NONLINEAR FREE VIBRATION OF CARBON NANOTUBE REINFORCED COMPOSITE BEAMS BASED ON THE THIRD ORDER BEAM THEORY

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ABSTRACT

This paper investigates the linear and nonlinear free vibration of nanocomposite beams reinforced by single-walled carbon nanotubes (SWCNTs) aligned in the beam axial direction. Two types of CNT distributions over the beam cross section are considered, i.e. functionally graded type Λ and V distributions. The third order shear deformable beam theory is employed and the geometric nonlinearity of von Kármán sense is included in this study. The energy functional of CNT composite beams is obtained using the classical variational method of Hamilton's principle and then is solved by the p-Ritz method. Nonlinear vibration characteristics of the CNT beams are presented and the effects of CNT filler volume fraction, distribution, beam length to depth ratio and end support conditions on nonlinear to linear frequency ratios are discussed. It is found that there are significant differences between the results using the first order beam theory and the results using the third order beam theory when both ends of the beam are soft-clamped.

Keywords: First order beam theory, third order beam theory, functionally graded materials, nanocomposites, Ritz method.

1. INTRODUCTION

Carbon nanotubes possess extraordinary mechanical, thermal and electrical properties. There have been tremendous research efforts on the potential use of CNT reinforced composites, and progress has been made in various fields (Coleman et al. 2006). New multifunctional materials or "smart materials" have emerged from research labs as a result of the overall development of CNTs composites. Unlike their micron scale counterparts, the physical properties of nanocomposites can be altered at an extremely low weight percentage of nanofillers. While for a given volume fraction, the interfacial regions between nanoparticles and matrix are significantly higher (Ma and Kim 2011).

With the rapid progress of advanced manufacturing techniques, especially the technique of controllable growth of aligned CNTs forest within nanoscale, vertically aligned CNTs arrays can now be fabricated in a large scale at a reasonable low cost. It's been reported that the super-aligned

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CNT arrays has been produced by continuous spinning of the normal aligned CNTs arrays (Di et al. 2012).

Recently, Shen (Shen 2009) incorporated the concept of functionally graded materials (FGM) to the nanocomposites by varying the volume fraction of aligned CNTs along the thickness direction of plates. It can be foreseen by this mean the final structural element will have its unique properties that can be tailored in terms of vibration control, durability, electrical or thermal conductivities. Researches on the bending, buckling, linear and nonlinear vibration of CNT composite plates and beams have appeared in recent years (Ke et al. 2010, Rafiee et al. 2013, Shen and Xiang 2012, Yas and Samadi 2012) and reported the influence of CNT distributions on the mechanical properties of the CNT structures. However, it is apparent that no study exists in the open literature on the nonlinear vibration analysis of CNT composite beams based on the third order beam theory and this paper aims to fill in this gap.

The mathematic modelling based on the third order beam theory is employed in this study. The geometric nonlinearity of the von Kármán sense is also included. Applying the Hamilton's principle, the virtual energy functional of CNT composite beams is obtained. The p-Ritz method (Liew et al. 1998) is ultilised to obtain the eigenvalue equation for the CNT composite beams. The direct iteration approach is used to search for the nonlinear frequencies at a given vibration amplitude. The influences of CNT distribution, volume fraction, beam boundary conditions and beam length to depth ratios on the nonlinear vibration characteristics of the beams are discussed. The results based on the third order beam theory are compared with the ones based on the first order beam theory and some interesting findings are observed.

