第13回偏微分方程式論

札幌シンポジウム

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下記の要領でシンポジウムを行ないますので、ご案内申しあげます。

代表者　久保田　幸　次

記

1. 日　時　　１９８８年８月１８日（木）　～　８月２０日（土）

2. 場　所　　北海道大学理学部数学教室　４－５０８室

3. 講　演

８月１８日（木）

＊　この時間は講演者を囲んでの自由な質問の時間とする予定です。

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MHD近似によるconfined plasmaに関する解のregularityについて

白田平

1. 電気抵抗0, 圧縮性MHD 及び境界条件を

(1) $$A_0(U)\frac{\partial U}{\partial t} + A_i(U)\frac{\partial U}{\partial x_i} = 0 \quad \text{in } [0,T]\times\Omega,$$

(2) $$U(0,x) = U_0(x).$$

ここで $U=(q, u, H, s)$, $(\mathrm{div}\, H = 0)$,

(3) $$(u, n) = (H, n) = 0 \quad \text{on } \partial\Omega,$$

$q = p + \frac{|H|^2}{2}$, n: unit normal to $\partial\Omega \in C^m$ $(m \gg 1)$.

MHD方程式は気体分子運動論的analogyより高温、粒子数十分大という仮定の下で—これはプラズマ閉じ込めの許容範囲を超えているが—物理的に導かれた1つのプラズマModel。この導出の故の方程式の無意味さにもかかわらず，応用数学、工学関係でこのModelがよく用いられるのは，経験的根拠より，プラズマの大雑把な巨視的imageを把握するのにsimpleで独立な系であるこのModelが適しているからと云われている。[1]

実際，ごく最近にも数値計算上，磁気面を

固定境界と考へた静止平衡解の Simulation やその線型，非線型安定性に関する考察も数多い。但し直接MHD方程式を用いるのでなく，Variation Method による様子。

一方前述の無意味さと物理的理論の大雑把さのため，数学者のこの non-linear problem への approach は皆無といってもよい。僅かに Symmetric hyperbolic system の研究の一つとして，Chen Shuxing（陳恕行）[2] により1982年に得られた一般論的存在定理を改良修正して上の Confined plasma の方程式に適用して（2-1）-regular 解：$\exists T>0$,

$$D_t^k D_{tan}^{\alpha} D_{normal}^{\beta} U \in L^{\infty}([0,T]; L^2(\Omega))$$

$$k+|\alpha|+2|\beta| \leq p,$$

$$p \geq 8, \quad \Omega \subset R^n \quad (n=2,3)$$

を全く"単純な" Iteration scheme により得たのに過ぎない（柳沢，白田（1987））。但し前述した理論の大雑把さから云へば，この様な解の存在は，一部の応用数学者にとっては陳恕行の理論で十分であった様にさえ思われる。[3]

又 stability の概念は時間的に Global な解が得られそうにない時，大変難かしいものとなる。より身近かな compressible flow に関してさへ stability の数学的研究は今後に待たねばならぬ様に思われる。

さて以上の状況をふまえて次の問題を考へよう：この ideal MHD 方程式の初期一境界値問題の regular 解：$\exists T>0$,

(4) $$D_t^k D_x^\alpha U \in L^\infty([0,T],\ L^2(\Omega)) \qquad k+|\alpha|\le p \quad (p\ge 3)$$

を構成出来ないか？勿論初期 data は p 次の compatibility 条件を満すものとする。

この問題は全く数学 side のものであるが，数値計算に対する好奇心からも出たものである。もう少しこの問題に対して数学的背景を記そう：

i) Incompressible の時には Г. В. Алексеев[4] (1982) により O.K.

ii) non conductor を2ツ以上内蔵する non—

degenerate な境界値問題を設定出来る時にはOK.（但しこの問題についての数値計算的背景を知らない。）

これらを考慮すると問題は方程式自身の非等方性と境界条件のバランスの問題と考へる事が出来る。尚 MHD の静止平衡解の linear stability の取り扱いと異なり, t－方向微分も tangential 微分として p 階まで同時に考へねばならぬ事情にあることを注意する。

こゝで compressible Euler equation の mixed problem では成立する regular solution の時間局所解の構成に反し, 前記の問題は, Linearized equations を媒介とする普通 hyperbolic system に利用される方法では否定される事を示そう. 即ち一般性を失わないその Linearized problem の解の発散の理由を明らかにしよう. 方法は辻 (1972)[5] を参照にした. 以下, その方針を記す。

2. Reduction.

$\bar{U}=(\bar{q},\bar{u},\bar{H},\bar{s})$：任意の(3),(4)を充す 8-vector valued fun.

$\bar{U}$を中心として(1)の linearized problem：

$$
(5)\quad
\begin{cases}
\bar{d}(\frac{D}{Dt}q)-\bar{d}\bar{H}\frac{D}{Dt}H+\operatorname{div}u\equiv 0,\\
\bar{\rho}(\frac{D}{Dt}u)+\nabla q-(\bar{H},\nabla)H\equiv 0,\\
\frac{D}{Dt}H-(\bar{H},\nabla)u+\bar{H}\operatorname{div}u\equiv 0,\\
\frac{D}{Dt}s\equiv 0,\quad \operatorname{div}H=0 \qquad \text{in } [0,T]\times\Omega,
\end{cases}
$$

$$
(3)\quad (u,n)=(H,n)=0 \qquad \text{on } [0,T]\times\partial\Omega.
$$

こゝで $q=p+|H|^2/2$，$d=\frac{p_\rho}{\rho}$，$\frac{D}{Dt}=\frac{\partial}{\partial t}+(\bar{u},\nabla)$，

$e^{s}=p\rho^{-\gamma}\ (\gamma>1)$，$\equiv 0$ は mod 低階項 w.r.t. U.

「∀初期data $U_0(\in H^p(\Omega))$：p次の compatible cond. w.r.t (5) に対し∃，(5)の(4)を充す解」の時，(5)を H^p-well posed と呼ぶ事にしよう.

但し低階項に一般に $\bar{u}$ の1階微分も入って来る．双曲型ではこれを除く事が多いが，必要なら $\bar{U}$ を regularize して又(3),(4)を充す様にしておくものとし，更に解を得るため低階項に適当な perturbation が加えられたものとする．

今，1つの上述の問題(5)が H^p-well posed とする．(5)は Rauch の Symmetric hyperbolic system with characteristic boundary of constant multiplicity.

依て $\Omega=\{x_1>0\}$, U compact support と考へて localize した問題を,原点で Freezing して Δ を除いても対応する constant coeff. の問題(5')は HP-well posed となる. 但し lower oder term $=0$.

(仮定) (5')で $\bar H=(0,\bar h,0)$, $\bar h>0$, $\bar\sigma>0$, $\bar\rho>0$.

更に,こゝで tangential 方向だけの座標変換を行ひ, $\frac{\partial}{\partial t}$ を $\frac{\partial}{\partial t}$ とする. これ(3)も含めて(5'')とする.

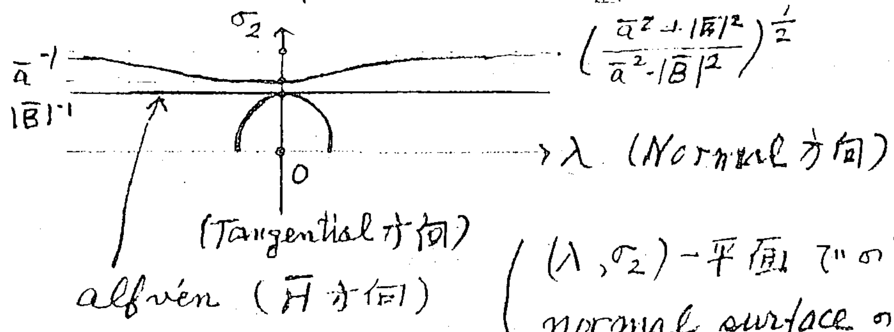
(5'')の特性方程式:

$$\tau(\tau^2-(\bar B,\sigma)^2)\{\tau^4-(\bar a^2+|\bar B|^2)\tau^2(\lambda^2+|\sigma|^2)+\bar a^2(\bar B,\sigma)^2(\lambda^2+|\sigma|^2)\}=0,$$

こゝで $\bar B=\bar H/\bar\rho^{\frac12}$, $\bar a=\bar p_\rho^{\frac12}=(\bar g_\rho)^{\frac12}$, $\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\leftrightarrow(\tau,\lambda,\sigma)$.

方程式の非等方性を明確にして置く. 例へば

$|\bar B|^2>\bar a^2$ →



((λ, σ_2)-平面での normal surface の切り口 ($\tau=1$)).

(cf. Courant-Hilbert, p615.)

特性 Matrix $A(\tau,\lambda,\sigma_2,\sigma_3)$

$$(6)=\left(\begin{array}{c|ccc|ccc}
\bar a\tau & \lambda & \sigma_2 & \sigma_3 & 0 & -\bar a\bar h\tau & 0\\
\hline
\lambda & \bar\rho\tau & & & -\bar h\sigma_2 & & \\
\sigma_2 & & \bar\rho\tau & & & -\bar h\sigma_2 & \\
\sigma_3 & & & \bar\rho\tau & & & -\bar h\sigma_2\\
\hline
0 & -\bar h\sigma_2 & 0 & 0 & \tau & & \\
0 & \bar h\lambda & 0 & \bar h\sigma_3 & & \tau & \\
0 & 0 & 0 & -\bar h\sigma_2 & & & \tau
\end{array}\right).$$

(以後の計算の単純化のために) 3行目, 6行目 $-\bar{h}\times$ 1行目は夫々

$$\bar{\rho}\frac{\partial u_2}{\partial t}-\bar{h}\frac{\partial H_2}{\partial x_2}=-\frac{\partial g}{\partial x_2},$$

$$(1+\bar{a}\bar{h}^2)\frac{\partial H_2}{\partial t}-\bar{h}\frac{\partial u_2}{\partial x_2}=\bar{a}\bar{h}\frac{\partial g}{\partial t}$$

であるが,

これを対角化して更に座標変換

$$\bar{t}=t,\quad \bar{x}_i=x_i\quad (i=1,3),$$

$$\bar{x}_2=\left(\bar{h}/(1+\bar{a}\bar{h}^2)^{\frac{1}{2}}\right)t+x_2$$

を行へば,

$$A(\tau,\lambda,\sigma)=A\left(\tilde{\tau}+\left(\frac{a^2|B|^2}{a^2+|B|^2}\right)^{\frac{1}{2}}\sigma_2,\lambda,\sigma\right)$$

$$\equiv\tilde{A}(\tilde{\tau},\lambda,\sigma)\ \text{---}\ ((5''')\text{の特性 Matrix})$$

[補題 1] (5') が H^1-well posed $\Rightarrow$ (a priori 評価)

$$\int_0^{\bar{t}_0}\frac{\partial^2 g}{\partial x_2\partial x_1}d\bar{t}\in L^2(x_1>0)\qquad \forall\,\bar{t}_0>0.$$

更に (6) の 2行目, 5行目

$$\frac{\partial g}{\partial x_1}+\bar{\rho}\frac{\partial u_1}{\partial t}-\bar{h}\frac{\partial}{\partial x_2}H_1=0,$$

$$\frac{\partial H_1}{\partial t}-\bar{h}\frac{\partial u_1}{\partial x_2}=0$$

より

[補題 2] 補題 1 の仮定の下で変換された方程式系 (5''') も H^1-well posed で ($p=1$ の場合)

$$\int_0^{\bar{t}_0}\frac{\partial^2 u_1}{\partial x_2^2}d\bar{t}\in L^2(x_1>0)\qquad \forall\,\bar{t}_0>0.$$

これは矛盾することを示すために以下で (5''') ((3) は勿論仮定) に対する Green 函数を作る。

3. Green function 及び $\frac{1}{4}$-loss.

$M=(0,1,0,0,0,0,0)$: Boundary matrix,

$$M\hat{U}(t,x,\sigma)=\left(\frac{1}{2\pi}\right)^2\iint M\,\frac{Q}{P}(\tilde{\tau},\lambda,\sigma)(iA_0)\,\hat{u}_0(\lambda,\sigma)\,e^{i(\lambda x+\tilde{\tau}t)}\,d\lambda d\tilde{\tau}$$

$$(7)\qquad -\left(\frac{1}{2\pi}\right)^2\iint M\,\frac{Q}{P}(\tilde{\tau},\lambda,\sigma)(iA_0)\,\hat{u}_0(\lambda,\sigma)\,e^{i(\lambda^+(\tilde{\tau},\sigma)x+\tilde{\tau}t)}\,d\lambda d\tilde{\tau}$$

(Distribution sense).

ここで $\operatorname{Im}\tilde{\tau}<0$, $P=\det$ of $\hat{A}(\tilde{\tau},\lambda,\sigma)$, $Q=$ Matrix of cofactors of $\hat{A}$, $\lambda^+(\tilde{\tau},\sigma)$ は $|\hat{A}(\tilde{\tau},\lambda,\sigma)|=0$ の $\operatorname{Im}>0$ な根.

[補題3] $\sigma_2>0$ とすれば, $(B,\sigma)\neq 0$. このとき, 根 $\tilde{\tau}(\lambda,\sigma)$ は distinct. $|\sigma'|=|\sigma|/(\lambda^2+|\sigma|^2)^{\frac{1}{2}}\ll 1$ かつ $\sigma_2\sim\sigma_3$ とする. $\tilde{\tau}^{(1)}=-\left(\frac{\bar{a}^2|B|^2}{\bar{a}^2+|B|^2}\right)^{\frac{1}{2}}\sigma_2$,

$\tilde{\tau}^{(2)}=O(|\sigma_2|^3/(\lambda^2+|\sigma|^2))$ $>$ $\tilde{\tau}^{(3)}=O(|\sigma|)$,

$\tilde{\tau}^{(4)}$, $\tilde{\tau}^{(5)}=O((\lambda^2+|\sigma|^2)^{\frac{1}{2}})$, $\tilde{\tau}^{(4)}>\tilde{\tau}^{5}$,

$\tilde{\tau}^{(6)}$, $\tilde{\tau}^{(7)}=-\left(\frac{\bar{a}^2|B|^2}{\bar{a}^2+|B|^2}\right)^{\frac{1}{2}}\sigma_2\pm|B|\sigma_2=O(|\sigma|)$

(Alfvén wave).

対応する固有 vectors (homogeneous degree 0) について:

$MU^{(1)}=0$,

$U^{(2)}$: $u_2^{(2)}$, $H_2^{(2)}\sim 1$, $u_1^{(2)}$, $H_1^{(2)}\sim O(|\sigma'|)$, 他 $\sim O(|\sigma'|^2)$.

$U^{(i)}(i=6,7)$: $u_2^{(i)}, H_2^{(i)}\sim O\left(\left|\frac{\lambda}{\sigma}\right| u_1^{(i)}\right)$, $H_1^{(i)}\sim u_1^{(i)}$, 他 $=0$,

(但し $U^{(6)}, U^{(7)}$ については $\sigma_2\sim\sigma_3$ の範囲で考える).

(7) における $\frac{1}{2\pi i}\int d\tau$ に留数定理を用いて

$$\tau = \tau^{(j)}(\lambda, \sigma),$$

$$\frac{Q}{P}(\tau, \lambda, \sigma) \longrightarrow \frac{Q}{P_\tau}(\tau^{(j)}(\lambda, \sigma), \lambda, \sigma)$$

におきかえる。

このとき，$j=1$ については $MU^{(1)}=0$ だから除ける，$j=2, 3, 4, 5$ については

$$(8) \quad \frac{1}{2\pi}\int M\frac{Q}{P_\tau}(\tau^{(j)}(\lambda,\sigma),\lambda,\sigma)A_0\hat{U}(\lambda,\sigma)\times \times\left\{e^{i\lambda^-(\tau^{(j)}(\lambda,\sigma),\sigma)x}-e^{i\lambda^+(\tau^{(j)}(\lambda,\sigma),\sigma)x}\right\}e^{i\tau^{(j)}t}d\lambda$$

こゝで $\int d\lambda = \int_0^\infty d\lambda \quad (j=2, 4), = \int_{-\infty}^0 d\lambda \quad (j=3, 5)$。

$j=6, 7$ では $\lambda^+(\tau^{(j)}(\lambda,\sigma),\sigma)=|\sigma_3|\sqrt{-1}$ となり (8) の
相ん記せないので補題3を用いて直接評価する：補題2の

$$(9) \quad \left\| \int_0^{t_0} \frac{\partial^2 u}{\partial x_2^2}\, dt \right\|_{L^2(x_1>0)} = \left\| \int_0^{t_0} \sigma^2 M\hat{U}(t, x_1, \sigma)\, dt \right\|_{L^2(x_1>0,\ |\sigma|<\infty)}$$

こゝで (8) においては例へば $j=2, 4$ では

$$\lambda^-(\tau^{(j)}(\lambda,\sigma),\sigma)=\lambda, \quad \lambda^+(\tau^{(j)}(\lambda,\sigma),\sigma)=-\lambda$$

だから sin-Fourier 変換を考へて

$$(9) \geq \left\| \sum_{i=2}^{5}{}' \sigma_2^2 M\frac{Q}{P_\tau}(\tau^{(i)}(\lambda,\sigma),\lambda,\sigma)A_0\hat{U}_0(\lambda,\sigma)\int_0^{t_0} e^{i\tau^{(i)}(\lambda,\sigma)t}dt \right\|$$

$$- \| i=6, 7 \text{ に対応する項} \|$$

ここで $\|\ \|$ は $L^2(\frac{1}{2}\sigma_3<\sigma_2<2\sigma_3,\ |\lambda|<\infty)$. 更に右辺オ1項の $\|\ \|$ は $L^2(\frac{1}{2}\sigma_3<\sigma_2<2\sigma_3, \varepsilon|\lambda|>|\sigma|)$ と縮める。

ここで $u_{10}=H_{10}=0$, $u_{20}\neq 0$, $H_{20}\neq 0$ とし補題 3 を用いて，任意 $\delta>0$ $\exists C_\delta$

$$(9)^2 + C_\delta\|u_0\|^2_{1+\delta} \geq \left\|\sigma_2^2((U^{(2)}, \hat{u}_0) M U^{(2)})(\lambda,\sigma)\int_0^{t_0} e^{i\tau^{(2)}t} dt\right\|^2,$$

右辺 ‖ ‖ の内 $\sim \sigma_2^2 (U^{(2)}, \hat{u}_0)|\sigma| \frac{1}{\tau^{(2)}}(1-\cos(\tau^{(2)}t_0))^{\frac{1}{2}}$

$\sim \lambda(U^{(2)}, \hat{u}_0)(1-\cos(\tau^{(2)}(\lambda,\sigma)t_0))$.

