- (2) $\theta (= C^i \xi_i)$ has fixed sign on V^n ,
- (3) Ψ is not constant along V^n .

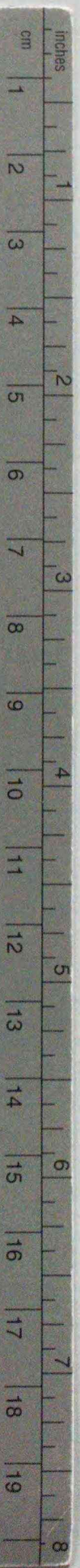
Then V^n is isometric to a sphere.

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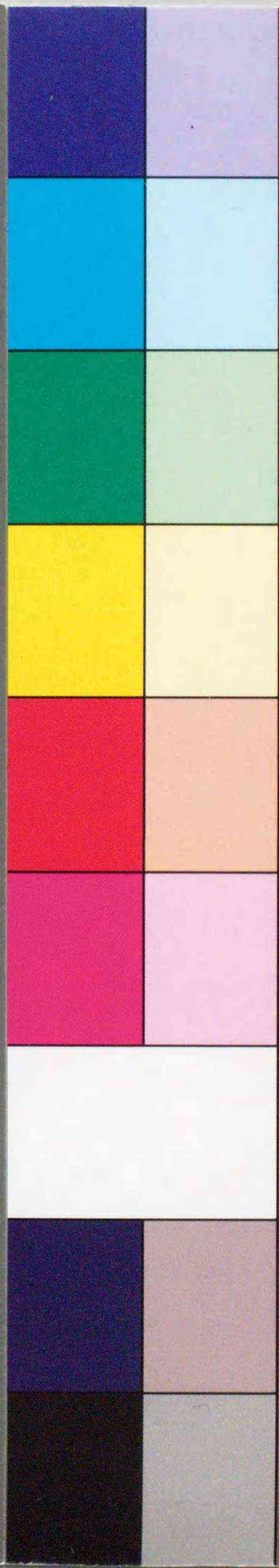




Kodak Color Control Patches

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Blue Cyan Green Yellow Red Magenta White 3/Color Black



Kodak Gray Scale



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A 1 2 3 4 5 6 M 8 9 10 11 12 13 14 15 B 17 18 19