2. MATHEMATICAL FORMULATION

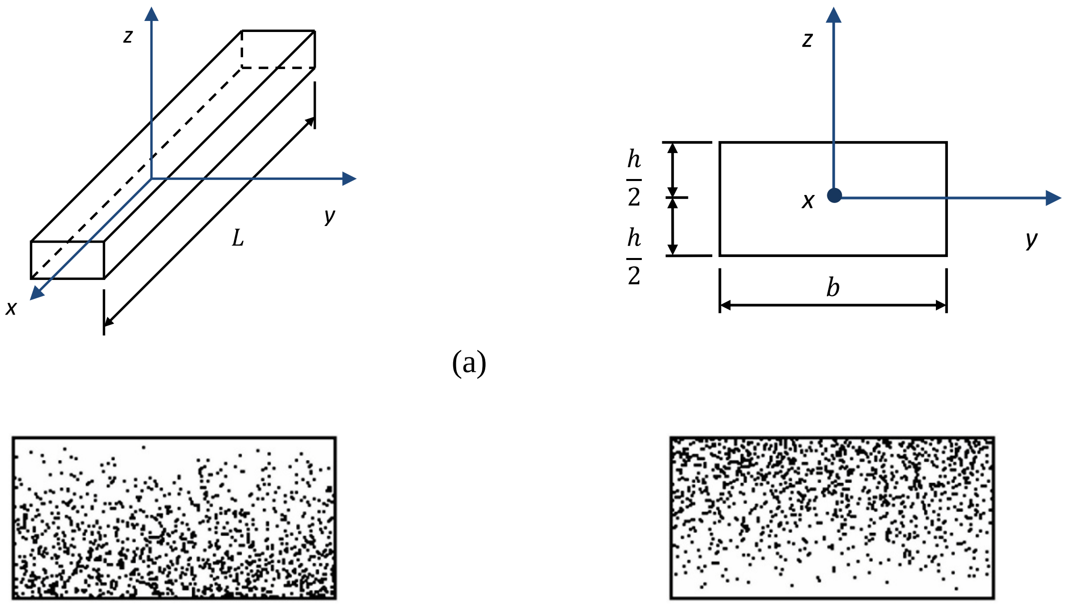
2.1. Problem definition

As shown in Fig. 1(a), consider a CNT composite beam system of length *L*, width *b*, thickness *h* along *x*, *y* and *z* directions, respectively. Two types of aligned CNT reinforced beams are considered, namely the functionally graded CNT beams type Λ (FGΛ-CNT) which features CNT concentrated at bottom region, and the functionally graded CNT beams type V (FGV-CNT) which features CNT concentrated at top region, as shown in Fig. 1(b). The reinforcing CNT alignments are along the length direction, which is the *x* axis.

For FGΛ-and FGV-CNT beams, the CNT distributions along the thickness of the beam follow the power laws. However from the practical perspective, only linear distribution is considered in the current study.

Therefore the distribution of FGΛ-CNT as a function of *z* coordinate is

$$f_{(z)} = \left(\frac{h-2z}{2h}\right) \qquad \left(-\frac{h}{2} \le z \le \frac{h}{2}\right) \tag{1a}$$



(a)

FGΛ-CNT FGV-CNT

(b)

Figure 1: Functionally graded carbon nanotube reinforced composite beams: (a) Coordinate system of beam and (b) Cross sections of FGΛ- and FGV-CNT beams.

and the distribution of FGV-CNT is

$$f_{(z)} = \left(\frac{h + 2z}{2h}\right) \qquad \left(-\frac{h}{2} \leq z \leq \frac{h}{2}\right) \tag{1b}$$

The effective Young's modulus and shear modulus of the CNT composite beams are determined based on matching the results from the molecular dynamics simulation with the extended rule of mixture. The expressions are as follows (Ke et al. 2010, Shen and Xiang 2012)

$$E_{11}(z) = \eta_1 V_{CNT}(z) E_{11}^{cnt} + V_m(z) E^m, \qquad \frac{\eta_2}{E_{22}(z)} = \frac{V_{CNT}(z)}{E_{22}^{cnt}} + \frac{V_m(z)}{E^m}, \qquad \frac{\eta_3}{G_{12}(z)} = \frac{V_{CNT}(z)}{G_{12}^{cnt}} + \frac{V_m(z)}{G^m},$$