$(\widehat{\frac{\partial}{\partial x_1} u_0|_{x_1>0}})(\lambda,\sigma) = i\lambda\hat{u}_0(\lambda,\sigma) - \hat{u}_0(x_1=0,\sigma)$ だから

$$k_2(\sigma) \equiv (U^{(2)}((1-|\sigma|^2)^{\frac{1}{2}},\sigma), \hat{u}_0(x_1=0,\sigma))$$ とし，

$$\iint |K_2(\sigma)|^2 (1-\cos(\tau^{(2)}(\lambda,\sigma)t_0))\, d\lambda\, d\sigma$$

を下より評価する。ここで積分は

$$\tfrac{1}{2}\sigma_3 < \sigma_2 < 2\sigma_3, \quad |\sigma|/\varepsilon < \lambda < |\sigma|^{\frac{3}{2}} \quad (0<\varepsilon\ll 1)$$

とすれば，ε を fix して，

$$\int (1-\cos(\tau^{(2)}(\lambda,\sigma)t_0))d\lambda = O(|\sigma|^{\frac{3}{2}}),$$

$$\int |K_2(\sigma)|^2 |\sigma|^{\frac{3}{2}} d\sigma \leqq C_1 + C_\delta \|U_0\|^2_{1+\frac{\delta}{2}}.$$

依って

[補題 4] $(5''')$ が H^1-well posed。更に $U_0 \in H^{1+\delta}(x_1\geq 0)$ $(0<\delta\ll 1)$ とせよ。 $U_{10}=H_{10}=0$ in $(x_1\geq 0)$ ならば

$$K_2(\sigma)|\sigma|^{\frac{3}{4}} \in L^2(\tfrac{1}{2}\sigma_3 < \sigma_2 < 2\sigma_3).$$

一般に $U_0 \in H^{1+\frac{1}{4}}(x_1>0)$ の時，$K_2(\sigma)|\sigma|^{\frac{3}{4}} \in L^2$.

$p>1$ のときは $D_{x_2}^{p-1}U$ を U の代りに用いる。

〔注意〕(i) compatibility condition については、U_0 自身は $u_{10}=H_{10}=0\ (x_1\geq 0)$ でも他の component は自由にとれる。但し $\dfrac{\partial^i U_0}{\partial x_1^i}(x_1=0)$ は制約される。

(ii) 以上は $n=2$ のときには Alfvén 波は走れないので、$n=3$ のときにもこれを上より評価した。又 $n=2$ のとき、補題4で $u_{10}=H_{10}=0$ on $(x_1>0)$ は不用、$K_2(0)|0|$係数 $L^2(R_0^1)$ と出来る。

(iii) 尚、Majda-Osher[6] は入射波の全く存在しない例を考えた。ここでは反射波、入射波共に存在する大変複雑な wave を持った structure を持つ。

以上より

〔定理〕 Linearized problem (3), (5) は、$\bar{H}|_{t=0,\partial\Omega}\neq 0$ ならば、H^p - well posed でない $(p\geq 3)$。
但し、$n=2$ のとき、$\bar{H}$ は考へている座標空間に含まれているものとする。

尚参考文献に数値計算に関するものは、数が多いので、すべて省いた。

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Effectively hyperbolic operators の零階特性帯の接続

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1. 零階特性帯の二対をつなぐこと.

この報告は，*effectively hyperbolic* な微分作用素または擬微分作用素の零階特性帯の *regular* — C^∞ または解析的 — な接続に関するものである．主表象 $p = p(\rho) = p(x, \xi)$ が *effectively hyperbolic* であるような多重特性的な点 $\hat{\rho}$ が与えられたとき，零階特性帯 $\rho = \rho(s)$ であって，$s \uparrow +\infty$ または $s \downarrow -\infty$ のとき $\hat{\rho}$ に近づくものを考える．このような零階特性帯はちょうど四本あることが知られている．そのうちの二本は点 $\hat{\rho}$ に向かって（パラメーター s に関して）はいってきて，他の二本は出ていく．報告したい結果は，次の様に述べられる：

定理 1. はいってくる（または出ていく）二本の零階特性帯のうちの一本は他の一本と自然につながり，こうして得られた二本の曲線は $\rho = \hat{\rho}$ の近傍で（*cotangent bundle* の部分多様体として）C^∞ *regular* である．もし $p = p(\rho)$ が（実）解析的であれば，これら二本の曲線も解析的である．

次のことが知られている： 双曲型作用素に対する Cauchy 問題が低階の項によらずに C^∞-適切であるための必要十分条件は，主表象が任意の多重（必然的に二重）特性的な点で effectively hyperbolic なことである． Effective hyperbolicity —— Ivrii と Petkov によって導入された —— とは，Hamilton 写像が零でない実の固有値を持つ（必然的に二個で，$\pm\lambda$ という形をしている）ということである． （ここで Hamilton 写像というのは Hamilton（ベクトル）場を線型化して得られるいわゆる“基本行列”のことである．） このような線型代数的な条件が，考えている点の近くでの零陪特性帯の力学系にどのように反映しているかを考えることは自然であろう． 定理 1 の証明は，次のことを示唆している：はいってくる（または出ていく）零陪特性帯を考える限り，このような力学系は，最も簡単なモデル $p^0(y,\eta)=\eta_0^2-y_0^2$ in T^*R^{n+1} の力学系の摂動である． このモデルは自然に (y_0,η_0)-平面に制限されるが，そこでは原点は鞍点（saddle point）である． ただし $y=(y_0,y')$, $\eta=(\eta_0,\eta')$.

零陪特性帯が C^∞ 級に接続されることは，すでに岩崎敷久氏によって示されている [1] —— そこでは，主表象の因数分解に関する深い結果 [2] が用いられている． 岩崎氏が [2] で用いた方法は Nash-Moser の陰函数定理 —— というよりむしろその証明であるが，それは複雑である． 以下に述べる方法は完全に初等的であり，C^∞ 級の場合と同時に解析的な場合にも同様に適用される．

2. どのように零階特性帯をつなぐか.

$\rho=\hat{\rho}$ のまわりで symplectic な座標変換 $(x,\xi)\to(y,\eta)$ を行なう. 新座標 $\rho=(y,\eta)$ with $\hat{\rho}=(0,0)$ に関して, 主表象は次の形をしている:

$$p(y,\eta)=\{\eta_0-\varphi(y,\eta')\}^2-\psi(y,\eta'),\quad \psi(y,\eta')\geqq 0$$

(零でない因子を掛けることを除いて).

ただしここで, φ と ψ は二次以上の項である(よって $\eta_0^2-\psi$ の quadratic part が Hamilton 写像に対応している); さらに, $\psi=E+O^3$ であって,

$$E(y,\eta')=y_0^2+\sum_{j=1}^{n_1}\mu_j(y_j^2+\eta_j^2)+\mu_0\sum_{j=n_1+1}^{n_2}y_j^2$$

($\mu_j,\mu_0>0$ は定数で, $0\leqq n_1\leqq n_2\leqq n$).

以上のことを用いると, なめらか(C^∞ または解析的)な函数 $\pi^\pm$ を構成することができて, 次の分解を得る:

$$p=p^+p^-\ \text{with}\ p^\pm=\eta_0-y_0\pi^\pm(y_0,v/y_0),\quad v=(y',\eta');$$

この因数分解は $|v|/|y_0|<\text{constant}$ という形の集合の上で成立する. この制限があっても問題ないのであるが, それは, はいってくる(または出ていく)零階特性帯が $|v|\leqq C|y_0|^{3/2}$ という評価をみたすからである. (定理1の結論から最終的には $v=O(y_0^2)$ という

評価が従う.)　こうして, 問題は p の Hamilton 場から $p^{\pm}$ の Hamilton 場へと置き換えられるが, ここでパラメーターも s から $t=t^{\pm}$ へとかわる. このとき, $\pm$ の何れの場合にも $dy_0/dt=1$ であるから, y_0 をパラメーター $t=t^{\pm}$ として採用してよい. 従って, ($\pm$ の何れの場合にも) 次の形の初期値問題へと導かれる:

$$(IVP)\qquad \frac{dv}{dt}(t)=F\left(t,\frac{v(t)}{t}\right),\qquad \frac{v(t)}{t}\longrightarrow 0\in R^{2n}\ \text{as}\ t\longrightarrow 0;$$

ただしここで, $F=F(t,u)$ はなめらかな函数であって $F(0,0)=0$ をみたし, さらに $(\partial F/\partial u)(0,0)$ の固有値はすべて虚軸上に乗っている. よって, 次のことを示せば定理1が証明されたことになる:

<u>定理2</u>.　上述の仮定の下で, 両側初期値問題 (IVP) は $t=0$ の近傍でなめらか (C^{∞} または解析的) な解 $v=v(t)$ を持つ. さらに, 解の一意性は ($t \gtrless 0$ に対応する) 片側問題の各々について成り立つ (端点 $t=0$ を除いて C^1 級な解の範囲で).

実際, いったん $v=v(t)$ が定まったならば, $z_0=z_0(t)$ は不定積分することによって求まる; こうして零階特性帯の各対を regular につなぐことができる. このとき, 容易にわかるように, 結果として得られた二本の曲線の点 ρ における接線は, 零でない実の固有値 $\pm\lambda$ に付随する固有ベクトルによって張られる. 前節で触れた最も簡単なモデルにおいては, これらの二本の曲線は直線

$z_0 = \pm y_0$ with $v = 0$ となり，パラメーターの変換 $s \to t = t^{\pm}$ は $|t| = e^{-2|s|}$ ($t \to 0$ のとき) をみたす —— これは一般の場合には次のように摂動される：

$$C_- \exp\left(\frac{-2}{1-\varepsilon}|s|\right) \leqq |t| \leqq C_+ \exp\left(\frac{-2}{1+\varepsilon}|s|\right).$$

実際，$0 < {}^\forall\varepsilon < 1$ に対して $Y_\pm(s) = |y_0(s)| \exp[2|s|/(1\pm\varepsilon)]$ とおくと，s_0 を固定したとき，

$$Y_-(s_0) \leqq Y_-(s) \leqq Y_+(s) \leqq Y_+(s_0) \quad \text{as} \quad \rho(s) \to \hat{\rho}.$$

3. Briot-Bouquet の特異点.

解析的な場合には，定理２（の解の存在に関する部分）は，Briot と Bouquet による著名な結果 [3] から直ちに導かれる．実際，$u(t) = v(t)/t$ とおけば，問題 (IVP) を

$$(BB) \qquad t\frac{du}{dt}(t) = f(t, u(t)), \qquad u(0) = 0 \in \mathbb{C}^N$$

と書くことができる．ただしここで，$f(t,u) = F(t,u) - u$ であり，$N = 2n$．このとき，

<u>Briot-Bouquetの定理</u>. もし $f=f(t,u)$ が $(t,u)=0 \in \mathbb{C}^{N+1}$ の近傍で正則であって, $M=(\partial f/\partial u)(0,0)$ と書いたとき,

$$f(0,0)=0, \quad \sigma(M)\cap \mathbb{N}=\phi \quad (\mathbb{N}=\{1,2,\cdots\})$$

をみたせば, 特異初期値問題 (BB) は $t=0\in\mathbb{C}$ の近傍で一意的な正則解 $u=u(t)$ を持つ. ただしここで $\sigma(M)$ は M の固有値全体の集合 (*spectrum*) をあらわす.

さらに, Briot-Bouquetの定理の C^∞ 版が de Hoog と Weiss [4], [5] によって与えられている (Russell [6] をも参照). これは, 片側問題として定式化されている:

$$(BB)_T \qquad t\frac{du}{dt}(t)=f(t,u(t)) \quad \text{for} \quad 0<t\leqq T.$$

ただしここで $f, \partial f/\partial u \in C^0([0,T_0]\times B(R_0))$ with $0<T\leqq T_0$ かつ $B(R_0)=\{v\in\mathbb{C}^N;\ |v|\leqq R_0\}$, $R_0>0$ である. さらに

$$\mathcal{C}^{r,1}_{T,R}=C^r([0,T],B(R))\cap C^{r+1}(0,T]$$

とおくと,

<u>Briot-Bouquetの定理の C^∞ 版</u>. 次のことを仮定する:

$$f(0,0)=0, \quad \sigma(M)\subset\{\lambda\in\mathbb{C};\ \mathrm{Re}\,\lambda<0\};$$

ただし $M=(\partial f/\partial u)(0,0)$. このとき, 定数 $T\in(0,T_0]$ と

$R\in(0,R_0]$ が存在して，方程式 $(BB)_T$ は一意的な解 $u\in\mathcal{C}^{0,1}_{T,R}$ を持つ —— これは初期条件 $u(0)=0$ を自動的にみたす．さらに，もし $f,\ \partial f/\partial u\in C^r([0,T]\times B(R))$ ならば $u\in\mathcal{C}^{r,1}_{T,R}$ が成り立つ．

定理2の証明を完結させるためには，次のことを示せばよい：一方の片側問題の C^∞ 解は，$t=0$ を越えて他方の問題の解となめらかにつながる．即ち，証明すべきことは，

(4) $$u^{(r)}(+0)=u^{(r)}(-0) \qquad \text{for } r\in\mathbb{N}_0=\{0,1,2,\cdots\}.$$

このことは，次の線型方程式の解を積分表示することによって示される：

$$t\frac{du}{dt}(t)-Mu(t)=g(t) \qquad \text{for } 0<t\leqq T;$$

一意的な解 $u\in C^0[0,T]\cap C^1(0,T]$ は次のように与えられる：

$$u(t)=\mathcal{E}_M g(t)=\int_0^t\left(\frac{t}{s}\right)^M g(s)\frac{ds}{s}=\int_0^1\left(\frac{1}{s}\right)^M g(ts)\frac{ds}{s}.$$

このとき，容易にわかるように，

$$(\mathcal{E}_M g)^{(r)}=\left(\frac{d}{dt}\right)^r\mathcal{E}_M g=\mathcal{E}_{M-rI}\,g^{(r)} \qquad (I\text{ は単位行列});$$

特に

$$(\mathcal{E}_M g)^{(r)}(0)=(rI-M)^{-1}g^{(r)}(0).$$

この公式を用いれば，(4) の成立を確めることができて，定理2の証明がおわる．

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DISSIPATIVE STRUCTURE OF PLASMAS
- remarks on electrostatic potential distribution in plasmas -

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1. Introduction

This paper studies one-dimensional electrostatic potential distributions in plasmas. The object of our analysis is a classical problem of plasma physics, which has been originally studied by Bohm [1] for the plasma sheath formation, however, some subjects are of up-to-date interest related to the theory of dissipative structures. We study mathematical classifications of equilibrium models with discussing real characteristics of governing partial differential equations (PDE's). We show the dissipative structure of dynamical systems is strongly related to characteristics. We start with reviewing some explicit examples of various type of static (temporally homogeneous) problems in physics. The electrostatic potential problem is an example of elliptic-hyperbolic PDE's, which is an important class of static problems that exhibits significant dissipative structures Structural instabilities of the plasma potential distributions are discussed in Sec.4.

2. Preliminary Examples and Classification of Steady-State Equations

We review some examples of steady-state problems in mathematical physics, and make a basic remark on the characteristics of PDE's from the view-point of dissipative structures. We cite simple and typical examples.

A. Elliptic Systems

The Laplace equation is the most typical steady-state PDE's. Physis examples described by this class of PDE's are electrostatic potential distributions, vector-potential distributions for static magnetic fields, steady-state stress distributions, steady-state temperature, density, and probability distributions in diffusion systems, etc. These are linear elliptic PDE's. Standard boundary-value problems are uniquely solvable. The steady states of this class of problems are characterized by the boundary data.