$$\nu_{12}(z) = V_{CNT}(z)\nu_{12}^{cnt} + V_m(z)\nu^m, \qquad \nu_{21}(z) = \frac{\nu_{12}(z)}{E_{11}(z)} E_{22}(z), \qquad \rho_{(z)} = V_{CNT}(z)\rho^{cnt} + V_m(z)\rho^m,$$

$$V_m(z) = 1 - V_{CNT}(z) \tag{2a)-(2g}$$

where $E_{11}^{cnt}, E_{22}^{cnt}, G_{12}^{cnt}, E^m$ and G^m are Young's modulus and shear modulus of CNT and matrix, η_1, η_2 and η_3 are CNT/matrix efficiency parameters, ν_{12}^{cnt} and ν^m are Poisson's ratios of CNT and matrix, and ρ^{cnt} and ρ^m are mass densities of CNT and matrix, respectively.

2.2. Variational formulation

The displacements along the *x* and *z* directions in the third order beam theory are given by (Reddy 2004)

$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t) - \frac{4}{3h^2}z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \tag{3a}$$

$$w(x,y,z,t) = w_0(x,y,t) \tag{3b}$$

in which u_0 and w_0 are the displacements along the x and z directions in the mid plane of the beam, and ϕ_x is the rotation of the cross section. The nonlinear virtual energy functional of von Kármán sense for the CNT composite beam is given by

$$\delta U - \delta K = \int_{\Omega_0} \Big\{ N_{xx}\left(\frac{\partial \delta u_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x}\right) + M_{xx}\frac{\partial \delta \phi_x}{\partial x} - c_1 P_{xx}\left(\frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2}\right) + Q_{xz}\left(\delta\phi_x + \frac{\partial \delta w_0}{\partial x}\right) - -c_2 R_{xz}\left(\delta\phi_x + \frac{\partial \delta w_0}{\partial x}\right) - \Big\{\delta u_0\left[I_0\ddot{u}_0 + I_1\ddot{\phi}_x - c_1 I_3\left(\ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x}\right)\right] + \delta\phi_x\Big[I_1\ddot{u}_0 + I_2\ddot{\phi}_x - c_1\Big(I_3\ddot{u}_0 + 2I_4\ddot{\phi}_x + I_4\frac{\partial \ddot{w}_0}{\partial x}\Big) + c_1^2 I_6\left(\ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x}\right)\Big] + \frac{\partial \delta w_0}{\partial x}\left[-c_1 I_3\ddot{u}_0 - c_1 I_4\ddot{\phi}_x + c_1^2 I_6\left(\ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x}\right)\right] + I_0\ddot{w}_0\delta w_0\Big\}\Big\}\,dx \tag{4}$$

in which the definitions of the internal force terms (N, M, P, Q, R), the inertia terms (I) and the parameters c_1 and c_2 can be found in Reddy (Reddy 2004). The third order shear deformation theory can be reduced to first order shear deformation theory by setting the third order shear stiffness and inertia terms to zero, meanwhile accounts for the shear correction factor (5/6).

2.3. Ritz method

The p-Ritz method (Liew et al. 1998) is employed to derived the nonlinear eigenvalue equation for the CNT composite beams. Three end support conditions are considered in this study, i.e. (1) Pin (P) end with $u_0 = 0$, $w_0 = 0$; (2) Soft-Clamped (SC) with $u_0 = 0$, $w_0 = 0$, $\phi_x = 0$; and (3) Hard-Clamped (HC) with $u_0 = 0$, $w_0 = 0$, $dw_0/dx = 0$, $\phi_x = 0$, respectively. The p-Ritz basic functions (Liew et al. 1998) are used to impose the boundary conditions. The number of Ritz polynomial terms is set to be 15 for all subsequent studies.

The direct iteration approach is applied to search for the nonlinear fundamental frequency of the beam at a given downward vibration amplitude. It is noted that for a FGΛ- or FGV-CNT beam in a vibration cycle, the maximum downward and upward amplitudes are different due to the unsymmetirc distributions of CNTs. The upward amplitude is determined through equating the energy of the beam for the downward half vibration cycle and the upward half vibration cycle. Detailed procedure can be referred to (Ke et al. 2010).

3. NUMERICAL RESULTS AND DISCUSSION

The following material constants are used in this study unless otherwise stated (Shen and Xiang 2012): E_{11}^{cnt} = 5646.6 GPa, E_{22}^{cnt} = 7080 GPa, G_{12}^{cnt} = 1944.5 GPa, E^m = 2.5 GPa, ν_{12}^{cnt} = 0.175, ν^m = 0.3, ρ^{cnt} = 2100 kg/m^3, and ρ^m = 1190 kg/m^3, respectively. By matching with the results from molecular dynamics simulations (Griebel and Hamaekers 2004, Han and Elliott 2007), the CNT efficiency parameters are given in Table 1 (Shen and Xiang 2012).

Table 1: CNT efficiency parameters

CNT efficiency parameters	V_{tcnt}		
	0.12	0.17	0.28
η_1	0.137	0.142	0.141
η_2	1.022	1.626	1.585
η_3	0.715	1.138	1.109

3.1. Analysis of first and third order beam solutions

Table 2 lists the linear fundamental frequency parameters ω_l and the nonlinear to linear fundamental frequency parameter ratios ω_{nl}/ω_l of FGΛ-CNT and FGV-CNT beams with different end support conditions based on the first order and the third order beam theories. The beam *L/h* ratio is set to be 10 and the total CNT volume fraction is 0.12. The nondimensional vibration parameter is of the form of $\omega_l = \Omega L^2\sqrt{\rho_m/(E_m h^2)}$, where Ω is the angular frequency of the beam. The maximum nondimensional amplitude is defined as $W_{max} = w_{max}/h$, where w_{max} is the maximum downward displacement in the beam.

Table 2: Nonlinear fundamental frequency parameter ratio ω_{nl}/ω_l

Boundary	Theory	ω_l	Beam type	W_{max} 0.1	0.2	0.3	0.4	0.5
P-P	3rd order	11.1601	FGΛ	1.0102	1.0778	1.2167	1.3925	1.5858
			FGV	1.0066	1.0379	1.1077	1.2188	1.3707
	1st order	11.1295	FGΛ	1.0104	1.0786	1.2182	1.3943	1.5888
			FGV	1.0067	1.0381	1.1087	1.2205	1.3666
SC-SC	3rd order	11.9480	FGΛ	1.0253	1.1086	1.2420	1.4072	1.5924
			FGV	1.0217	1.0862	1.1932	1.3373	1.5082
	1st order	13.8612	FGΛ	1.0212	1.0823	1.1777	1.3002	1.4431
			FGV	1.0212	1.0823	1.1778	1.3002	1.4432
HC-HC	3rd order	14.8064	FGΛ	1.0206	1.0803	1.1719	1.2901	1.4264
			FGV	1.0206	1.0800	1.1721	1.2901	1.4256
	1st order	14.0623	FGΛ	1.0222	1.0860	1.1848	1.3116	1.4588
			FGV	1.0222	1.0860	1.1851	1.3116	1.4585

We observe in Table 2 that firstly, for the linear fundamental frequency parameters ω_l, there are negligible differences between the results of the first order and third order theories for the beam with P-P boundary conditions. However the first order result is substantially larger than the corresponding third order one for the beam with SC-SC boundary conditions by about 16%. On the other hand, the first order linear frequency parameter for the HC-HC CNT beam is smaller than the corresponding one based on the third order beam theory by about 5.1%.

Secondly, for the nonlinear to linear frequency ratios ω_{nl}/ω_l, there are no significant differences between the first and third order results for the beams with P-P boundary conditions. But for beams with SC-SC boundary conditions, the ratios obtained from the third order theory are larger than the results from the first order theory. As for beams with HC-HC boundary conditions, the results based on the third order theory are slightly smaller than the ones from the first order theory.