B. Hyperbolic Systems

When the steady-state equations are hyperbolic type, we should supply "initial data" to integrate the equations. Here, the term "initial" might be confusing. It implies purely mathematical classification of integration data, and is not necessarily related to the physical time. The Hamilton-Jacobi PDE's of classical dynamics,

$$\partial_t \phi + H(x, \partial_x \phi) = 0$$

with a temporally-homogeneous Hamiltonian, is the most important example. The real characteristics are described by the corresponding characteristic and bi-characteristic ordinary differential equations (ODE's), viz., the Hamilton canonical ODE's

$$dx/dt = \partial_p H(x,p), \quad dp/dt = -\partial_x H(x,p).$$

When the Hamiltonian is not explicitly time-dependent, and when we supply temporally-homogeneous "initial condition", viz., constant particle source at the initial position, ∂_t drops, and the system describes steady particle flows. The term "initial" then implies the start points of the particle orbits that are the characteristics. To avoid confusion, let us address such temporally-homogeneous initial date to "I-data", while let us call the noncharacteristic boundary data "B-data".

The Liouville equation

$$\partial_t f + \{f, H\} = 0$$

is an alternative and equivalent expression of the classical dynamics. The dependent variable f stands for the particle density in the phase space (x,p). The structure of the equilibrium is perfectly correspondent to the I-data.

C. Elliptic-Hyperbolic Systems

Elliptic-Hyperbolic mixed systems are mostly general and interesting model equations. The equilibrium is not fully determined by B-data, but I-data should be also prescribed to find an equilibrium. It is usually very hard to develop a general mathematical theory for an elliptic-hyperbolic system when the characteristics is dependent to the dependent variables. The steady-state Euler equations of ideal incompressible flow is a typical example of such equilibrium problems;

$$(v,\mathrm{grad}\,)v + \mathrm{grad}\,p = 0, \quad \mathrm{div}\,v = 0.$$

Characteristic equation for this system is

$$(\mathrm{grad}\,\psi)^2 \cdot (v,\mathrm{grad}\,\psi)^2 = 0,$$

which says the system is two-elliptic and two-hyperbolic, and the characteristics if the flow curves themselves.

There are also some examples of elliptic- hyperbolic equilibrium problems in plasma physics. The magnetohydrodynamic equilibrium equation

$$(\mathrm{rot}\,B)\times B = \mathrm{grad}\,p, \quad \mathrm{div}\,B = 0$$

is a system of two-hyperbolic and two-elliptic PDE's. We should supply two of independent I-data. Physically they correspond to the distributions of the pressure and the force-free field. When we consider two-dimensional ($\partial/\partial z = 0$) problems, the magnetostatic equation reduces to the Grad-Shafranov equation:

$$L\phi = (I(\phi)^2)' + p(\phi)'$$

where L is an elliptic differential operator (Laplacian), ϕ is the flux function, $I(\phi)$ is the distribution of the z-component of the magnetic field, and $p(\phi)$ is the distribution of the pressure.

Another important example is the ion-sheath and ion-acoustic-shock problems of electrostatic plasmas. When we consider that the dynamics of ions is fully conservative and the dynamics of electron is fully relaxed (dissipative), and when we consider one-dimensional problems, the self-consistent ion-sheath equation reduces to the Bohm sheath equation (see Sec.3);

$$L\psi = V'(\psi - P),$$

where $V(\psi^*)$ is the so-called Sagdeev potential (see Eq.(2)), and P is a constant. Formal analogy of both equations is worthwhile noting. Structures are subject to the characteristic functions; for the Grad-Shafranov equation, $p(\phi)$ and $I(\phi)$, and for the Bohm equation, $V(\psi - P)$. These characteristic functions are related to the dissipative structure of the systems. In the next section, we discuss the physical background of the Bohm equation.

3. Bohm Equation

We consider an electrostatic plasma in the one-dimensional half space $(0, +\infty)$. The stationary electro-static potential distribution in a plasma is governed by a model equation called Bohm equation:

$$L\psi = (1 - 2M^{-2}\psi^*)^{-1/2} - e^{\psi^*}, \tag{1}$$

where

$$\psi^* := \psi - P,$$

ψ is the electrostatic potential normalized by the thermal energy of electrons, and $L = -d^2/dx^2$ is an elliptic differential operator of one dimension. The coordinate x is normalized by the Debye length. The model was firstly studied by Bohm for the sheath potential of plasmas contacting with walls [1]. The model is also related to the ion acoustic shock. Sagdeev introduced the so-called Sagdeev potential to study oscillating solutions for the model equation [2].

The Bohm equation has two independent parameters; M is the Mach number defined by

$$M := [(\text{kinetic energy of ion})/(\text{thermal energy of electrons})]^{1/2},$$

and P is the potential deep inside the plasma. The first and the second terms in the right-hand-side of Eq.(1) correspond to the densities of ions and electrons, respectively. Using the Sagdeev potential, we write Eq.(1) as

$$L\psi = V'(\psi^*), \tag{1'}$$

where the Sagdeev potential $V(\psi^*)$ is defined by

$$V(\psi^*) = 1-e^{\psi^*}+M^2[1-(1-\frac{2}{M^2}\psi^*)^{1/2}]. \tag{2}$$

The Bohm equation is a nonlinear elliptic differential equation. We consider boundary-value problems for the equation. We set

$$\psi(0) = 0, \tag{3}$$

$$\lim_{n\to\infty} \psi(x_n) = P, \tag{4}$$

where the limit is taken for a certain sequence $\{x_n\ ; n=1,2,\cdots\}$ which satisfies $\lim_{n\to\infty} x_n = +\infty$. This limit is generaly dependent on the choice of the sequence. The condition (4) implies that the potential $\psi(x)$ should not deviate from the value P, but may oscillate around P. Physically P corresponds to the potential inside the plasma where the ion is originated. Therefore, the value $\psi^* = \psi - P$ is the potential difference that the ion feels. We do not consider that the ion is originated at the mathematical infinity. The weak boundary condition (4) permits a large variety of solutions for the model equation (see Sec.4).

4. Mathematical Technique to Solve Bohm Equation

In this section, we prepare mathematical background and note some non-trivialities. The Bohm equation is a nonlinear elliptic differential equation, so the solvability, the uniqueness, bifurcations of solutions, and the stabilities are subjects of mathematical considerations. Let us start with reviewing the simplest solution for the Bohm equation, that is the Bohm-sheath solution, and then discuss mathematical problems concerning the structural stability of the sheath solutions.

When we linearize the Bohm equation, we have

$$L\psi = \alpha(\psi-P), \quad (\alpha := M^2-1). \tag{5}$$

Equation (5) has different characters for $M>1$ and $M<1$ regimes. For $M>1$, Eq.(5) has a unique solution for every $P \in \mathrm{R}$:

$$\psi(x) = P[1 - e^{-\sqrt{\alpha}x}]. \qquad (6)$$

A mathematical question is the solvability of the original nonlinear equation (1) for given P and $M>1$. Another question is the structural stability of the solution. The solution (6) for the linearized equation (5) satisfies the boundary condition (4) in a stronger sense; viz., the function asymptotically converges to P as $x \to +\infty$. Since we set a weaker condition, we may expect a wider class of solutions for the nonlinear equation. The ion-acoustic-shock solutions are given by the structural instability of the asymptotic solutions. These points will be discussed in the next section.

In the region $M<1$, the linearized equation (5) is of the Helmholtz type, and the equation has non-trivial solutions for $P=0$. This implies the possibility of bifurcation of solutions for the original nonlinear equation (1). To answer the above-mentioned questions, the Sagdeev method using a formulation of initial-value problems is useful.

Although the formulation is essentially a boundary-value problem, we may take the advantage of one-dimensional differential equations, and convert the boundary-value problem to an initial-value problem (IVP) for an ordinary differential equation. By this technique, we easily find structurally unstable solutions; see section 4.

We consider initial values at $x = 0$;

$$\psi(0) = 0,$$

$$\psi'(0) = v,$$

where v is a certain number that should be determined to meet the boundary condition (4). Using the analogy of Newton's equation (x: time, ψ: position), $V(\psi)$: potential energy, $\psi'^2/2$: kinetic energy), we may easily find v that matches the boundary condition (3). This IVP technique for solving the Bohm equation has been given by Sagdeev to find oscillating solutions. In the next section, we will study the mathematical structure of the Bohm equation using the IVP method.

5. Solvability and Structural Stability

The IVP method has an advantage in studying the structural stability for the Bohm equation. Figure 1 shows the numerically calculated Sagdeev potential. We can construct solutions by starting from x(considered to be the time) $= 0$ with $\psi^*(0)$(considered to be the initial position) $= -P$ and $\psi^{*\prime}(0)$(considered to be the initial velocity) $= v$. The boundary condition (4) should be finally satisfied. Therefore, the curve $\psi^*(x)$ should stay around 0 in the sense of the weak convergence of the condition (4). The initial value v is chosen to satisfy this condition. The analogy of one-dimensional Newtonian dynamics easily explains the method to find the solution.

First let us consider the case of $M \geq 1$. We should set

$$v^2/2 = V(-P), \qquad (7)$$

to get the asymptotic convergence of $\psi^*(x)$ to 0 at $x \to +\infty$. Figure 2(a) shows the asymptotic solution that corresponds to the Bohm sheath. The asymptotics $\psi^* = 0$, however, is top of the potential, so that the structural instability may give bifurcated solutions. When we start with a little bit larger velocity v, we get oscillations in the region of $\psi^* \geq 0$, if we may include some dissipative structure; Fig.2(b). The oscillating solution is the ion-acoustic shock, which has been given by Sagdeev.

Next, let us discuss the regime of $M<1$. When $M<1$, the position $\psi^* = 0$ is the bottom of the Sagdeev potential. Because of this structural change in the Sagdeev potential, we see a drastically different behavior of solutions. There is no asymptotic solution. Only oscillating solutions may exist. The weak boundary condition (4) retains such pathological solutions. Figure 2(c) shows a typical oscillatory solution in the $M<1$ regime.

Figure 3 shows the solvability and classification of solutions for the Bohm equation. The Sagdeev potential is not defined in the regime $\psi^* \geq \psi^*_c := M^2/2$. The potential difference $\psi^* \geq \psi^*_c$ is large enough to stop the transit motion of ions. When ions are stopped by the potential barrier, positive charge accumulates, so that steady solutions do not exist. The no-solution region 'C' in Fig.3 is given by the positive potential barrier. In the region 'A', the Bohm equation (1) has normal positive-ion-sheath solutions (Fig.2(a)) and ion-acoustic-shock solutions (Fig.2(b)). In the region 'B', only oscillating solutions (Fig.2(c)) exist.

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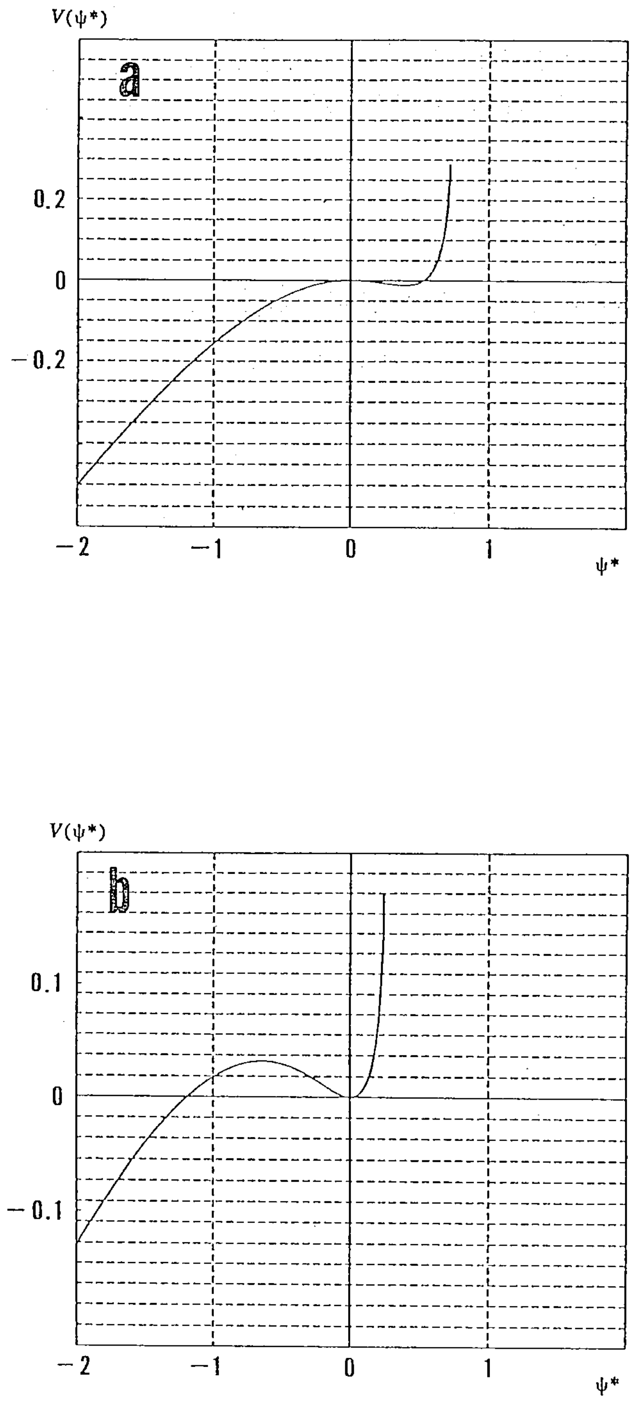

Fig.1 Sagdeev potentials for (a)$M = 1.2$ and (b)$M = 0.7$.

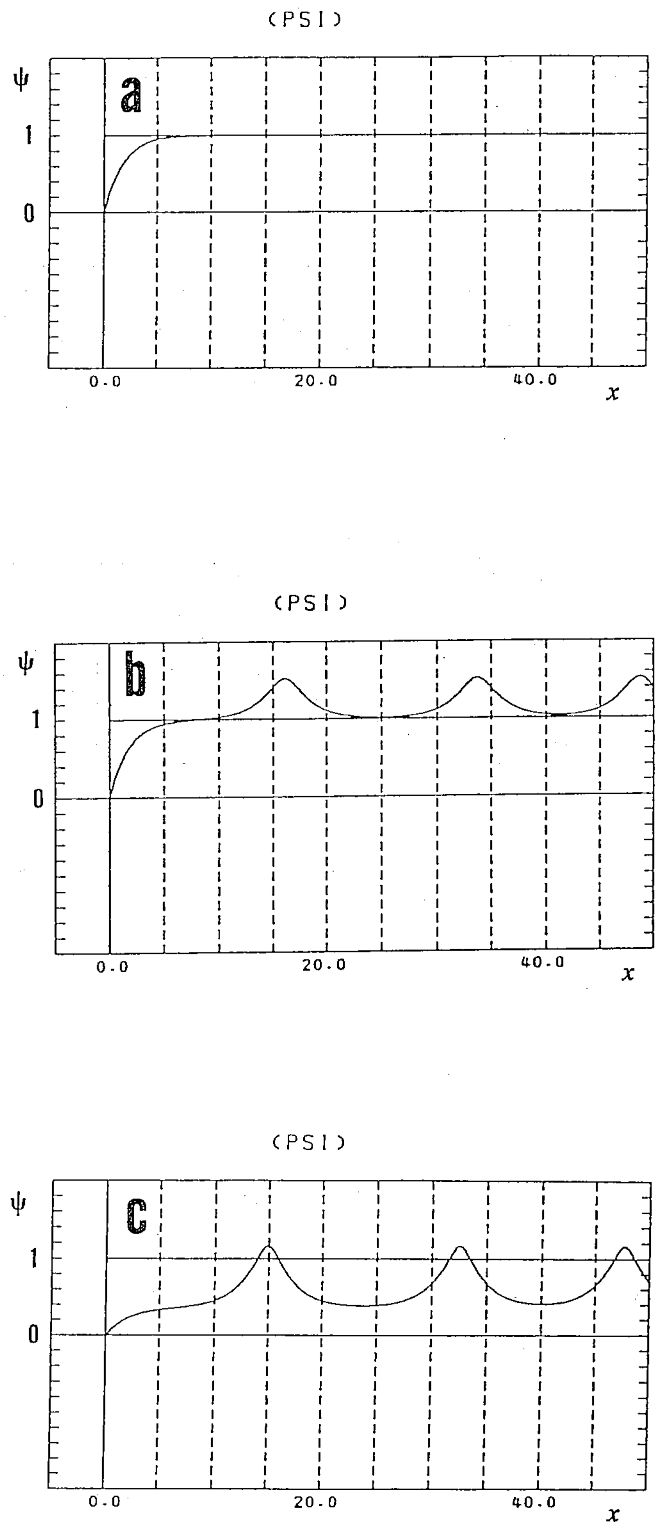

Fig.2 Solutions for the Bohm equation.
(a) Asymptotic solution for $M = 1.2$, $P = 1$.
(b) Oscillating solution for $M = 1.2$, $P = 1$.
(c) Oscillating solution for $M = 0.7$, $P = 1$.