Thirdly, we observe the influence of carbon nanotube distributions on the nonlinear vibration of the beams. For HC-HC beams, the nonlinear to linear frequency ratios ω_{nl}/ω_l are almost identical for FGΛ and FGV distributions based on the first order or the third order theory. For the P-P case, the nonlinear to linear frequency ratios ω_{nl}/ω_l show moderate differences for FGΛ and FGV distributions when either the first order or the third order theory is used. However, for SC-SC beams, the results based on the first order theory are almost identical, while the results based on the third order theory show moderate discrepancies for FGΛ and FGV distributions.

3.2. Parametric studies

The following parametric studies are all based on the third order beam theory. Table 3 shows the nonlinear to linear fundamental frequency parameter ratio ω_{nl}/ω_l of FGΛ-CNT beams with different length to depth ratios *L/h*. The boundary conditions are SC-SC and the total CNT volume fraction is 0.17. The results show that for given amplitude, the nonlinear to linear frequency ratio decreases as the *L/h* ratio increases.

Table 3: Nonlinear to linear fundamental frequency parameter ratio ω_{nl}/ω_l for FGΛ-CNT beams with different length to depth ratio *L/h* (SC-SC, V_{tcnt}= 0.17)

		W_{max}				
L/h	ω_l	0.1	0.2	0.3	0.4	0.5
10	14.9585	1.0228	1.0977	1.2186	1.3711	1.5425
20	23.2781	1.0101	1.0414	1.0936	1.1642	1.2493
30	28.1484	1.0073	1.0292	1.0656	1.1151	1.1757
40	31.0400	1.0063	1.0250	1.0559	1.0979	1.1496

Figure 2 presents the nonlinear to linear fundamental frequency parameter ratio ω_{nl}/ω_l against the nondimensional maximum amplitude for FGΛ- and FGV-CNT beams. For both beams, the length to depth ratio *L/h* is 10, and with total volume fraction being 0.17. Results show that for given downward amplitude, the frequency parameters ratio ω_{nl}/ω_l is symmetrical for FGΛ- and FGV-CNT beams with HC-HC boundary conditions, but unsymmetrical for FGΛ- and FGV-CNT beams with P-P and SC-SC boundary conditions.

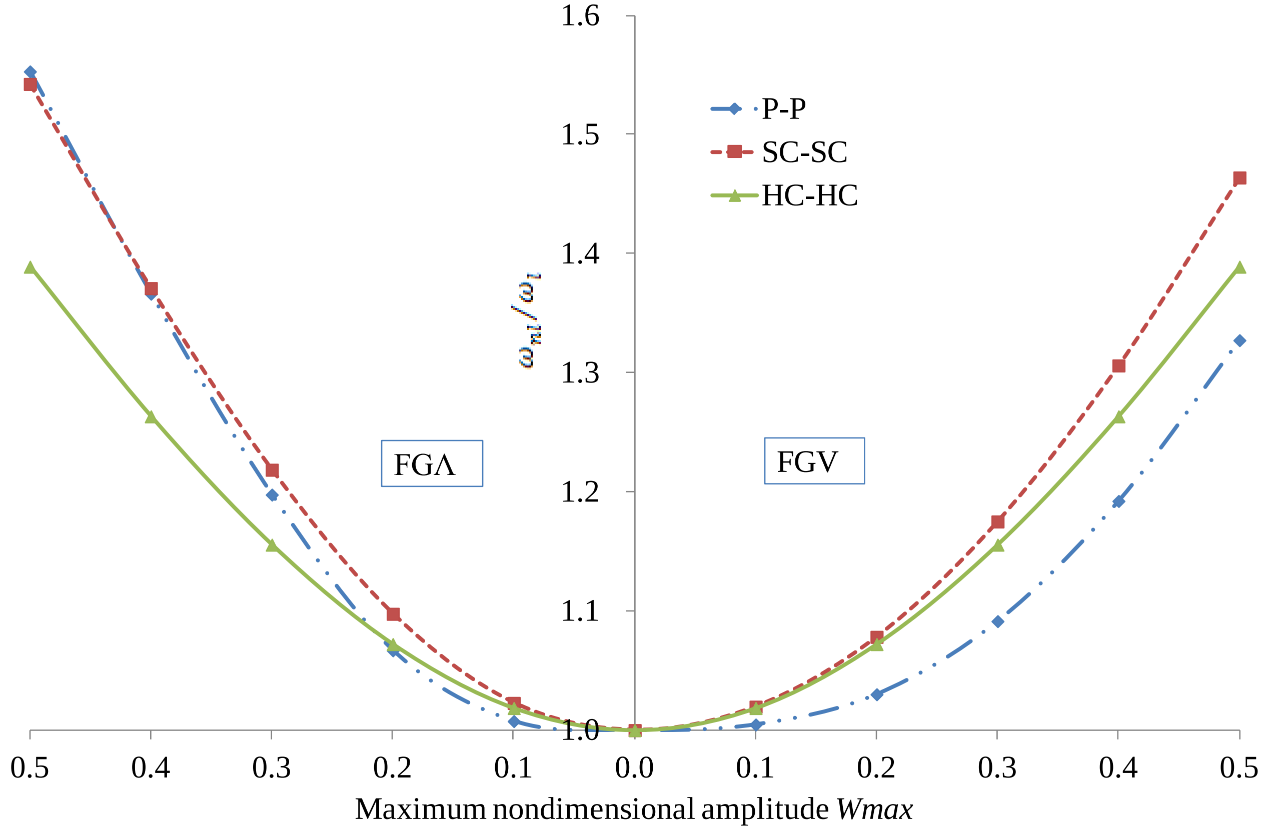


Figure 2: Nonlinear to linear frequency ratios ω_{nl}/ω_l versus nondimensional downward maximum amplitude W_{max} for CNT beams with FGΛ and FGV distributions (*L/h* = 10, V_{tcnt}= 0.17).

Table 4 lists the results of nonlinear to linear fundamental frequency parameter ratio ω_{nl}/ω_l of the FGΛ- and FGV-CNT beams. The *L/h* is 10 and total CNT volume fraction is 0.28. Similarly, it can be seen that the results of FGΛ- and FGV-CNT beams are symmetrical for HC-HC boundary conditions but unsymmetrical for P-P and SC-SC boundary conditions.

Table 4: Nonlinear to linear fundamental frequency parameter ratio ω_{nl}/ω_l for FGΛ- and FGV-CNT beams with CNT volume fraction V_{tcnt}= 0.28

Boundary	Theory	ω_l	Beam type	W_{max}				
				0.1	0.2	0.3	0.4	0.5
P-P	3rd order	15.4046	FGΛ	1.0127	1.0896	1.2401	1.4262	1.6303
			FGV	1.0083	1.0450	1.1248	1.2482	1.4056
SC-SC	3rd order	16.1615	FGΛ	1.0289	1.1249	1.2767	1.4615	1.6674
			FGV	1.0243	1.0972	1.2187	1.3810	1.5725
HC-HC	3rd order	20.0601	FGΛ	1.0240	1.0927	1.1978	1.3309	1.4840
			FGV	1.0240	1.0927	1.1978	1.3308	1.4837

4. CONCLUSIONS

This paper presents a numerical study on the nonlinear vibration of functionally graded (FG) aligned carbon nanotube (CNT) reinforced composite beams. FGΛ- and FGV-CNT beams are considered in the current study. Employing the third order beam theory in associate with the geometric nonlinearity of von Kármán sense, the nonlinear eigenvalue equation for the CNT beams is derived using the p-Ritz method. The differences of the nonlinear to linear frequency ratios based on the first and third order beam theories are discussed in details for beams with various boundary conditions. The influence of the beam boundary conditions, beam length to depth ratios, CNT volume fractions and CNT distributions on the nonlinear vibration characteristics of the CNT beams are investigated.

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