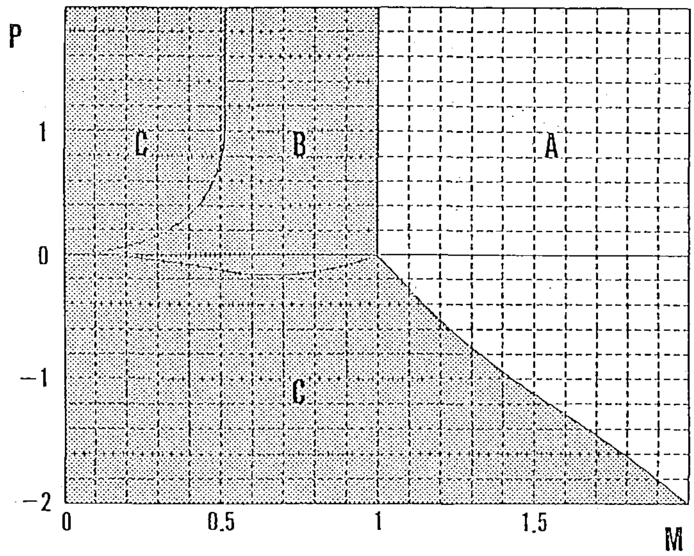

Fig.3 Solvability and classification of solutions for the Bohm equation in the M/P plane.

Region A : asymptotic solutions (Bohm-sheath solutions) and oscillating solutions (ion-acoustic-shock solutions) exist.

Region B : oscillating solutions exist.

Region C : no solution exists.

磁気流体力学の方程式系の境界値問題.

奈良女子大　理　柳沢　卓.

磁場の下で電気伝導性の流体が行なう運動を論ずる分野を磁気流体力学(Magnetohydrodynamics, MHD)という。特に、対象とする電気伝導性流体の電気抵抗及び粘性を0としたものを理想磁気流体力学(Ideal MHD)という。

以下、この理想磁気流体力学の境界値問題を論ずるので、電気伝導性流体(プラズマ)は固定境界壁Γで囲まれた領域Ωに満たされているものとする。

このときの基礎方程式は次のものである.

$$
\left.
\begin{aligned}
&(a)\quad \rho_p(\partial_t + (u\cdot\nabla))p + \rho\,\nabla\cdot u = 0 \\
&(b)\quad \rho(\partial_t + (u\cdot\nabla))u + \nabla p + \mu H\times(\nabla\times H) = 0 \\
&(c)\quad (\partial_t + (u\cdot\nabla))H - (H\cdot\nabla)u + H(\nabla\cdot u) = 0 \\
&(d)\quad (\partial_t + (u\cdot\nabla))S = 0 \\
&(e)\quad \nabla\cdot H = 0 \qquad \text{in } [0,T]\times\Omega,
\end{aligned}
\right\} \quad (1)
$$

ここに $p=p(t,x)$, $u=u(t,x)=(u^1,u^2,u^3)$, $H=H(t,x)=(H^1,H^2,H^3)$,

$S=S(t,x)$ が未知関数で、それぞれ時刻 t，空間座標、$x=(x_1,x_2,x_3)$ における圧力、速度ベクトル、磁場ベクトル、エントロピーを表わす。これらをあわせて $U={}^t(p,u,H,S)$ と書くことにする。 ρ は密度を表わし、状態方程式; $\rho=\rho(p,S)>0$ for $p>0$，により p と S より決められる。$\rho_p\equiv\frac{\partial\rho}{\partial p}$。 μ は透磁率を表わし、ここでは正定数と仮定する。

初期条件としては次を課す。

$$U|_{t=0}={}^t(p_0,u_0,H_0,S_0)\equiv U_0 \qquad \text{in } \Omega \ . \qquad (2)$$

この時、MHD発電等の関連でも興味深い Γ が完全導体壁の時の境界条件の下で、(1),(2) の時間的局所解が一意的に存在するかどうかを考えたい。

Γ が完全導体壁の時の境界条件とは次のものである(cf.[2])。まず、速度ベクトル u に対しては通常の流体力学におけるものと同じく、

$$u\cdot n=0 \qquad \text{on } [0,T]\times\Gamma, \qquad (3)$$

なる条件を課する。ここに、$n=n(x)={}^t(n_1,n_2,n_3)$ は $x\in\Gamma$ における外向き単位法線ベクトルを表わす。次に、境界

が完全導体であることより、電場Eの接線成分が0，すなわち

$$E \times n = 0 \qquad \text{on } [0,T]\times\Gamma, \qquad (4)$$

である。これとOhmの法則

$$E = -u\times(\mu H)$$

より、(3)を考慮すれば次が従う。

$$u(H\cdot n) = 0 \qquad \text{on } [0,T]\times\Gamma. \qquad (5)$$

(3)と(5)をあわせて完全導体壁の境界条件とよぶ。これらは次の様に書きかえることができる。

$$u = 0 \qquad \text{on } \Gamma_{0,T}, \qquad (6)$$

$$u\cdot n = 0, \quad H\cdot n = 0 \text{ on } \Gamma_{1,T}, \qquad (7)$$

ここに $[0,T]\times\Gamma = \Gamma_{0,T}\cup\Gamma_{1,T}$, $\Gamma_{0,T}\cap\Gamma_{1,T} = \phi$,

$\Gamma_{0,T} \equiv \{(t,x)\in[0,T]\times\Gamma \mid H\cdot n \neq 0\}$, $\Gamma_{1,T} \equiv \{(t,x)\in[0,T]\times\Gamma \mid H\cdot n = 0\}$.

特に $[0,T]\times\Gamma = \Gamma_{0,T}$ 及び $[0,T]\times\Gamma = \Gamma_{1,T}$ の場合に対して、我々の得た結果は次のものである。

<u>Case I.</u> $[0,T]\times\Gamma = \Gamma_{0,T}$ の場合.

<u>定理1.</u> (cf. [4]) Ωを$\mathbb{R}^3$における有界領域とし、その境界Γは滑らか、かつコンパクトな2つ以上の成分よりなるものと

する。m (整数) $\geqq 3$ とし初期値に次を仮定する：

$U_0 \in H^m(\Omega)$ かつ

$$p_0 > 0, \quad \nabla \cdot H_0 = 0 \quad \text{in } \Omega, \quad H_0 \cdot n \neq 0 \quad \text{on } \Gamma, \qquad (8)$$

及び $m-1$ 次の Compatibility 条件

$$\partial_t^k U(0) = 0 \qquad \text{on } \Gamma, \ k = 0, 1, \cdots, m-1. \qquad (9)$$

このとき $^\exists T_1 > 0$ s.t. (1),(2),(6) は唯1つの解 $U \in \bigcap_{j=0}^{m} C^j([0, T_1]; H^{m-j}(\Omega))$ を持つ。

<u>Case II.</u> $[0, T] \times \Gamma = \Gamma_{1,T}$ の場合.

<u>定理2.</u> ([3]) Ω を $\mathbb{R}^3$ における有界領域とし、その境界 Γ は滑らか、かつコンパクトであるとする。m(整数) $\geqq 8$ とし初期値に次を仮定する：

$U_0 \in H^m(\Omega)$ かつ

$$p_0 > 0, \quad \nabla \cdot H_0 = 0 \quad \text{in } \Omega, \quad H_0 \cdot n = 0 \quad \text{on } \Gamma, \qquad (10)$$

及び $m-1$ 次の Compatibility 条件

$$\partial_t^k U(0) \cdot n = 0 \qquad \text{on } \Gamma, \ k = 0, 1, \cdots, m-1. \qquad (11)$$

このとき $^\exists T_2 > 0$ s.t. (1),(2),(7) は唯1つの解 $U \in X_m(T, \Omega)$ を持つ。

ここに $X_m(T, \Omega)$ は次の様な関数全体を表わす：

β(整数)≥ 0とし、$\Lambda_1, \cdots, \Lambda_\beta$を境界に対して tangential な、滑らか、かつ有界なベクトル場とする、i.e. $\langle \Lambda_i(x), n(x) \rangle = 0$ for $x \in \Gamma$, $i=1, \cdots, \beta$. このとき $\alpha + \beta \leq m - 2k$, $k = 0, 1, \cdots, [\frac{m}{2}]$ に対して、$\partial_t^\alpha \Lambda_1 \cdots \Lambda_\beta \partial_n^k U(t,x) \in L^\infty(0,T; L^2(\Omega))$ なる関数全体(∂_nは法線方向微分を表す).

境界が完全導体壁でない時の境界条件等については、[1],[2]を参照されたい.

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退化した放物型方程式の自由境界の挙動

東大理　保野博

次の形の初期値問題を考える。

$$(1)\quad \begin{cases} u_t = (u^m)_{xx} - \lambda u^p & (x \in \mathbb{R},\ t>0) \\ u(x,0) = u_0(x) & (x \in \mathbb{R}) \end{cases}$$

ここで $m \geq 1$, $\lambda \geq 0$, $p > 0$ は定数であり、u_0 は非負連続関数とする。 $m=1, \lambda=0$ のときは (1) は古典的な熱方程式にほかならず、$m>1, \lambda=0$ のときは所謂 porous medium 方程式として知られているものである。

$m>1$ の場合、(1) は $u=0$ となる点で退化した放物型方程式に対する初期値問題であり、よく知られているように、u_0 が有界な台を持てば各 $t>0$ に対して $u(\cdot,t)$ も有界な台をもつ。更に次の結果が知られている。以下、

$$\Omega(t) = \{x \in \Omega \mid u(x,t) > 0\}$$

とおき、$\Omega(0)$ は有界であると仮定すると($\lambda > 0$ も仮定)、

(i) $1 < m \leq p$ のとき、$\Omega(t)$ は t に関して単調に増大し

$$\bigcup_{t \geq 0} \Omega(t) = \mathbb{R}$$

が成り立つ (Kalashnikov 1974, Herrero-Vazquez 1987).

(ii) $1 \leq p < m$ のとき、$\Omega(t)$ は t に関して単調に増大し、しかも $\bigcup_{t \geq 0} \Omega(t)$ は有界集合である (Kalashnikov 1974).

(iii) $m \geq 1$, $0 < p < 1$ のとき、$\Omega(t)$ は有界集合で、ある $T \geq 0$ が存在して

$$\Omega(t) = \phi \quad ({}^\forall t \geq T)$$

(Evans-Knerr 1979, Kersner 1980).

本講演では上記(iii)の場合の$\Omega(t)$の挙動を詳しく調べるのが目的である。この場合は自由境界$\partial\Omega(t)$の満たす方程式(interface equation)がRosenau-Kamin (1983)により与えられているが、彼らの結果は形式的計算に基いたもので厳密な結果ではない。一般に(iii)の場合は(i)(ii)と較べて解析が格段に困難であると思われる.

以下、解uの「消滅時刻」を

$$T_0 = \min\{t \geq 0 \mid \Omega(t) = \phi\}$$

によって定義し、また、各$t>0$に対し

$$E(t) = \{x \notin \overline{\Omega}(t) \mid {}^{\exists}t_n, {}^{\exists}x_n\ (n=1,2,\cdots) \text{ such that } t_n \uparrow t,\ x_n \to x\ (n\to\infty),\ x_n \in \Omega(t_n)\ (n=1,2,\cdots)\}$$

とおく.

<u>定理1</u>　$m \geq 1$, $0<p<1$, $\lambda>0$ とし、$u_0(x)$は$\mathbb{R}$上の非負連続関数で有界な台をもつと仮定する。

(i) ${}^{\forall}t>0$に対し、$\Omega(t)$の連結成分の個数は有限で、それは$u_0(x)$の極大点の個数を越えない。

(ii) μをルベーグ測度とすると、$\mu(\Omega(t))$は$t>0$で連続.

<u>定理2</u>　m, p, λ, u_0は前定理の通りとする.

(i) $$\lim_{s\uparrow t}\Omega(s) = \Omega(t) \cup E(t) \qquad ({}^{\forall}t>0)$$

(ただし極限はハウスドルフの距離に関するもの)

(ii) $\bigcup_{t>0}(E(t)\times\{t\})$の要素の個数は$u_0(x)$の極大点の個数を越えない。しかも、仮りに$u_0(x)$の極大点が無数に存在しても、各$\tau>0$に対し$\bigcup_{t\geq\tau}(E(t)\times\{t\})$は有限集合である。

系3　T_0を解u_0の消滅時刻とすると、$E(T_0)$は有限集合であり、

$$\lim_{t\uparrow T_0}\Omega(t)=E(T_0)\quad(\text{ハウスドルフの距離に関して})$$

が成立する。しかも$E(T_0)$の要素の個数は$u_0(x)$の極大点の数を越えない。

上の定理から自由境界$\partial\Omega(t)$はxt－平面上の連続曲線をなすことがわかる。porous medium方程式（すなわち$\lambda=0$の場合）はBarenblattの特殊解と比較定理を用いて$\Omega(t)$の連続性（従って自由境界の連続性）を証明するのはやさしいし、$\lambda>0, p\geq 1$の場合も容易である。しかし我々の問題（$\lambda>0$, $0<p<1$の場合）に同じ方法は適用できない。上記定理の証明には、porous medium方程式の場合とは本質的に異なる方法が用いられる。その際、重要な役割を演じるのが次の補題である。

補題　任意の$\tau>0$および殆どすべての$x\in\mathbb{R}$に対し、

$$\lim_{t\uparrow\tau}\operatorname{sgn}(u_x(x,t))$$

が存在する。ここで$\operatorname{sgn}(\cdot)$は符号関数である。

上の補題は方程式(1)が反転$x\mapsto 2a-x$に関して同変であること、および最大値原理とJordanの曲線定理を用いて証明される。同様の補題は半線型放物型方程式の爆発問題にも適用でき、爆発集合が常に有限集合になることを証明することができる。

(1)を多孔性媒質中を拡散する流体のモデルと考えると、$-\lambda u^p$は蒸発の効果を表わしている。このとき、系3の結果は次のように解釈できる。すなわち、$\Omega(t)$を時刻tにおける流体の『しみ』と考えると、蒸発によって流体が消滅する際、密度$u(x,t)$の値そのものが0に収束する（すなわち「しみ」が薄くなっていく）ことは当然だが、同時に、「しみ」の各連結成分の直径も0に近づく。

なお、上の一連の結果は広島大学の陳旭彦、三村昌泰両氏との共同研究の成果である。

準線型放物型方程式の安定多様体と幾何学における変分問題

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In this talk, we investigate the existence theorem of a stable manifold for a quasi-linear parabolic equation and variational problems in geometry. First we prepare some notations:

- (M, g): a closed Riemannian manifold,
- H: a vector bundle over M with projection π,
- $H^m(M, H)$: the m-th order Sobolev space on the sections of H.

We consider a class of quasi-linear parabolic equations of the following type:

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -J(u) + N(u), & \text{for} \quad u(t,\cdot) \in \Gamma(H) \\ u(0) = u_0, \end{cases}$$

where J is an elliptic operator of $2k$-th order and $N(u)$ represents the non-linear part of the equation. Moreover, the equation satisfies the following three conditions:

(C1) The zero is a stationary solution of (1),

$$\text{i.e.,} \quad -J(0) + N(0) = 0.$$

(C2) $\langle J(u), u\rangle_{L^2} \geq \Lambda\langle u, u\rangle_{L^2}$ for some $\Lambda \in \mathbf{R}$, and J is self adjoint with respect to L^2.

(C3) For $m > \frac{1}{2}\dim M + 2k$ and for $u, v \in H^{m+k}(M, H)$ such that $\|u\|_{H^{m+k}}, \|v\|_{H^{m+k}} < 1$ we have

$$\|N(u) - N(v)\|_{H^{m-k}} \leq C[\|u\|_{H^m}\|u-v\|_{H^{m+k}} + \|u-v\|_{H^m}\|v\|_{H^{m+k}}],$$

and

$$N(0) = 0.$$

The main result on the existence of stable and unstable manifolds, briefly stated, is:

Theorem A *For the stationary solution* 0 *of* (1), *there exist*

(a) *a finite codimensional stable invariant manifold whose elements are close to* 0,

(b) *a finite dimensional unstable invariant manifold whose elements are close to* 0.

Remark. In the above theorem,

(a) the codimension of the stable manifold is equal to the dimension of negative and zero eigenspaces of J,

(b) the dimension of the unstable manifold is equal to the dimension of negative eigenspaces of J.

Another purpose of this talk is to introduce the asymptotic stability of the gradient flow of a variational problem in geometry.

Theorem B *For the functional*

$$\mathcal{L}(s) = \int_M L(s)(x)\, d\mu_M \qquad s \in C^\infty(E),$$

where E is a smooth fiber bundle over (M, g), we suppose that s_1 is a weakly stable critical point and that the connected component of critical set which contains s_1 is non-degenerate. Then the equation of the gradient flow of $\mathcal{L}$:

$$\begin{cases} \dfrac{\partial s}{\partial t} = -\mathcal{E}\mathcal{L}(s) \\ s(0) = s_0 \end{cases}$$

has unique solution provided that Euler-Lagrange operator of $\mathcal{L}$: $\mathcal{E}\mathcal{L}$ is elliptic and that s_1 is close to s_0. Moreover the solution tends to a critical point as $t \to \infty$ with exponential order.

One of the most important example of variational problems is harmonic maps. Let (M, g) and (N, h) be closed Riemannian manifolds. For smooth map $f : M \to N$, the energy functional is given by

$$E(f) = \int_M |df|^2\, d\mu_M.$$

A harmonic maps is characterized by a critical point of the functional. The Euler-Lagrange equation of the functional is a second order semi-linear elliptic equation. Applying Theorem B, we obtain a stable manifold theorem for the equation of the gradient flow (which is called the Eells-Sampson equation).

Another important example of variational problems is the Yang-Mills connection. The Euler-Lagrange equation of the Yang-Mills functional is not elliptic. However, we can avoid this difficulty and obtain a stable manifold theorem. (See [**MK,KMN**]).

Finaly, we note that this talk is based on [**N3**].

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ストークス乗数と弱双曲型作用素

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0．序．

a>0は定数，j,k は整数で $0 < k < j-1$ とする．$s < 0$ としコーシー問題

$$(0.1)\quad \begin{cases} (\partial_t^2 - t^{2j}\partial_x^2 - at^k\partial_x)u = 0, \\ u(s,x) = 0, \\ u_t(s,x) = u_0(x) \end{cases}$$

の基本解 $e(t,s,\xi)$ を構成する。即ち

$$(0.2)\quad \begin{cases} (\partial_t^2 + t^{2j}\xi^2 - iat^k\xi)e(t,s,\xi) = 0, \\ e(t,s,\xi) = 0, \\ e_t(t,s,\xi) = 1 \end{cases}$$

を $s < 0 < t$ として解く．そして $\xi \to +\infty$ としたときの漸近挙動を求める．結果は次の通り．

定理． $d = (j-k-1)/(2j-k)$ とする．
常微分方程式の初期値問題 (0.2) の解 $e(t,s,\xi)$ は $s < 0 < t$ のとき次のような漸近展開を持つ．

(0.3)　$e(t,s,\xi) =$

$$= \Sigma_{m,n=1}^{2} a_{m,n}(t,s)\xi^{-1}\exp[C_{m,n}\xi^d]\exp[i\phi_{m,n}(t,s,\xi)](1+o(1)), \quad \xi \to +\infty.$$

ここに

$$a_{m,n}(t,s) \neq 0,$$

$$(0.4)\quad \phi_{m,n}(t,s,\xi) = \{(-1)^m t^{j+1} + (-1)^n s^{j+1}\}\xi/(j+1),$$

そして $C_{m,n}$ は定数で

$$(0.5)\quad \begin{cases} 2k+3 < j \text{ のとき:} & \text{すべての } \mathrm{Re}\, C_{m,n} > 0, \\ 2k+3 = j \text{ のとき:} & 1\text{つの } \mathrm{Re}\, C_{m,n} = 0,\ \text{残りは正}, \\ k+1 < j < 2k+3 \text{ のとき:} & 1\text{つの } \mathrm{Re}\, C_{m,n} < 0,\ \text{残りは正}, \end{cases}$$

となる．

定理から解ること．$\mathrm{Re}\, C_{m,n} > 0$ のとき $C_{m,n}$ を含む項は無限階のフーリエ

積分作用素で，(0.4) から定まる trajectory に沿って (1/d)-UWF (ultra wave front set) が伝播する．したがって $u_0(x) = \delta(x)$ (Dirac 関数) とすると，

k+1 < j < 2k+1 の場合は４本の trajectories のうち３本に沿っては (1/d)-UWF が伝播し１本に沿っては Gevrey class of order 1/d となる．

2k+3 < j の場合は４本共に沿って (1/d)-UWF が伝播する．

即ち，t = 0 で分岐が起こるかどうかが解る．また，どの Gevrey class で well posed かということも (0.3) 式から解る．初期値が Gevrey class g であっても g > 1/d であれば t > 0 になれば ultra distribution of order 1/d となる．Infinite order の擬微分作用素については，たとえば L. Zanghirati [4] を見よ．k =j-1 の場合は Taniguchi and Tozaki [3] に詳しい．

１．証明の方針．

(1.1) $\omega = \exp[i\pi/(j+1)]$

とし

(1.2) $z(t) = \omega^{1/2}\xi^{1/(2j-k)}t$

とおくと，

$$\lambda = \xi^{(j-k-1)/(2j-k)} > 0, \qquad b = \exp[i\pi(j-k-1)/(2j+2)]a$$

として(0.2) は

$$(1.3)\quad \begin{cases} Ly := \{\partial_z^2 - \lambda^2(z^{2j}+bz^k)\}y = 0, \\ y(z(s)) = 0, \\ y'(z(s)) = \omega^{-1/2}\xi^{-1/(2j-k)} \end{cases}$$

となる．

(1.4) $Ly = 0$

については Sibuya [2] と同様にして次の結果をうる:

$$S_m = \{z;\ |\arg[z] - m\pi/(j+1)| < \pi/(2j+2)\},\quad m=0,1,\dots,2j+1$$

とする．(1.4) の解 y で，$y\to 0$ as $z\to\infty$ in S_mを subdominant solution in S_m という．

<u>命題１．</u>(Sibuya's rotation formula) $Ly(b,\lambda,z) = 0$ とし

(1.5) $y_m(b,\lambda,z) = y(\omega^{-(2j-k)m}b,\lambda,\omega^{-m}z)$

とすると $Ly_m(b,\lambda,z) = 0$ である．

<u>命題2.</u> (1.4) の S_0 における subdominant solution $f(b,\lambda,z)$ で，次のようなものがある．

(i)　$f(b,\lambda,z)$ は $(\lambda^{1/(j+1)}z, \lambda^{(k-2j)/(j+1)}b)$ の整関数．

(ii)　$0 < \lambda_0 \leqq \lambda < +\infty$ とする．次のような漸近展開を持つ．

(1.6)　$f(b,\lambda,z) \sim \lambda^{-1/2} z^{-j/2}(1+\Sigma_{n=1}^{\infty} B_n z^{-n/2})\exp[-\lambda z^{j+1}/(j+1)]$,

(1.7)　$f'(b,\lambda,z) \sim \lambda^{1/2} z^{j/2}(-1+\Sigma_{n=1}^{\infty} B'_n z^{-n/2})\exp[-\lambda z^{j+1}/(j+1)]$,

$$S_{2j+1} \cup \bar{S}_0 \cup S_1 \ni z \to \infty$$

<u>命題3.</u>

(1.8)　$f_m(v,\lambda,z) = f(\omega^{-(2j-k)m}b, \lambda, \omega^{-m}z)$,　$m = 0,1,\ldots,2j+1$

とすると $f_m(b,\lambda,z)$ は S_m での subdominant solution で，(1.6), (1.7) と同様の漸近展開を持つ．

$W_{m,n}$で $f_m(b,\lambda,z)$ と $f_n(b,\lambda,z)$ とのロンスキアンを表すことにし，

$f_m(t) = f_m(b,\lambda,z(t))$,　　$z(t)$ は (1.2) のもの，

とする．

コーシー問題 (1.3) の解を $y(z)$ とすると，(0.2) の解は

$e(t,s,\xi) = y(z(t))$

とあらわされる．また，初期条件と1次独立性とから

$$y(z(t)) = \omega^{-1/2}\xi^{-1/(2j-k)}W^{-1}_{j+1,j+2}\{f_{j+2}(s)f_{j+1}(t) - f_{j+1}(s)f_{j+2}(t)\}$$

が得られる．

$$f_{j+1}(t) = W_{j+1,1}W^{-1}_{0,1}f_0(t) - W_{j+1,0}W^{-1}_{0,1}f_1(t)$$

$$f_{j+2}(t) = W_{j+2,1}W^{-1}_{0,1}f_0(t) - W_{j+2,1}W^{-1}_{0,1}f_1(t)$$

も1次独立性から得られる．この2式が connection formula で，係数がストークス乗数である．これらから $s < 0 < t$ のとき

(1.9)　$e(t,s,\xi) =$

$$= \omega^{-1/2}\xi^{-1/(2j-k)}W^{-1}_{0,1}W^{-1}_{j+1,j+2}\{W_{j+1,1}f_{j+2}(s)f_0(t) - W_{j+1,0}f_{j+2}(s)f_1(t) - W_{j+2,1}f_{j+1}(s)f_0(t) + W_{j+2,0}f_{j+1}(s)f_1(t)\}$$

をうる．

(1.10)　$W_{0,1} = -W_{j+1,j+2} = 2\omega^{j/2}$

は命題１，２，３から得られる．

2． $f_0(0)$, $f_1(0)$, $f_{j+1}(0)$, $f_{j+2}(0)$ の漸近挙動．

$W_{j+1,0}$, $W_{j+1,1}$, $W_{j+2,0}$, $W_{j+2,1}$ の漸近挙動をもとめるには，同じ z に対する f_0, f_1, f_{j+1}, f_{j+2} の漸近挙動が必要となる．

F. W. J. Olver [1] によると，S_0 での subdominant solution で次のようなものが存在する．

$$(2.1)\quad \begin{cases} g(b,\lambda,0) = \text{const.}(1+o(1)), \\ g'(b,\lambda,0) = \text{const.}\lambda^{2/(k+2)}(1+o(1)), \quad \lambda\to+\infty, \end{cases}$$

$$(2.2)\quad g(b,\lambda,z) = \text{const.}\lambda^{-k/(2k+4)} z^{-j/2} \exp[-\lambda\xi^{(k+2)/2}],\quad \lambda\to+\infty,$$

但し，

$$(2.3)\quad \xi = \{\textstyle\int_0^z (t^{2j}+bt^k)^{1/2}dt\}^{2/(k+2)}.$$

この解と $f(b,\lambda,z)$ との比を求め，rotation formula を用いて $f_m(0)$ の漸近挙動がもとめられる．即ち

$$\lim_{z\to+\infty} f(b,\lambda,z)/g(b,\lambda,z)$$

を計算すると

$$(2.4)\quad f(b,\lambda,z) = C\exp[Kb^{(j+1)/(2j-k)}\lambda]g(b,\lambda,z)(1+o(1)),\quad \lambda\to+\infty$$

但し

$$K = \textstyle\int_0^\infty \{(t^{2j}+t^k)^{1/2}-t^j\}dt > 0.$$

これと (2.1) と rotation formula とから，

$$(2.6)\quad f_m(b,\lambda,0) = C'\lambda^{-1/(k+2)}\exp[c_m\lambda](1+o(1)),$$

$$(2.7)\quad f'_m(b,\lambda,0) = C''\lambda^{1/(k+2)}\exp[c_m\lambda](1+o(1)),\quad \lambda\to+\infty$$

を得る．ここに

$$(2.8)\quad |c_m| = a^{(j+1)/(2j-k)}K,$$

$$(2.9)\quad \arg[c_0] = \pi(j-k-1)/(4j-2k),$$

$$(2.10)\quad \arg[c_1] = \pi(j+k+3)/(4j-2k),$$

$$(2.11)\quad \arg[c_{j+1}] = \begin{cases} \arg[c_0], & k = \text{even}, \\ -\arg[c_1], & k = \text{odd}, \end{cases}$$

そして

$$(2.12)\quad \arg[c_{j+2}] = \begin{cases} \arg[c_1], & k = \text{even}, \\ -\arg[c_0], & k = \text{odd} \end{cases}$$

である．ここで重要なことは rotation formula を使うとき

$$(2.13)\quad |\arg[\omega^* b]| < \pi$$

となるようにω^*の偏角を選ばねばならぬことである．

(2.6) と (2.7) とから $W_{m,n}$ の漸近挙動がもとまり，それから定理が得られる．

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（於北大 1988.8）

退化する準線形波動方程式に対する局所古典解の存在と一意性について

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次の混合問題を考える。

$$
(\mathrm{P})\begin{cases} u_{tt}=(a(u)u_x)_x & ,\ 0<x<1,\ t>0 \\ u|_{t=0}=\rho(x),\ u_t|_{t=0}=\sigma(x) & ,\ 0<x<1 \\ u(t,0)=u(t,1)=0 & ,\ t\geqq 0 \end{cases}
$$

ここで，$a(u)$ は C^{d} 級（$d\geqq 4$）であり，次を満たすものとする。

$$
\begin{cases} \text{i) } u>0 \text{ ならば } a(u)>0 \\ 3\leqq m\leqq d \text{ なる } m \text{ があって} \\ \text{ii) } a(0)=a'(0)=\cdots\cdots=\partial_u^{m-2}a(0)=0 \\ \text{iii) } \partial_u^{m-1}a(0)>0 \end{cases}
$$

初期値 $\rho(x)$，$\sigma(x)$ について，次の条件を仮定する。

(I) $\rho(x)\in H^{k}(0,1)$，$\rho^{\frac{m-1}{2}}\partial_x^{k+1}\rho(x)\in L_2(0,1)$ （ただし，$3\leqq k\leqq d-1$）

$\rho(0)=\rho(1)=0$ ，$\rho(x)>0$; $0<x<1$

$\rho'(0)>0$ ，$\rho'(1)<0$

(II) $\sigma(x)\in H^{k}(0,1)$，$\sigma(0)=\sigma(1)=0$

ここで，次の関数空間を導入する。

$$
Y_T^{s}=L_\infty([0,T];\mathcal{H}^{s}(0,1))\cap\bigcap_{i=1}^{s}L_\infty^{i}([0,T];H^{s-i}(0,1))
$$

ただし，$L_\infty^{i}([0,T];H^{s-i}(0,1))=\{u(t,x)\,;\,\partial_t^{i}u\in L_\infty([0,T];H^{s-i}(0,1))\}$

$\mathcal{H}^{s}(0,1)=\{u\,;\,u\in H^{s-1}(0,1),\ \rho^{\frac{m-1}{2}}\partial_x^{s}u\in L_2(0,1)\}$ $(m\geqq 3)$

Y_T^s は、ノルム

$$|||u|||_{Y,s,T} = \sup_{0\leqq t\leqq T}|||u(t)|||_{Y,s} = \sup_{0\leqq t\leqq T}\left(\|u\|_{s-1}^2 + \sum_{i=1}^{s}\|\partial_t^i u\|_{s-i}^2 + \|\rho^{\frac{m-1}{2}}\partial_x^s u\|_0^2\right)^{\frac{1}{2}}$$

によって Banach 空間をなす。

ところで、$a(u) = mu^{m-1}\ (m \geqq 5)$ の場合は、Ebihara [1] において論じられている。Ebihara は、方程式に粘性項をつけ加えた方程式に対する大域解の存在より近似解を構成し、その収束性から局所解の存在を示している。

ここでは、線形化問題に対する大域解の存在より近似解を構成し、その収束性から局所解の存在を示す。

定理 1

(I), (II) を仮定する。このとき正の数 $T_k = T_k(\|\rho\|_k, \|\sigma\|_k, \|\rho^{\frac{m-1}{2}}\partial_x^{k+1}\rho\|_0)$ が存在して、(P) は $u(t,x) \in Y_{T_k}^{k+1}$ なる一意的な解をもち、さらに

$$\begin{cases} u(t,x) > 0 \ ;\ 0 < x < 1,\ 0 \leqq t \leqq T_k \\ u_x(t,0) > 0,\ u_x(t,1) < 0 \ ;\ 0 \leqq t \leqq T_k \end{cases}$$

を満たす。

注意

Sobolev の補題を用いると、定理 1 の $u(t,x)$ は

$$u(t,x) \in C^2([0,T_k]; C^{k-3}[0,1]) \cap C^1([0,T_k]; C^{k-2}[0,1]) \cap C^0([0,T_k]; C^{k-1}[0,1])$$

となり、$u(t,x)$ は Classical な意味で (P) を満たす。

定理 2

(I), (II) を仮定する。(P) の解 $u(t,x) \in Y_{T_0}^{k_0+1}\ (k_0 < k)$ があって、さらに

$$\begin{cases} u(t,x) > 0 \ ;\ 0 < x < 1,\ 0 \leqq t \leqq T_0 \\ u_x(t,0) > 0,\ u_x(t,1) < 0 \ ;\ 0 \leqq t \leqq T_0 \end{cases}$$

を満たすならば $u(t,x) \in Y_{T_0}^{k+1}$ となる。

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Singular Limit of the Alfven Number for Incompressible Ideal Magneto-Fluid Motion

Shun'ichi Gotoh

We discuss the singular limit with respect to the Alfven number for the incompressible ideal magneto-fluid motion in the three dimensional Euclidean space $\mathbb{R}^3$ or the torus $\mathbb{T}^3$ (i.e., the periodic motion) which is denoted by G.

In the fluid dynamics there appear many systems of non-linear differential equations involving parameters such as the Mach number and the Alfven number etc.. One problem on the singular limit is to determine the limiting system which has a completely different property comparing with the original system, as such a parameter tends to some value.

When the system is hyperbolic, this problem has been studied in G.Browning - H.-O.Kreiss [2], S.Klainerman - A.Majda [5], A.Majda [6] and S.Schochet [7]. In particular, Browning and Kreiss studied the Alfven limit for the compressible magneto-fluid motion as an example of their theorem. However, to show this, they needed more assumptions on the initial data than those in other papers above.

The purpose of this note is to determine the limiting system for the incompressible magneto-fluid motion under the natural assumptions on the initial data. In $\mathbb{R}^3$ the limiting system becomes the equations of the magneto-static field (see (1.8)). And, in $\mathbb{T}^3$ it becomes the system involving two dimensional convective derivative (see (1.6)) but the author does not know whether its physical meaning has been clarified.

We consider the following system involving a large parameter α,

(1.1.a) $(\partial_t+(v^\alpha,\nabla))v^\alpha+\nabla p^\alpha+\alpha^2 H^\alpha\times \mathrm{rot} H^\alpha=0$

(1.1.b) $(\partial_t+(v^\alpha,\nabla))H^\alpha-(H^\alpha,\nabla)v^\alpha=0$ in $[0,T^\alpha]\times G$

(1.1.c) $\mathrm{div}\ v^\alpha=\mathrm{div}\ H^\alpha=0$

(1.1.d) $v^\alpha(0)=v_0^\alpha,\ H^\alpha(0)=H_0^\alpha$ on G.

Here the fluid velocity $v^\alpha=v^\alpha(t,x)={}^t(v_1^\alpha,v_2^\alpha,v_3^\alpha)$, the magnetic field $H^\alpha=H^\alpha(t,x)={}^t(H_1^\alpha,H_2^\alpha,H_3^\alpha)$ and the pressure $p^\alpha=p^\alpha(t,x)$ are unknowns depending on α. The reciprocal of α is the Alfven number which is in proportion to $|v_m|/|H_m|$, where $|v_m|,|H_m|$ are typical mean values of these quantities.

We assume that the initial data (1.1.d) satisfy

(1.2) $H_0^\alpha=\bar{H}+\alpha^{-1}K_0^\alpha,\ (v_0^\alpha,K_0^\alpha)\in H_\sigma^s(G)$,

where $\bar{H}$ is a non zero constant vector and $s\geq 3$ is an integer. Throughout this note, $H^r(G)$ denotes the Sobolev space of the L^2-type with inner product $(\cdot,\cdot)_r$ and norm $\|\cdot\|_r$ and $H_\sigma^r(G)$ denotes the solenoidal subspace of $H^r(G)$.

Setting $K^\alpha=\alpha(H^\alpha-\bar{H})$, we can write (1.1.a)-(1.1.d) in the form

(1.3.a) $(\partial_t+(v^\alpha,\nabla))v^\alpha+K^\alpha\times \mathrm{rot} K^\alpha+\nabla(p^\alpha+\alpha\bar{H}\cdot K^\alpha)-\alpha(\bar{H},\nabla)K^\alpha=0$

(1.3.b) $(\partial_t+(v^\alpha,\nabla))K^\alpha-(K^\alpha,\nabla)v^\alpha-\alpha(\bar{H},\nabla)v^\alpha=0$ in $[0,T^\alpha]\times G$

(1.3.c) $\mathrm{div}\ v^\alpha=\mathrm{div}\ K^\alpha=0$

(1.3.d) $v^\alpha(0)=v_0^\alpha,\ K^\alpha(0)=K_0^\alpha$ on G.

It is known that, for fixed α, there exists a local in time unique classical solution of (1.3.a)-(1.3.d) (for example, see [1],[4]). The solution belongs to the following function space

(1.4.a) $(v^\alpha,K^\alpha)\in C([0,T^\alpha];H^s(G))\cap C^1([0,T^\alpha];H^{s-1}(G))$, (1.4.b) $\nabla p^{\alpha} \in C([0,T^{\alpha}];H^{s-1}(G))$.

In addition to (1.2) we require the following assumptions on the initial data: there exist vector field $(v_0^{\infty},K_0^{\infty}) \in H_{\sigma}^{s}(G)$ and a constant $\Delta_0>0$ such that

(1.5.a) $(v_0^{\alpha},K_0^{\alpha}) \to (v_0^{\infty},K_0^{\infty})$ in $H^s(G)$, as $\alpha \to \infty$,

(1.5.b) $\alpha\|(\bar{H},\nabla)v_0^{\alpha}\|_{s-1}+\alpha\|(\bar{H},\nabla)K_0^{\alpha}\|_{s-1} \le \Delta_0$.

We note that (1.5.a) and (1.5.b) imply that $(\bar{H},\nabla)v_0^{\infty}=(\bar{H},\nabla)K_0^{\infty}=0$ and there exists a constant $\Delta_1>0$ such that

(1.5.c) $\|v_0^{\alpha}\|_s+\|K_0^{\alpha}\|_s \le \Delta_1$.

Now, our main results are the following

THEOREM Assume that (1.2),(1.5.a) and (1.5.b) hold. Then there exist a constant $T_*>0$, independent of α, and vecter fields

$$(v^{\infty},K^{\infty}) \in C([0,T_*];H^s(G)) \cap C^1([0,T_*];H^{s-1}(G))$$

such that

$$(v^{\alpha},K^{\alpha}) \to (v^{\infty},K^{\infty}) \quad \text{weak}^* \text{ in } L^{\infty}([0,T_*];H^s(G)), \text{ as } \alpha \to \infty,$$

and (v^{∞},K^{∞}) is a unique solution of the following system

(1.6.a) $(\partial_t+(v^{\infty},\nabla))v^{\infty}+K^{\infty}\times \mathrm{rot}K^{\infty}+\nabla q^{\infty}-(\bar{H},\nabla)L^{\infty}=0$

(1.6.b) $(\partial_t+(v^{\infty},\nabla))K^{\infty}-(K^{\infty},\nabla)v^{\infty}-(\bar{H},\nabla)u^{\infty}=0$

(1.6.c) $\mathrm{div}\ v^{\infty}=\mathrm{div}\ K^{\infty}=0,\ (\bar{H},\nabla)v^{\infty}=(\bar{H},\nabla)K^{\infty}=0$ in $[0,T_*]\times G$

(1.6.d) $\mathrm{div}\ u^{\infty}=\mathrm{div}\ L^{\infty}=0$

(1.6.e) $v^{\infty}(0)=v_0^{\infty},\ K^{\infty}(0)=K_0^{\infty}$ on G.

Here $(\nabla q^{\infty},(\bar{H},\nabla)u^{\infty},(\bar{H},\nabla)L^{\infty})$ is uniquely determined by

(1.7) $(\nabla(p^{\alpha}+\alpha\bar{H}\cdot K^{\alpha}),\alpha(\bar{H},\nabla)v^{\alpha},\alpha(\bar{H},\nabla)K^{\alpha}) \to (\nabla q^{\infty},(\bar{H},\nabla)u^{\infty},(\bar{H},\nabla)L^{\infty})$ weak* in $L^{\infty}([0,T_*];H^{s-1}(G))$.

<u>Remarks</u> (1) It follows that $(\nabla q^{\infty},(\bar{H},\nabla)u^{\infty},(H,\nabla)L^{\infty}) \in C([0,T_*];H^{s-1}(G))$

and $(q^{\infty},u^{\infty},L^{\infty})\in L^{\infty}([0.T_{*}];L^2_{loc}(\mathbb{R}^3))$ or $L^{\infty}([0,T_{*}];L^2(\mathbb{T}^3))$.

(2) If $G=\mathbb{R}^3$, then $(\bar{H},\nabla)v^{\infty}=(\bar{H},\nabla)K^{\infty}=0$ imply $v^{\infty}=K^{\infty}=0$. Therefore, (1.6.a)-(1.6.d) are simply described by

(1.8.a) $\nabla q^{\infty}-(\bar{H},\nabla)L^{\infty}=0$

(1.8.b) $(\bar{H},\nabla)u^{\infty}=0$ in $\mathbb{R}^3$.

(1.8.c) $\operatorname{div} u^{\infty}=\operatorname{div} L^{\infty}=0$

The sketch of the proof to our theorem is as follows. We show the energy estimates of the solutions to (1.3.a)-(1.3.d), which are uniformly to α. Next, it is proved similar to [6] that the solutions converge as $\alpha \to \infty$. Finally, to determine the limiting system, we essentially employ the following

<u>LEMMA</u> Let $\{V^{\alpha}(t,x)\}$ be the sequence of functions satisfying the following assumptions:

(1.9.a) $V^{\alpha}\in C([0,T_{*}];H^s(G))$

and there exists a constant $\Delta_2>0$, independent of α, such that

(1.9.b) $\|(\bar{H},\nabla)V^{\alpha}(t)\|_{s-1}\le \Delta_2$ for any $t\in[0,T_{*}]$.

Then, by passing to a subsequence, there exists a function $V^{\infty}(t,x)$ such that, as $\alpha \to \infty$,

(1.10.a) $\hat{V}^{\alpha} \to V^{\infty}$ weak* in $L^{\infty}([0,T_{*}];L^2_{loc}(G))$,

(1.10.b) $(\bar{H},\nabla)\hat{V}^{\alpha}=(\bar{H},\nabla)V^{\alpha} \to (\bar{H},\nabla)V^{\infty}$ weak* in $L^{\infty}([0,T_{*}];H^{s-1}(G))$,

where $\hat{V}^{\alpha}(t,x)=V^{\alpha}(t,x)-V^{\alpha}(t,x-(\bar{H}/|\bar{H}|,x)\bar{H}/|\bar{H}|)$.

<u>Remark</u> In the case of $G=\mathbb{R}^3$, L^2_{loc} means localization in the direction to $\bar{H}$.

For the detailes of the proof see [3].

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SINGULAR LIMIT APPROACH TO STABILITY AND BIFURCATION FOR REACTION DIFFUSION SYSTEMS

— The SLEP method and its applications —

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Patterns with sharp transition layers appear in various fields such as patchiness and segregation in eco-systems, travelling waves in excitable media, striking patterns in morphogenesis models, dendric patterns in solidification problem, and so on.

The most simple but substantial model system, to which most of the above ones fall, is given by the following reaction-diffusion equations:

$$(P) \qquad \begin{aligned} u_s &= d_1 \Delta u + f(u,v) \\ v_s &= d_2 \Delta v + \delta g(u,v) \end{aligned} \qquad \text{on } \Omega,$$

where d_1 and d_2 are the diffusion rates of u and v and δ is the ratio of the reaction rates. The region Ω is either $(-\ell, \ell)$, $\mathbb{R}$, a rectangle, or a channel. The Neumann boundary conditions is added to (P), if necessary. It is usually assumed in (P) that one of the following conditions holds:

(a) Difference in the diffusion rates of u and v;

(b) Difference in the reaction rates of u and v;

(c) Combination of (a) and (b).

Most of the symmetry breaking stationary patterns in the framework of Turing's diffusion driven instability fall into the first category. One of the well known models is the Gierer and Meinhardt equation describing morphogenetic patterns. Propagator-controller systems including a simple skelton model for the B-Z reaction lie in the second category. It is essential for such systems that one of the components reacts much faster than the other. Formally speaking, the FitzHugh-Nagumo equations belong to the third category in which the second component v does not diffuse and reacts much slower than the first one. However, the qualitative behavior of solutions of the FHN equations is almost similar to those of propagator-controller systems. For this reason, the FHN equations fall essentially into the second category. Moreover, for this specific model, there are already several results for the stability of the travelling pulses.

Here, we focus on the third category where the first component u reacts much faster than the second one v, although u diffuses slower than v (See [3] and [4] for the first category case). More specifically, we use the new paremeters

$$\varepsilon = \sqrt{d_1}\ , \quad \tau = \delta/\sqrt{d_1}\ , \quad D = d_2/\delta$$

and rewrite (P) as

$$(P)_{\varepsilon,\tau} \quad \begin{cases} \varepsilon\tau u_t = \varepsilon^2\Delta u + f(u,v) \\ v_t = D\Delta v + g(u,v)\ , \end{cases}$$

where we used the new time variable $t = \delta s$. We assume that ε (> 0) is sufficiently small. Therefore, we can use the singular perturbation method to obtain, for instance, stationary solutions and travelling waves. The most simple example for the nonlinearities f and g is that f is cubic-like and g is a linear function of u and v. The parameter τ and ε are called the *relaxation* and *layer* parameters, respectively, since τ controls the ratio of the reaction rates and ε represents the width of the transition layer. Note that $(P)_{\varepsilon,\tau}$ covers all the above three categories, when τ varies in R_+. In fact, when $\tau = O(1/\varepsilon)$ (resp. $O(\varepsilon)$), it belongs to the class (a)(resp. (b)), and when $\tau = O(1)$, it falls into the class (c). In this sense, we may say that the category (c) is the intermediate one between (a) and (b). It should be noted here that the typical patterns observable in the classes (a) and (b) are different: stationary patterns are stable in the category (a), while propagating waves are more common in the category (b). Roughly speaking, the fronts always settle down somewhere in (a), however they move to some direction in (b). Therefore, one can imagine that some kind of transition process might happen to the structure of solutions, when the system shifts from (a) to (b) (i.e., τ decreases in $(P)_{\varepsilon,\tau}$).

In this talk, we mainly focus on the one-dimensional case (i.e., $\Omega = I \subset \mathbb{R}$), and consider the following problems.

(1) Find the observable patterns in the category (c), and clarify their stability properties.

(2) Describe the transition process when the system shifts from (a) to (b), and, especially, what kind of bifurcation phenomena occurs when τ decreases ?

Before stating the third problem, we remark the following fact. For an appropriately fixed τ, the stable patterns in the category (c) drastically change depending on whether I is finite or not. Namely, we have the layer oscillation (spacially inhomogeneous and time periodic solutions) for the finite interval case, while the travelling front solutions appear for the infinite line case. In other words, there appears a *Hopf* bifurcation for the finite case, but a *static* bifurcation of travelling type for the infinite case. Thus, the last problem is

(3) How the structure of the bifurcating solutions in the category (c) is deformed when the length ℓ of the interval becomes infinite.

The answers for the problems (1) and (2) are already obtained ,at least partially, in [1], [2], [6], and [7]. For the third problem, it has been proved in [2] that the behavior of the critical eigenvalues causing the bifurcation is deformed continuously when the length ℓ tends to infinity.

In order to deal with the above problems rigorously, we always face to the difficulties coming from the *largeness* of the amplitude (because of the existence of sharp transition layers) and the *smallness* of ε. Nevertheless, at least for the existence of layered solutions, several methods have been developped systematically. For instance, singular perturbation method (or matched asymptotic method) is one of the most powerful and constructive methods. However, there have been very few *unified* approach to treat the stability and bifurcation problems for large amplitude singularly perturbed solutions. There are several reasons for this. First of all, since the singularly perturbed solution has a sharp front at each layer position which becomes a discontinuous point as $\varepsilon \downarrow 0$, the eigenfunctions of the linearized problem at the singularly perturbed solution in general do *not* remain as usual functions when $\varepsilon \downarrow 0$. Also one of the linearized equations becomes an algebraic one, since the the derivative of the second order vanishes as $\varepsilon \downarrow 0$. Secondly, for the bifurcation problem as in (2), how one can control critical eigenvalues (i.e., Re-parts of them are close to zero) of the linearized problem at a singularly perturbed solution *uniformly* with respect to small ε. Despite these degeneracies, the most desirable thing is to find a nice limiting system from which one can extract necessary information on the behavior of the spectrum for small ε.

For that purpose, some blowing up technique is necessary to take advantage of the smallness ε, otherwise just formal limiting arguments (when $\varepsilon = 0$, not $\varepsilon \downarrow 0$) bring us insufficient information on stability properties of sin~ularly perturbed solutions.

The basic tool, which will be employed here, to overcome the above difficulties is *the Singular Limit Eigenvalue Problem* (SLEP) method which enables us to study stability and bifurcation problems of singularly perturbed solutions (see [2], [3], [4], [5], [6], and [7]). The key idea of the SLEP method is to reserve the information coming from layers in the form of the *distribution along the interface* (the *Dirac's point mass* distribution in one-dimensional case) of the linearized problem as $\varepsilon \downarrow 0$. The weight function of mass distribution plays an important role to determine the behavior of critical eigenvalues. An appropriate ε-scaling to eigenfunctions of the scalar operator $\varepsilon^2\Delta + f_u^\varepsilon$ ((1,1)-component of the linearized problem) is crucial to derive the SLEP system. It is expected that the SLEP method could solve the similar problems in higher and general domains (see also [5]).

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L^2-concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical power nonlinearity

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In the present paper we consider the solution of the Schrödinger equation:

(1) $i\frac{\partial u}{\partial t} = -\Delta u + f(u), \qquad u(0,x) = \varphi(x),$

where Δ is the Laplace operator on $\mathbb{R}^N$, $u:[0,T)\times\mathbb{R}^N \to \mathbb{C}$, f is a complex-valued function such that $f(z) \simeq -|z|^{N/4}z$ as $|z| \to \infty$ and $\varphi \in H^1$.

In the special case of $f(z) = -|z|^{p-1}z$ with $p \in (1, 2^*-1)$ (where $2^* = 2N/(N-2)$ if $N \geq 3$, otherwise $2^* = +\infty$), it is well known that for $p \geq 1 + 4/N$, there are singular solutions of the equation (1) for suitable initial data (See Zakharov, Sobolev and Synackh [16], Glassey [5] and M. Tsutsumi[12]). That is, there are some solutions $u(t)$ of the equation (1) such that $u(\cdot) \in C([0,T); H^1)$ and $\lim_{t\to T} \|u(t)\|_{H^1} = +\infty$. We study the behavior at time T of blow-up solutions $u(t)$ in the critical case $p = 1 + 4/N$. For $N = 2$, this case has a physical interest: it can be considered as the first approximation of a model of a planar beam which is propagating along a single direction t in $\mathbb{R}^3$.

The phenomena which occur in the case where $p = 1 + 4/N$ seem to be quite different from the other cases. Indeed, for $p < 1 + 4/N$, blowing-up in finite time never occurs (Ginibre and Velo [3]). For $p = 1 + 4/N$, there are some examples of explicit blow-up solutions without a strong limit in L^2 at blow-up time (see Weinstein [15] and Nawa and M. Tsutsumi [9]). In the supercritical case $1 + 4/N < p < 2^*-1$, the numerical computations (Lemesurier, Papanicolaou, C. Sulem and P.L. Sulem [7]) and some mathematical analysis (Merle [8]) suggest that every blow-up solution has a strong limit in L^2 at the blow-up time.

In addition, in the critical case the explicit examples of blow-up solutions lose their L^2 continuity because of a "mass concentration" phenomenon. That is, the L^2 density concentrates at the blow-up point (see Weinstein [15] and Nawa and M. Tsutsumi [9]).

In this paper, in the case where $f(z)$ behaves like the critical power $-|z|^{4/N}z$ as $|z| \to +\infty$, we consider the following two questions:

Are there some blow-up solutions with a strong limit in L^2 at blow-up time?

Does the L^2 concentration occur for the other blow-up solutions than the already known explicit blow-up solutions?

We assume that $f:\mathbb{C} \to \mathbb{C}$ satisfies the following assumptions:

(F.1) $f(0) = 0$.

(F.2) f is a continuously differentiable function such that there is $g: \mathbb{R} \to \mathbb{R}$ such that $f(z) = g(|z|^2)z$.

(F.3) $f(z) = h(z) - |z|^{4/N}z$ where $|h'(z)| \leq c|z|^{r-1}$ for some $c > 0$ and $1 \leq r < 1 + 4/N$.

Under the assumptions (F.1)-(F.3) the equation (1) has a unique solution $u(t)$ in H^1 and there exists T such that $u(t) \in C([0,T);H^1)$ and either $T = +\infty$ or $\lim_{t\to T} \|u(t)\|_{H^1} = +\infty$ (see Ginibre and Velo [3][4], Kato [5] and Cazenave and Weissler [1][2]). In addition, we have for all $t \in [0,T)$

(2) $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$,

(3) $E(u(t)) = \|\nabla u(t)\|_{L^2}^2 + \int F(|u(t,x)|)dx = E(\varphi)$,

where $F(\rho) = 2\int_0^\rho f(s)ds$.

We have the following theorem which answers to the first question:

Theorem 1. Assume that f satisfies (F.1)-(F.3) and $\varphi \in H^1$. If the solution $u(t)$ in $C([0,T);H^1)$ of the equation (1) blows up at $t = T$, then there is no sequence $\{t_n\}$ such that $t_n \to T$ and $u(t_n)$ converges strongly in L^2 as $t_n \to T$.

Remark 1. It is already known that when $f(z) = -|z|^{4/N}z$ we can construct a local L^2 solution of the equation (1) for all initial data in L^2 (see Cazenave and Weissler [2], Y. Tsutsumi [13] and Strauss [11]). Therefore Theorem 1 implies that for an initial datum $\varphi \in H^1$ the blow-up solution can not be extended beyond the blow-up time in the strong topology of L^2. That is, the blow-up time in the H^1 framework is the same as the blow-up time in the L^2 framework.

Remark 2. Theorem 1 is the first result of nonexistence of a strong limit in L^2 in the case where $h \neq 0$. Indeed, we do not have to assume the pseudoconformal invariance which is essentially used to find explicit examples of blow-up solutions.

Assume now that φ has a spherical symmetry and so $u(t)$ has the same symmetry. This symmetry implies that the L^2 concentration occurs at the origin at the blow-up time.

<u>Theorem 2</u>. Assume that f satisfies (F.1)-(F.3), φ has a spherical symmetry and $N \geq 2$. If the solution $u(t)$ in $C([0,T);H^1)$ of the equation (1) blows up at time $t = T$, then the origin 0 is a blow-up point and for each $R > 0$

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<R)} \geq \|Q\|_{L^2},$$

where Q is a ground state solution of the equation:

$$(4) \qquad -\Delta u + u - |u|^{4/N}u = 0 \quad \text{in } \mathbb{R}^N.$$

<u>Remark 4</u>. If $x_0 \in \mathbb{R}^N$ satisfies

$$\|\nabla u(t)\|_{L^2(|x|<R)} \to +\infty \quad (t \to T)$$

for any $R > 0$, then we say that x_0 is a blow-up point.

<u>Remark 5</u>. It is already known that there is a ground state solution of (4), that is, the non-trivial least energy solution of (4) (see, e.g., Weinstein [14]). In addition, we remark that the lower bound $\|Q\|_{L^2}$ in Theorem 2 depends only on the behavior of f at infinity.

<u>Remark 6</u>. In the case where $f(z) = -|z|^{4/N}z$ and $\|\varphi\|_{L^2} = \|Q\|_{L^2}$, Weinstein proved Theorem 2. Theorem 2 covers the more general

nonlinearity and the spherically symmetric initial data with $\|\varphi\|_{L^2} \neq \|Q\|_{L^2}$.

Remark 7. In Theorem 2 we do not have to assume the pseudoconformal invariance.

Remark 8. Theorem 2 is optimal in the following sense: there is an explicit example of blow-up solution such that for all R > 0

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<R)} = \|Q\|_{L^2} .$$

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$L_q - L_r$ estimates for solutions of the nonstationary Stokes equation in an exterior domain and the Navier-Stokes initial value problems in L_q spaces

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1. INTRODUCTION

Let Ω be an exterior domain in R^n, $n \geq 3$, with smooth boundary $\partial\Omega$. Consider the exterior nonstationary problems for the Navier-Stokes equation:

$$\text{(NS)}\quad \begin{aligned} \partial_t u - \Delta u + (u\cdot\nabla)u &= -\nabla p && \text{in } (0,\infty)\times\Omega,\\ \operatorname{div} u &= 0 && \text{in } (0,\infty)\times\Omega,\\ u|_{\partial\Omega} &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\ u(x,0) &= a(x) && \text{in } \Omega. \end{aligned}$$

In this note we show that if the initial data $a(x)$ is a small solenoidal vector function in $L_n(\Omega)^n$, then there exists a unique global strong solution $u(t,x)$ to (NS) with some decay properties. To do this, we shall follow the argument of Kato [7] so that the crucial step consists in obtaining $L_q - L_r$ estimates for the semigroup generated by the Stokes operator in the exterior domain Ω and its derivatives of first order. We arrive at the $L_q - L_r$ estimates by studying the resolvent expansions. The basic ideas are similar to those of Iwashita-Shibata [5] and Shibata [10] (see also Vainberg [11]). All the details in this note are given in Iwashita [6].

2. STATEMENT OF THE MAIN RESULTS

Let $1 < q < \infty$ and set

$$J_q(\Omega) = \text{the completion in } L_q(\Omega)^n \text{ of } \{u \in C_0^\infty(\Omega)^n; \operatorname{div} u = 0\}.$$

We denote by P the projection from $L_q(\Omega)^n$ onto $J_q(\Omega)$. The Stokes operator A is defined by $A = -P\Delta$ with dense domain

$$\mathcal{D}_q(A) = \{u \in W_q^2(\Omega)^n;\ u|_{\partial\Omega} = 0\} \cap J_q(\Omega),$$

where $W_q^2(\Omega)$ stands for the Sobolev space of order two. Recently, it has been proved by Borchers and Sohr [2] that $-A$ generates a bounded analytic semigroup e^{-tA} on $J_q(\Omega)$.

The following theorem is a fundamental result in this note.

THEOREM 2.1. *Let $n \geq 3$ and $1 < q < \infty$. Let s and s' be real numbers such that $s > n(1-1/q)$ and $s' < -n/q$. Then there exists a positive constant $C = C(q,s,s')$ such that the inequality*

$$\|e^{-tA}f\|_{L_q^{s'}(\Omega)} \leq C(1+t)^{-n/2}\|f\|_{L_q^s(\Omega)}, \quad t \geq 0$$

is valid for any $f \in J_q(\Omega) \cap L_q^s(\Omega)^n$, where $L_q^s(\Omega)$ is a weighted L_q space defined by

$$L_q^s(\Omega) = \{(1+|x|)^s f(x) \in L_q(\Omega)\}.$$

Based on this result, we obtain

THEOREM 2.2. *Let $n \geq 3$ and $1 < q < \infty$.*
(i)(Local energy decay) For any $R \gg 1$, there exists a constant $C = C(q,R) > 0$ such that

$$\|e^{-tA}f\|_{L_q(\Omega_R)} \leq C(1+t)^{-n/2q}\|f\|_{L_q(\Omega)}, \quad t \geq 0 \tag{2.1}$$

for any $f \in J_q(\Omega)$, where $\Omega_R = \{x \in \Omega;\ |x| < R\}$.
(ii)($L_q - L_r$ estimate) Let $q < r < \infty$ and put $\sigma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$. Then the estimate

$$\|e^{-tA}f\|_{L_r(\Omega)} \leq Ct^{-\sigma}\|f\|_{L_q(\Omega)}, \quad t > 0 \tag{2.2}$$

holds for any $f \in J_q(\Omega)$.

For the global solvability of (NS) with small initial data, we require, in addition to (2.3), the $L_q - L_r$ estimates for the first derivatives of e^{-tA}.

THEOREM 2.3. *Let $n \geq 3$ and $1 < q \leq r \leq n$. Then,*

$$\|\partial e^{-tA}f\|_{L_r(\Omega)} \leq Ct^{-\sigma-1/2}\|f\|_{L_q(\Omega)}, \quad t > 0 \tag{2.3}$$

for any $f \in J_q(\Omega)$, where ∂u stands for any of $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$ and $\sigma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$.

REMARK 2.4.. In [4] Giga and Sohr study the fractional powers of the Stokes operator and, as an application, prove the estimates (2.2) and (2.3) when $1 < q < n/2$, $0 \leq \sigma \leq 1$ or $1 < q \leq 2$, $0 \leq \sigma \leq 1/2$.

The restriction $r \leq n$ in Theorem 2.3 is a demand for the completely technical condition $\sigma + 1/2 \leq n/2q$, although it seems that the elimination of the restriction is left outside of our method. Nevertheless, the $L_q - L_r$ estimates (2.2) and (2.3) enable us to obtain

THEOREM 2.5. *Let $n \geq 3$. There exists a constant $\varepsilon > 0$ such that if $a \in J_n(\Omega)$ and $\|a\|_{L_n(\Omega)} < \varepsilon$, then a unique global strong solution u to (NS) exists possessing the following properties:*

$$t^{(1-n/q)/2}u \in \mathcal{B}([0,\infty);\ J_q(\Omega)) \qquad \text{for any } q,\ n \leq q < \infty, \tag{2.4}$$

$$t^{1/2}\partial u \in \mathcal{B}([0,\infty);\ L_n(\Omega)), \tag{2.5}$$

$$t^{1-n/2q}\partial u \in C([0,\infty);\ L_q(\Omega)) \qquad \text{for any } q,\ n < q < \infty, \tag{2.6}$$

where $\mathcal{B}$ denotes the class of bounded continuous functions. All the values in (2.4)-(2.6) vanish at $t = 0$ except for $q = n$ in (2.4) and,in case $q = n$, then $u(0,x) = a(x)$.

The proof of Theorem 2.5 is done along the same line as in Fujita-Kato [3] and Kato [7].

3. PROOF OF THEOREM 2.1.

Let

$$\Sigma(\delta) = \{z \in C^1 \setminus \{0\};\ \delta < \arg z < 2\pi - \delta\}, \quad 0 < \delta < \pi,$$

and let $B(\mathcal{H}_1, \mathcal{H}_2)$ denote the totality of continuous linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

THEOREM 3.1. *Let $1 < q < \infty$, $0 < \delta < \pi$, $s > n(1-1/q)$, and $s' < -n/q$. Then there exists $\varepsilon_0 > 0$ such that the resolvent $(A-z)^{-1}$ with $z \in \Sigma(\delta)$, $|z| < \varepsilon_0$ has the following expansion as an operator in $B(L_q^s(\Omega)^n \cap J_q(\Omega), W_q^{2,s'}(\Omega)^n)$:*

$$(A - z)^{-1} = z^{n/2-1}(\log z)^{\varepsilon(n)} G_1 + G_2(z) + z^{n/2-1} G_3(z), \tag{3.1}$$

where $\varepsilon(n) = 0$ for n odd and $= 1$ for n even; $G_2(z)$ is holomorphic in z and $G_3(z)$ tends to zero as $t \to 0$.

Theorem 2.1 is an immediate consequence of the expansion (3.1). We shall outline the proof of Theorem 3.1. Take numbers b and d so that $d > b \gg 1$. Let L be an operator $\in B(L_q(\Omega_d)^n, W_q^2(\Omega_d)^n)$ such that for any $f \in L_q(\Omega_d)^n$, Lf satisfies with some $p \in W_q^1(\Omega_d)$

$$\begin{aligned} -\Delta Lf + \nabla p &= f \quad \text{in } \Omega_d, \\ \operatorname{div} Lf &= 0 \quad \text{in } \Omega_d, \\ Lf|_{\partial\Omega_d} &= 0 \quad \text{on } \partial\Omega_d \end{aligned}$$

(cf. e.g., Ladyzhenskaya [8]). Let φ and ψ be C^∞-functions in R^n such that $\varphi = 1$ for $|x| \geq b$ and $= 0$ for $|x| \leq b-1$; $\psi = 1$ for $|x| \geq b-2$ and $= 0$ for $|x| \leq b-3$. For $f \in L_q^s(\Omega)^n$, let $f_d = f|_{\Omega_d}$ and let $f_0 = f$ in Ω and $= 0$ in $R^n \setminus \Omega$. We define a regularizer $R_1(z)$ by

$$R_1(z)f = \varphi R_0(z)(\psi f_0) + (1-\varphi)Lf_d - Q(z)f, \quad f \in L_q^s(\Omega)^n,$$

where $R_0(z)$ is the resolvent of the Stokes operator in R^n:

$$R_0(z)g(x) = \mathcal{F}^{-1}\left[\frac{P_0(\xi)\hat{g}(\xi)}{|\xi|^2 - z}\right](x), \quad P_0(\xi) = \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right),$$

and $R_0(z)$ has the same type of expansion as in (3.1), which is derived from the resolvent expansion for the Laplacian by Murata [9]. The operator $Q(z)$ is a modifier so that $\operatorname{div} R_1(z)f = 0$ in Ω, and its existence is assured by the following result due to Bogovskii [1].

PROPOSITION 3.2. *Let D be a bounded domain in R^n, $n \geq 2$, with smooth boundary. Let $1 < q < \infty$ and let m be a nonnegative integer. For any $f \in \overset{\circ}{W}{}_q^{m+2}(D)$ with*

$$\int_D f(x)\,dx = 0,$$

there exists a vector field $u \in \overset{\circ}{W}{}_q^{m+2}(D)^n$ *such that* u *fulfills* $\operatorname{div} u = f$ in D and

$$\|u\|_{W_q^{m+1}(D)} \le C\|f\|_{W_q^m(D)}.$$

The operator $R_1(z)$ also satisfies the Dirichlet boundary condition: $R_1(z)f|_{\partial\Omega} = 0$ if $f \in L_q^s(\Omega)^n$. A pressure Πf associated with $R_1(z)f$ is defined as follows:

$$\Pi f = \varphi p_0 + (1-\varphi)p_1, \quad f \in L_q^s(\Omega)^n,$$

where

$$p_0(x) = \mathcal{F}^{-1}\left[\frac{\xi \cdot \mathcal{F}(\psi f_0)(\xi)}{i|\xi|^2}\right](x)$$

and p_1 is a pressure associated with Lf_d and satisfies

$$\int_{\Omega_d} p_1(x)dx = \int_{|x|\le d} p_0(x)dx.$$

Then it turns out that $\Pi \in B(L_q^s(\Omega)^n, W_q^{1,s'}(\Omega))$ and $\nabla\Pi \in B(L_q^s(\Omega)^n, L_q(\Omega)^n)$. The operators $R_1(z)$ and Π thus defined obey

$$(-\Delta - z)R_1(z)f + \nabla\Pi f = f + S(z)f \quad \text{in } \Omega,$$

where $S(z)$ is a compact operator in $L_q^s(\Omega)^n$ and is holomorphic in $z \in \Sigma(\delta)$.

LEMMA 3.3. *The inverse* $(I+S(z))^{-1}$ *of* $I+S(z)$ *exists as a* $B(L_q^s(\Omega)^n, L_q^s(\Omega)^n)$*-valued meromorphic function of* $z \in \Sigma(\delta)$ *and has no poles in* $\Sigma(\delta) \cap \{|z| < \varepsilon_0\}$ *for some* $\varepsilon_0 > 0$. *The same type of expansion as (3.1) is valid for* $(I+S(z))^{-1}$.

This lemma is an immediate consequence of analytic Fredholm's alternative and the fact that the inverse $(I+S(0))^{-1}$ does exist in $B(L_q^s(\Omega)^n, L_q^s(\Omega)^n))$. On the way to this result, a crucial role is played by the following result on uniqueness.

PROPOSITION 3.4. *Let* $n \ge 3$ *and* $1 < q < \infty$. *Suppose that* $u \in W_q^{1,\tau}(\Omega)^n$ *with* $\partial^\alpha u \in L_q(\Omega)^n$, $|\alpha| = 2$, *and* $p \in L_q^{\tau'}(\Omega)$ *with* $\nabla p \in L_q(\Omega)^n$ *for some* $\tau, \tau' \in R^1$, *where* $W_q^{1,\tau}(\Omega)$ *is a weighted Sobolev space, and assume that* u *and* p *satisfy*

$$-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega,$$

and

$$\lim_{R\to\infty} \frac{1}{R^n}\int_{R<|x|<2R} |u(x)|^q dx = \lim_{R\to\infty} \frac{1}{R^n}\int_{R<|x|<2R} |p(x)|^q dx = 0.$$

Then, $u = 0$ *and* $p = 0$.

By Lemma 3.3, we conclude that $(A-z)^{-1} = R_1(z)(I+S(z))^{-1}$, which proves Theorem 3.1.

4. Proofs of Theorems 2.2 and 2.3

The estimates (2.3) and (2.4) for small $t > 0$ are obtained by Sobolev's embedding theorem and a real interpolation method from the estimate

$$\|u\|_{W_q^{2m}(\Omega)} \le C_m \left(\|A^m u\|_{L_q(\Omega)} + \|u\|_{L_q(\Omega)}\right)$$

for any $u \in \mathcal{D}_q(A^m)$.

For large time estimates, we use the result in Theorem 2.1. We may take $u_0 = e^{-A} f$ as the initial value. Let v_0 be an extension of u_0 to R^n such that $\mathrm{div} v_0 = 0$ in R^n. We define an operator $E(t)$ by

$$E(t)f = (4\pi t)^{-n/2} \int_{R^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)dy, \quad f \in L_q(R^n).$$

Then it is well known that the $W_q^{2\sigma} - L_r$ estimates for $E(t)$ and $\partial E(t)$ are given by $C(1+t)^{-\sigma}$ and $C(1+t)^{-\sigma-1/2}$, respectively. Choose $\varphi \in C_0^\infty(R^n)$ so that $\varphi = 1$ on a neighbourhood of Ω. Then, Proposition 3.2 permits us to have a modifier $v_1(t)$, which satisfies

$$\mathrm{div}\,[(1-\varphi)E(t)v_0 - v_1(t)] = 0 \quad \text{in}\, R^n.$$

Put

$$\begin{aligned} v_2(t) &= e^{-tA}u_0 - (1-\varphi)E(t)v_0 - v_1(t) && \text{in}\, \Omega, \\ v_2 &= v_2(0) && \text{in}\, \Omega. \end{aligned}$$

Then, $v_2 = 0$ for large $|x|$, and there exists $p(t)$ such that

$$\begin{aligned} (\partial_t - \Delta)v_2(t) + \nabla p(t) &= g(t) && \text{in}\,(0,\infty)\times\Omega, \\ \mathrm{div}\, v_2(t) &= 0 && \text{in}\,(0,\infty)\times\Omega, \\ v_2(t)|_{\partial\Omega} &= 0 && \text{on}\,(0,\infty)\times\partial\Omega, \\ v_2(0) &= v_2 && \text{in}\,\Omega, \end{aligned}$$

where $g(t)$ satisfies div $g(t) = 0$ and supp $g(t)$ is compact in Ω. Therefore we have

$$v_2(t) = e^{-tA}v_2 + \int_0^t e^{-(t-s)A} g(s)ds.$$

This identity and Theorem 2.1 lead us to the $L_q(\Omega_R)$- estimates for $v_2(t)$, which combined with the $L_q(\Omega_R)$- estimates for $E(t)v_0$ and $v_1(t)$ to give Theorem 2.2, (i). Similarly, we have the same bound of $e^{-tA}u_0$ in $W_q^m(\Omega_R)^n$.

Next, we take $\psi \in C^\infty(R^n)$ such that $\psi = 1$ for $|x|$ large and $= 0$ in a neighbourhood of Ω. For some modifier $v_3(t)$, we set $v_4(t) = \psi u(t) - v_3(t)$ and then have

$$\begin{aligned} (\partial_t - \Delta)v_4(t) + \nabla(\psi p(t)) &= h(t) && \text{in}\,(0,\infty)\times R^n, \\ \mathrm{div}\, v_4(t) &= 0 && \text{in}\,(0,\infty)\times R^n, \\ v_4(0) &= 0 && \text{in}\,\Omega, \end{aligned}$$

where supp $h(t)$ is compact. Hence we have

$$v_4(t) = E(t)v_4(0) + \int_0^t E(t-s)P_0 h(s)ds,$$

where $P_0 = \mathcal{F}^{-1}[P_0(\xi)\mathcal{F}\cdot]$. From this we obtain the L_r-estimates for $v_4(t)$ and $\partial v_4(t)$ and, hence, for $\psi e^{-tA}u_0$ and $\partial(\psi e^{-tA}u_0)$. We combine this with the $L_r^{loc}(\bar{\Omega})$ estimates to verify Theorems 2.2 and 2.3.

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On Nonlinear Small Data Scattering

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Let X be a Hilbert space with norm $\| \ \|_X$, and A be a selfadjoint operator in X with dense domain $\mathcal{D}(A) \subset X$. We consider the evolution equation

$$(2.1) \qquad \begin{cases} i\partial_t u = Au + F(u), & t \in R \\ u|_{t=-\infty} = \varphi_- \in X. \end{cases}$$

It is convenient to rewrite (2.1) into the integral form. Let

$$(2.2) \qquad U_0(t) = \exp\{-iAt\}, \quad t \in R.$$

Then we have from (2.1)

$$(2.3) \qquad u(t) = U_0(t)\varphi_- + \int_{-\infty}^{t} U_0(t-\tau)F(u(\tau))d\tau.$$

We make the following hypotheses.

(I) There exist Banach spaces Y and Z such that X, Y and Y' are continuously embedded in Z, and Z' is dense in each X, Y and Y'. Here Y' and Z' are the dual spaces, with respect to X, of Y and Z, respectively.

(II) $U_0(t)$ restricted to $X \cap Y'$ has a continuous linear extension (still denoted $U_0(t)$) which maps Y' to Y, and there exist $c > 0$ and $0 < d < 1$ such that

$$(2.4) \qquad \|U_0(t)f\|_Y \le c|t|^{-d}\|f\|_{Y'} \quad \text{for } t \neq 0 \text{ and } f \in Y'.$$

$U_0(t)$ also has a continuous extension from Y to Z such that

$$(2.5) \qquad U_0(t)U_0(s)f = U_0(t+s)f \quad \text{for } f \in Y'.$$

(III) F maps $X \cap Y$ to Y', $F(0) = 0$ and we have

$$(2.6) \qquad \|F(u)-F(v)\|_{Y'} \le c\|u-v\|_X\{\|u\|_Y^{s-1}+\|v\|_Y^{s-1}\}$$
$$+ c\{\|u\|_X+\|v\|_X\}\|u-v\|_Y\{\|u\|_Y^{s-2}+\|v\|_Y^{s-2}\}$$

for u, $v \in X \cap Y$, where $s = \frac{2}{d}$.

(IV) Moreover, F maps Y into X and we have

$$(2.7) \qquad \|F(u)-F(v)\|_X \le c\|u-v\|_Y\{\|u\|_Y^{s-1}+\|v\|_Y^{s-1}\} \quad \text{for } u,\ v \in Y.$$

The integral equation (2.3) will be considered in the following space of functions $u(t)$:

$$(2.8) \qquad W = L^s(R;\ Y) \cap L^\infty(R;\ X).$$

Theorem 2.1. *Under (I) ~ (IV) there exists a $\delta > 0$ with the following properties: If $\varphi_- \in X$ and $\|\varphi_-\|_X \le \delta$, then there exists a unique solution $u(t) \in W$ of the integral equation (2.3) such that*

$$(2.9) \qquad \|u\|_W \le \frac{4}{3}\ \|U_0(t)\varphi_-\|_W \le \frac{4c}{3}\ \|\varphi_-\|_X\ ;$$

$$(2.10) \qquad \|u(t)-\ U_0(t)\varphi_-\|_X \longrightarrow 0 \quad \text{as} \quad t \longrightarrow -\infty\ .$$

Furthermore, there exists a unique $\varphi_+ \in X$ such that

$$(2.11) \qquad \|u(t)-\ U_0(t)\varphi_+\|_X \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty\ .$$

Thus, we can define the scattering operator S: $\varphi_- \longrightarrow \varphi_+$ on a neighborhood of 0 in X.

The proof of this theorem will be done based on a contraction mapping principle. For this aim we prepare three propositions.

Throughout this paper the following Hardy-Littlewood-Sobolev inequality will play an important role. As for the proof see e.g., Hörmander [ ], Theorem 4.5.3 .

Lemma 2.2. *If* $1 < p < q < \infty$ *and*

$$\frac{1}{q} = \frac{1}{p} - \frac{\nu}{n}, \tag{2.12}$$

then

$$\| |x|^{-n+\nu} * f \|_{L^q} \le c(p,\nu) \|f\|_{L^p} \quad \text{for } f \in L^p(R^n). \tag{2.13}$$

Proposition 2.3. $f(t) \longrightarrow \int_{-\infty}^{\infty} U_0(-t) f(t) dt$ *is a continuous map of* $L^{s'}(R;\ Y')$, *where* $\frac{1}{s'} = 1 - \frac{1}{s}$, *to* X. *Namely, there exists a* $C_1 > 0$ *such that*

$$\left\| \int_{-\infty}^{\infty} U_0(-t) f(t) dt \right\|_X \le C_1 \|f\|_{L^{s'}(Y')} \quad \text{for } f(t) \in L^{s'}(R;\ Y'). \tag{2.14}$$

Proof. We have only to prove (2.14) for $f(t)$ in a dense set of $L^{s'}(R;\ Y')$. Let $f(t) \in C_0^{\infty}(R;\ Z')$. In this case we can change the order of integrations to obtain

$$\left\| \int_{-\infty}^{\infty} U_0(-t) f(t) dt \right\|_X^2 = \left| \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} U_0(t-\tau) f(\tau) d\tau,\ f(t) \right)_X dt \right|.$$

Here $(\ ,\ )_X$ denotes the innerproduct in X, or more generally, the duality between Z and Z'. Using (I), (II) and the Hölder inequality, we then have

$$\le \int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} U_0(t-\tau) f(\tau) d\tau \right\|_Y \|f(t)\|_{Y'} dt$$

$$\le \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} c|t-\tau|^{-d} \|f(\tau)\|_{Y'} d\tau \right] \|f(t)\|_{Y'} dt$$

$$\le c \left\| \int_{-\infty}^{\infty} |t-\tau|^{-d} \|f(\tau)\|_{Y'} d\tau \right\|_{L^s} \|f\|_{L^{s'}(Y')}.$$

The requirement $\frac{1}{s} = \frac{d}{2}$ implies that $\frac{1}{s} = \frac{1}{s'} - (1-d)$. Thus, we can apply Lemma 2.2 with $n = 1$ and $\nu = 1-d$ to obtain

$$\leq cc(s',1-d)\|f\|^2_{L^{s'}(Y')}.$$

This proves (2.14) if we put $C_1 = \sqrt{cc(s',1-d)}$. □

Proposition 2.4. *Let $\varphi \in X$. Then $U_0(t)\varphi \in L^s(\mathrm{R}; Y)$ and we have*

$$(2.15) \qquad \|U_0(t)\varphi\|_{L^s(Y)} \leq C_1\|\varphi\|_X \quad \text{for } \varphi \in X,$$

where C_1 is the constant given in (2.14).

Proof. Let $\varphi \in X$ and $f(t) \in C_0^\infty(\mathrm{R}; Z')$. Then we have from the above proposition

$$\left|\int_{-\infty}^{\infty} (f(t), U_0(t)\varphi)_X dt\right| = \left|\left(\int_{-\infty}^{\infty} U_0(-t)f(t)dt, \varphi\right)_X\right|$$
$$\leq C_1\|f\|_{L^{s'}(Y')}\|\varphi\|_X.$$

This proves (2.15) since $C_0^\infty(\mathrm{R}; Z')$ is dense in $L^{s'}(\mathrm{R}; Y')$. □

Proposition 2.5. *There exists a $C_2 > 0$ such that*

$$(2.16) \quad \left\|\int_{-\infty}^{t} U_0(t-\tau)\{F(u(\tau))-F(v(\tau))\}d\tau\right\|_W \leq C_2\|u-v\|_W\{\|u\|_W^{s-1}+\|v\|_W^{s-1}\}$$

for $u(t)$, $v(t) \in W$.

Proof. By (II) and (III)

$$\left\|\int_{-\infty}^{t} U_0(t-\tau)\{F(u(\tau))-F(v(\tau))\}d\tau\right\|_{L^s(Y)}$$
$$\leq \left\|\int_{-\infty}^{\infty} c|t-\tau|^{-d}\|F(u(\tau))-F(v(\tau))\|_{Y'}d\tau\right\|_{L^s}$$
$$\leq \left\|\int_{-\infty}^{\infty} c|t-\tau|^{-d}\|u(\tau)-v(\tau)\|_X\{\|u(\tau)\|_Y^{s-1}+\|v(\tau)\|_Y^{s-1}\}d\tau\right\|_{L^s}$$
$$+ \left\|\int_{-\infty}^{\infty} c|t-\tau|^{-d}\{\|u(\tau)\|_X+\|v(\tau)\|_X\}\|u(\tau)-v(\tau)\|_Y \times\{\|u(\tau)\|_Y^{s-2}+\|v(\tau)\|_Y^{s-2}\}d\tau\right\|_{L^s}.$$

Noting $\frac{1}{s} = \frac{1}{s'}-(1-d)$, we can apply Lemma 2.2 to obtain

$$\leq C\|u-v\|_{L^\infty(X)}\{\|u\|^{s-1}_{L^s(Y)}+\|v\|^{s-1}_{L^s(Y)}\}$$

$$+ C\{\|u\|_{L^\infty(X)}+\|v\|_{L^\infty(X)}\}\|u-v\|_{L^s(Y)}\{\|u\|^{s-2}_{L^s(Y)}+\|v\|^{s-2}_{L^s(Y)}\}.$$

On the other hand, by (IV)

$$\left\|\int_{-\infty}^{t} U_0(t-\tau)\{F(u(\tau))-F(v(\tau))\}d\tau\right\|_{L^\infty(X)}$$

$$\leq \int_{-\infty}^{\infty} \|F(u(\tau))-F(v(\tau))\|_X d\tau$$

$$\leq \tilde{C}\int_{-\infty}^{\infty} \|u(\tau)-v(\tau)\|_Y\{\|u(\tau)\|_Y^{s-1}+\|v(\tau)\|_Y^{s-1}\}d\tau$$

$$\leq \tilde{C}\|u-v\|_{L^s(Y)}\{\|u\|^{s-1}_{L^s(Y)}+\|v\|^{s-1}_{L^s(Y)}\}.$$

Summarizing these inequalities, we obtain (2.16) with $C_2 \leq 2C\tilde{C}$. □

Proof of Theorem 2.1. We put

$$(\Phi u)(t) = U_0(t)\varphi_- + \int_{-\infty}^{t} U_0(t-\tau)F(u(\tau))d\tau, \tag{2.17}$$

and consider it in the ball $\mathcal{B}(\delta_1) = \{u \in W;\ \|u\|_W \leq \delta_1\}$, where the constants $\delta_1 > 0$ and $\delta > 0$ in the theorem are chosen to satisfy

$$2C_2\delta_1^{s-1} \leq \frac{1}{2} \quad \text{and} \quad (1+C_1)\delta \leq \frac{3}{4}\delta_1. \tag{2.18}$$

Let $u \in \mathcal{B}(\delta_1)$. Then by Propositon 2.4 with $\varphi = \varphi_-$ and Proposition 2.5 with $v = 0$,

$$\|\Phi u\|_W \leq (1+C_1)\|\varphi_-\|_X + C_2\|u\|_W^s \leq \delta_1. \tag{2.19}$$

On the other hand, it follows from Proposition 2.5 that

$$\|\Phi u - \Phi v\|_W \leq C_2\{\|u\|_W^{s-1}+\|v\|_W^{s-1}\}\|u-v\|_W \leq \frac{1}{2}\|u-v\|_W \tag{2.20}$$

for any u, v $\in \mathcal{B}(\delta_1)$. (2.19) and (2.20) show that Φ defines a contractiuon map of $\mathcal{B}(\delta_1)$ into itself. Thus, there exists a unique fixed point $u \in \mathcal{B}(\delta_1)$ which solves (2.3). (2.19) and the first inequality of (2.18) imply that this u satisfies (2.9) also. Moreover, we have

$$\|u(t)-U_0(t)\varphi_-\|_X \leq \int_{-\infty}^{t} c\|u(\tau)\|_Y^s d\tau \longrightarrow 0 \quad \text{as } t \longrightarrow -\infty,$$

and (2.10) follows. Next, put

$$\varphi_+ = \varphi_- + \int_{-\infty}^{\infty} U_0(-\tau)F(u(\tau))d\tau. \tag{2.21}$$

Then $\varphi_+ \in X$ by Proposition 2.3, and we have noting (II),

$$U_0(t)\varphi_+ = u(t) + \int_t^{\infty} U_0(t-\tau)F(u(\tau))d\tau.$$

Thus, letting $t \longrightarrow +\infty$, we obtain (2.11). □